ON VARIABLE LEBESGUE SPACES

by

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B.A., Saint Louis University, 2003
M.S., Saint Louis University, 2005

AN ABSTRACT OF A DISSERTATION

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Abstract

The reader will recall that the classical p-Lebesgue spaces are those functions defined on a measure space (X, μ) whose modulus raised to the p^{th} power is integrable. This condition gives many quantitative measurements on the growth of the function, both locally and globally. Results and applications pertaining to such functions are ubiquitous. That said, the constancy of the exponent p when computing $\int_X |f|^p \, d\mu$ is limiting in the sense that it is intrinsically uniform in scope. Speaking loosely, there are instances in which one is concerned with the p growth of a function in a region A and its q growth in another region B. As such, allowing the exponent to vary from region to region (or point to point) is a reasonable course of action.

The task of developing such a theory was first taken up by Wladyslaw Orlicz in the 1930's. The theory he developed, of which variable Lebesgue spaces are a special case, was only intermittently studied and analyzed through the end of the century. However, at the turn of the millennium, several results and their applications sparked a focused and intense interest in variable L^p spaces. It was found that with very few assumptions on the exponent function many of the classical structure and density theorems are valid in the variable-exponent case. Somewhat surprisingly, these results were largely proved using intuitive adaptations of well-established methods. In fact, this methodology set the tone for the first part of the decade, where a multitude of "affirmative" results emerged. While the successful adaptation of classical results persists to a large extent today, there are nontrivial situations in which one cannot hope to extend a result known for constant L^p.

In this paper, we wish to explore both of the aforementioned directions of research. We will first establish the fundamentals for variable L^p. Afterwards, we will apply these fundamentals to some classical L^p results that have been extended to the variable setting.

We will conclude by shifting our attention to Littlewood-Paley theory, where we will furnish an example for which it is impossible to extend constant-exponent results to the variable case.

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Chapter 1

The Fundamentals of Variable Lebesgue Spaces

1.1 Preliminaries

Let (X, μ) be a measure space. Denote by $\mathscr{M} = \mathscr{M}(X, \mu)$ the set of μ-measurable functions from X into the complex plane, $\mathbb{C}$, or into the extended real numbers, $[-\infty, \infty]$. An exponent for (X, μ), is any element of $\mathscr{M}$ taking values in $[1, \infty]$. We denote the set of exponents by $\mathscr{E} = \mathscr{E}(X, \mu)$. It is worth noting that some authors allow their exponents to take values below one. Most, if not all, of these authors still insist that the exponent be positive. In the rare event we consider such exponents, we will call them generalized exponents.

Fix $p \in \mathscr{E}$. Given any $f \in \mathscr{M}$, the identity

$$|f(x)|^{p(x)} = \exp\left[p(x) \log |f(x)|\right]$$

shows that $|f|^p$ is measurable. Since it is also nonnegative, the integral $\int_X |f(x)|^{p(x)}\, d\mu(x)$

is well-defined. Thus, the functional $\rho_p : \mathscr{M} \to [0, \infty]$ given by

$$\rho_p(f) := \int_{F_p} |f(x)|^{p(x)}\, d\mu(x) + \operatorname*{ess\,sup}_{x \in F_p^c} |f(x)|$$

is also well-defined, where $F_p := \{p < \infty\}$. In the event that $\mu(F_p^c) = 0$, we are taking the second term in the definition of $\rho_p(f)$ to be zero. The functional ρ_p is called the p-modular for (X, μ). We take a moment to record some useful properties of this functional.

Proposition 1.1. *Let $p \in \mathscr{E}$ and $f, g \in \mathscr{M}$. Then the following hold:*

(a) $\rho_p(f) \geq 0$, *with equality if and only if* $f = 0$, μ *a.e.*

(b) $\rho_p(f) \leq \rho_p(g)$, *whenever* $|f| \leq |g|$, μ *a.e.*

(c) ρ_p *is convex.*

(d) *Suppose* $\rho_p(f/t_0) < \infty$, *for some* $t_0 > 0$. *Define* $F(t) := \rho_p(f/t)$, *for* $t \geq t_0$. *Then* F *is bounded by* $\rho_p(f/t_0)$, *non-increasing, continuous and has zero limit at infinity.*

(e) (Fatou's lemma) *For any* $\{f_n\} \subset \mathscr{M}$,

$$\rho_p\left(\liminf_{n\to\infty} |f_n|\right) \leq \liminf_{n\to\infty} \rho_p(f_n)$$

Proof. Parts (a) and (b) follow immediately from the definition of ρ_p.

