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Implicit Function Theorem via the DSM

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Abstract

Sufficient conditions are given for an implicit function theorem to hold. The result is established by an application of the Dynamical Systems Method (DSM). It allows one to solve a class of nonlinear operator equations in the case when the Fréchet derivative of the nonlinear operator is a smoothing operator, so that its inverse is an unbounded operator.

MSC: 47J05, 47J25,
Key words: Dynamical Systems Method (DSM); Hard implicit function theorem; Newton's method

1 Introduction

The aim of this paper is to demonstrate the power of the Dynamical Systems Method (DSM) as a tool for proving theoretical results. The DSM was systematically developed in [6] and applied to solving nonlinear operator equations in [6] (see also [7]), where the emphasis was on convergence and stability of the DSM-based algorithms for solving operator equations, especially nonlinear and ill-posed equations. The DSM for solving an operator equation $F(u) = h$ consists of finding a nonlinear map $u \mapsto \Phi(t, u)$, depending on a parameter $t \in [0, \infty)$, that has the following three properties:

(1) the Cauchy problem

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0 \quad (\dot{u} := \frac{du(t)}{dt})$$

has a unique global solution $u(t)$ for a given initial approximation u_0;

(2) the limit $u(\infty) = \lim_{t\to\infty} u(t)$ exists; and

(3) this limit solves the original equation $F(u) = h$, i.e., $F(u(\infty)) = h$.

The operator $F : H \to H$ is a nonlinear map in a Hilbert space H. It is assumed that the equation $F(u) = h$ has a solution, possibly nonunique.

The problem is to find a Φ such that the properties (1), (2), and (3) hold. Various choices of Φ for which these properties hold are proposed in [6], where the DSM is justified for wide classes of operator equations, in particular, for some classes of nonlinear ill-posed equations (i.e., equations $F(u) = 0$ for which the linear operator $F'(u)$ is not boundedly invertible). By $F'(u)$ we denote the Fréchet derivative of the nonlinear map F at the element u.

In this note the DSM is used as a tool for proving a "hard" implicit function theorem.

Let us first recall the usual implicit function theorem. Let U solve the equation $F(U) = f$.

Proposition: If $F(U) = f$, F is a C^1-map in a Hilbert space H, and $F'(U)$ is a boundedly invertible operator, i.e., $\|[F'(U)]^{-1}]\| \leq m$, then the equation

$$F(u) = h \tag{1.1}$$

is uniquely solvable for every h sufficiently close to f.

For convenience of the reader we include a proof of this known result.

Proof of the Proposition. First, one can reduce the problem to the case $u = 0$ and $h = 0$. This is done as follows. Let $u = U + z$, $h - f = p$, $F(U + z) - F(U) := \phi(z)$. Then $\phi(0) = 0$, $\phi'(0) = F'(U)$, and equation (1.1) is equivalent to the equation

$$\phi(z) = p, \tag{1.2}$$

with the assumptions

$$\phi(0) = 0, \quad \lim_{z\to 0} \|\phi'(z) - \phi'(0)\| = 0, \quad \|[\phi'(0)]^{-1}\| \leq m. \tag{1.3}$$

We want to prove that equation (1.2) under the assumptions (1.3) has a unique solution $z = z(p)$, such that $z(0) = 0$, and $\lim_{p\to 0} z(p) = 0$. To prove this, consider the equation

$$z = z - [\phi'(0)]^{-1}(\phi(z) - p) := B(z), \tag{1.4}$$

and check that the operator B is a contraction in a ball $\mathcal{B}_\epsilon := \{z : \ \|z\| \leq \epsilon\}$ if $\epsilon > 0$ is sufficiently small, and B maps $\mathcal{B}_\epsilon$ into itself. If this is proved, then the desired result follows from the contraction mapping principle.

One has

$$\|B(z)\| = \|z - [\phi'(0)]^{-1}(\phi'(0)z + \eta - p)\| \leq m\|\eta\| + m\|p\|, \tag{1.5}$$

where $\|\eta\| = o(\|z\|)$. If ϵ is so small that $m\|\eta\| < \frac{\epsilon}{2}$ and p is so small that $m\|p\| < \frac{\epsilon}{2}$, then $\|B(z)\| < \epsilon$, so $B : \mathcal{B}_\epsilon \to \mathcal{B}_\epsilon$.

