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ON THE DISCRETE HILBERT TRANSFORM

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CHAPTER I

INTRODUCTION

In almost every field where Fourier techniques are used to represent and analyze physical processes, one finds that there are situations where there exist relationships between the real and imaginary parts or the magnitude and phase of the Fourier Transform. These relationships are known by different names, depending upon the field of interest, but often they are called Hilbert transform relations. In this respect, the field of digital signal processing is no exception.

This report concerns the discrete form of the Hilbert transform. It is known as the discrete Hilbert transform, and is conveniently represented in terms of a matrix equation. Our main objective is to implement the discrete Hilbert transform on a minicomputer system, and illustrate its frequency characteristics via examples. To this end, experimental results related to the following type of discrete Hilbert transform are included: lowpass, bandpass, highpass, and bandstop.

CHAPTER II

THE DISCRETE HILBERT TRANSFORM

2.1 Introduction

In this chapter, a development of the discrete Hilbert transform (DHT) is presented. It is shown that the DHT can be expressed in the form of a matrix equation.

2.2 Derivation of the DHT

The Hilbert transform (HT) of a continuous signal f(t) is defined as [3]:

$$g(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - t} d\tau \qquad (2\text{-}1)$$

where g(t) denotes the HT.

An alternate form of writing (2-1) is:

$$g(t) = \frac{1}{\pi t} * f(t) \qquad (2\text{-}2)$$

where the symbol "*" denotes convolution. Thus, the HT of f(t) can be interpreted as a convolution between f(t) and $1/\pi t$ to produce the desired signal g(t).

Taking the Fourier transform (FT) of both sides of (2-2) yields:

$$G(jw) = H(jw)\ F(jw). \qquad (2\text{-}3)$$

In (2-3), F(jw) and G(jw) denote the Fourier transforms of f(t) and g(t), respectively, and

$$H(jw) = FT(1/\pi t) = -j\ \mathrm{sgn}(w) \qquad (2\text{-}4)$$

with

$$\operatorname{sgn}(w) = \begin{cases} 1, & w > 0 \\ 0, & w = 0 \\ -1, & w < 0 \end{cases}$$

We now consider the case when the input is a discrete signal. Then corresponding to the input f(t), we have a data sequence fn, n = 0, 1, 2, ..., (N-1). Again, corresponding to H(jw) in (2-4), one has a sequence of complex numbers, H_k, k = 0, 1, 2, ... (N-1). We denote the resulting DHT sequence by g_n, n = 0, 1, ..., (N-1). In what follows, the cases when N is even or odd, one is treated separately.

Case 1: N is even

Equation (2-4) yields

$$H_k = \begin{cases} -j, & k = 1, 2, \ldots, (N/2-1) \\ 0, & k = N/2, 0. \\ j, & k = (N/2 + 1), (N/2 + 2), \ldots, (N-1). \end{cases} \tag{2-5}$$

Next, let F_k and G_k, k = 0, 1, ..., (N-1) denote the discrete Fourier transform (DFT) coefficients of the input data sequence {fn} i.e.,

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n W^{-nk}, \quad k = 0, 1, \ldots, (N-1) \tag{2-6}$$

and

$$G_k = \frac{1}{N} \sum_{n=0}^{N-1} g_n W^{-nk}, \quad k = 0, 1, \ldots, (N-1) \tag{2-7}$$

where $W = e^{-j2\pi/N}$.

Thus, corresponding to (2-3) we have

$$G_k = H_k F_k, \quad k = 0, 1, \ldots, (N-1). \tag{2-8}$$

Substituting (2-6) in (2-8) and then taking the inverse DFT, one obtains

$$g_n = \frac{1}{N} \sum_{k=0}^{N-1} H_k \sum_{m=0}^{N-1} f_m W^{km} W^{-kn}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_m \sum_{k=0}^{N-1} H_k W^{(m-n)k}, \quad n=0, 1, \ldots, (N-1) \tag{2-9}$$

Substitution for the H_k in (2-9) using (2-5) results in

$$g_n = \frac{-j}{N} \sum_{m=0}^{N-1} f_m [1-(-1)^{n-m}] \left\{ \sum_{k=1}^{\frac{N-1}{2}} e^{-jk(n-m)\ 2\pi/N} \right\}, \tag{2-10}$$

for n = 0, 1, 2, ..., (N-1).