(c) To prove this part, let $\lambda \in [0, 1]$. Recall that $t \mapsto t^q$ is convex on $[0, \infty)$, for every

$q \geq 1$. Thus,

$$\begin{aligned}
\rho_p((1-\lambda)f + \lambda g) &= \int_{F_p} |(1-\lambda)f(x) + \lambda g(x)|^{p(x)} \, d\mu(x) + \operatorname*{ess\,sup}_{x \in F_p^c} |(1-\lambda)f(x) + \lambda g(x)| \\
&\leq \int_{F_p} [(1-\lambda)\,|f(x)| + \lambda\,|g(x)|]^{p(x)} \, d\mu(x) \\
&\qquad + \operatorname*{ess\,sup}_{x \in F_p^c} [(1-\lambda)\,|f(x)| + \lambda\,|g(x)|] \\
&\leq \int_{F_p} \left[(1-\lambda)\,|f(x)|^{p(x)} + \lambda\,|g(x)|^{p(x)}\right] d\mu(x) \\
&\qquad + (1-\lambda) \operatorname*{ess\,sup}_{x \in F_p^c} |f(x)| + \lambda \operatorname*{ess\,sup}_{x \in F_p^c} |g(x)| \\
&= (1-\lambda)\rho_p(f) + \lambda\rho_p(g).
\end{aligned}$$

(d) Assume the additional hypothesis stated for this part of the proposition. That F is bounded by $\rho_p(f/t_0)$ and non-increasing follow from part (c) above. That F is continuous on its domain of definition and has zero limit at infinity both follow from the dominated convergence theorem.

(e) This follows immediately from the usual Fatou lemma:

$$\begin{aligned}
\rho_p\left(\liminf_{n\to\infty} |f_n|\right) &= \int_{F_p} \left[\liminf_{n\to\infty} |f_n(x)|\right]^{p(x)} d\mu(x) + \operatorname*{ess\,sup}_{x \in F_p^c} \left[\liminf_{n\to\infty} |f_n(x)|\right] \\
&\leq \liminf_{n\to\infty} \int_{F_p} |f_n(x)|^{p(x)} \, d\mu(x) + \liminf_{n\to\infty} \left[\operatorname*{ess\,sup}_{x \in F_p^c} |f_n(x)|\right] \\
&= \liminf_{n\to\infty} \rho_p(f_n).
\end{aligned}$$

■

Now that we have established the most basic properties for the p-modular, we can define the function spaces that we wish to investigate throughout this paper. Given $p \in \mathscr{E}$, let

$L^p = L^p(X,\mu)$ be set of all μ-measurable functions f for which the p-modular of some (inverse) scalar multiple is finite. In symbols,

$$L^p(X,\mu) := \{f \in \mathscr{M} : \rho_p(f/t_0) < \infty, \exists\, t_0 > 0\}.$$

We define a functional on L^p as follows:

$$\|f\|_p := \inf\{t > 0 : \rho_p(f/t) \leq 1\}.$$

By Proposition 1.1(e), the set $\{t > 0 : \rho_p(f/t) \leq 1\}$ is nonempty, for each $f \in L^p$. Hence, the functional specified above is well-defined and finite, for every $f \in L^p$. As the notation suggests, $\|\cdot\|_p$ will turn out to be a seminorm, and furthermore, under the usual convention of identifying functions that are equal μ a.e., it will actually be a norm. Of course, this needs to be verified. To this end, we gather some preliminary facts about $\|\cdot\|_p$ that are useful in their own right.