Let us check that B is a contraction mapping in $\mathcal{B}_\epsilon$. One has:

$$\begin{aligned} \|Bz - By\| &= \|z - y - [\phi'(0)]^{-1}(\phi(z) - \phi(y))\| \\ &= \|z - y - [\phi'(0)]^{-1} \int_0^1 \phi'(y + t(z - y))dt(z - y)\| \\ &\leq m \int_0^1 \|\phi'(y + t(z - y)) - \phi'(0)\|dt\|z - y\|. \end{aligned} \tag{1.6}$$

If $y, z \in \mathcal{B}_\epsilon$, then

$$\sup_{0 \le t \le 1} \|\phi'(y + t(z - y)) - \phi'(0)\| = o(1), \qquad \epsilon \to 0.$$

Therefore, if ϵ is so small that $mo(1) < 1$, then B is a contraction mapping in $\mathcal{B}_\epsilon$, and equation (1.2) has a unique solution $z = z(p)$ in $\mathcal{B}_\epsilon$, such that $z(0) = 0$. The proof is complete. $\Box$

The crucial assumptions, on which this proof is based, are assumptions (1.3).

Suppose now that $\phi'(0)$ *is not boundedly invertible, so that the last assumption in* (1.3) *is not valid.* Then a theorem which still guarantees the existence of a solution to equation (1.2) for some set of p is called a "hard" implicit function theorem. Examples of such theorems one may find, e.g., in [1], [2], [3], and [4].

Our goal in this paper is to establish a new theorem of this type using a new method of proof, based on the Dynamical Systems Method (DSM). In [8] we have demonstrated a theoretical application of the DSM by establishing some surjectivity results for nonlinear operators.

The result, presented in this paper, is a new illustration of the applicability of the DSM as a tool for proving theoretical results.

To formulate the result, let us introduce the notion of a scale of Hilbert spaces H_a (see [5]). Let $H_a \subset H_b$ and $\|u\|_b \le \|u\|_a$ if $a \ge b$. Example of spaces H_a is the scale of Sobolev spaces $H_a = W^{a,2}(D)$, where $D \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary.

Consider equation (1.1). Assume that

$$F(U) = f; \qquad F : H_a \to H_{a+\delta}, \qquad u \in B(U, R) := B_a(U, R), \tag{1.7}$$

where $B_a(U, R) := \{u \ : \ \|u - U\|_a \le R\}$ and $\delta = \mathit{const} > 0$, and the operator $F : H_a \to H_{a+\delta}$ is continuous. Furthermore, assume that $A := A(u) := F'(u)$ exists and is an isomorphism of H_a onto $H_{a+\delta}$:

$$c_0\|v\|_a \le \|A(u)v\|_{a+\delta} \le c_0'\|v\|_a, \qquad u, v \in B(U, R), \tag{1.8}$$

that

$$\|A^{-1}(v)A(w)\|_a \le c, \qquad v, w \in B(U, R), \tag{1.9}$$

and

$$\|A^{-1}(u)[A(u) - A(v)]\|_a \le c||u - v||_a, \qquad u, v \in B(U, R). \tag{1.10}$$

Here and below we denote by $c > 0$ various constants. Note that (1.8) implies

$$||A^{-1}(u)\psi||_a \le c_0^{-1}||\psi||_{a+\delta}, \qquad \psi = A(u)[F(v) - h], \quad v \in B(U, R).$$

Assumption (1.8) implies that $A(u)$ is a smoothing operator similar to a smoothing integral operator, and its inverse is similar to the differentiation operator of order $\delta > 0$. Therefore, the operator $A^{-1}(u) = [F'(u)]^{-1}$ causes the "loss of the derivatives". In general, this may lead to a breakdown of the Newton process (method) in a finitely many steps. Our assumptions (1.7)-(1.10) guarantee that this will not happen.

Assume that

$$u_0 \in B_a(U, \rho), \qquad h \in B_{a+\delta}(f, \rho), \tag{1.11}$$

where $\rho > 0$ is a sufficiently small number:

$$\rho \leq \rho_0 := \frac{R}{1 + c_0^{-1}(1 + c_0')},$$

and c_0, c_0' are the constants from (1.8). Then $F(u_0) \in B_{a+\delta}(f, c_0'\rho)$, because $||F(u_0) - F(U)|| \leq c_0'||u_0 - U|| \leq c_0'\rho$.

Consider the problem

$$\dot{u} = -[F'(u)]^{-1}(F(u) - h), \quad u(0) = u_0. \tag{1.12}$$

Our basic result is:

Theorem 1.1. *If the assumptions* (1.7)-(1.11) *hold, and* $0 < \rho \leq \rho_0 := \frac{R}{1+c_0^{-1}(1+c_0')}$, *where* c_0, c_0' *are the constants from* (1.8), *then problem* (1.12) *has a unique global solution* $u(t)$, *there exists* $V := u(\infty)$,

$$\lim_{t\to\infty} \|u(t) - V\|_a = 0, \tag{1.13}$$

and

$$F(V) = h. \tag{1.14}$$

Theorem 1.1 says that if $F(U) = f$ and $\rho \leq \rho_0$, then for any $h \in B_{a+\delta}(f, \rho)$ equation (1.1) is solvable and a solution to (1.1) is $u(\infty)$, where $u(\infty)$ solves problem (1.12).

