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THE DYNAMICAL SYSTEMS METHOD FOR SOLVING NONLINEAR EQUATIONS WITH MONOTONE OPERATORS

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A review of the authors' results is given. Several methods are discussed for solving nonlinear equations $F(u) = f$, where F is a monotone operator in a Hilbert space, and noisy data are given in place of the exact data. A discrepancy principle for solving the equation is formulated and justified. Various versions of the Dynamical Systems Method (DSM) for solving the equation are formulated. These versions of the DSM include a regularized Newton-type method, a gradient-type method, and a simple iteration method. A priori and a posteriori choices of stopping rules for these methods are proposed and justified. Convergence of the solutions, obtained by these methods, to the minimal norm solution to the equation $F(u) = f$ is proved. Iterative schemes with a posteriori choices of stopping rule corresponding to the proposed DSM are formulated. Convergence of these iterative schemes to a solution to equation $F(u) = f$ is justified. New nonlinear differential inequalities are derived and applied to a study of large-time behavior of solutions to evolution equations. Discrete versions of these inequalities are established.

Keywords: Ill-posed problems, nonlinear operator equations, monotone operators, nonlinear inequalities, Dynamical systems method.

AMS Subject Classification: 47H05, 47J05, 47N20, 65J20, 65M30.

1. Introduction

Consider equation

$$F(u) = f, \tag{1.1}$$

where F is an operator in a Hilbert space H. Throughout this paper we assume that F is a monotone continuous operator. Monotonicity is understood as follows:

$$\langle F(u) - F(v), u - v\rangle \geq 0, \qquad \forall u, v \in H. \tag{1.2}$$

We assume that equation (1.1) has a solution, possibly non-unique. Assume that f is not known but f_δ, the "noisy data", $\|f_\delta - f\| \le \delta$, are known.

There are many practically important problems which are ill-posed in the sense of J. Hadamard. Problem (1.1) is well-posed in the sense of Hadamard if and only if F is injective, surjective, and the inverse operator F^{-1} is continuous. To solve ill-posed problem (1.1), one has to use regularization methods rather than the classical Newton's or Newton-Kantorovich's methods. Regularization methods for stable solution of linear ill-posed problems have been studied extensively (see [14], [16], [38] and references therein). Among regularization methods, the *variational regularization* (VR) is one of the frequently used methods. When $F = A$, where A is a linear operator, the VR method consists of minimizing the following functional:

$$\|Au - f_\delta\|^2 + \alpha\|u\|^2 \to \min. \tag{1.3}$$

The minimizer $u_{\delta,a}$ of problem (1.3) can be found from the Euler equation:

$$(A^*A + \alpha I)u_{\delta,\alpha} = A^*f_\delta.$$

In the VR method the choice of the regularization parameter α is important. Various choices of the regularization parameter have been proposed and justified. Among these, the *discrepancy principle* (DP) appears to be the most efficient in practice (see [14]). According to the DP one chooses α as the solution to the following equation:

$$\|Au_{\delta,\alpha} - f_\delta\| = C\delta, \qquad 1 < C = const. \tag{1.4}$$

When the operator F is nonlinear, the theory is less complete (see [2], [37]). In this case, one may try to minimize the functional

$$\|F(u) - f_\delta\|^2 + \alpha\|u\|^2 \to \min \tag{1.5}$$

as in the case of linear operator F. The minimizer to problem (1.5) solves the following Euler equation

$$F'(u_{\delta,\alpha})^*F(u_{\delta,\alpha}) + \alpha u_{\delta,\alpha} = F'(u_{\delta,\alpha})^*f_\delta. \tag{1.6}$$

However, there are several principal difficulties in nonlinear problems: there are no general results concerning the solvability of (1.6), and the notion of minimal-norm solution does not make sense, in general, when F is nonlinear. Other methods for solving (1.1) with nonlinear F have been studied. Convergence proofs of these methods often rely on the source-type assumptions. These assumptions are difficult to verify in practice and they may not hold.

Equation (1.1) with a monotone operator F is of interest and importance in many applications. Every solvable linear operator equation $Au = f$ can be reduced to solving operator equation with a monotone operator A^*A. For equations with a bounded operator A this is a simple fact, and for unbounded, closed, densely defined linear operators A it is proved in [29], [31], [32], [16].

Physical problems with dissipation of energy often can be reduced to solving equations with monotone operators (see, e.g., [35]). For simplicity we present the results for equations in Hilbert spaces, but some results can be generalized to the operators in Banach spaces.

When F is monotone then the notion minimal-norm solution makes sense (see, e.g., [16], p. 110). In [36], Tautenhahn studied a discrepancy principle for solving equation (1.1). The discrepancy principle in [36] requires solving for α the following equation:

$$\|(F'(u_{\delta,\alpha}) + \alpha I)^{-1}(F(u_{\delta,\alpha}) - f_\delta)\| = C\delta, \qquad 1 < C = const, \tag{1.7}$$

where $u_{\delta,\alpha}$ solves the equation:

$$F(u_{\delta,\alpha}) + \alpha u_{\delta,\alpha} = f_\delta.$$

For this discrepancy principle optimal rate of convergence is obtained in [36]. However, the convergence of the method is justified under source-type assumptions and other restrictive assumptions. These assumptions often do not hold and some of them cannot be verified, in general. In addition, equation (1.7) is difficult to solve numerically.

A continuous analog of the Newton method for solving well-posed operator equations was proposed in [3], in 1958. In [1], [4]–[34], and in the monograph [16] the Dynamical Systems Method for solving operator equations is studied systematically. The DSM consists of finding a nonlinear map $\Phi(t, u)$ such that the Cauchy problem

$$\dot{u} = \Phi(t, u), \qquad u(0) = u_0, \tag{1.8}$$

has a unique solution for all $t \geq 0$, there exists $\lim_{t\to\infty} u(t) := u(\infty)$, and $F(u(\infty)) = f$,

$$\exists!\ u(t) \quad \forall t \geq 0; \qquad \exists u(\infty); \qquad F(u(\infty)) = f. \tag{1.9}$$

Various choices of Φ were proposed in [16] for (1.9) to hold. Each such choice yields a version of the DSM.

In this paper, several methods developed by the authors for solving stably equation (1.1) with a monotone operator F in a Hilbert space H and noisy data f_δ, given in place of the exact data f, are presented. A discrepancy principle (DP) for solving equation (1.1) stably is formulated and justified. In this DP the only assumptions on F are the continuity and monotonicity. Thus, our result is quite general and can be applied for a wide range of problems. Several versions of the Dynamical Systems Method (DSM) for solving equation (1.1) are formulated. These versions of the DSM are a Newton-type method, a gradient-type method and a simple iterations method. A priori and a posteriori choices of stopping rules for several versions of the DSM and for the corresponding iterative schemes are proposed and justified. Convergence of the solutions of these versions of the DSM to the minimal-norm solution to the equation $F(u) = f$ is proved. Iterative schemes, corresponding to the proposed versions of the DSM, are formulated. Convergence of these iterative schemes to a solution to equation $F(u) = f$ is established. When one uses these iterative schemes one does not have to solve a nonlinear equation for the regularization parameter. The stopping time is chosen automatically in the course of calculations. Implementation of these methods is illustrated in Section 6 by a numerical experiment. In Sections 2 and 3 basic and auxiliary results are formulated, in Section 4 proofs are given, in Section 5 ideas of applications of the basic nonlinear inequality (2.87) are outlined.

2. Basic results

2.1. *A discrepancy principle*

Let us consider the following equation

$$F(V_{\delta,a}) + aV_{\delta,a} - f_\delta = 0, \qquad a > 0, \tag{2.1}$$

where $a = const$. It is known (see, e.g., [16], p.111) that equation (2.1) with a monotone continuous operator F has a unique solution $V_{\delta,a}$ for any $f_\delta \in H$.

Assume that equation (1.1) has a solution. It is known that the set of solution $\mathcal{N} := \{u : F(u) = f\}$ is convex and closed if F is monotone and continuous (see, e.g., [16], p.110). A closed and convex set $\mathcal{N}$ in H has a unique minimal-norm element. This minimal-norm solution to (1.1) is denoted by y.

Theorem 1. *Let $\gamma \in (0, 1]$ and $C > 0$ be some constants such that $C\delta^\gamma > \delta$. Assume that $\|F(0) - f_\delta\| > C\delta^\gamma$. Let y be the minimal-norm solution to equation* (1.1)*. Then there exists a unique $a(\delta) > 0$ such that*

$$\|F(V_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma, \tag{2.2}$$

where $V_{\delta,a(\delta)}$ solves (2.1) *with $a = a(\delta)$.*

If $0 < \gamma < 1$ then

$$\lim_{\delta\to 0} \|V_{\delta,a(\delta)} - y\| = 0. \tag{2.3}$$

Instead of using (2.1), one may use the following equation:

$$F(V_{\delta,a}) + a(V_{\delta,a} - \bar{u}) - f_\delta = 0, \qquad a > 0, \tag{2.4}$$

where $\bar{u}$ is an element of H. Denote $F_1(u) := F(u + \bar{u})$. Then F_1 is monotone and continuous. Equation (2.4) can be written as:

$$F_1(U_{\delta,a}) + aU_{\delta,a} - f_\delta = 0, \qquad U_{\delta,a} := V_{\delta,a} - \bar{u}, \quad a > 0. \tag{2.5}$$

Applying Theorem 1 with $F = F_1$ one gets the following result:

Corollary 2. *Let $\gamma \in (0, 1]$ and $C > 0$ be some constants such that $C\delta^\gamma > \delta$. Let $\bar{u} \in H$ and z be the solution to* (1.1) *with minimal distance to $\bar{u}$. Assume that $\|F(\bar{u}) - f_\delta\| > C\delta^\gamma$. Then there exists a unique $a(\delta) > 0$ such that*

$$\|F(\tilde{V}_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma, \tag{2.6}$$

where $\tilde{V}_{\delta,a(\delta)}$ solves the following equation:

$$F(\tilde{V}_{\delta,a}) + a(\delta)(\tilde{V}_{\delta,a} - \bar{u}) - f_\delta = 0. \tag{2.7}$$

If $\gamma \in (0,1)$ then this $a(\delta)$ satisfies

$$\lim_{\delta\to 0} \|\tilde{V}_{\delta,a(\delta)} - z\| = 0. \tag{2.8}$$

The following result is useful for the implementation of our DP.

Theorem 3. *Let δ, F, f_δ, and y be as in Theorem 1 and $0 < \gamma < 1$. Assume that $v_\delta \in H$ and $\alpha(\delta) > 0$ satisfy the following conditions:*

$$\|F(v_\delta) + \alpha(\delta)v_\delta - f_\delta\| \le \theta\delta, \qquad \theta > 0, \tag{2.9}$$

and

$$C_1\delta^\gamma \le \|F(v_\delta) - f_\delta\| \le C_2\delta^\gamma, \qquad 0 < C_1 < C_2. \tag{2.10}$$

Then one has:

$$\lim_{\delta\to 0} \|v_\delta - y\| = 0. \tag{2.11}$$

Remark 1. Based on Theorem 3 an algorithm for solving nonlinear equations with monotone Lipchitz continuous operators is outlined in [12].

Remark 2. It is an open problem to choose γ and C_1, C_2, optimal in some sense.

Remark 3. Theorem 1 and Theorem 3 do not hold, in general, for $\gamma = 1$. Indeed, let $Fu = \langle u, p\rangle p$, $\|p\| = 1$, $p \perp \mathcal{N}(F) := \{u \in H : Fu = 0\}$, $f = p$, $f_\delta = p + q\delta$, where $\langle p, q\rangle = 0$, $\|q\| = 1$, $Fq = 0$, $\|q\delta\| = \delta$. For linear operator F we write Fu, rather than $F(u)$. One has $Fy = p$, where $y = p$, is the minimal-norm solution to the equation $Fu = p$. Equation $FV_{\delta,a} + aV_{\delta,a} = p + q\delta$, has the unique solution $V_{\delta,a} = q\delta/a + p/(1 + a)$. Equation (2.2) is $C\delta = \|q\delta + (ap)/(1 + a)\|$. This equation yields $a = a(\delta) = c\delta/(1 - c\delta)$, where $c := (C^2 - 1)^{1/2}$, and we assume $c\delta < 1$. Thus, $\lim_{\delta\to 0} V_{\delta,a(\delta)} = p + c^{-1}q := v$, and $Fv = p$. Therefore $v = \lim_{\delta\to 0} V_{\delta,a(\delta)}$ is not p, i.e., is not the minimal-norm solution to the equation $Fu = p$. This argument is borrowed from [15], p. 29.

If equation (1.1) has a unique solution, then one can prove convergence results (2.3) and (2.11) for $\gamma = 1$.

2.2. *The Dynamical Systems Method*

Let $a(t) \searrow 0$ be a positive and strictly decreasing function. Let $V_\delta(t)$ solve the following equation:

$$F(V_\delta(t)) + a(t)V_\delta(t) - f_\delta = 0. \tag{2.12}$$

Throughout the paper we assume that equation $F(u) = f$ has a solution in H, possibly nonunique, and y is the minimal-norm solution to this equation. Let f be unknown, but f_δ be given, $\|f_\delta - f\| \le \delta$.

2.2.1. *The Newton-type DSM*

In this section we assume that F is a monotone operator, twice Fréchet differentiable, and

$$\|F^{(j)}(u)\| \le M_j(R, u_0), \qquad \forall u \in B(u_0, R), \quad 0 \le j \le 2. \tag{2.13}$$

This assumption is satisfied in many applications.

Denote

$$A := F'(u_\delta(t)), \quad A_a := A + aI, \tag{2.14}$$

where I is the identity operator. Let $u_\delta(t)$ solve the following Cauchy problem:

$$\dot{u}_\delta = -A_{a(t)}^{-1}[F(u_\delta) + a(t)u_\delta - f_\delta], \quad u_\delta(0) = u_0. \tag{2.15}$$

We assume below that $||F(u_0) - f_\delta|| > C_1\delta^\zeta$, where $C_1 > 1$ and $\zeta \in (0,1]$ are some constants. We also assume without loss of generality that $\delta \in (0,1)$. Assume that equation (1.1) has a solution, possibly nonunique, and y is the minimal norm solution to equation (1.1). Recall that we are given the noisy data f_δ, $\|f_\delta - f\| \le \delta$.

Lemma 4 ([9] Lemma 2.7). *Suppose M_1, c_0, and c_1 are positive constants and $0 \ne y \in H$. Then there exist $\lambda > 0$ and a function $a(t) \in C^1[0,\infty)$, $0 < a(t) \searrow 0$, such that the following conditions hold*

$$\frac{M_1}{\|y\|} \le \lambda, \tag{2.16}$$

$$\frac{c_0}{a(t)} \le \frac{\lambda}{2a(t)}\left[1 - \frac{|\dot{a}(t)|}{a(t)}\right], \tag{2.17}$$

$$c_1\frac{|\dot{a}(t)|}{a(t)} \le \frac{a(t)}{2\lambda}\left[1 - \frac{|\dot{a}(t)|}{a(t)}\right], \tag{2.18}$$

$$\|F(0) - f_\delta\| \le \frac{a^2(0)}{\lambda}. \tag{2.19}$$

In the proof of Lemma 2.7 in [9] we have demonstrated that conditions (2.17)–(2.19) are satisfied for $a(t) = \frac{d}{(c+t)^b}$, where $b \in (0,1]$, $c, d > 0$ are constants, $c > 6b$, and d is sufficiently large.

Theorem 5. *Assume $a(t) = \frac{d}{(c+t)^b}$, where $b \in (0,1]$, $c, d > 0$ are constants, $c > 6b$, and d is sufficiently large so that conditions (2.17)–(2.19) hold. Assume that $F : H \to H$ is a twice Fréchet differentiable monotone operator, (2.13) holds, u_0 is an element of H satisfying inequalities*

$$\|u_0 - V_0\| \le \frac{\|F(0) - f_\delta\|}{a(0)}, \quad h(0) = \|F(u_0) + a(0)u_0 - f_\delta\| \le \frac{1}{4}a(0)\|V_\delta(0)\|. \tag{2.20}$$

Then the solution $u_\delta(t)$ to problem (2.15) exists on an interval $[0, T_\delta]$, $\lim_{\delta\to 0} T_\delta = \infty$, and there exists a unique t_δ, $t_\delta \in (0, T_\delta)$ such that $\lim_{\delta\to 0} t_\delta = \infty$ and

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C_1\delta^\zeta, \quad \|F(u_\delta(t)) - f_\delta\| > C_1\delta^\zeta, \quad \forall t \in [0, t_\delta), \tag{2.21}$$

where $C_1 > 1$ *and* $0 < \zeta \le 1$. *If* $\zeta \in (0,1)$ *and* t_δ *satisfies* (2.21), *then*

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0. \tag{2.22}$$

Remark 4. One can choose u_0 satisfying inequalities (2.20). Indeed, if u_0 is a sufficiently close approximation to $V_\delta(0)$ the solution to equation (2.12) then inequalities (2.20) are satisfied. Note that the second inequality in (2.20) is a sufficient condition for the following inequality (see also (4.55))

$$e^{-\frac{t}{2}}h(0) \le \frac{1}{4}a(t)\|V_\delta(t)\|, \quad t \ge 0, \tag{2.23}$$

to hold. In our proof inequality (2.23) (or inequality (4.55)) is used at $t = t_\delta$. The stopping time t_δ is often sufficiently large for the quantity $e^{\frac{t_\delta}{2}}a(t_\delta)$ to be large. Note that $\|V_\delta(t)\|$ is a strictly increasing function of $t \in (0,\infty)$ (see Lemma 20). In this case inequality (2.23) with $t = t_\delta$ is satisfied for a wide range of u_0.

Condition $c > 6b$ is used in the proof of Lemma 27 (see below).