Proposition 1.2. *Let $p \in \mathscr{E}$ and $f, g \in L^p$.*

(a) $\|f\|_p \geq 0$, *with equality if and only if* $f = 0$, μ *a.e.*

(b) $\|f\| \leq \|g\|$, *whenever* $|f| \leq |g|$, μ *a.e.*

(c) $\rho_p\left(f/\|f\|_p\right) \leq 1$, *whenever* $f \neq 0$ μ *a.e. In other words, the infimum in the definition for the seminorm of* f *is achieved, whenever* $f \neq 0$ μ *a.e.*

(d) *Suppose, in addition, that* p *is bounded on its finiteness set; i.e.* $\operatorname{ess\,sup}_{x\in F_p} p(x) < \infty$. *Then* $\rho_p\left(f/\|f\|_p\right) = 1$, *whenever* $f \neq 0$ μ *a.e.*

(e) (unit ball property) $\rho_p(f) \leq 1$ *if and only if* $\|f\|_p \leq 1$.

(f) *If* $\rho_p(f) \leq 1$ *or* $\|f\|_p \leq 1$*, then* $\rho_p(f) \leq \|f\|_p$*.*

(g) *If* $\rho_p(f) > 1$ *or* $\|f\|_p > 1$*, then* $\|f\|_p \leq \rho_p(f)$*.*

Proof. (a) If $f = 0$ μ a.e., then $\rho_p(f/t) = 0$, for every $t > 0$, whence $\|f\|_p = 0$. Now, assume that $f \neq 0$, μ a.e. Then there exists $\epsilon > 0$ such that $\mu(E) > 0$, where $E := \{|f| \geq \epsilon\}$. Suppose, in order to reach a contradiction, that $\|f\|_p = 0$. By the definition of $\|f\|_p$, we may then find a sequence $\{t_n\}$ such that $t_n < \epsilon$, $t_n \searrow 0$ and $\rho_p(f/t_n) \leq 1$. So, for all n, we have

$$\begin{aligned} 1 &\geq \rho_p(f/t_n) \\ &\geq \int_{E\cap F_p} \left|\frac{f(x)}{t_n}\right|^{p(x)} d\mu(x) + \operatorname*{ess\,sup}_{x\in E\cap F_p^c} \left|\frac{f(x)}{t_n}\right| \\ &\geq \frac{\epsilon}{t_n}\left[\mu(E\cap F_p) + \operatorname*{ess\,sup}_X \mathbf{1}_{E\cap F_p^c}\right]. \end{aligned} \tag{1.1}$$

Letting $n \to \infty$ in (1.1) yields the desired contradiction, which completes the proof of (a).

(b) If $|f| \leq |g|$, μ a.e., then $\rho_p(f/t) \leq \rho_p(g/t)$, for every $t > 0$, by Proposition 1.1(c). Therefore, $\{t > 0 : \rho_p(g/t) \leq 1\} \subset \{t > 0 : \rho_p(f/t) \leq 1\}$, whence $\|f\|_p \leq \|g\|_p$.

(c) Choose $t_n > 0$ such that $t_n \searrow \|f\|_p$ and $\rho_p(f/t_n) \leq 1$. Now, apply Fatou's lemma (Proposition 1.1(e)) to the sequence $\{f_n\}$, where $f_n := f/t_n$.

(d) Put $p_0 := \operatorname{ess\,sup}_{x\in F_p} p(x)$. Assume, with a view to reach a contradiction, that $\rho_p\left(f/\|f\|_p\right) < 1$. Under this assumption, we may choose $0 < t < \|f\|_p$ so close to $\|f\|_p$ so that $\left(\|f\|_p/t\right)^{p_0}\rho_p\left(f/\|f\|_p\right)\leq 1$. For such a t, observe that

$$\begin{aligned}\rho_p\left(\frac{f}{t}\right) &= \int_{F_p}\left(\frac{\|f\|_p}{t}\right)^{p(x)}\left|\frac{f(x)}{\|f\|_p}\right|^{p(x)}d\mu(x)+\left(\frac{\|f\|_p}{t}\right)\operatorname*{ess\,sup}_{x\in F_p^c}\left|\frac{f(x)}{\|f\|_p}\right| \\ &\leq \left(\frac{\|f\|_p}{t}\right)^{p_0}\rho_p\left(\frac{f}{\|f\|_p}\right) \\ &\leq 1.\end{aligned}$$

By the infimal definition of $\|f\|_p$, it must be that $\|f\|_p\leq t$, contradicting the choice of t. Hence, our assumption is false; i.e. $\rho_p\left(f/\|f\|_p\right)\geq 1$. In view of part (c) of this proposition, the proof of (d) is complete.