In Section 2 we prove Theorem 1.1.

2 Proof

Let us outline the ideas of the proof. The local existence and uniqueness of the solution to (1.12) will be established if one verifies that the operator $A^{-1}(u)[F(u) - h]$ is locally Lipschitz in H_a. The global existence of this solution $u(t)$ will be established if one proves the uniform boundedness of $u(t)$:

$$\sup_{t\geq 0} \|u(t)\|_a \leq c. \tag{2.1}$$

Let us first prove (in paragraph a) below) estimate (2.1), the existence of $u(\infty)$, and the relation (1.14), assuming the local existence of the solution to (1.12).

In paragraph b) below the local existence of the solution to (1.12) is proved.

a) If $u(t)$ exists locally, then the function

$$g(t) := \|\phi\|_{a+\delta} := \|F(u(t)) - h\|_{a+\delta} \tag{2.2}$$

satisfies the relation

$$g\dot{g} = (F'(u(t))\dot{u}, \phi)_{a+\delta} = -g^2, \tag{2.3}$$

where equation (1.12) was used. Since $g \geq 0$, it follows from (2.3) that

$$g(t) \leq g(0)e^{-t}, \qquad g(0) = \|F(u_0) - h\|_{a+\delta}. \tag{2.4}$$

From (1.12), (2.3) and (1.8) one gets:

$$\|\dot{u}\|_a \leq \frac{1}{c_0}\|\phi\|_{a+\delta} = \frac{g(0)}{c_0}e^{-t} := re^{-t}, \qquad r := \frac{\|F(u_0) - h\|_{a+\delta}}{c_0}. \tag{2.5}$$

Therefore,

$$\lim_{t\to\infty} \|\dot{u}(t)\|_a = 0, \tag{2.6}$$

and

$$\int_0^\infty \|\dot{u}(t)\|_a dt < \infty. \tag{2.7}$$

This inequality implies

$$||u(\tau) - u(s)|| \leq \int_s^\tau ||\dot{u}(t)dt|| < \epsilon, \qquad \tau > s > s(\epsilon),$$

where $\epsilon > 0$ is an arbitrary small fixed number, and $s(\epsilon)$ is a sufficiently large number. Thus, the limit $V := \lim_{t\to\infty} u(t) := u(\infty)$ exists by the Cauchy criterion, and (1.13) holds. Assumptions (1.7) and (1.8) and relations (1.12), (1.13), and (2.6) imply (1.14).

Integrating inequality (2.5) yields

$$\|u(t) - u_0\|_a \leq r, \tag{2.8}$$

and

$$\|u(t) - u(\infty)\|_a \leq re^{-t}. \tag{2.9}$$

Inequality (2.8) implies (2.1).

b) Let us now prove the local existence of the solution to (1.12).

We prove that the operator in (1.12) $A^{-1}(u)[F(u) - h]$ is locally Lipschitz in H_a. This implies the local existence of the solution to (1.12).

One has

$$\begin{aligned} &\|A^{-1}(u)(F(u) - h) - A^{-1}(v)(F(v) - h)\|_a \leq \|[A^{-1}(u) - A^{-1}(v)](F(u) - h)\|_a \\ &+ \|A^{-1}(v)(F(u) - F(v))\|_a := I_1 + I_2. \end{aligned} \tag{2.10}$$

Write

$$F(u) - F(v) = \int_0^1 A(v + t(u - v))(u - v)dt, \tag{2.11}$$

and use assumption (1.9) with $w = v + t(u - v)$ to conclude that

$$I_2 \le c\|u - v\|_a. \tag{2.12}$$

Write

$$A^{-1}(u) - A^{-1}(v) = A^{-1}(u)[A(v) - A(u)]A^{-1}(v), \tag{2.13}$$

and use the estimate

$$\|A^{-1}(v)[F(u) - h]\|_a \le c, \tag{2.14}$$

which is a consequence of assumptions (1.7) and (1.8). Then use assumption (1.10) to conclude that

$$I_1 \le c\|u - v\|_a. \tag{2.15}$$

From (2.10), (2.12) and (2.15) it follows that the operator $A^{-1}(u)[F(u) - h]$ is locally Lipschitz.