2.2.2. *The Dynamical Systems Gradient Method*

In this section we assume that F is a monotone operator, twice Fréchet differentiable, and estimates (2.13) hold.

Denote

$$A := F'(u_\delta(t)), \quad A_a := A + aI, \quad a = a(t),$$

where I is the identity operator. Let $u_\delta(t)$ solve the following Cauchy problem:

$$\dot u_\delta = -A^*_{a(t)}[F(u_\delta) + a(t)u_\delta - f_\delta], \quad u_\delta(0) = u_0. \tag{2.24}$$

Let us recall the following result:

Lemma 6 ([10] Lemma 11). *Suppose* M_1, c_0, *and* c_1 *are positive constants and* $0 \ne y \in H$. *Then there exist* $\lambda > 0$ *and a function* $a(t) \in C^1[0,\infty)$, $0 < a(t) \searrow 0$, *such that*

$$|\dot a(t)| \le \frac{a^3(t)}{4}, \tag{2.25}$$

and the following conditions hold

$$\frac{M_1}{\|y\|} \le \lambda, \tag{2.26}$$

$$c_0(M_1 + a(t)) \le \frac{\lambda}{2a^2(t)}\left[a^2(t) - \frac{2|\dot a(t)|}{a(t)}\right], \tag{2.27}$$

$$c_1\frac{|\dot a(t)|}{a(t)} \le \frac{a^2(t)}{2\lambda}\left[a^2(t) - \frac{2|\dot a(t)|}{a(t)}\right], \tag{2.28}$$

$$\frac{\lambda}{a^2(0)}g(0) < 1. \tag{2.29}$$

We have demonstrated in the proof of Lemma 11 in [10] that conditions (2.25)–(2.29) are satisfied with $a(t) = \frac{d}{(c+t)^b}$, where $b \in (0, \frac{1}{4}]$, $c \geq 1$, and $d > 0$ are constants, and d is sufficiently large.

Theorem 7. *Let* $a(t) = \frac{d}{(c+t)^b}$, *where* $b \in (0, \frac{1}{4}]$, $c \geq 1$, *and* $d > 0$ *are constants, and* d *is sufficiently large so that conditions* (2.25)–(2.29) *hold. Assume that* $F : H \to H$ *is a twice Fréchet differentiable monotone operator,* (2.13) *holds,* u_0 *is an element of* H, *satisfying inequality*

$$h(0) = \|F(u_0) + a(0)u_0 - f_\delta\| \leq \frac{1}{4}a(0)\|V_\delta(0)\|. \tag{2.30}$$

Then the solution $u_\delta(t)$ *to problem* (2.24) *exists on an interval* $[0, T_\delta]$, $\lim_{\delta \to 0} T_\delta = \infty$, *and there exists* t_δ, $t_\delta \in (0, T_\delta)$, *not necessarily unique, such that*

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C_1\delta^\zeta, \quad \lim_{\delta \to 0} t_\delta = \infty, \tag{2.31}$$

where $C_1 > 1$ *and* $0 < \zeta \leq 1$ *are constants. If* $\zeta \in (0, 1)$ *and* t_δ *satisfies* (2.31), *then*

$$\lim_{\delta \to 0} \|u_\delta(t_\delta) - y\| = 0. \tag{2.32}$$

Remark 5. One can easily choose u_0 satisfying inequality (2.30). Note that inequality (2.30) is a sufficient condition for the following inequality (cf. (4.95))

$$e^{-\varphi(t)}h(0) \leq \frac{1}{4}a(t)\|V_\delta(t)\|, \quad t \geq 0, \tag{2.33}$$

to hold. In our proof inequality (2.33) (see also (4.95)) is used at $t = t_\delta$. The stopping time t_δ is often sufficiently large for the quantity $e^{\varphi(t_\delta)}a(t_\delta)$ to be large. In this case inequality (2.33) (cf. (4.95)) with $t = t_\delta$ is satisfied for a wide range of u_0. The parameter ζ is not fixed in (2.31). While we could fix it, for example, by setting $\zeta = 0.9$, it is an interesting *open problem* to propose an optimal in some sense criterion for choosing ζ.

2.2.3. *The simple iteration DSM*

In this section we assume that F is a monotone operator, Fréchet differentiable, and

$$\sup_{u \in B(u_0, R)} \|F'(u)\| \leq M_1(u_0, R). \tag{2.34}$$

Let us consider a version of the DSM for solving equation (1.1):

$$\dot{u}_\delta = -\big(F(u_\delta) + a(t)u_\delta - f_\delta\big), \quad u_\delta(0) = u_0, \tag{2.35}$$

where F is a monotone operator.

The advantage of this version compared with (2.15) is the absence of the inverse operator in the algorithm, which makes the algorithm (2.35) less expensive than (2.15). On the other hand, algorithm (2.15) converges faster than (2.35) in many cases. The algorithm (2.35) is cheaper than the DSM gradient algorithm proposed in (2.24).

The advantage of method (2.35), a modified version of the simple iteration method, over the Gauss-Newton method and the version (2.15) of the DSM is the following: neither inversion of matrices nor evaluation of F' is needed in a discretized version of (2.35). Although the convergence rate of the DSM (2.35) maybe slower than that of the DSM (2.15), the DSM (2.35) might be faster than the DSM (2.15) for large-scale systems due to its lower computation cost.

In this Section we investigate a stopping rule based on a discrepancy principle (DP) for the DSM (2.35). The main results of this Section is Theorem 9 in which a DP is formulated, the existence of a stopping time t_δ is proved, and the convergence of the DSM with the proposed DP is justified under some natural assumptions.

Lemma 8 ([11] **Lemma 11).** *Suppose M_1 and c_1 are positive constants and $0 \neq y \in H$. Then there exist a number $\lambda > 0$ and a function $a(t) \in C^1[0,\infty)$, $0 < a(t) \searrow 0$, such that*

$$|\dot{a}(t)| \le \frac{a^2(t)}{2}, \tag{2.36}$$

and the following conditions hold

$$\frac{M_1}{\|y\|} \le \lambda, \tag{2.37}$$

$$0 \le \frac{\lambda}{2a(t)}\left[a(t) - \frac{|\dot{a}(t)|}{a(t)}\right], \tag{2.38}$$

$$c_1\frac{|\dot{a}(t)|}{a(t)} \le \frac{a(t)}{2\lambda}\left[a(t) - \frac{|\dot{a}(t)|}{a(t)}\right], \tag{2.39}$$

$$\frac{\lambda}{a(0)}g(0) < 1. \tag{2.40}$$

It is shown in the proof of Lemma 11 in [11] that conditions (2.36)–(2.40) hold for the function $a(t) = \frac{d}{(c+t)^b}$, where $b \in (0, \frac{1}{2}]$, $c \ge 1$ and $d > 0$ are constants, and d is sufficiently large.

Theorem 9. *Let $a(t) = \frac{d}{(c+t)^b}$, where $b \in (0, \frac{1}{2}]$, $c \ge 1$ and $d > 0$ are constants, and d is sufficiently large so that conditions (2.36)–(2.40) hold. Assume that $F : H \to H$ is a Fréchet differentiable monotone operator, condition (2.34) holds, and u_0 is an element of H, satisfying inequality*

$$h(0) = \|F(u_0) + a(0)u_0 - f_\delta\| \le \frac{1}{4}a(0)\|V_\delta(0)\|, \tag{2.41}$$

Assume that equation $F(u) = f$ has a solution $y \in B(u_0, R)$, possibly nonunique, and y is the minimal-norm solution to this equation. Then the solution $u_\delta(t)$ to problem (2.35) exists on an interval $[0, T_\delta]$, $\lim_{\delta\to 0} T_\delta = \infty$, and there exists t_δ, $t_\delta \in (0, T_\delta)$, not necessarily unique, such that

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C_1\delta^\zeta, \quad \lim_{\delta\to 0} t_\delta = \infty, \tag{2.42}$$

where $C_1 > 1$ and $0 < \zeta \le 1$ are constants. If $\zeta \in (0,1)$ and t_δ satisfies (2.42)*, then*

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0. \tag{2.43}$$

Remark 6. One can easily choose u_0 satisfying inequality (2.41). Again, inequality (2.41) is a sufficient condition for (2.33) (cf. (4.131)) to hold. In our proof inequality (2.33) is used at $t = t_\delta$. The stopping time t_δ is often sufficiently large for the quantity $e^{\varphi(t_\delta)}a(t_\delta)$ to be large. In this case inequality (2.33) with $t = t_\delta$ is satisfied for a wide range of u_0.

2.3. *Iterative schemes*

Let $0 < a_n \searrow 0$ be a positive strictly decreasing sequence. Denote $V_n := V_{n,\delta}$ where $V_{n,\delta}$ solves the following equation:

$$F(V_{n,\delta}) + a_n V_{n,\delta} - f_\delta = 0. \tag{2.44}$$

Note that if $a_n := a(t_n)$ then $V_{n,\delta} = V_\delta(t_n)$.

2.3.1. *Iterative scheme of Newton-type*

In this section we assume that F is a monotone operator, twice Fréchet differentiable, and

$$\|F^{(j)}(u)\| \le M_j(R, u_0), \qquad \forall u \in B(u_0, R), \quad 0 \le j \le 2, \tag{2.45}$$

and do not repeat this assumption. Consider the following iterative scheme:

$$u_{n+1} = u_n - A_n^{-1}[F(u_n) + a_n u_n - f_\delta], \quad A_n := F'(u_n) + a_n I, \quad u_0 = u_0, \tag{2.46}$$

where u_0 is chosen so that inequality (2.52) holds. Note that $F'(u_n) \ge 0$ since F is monotone. Thus, $\|A_n^{-1}\| \le \frac{1}{a_n}$.

Lemma 10 ([8] **Lemma 2.5).** *Suppose M_1, c_0, and c_1 are positive constants and $0 \ne y \in H$. Then there exist $\lambda > 0$ and a sequence $0 < (a_n)_{n=0}^\infty \searrow 0$ such that the following conditions hold*

$$a_n \le 2a_{n+1}, \tag{2.47}$$

$$\|f_\delta - F(0)\| \le \frac{a_0^2}{\lambda}, \tag{2.48}$$

$$\frac{M_1}{\lambda} \le \|y\|, \tag{2.49}$$

$$\frac{a_n - a_{n+1}}{a_{n+1}^2} \le \frac{1}{2c_1\lambda}, \tag{2.50}$$

$$c_0\frac{a_n}{\lambda^2} + \frac{a_n - a_{n+1}}{a_{n+1}}c_1 \le \frac{a_{n+1}}{\lambda}. \tag{2.51}$$

It is shown in the proof of Lemma 2.5 in [8] that conditions (2.47)–(2.51) hold for the sequence $a_n = \frac{d}{(c+n)^b}$, where $c \ge 1$, $0 < b \le 1$, and d is sufficiently large.

Remark 7. In Lemmas 10–14, one can choose a_0 and λ so that $\frac{a_0}{\lambda}$ is uniformly bounded as $\delta \to 0$ even if $M_1(R) \to \infty$ as $R \to \infty$ at an arbitrary fast rate. Choices of a_0 and λ, satisfying this condition, are discussed in [8], [10] and [11].

Let a_n and λ satisfy conditions (2.47)–(2.51). Assume that equation $F(u) = f$ has a solution $y \in B(u_0, R)$, possibly nonunique, and y is the minimal-norm solution to this equation. Let f be unknown but f_δ be given, and $\|f_\delta - f\| \le \delta$. We have the following result:

Theorem 11. *Assume* $a_n = \frac{d}{(c+n)^b}$ *where* $c \ge 1$, $0 < b \le 1$, *and* d *is sufficiently large so that conditions* (2.47)–(2.51) *hold. Let* u_n *be defined by* (2.46). *Assume that* u_0 *is chosen so that* $\|F(u_0) - f_\delta\| > C_1\delta^\gamma > \delta$ *and*

$$g_0 := \|u_0 - V_0\| \le \frac{\|F(0) - f_\delta\|}{a_0}. \tag{2.52}$$

Then there exists a unique n_δ, *depending on* C_1 *and* γ *(see below), such that*

$$\|F(u_{n_\delta}) - f_\delta\| \le C_1\delta^\gamma, \quad C_1\delta^\gamma < \|F(u_n) - f_\delta\|, \quad \forall n < n_\delta, \tag{2.53}$$

where $C_1 > 1$, $0 < \gamma \le 1$.

Let $0 < (\delta_m)_{m=1}^\infty$ *be a sequence such that* $\delta_m \to 0$. *If* N *is a cluster point of the sequence* n_{δ_m} *satisfying* (2.53), *then*

$$\lim_{m\to\infty} u_{n_{\delta_m}} = u^*, \tag{2.54}$$

where u^* *is a solution to the equation* $F(u) = f$. *If*

$$\lim_{m\to\infty} n_{\delta_m} = \infty, \tag{2.55}$$

and $\gamma \in (0, 1)$, *then*

$$\lim_{m\to\infty} \|u_{n_{\delta_m}} - y\| = 0. \tag{2.56}$$

Note that by Remark 9, inequality (2.52) is satisfied with $u_0 = 0$.

2.3.2. *Iterative scheme of gradient-type*

In this section we assume that F is a monotone operator, twice Fréchet differentiable, and estimates (2.45) hold.

Consider the following iterative scheme:

$$u_{n+1} = u_n - \alpha_n A_n^*[F(u_n) + a_n u_n - f_\delta], \quad A_n := F'(u_n) + a_n I, \quad u_0 = u_0, \tag{2.57}$$

where u_0 is chosen so that inequality (2.65) holds, and $\{\alpha_n\}_{n=1}^\infty$ is a positive sequence such that

$$0 < \tilde{\alpha} \le \alpha_n \le \frac{2}{a_n^2 + (M_1 + a_n)^2}, \qquad ||A_n|| \le M_1 + a_n. \tag{2.58}$$

It follows from this condition that

$$\|1 - \alpha_n A_{a_n}^* A_{a_n}\| = \sup_{a_n^2 \le \lambda \le (M_1+a_n)^2} |1 - \alpha_n\lambda| \le 1 - \alpha_n a_n^2. \tag{2.59}$$

Note that $F'(u_n) \geq 0$ since F is monotone.

Lemma 12 ([10] Lemma 12). *Suppose M_1, c_0, c_1 and $\tilde{\alpha}$ are positive constants and $0 \neq y \in H$. Then there exist $\lambda > 0$ and a sequence $0 < (a_n)_{n=0}^{\infty} \searrow 0$ such that the following conditions hold*

$$\frac{a_n}{a_{n+1}} \leq 2, \tag{2.60}$$

$$\|f_\delta - F(0)\| \leq \frac{a_0^3}{\lambda}, \tag{2.61}$$

$$\frac{M_1}{\lambda} \leq \|y\|, \tag{2.62}$$

$$\frac{c_0(M_1 + a_0)}{\lambda} \leq \frac{1}{2}, \tag{2.63}$$

$$\frac{a_n^2}{\lambda} - \frac{\tilde{\alpha} a_n^4}{2\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}} c_1 \leq \frac{a_{n+1}^2}{\lambda}. \tag{2.64}$$

It is shown in the proof of Lemma 12 in [10] that the sequence $(a_n)_{n=0}^{\infty}$ satisfying conditions (2.60)–(2.64) can be chosen of the form $a_n = \frac{d}{(c+n)^b}$, where $c \geq 1$, $0 < b \leq \frac{1}{4}$, and d is sufficiently large.

Assume that equation $F(u) = f$ has a solution in $B(u_0, R)$, possibly nonunique, and y is the minimal-norm solution to this equation. Let f be unknown but f_δ be given, and $\|f_\delta - f\| \leq \delta$. We prove the following result:

Theorem 13. *Assume $a_n = \frac{d}{(c+n)^b}$ where $c \geq 1$, $0 < b \leq \frac{1}{4}$, and d is sufficiently large so that conditions (2.60)–(2.64) hold. Let u_n be defined by (2.57). Assume that u_0 is chosen so that $\|F(u_0) - f_\delta\| > C_1\delta^\zeta > \delta$ and*

$$g_0 := \|u_0 - V_0\| \leq \frac{\|F(0) - f_\delta\|}{a_0}. \tag{2.65}$$

Then there exists a unique n_δ such that

$$\|F(u_{n_\delta}) - f_\delta\| \leq C_1\delta^\zeta, \quad C_1\delta^\zeta < \|F(u_n) - f_\delta\|, \quad \forall n < n_\delta, \tag{2.66}$$

where $C_1 > 1$, $0 < \zeta \leq 1$.

Let $0 < (\delta_m)_{m=1}^{\infty}$ be a sequence such that $\delta_m \to 0$. If the sequence $\{n_m := n_{\delta_m}\}_{m=1}^{\infty}$ is bounded, and $\{n_{m_j}\}_{j=1}^{\infty}$ is a convergent subsequence, then

$$\lim_{j\to\infty} u_{n_{m_j}} = \tilde{u}, \tag{2.67}$$

where $\tilde{u}$ is a solution to the equation $F(u) = f$. If

$$\lim_{m\to\infty} n_m = \infty, \tag{2.68}$$

and $\zeta \in (0, 1)$, then

$$\lim_{m\to\infty} \|u_{n_m} - y\| = 0. \tag{2.69}$$

It is pointed out in Remark 9 that inequality (2.65) is satisfied with $u_0 = 0$.

2.3.3. *A simple iteration method*

In this section we assume that F is a monotone operator, Fréchet differentiable.