(e) If $\rho_p(f)\leq 1$, then $1\in\{t>0:\rho_p(f/t)\leq 1\}$, whence $\|f\|_p\leq 1$. This shows that $\rho_p(f)\leq 1$ implies $\|f\|_p\leq 1$.

Now, assume that $\|f\|_p\leq 1$. Note first that we may assume additionally that $\|f\|_p\neq 0$, for otherwise both $\rho_p(f)=\|f\|_p=0$. Since $0<\|f\|_p\leq 1$, we have that $|f(x)|^{p(x)}/\|f\|_p\leq \left|f(x)/\|f\|_p\right|^{p(x)}$, for every $x\in X$. Thus,

$$\begin{aligned}\frac{\rho_p(f)}{\|f\|_p} &= \int_{F_p}\frac{|f(x)|^{p(x)}}{\|f\|_p}d\mu(x)+\operatorname*{ess\,sup}_{x\in F_p^c}\left|\frac{f(x)}{\|f\|_p}\right| \\ &\leq \int_{F_p}\left|\frac{f(x)}{\|f\|_p}\right|^{p(x)}d\mu(x)+\operatorname*{ess\,sup}_{x\in F_p^c}\left|\frac{f(x)}{\|f\|_p}\right| \\ &= \rho_p\left(\frac{f}{\|f\|_p}\right) \\ &\leq 1,\end{aligned}$$

so that $\rho_p(f)\leq\|f\|_p\leq 1$, as desired

(f) We have already shown in the proof of (e) that $\|f\|_p\leq 1$ implies that $\rho_p(f)\leq\|f\|_p$. Pairing this with the new found fact that $\rho_p(f)\leq 1$ is equivalent to $\|f\|_p\leq 1$ proves (f).

(g) Since $\rho_p(f) \leq 1$ is equivalent to $\|f\|_p \leq 1$, we may assume $\rho_p(f) > 1$ to prove this part. By convexity, we have $\rho_p(f/\rho_p(f)) \leq \rho_p(f)/\rho_p(f) = 1$, whence $\|f\|_p \leq \rho_p(f)$. ■

As stated above, we will use these facts to argue that L^p, under the usual identification, is a normed vector space. In fact, more can be said. This is the content of the following:

Theorem 1.3. *L^p is a complex vector space and the functional $\|\cdot\|_p$ is a seminorm on L^p. If one identifies functions in L^p that are equal μ a.e., then L^p is Banach space.*

Proof. Clearly, L^p is nonempty: it contains the zero function. Positive definiteness is the content of Proposition 1.2(a). Since $\rho_p(f/t) = \rho((\alpha f)/(|\alpha|\, t))$, for all $t > 0$, we see that $\alpha f \in L^p$, whenever f is. To show the homogeneity of $\|\cdot\|_p$, let $f \in L^p$ and $\alpha \in \mathbb{C}$. We may assume that $\alpha \neq 0$ and that $f \neq 0$ μ a.e. Under these additional assumptions, Proposition 1.2(c) applies and we find that

$$\rho_p\left(\frac{\alpha f}{|\alpha|\, \|f\|_p}\right) = \rho_p\left(\frac{f}{\|f\|_p}\right) \leq 1,$$

whence $\|\alpha f\| \leq |\alpha|\, \|f\|_p$. Analogously,

$$\rho_p\left(\frac{f}{\frac{\|\alpha f\|_p}{|\alpha|}}\right) = \rho_p\left(\frac{\alpha f}{\|\alpha f\|}\right) \leq 1,$$

whence $\|f\|_p \leq \|\alpha f\|_p / |\alpha|$, or equivalently, $|\alpha|\, \|f\|_p \leq \|\alpha f\|_p$. By antisymmetry, $\|\alpha f\|_p = |\alpha|\, \|f\|_p$.