Note that

$$\|u(t) - U\|_a \le \|u(t) - u_0\|_a + \|u_0 - U\|_a \le r + \rho, \tag{2.16}$$

$$\|F(u(t)) - h\|_{a+\delta} \le \|F(u_0) - h\|_{a+\delta} \le \|F(u_0) - f\|_{a+\delta} + \|f - h\|_{a+\delta} \le (1 + c_0')\rho, \tag{2.17}$$

so, from (2.5) one gets

$$r \le \frac{(1 + c_0')\rho}{c_0}. \tag{2.18}$$

Choose

$$R \ge r + \rho. \tag{2.19}$$

Then the trajectory $u(t)$ stays in the ball $B(U, R)$ for all $t \ge 0$, and, therefore, assumptions (1.7)-(1.10) hold in this ball for all $t \ge 0$.

Condition (2.19) and inequality (2.18) imply

$$\rho \le \rho_0 = \frac{R}{1 + c_0^{-1}(1 + c_0')}. \tag{2.20}$$

This is the "smallness" condition on ρ.
Theorem 1.1 is proved. $\square$

3 Example

Let

$$F(u) = \int_0^x u^2(s)ds, \qquad x \in [0, 1].$$

Then

$$A(u)q = 2\int_0^x u(s)q(s)ds.$$

Let $f = x$ and $U = 1$. Then $F(U) = x$. Choose $a = 1$ and $\delta = 1$. Denote by $H_a = H_a(0, 1)$ the usual Sobolev space. Assume that

$$h \in B_2(x, \rho) := \{h \ : \ \|h - x\|_2 \le \rho\},$$

and $\rho > 0$ is sufficiently small. One can verify that

$$A^{-1}(u)\psi = \frac{\psi'(x)}{2u(x)}$$

for any $\psi \in H_1$.

Let us check conditions (1.7)-(1.11) for this example.

Condition (1.7) holds, because if $u_n \to u$ in H_1, then

$$\int_0^x u_n^2(s)ds \to \int_0^x u^2(s)ds$$

in H_2. To verify this, it is sufficient to check that

$$\frac{d^2}{dx^2}\int_0^x u_n^2(s)ds \to 2uu',$$

where $\to$ means the convergence in $H := H_0 := L^2(0,1)$. In turn, this is verified if one checks that $u_n' u_n \to u'u$ in $L^2(0,1)$, provided that $u_n' \to u'$ in $L^2(0,1)$.

One has

$$I_n := \|u_n' u_n - u'u\|_0 \le \|(u_n' - u')u_n\|_0 + \|u'(u_n - u)\|_0.$$

Since $\|u_n'\|_0 \le c$, one concludes that $\|u_n\|_{L^\infty(0,1)} \le c_1$ and $\lim_{n\to\infty} \|u_n - u\|_{L^\infty} = 0$. Thus,

$$\lim_{n\to\infty} I_n = 0.$$

Condition (1.8) holds because $\|u\|_{L^\infty(0,1)} \le c\|u\|_1$, and

$$\|\int_0^x u(s)q(s)ds\|_2 \le c\|u'q + uq'\|_0 \le c(\|q\|_{L^\infty(0,1)}\|u\|_1 + \|u\|_{L^\infty(0,1)}\|q\|_1),$$

so

$$\|\int_0^x u(s)q(s)ds\|_2 \le c_0'\|u\|_1\|q\|_1,$$

and

$$\|\int_0^x uqds\|_2 \ge \|uq\|_1 \ge c_0\|q\|_1,$$

provided that $u \in B_1(1,\rho)$ and $\rho > 0$ is sufficiently small.

Condition (1.9) holds because

$$\|A^{-1}(v)A(w)q\|_1 = \|\frac{1}{v(x)}w(x)q\|_1 \le c\|q\|_1,$$

provided that $u, w \in B_1(1,\rho)$ and $\rho > 0$ is sufficiently small.

Condition (1.10) holds because

$$\|A^{-1}(u)\int_0^x (u-v)qds\|_1 = \|\frac{u-v}{2u}q\|_1 \le c\|u-v\|_1\|q\|_1,$$

provided that $u, v \in B_1(1, \rho)$ and $\rho > 0$ is sufficiently small.

By Theorem 1.1 the equation

$$F(u) := \int_0^x u^2(s)ds = h,$$

where $\|h - x\|_2 \le \rho$ and $\rho > 0$ is sufficiently small, has a solution V,

$$F(V) = h.$$

This solution can be obtained as $u(\infty)$, where $u(t)$ solves problem (1.12) and conditions (1.11) and (2.20) hold.

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