Consider the following iterative scheme:

$$u_{n+1} = u_n - \alpha_n [F(u_n) + a_n u_n - f_\delta], \quad u_0 = u_0, \tag{2.70}$$

where u_0 is chosen so that inequality (2.77) holds, and $\{\alpha_n\}_{n=1}^\infty$ is a positive sequence such that

$$0 < \tilde{\alpha} \le \alpha_n \le \frac{2}{a_n + (M_1 + a_n)}, \qquad M_1(u_0, R) = \sup_{u \in B(u_0,R)} \|F'(u)\|. \tag{2.71}$$

It follows from this condition that

$$\|1 - \alpha_n(J_n + a_n)\| = \sup_{a_n \le \lambda \le M_1 + a_n} |1 - \alpha_n \lambda| \le 1 - \alpha_n a_n. \tag{2.72}$$

Here, J_n is an operator in H such that $J_n = J_n^* \ge 0$ and $\|J_n\| \le M_1$, $\forall u \in B(u_0, R)$. A specific choice of J_n is made in formula (4.186) below.

Lemma 14 ([11] **Lemma 12).** *Suppose M_1, c_1 and $\tilde{\alpha}$ are positive constants and $0 \ne y \in H$. Then there exist a number $\lambda > 0$ and a sequence $0 < (a_n)_{n=0}^\infty \searrow 0$ such that the following conditions hold*

$$\frac{a_n}{a_{n+1}} \le 2, \tag{2.73}$$

$$\|f_\delta - F(0)\| \le \frac{a_0^2}{\lambda}, \tag{2.74}$$

$$\frac{M_1}{\lambda} \le \|y\|, \tag{2.75}$$

$$\frac{a_n}{\lambda} - \frac{\tilde{\alpha} a_n^2}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}} c_1 \le \frac{a_{n+1}}{\lambda}. \tag{2.76}$$

It is shown in the proof of Lemma 12 in [11] that conditions (2.73)–(2.76) hold for the sequence $a_n = \frac{d}{(c+n)^b}$, where $c \ge 1$, $0 < b \le \frac{1}{2}$, and d is sufficiently large.

Let a_n and λ satisfy conditions (2.73)–(2.76). Assume that equation $F(u) = f$ has a solution $y \in B(u_0, R)$, possibly nonunique, and y is the minimal-norm solution to this equation. Let f be unknown but f_δ be given, and $\|f_\delta - f\| \le \delta$. We prove the following result:

Theorem 15. *Assume that F is a Fréchet differentiable monotone operator and F' is selfadjoint. Assume $a_n = \frac{d}{(c+n)^b}$ where $c \ge 1$, $0 < b \le \frac{1}{2}$, and d is sufficiently large so that conditions* (2.73)–(2.76) *hold. Let u_n be defined by* (2.70). *Assume that u_0 is chosen so that $\|F(u_0) - f_\delta\| > C_1 \delta^\zeta > \delta$ and*

$$g_0 := \|u_0 - V_0\| \le \frac{\|F(0) - f_\delta\|}{a_0}. \tag{2.77}$$

Then there exists a unique n_δ such that

$$\|F(u_{n_\delta}) - f_\delta\| \le C_1 \delta^\zeta, \quad C_1 \delta^\zeta < \|F(u_n) - f_\delta\|, \quad \forall n < n_\delta, \tag{2.78}$$

where $C_1 > 1$, $0 < \zeta \leq 1$.

Let $0 < (\delta_m)_{m=1}^{\infty}$ *be a sequence such that* $\delta_m \to 0$. *If the sequence* $\{n_m := n_{\delta_m}\}_{m=1}^{\infty}$ *is bounded, and* $\{n_{m_j}\}_{j=1}^{\infty}$ *is a convergent subsequence, then*

$$\lim_{j\to\infty} u_{n_{m_j}} = \tilde{u}, \tag{2.79}$$

where $\tilde{u}$ *is a solution to the equation* $F(u) = f$. *If*

$$\lim_{m\to\infty} n_m = \infty, \tag{2.80}$$

and $\zeta \in (0, 1)$, *then*

$$\lim_{m\to\infty} \|u_{n_m} - y\| = 0. \tag{2.81}$$

Remark 8. If H is a complex Hilbert space, then a bounded nonnegative-definite operator $A = F'$, the Fréchet derivative of a monotone operator F, is selfadjoint; if A is a bounded linear operator defined on all of H and $\langle Au, u\rangle \geq 0$ for all $u \in H$, then A is selfadjoint. This is not true, in general, if H is a real Hilbert space. Example: $H = R^2$, A is matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then A is not selfadjoint, but $\langle Au, u\rangle = u_1^2 + u_1u_2 + u_2^2 \geq 0$ for all $u_1, u_2 \in \mathbb{R}$.

Remark 9. In theorems 11–15 we choose $u_0 \in H$ such that

$$g_0 := \|u_0 - V_0\| \leq \frac{\|F(0) - f_\delta\|}{a_0}. \tag{2.82}$$

It is easy to choose u_0 satisfying this condition. Indeed, if, for example, $u_0 = 0$, then by Lemma 20 in Section 3.2 (see below) one gets

$$g_0 = \|V_0\| = \frac{a_0\|V_0\|}{a_0} \leq \frac{\|F(0) - f_\delta\|}{a_0}. \tag{2.83}$$

If (2.82) and either (2.48) or (2.74) hold then

$$g_0 \leq \frac{a_0}{\lambda}. \tag{2.84}$$

This inequality is used in the proof of Theorem 11 and 15.

If (2.82) and (2.61) hold, then

$$g_0 \leq \frac{a_0^2}{\lambda}. \tag{2.85}$$

This inequality is used in the proof of Theorem 13.

2.4. *Nonlinear inequalities*

2.4.1. *A nonlinear differential inequality*

In [16] the following differential inequality

$$\dot{g}(t) \leq -\gamma(t)g(t) + \alpha(t)g^2(t) + \beta(t), \qquad t \geq \tau_0, \tag{2.86}$$

was studied and applied to various evolution problems. In (2.86) $\alpha(t), \beta(t), \gamma(t)$ and $g(t)$ are continuous nonnegative functions on $[\tau_0, \infty)$ where τ_0 is a fixed number. In [16], an upper bound for $g(t)$ is obtained under some conditions on α, β, γ. In [13] the following generalization of (2.86):

$$\dot{g}(t) \le -\gamma(t)g(t) + \alpha(t)g^p(t) + \beta(t), \qquad t \ge \tau_0, \quad p > 1, \tag{2.87}$$

is studied.

We have the following result:

Theorem 16 ([13] Theorem 1). *Let $\alpha(t), \beta(t)$ and $\gamma(t)$ be continuous functions on $[\tau_0, \infty)$ and $\alpha(t) > 0, \forall t \ge \tau_0$. Suppose there exists a function $\mu(t) > 0$, $\mu \in C^1[\tau_0, \infty)$, such that*

$$\frac{\alpha(t)}{\mu^p(t)} + \beta(t) \le \frac{1}{\mu(t)}\left[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right]. \tag{2.88}$$

Let $g(t) \ge 0$ be a solution to inequality (2.87) such that

$$\mu(\tau_0)g(\tau_0) < 1. \tag{2.89}$$

Then $g(t)$ exists globally and the following estimate holds:

$$0 \le g(t) < \frac{1}{\mu(t)}, \qquad \forall t \ge \tau_0. \tag{2.90}$$

Consequently, if $\lim_{t\to\infty} \mu(t) = \infty$, then

$$\lim_{t\to\infty} g(t) = 0. \tag{2.91}$$

Theorem 16 remains valid if the sign "$<$" in (2.89) and (2.90) is replaced by the sign "$\le$" (see Theorem 2 in [13]).

When $p = 2$ we have the following corollary:

Corollary 17 ([16] p. 97). *Suppose there exists a monotonically growing function $\mu(t)$,*

$$\mu \in C^1[\tau_0, \infty), \quad \mu > 0, \quad \lim_{t\to\infty} \mu(t) = \infty,$$

such that

$$0 \le \alpha(t) \le \frac{\mu(t)}{2}\left[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right], \qquad \dot{u} := \frac{du}{dt}, \tag{2.92}$$

$$\beta(t) \le \frac{1}{2\mu(t)}\left[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right], \tag{2.93}$$

where $\alpha(t), \beta(t)$ and $\gamma(t)$ are continuous nonnegative functions on $[\tau_0, \infty)$, $\tau_0 \ge 0$. Let $g(t) \ge 0$ be a solution to inequality (2.87) such that

$$\mu(\tau_0)g(\tau_0) < 1, \tag{2.94}$$

Then $g(t)$ exists globally and the following estimate holds:

$$0 \le g(t) < \frac{1}{\mu(t)}, \qquad \forall t \ge \tau_0. \tag{2.95}$$

Consequently, if $\lim_{t\to\infty}\mu(t)=\infty$, *then*

$$\lim_{t\to\infty} g(t)=0.$$

2.4.2. *A discrete version of the nonlinear inequality*

Theorem 18 ([13] Theorem 4). *Let* α_n, γ_n *and* g_n *be nonnegative sequences of numbers, and the following inequality holds:*

$$\frac{g_{n+1}-g_n}{h_n}\le -\gamma_n g_n+\alpha_n g_n^p+\beta_n, \qquad h_n>0, \quad 0<h_n\gamma_n<1, \tag{2.96}$$

or, equivalently,

$$g_{n+1}\le g_n(1-h_n\gamma_n)+\alpha_n h_n g_n^p+h_n\beta_n, \qquad h_n>0, \quad 0<h_n\gamma_n<1. \tag{2.97}$$

If there is a monotonically growing sequence of positive numbers $(\mu_n)_{n=1}^{\infty}$, *such that the following conditions hold:*

$$\frac{\alpha_n}{\mu_n^p}+\beta_n\le\frac{1}{\mu_n}\left(\gamma_n-\frac{\mu_{n+1}-\mu_n}{\mu_n h_n}\right), \tag{2.98}$$

$$g_0\le\frac{1}{\mu_0}, \tag{2.99}$$

then

$$0\le g_n\le\frac{1}{\mu_n}\qquad \forall n\ge 0. \tag{2.100}$$

Therefore, if $\lim_{n\to\infty}\mu_n=\infty$, *then* $\lim_{n\to\infty} g_n=0$.

3. Auxiliary results

3.1. *Auxiliary results from the theory of monotone operators*

Recall the following result (see e.g., [16], p. 112):

Lemma 19. *Assume that equation* (1.1) *is solvable,* y *is its minimal-norm solution, assumption* (1.2) *holds, and* F *is continuous. Then*

$$\lim_{a\to 0}\|V_a-y\|=0, \tag{3.1}$$

where $V_a:=V_{0,a}$ *solves equation* (2.1) *with* $\delta=0$.

3.2. *Auxiliary results for the regularized equation* (2.1)

Lemma 20 ([12] Lemma 2). *Assume* $\|F(0)-f_\delta\|>0$. *Let* $a>0$, *and* F *be monotone. Denote*

$$\psi(a):=\|V_{\delta,a}\|, \qquad \phi(a):=a\psi(a)=\|F(V_{\delta,a})-f_\delta\|,$$

where $V_{\delta,a}$ *solves* (2.1). *Then* $\psi(a)$ *is decreasing, and* $\phi(a)$ *is increasing (in the strict sense).*

Lemma 21 ([12] **Lemma 3**). *If F is monotone and continuous, then $\|V_{\delta,a}\| = O(\frac{1}{a})$ as $a \to \infty$, and*

$$\lim_{a\to\infty} \|F(V_{\delta,a}) - f_\delta\| = \|F(0) - f_\delta\|. \tag{3.2}$$

Lemma 22 ([12] **Lemma 4**). *Let $C > 0$ and $\gamma \in (0, 1]$ be constants such that $C\delta^\gamma > \delta$. Suppose that $\|F(0) - f_\delta\| > C\delta^\gamma$. Then, there exists a unique $a(\delta) > 0$ such that $\|F(V_{\delta,a(\delta)}) - f_\delta\| = C\delta^\gamma$.*

Lemma 23. *If F is monotone and $a \geq 0$, then*

$$\max\Big(\|F(u) - F(v)\|, a\|u - v\|\Big) \leq \|F(u) - F(v) + a(u - v)\|, \qquad \forall u, v \in H. \tag{3.3}$$

Proof. Denote

$$w := F(u) - F(v) + a(u - v), \qquad h := \|w\|. \tag{3.4}$$

Since $\langle F(u) - F(v), u - v\rangle \geq 0$, one obtains from two equations

$$\langle w, u - v\rangle = \langle F(u) - F(v) + a(u - v), u - v\rangle, \tag{3.5}$$

and

$$\langle w, F(u) - F(v)\rangle = \|F(u) - F(v)\|^2 + a\langle u - v, F(u) - F(v)\rangle, \tag{3.6}$$

the following inequalities:

$$a\|u - v\|^2 \leq \langle w, u - v\rangle \leq \|u - v\|h, \tag{3.7}$$

and

$$\|F(u) - F(v)\|^2 \leq \langle w, F(u) - F(v)\rangle \leq h\|F(u) - F(v)\|. \tag{3.8}$$

Inequalities (3.7) and (3.8) imply

$$a\|u - v\| \leq h, \quad \|F(u) - F(v)\| \leq h. \tag{3.9}$$

Lemma 23 is proved. □

Lemma 24. *Let $t_0 > 0$ satisfy*

$$\frac{\delta}{a(t_0)} = \frac{1}{C-1}\|y\|, \qquad C > 1. \tag{3.10}$$

Then,

$$\|F(V_\delta(t_0)) - f_\delta\| \leq C\delta, \tag{3.11}$$

and

$$\|\dot{V}_\delta\| \leq \frac{|\dot{a}|}{a}\|y\|\left(1 + \frac{1}{C-1}\right), \quad \forall t \leq t_0. \tag{3.12}$$

Proof. This t_0 exists and is unique since $a(t) > 0$ monotonically decays to 0 as $t \to \infty$. Since $a(t) > 0$ monotonically decays, one has:

$$\frac{\delta}{a(t)} \leq \frac{1}{C-1}\|y\|, \qquad 0 \leq t \leq t_0. \tag{3.13}$$

By Lemma 22 there exists $t_1 > 0$ such that

$$\|F(V_\delta(t_1)) - f_\delta\| = C\delta, \quad F(V_\delta(t_1)) + a(t_1)V_\delta(t_1) - f_\delta = 0. \tag{3.14}$$

We claim that $t_1 \in [0, t_0]$.

Indeed, from (3.14) and (3.30) one gets

$$C\delta = a(t_1)\|V_\delta(t_1)\| \leq a(t_1)\left(\|y\| + \frac{\delta}{a(t_1)}\right) = a(t_1)\|y\| + \delta, \quad C > 1,$$

so

$$\delta \leq \frac{a(t_1)\|y\|}{C-1}.$$

Thus,

$$\frac{\delta}{a(t_1)} \leq \frac{\|y\|}{C-1} = \frac{\delta}{a(t_0)}.$$

Since $a(t) \searrow 0$, one has $t_1 \leq t_0$. It follows from the inequality $t_1 \leq t_0$, Lemma 20 and the first equality in (3.14) that

$$\|F(V_\delta(t_0)) - f_\delta\| \leq \|F(V_\delta(t_1)) - f_\delta\| = C\delta.$$

Differentiating both sides of (2.12) with respect to t, one obtains

$$A_{a(t)}\dot{V_\delta} = -\dot{a}V_\delta.$$

This and the relations

$$A_a := F'(u) + aI, \quad F'(u) := A \geq 0,$$

imply

$$\|\dot{V_\delta}\| \leq |\dot{a}|\|A_{a(t)}^{-1}V_\delta\| \leq \frac{|\dot{a}|}{a}\|V_\delta\| \leq \frac{|\dot{a}|}{a}\left(\|y\| + \frac{\delta}{a}\right) \leq \frac{|\dot{a}|}{a}\|y\|\left(1 + \frac{1}{C-1}\right), \quad \forall t \leq t_0. \tag{3.15}$$

Lemma 24 is proved. □

Lemma 25. *Let* $n_0 > 0$ *satisfy the inequality:*

$$\frac{\delta}{a_{n_0+1}} > \frac{1}{C-1}\|y\| \geq \frac{\delta}{a_{n_0}}, \qquad C > 1. \tag{3.16}$$

Then,

$$\|F(V_{n_0+1}) - f_\delta\| \leq C\delta, \tag{3.17}$$

$$\|V_n\| \leq \|y\|\left(1 + \frac{2}{C-1}\right), \qquad 0 \leq n \leq n_0 + 1, \tag{3.18}$$

and

$$\|V_n - V_{n+1}\| \le \frac{a_n - a_{n+1}}{a_{n+1}}\|y\|\left(1 + \frac{2}{C-1}\right), \qquad 0 \le n \le n_0 + 1. \tag{3.19}$$

Proof. The number n_0, satisfying (3.16), exists and is unique since $a_n > 0$ monotonically decays to 0 as $n \to \infty$.