We will now show that L^p is closed under addition and that $\|\cdot\|_p$ satisfies the triangle inequality: Let $f, g \in L^p$. If either f or g vanished μ a.e., then the desired results would be trivial. So, we may safely assume that both f and g do not vanish μ a.e. Thus, the following computations are valid:

$$\begin{aligned}
\rho_p\left(\frac{f+g}{\|f\|_p+\|g\|_p}\right) &= \rho_p\left(\frac{\|f\|_p}{\|f\|_p+\|g\|_p}\frac{f}{\|f\|_p}+\frac{\|g\|_p}{\|f\|_p+\|g\|_p}\frac{g}{\|g\|_p}\right) \\
&\leq \frac{\|f\|_p}{\|f\|_p+\|g\|_p}\rho_p\left(\frac{f}{\|f\|_p}\right)+\frac{\|g\|_p}{\|f\|_p+\|g\|_p}\rho_p\left(\frac{g}{\|g\|_p}\right) \\
&\leq \frac{\|f\|_p}{\|f\|_p+\|g\|_p}+\frac{\|g\|_p}{\|f\|_p+\|g\|_p} \\
&= 1.
\end{aligned}$$

This simultaneously shows that L^p is closed under addition and that $\|\cdot\|_p$ satisfies the triangle inequality. Note that the first and second inequalities in the computation above are justified by Propositions 1.1(c) and 1.2(c), respectively.

It remains show that L^p is a Banach space under the specified identification. Let $\{f_n\}$ be a Cauchy sequence in L^p. For each $k \in \mathbb{N}$, choose f_{n_k} such that

$$\left\|f_{n_{k+1}}-f_{n_k}\right\|_p \leq 2^{-k}.$$

Consider the finite sum $s_m := \sum_{k=1}^{m}\left|f_{n_{k+1}}-f_{n_k}\right|$. By the triangle inequality, we see that $s_m \in L^p$ and $\|s_m\|_p < \sum_{k=1}^{m} 2^{-k} < 1$, whence $\rho_p(s_m) \leq 1$. Since s_m is a monotonically increasing sequence, it makes sense to define $s := \lim_{m\to\infty} s_m$. By Fatou's lemma, we have that $\rho_p(s) \leq \liminf_{m\to\infty}\rho_p(s_m) < 1$. Thus, $s \in L^p$, and therefore, is finite μ a.e. Consequently, the series $f := f_{n_1} + \sum_{k=1}^{\infty} f_{n_{k+1}} - f_{n_k}$ converges absolutely μ a.e and is an element of L^p. Note that $f = \lim_{k\to\infty} f_{n_k}$, μ a.e. We claim that $f_n \to f$ in L^p. To this end, let $\epsilon > 0$. Choose $N \in \mathbb{N}$ so large so that $m, n \geq N$ implies $\|f_m - f_n\|_p < \epsilon$, or equivalently,

$\|(f_m - f_n)/\epsilon\|_p < 1$. Then, for $n \geq N$, Fatou's lemma and the unit ball property imply that

$$\begin{aligned}\rho_p\left(\frac{f-f_n}{\epsilon}\right) &\leq \liminf_{k\to\infty} \rho_p\left(\frac{f_{n_k}-f_n}{\epsilon}\right) \\ &\leq \liminf_{k\to\infty} \left\|\frac{f_{n_k}-f_n}{\epsilon}\right\|_p \\ &< 1,\end{aligned}$$

whence $\|f - f_n\| \leq \epsilon$, by the definition of $\|\cdot\|_p$. In summary, given $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $n \geq N$ implies $\|f - f_n\|_p \leq \epsilon$; i.e. $f_n \to f$ in L^p. ■

As the discussion thus far has been in a general setting, we now take a moment to discuss how variable Lebesgue spaces coincide and differ from the usual (constant) Lebesgue spaces. It is natural to first point out that variable Lebesgue spaces with the relevant norms, as defined above, are indeed generalizations of the usual Lebesgue spaces with the usual norm. To see this, let $p \in [1, \infty]$ be a constant.