In (2-10) we use the result that

$$\sum_{k=1}^{\frac{N-1}{2}} e^{jk(n-m)\ 2\pi/N} = j\cot\ [(m-n)\pi/N] \tag{2-11}$$

to obtain

$$g_n = \frac{1}{N} \sum_{m=0}^{N-1} f_m [1-(-1)^{n-m}] \cot\ [(m-n)\pi/N], \tag{2-12}$$

for n = 0, 1, ..., (N-1).

A more convenient way of writing (2-12) is as follows:

$$g_n = \frac{1}{N} \sum_{m=0}^{N-1} f_m c_{n,m}, \quad n=0, 1, \ldots, (N-1) \tag{2-13}$$

where

$$c_{n,m} = \begin{cases} 0, & n=m, \text{ and } (n-m) \text{ even} \\ 2 \cot\ [(n-m)\pi/N], & (n-m) \text{ odd.} \end{cases}$$

Case 2: N is odd

Here (2-4) yields

$$H_k = \begin{cases} -j, & k = 1, 2, \ldots (N-1)/2 \\ 0, & k = 0 \\ j, & k = (N+1)/2, (N+3)/2, \ldots, (N-1). \end{cases} \quad (2\text{-}14)$$

Proceeding in a manner similar to case 1, we obtain the following result

$$g_n = \frac{1}{N} \sum_{m=0}^{N-1} f_m \, c_{n,m}, \quad n = 0, 1, \ldots, (N-1) \quad (2\text{-}15)$$

where

$$c_{n-m} = \begin{cases} \cot\,(n-m)\pi/N - \dfrac{(-1)^{n-m}}{\sin\,[(n-m)\pi/N]}, & n \neq m \\ 0, & n = m \end{cases}$$

Equations (2-13) and (2-15) are defined as the DHT of a given data sequence f_n, $n = 0, 1, \ldots, (N-1)$, depending upon whether N is even or odd, respectively.

2.3 Matrix Representation

Either form of the DHT in (2-13) and (2-15) can be written in terms of a matrix notation as follows:

$$\underline{G}_N = \underline{H}_N \, \underline{F}_N$$

where:

$\underline{F}'_N = [f_0, f_1, \ldots, f_{N-1}]$ is the input data vector, prime denoting transpose;

$\underline{G}'_N = [g_0, g_1, \ldots, g_{N-1}]$ is the corresponding DHT vector, and

$\underline{H}_N$ is an (NXN) DHT matrix.

To illustrate, consider the case when N is even. Then (2-13) yields the following DHT matrix:

$$\underline{H}_N = \begin{bmatrix} 0 & -c_1 & 0 & -c_3 & 0 & -c_5 & \cdots & -c_{N-1} \\ c_1 & 0 & -c_1 & 0 & -c_3 & 0 & \cdots & 0 \\ 0 & c_1 & 0 & -c_1 & 0 & -c_3 & \cdots & -c_{N-3} \\ & & \cdot & & \cdot & & & \\ & & \cdot & & \cdot & & & \\ c_{N-1} & 0 & c_{N-3} & 0 & c_{N-5} & 0 & & 0 \end{bmatrix} \qquad (2\text{-}17)$$

where $c_k = 2 \cot (K\pi/N)$.

An examination of (2-17) reveals the following properties of the DHT matrix:

1) It is a square circulant matrix of order N, with only P distinct elements, (ignoring differences in sign), where

$$P = \text{integer value of } N/4. \qquad (2\text{-}18)$$

2) It is a skew symmetric matrix.

3) Because of 1) and 2) above, only the first N/2 elements of the first row are sufficient to construct the entire matrix.

One may verify that properties very similar to the above are valid for the case when N is odd, by starting with (2-15).