One has $\frac{a_n}{a_{n+1}} \le 2$, $\forall\, n \ge 0$. This and inequality (3.16) imply

$$\frac{2}{C-1}\|y\| \ge \frac{2\delta}{a_{n_0}} > \frac{\delta}{a_{n_0+1}} > \frac{1}{C-1}\|y\| \ge \frac{\delta}{a_{n_0}}, \qquad C > 1. \tag{3.20}$$

Thus,

$$\frac{2}{C-1}\|y\| > \frac{\delta}{a_n}, \quad \forall n \le n_0 + 1. \tag{3.21}$$

It follows from Lemma 22 that there exists $n_1 > 0$ such that

$$\|F(V_{n_1+1}) - f_\delta\| \le C\delta < \|F(V_{n_1}) - f_\delta\|, \tag{3.22}$$

where V_n solves the equation $F(V_n) + a_n V_n - f_\delta = 0$. *We claim that* $n_1 \in [0, n_0]$. Indeed, one has $\|F(V_{n_1}) - f_\delta\| = a_{n_1}\|V_{n_1}\|$, and $\|V_{n_1}\| \le \|y\| + \frac{\delta}{a_{n_1}}$ (cf. (3.30)), so

$$C\delta < a_{n_1}\|V_{n_1}\| \le a_{n_1}\left(\|y\| + \frac{\delta}{a_{n_1}}\right) = a_{n_1}\|y\| + \delta, \quad C > 1. \tag{3.23}$$

Therefore,

$$\delta < \frac{a_{n_1}\|y\|}{C-1}. \tag{3.24}$$

From (3.24) and (3.16) one gets

$$\frac{\delta}{a_{n_1}} < \frac{\|y\|}{C-1} < \frac{\delta}{a_{n_0+1}}. \tag{3.25}$$

Since a_n decreases monotonically, inequality (3.25) implies $n_1 \le n_0$. This, the first inequality in (3.22) and Lemma 20 imply

$$\|F(V_{n_0+1}) - f_\delta\| \le \|F(V_{n_1+1}) - f_\delta\| \le C\delta. \tag{3.26}$$

One has

$$\begin{aligned} a_{n+1}\|V_n - V_{n+1}\|^2 &= \langle (a_{n+1} - a_n)V_n - F(V_n) + F(V_{n+1}), V_n - V_{n+1}\rangle \\ &\le \langle (a_{n+1} - a_n)V_n, V_n - V_{n+1}\rangle \\ &\le (a_n - a_{n+1})\|V_n\|\|V_n - V_{n+1}\|, \qquad \forall n \ge 0. \end{aligned} \tag{3.27}$$

By (3.30), $\|V_n\| \le \|y\| + \frac{\delta}{a_n}$, and, by (3.21), $\frac{\delta}{a_n} \le \frac{2\|y\|}{C-1}$ for all $n \le n_0 + 1$. This implies (3.18).

From (3.18) and (3.27) one obtains

$$\|V_n - V_{n+1}\| \le \frac{a_n - a_{n+1}}{a_{n+1}}\|V_n\| \le \frac{a_n - a_{n+1}}{a_{n+1}}\|y\|\left(1 + \frac{2}{C-1}\right), \quad \forall n \le n_0 + 1. \tag{3.28}$$

Lemma 25 is proved. □

Lemma 26. *Let $V_a := V_{\delta,a}|_{\delta=0}$, so $F(V_a)+aV_a-f=0$. Let y be the minimal-norm solution to equation* (1.1). *Then*

$$\|V_{\delta,a}-V_a\| \le \frac{\delta}{a}, \quad \|V_a\| \le \|y\|, \qquad a>0, \tag{3.29}$$

and

$$\|V_{\delta,a}\| \le \|V_a\| + \frac{\delta}{a} \le \|y\| + \frac{\delta}{a}, \qquad a>0. \tag{3.30}$$

Proof. From (2.1) one gets

$$F(V_{\delta,a}) - F(V_a) + a(V_{\delta,a}-V_a) = f_\delta - f.$$

Multiply this equality by $(V_{\delta,a}-V_a)$ and use (1.2) to obtain

$$\begin{aligned}\delta\|V_{\delta,a}-V_a\| &\ge \langle f_\delta - f, V_{\delta,a}-V_a\rangle \\ &= \langle F(V_{\delta,a}) - F(V_a) + a(V_{\delta,a}-V_a), V_{\delta,a}-V_a\rangle \\ &\ge a\|V_{\delta,a}-V_a\|^2.\end{aligned}$$

This implies the first inequality in (3.29).

Let us derive a uniform with respect to a bound on $\|V_a\|$. From the equation

$$F(V_a) + aV_a - F(y) = 0,$$

and the monotonicity of F one gets

$$0 = \langle F(V_a) + aV_a - F(y), V_a - y\rangle \ge a\langle V_a, V_a - y\rangle.$$

This implies the desired bound:

$$\|V_a\| \le \|y\|, \qquad \forall a>0. \tag{3.31}$$

Similar arguments one can find in [16], p. 113.

Inequalities (3.30) follow from (3.29) and (3.31) and the triangle inequality.

Lemma 26 is proved. □

Lemma 27 ([9] **Lemma 2.11).** *Let $a(t) = \frac{d}{(c+t)^b}$ where $d,c,b>0$, $c \ge 6b$. One has*

$$e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} |\dot a(s)| \|V_\delta(s)\| ds \le \frac{1}{2} a(t) \|V_\delta(t)\|, \qquad t \ge 0. \tag{3.32}$$

Lemma 28 ([10] **Lemma 9).** *Let $a(t) = \frac{d}{(c+t)^b}$ where $b \in (0, \frac{1}{4}]$, $d^2c^{1-2b} \ge 6b$. Define $\varphi(t) = \int_0^t \frac{a^2(s)}{2} ds$. Then, one has*

$$e^{-\varphi(t)} \int_0^t e^{\varphi(t)} |\dot a(s)| \|V_\delta(s)\| ds \le \frac{1}{2} a(t) \|V_\delta(t)\|. \tag{3.33}$$

Lemma 29 ([11] **Lemma 9).** *Let* $a(t) = \frac{d}{(c+t)^b}$ *where* $b \in (0, \frac{1}{2}]$, $dc^{1-b} \geq 6b$. *Define* $\varphi(t) = \int_0^t \frac{a(s)}{2} ds$. *Then, one has*

$$e^{-\varphi(t)} \int_0^t e^{\varphi(t)} |\dot{a}(s)| \|V_\delta(s)\| ds \leq \frac{1}{2} a(t) \|V_\delta(t)\|. \tag{3.34}$$

4. Proofs of the basic results

4.1. *Proofs of the Discrepancy Principles*

4.1.1. *Proof of Theorem 1*

Proof. The existence and uniqueness of $a(\delta)$ follow from Lemma 22. Let us show that

$$\lim_{\delta \to 0} a(\delta) = 0. \tag{4.1}$$

The triangle inequality, the first inequality in (3.29), equality (2.2) and equality (2.1) imply

$$\begin{aligned} a(\delta)\|V_{a(\delta)}\| &\leq a(\delta)\big(\|V_{\delta,a(\delta)} - V_{a(\delta)}\| + \|V_{\delta,a(\delta)}\|\big) \\ &\leq \delta + a(\delta)\|V_{\delta,a(\delta)}\| = \delta + C\delta^\gamma, \end{aligned} \tag{4.2}$$

where V_a solves (2.1) with $\delta = 0$. From inequality (4.2), one gets

$$\lim_{\delta \to 0} a(\delta)\|V_{a(\delta)}\| = 0. \tag{4.3}$$

It follows from Lemma 20 with $f_\delta = f$, i.e., $\delta = 0$, that the function $\phi_0(a) := a\|V_a\|$ is nonnegative and strictly increasing on $(0, \infty)$. This and relation (4.3) imply:

$$\lim_{\delta \to 0} a(\delta) = 0. \tag{4.4}$$

From (2.2) and (3.30), one gets

$$C\delta^\gamma = a\|V_{\delta,a}\| \leq a(\delta)\|y\| + \delta. \tag{4.5}$$

Thus, one gets:

$$C\delta^\gamma - \delta \leq a(\delta)\|y\|. \tag{4.6}$$

If $\gamma < 1$ then $C - \delta^{1-\gamma} > 0$ for sufficiently small δ. This implies:

$$0 \leq \lim_{\delta \to 0} \frac{\delta}{a(\delta)} \leq \lim_{\delta \to 0} \frac{\delta^{1-\gamma}\|y\|}{C - \delta^{1-\gamma}} = 0. \tag{4.7}$$

By the triangle inequality and the first inequality (3.29), one has

$$\|V_{\delta,a(\delta)} - y\| \leq \|V_{a(\delta)} - y\| + \|V_{a(\delta)} - V_{\delta,a(\delta)}\| \leq \|V_{a(\delta)} - y\| + \frac{\delta}{a(\delta)}. \tag{4.8}$$

Relation (2.3) follows from (4.4), (4.7), (4.8) and Lemma 19. □

4.1.2. *Proof of Theorem 3*

Proof. By Lemma 23

$$a\|u-v\| \le \|F(u)-F(v)+au-av\|, \qquad \forall v,u\in H, \quad \forall a>0. \tag{4.9}$$

Using inequality (4.9) with $v=v_\delta$ and $u=V_{\delta,\alpha(\delta)}$, equation (1.4) with $a=\alpha(\delta)$, and inequality (2.9), one gets

$$\begin{aligned}\alpha(\delta)\|v_\delta - V_{\delta,\alpha(\delta)}\| &\le \|F(v_\delta)-F(V_{\delta,\alpha(\delta)})+\alpha(\delta)v_\delta-\alpha(\delta)V_{\delta,\alpha(\delta)}\| \\ &= \|F(v_\delta)+\alpha(\delta)v_\delta - f_\delta\| \le \theta\delta.\end{aligned} \tag{4.10}$$

Therefore,

$$\|v_\delta - V_{\delta,\alpha(\delta)}\| \le \frac{\theta\delta}{\alpha(\delta)}. \tag{4.11}$$

Using the triangle inequality, (3.30) and (4.11), one gets:

$$\alpha(\delta)\|v_\delta\| \le \alpha(\delta)\|V_{\delta,\alpha(\delta)}\| + \alpha(\delta)\|v_\delta - V_{\delta,\alpha(\delta)}\| \le \alpha(\delta)\|y\|+\delta+\theta\delta. \tag{4.12}$$

From the triangle inequality and inequalities (2.9) and (2.10) one obtains:

$$\alpha(\delta)\|v_\delta\| \ge \|F(v_\delta)-f_\delta\| - \|F(v_\delta)+\alpha(\delta)v_\delta - f_\delta\| \ge C_1\delta^\gamma - \theta\delta. \tag{4.13}$$

Inequalities (4.12) and (4.13) imply

$$C_1\delta^\gamma - \theta\delta \le \theta\delta + \alpha(\delta)\|y\| + \delta. \tag{4.14}$$

This inequality and the fact that $C_1 - \delta^{1-\gamma} - 2\theta\delta^{1-\gamma} > 0$ for sufficiently small δ and $0<\gamma<1$ imply

$$\frac{\delta}{\alpha(\delta)} \le \frac{\delta^{1-\gamma}\|y\|}{C_1 - \delta^{1-\gamma} - 2\theta\delta^{1-\gamma}}, \qquad 0<\delta\ll 1. \tag{4.15}$$

Thus, one obtains

$$\lim_{\delta\to 0}\frac{\delta}{\alpha(\delta)} = 0. \tag{4.16}$$

From the triangle inequality and inequalities (2.9), (2.10) and (4.11), one gets

$$\begin{aligned}\alpha(\delta)\|V_{\delta,\alpha(\delta)}\| &\le \|F(v_\delta)-f_\delta\| + \|F(v_\delta)+\alpha(\delta)v_\delta - f_\delta\| + \alpha(\delta)\|v_\delta - V_{\delta,\alpha(\delta)}\| \\ &\le C_2\delta^\gamma + \theta\delta + \theta\delta.\end{aligned}$$

This inequality implies

$$\lim_{\delta\to 0}\alpha(\delta)\|V_{\delta,\alpha(\delta)}\| = 0. \tag{4.17}$$

The triangle inequality and inequality (3.29) imply

$$\begin{aligned}\alpha\|V_\alpha\| &\le \alpha\big(\|V_{\delta,\alpha}-V_\alpha\| + \|V_{\delta,\alpha}\|\big) \\ &\le \delta + \alpha\|V_{\delta,\alpha}\|.\end{aligned} \tag{4.18}$$

From (4.18) and (4.17), one gets

$$\lim_{\delta\to 0} \alpha(\delta)\|V_{\alpha(\delta)}\| = 0. \tag{4.19}$$

It follows from Lemma 20 with $f_\delta = f$, i.e., $\delta = 0$, that the function $\phi_0(a) := a\|V_a\|$ is nonnegative and strictly increasing on $(0, \infty)$. This and relation (4.19) imply

$$\lim_{\delta\to 0} \alpha(\delta) = 0. \tag{4.20}$$

From the triangle inequality and inequalities (4.11) and (3.29) one obtains

$$\begin{aligned} \|v_\delta - y\| &\le \|v_\delta - V_{\delta,\alpha(\delta)}\| + \|V_{\delta,\alpha(\delta)} - V_{\alpha(\delta)}\| + \|V_{\alpha(\delta)} - y\| \\ &\le \frac{\theta\delta}{\alpha(\delta)} + \frac{\delta}{\alpha(\delta)} + \|V_{\alpha(\delta)} - y\|, \end{aligned} \tag{4.21}$$

where $V_{\alpha(\delta)}$ solves equation (3) with $a = \alpha(\delta)$ and $f_\delta = f$.

The conclusion (2.11) follows from (4.16), (4.20), (4.21) and Lemma 19. Theorem 3 is proved. □

4.2. *Proofs of convergence of the Dynamical Systems Method*

4.2.1. *Proof of Theorem 5*

Proof. Denote

$$C := \frac{C_1 + 1}{2}. \tag{4.22}$$

Let

$$w := u_\delta - V_\delta, \quad g(t) := \|w\|. \tag{4.23}$$

From (4.23) and (2.15) one gets

$$\dot{w} = -\dot{V}_\delta - A_{a(t)}^{-1}\big[F(u_\delta) - F(V_\delta) + a(t)w\big]. \tag{4.24}$$

We use Taylor's formula and get:

$$F(u_\delta) - F(V_\delta) + aw = A_a w + K, \quad \|K\| \le \frac{M_2}{2}\|w\|^2, \tag{4.25}$$

where $K := F(u_\delta) - F(V_\delta) - Aw$, and M_2 is the constant from the estimate (2.13) and $A_a := A + aI$. Multiplying (4.24) by w and using (4.25) one gets

$$g\dot{g} \le -g^2 + \frac{M_2}{2}\|A_{a(t)}^{-1}\|g^3 + \|\dot{V}_\delta\|g. \tag{4.26}$$

Since $0 < a(t) \searrow 0$, there exists $t_0 > 0$ such that

$$\frac{\delta}{a(t_0)} = \frac{1}{C-1}\|y\|, \qquad C > 1. \tag{4.27}$$

This and Lemma 24 imply that inequalities (3.11) and (3.12) hold. Since $g \ge 0$, inequalities (4.26) and (3.12) imply, for all $t \in [0, t_0]$, that

$$\dot{g} \le -g(t) + \frac{c_0}{a(t)}g^2 + \frac{|\dot{a}|}{a(t)}c_1, \quad c_0 = \frac{M_2}{2}, \quad c_1 = \|y\|\left(1 + \frac{1}{C-1}\right). \tag{4.28}$$

Inequality (4.28) is of the type (2.87) with

$$\gamma(t) = 1, \quad \alpha(t) = \frac{c_0}{a(t)}, \quad \beta(t) = c_1 \frac{|\dot{a}|}{a(t)}. \tag{4.29}$$

Let us check assumptions (2.92)–(2.94). Take

$$\mu(t) = \frac{\lambda}{a(t)}, \tag{4.30}$$

where $\lambda = const > 0$ and satisfies conditions (2.16)–(2.19) in Lemma 4. It follows that inequalities (2.92)–(2.94) hold. Since u_0 satisfies the first inequality in (2.20), one gets $g(0) \le \frac{a(0)}{\lambda}$, by Remark 9. This, inequalities (2.92)–(2.94), and Corollary 17 yield

$$g(t) < \frac{a(t)}{\lambda}, \quad \forall t \le t_0, \qquad g(t) := \|u_\delta(t) - V_\delta(t)\|. \tag{4.31}$$

Therefore,

$$\begin{aligned} \|F(u_\delta(t)) - f_\delta\| \le& \|F(u_\delta(t)) - F(V_\delta(t))\| + \|F(V_\delta(t)) - f_\delta\| \\ \le& M_1 g(t) + \|F(V_\delta(t)) - f_\delta\| \\ \le& \frac{M_1 a(t)}{\lambda} + \|F(V_\delta(t)) - f_\delta\|, \qquad \forall t \le t_0. \end{aligned} \tag{4.32}$$

From (3.11) one gets

$$\|F(V_\delta(t_0)) - f_\delta\| \le C\delta. \tag{4.33}$$

This, inequality (4.32), the inequality $\frac{M_1}{\lambda} \le \|y\|$ (see (2.16)), the relation (4.27), and the definition $C_1 = 2C - 1$ (see (4.22)), imply

$$\begin{aligned} \|F(u_\delta(t_0)) - f_\delta\| \le& \frac{M_1 a(t_0)}{\lambda} + C\delta \\ \le& \frac{M_1 \delta (C-1)}{\lambda \|y\|} + C\delta \le (C-1)\delta + C\delta = C_1 \delta. \end{aligned} \tag{4.34}$$

Thus, if

$$\|F(u_\delta(0)) - f_\delta\| > C_1 \delta^\zeta, \quad 0 < \zeta \le 1, \tag{4.35}$$

then, by the continuity of the function $t \to \|F(u_\delta(t)) - f_\delta\|$ on $[0, \infty)$, there exists $t_\delta \in (0, t_0)$ such that

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C_1 \delta^\zeta \tag{4.36}$$

for any given $\zeta \in (0, 1]$, and any fixed $C_1 > 1$.

Let us prove (2.22).