If $1 \leq p < \infty$, then $\rho_p(f)$ is the usual p-integral of $|f|$. That is, $\rho_p(f) = \int_X |f|^p \, d\mu$. In this case, $\rho_p(f)$ is finite if and only if $\rho_p(f/t)$ is finite for any $t \neq 0$. Thus, L^p, as defined above, coincides with the usual L^p. Here, the norms defined on the spaces coincide because

$$\rho_p\left(\frac{f}{t}\right) \leq 1 \Leftrightarrow \frac{1}{t^p}\int_X |f|^p \, d\mu \leq 1 \Leftrightarrow \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \leq t.$$

In the event that $p = \infty$, then $\rho_p(f)$ is the essential supremum of $|f|$ over all of X. Therefore, L^p, as defined above, coincides with the usual L^∞ in that it is the set of all essentially bounded μ-measurable functions on X. The norms agree in this scenario, since

$$\rho_p\left(\frac{f}{t}\right) \leq 1 \Leftrightarrow \operatorname*{ess\,sup}_X |f| \leq t.$$

To briefly illustrate that variable Lebesgue spaces are fundamentally different from the classical Lebesgue spaces, we provide two examples in which these spaces exhibit behaviors that are somewhat counterintuitive. For our first example, let $X := [1, \infty)$, $d\mu := dx$ and $p(x) := x$. Let $\alpha \in \mathbb{C}\backslash\{0\}$ denote the constant function $x \mapsto \alpha$. If $t_0 > |\alpha|$, the computation

$$\int_1^\infty \left|\frac{\alpha}{t_0}\right|^x dx = \lim_{n\to\infty} \frac{\left(|\alpha|\, t_0^{-1}\right)^n - |\alpha|\, t_0^{-1}}{\ln\left(|\alpha|\, t_0^{-1}\right)} \tag{1.2}$$

shows that $\alpha \in L^x$. This is to be held in contrast to the fact that the only constant function contained in any of the constant-exponent Lebesgue spaces on $[1, \infty)$ is the zero function. As a side remark, it is worth noting that if we omit the scaling factor t_0 in (1.2) we could only conclude that $\alpha \in L^p$ if and only if $|\alpha| < 1$. So, in general, it is not sufficient to take L^p to be the set $\{f \in \mathscr{M} : \rho_p(f) < \infty\}$ in our definition.

For our next example, we consider $X = \mathbb{R}^n$, $d\mu = dx$. The translation invariance of $L^p(\mathbb{R}^n)$ for constant p is so fundamental a property that is often taken for granted. Unfortunately, such liberties are not permissible in the variable setting. More precisely, in [1], Diening proves the existence of a nonzero translation that is not continuous on $L^p(\mathbb{R}^n)$, whenever the p is nonconstant. More precisely, if p is essentially non-constant (i.e. $\operatorname{ess\,sup}_X p > \operatorname{ess\,inf}_X p$), then there exists $t \in \mathbb{R}^n\backslash\{0\}$ such that the translation operator $f(x) \mapsto f(x-t)$ is not continuous. While elementary, these two examples already provide a glimpse of what has motivated much of the research recently put forth from the mathematical community regarding this subject: to what extent can the results and methods established for classical Lebesgue (and other exponent) spaces be extended to variable Lebesgue spaces. This is a theme that we will revisit extensively throughout this dissertation.

Before moving on to the next section, it is worth mentioning that there are several alternative definitions of the modular used when defining the variable Lebesgue function spaces. The modular introduced at the outset of this section was first formulated by Kováčik and Rákosník in [6]. Two different modulars are given and predominantly used in [2]. In that text, the authors show that these various modulars all define the same function space and, in fact, that all of the resulting norms are equivalent.

1.2 Properties of Bounded Exponents

Even in the constant setting, one must often separate the extreme values of 1 and ∞ when formulating results for L^p spaces. This trend persists to the variable setting. In this section, we limit ourselves to the discussion of bounded exponents; i.e. exponents such that $p^+ := \text{ess sup}_X\, p < \infty$. With this added hypothesis, the two main results of this section are that modular and norm convergence coincide and that simple functions that vanish outside a set of finite measure are dense. We start by proving the first of these two claims:

Proposition 1.4. *If p is a bounded exponent, then ρ_p-convergence and $\|\cdot\|_p$-convergence in L^p are equivalent.*

Proof. By Proposition 1.2(f), we see that norm convergence always implies modular convergence, even when p has an infinite part or is unbounded. So, we need only show modular convergence implies norm convergence, whenever $p^+ < \infty$. To this end, let $\{f_n\} \subset L^p$ with $\rho_p(f_n) \to 0$, as $n \to \infty$. For any $0 < \epsilon < 1$, choose N so large so that $n \geq N$ implies

$\rho_p(f_n) \le \epsilon^{p^+}$. Then

$$\rho_p\left(\frac{f_n}{\epsilon}\right) = \int_X |f_n|^p \left(\frac{1}{\epsilon}\right)^p \le \frac{1}{\epsilon^{p^+}}\rho_p(f_n) \le 1,$$

for all $n \ge N$. Consequently, $\|f_n\|_p \le \epsilon$, for $n \ge N$. Since $0 < \epsilon < 1$ is arbitrary, it follows that $\|f_n\|_p \to 0$, as $n \to \infty$. ■

Theorem 1.5. *The class of simple functions that vanish outside a set of finite measure is dense in L^p, whenever p is a bounded exponent.*

Proof. Let $\mathscr{S}_0$ denote the class of simple functions that vanish outside a set of finite measure. If $E \subset X$ is measurable and $\mu(E) < \infty$, then

$$\rho_p(\mathbf{1}_E) \le \mu(E) < \infty,$$

whence $\mathbf{1}_E \in L^p$. Since every element of $\mathscr{S}_0$ is a finite linear combination of indicator functions on finitely μ-measured sets, it follows that $\mathscr{S}_0 \subset L^p$. Now, let $f \in L^p$ with $f \ge 0$. There exists a sequence $\{s_n\}$ of simple functions such that $0 \le s_n \le f$ and $s_n \nearrow f$. (See [10].) Choose $t > 0$ such that $\rho_p(f/t) < \infty$. As $0 \le s_n \le f$, we have

$$\rho_p(s_n) \le \rho_p(f) \le \max\left\{1, t^{p^+}\right\}\rho_p(f/t) < \infty.$$

From this, it follows that each $s_n \in L^p$. Since $|f - s_n|^p \to 0$ pointwise and

$$|f - s_n|^p \le (2f)^p = \left(2t\frac{f}{t}\right)^p \le \max\left\{1, (2t)^{p^+}\right\}\left(\frac{f}{t}\right)^p \in L^1,$$

Lebesgue's dominated convergence theorem implies that $\rho_p(f - s_n) \to 0$, whence $\|f - s_n\|_p \to 0$, by Theorem 1.4 above. In summary, we have shown that any $f \ge 0$ in L^p is in the closure (in L^p) of $\mathscr{S}_0$. For general (i.e. complex) $f \in L^p$, write $f = (a^+ - a^-) + (b^+ - b^-)i$, where $a^\pm$ are the positive and negative parts of the real part of f and similarly for $b^\pm$. Use the above to get sequences $\{a_n^\pm\}$ and $\{b_n^\pm\}$ in $\mathscr{S}_0$ such that $a_n^\pm \to a^\pm$ and $b_n^\pm \to b^\pm$ in L^p, respectively. Put $s_n := (a_n^+ - a_n^-) + (b_n^+ - b_n^-)i$. Then each s_n is a (complex-valued) simple function, and furthermore,

$$\|f - s_n\|_p \leq \left\|a^+ - a_n^+\right\|_p + \left\|a^- - a_n^-\right\|_p + \left\|b^+ - b_n^+\right\|_p + \left\|b^- - b_n^-\right\|_p \to 0,$$

as $n \to \infty$. ■

1.3 Hölder's Inequality and Embeddings

In this section, we wish to provide one of the ubiquitous tools of classical analysis: Hölder's inequality. This inequality will allow us to see how many of the L^p spaces are nested within each other. Moreover, Hölder's inequality will motivate the introduction of the conjugate norm in the next section, which ultimately leads to the very useful norm conjugacy inequality.

For the proofs of many of the inequalities to follow, it will be convenient to introduce the following function defined on the μ-measurable subsets of X:

$$k(E) := \operatorname*{ess\,sup}_X \mathbf{1}_E = \begin{cases} 0 & \mu(E) = 0 \\ 1 & \mu(E) \neq 0 \end{cases}.$$

Theorem 1.6 (Hölder's inequality)**.** *Let p and q be exponents. If the function r defined by the equation*

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \tag{1.3}$$

is an exponent (i.e. $r \geq 1$), then there exists a constant $K = K(p,q) \in [1,5]$ such that

$$\|fg\|_r \leq K \|f\|_p \|g\|_q , \tag{1.4}$$

for every $f, g \in \mathscr{M}$. If p and q are constant exponents, then $K = 1$.