From (4.32) with $t = t_\delta$, and from (3.30), one gets

$$\begin{aligned} C_1 \delta^\zeta \le& M_1 \frac{a(t_\delta)}{\lambda} + a(t_\delta) \|V_\delta(t_\delta)\| \\ \le& M_1 \frac{a(t_\delta)}{\lambda} + \|y\| a(t_\delta) + \delta. \end{aligned} \tag{4.37}$$

Thus, for sufficiently small δ, one gets

$$\tilde{C}\delta^{\zeta} \le a(t_\delta)\left(\frac{M_1}{\lambda} + \|y\|\right), \quad \tilde{C} > 0, \tag{4.38}$$

where $\tilde{C} < C_1$ is a constant. Therefore,

$$\lim_{\delta\to 0} \frac{\delta}{a(t_\delta)} \le \lim_{\delta\to 0} \frac{\delta^{1-\zeta}}{\tilde{C}}\left(\frac{M_1}{\lambda} + \|y\|\right) = 0, \quad 0 < \zeta < 1. \tag{4.39}$$

We claim that

$$\lim_{\delta\to 0} t_\delta = \infty. \tag{4.40}$$

Let us prove (4.40). Using (2.15), one obtains:

$$\frac{d}{dt}\big(F(u_\delta) + au_\delta - f_\delta\big) = A_a\dot{u}_\delta + \dot{a}u_\delta = -\big(F(u_\delta) + au_\delta - f_\delta\big) + \dot{a}u_\delta. \tag{4.41}$$

This and (2.12) imply:

$$\frac{d}{dt}\big[F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta)\big] = -\big[F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta)\big] + \dot{a}u_\delta. \tag{4.42}$$

Denote

$$v := v(t) := F(u_\delta(t)) - F(V_\delta(t)) + a(t)(u_\delta(t) - V_\delta(t)), \qquad h := h(t) := \|v\|. \tag{4.43}$$

Multiplying (4.42) by v, one obtains

$$\begin{aligned} h\dot{h} &= -h^2 + \langle v, \dot{a}(u_\delta - V_\delta)\rangle + \dot{a}\langle v, V_\delta\rangle \\ &\le -h^2 + h|\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|h\|V_\delta\|, \qquad h \ge 0. \end{aligned} \tag{4.44}$$

Thus,

$$\dot{h} \le -h + |\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|\|V_\delta\|. \tag{4.45}$$

Note that from inequality (3.3) one has

$$a\|u_\delta - V_\delta\| \le h, \quad \|F(u_\delta) - F(V_\delta)\| \le h. \tag{4.46}$$

Inequalities (4.45) and (4.46) imply

$$\dot{h} \le -h\left(1 - \frac{|\dot{a}|}{a}\right) + |\dot{a}|\|V_\delta\|. \tag{4.47}$$

Since $1 - \frac{|\dot{a}|}{a} \ge \frac{1}{2}$ because $c \ge 2b$, it follows from inequality (4.47) that

$$\dot{h} \le -\frac{1}{2}h + |\dot{a}|\|V_\delta\|. \tag{4.48}$$

Inequality (4.48) implies:

$$h(t) \le h(0)e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot{a}|\|V_\delta\|ds. \tag{4.49}$$

From (4.49) and the second inequality in (4.46), one gets

$$\|F(u_\delta(t)) - F(V_\delta(t))\| \le h(0)e^{-\frac{t}{2}} + e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}}|\dot{a}|\|V_\delta\|ds. \tag{4.50}$$

This and the triangle inequality imply

$$\begin{aligned} \|F(u_\delta(t)) - f_\delta\| &\ge \|F(V_\delta(t)) - f_\delta\| - \|F(V_\delta(t)) - F(u_\delta(t))\| \\ &\ge a(t)\|V_\delta(t)\| - h(0)e^{-\frac{t}{2}} - e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}}|\dot{a}|\|V_\delta\|ds. \end{aligned} \tag{4.51}$$

By Lemma 27 one gets

$$\frac{1}{2}a(t)\|V_\delta(t)\| \ge e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}}|\dot{a}|\|V_\delta(s)\|ds. \tag{4.52}$$

From the second inequality in (2.20), one gets

$$h(0)e^{-\frac{t}{2}} \le \frac{1}{4}a(0)\|V_\delta(0)\|e^{-\frac{t}{2}}, \qquad t \ge 0. \tag{4.53}$$

Since $a(t) = \frac{d}{(c+t)^b}$, $b \in (0, 1]$, $c \ge 1$, $2b < c$, one gets

$$e^{-\frac{t}{2}}a(0) \le a(t). \tag{4.54}$$

Therefore,

$$e^{-\frac{t}{2}}h(0) \le \frac{1}{4}a(t)\|V_\delta(0)\| \le \frac{1}{4}a(t)\|V_\delta(t)\|, \quad t \ge 0, \tag{4.55}$$

where we have used the inequality $\|V_\delta(t)\| \le \|V_\delta(t')\|$ for $t < t'$, established in Lemma 20. From (4.36) and (4.51), (4.52), (4.55), one gets

$$C_1\delta^\zeta = \|F(u_\delta(t_\delta)) - f_\delta\| \ge \frac{1}{4}a(t_\delta)\|V_\delta(t_\delta)\|. \tag{4.56}$$

It follows from the triangle inequality and the first inequality in (3.29) that

$$a(t)\|V(t)\| \le a(t)\|V_\delta(t)\| + \delta.$$

This and (4.56) imply

$$0 \le \lim_{\delta\to 0} a(t_\delta)\|V(t_\delta)\| \le \lim_{\delta\to 0}\left(4C_1\delta^\zeta + \delta\right) = 0. \tag{4.57}$$

Since $\|V(t)\|$ increases (see Lemma 20), the above formula implies $\lim_{\delta\to 0} a(t_\delta) = 0$. Since $0 < a(t) \searrow 0$, it follows that $\lim_{\delta\to 0} t_\delta = \infty$, i.e., (4.40) holds.

It is now easy to finish the proof of the Theorem 5.

From the triangle inequality and inequalities (4.31) and (3.29) one obtains

$$\begin{aligned} \|u_\delta(t_\delta) - y\| &\le \|u_\delta(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - y\| \\ &\le \frac{a(t_\delta)}{\lambda} + \frac{\delta}{a(t_\delta)} + \|V(t_\delta) - y\|. \end{aligned} \tag{4.58}$$

Note that $V(t) := V_\delta(t)|_{\delta=0}$ and $V_\delta(t)$ solves (2.12). Note that $V(t_\delta) = V_{0,a(t_\delta)}$ (see equation (2.12)). From (4.39), (4.40), inequality (4.58) and Lemma 19, one obtains (2.22). Theorem 5 is proved. $\square$

Remark 10. The trajectory $u_\delta(t)$ remains in the ball $B(u_0, R) := \{u : \|u - u_0\| < R\}$ for all $t \le t_\delta$, where R does not depend on δ as $\delta \to 0$. Indeed, estimates (4.31), (3.30) and (3.13) imply:

$$\begin{aligned}\|u_\delta(t) - u_0\| &\le \|u_\delta(t) - V_\delta(t)\| + \|V_\delta(t)\| + \|u_0\| \\ &\le \frac{a(0)}{\lambda} + \frac{C\|y\|}{C-1} + \|u_0\| := R, \qquad \forall t \le t_\delta.\end{aligned} \tag{4.59}$$

Here we have used the fact that $t_\delta < t_0$ (see Lemma 24). Since one can choose $a(t)$ and λ so that $\frac{a(0)}{\lambda}$ is uniformly bounded as $\delta \to 0$ regardless of the growth of M_1 (see Remark 7) one concludes that R can be chosen independent of δ and M_1.

4.2.2. *Proof of Theorem 7*

Proof. Denote

$$C := \frac{C_1 + 1}{2}. \tag{4.60}$$

Let

$$w := u_\delta - V_\delta, \quad g(t) := \|w\|. \tag{4.61}$$

From (4.61) and (2.24) one gets

$$\dot{w} = -\dot{V}_\delta - A^*_{a(t)}\big[F(u_\delta) - F(V_\delta) + a(t)w\big]. \tag{4.62}$$

We use Taylor's formula and get:

$$F(u_\delta) - F(V_\delta) + aw = A_a w + K, \quad \|K\| \le \frac{M_2}{2}\|w\|^2, \tag{4.63}$$

where $K := F(u_\delta) - F(V_\delta) - Aw$, and M_2 is the constant from the estimate (2.13) and $A_a := A + aI$. Multiplying (4.62) by w and using (4.63) one gets

$$g\dot{g} \le -a^2 g^2 + \frac{M_2(M_1 + a)}{2} g^3 + \|\dot{V}_\delta\| g, \quad g := g(t) := \|w(t)\|, \tag{4.64}$$

where the estimates: $\langle A_a^* A_a w, w\rangle \ge a^2 g^2$ and $\|A_a\| \le M_1 + a$ were used. Note that the inequality $\langle A_a^* A_a w, w\rangle \ge a^2 g^2$ is true if $A \ge 0$. Since F is monotone and differentiable (see (1.2)), one has $A := F'(u_\delta) \ge 0$.

Let $t_0 > 0$ be such that

$$\frac{\delta}{a(t_0)} = \frac{1}{C-1}\|y\|, \qquad C > 1, \tag{4.65}$$

as in (3.10). It follows from Lemma 24 that inequalities (3.11) and (3.12) hold.

Since $g \geq 0$, inequalities (4.64) and (3.12) imply, for all $t \in [0, t_0]$, that

$$\dot{g}(t) \leq -a^2(t)g(t) + c_0(M_1 + a(t))g^2(t) + \frac{|\dot{a}(t)|}{a(t)}c_1, \quad c_0 = \frac{M_2}{2},\ c_1 = \|y\|\left(1 + \frac{1}{C-1}\right). \tag{4.66}$$

Inequality (4.66) is of the type (2.87) with

$$\gamma(t) = a^2(t), \quad \alpha(t) = c_0(M_1 + a(t)), \quad \beta(t) = c_1\frac{|\dot{a}(t)|}{a(t)}. \tag{4.67}$$

Let us check assumptions (2.92)–(2.94). Take

$$\mu(t) = \frac{\lambda}{a^2(t)}, \quad \lambda = \text{const.} \tag{4.68}$$

By Lemma 6 there exist λ and $a(t)$ such that conditions (2.25)–(2.29) hold. This implies that inequalities (2.92)–(2.94) hold. Thus, Corollary 17 yields

$$g(t) < \frac{a^2(t)}{\lambda}, \quad \forall t \leq t_0. \tag{4.69}$$

Note that inequality (4.69) holds for $t = 0$ since (2.29) holds. Therefore,

$$\begin{aligned} \|F(u_\delta(t)) - f_\delta\| \leq& \|F(u_\delta(t)) - F(V_\delta(t))\| + \|F(V_\delta(t)) - f_\delta\| \\ \leq& M_1 g(t) + \|F(V_\delta(t)) - f_\delta\| \\ \leq& \frac{M_1 a^2(t)}{\lambda} + \|F(V_\delta(t)) - f_\delta\|, \qquad \forall t \leq t_0. \end{aligned} \tag{4.70}$$

It follows from Lemma 20 that $\|F(V_\delta(t)) - f_\delta\|$ is decreasing. Since $t_1 \leq t_0$, one gets

$$\|F(V_\delta(t_0)) - f_\delta\| \leq \|F(V_\delta(t_1)) - f_\delta\| = C\delta. \tag{4.71}$$

This, inequality (4.70), the inequality $\frac{M_1}{\lambda} \leq \|y\|$ (see (2.26)), the relation (4.65), and the definition $C_1 = 2C - 1$ (see (4.60)) imply

$$\begin{aligned} \|F(u_\delta(t_0)) - f_\delta\| \leq& \frac{M_1 a^2(t_0)}{\lambda} + C\delta \\ \leq& \frac{M_1 \delta(C-1)}{\lambda\|y\|} + C\delta \leq (2C-1)\delta = C_1\delta. \end{aligned} \tag{4.72}$$

We have used the inequality

$$a^2(t_0) \leq a(t_0) = \frac{\delta(C-1)}{\|y\|} \tag{4.73}$$

which is true if δ is sufficiently small, or, equivalently, if t_0 is sufficiently large. Thus, if

$$\|F(u_\delta(0)) - f_\delta\| > C_1\delta^\zeta, \quad 0 < \zeta \leq 1, \tag{4.74}$$

then there exists $t_\delta \in (0, t_0)$ such that

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C_1\delta^\zeta \tag{4.75}$$

for any given $\zeta \in (0, 1]$, and any fixed $C_1 > 1$.

Let us prove (2.32)*. If this is done, then Theorem 7 is proved.*

First, we prove that $\lim_{\delta\to 0}\frac{\delta}{a(t_\delta)} = 0$.

From (4.70) with $t = t_\delta$, (2.12) and (3.30), one gets

$$\begin{aligned} C_1\delta^\zeta &\le M_1\frac{a^2(t_\delta)}{\lambda} + a(t_\delta)\|V_\delta(t_\delta)\| \\ &\le M_1\frac{a^2(t_\delta)}{\lambda} + \|y\|a(t_\delta) + \delta. \end{aligned} \tag{4.76}$$

Thus, for sufficiently small δ, one gets

$$\tilde{C}\delta^\zeta \le C_1\delta^\zeta - \delta \le a(t_\delta)\left(\frac{M_1 a(0)}{\lambda} + \|y\|\right), \quad \tilde{C} > 0, \tag{4.77}$$

where $\tilde{C} < C_1$ is a constant. Therefore,

$$\lim_{\delta\to 0}\frac{\delta}{a(t_\delta)} \le \lim_{\delta\to 0}\frac{\delta^{1-\zeta}}{\tilde{C}}\left(\frac{M_1 a(0)}{\lambda} + \|y\|\right) = 0, \quad 0 < \zeta < 1. \tag{4.78}$$

Secondly, we prove that

$$\lim_{\delta\to 0} t_\delta = \infty. \tag{4.79}$$

Using (2.24), one obtains:

$$\frac{d}{dt}\big(F(u_\delta) + au_\delta - f_\delta\big) = A_a\dot{u}_\delta + \dot{a}u_\delta = -A_aA_a^*\big(F(u_\delta) + au_\delta - f_\delta\big) + \dot{a}u_\delta. \tag{4.80}$$

This and (2.12) imply:

$$\frac{d}{dt}\big[F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta)\big] = -A_aA_a^*\big[F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta)\big] + \dot{a}u_\delta. \tag{4.81}$$

Denote

$$v := F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta), \quad h := h(t) := \|v(t)\|. \tag{4.82}$$

Multiplying (4.81) by v and using monotonicity of F, one obtains

$$\begin{aligned} h\dot{h} &= -\langle A_aA_a^*v, v\rangle + \langle v, \dot{a}(u_\delta - V_\delta)\rangle + \dot{a}\langle v, V_\delta\rangle \\ &\le -h^2a^2 + h|\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|h\|V_\delta\|, \qquad h \ge 0. \end{aligned} \tag{4.83}$$

Again, we have used the inequality $A_aA_a^* \ge a^2$, which holds for $A \ge 0$, i.e., monotone operators F. Thus,

$$\dot{h} \le -ha^2 + |\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|\|V_\delta\|. \tag{4.84}$$

From inequality (3.3) we have

$$a\|u_\delta - V_\delta\| \le h, \quad \|F(u_\delta) - F(V_\delta)\| \le h. \tag{4.85}$$

Inequalities (4.84) and (4.85) imply

$$\dot{h} \le -h\left(a^2 - \frac{|\dot{a}|}{a}\right) + |\dot{a}|\|V_\delta\|. \tag{4.86}$$

Since $a^2 - \frac{|\dot a|}{a} \ge \frac{3a^2}{4} > \frac{a^2}{2}$ by inequality (2.25), it follows from inequality (4.86) that

$$\dot h \le -\frac{a^2}{2}h + |\dot a|\|V_\delta\|. \tag{4.87}$$

Inequality (4.87) implies:

$$h(t) \le h(0)e^{-\int_0^t \frac{a^2(s)}{2}ds} + e^{-\int_0^t \frac{a^2(s)}{2}ds}\int_0^t e^{\int_0^s \frac{a^2(\xi)}{2}d\xi}|\dot a(s)|\|V_\delta(s)\|ds. \tag{4.88}$$

Denote

$$\varphi(t) := \int_0^t \frac{a^2(s)}{2}ds. \tag{4.89}$$

From (4.88) and (4.85), one gets

$$\|F(u_\delta(t)) - F(V_\delta(t))\| \le h(0)e^{-\varphi(t)} + e^{-\varphi(t)}\int_0^t e^{\varphi(s)}|\dot a(s)|\|V_\delta(s)\|ds. \tag{4.90}$$

This and the triangle inequality imply

$$\begin{aligned}\|F(u_\delta(t)) - f_\delta\| &\ge \|F(V_\delta(t)) - f_\delta\| - \|F(V_\delta(t)) - F(u_\delta(t))\| \\ &\ge a(t)\|V_\delta(t)\| - h(0)e^{-\varphi(t)} - e^{-\varphi(t)}\int_0^t e^{\varphi(s)}|\dot a|\|V_\delta\|ds.\end{aligned} \tag{4.91}$$

From Lemma 28 it follows that

$$\frac{1}{2}a(t)\|V_\delta(t)\| \ge e^{-\varphi(t)}\int_0^t e^{\varphi(s)}|\dot a|\|V_\delta(s)\|ds. \tag{4.92}$$

From (2.30) one gets

$$h(0)e^{-\varphi(t)} \le \frac{1}{4}a(0)\|V_\delta(0)\|e^{-\varphi(t)}, \qquad t \ge 0. \tag{4.93}$$

If $c \ge 1$ and $2b \le c_1^2$, then it follows that

$$e^{-\varphi(t)}a(0) \le a(t). \tag{4.94}$$

Indeed, inequality $a(0) \le a(t)e^{\varphi(t)}$ is obviously true for $t = 0$, and $\big(a(t)e^{\varphi(t)}\big)'_t \ge 0$, provided that $c \ge 1$ and $2b \le c_1^2$.

Inequalities (4.93) and (4.94) imply

$$e^{-\varphi(t)}h(0) \le \frac{1}{4}a(t)\|V_\delta(0)\| \le \frac{1}{4}a(t)\|V_\delta(t)\|, \quad t \ge 0. \tag{4.95}$$

where we have used the inequality $\|V_\delta(t)\| \le \|V_\delta(t')\|$ for $t \le t'$, established in Lemma 20. From (4.75) and (4.91), (4.92), (4.95), one gets

$$C\delta^\zeta = \|F(u_\delta(t_\delta)) - f_\delta\| \ge \frac{1}{4}a(t_\delta)\|V_\delta(t_\delta)\|. \tag{4.96}$$

It follows from the triangle inequality and the first inequality in (3.29) that

$$a(t)\|V(t)\| \le a(t)\|V_\delta(t)\| + \delta. \tag{4.97}$$

From (4.97) and (4.96) one gets

$$0 \le \lim_{\delta\to 0} a(t_\delta)\|V(t_\delta)\| \le \lim_{\delta\to 0} \big(4C\delta^\zeta + \delta\big) = 0. \tag{4.98}$$

Since $\|V(t)\|$ is increasing, this implies $\lim_{\delta\to 0} a(t_\delta) = 0$. Since $0 < a(t) \searrow 0$, it follows that (4.79) holds.

From the triangle inequality and inequalities (4.69) and (3.29) one obtains

$$\begin{aligned}\|u_\delta(t_\delta) - y\| &\le \|u_\delta(t_\delta) - V_\delta\| + \|V(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - y\| \\ &\le \frac{a^2(t_\delta)}{\lambda} + \frac{\delta}{a(t_\delta)} + \|V(t_\delta) - y\|,\end{aligned} \tag{4.99}$$

where $V(t) := V_\delta(t)|_{\delta=0}$ and $V_\delta(t)$ solves (2.12). From (4.78), (4.79), inequality (4.99) and Lemma 19, one obtains (2.32). Theorem 7 is proved. $\square$

By the arguments, similar to the ones in the proof of Theorem 11–15 or in Remark 10, one can show that the trajectory $u_\delta(t)$ remains in the ball $B(u_0, R) := \{u : \|u - u_0\| < R\}$ for all $t \le t_\delta$, where R does not depend on δ as $\delta \to 0$.

4.2.3. *Proof of Theorem 9*

Proof. Denote

$$C := \frac{C_1 + 1}{2}. \tag{4.100}$$

Let

$$w := u_\delta - V_\delta, \qquad g := g(t) := \|w(t)\|. \tag{4.101}$$

From (4.101) and (2.35) one gets

$$\dot{w} = -\dot{V}_\delta - \big[F(u_\delta) - F(V_\delta) + a(t)w\big]. \tag{4.102}$$

Multiplying (4.102) by w and using (1.2) one gets

$$g\dot{g} \le -ag^2 + \|\dot{V}_\delta\| g. \tag{4.103}$$

Let $t_0 > 0$ be such that

$$\frac{\delta}{a(t_0)} = \frac{1}{C-1}\|y\|, \qquad C > 1. \tag{4.104}$$

This t_0 exists and is unique since $a(t) > 0$ monotonically decays to 0 as $t \to \infty$. It follows from inequality (4.104) and Lemma 24 that inequalities (3.11) and (3.12) hold.

Since $g \ge 0$, inequalities (4.103) and (3.12) imply

$$\dot{g} \le -a(t)g(t) + \frac{|\dot{a}(t)|}{a(t)}c_1, \quad c_1 = \|y\|\left(1 + \frac{1}{C-1}\right). \tag{4.105}$$

Inequality (4.105) is of the type (2.87) with

$$\gamma(t) = a(t), \quad \alpha(t) = 0, \quad \beta(t) = c_1 \frac{|\dot{a}(t)|}{a(t)}. \tag{4.106}$$

Let us check assumptions (2.92)–(2.94). Take

$$\mu(t) = \frac{\lambda}{a(t)}, \quad \lambda = \text{const}. \tag{4.107}$$

By Lemma 8 there exist λ and $a(t)$ such that conditions (2.37)–(2.40) hold. It follows that inequalities (2.92)–(2.94) hold. Thus, Corollary 17 yields

$$g(t) < \frac{a(t)}{\lambda}, \quad \forall t \le t_0. \tag{4.108}$$

The triangle inequality and inequality (4.108) imply

$$\begin{aligned} \|F(u_\delta(t)) - f_\delta\| &\le \|F(u_\delta(t)) - F(V_\delta(t))\| + \|F(V_\delta(t)) - f_\delta\| \\ &\le M_1 g(t) + \|F(V_\delta(t)) - f_\delta\| \\ &\le \frac{M_1 a(t)}{\lambda} + \|F(V_\delta(t)) - f_\delta\|, \qquad \forall t \le t_0. \end{aligned} \tag{4.109}$$

Inequality (3.11), inequality (4.109), the inequality $\frac{M_1}{\lambda} \le \|y\|$ (see (2.37)), the relation (4.104), and the definition $C_1 = 2C - 1$ (see (4.100)) imply

$$\begin{aligned} \|F(u_\delta(t_0)) - f_\delta\| &\le \frac{M_1 a(t_0)}{\lambda} + C\delta \\ &\le \frac{M_1 \delta (C-1)}{\lambda \|y\|} + C\delta \le (C-1)\delta + C\delta = C_1 \delta. \end{aligned} \tag{4.110}$$

Thus, if

$$\|F(u_\delta(0)) - f_\delta\| > C_1 \delta^\zeta, \quad 0 < \zeta \le 1, \tag{4.111}$$

then there exists $t_\delta \in (0, t_0)$ such that

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C_1 \delta^\zeta \tag{4.112}$$

for any given $\zeta \in (0, 1]$, and any fixed $C_1 > 1$.

Let us prove (2.43). *If this is done, then Theorem 9 is proved.*

First, we prove that $\lim_{\delta \to 0} \frac{\delta}{a(t_\delta)} = 0$.

From (4.109) with $t = t_\delta$, and from (3.30), one gets

$$\begin{aligned} C_1 \delta^\zeta &\le M_1 \frac{a(t_\delta)}{\lambda} + a(t_\delta) \|V_\delta(t_\delta)\| \\ &\le M_1 \frac{a(t_\delta)}{\lambda} + \|y\| a(t_\delta) + \delta. \end{aligned} \tag{4.113}$$

Thus, for sufficiently small δ, one gets

$$\tilde{C} \delta^\zeta \le a(t_\delta) \left(\frac{M_1}{\lambda} + \|y\| \right), \quad \tilde{C} > 0, \tag{4.114}$$

where $\tilde{C} < C_1$ is a constant. Therefore,

$$\lim_{\delta\to 0}\frac{\delta}{a(t_\delta)} \le \lim_{\delta\to 0}\frac{\delta^{1-\zeta}}{\tilde{C}}\left(\frac{M_1}{\lambda}+\|y\|\right) = 0, \quad 0<\zeta<1. \tag{4.115}$$

Secondly, we prove that

$$\lim_{\delta\to 0} t_\delta = \infty. \tag{4.116}$$

Using (2.35), one obtains:

$$\frac{d}{dt}\big(F(u_\delta)+au_\delta-f_\delta\big) = A_a\dot{u}_\delta + \dot{a}u_\delta = -A_a\big(F(u_\delta)+au_\delta-f_\delta\big)+\dot{a}u_\delta, \tag{4.117}$$

where $A_a := F'(u_\delta)+a$. This and (2.12) imply:

$$\frac{d}{dt}\big[F(u_\delta)-F(V_\delta)+a(u_\delta-V_\delta)\big] = -A_a\big[F(u_\delta)-F(V_\delta)+a(u_\delta-V_\delta)\big]+\dot{a}u_\delta. \tag{4.118}$$

Denote

$$v := F(u_\delta)-F(V_\delta)+a(u_\delta-V_\delta), \quad h=\|v\|. \tag{4.119}$$

Multiplying (4.118) by v and using monotonicity of F, one obtains

$$\begin{aligned} h\dot{h} &= -\langle A_a v, v\rangle + \langle v, \dot{a}(u_\delta-V_\delta)\rangle + \dot{a}\langle v, V_\delta\rangle \\ &\le -h^2 a + h|\dot{a}|\|u_\delta-V_\delta\| + |\dot{a}|h\|V_\delta\|, \qquad h\ge 0. \end{aligned} \tag{4.120}$$

Again, we have used the inequality $\langle F'(u_\delta)v,v\rangle \ge 0$ which follows from the monotonicity of F. Thus,

$$\dot{h} \le -ha + |\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|\|V_\delta\|. \tag{4.121}$$

Inequalities (4.121) and (3.3) imply

$$\dot{h} \le -h\left(a-\frac{|\dot{a}|}{a}\right) + |\dot{a}|\|V_\delta\|. \tag{4.122}$$

Since $a-\frac{|\dot{a}|}{a} \ge \frac{a}{2}$ by inequality (2.36), it follows from inequality (4.122) that

$$\dot{h} \le -\frac{a}{2}h + |\dot{a}|\|V_\delta\|. \tag{4.123}$$

Inequality (4.123) implies:

$$h(t) \le h(0)e^{-\int_0^t \frac{a(s)}{2}ds} + e^{-\int_0^t \frac{a(s)}{2}ds}\int_0^t e^{\int_0^s \frac{a(\xi)}{2}d\xi}|\dot{a}(s)|\|V_\delta(s)\|ds. \tag{4.124}$$

Denote

$$\varphi(t) := \int_0^t \frac{a(s)}{2}ds. \tag{4.125}$$

From (4.124) and (3.3), one gets

$$\|F(u_\delta(t))-F(V_\delta(t))\| \le h(0)e^{-\varphi(t)} + e^{-\varphi(t)}\int_0^t e^{\varphi(s)}|\dot{a}(s)|\|V_\delta(s)\|ds. \tag{4.126}$$

Therefore,

$$\begin{aligned}\|F(u_\delta(t)) - f_\delta\| &\geq \|F(V_\delta(t)) - f_\delta\| - \|F(V_\delta(t)) - F(u_\delta(t))\| \\ &\geq a(t)\|V_\delta(t)\| - h(0)e^{-\varphi(t)} - e^{-\varphi(t)} \int_0^t e^{\varphi(s)}|\dot{a}|\|V_\delta\|ds.\end{aligned} \tag{4.127}$$

From Lemma 29 it follows that

$$\frac{1}{2}a(t)\|V_\delta(t)\| \geq e^{-\varphi(t)} \int_0^t e^{\varphi(s)}|\dot{a}|\|V_\delta(s)\|ds. \tag{4.128}$$

From (2.41) one gets

$$h(0)e^{-\varphi(t)} \leq \frac{1}{4}a(0)\|V_\delta(0)\|e^{-\varphi(t)}, \qquad t \geq 0. \tag{4.129}$$

If $c \geq 1$ and $2b \leq d$, then it follows that

$$e^{-\varphi(t)}a(0) \leq a(t). \tag{4.130}$$

Indeed, inequality $a(0) \leq a(t)e^{\varphi(t)}$ is obviously true for $t = 0$, and $\big(a(t)e^{\varphi(t)}\big)'_t \geq 0$, provided that $c \geq 1$ and $2b \leq d$.

Inequalities (4.129) and (4.130) imply

$$e^{-\varphi(t)}h(0) \leq \frac{1}{4}a(t)\|V_\delta(0)\| \leq \frac{1}{4}a(t)\|V_\delta(t)\|, \quad t \geq 0, \tag{4.131}$$

where we have used the inequality $\|V_\delta(t)\| \leq \|V_\delta(t')\|$ for $t \leq t'$, established in Lemma 20. From (4.112) and (4.127), (4.128), (4.131), one gets

$$C\delta^\zeta = \|F(u_\delta(t_\delta)) - f_\delta\| \geq \frac{1}{4}a(t_\delta)\|V_\delta(t_\delta)\|. \tag{4.132}$$

It follows from the triangle inequality and the first inequality in (3.29) one obtains

$$a(t)\|V(t)\| \leq a(t)\|V_\delta(t)\| + \delta. \tag{4.133}$$

This and inequality (4.132) imply

$$0 \leq \lim_{\delta\to 0} a(t_\delta)\|V(t_\delta)\| \leq \lim_{\delta\to 0} \big(4C\delta^\zeta + \delta\big) = 0. \tag{4.134}$$

Since $\|V(t)\|$ is increasing, this implies $\lim_{\delta\to 0} a(t_\delta) = 0$. Since $0 < a(t) \searrow 0$, it follows that (4.116) holds.

From the triangle inequality and inequalities (4.108) and (3.29) one obtains:

$$\begin{aligned}\|u_\delta(t_\delta) - y\| &\leq \|u_\delta(t_\delta) - V_\delta\| + \|V(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - y\| \\ &\leq \frac{a(t_\delta)}{\lambda} + \frac{\delta}{a(t_\delta)} + \|V(t_\delta) - y\|,\end{aligned} \tag{4.135}$$

where $V(t) := V_\delta(t)|_{\delta=0}$ and $V_\delta(t)$ solves (2.12). From (4.115), (4.116), inequality (4.135) and Lemma 19, one obtains (2.43). Theorem 9 is proved. □

By the arguments, similar to the ones in the proof of Theorem 11–15 or in Remark 10, one can show that: the trajectory $u_\delta(t)$ remains in the ball $B(u_0, R) := \{u : \|u - u_0\| < R\}$ for all $t \leq t_\delta$, where R does not depend on δ as $\delta \to 0$.

4.3. *Proofs of convergence of the iterative schemes*

4.3.1. *Proof of Theorem 11*

Proof. Denote

$$C := \frac{C_1 + 1}{2}. \tag{4.136}$$

Let

$$z_n := u_n - V_n, \quad g_n := \|z_n\|. \tag{4.137}$$

We use Taylor's formula and get:

$$F(u_n) - F(V_n) + a_n z_n = A_{a_n} z_n + K_n, \quad \|K_n\| \le \frac{M_2}{2}\|z_n\|^2, \tag{4.138}$$

where $K_n := F(u_n) - F(V_n) - F'(u_n)z_n$ and M_2 is the constant from (2.13). From (2.46) and (4.138) one obtains

$$z_{n+1} = z_n - z_n - A_n^{-1}K(z_n) - (V_{n+1} - V_n). \tag{4.139}$$

From (4.139), the second inequality in (4.138), and the estimate $\|A_n^{-1}\| \le \frac{1}{a_n}$, one gets

$$g_{n+1} \le \frac{M_2 g_n^2}{2a_n} + \|V_{n+1} - V_n\|. \tag{4.140}$$

Since $0 < a_n \searrow 0$, there exists a unique n_0 such that

$$\frac{\delta}{a_{n_0+1}} > \frac{1}{C-1}\|y\| \ge \frac{\delta}{a_{n_0}}, \qquad C > 1. \tag{4.141}$$

It follows from (4.141) and Lemma 25 that inequalities (3.17)–(3.19) hold.

Inequalities (4.140) and (3.19) imply

$$g_{n+1} \le \frac{c_0}{a_n} g_n^2 + \frac{a_n - a_{n+1}}{a_{n+1}} c_1, \quad c_0 = \frac{M_2}{2}, \quad c_1 = \|y\|\left(1 + \frac{2}{C-1}\right), \tag{4.142}$$

for all $n \le n_0 + 1$.

Let us show by induction that

$$g_n < \frac{a_n}{\lambda}, \qquad 0 \le n \le n_0 + 1. \tag{4.143}$$

Inequality (4.143) holds for $n = 0$ by Remark 9 (see (2.84)). Suppose (4.143) holds for some $n \ge 0$. From (4.142), (4.143) and (2.51), one gets

$$\begin{aligned} g_{n+1} &\le \frac{c_0}{a_n}\left(\frac{a_n}{\lambda}\right)^2 + \frac{a_n - a_{n+1}}{a_{n+1}} c_1 \\ &= \frac{c_0 a_n}{\lambda^2} + \frac{a_n - a_{n+1}}{a_{n+1}} c_1 \\ &\le \frac{a_{n+1}}{\lambda}. \end{aligned} \tag{4.144}$$

Thus, by induction, inequality (4.143) holds for all n in the region $0 \le n \le n_0 + 1$.

The triangle inequality implies

$$\|u_0 - u_n\| \le \|u_0\| + \|z_n\| + \|V_n\|. \tag{4.145}$$

Inequalities (3.18), (4.143), and (4.145) guarantee that the sequence u_n, generated by the iterative process (2.46), remains in the ball $B(u_0, R)$ for all $n \le n_0+1$, where

$$R \le \|u_0\| + \frac{a_0}{\lambda} + \|y\|\frac{C+1}{C-1}. \tag{4.146}$$

By Remark 7, one can choose a_0 and λ so that $\frac{a_0}{\lambda}$ is uniformly bounded as $\delta \to 0$ even if $M_1(R) \to \infty$ as $R \to \infty$ at an arbitrary fast rate. Thus, the sequence u_n stays in the ball $B(u_0, R)$ for $n \le n_0 + 1$ when $\delta \to 0$. An upper bound on R is given above. It does not depend on δ as $\delta \to 0$.

One has:

$$\begin{aligned}\|F(u_n) - f_\delta\| \le& \|F(u_n) - F(V_n)\| + \|F(V_n) - f_\delta\| \\ \le& M_1 g_n + \|F(V_n) - f_\delta\| \\ \le& \frac{M_1 a_n}{\lambda} + \|F(V_n) - f_\delta\|, \qquad \forall n \le n_0 + 1,\end{aligned} \tag{4.147}$$

where (4.143) was used and M_1 is the constant from (2.13). By Lemma 25 one gets

$$\|F(V_{n_0+1}) - f_\delta\| \le C\delta. \tag{4.148}$$

From (2.49), (4.147), (4.148), the relation (4.141), and the definition $C_1 = 2C - 1$ (see (4.136)), one concludes that

$$\begin{aligned}\|F(u_{n_0+1}) - f_\delta\| \le& \frac{M_1 a_{n_0+1}}{\lambda} + C\delta \\ \le& \frac{M_1 \delta (C-1)}{\lambda \|y\|} + C\delta \le (2C-1)\delta = C_1\delta.\end{aligned} \tag{4.149}$$

Thus, if

$$\|F(u_0) - f_\delta\| > C_1\delta^\gamma, \quad 0 < \gamma \le 1, \tag{4.150}$$

then one concludes from (4.149) *that there exists* n_δ, $0 < n_\delta \le n_0 + 1$, *such that*

$$\|F(u_{n_\delta}) - f_\delta\| \le C_1\delta^\gamma < \|F(u_n) - f_\delta\|, \quad 0 \le n < n_\delta, \tag{4.151}$$

for any given $\gamma \in (0, 1]$, *and any fixed* $C_1 > 1$.

Let us prove (2.54). If $n > 0$ is fixed, then $u_{\delta,n}$ is a continuous function of f_δ. Denote

$$\tilde{u}_N = \lim_{\delta \to 0} u_{\delta,N}, \tag{4.152}$$

where $N < \infty$ is a cluster point of n_{δ_m}, so that there exists a subsequence of n_{δ_m}, which we denote by n_m, such that

$$\lim_{m \to \infty} n_m = N. \tag{4.153}$$

From (4.152) and the continuity of F, one obtains:

$$\|F(\tilde{u}_N) - f\| = \lim_{m\to\infty} \|F(u_{n_{\delta_m}}) - f_{\delta_m}\| \le \lim_{m\to\infty} C_1\delta_m^\gamma = 0. \tag{4.154}$$

Thus, $\tilde{u}_N$ is a solution to the equation $F(u) = f$, and (2.54) is proved.

Let us prove (2.56) *assuming that* (2.55) *holds.* From (4.151) and (4.147) with $n = n_\delta - 1$, and from (3.30), one gets

$$C_1\delta^\gamma \le M_1 \frac{a_{n_\delta - 1}}{\lambda} + a_{n_\delta-1}\|V_{n_\delta - 1}\| \le M_1\frac{a_{n_\delta-1}}{\lambda} + \|y\|a_{n_\delta-1} + \delta. \tag{4.155}$$

If $0 < \delta < 1$ and δ is sufficiently small, then it follows from (4.155) that

$$\tilde{C}\delta^\gamma \le a_{n_\delta-1}\left(\frac{M_1}{\lambda} + \|y\|\right), \quad \tilde{C} > 0, \tag{4.156}$$

where $\tilde{C}$ is a constant. Therefore, by (4.156),

$$0 \le \lim_{\delta\to 0}\frac{\delta}{2a_{n_\delta}} \le \lim_{\delta\to 0}\frac{\delta}{a_{n_\delta-1}} \le \lim_{\delta\to 0}\frac{\delta^{1-\gamma}}{\tilde{C}}\left(\frac{M_1}{\lambda} + \|y\|\right) = 0, \quad 0<\gamma<1. \tag{4.157}$$

In particular, for $\delta = \delta_m$, one gets

$$\lim_{\delta_m\to 0}\frac{\delta_m}{a_{n_{\delta_m}}} = 0. \tag{4.158}$$

From the triangle inequality, inequalities (3.29) and (4.143), one obtains

$$\begin{aligned}\|u_{n_{\delta_m}} - y\| &\le \|u_{n_{\delta_m}} - V_{n_{\delta_m}}\| + \|V_{n_{\delta_m}} - V_{n_{\delta_m},0}\| + \|V_{n_{\delta_m},0} - y\|\\ &\le \frac{a_{n_{\delta_m}}}{\lambda} + \frac{\delta_m}{a_{n_{\delta_m}}} + \|V_{n_{\delta_m},0} - y\|.\end{aligned} \tag{4.159}$$

Recall that $V_{n,0} = \tilde{V}_{a_n}$ (cf. (2.44) and (2.12)). From (2.55), (4.158), inequality (4.159) and Lemma 19, one obtains (2.56). Theorem 11 is proved. □

4.3.2. *Proof of Theorem 13*

Proof. Denote

$$C := \frac{C_1 + 1}{2}. \tag{4.160}$$

Let

$$z_n := u_n - V_n, \quad g_n := \|z_n\|. \tag{4.161}$$

We use Taylor's formula and get:

$$F(u_n) - F(V_n) + a_n z_n = A_n z_n + K_n, \quad \|K_n\| \le \frac{M_2}{2}\|z_n\|^2, \tag{4.162}$$

where $K_n := F(u_n) - F(V_n) - F'(u_n)z_n$ and M_2 is the constant from (2.13). From (2.57) and (4.162) one obtains

$$z_{n+1} = z_n - \alpha_n A_n^* A_n z_n - \alpha_n A_n^* K(z_n) - (V_{n+1} - V_n). \tag{4.163}$$

From (4.163), (4.162), (2.59), and the estimate $\|A_n\| \le M_1 + a_n$, one gets

$$\begin{aligned} g_{n+1} &\le g_n\|1-\alpha_n A_n^* A_n\| + \frac{\alpha_n M_2(M_1+a_n)}{2} g_n^2 + \|V_{n+1}-V_n\| \\ &\le g_n(1-\alpha_n a_n^2) + \frac{\alpha_n M_2(M_1+a_n)}{2} g_n^2 + \|V_{n+1}-V_n\|. \end{aligned} \tag{4.164}$$

Since $0 < a_n \searrow 0$, for any fixed $\delta > 0$ there exists n_0 such that

$$\frac{\delta}{a_{n_0+1}} > \frac{1}{C-1}\|y\| \ge \frac{\delta}{a_{n_0}}, \qquad C > 1. \tag{4.165}$$

This and Lemma 25 imply that inequalities (3.17)–(3.19) hold.

Inequalities (4.164) and (3.19) imply

$$g_{n+1} \le (1-\alpha_n a_n^2)g_n + \alpha_n c_0(M_1+a_n)g_n^2 + \frac{a_n - a_{n+1}}{a_{n+1}}c_1, \qquad \forall\, n \le n_0+1, \tag{4.166}$$

where $c_0 = \frac{M_2}{2}$ and $c_1 = \|y\|\big(1 + \frac{1}{C-1}\big)$ (cf. (4.66)).

Let us show by induction that

$$g_n < \frac{a_n^2}{\lambda}, \qquad 0 \le n \le n_0+1. \tag{4.167}$$

Inequality (4.167) holds for $n = 0$ by Remark 9 (see (2.85)). Suppose (4.167) holds for some $n \ge 0$. From (4.166), (4.167) and (2.64), one gets

$$\begin{aligned} g_{n+1} &\le (1-\alpha_n a_n^2)\frac{a_n^2}{\lambda} + \alpha_n c_0(M_1+a_n)\left(\frac{a_n^2}{\lambda}\right)^2 + \frac{a_n - a_{n+1}}{a_{n+1}}c_1 \\ &= \frac{a_n^4}{\lambda}\left(\frac{\alpha_n c_0(M_1+a_n)}{\lambda} - \alpha_n\right) + \frac{a_n^2}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}}c_1 \\ &\le -\frac{\alpha_n a_n^4}{2\lambda} + \frac{a_n^2}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}}c_1 \\ &\le \frac{a_{n+1}^2}{\lambda}. \end{aligned} \tag{4.168}$$

Thus, by induction, inequality (4.167) holds for all n in the region $0 \le n \le n_0+1$.

The triangle inequality implies

$$\|u_0 - u_n\| \le \|u_0\| + \|z_n\| + \|V_n\|. \tag{4.169}$$

Inequalities (3.18), (4.167), and (4.169) guarantee that the sequence u_n, generated by the iterative process (2.57), remains in the ball $B(u_0, R)$ for all $n \le n_0+1$, where

$$R \le \|u_0\| + \frac{a_0}{\lambda} + \|y\|\frac{C+1}{C-1}. \tag{4.170}$$

By Remark 7, one can choose a_0 and λ so that $\frac{a_0}{\lambda}$ is uniformly bounded as $\delta \to 0$ even if $M_1(R) \to \infty$ as $R \to \infty$ at an arbitrary fast rate. Thus, the sequence u_n stays in the ball $B(u_0, R)$ for $n \le n_0 + 1$ when $\delta \to 0$. An upper bound on R is given above. It does not depend on δ as $\delta \to 0$.

One has:

$$\begin{aligned}\|F(u_n) - f_\delta\| \le& \|F(u_n) - F(V_n)\| + \|F(V_n) - f_\delta\| \\ \le& M_1 g_n + \|F(V_n) - f_\delta\| \\ \le& \frac{M_1 a_n^2}{\lambda} + \|F(V_n) - f_\delta\|, \qquad \forall n \le n_0 + 1,\end{aligned} \tag{4.171}$$

where (4.167) was used and M_1 is the constant from (2.13). By Lemma 25 one gets

$$\|F(V_{n_0+1}) - f_\delta\| \le C\delta. \tag{4.172}$$

From (2.62), (4.171), (4.172), the relation (4.165), and the definition $C_1 = 2C - 1$ (see (4.160)), one concludes that

$$\begin{aligned}\|F(u_{n_0+1}) - f_\delta\| \le& \frac{M_1 a_{n_0+1}^2}{\lambda} + C\delta \\ \le& \frac{M_1 \delta(C-1)}{\lambda\|y\|} + C\delta \le (C-1)\delta + C\delta = C_1\delta.\end{aligned} \tag{4.173}$$

Thus, if

$$\|F(u_0) - f_\delta\| > C_1\delta^\zeta, \quad 0 < \zeta \le 1, \tag{4.174}$$

then one concludes from (4.173) *that there exists* n_δ, $0 < n_\delta \le n_0 + 1$, *such that*

$$\|F(u_{n_\delta}) - f_\delta\| \le C_1\delta^\zeta < \|F(u_n) - f_\delta\|, \quad 0 \le n < n_\delta, \tag{4.175}$$

for any given $\zeta \in (0, 1]$, *and any fixed* $C_1 > 1$.

Let us prove (2.67).

If $n > 0$ is fixed, then $u_{\delta,n}$ is a continuous function of f_δ. Denote

$$\tilde{u} := \tilde{u}_N = \lim_{\delta\to 0} u_{\delta, n_{m_j}}, \tag{4.176}$$

where

$$\lim_{j\to\infty} n_{m_j} = N. \tag{4.177}$$

From (4.176) and the continuity of F, one obtains:

$$\|F(\tilde{u}) - f\| = \lim_{j\to\infty} \|F(u_{n_{m_j}}) - f_{\delta_{m_j}}\| \le \lim_{j\to\infty} C_1\delta_{m_j}^\zeta = 0. \tag{4.178}$$

Thus, $\tilde{u}$ is a solution to the equation $F(u) = f$, and (2.67) is proved.

Let us prove (2.69) *assuming that* (2.68) *holds.*

From (4.175) and (4.171) with $n = n_\delta - 1$, and from (3.30), one gets

$$C_1\delta^\zeta \le M_1\frac{a_{n_\delta-1}^2}{\lambda} + a_{n_\delta-1}\|V_{n_\delta-1}\| \le M_1\frac{a_{n_\delta-1}^2}{\lambda} + \|y\|a_{n_\delta-1} + \delta. \tag{4.179}$$

If $\delta > 0$ is sufficiently small, then the above equation implies

$$\tilde{C}\delta^\zeta \le a_{n_\delta-1}\left(\frac{M_1 a_0}{\lambda} + \|y\|\right), \quad \tilde{C} > 0, \tag{4.180}$$

where $\tilde{C} < C_1$ is a constant, and the inequality $a^2_{n_\delta-1} \le a_{n_\delta-1}a_0$ was used. Therefore, by (2.60),

$$0 \le \lim_{\delta\to 0} \frac{\delta}{2a_{n_\delta}} \le \lim_{\delta\to 0} \frac{\delta}{a_{n_\delta-1}} \le \lim_{\delta\to 0} \frac{\delta^{1-\zeta}}{\tilde{C}} \left(\frac{M_1 a_0}{\lambda} + \|y\| \right) = 0, \quad 0 < \zeta < 1. \quad (4.181)$$

In particular, for $\delta = \delta_m$, one gets

$$\lim_{\delta_m\to 0} \frac{\delta_m}{a_{n_m}} = 0. \quad (4.182)$$

From the triangle inequality and inequalities (3.29) and (4.167) one obtains

$$\begin{aligned} \|u_{n_m} - y\| &\le \|u_{n_m} - V_{n_m}\| + \|V_n - V_{n_m,0}\| + \|V_{n_m,0} - y\| \\ &\le \frac{a^2_{n_m}}{\lambda} + \frac{\delta_m}{a_{n_m}} + \|V_{n_m,0} - y\|. \end{aligned} \quad (4.183)$$

Recall that $V_{n,0} = \tilde{V}_{a_n}$ (cf. (2.44) and (2.12)). From (2.68), (4.182), inequality (4.183) and Lemma 19, one obtains (2.69). Theorem 13 is proved. □

4.3.3. *Proof of Theorem 15*

Proof. Denote

$$C := \frac{C_1 + 1}{2}. \quad (4.184)$$

Let

$$z_n := u_n - V_n, \quad g_n := \|z_n\|. \quad (4.185)$$

One has

$$F(u_n) - F(V_n) = J_n z_n, \qquad J_n = \int_0^1 F'(u_0 + \xi z_n) d\xi. \quad (4.186)$$

It follows for our assumptions that $J_n = J_n^* \ge 0$ and $\|J_n\| \le M_1$. From (2.70), (4.185) and (4.186) one obtains

$$\begin{aligned} z_{n+1} &= z_n - \alpha_n[F(u_n) - F(V_n) + a_n z_n] - (V_{n+1} - V_n) \\ &= (1 - \alpha_n(J_n + a_n))z_n - (V_{n+1} - V_n). \end{aligned} \quad (4.187)$$

From the triangle inequality, (4.187) and (2.72), one gets

$$\begin{aligned} g_{n+1} &\le g_n\|1 - \alpha_n(J_n + a_n)\| + \|V_{n+1} - V_n\| \\ &\le g_n(1 - \alpha_n a_n) + \|V_{n+1} - V_n\|. \end{aligned} \quad (4.188)$$

Since $0 < a_n \searrow 0$, for any fixed $\delta > 0$ there exists n_0 such that

$$\frac{\delta}{a_{n_0+1}} > \frac{1}{C-1}\|y\| \ge \frac{\delta}{a_{n_0}}, \qquad C > 1. \quad (4.189)$$

This and Lemma 25 imply that inequalities (3.17)–(3.19) hold.

Inequalities (4.188) and (3.19) imply

$$g_{n+1} \le (1-\alpha_n a_n)g_n + \frac{a_n - a_{n+1}}{a_{n+1}}c_1, \qquad \forall\, n \le n_0+1, \tag{4.190}$$

where $c_1 = \|y\|\big(1+\frac{1}{C-1}\big)$ (cf. (4.105)).

Let us show by induction that

$$g_n < \frac{a_n}{\lambda}, \qquad 0 \le n \le n_0+1. \tag{4.191}$$

Inequality (4.191) holds for $n=0$ by Remark 9 (see (2.84)). Suppose (4.191) holds for some $n \ge 0$. From (4.190), (4.191) and (2.76), one gets

$$\begin{aligned} g_{n+1} &\le (1-\alpha_n a_n)\frac{a_n}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}}c_1 \\ &= -\frac{\alpha_n a_n^2}{\lambda} + \frac{a_n}{\lambda} + \frac{a_n - a_{n+1}}{a_{n+1}}c_1 \\ &\le \frac{a_{n+1}}{\lambda}. \end{aligned} \tag{4.192}$$

Thus, by induction, inequality (4.191) holds for all n in the region $0 \le n \le n_0+1$.

The triangle inequality implies

$$\|u_0 - u_n\| \le \|u_0\| + \|z_n\| + \|V_n\|. \tag{4.193}$$

Inequalities (3.18), (4.191), and (4.193) guarantee that the sequence u_n, generated by the iterative process (2.70), remains in the ball $B(u_0,R)$ for all $n \le n_0+1$, where

$$R \le \|u_0\| + \frac{a_0}{\lambda} + \|y\|\frac{C+1}{C-1}. \tag{4.194}$$

By Remark 7, one can choose a_0 and λ so that $\frac{a_0}{\lambda}$ is uniformly bounded as $\delta \to 0$ even if $M_1(R) \to \infty$ as $R \to \infty$ at an arbitrary fast rate. Thus, the sequence u_n stays in the ball $B(u_0,R)$ for $n \le n_0+1$ when $\delta \to 0$. An upper bound on R is given above. It does not depend on δ as $\delta \to 0$.

One has:

$$\begin{aligned} \|F(u_n) - f_\delta\| \le& \|F(u_n) - F(V_n)\| + \|F(V_n) - f_\delta\| \\ \le& M_1 g_n + \|F(V_n) - f_\delta\| \\ \le& \frac{M_1 a_n}{\lambda} + \|F(V_n) - f_\delta\|, \qquad \forall n \le n_0+1, \end{aligned} \tag{4.195}$$

where (4.191) was used and M_1 is the constant from (2.13). From (3.17) one gets

$$\|F(V_{n_0+1}) - f_\delta\| \le C\delta. \tag{4.196}$$

From (2.75), (4.195), (4.196), the relation (4.189), and the definition $C_1 = 2C-1$ (see (4.184)), one concludes that

$$\begin{aligned} \|F(u_{n_0+1}) - f_\delta\| \le& \frac{M_1 a_{n_0+1}}{\lambda} + C\delta \\ \le& \frac{M_1 \delta (C-1)}{\lambda \|y\|} + C\delta \le (C-1)\delta + C\delta = C_1\delta. \end{aligned} \tag{4.197}$$

Thus, if

$$\|F(u_0) - f_\delta\| > C_1\delta^\zeta, \quad 0 < \zeta \le 1, \tag{4.198}$$

then one concludes from (4.197) *that there exists* n_δ, $0 < n_\delta \le n_0 + 1$, *such that*

$$\|F(u_{n_\delta}) - f_\delta\| \le C_1\delta^\zeta < \|F(u_n) - f_\delta\|, \quad 0 \le n < n_\delta, \tag{4.199}$$

for any given $\zeta \in (0, 1]$, *and any fixed* $C_1 > 1$.

Let us prove (2.79).

If $n > 0$ is fixed, then $u_{\delta,n}$ is a continuous function of f_δ. Denote

$$\tilde{u} := \tilde{u}_N = \lim_{\delta\to 0} u_{\delta,n_{m_j}}, \tag{4.200}$$

where

$$\lim_{j\to\infty} n_{m_j} = N. \tag{4.201}$$

From (4.200) and the continuity of F, one obtains:

$$\|F(\tilde{u}) - f\| = \lim_{j\to\infty} \|F(u_{n_{m_j}}) - f_{\delta_{m_j}}\| \le \lim_{j\to\infty} C_1\delta_{m_j}^\zeta = 0. \tag{4.202}$$

Thus, $\tilde{u}$ is a solution to the equation $F(u) = f$, and (2.79) is proved.

Let us prove (2.81) *assuming that* (2.80) *holds.*

From (4.199) and (4.195) with $n = n_\delta - 1$, and from (3.30), one gets

$$C_1\delta^\zeta \le M_1\frac{a_{n_\delta-1}}{\lambda} + a_{n_\delta-1}\|V_{n_\delta-1}\| \le M_1\frac{a_{n_\delta-1}}{\lambda} + \|y\|a_{n_\delta-1} + \delta. \tag{4.203}$$

If $\delta > 0$ is sufficiently small, then the above equation implies

$$\tilde{C}\delta^\zeta \le a_{n_\delta-1}\left(\frac{M_1}{\lambda} + \|y\|\right), \quad \tilde{C} > 0, \tag{4.204}$$

where $\tilde{C} < C_1$ is a constant. Therefore, by (2.73),

$$\lim_{\delta\to 0}\frac{\delta}{2a_{n_\delta}} \le \lim_{\delta\to 0}\frac{\delta}{a_{n_\delta-1}} \le \lim_{\delta\to 0}\frac{\delta^{1-\zeta}}{\tilde{C}}\left(\frac{M_1}{\lambda} + \|y\|\right) = 0, \quad 0 < \zeta < 1. \tag{4.205}$$

In particular, for $\delta = \delta_m$, one gets

$$\lim_{\delta_m\to 0}\frac{\delta_m}{a_{n_m}} = 0. \tag{4.206}$$

From the triangle inequality, inequalities (3.29) and (4.191), one obtains

$$\begin{aligned}\|u_{n_m} - y\| &\le \|u_{n_m} - V_{n_m}\| + \|V_{n_m} - V_{n_m,0}\| + \|V_{n_m,0} - y\| \\ &\le \frac{a_{n_m}}{\lambda} + \frac{\delta_m}{a_{n_m}} + \|V_{n_m,0} - y\|.\end{aligned} \tag{4.207}$$

Recall that $V_{n,0} = \tilde{V}_{a_n}$ (cf. (2.44) and (2.12)). From (2.80), (4.206), inequality (4.207) and Lemma 19, one obtains (2.81). Theorem 15 is proved. □

4.4. *Proofs of the nonlinear inequalities*

Proof. [Proof of Theorem 16] Denote $w(t) := g(t)e^{\int_{\tau_0}^t \gamma(s)ds}$. Then inequality (2.87) takes the form

$$\dot{w}(t) \le a(t)w^p(t) + b(t), \qquad w(\tau_0) = g(\tau_0) := g_0, \tag{4.208}$$

where

$$a(t) := \alpha(t)e^{(1-p)\int_{\tau_0}^t \gamma(s)ds}, \qquad b(t) := \beta(t)e^{\int_{\tau_0}^t \gamma(s)ds}. \tag{4.209}$$

Denote

$$\eta(t) = \frac{e^{\int_{\tau_0}^t \gamma(s)ds}}{\mu(t)}. \tag{4.210}$$

From inequality (2.89) and relation (4.210) one gets

$$w(\tau_0) = g(\tau_0) < \frac{1}{\mu(\tau_0)} = \eta(\tau_0). \tag{4.211}$$

It follows from the inequalities (3.29), (4.208) and (4.211) that

$$\dot{w}(\tau_0) \le \alpha(\tau_0)\frac{1}{\mu^p(\tau_0)} + \beta(\tau_0) \le \frac{1}{\mu(\tau_0)}\left[\gamma - \frac{\dot{\mu}(\tau_0)}{\mu(\tau_0)}\right] = \frac{d}{dt}\frac{e^{\int_{\tau_0}^t \gamma(s)ds}}{\mu(t)}\bigg|_{t=\tau_0} = \dot{\eta}(\tau_0). \tag{4.212}$$

From the inequalities (4.211) and (4.212) it follows that there exists $\delta > 0$ such that

$$w(t) < \eta(t), \qquad \tau_0 \le t \le \tau_0 + \delta. \tag{4.213}$$

To continue the proof we need two Claims.

Claim 1. If

$$w(t) \le \eta(t), \qquad \forall t \in [\tau_0, T], \quad T > \tau_0, \tag{4.214}$$

then

$$\dot{w}(t) \le \dot{\eta}(t), \qquad \forall t \in [\tau_0, T]. \tag{4.215}$$

Proof of Claim 1.

It follows from inequalities (2.88), (4.208) and the inequality $w(T) \le \eta(T)$, that

$$\begin{aligned}\dot{w}(t) &\le e^{(1-p)\int_{\tau_0}^t \gamma(s)ds}\alpha(t)\frac{e^{p\int_{\tau_0}^t \gamma(s)ds}}{\mu^p(t)} + \beta(t)e^{\int_{\tau_0}^t \gamma(s)ds}\\ &\le \frac{e^{\int_{\tau_0}^t \gamma(s)ds}}{\mu(t)}\left[\gamma - \frac{\dot{\mu}(t)}{\mu(t)}\right]\\ &= \frac{d}{dt}\frac{e^{\int_{\tau_0}^t \gamma(s)ds}}{\mu(t)}\bigg|_{t=t} = \dot{\eta}(t), \qquad \forall t \in [\tau_0, T].\end{aligned} \tag{4.216}$$

Claim 1 is proved.

Denote

$$T := \sup\{\delta \in \mathbb{R}^+ : w(t) < \eta(t),\ \forall t \in [\tau_0, \tau_0 + \delta]\}. \tag{4.217}$$

Claim 2. One has $T = \infty$.

Claim 2 says that every nonnegative solution $g(t)$ to inequality (2.87), satisfying assumptions (2.88)–(2.89), is defined globally.

Proof of Claim 2.

Assume the contrary, i.e., $T < \infty$. From the definition of T and the continuity of w and η one gets

$$w(T) \le \eta(T). \tag{4.218}$$

It follows from inequality (4.218) and *Claim 1* that

$$\dot{w}(t) \le \dot{\eta}(t), \qquad \forall t \in [\tau_0, T]. \tag{4.219}$$

This implies

$$w(T) - w(\tau_0) = \int_{\tau_0}^{T} \dot{w}(s)ds \le \int_{\tau_0}^{T} \dot{\eta}(s)ds = \eta(T) - \eta(\tau_0). \tag{4.220}$$

Since $w(\tau_0) < \eta(\tau_0)$ by assumption (2.89), it follows from inequality (4.220) that

$$w(T) < \eta(T). \tag{4.221}$$

Inequality (4.221) and inequality (4.219) with $t = T$ imply that there exists an $\epsilon > 0$ such that

$$w(t) < \eta(t), \qquad T \le t \le T + \epsilon. \tag{4.222}$$

This contradicts the definition of T in (4.217), and the contradiction proves the desired conclusion $T = \infty$.

Claim 2 is proved.

It follows from the definitions of $\eta(t)$ and $w(t)$ and from the relation $T = \infty$ that

$$g(t) = e^{-\int_{\tau_0}^{t} \gamma(s)ds} w(t) < e^{-\int_{\tau_0}^{t} \gamma(s)ds} \eta(t) = \frac{1}{\mu(t)}, \qquad \forall t > \tau_0. \tag{4.223}$$

Theorem 16 is proved. □

4.4.1. *Proof of Theorem 18*

Proof. Let us prove (2.100) by induction. Inequality (2.100) holds for $n = 0$ by assumption (2.99). Suppose that (2.100) holds for all $n \le m$. From inequalities

(2.96), (2.98), and from the induction hypothesis $g_n \le \frac{1}{\mu_n}$, $n \le m$, one gets

$$\begin{aligned} g_{m+1} &\le g_m(1 - h_m\gamma_m) + \alpha_m h_m g_m^p + h_m\beta_m \\ &\le \frac{1}{\mu_m}(1 - h_m\gamma_m) + h_m\frac{\alpha_m}{\mu_m^p} + h_m\beta_m \\ &\le \frac{1}{\mu_m}(1 - h_m\gamma_m) + \frac{h_m}{\mu_m}\left(\gamma_m - \frac{\mu_{m+1} - \mu_m}{\mu_m h_m}\right) \\ &= \frac{1}{\mu_m} - \frac{\mu_{m+1} - \mu_m}{\mu_m^2} \\ &= \frac{1}{\mu_{m+1}} - (\mu_{m+1} - \mu_m)\Big(\frac{1}{\mu_m^2} - \frac{1}{\mu_m\mu_{m+1}}\Big) \\ &= \frac{1}{\mu_{m+1}} - \frac{(\mu_{m+1} - \mu_m)^2}{\mu_n^2\mu_{m+1}} \le \frac{1}{\mu_{m+1}}. \end{aligned} \tag{4.224}$$

Therefore, inequality (2.100) holds for $n = m + 1$. Thus, inequality (2.100) holds for all $n \ge 0$ by induction. Theorem 18 is proved. □

5. Applications of the nonlinear inequality (2.87)

Here we only sketch the idea for many possible applications of this inequality for a study of dynamical systems in a Hilbert space.

Let

$$\dot{u} = Au + h(t, u) + f(t), \qquad u(0) = u_0, \quad \dot{u} := \frac{du}{dt}, \quad t \ge 0, \tag{5.1}$$

where A is a selfadjoint operator in a Hilbert space, $h(t, u)$ is a nonlinear operator in H, which is locally Lipschitz with respect to u and Hölder-continuous with respect to $t \in \mathbb{R}_+ := [0, \infty)$, and f is a Hölder continuous function on $\mathbb{R}_+$ with values in H.

Assume that

$$\mathrm{Re}\langle Au, u\rangle \le -\gamma(t)\langle u, u\rangle, \quad \mathrm{Re}\langle h(t, u), u\rangle \le \alpha(t)\|u\|^{1+p} \qquad \forall u \in D(A), \tag{5.2}$$

where $\gamma(t)$ and $\alpha(t)$ are continuous functions on $\mathbb{R}_+$, $h(t, 0) = 0$, $p > 1$ is a constant.

Our aim is to estimate the behavior of solutions to (5.1) as $t \to \infty$, in particular, to give sufficient conditions for a global existence of the unique solution to (5.1). Our approach consists of a reduction of the problem to the inequality (2.87) and an application of Theorem 16.

Let $g(t) := \|u(t)\|$. Problem (5.1) has a unique local solution under our assumptions. Multiplying (5.1) by u from left, then from right, add, and use (5.2) to get

$$\dot{g}g \le -\gamma(t)g^2 + \alpha(t)g^{1+p} + \beta(t)g, \qquad \beta(t) := \|f(t)\|. \tag{5.3}$$

Since $g \ge 0$, one gets

$$\dot{g} \le -\gamma(t)g + \alpha(t)g^p(t) + \beta(t). \tag{5.4}$$

Now Theorem 16 is applicable. This Theorem yields sufficient conditions (2.88) and (2.89) for the global existence of the solution to (5.1) and estimate (2.90) for the behavior of $\|u(t)\|$ as $t \to \infty$.

The outlined scheme is widely applicable to stability problems, to semilinear parabolic problems, and to hyperbolic problems as well. It yields some novel results. For instance, if the operator A is a second-order elliptic operator with matrix $a_{ij}(x,t)$, then Theorem 16 allows one to treat degenerate problems, namely, it allows, for example, the minimal eigenvalue $\lambda(x,t)$ of a selfadjoint matrix $a_{ij}(x,t)$ to depend on time in such a way that $\min_x \lambda(x,t) := \lambda(t) \to 0$ as $t \to \infty$ at a certain rate.

6. A numerical experiment

Let us present results of a numerical experiment. We solve nonlinear equation (1.1) with

$$F(u) := B(u) + \big(\arctan(u)\big)^3 := \int_0^1 e^{-|x-y|} u(y)dy + \big(\arctan(u)\big)^3. \tag{6.1}$$

Since the function $u \to \arctan^3 u$ is increasing on $\mathbb{R}$, one has

$$\langle \big(\arctan(u)\big)^3 - \big(\arctan(v)\big)^3, u-v\rangle \geq 0, \qquad \forall\, u,v \in H. \tag{6.2}$$

Moreover,

$$e^{-|x|} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{1+\lambda^2} d\lambda. \tag{6.3}$$

Therefore, $\langle B(u-v), u-v\rangle \geq 0$, so

$$\langle F(u-v), u-v\rangle \geq 0, \qquad \forall\, u,v \in H. \tag{6.4}$$

The Fréchet derivative of F is:

$$F'(u)w = \frac{3\big(\arctan(u)\big)^2}{1+u^2} w + \int_0^1 e^{-|x-y|} w(y)dy. \tag{6.5}$$

If $u(x)$ vanishes on a set of positive Lebesgue's measure, then $F'(u)$ is not boundedly invertible in H. If $u \in C[0,1]$ vanishes even at one point x_0, then $F'(u)$ is not boundedly invertible in H.

We use the following iterative scheme

$$\begin{aligned} u_{n+1} &= u_n - (F'(u_n) + a_n I)^{-1}(F(u_n) + a_n u_n - f_\delta), \\ u_0 &= 0, \end{aligned} \tag{6.6}$$

and stop iterations at $n := n_\delta$ such that the following inequality holds

$$\|F(u_{n_\delta}) - f_\delta\| < C\delta^\gamma, \quad \|F(u_n) - f_\delta\| \geq C\delta^\gamma, \quad n < n_\delta, \quad C > 1, \quad \gamma \in (0,1). \tag{6.7}$$

The existence of the stopping time n_δ is proved and the choice $u_0 = 0$ is also justified in this paper. The drawback of the iterative scheme (6.6) compared to the

DSM in this paper is that the solution u_{n_δ} may converge not to the minimal-norm solution to equation (1.1) but to another solution to this equation, if this equation has many solutions. There might be other iterative schemes which are more efficient than scheme (6.6), but this scheme is simple and easy to implement.

Integrals of the form $\int_0^1 e^{-|x-y|}h(y)dy$ in (6.1) and (6.5) are computed by using the trapezoidal rule. The noisy function used in the test is

$$f_\delta(x) = f(x) + \kappa f_{noise}(x), \quad \kappa > 0.$$

The noise level δ and the relative noise level are defined by the formulas:

$$\delta = \kappa\|f_{noise}\|, \quad \delta_{rel} := \frac{\delta}{\|f\|}.$$

In the test κ is computed in such a way that the relative noise level δ_{rel} equals to some desired value, i.e.,

$$\kappa = \frac{\delta}{\|f_{noise}\|} = \frac{\delta_{rel}\|f\|}{\|f_{noise}\|}.$$

We have used the relative noise level as an input parameter in the test.

In the test we took $h = 1$, $C = 1.01$, and $\gamma = 0.99$. The exact solution in the first test is $u(x) = 1$, and the right-hand side is $f = F(1)$.

It is proved that one can take $a_n = \frac{d}{1+n}$, and d is sufficiently large. However, in practice, if we choose d too large, then the method will use too many iterations before reaching the stopping time n_δ in (6.7). This means that the computation time will be large in this case. Since

$$\|F(V_\delta) - f_\delta\| = a(t)\|V_\delta\|,$$

and $\|V_\delta(t_\delta) - u_\delta(t_\delta)\| = O(a(t_\delta))$, we have

$$C\delta^\gamma = \|F(u_\delta(t_\delta)) - f_\delta\| \le a(t_\delta)\|V_\delta\| + O(a(t_\delta)),$$

and we choose

$$d = C_0\delta^\gamma, \qquad C_0 > 0.$$

In the experiments our method works well with $C_0 \in [3, 10]$. In the test we chose a_n by the formula $a_n := C_0\frac{\delta^{0.99}}{n+1}$. The number of nodal points, used in computing integrals in (6.1) and (6.5), was $N = 50$. The accuracy of the solutions obtained in the tests with $N = 20$ and $N = 30$ was about the same as for $N = 50$.

Numerical results for various values of δ_{rel} are presented in Table 1. In this experiment, the noise function f_{noise} is a vector with random entries normally distributed, with mean value 0 and variance 1. Table 1 shows that the iterative scheme yields good numerical results.

Remark 11. During the time this paper was waiting publication, the following paper has appeared:

Table 1. Results when $C_0 = 4$ and $N = 50$.

δ_{rel}	0.05	0.03	0.02	0.01	0.003	0.001
Number of iterations	28	29	28	29	29	29
$\frac{\|u_{DSM}-u_{exact}\|}{\|u_{exact}\|}$	0.0770	0.0411	0.0314	0.0146	0.0046	0.0015

N.S.Hoang, A.G.Ramm, *Existence of solution to an evolution equation and a justification of the DSM for equations with monotone operators*, Comm. Math.Sci., 7, N4, (2009), 1073-1079.

In this paper the smoothness assumption on $F'(u)$ is replaced by the continuity of $F'(u)$. This allows one to treat Newton-type versions of the DSM under minimal natural assumptions.

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