Proof. Decompose the finiteness set of r, F_r, into the following three sets:

$$A := \{r = p < \infty\} = \{q = \infty, p < \infty\},$$

$$B := \{r = q < \infty\} = \{p = \infty, q < \infty\},$$

$$C := \{r < p < \infty\} = \{r < q < \infty\}$$

Let us assume first that $\|f\|_p , \|g\|_q \leq 1$ so that $\rho_p(f), \rho_q(g) \leq 1$. Since $A \subset F_q^c$, we have

$$\operatorname*{ess\,sup}_A |g| \leq \operatorname*{ess\,sup}_{F_q^c} |g| \leq \rho_q(g) \leq 1.$$

It follows that $|g| \leq 1$ a.e. on A, so that $|g|^r \leq 1$ a.e. on A, which in turn implies that $\operatorname{ess\,sup}_A |g|^r \leq K(A)$. Hence,

$$\int_A |fg|^r \leq \operatorname*{ess\,sup}_A |g|^r \int_A |f|^p \leq k(A)\rho_p(f) \leq k(A). \tag{1.5}$$

In a symmetric manner, we also find that

$$\int_B |fg|^r \leq k(B). \tag{1.6}$$

To estimate on C, we first recall that the convexity of the exponential function justifies the following:

$$|fg|^r = \exp\left[r \log\left(|f| \cdot |g|\right)\right] = \exp\left[\frac{r}{p}\log\left(|f|^p\right) + \frac{r}{q}\log\left(|g|^q\right)\right] \le \frac{r}{p}|f|^p + \frac{r}{q}|g|^q,$$

on C. Hence, we have

$$\begin{aligned}
\int_C |fg|^r &\le \int_C \frac{r}{p}|f|^p + \frac{r}{q}|g|^q \\
&\le \left(\operatorname*{ess\,sup}_C \frac{r}{p}\right)\int_C |f|^p + \left(\operatorname*{ess\,sup}_C \frac{r}{q}\right)\int_C |g|^q \\
&\le \left(\operatorname*{ess\,sup}_C \frac{r}{p}\right)\rho_p(f) + \left(\operatorname*{ess\,sup}_C \frac{r}{q}\right)\rho_q(g) \\
&\le \operatorname*{ess\,sup}_C \frac{r}{p} + \operatorname*{ess\,sup}_C \frac{r}{q}.
\end{aligned} \tag{1.7}$$

Since $F_r^c = F_p^c \cap F_q^c$, we see that

$$\begin{aligned}
\operatorname*{ess\,sup}_{F_r^c} |fg| &\le \left(\operatorname*{ess\,sup}_{F_r^c} |f|\right)\left(\operatorname*{ess\,sup}_{F_r^c} |g|\right) \\
&\le \left(\operatorname*{ess\,sup}_{F_p^c} |f|\right)\left(\operatorname*{ess\,sup}_{F_q^c} |g|\right) \\
&\le \left[k(F_p^c)\rho_p(f)\right]\left[k(F_q^c)\rho_q(g)\right] \\
&\le k(F_p^c)k(F_q^c).
\end{aligned} \tag{1.8}$$

At last, we may combine inequalities (1.5) through (1.8) to obtain

$$\begin{aligned}
\rho_r(fg) &= \int_{F_r} |fg|^r + \operatorname*{ess\,sup}_{F_r^c} |fg| \\
&= \int_A + \int_B + \int_C |fg|^r + \operatorname*{ess\,sup}_{F_r^c} |fg| \\
&\le k(A) + k(B) + \operatorname*{ess\,sup}_C \frac{r}{p} + \operatorname*{ess\,sup}_C \frac{r}{q} + k(F_p^c)k(F_q^c) \\
&=: K.
\end{aligned}$$

Considering the definition (1.3) of r, it is straight-forward to verify that $1 \le K \le 5$. Since $K \ge 1$, we may conclude, by the convexity[1] of ρ_r, that

$$\rho_r\left(\frac{fg}{K}\right) \le \frac{\rho_r(fg)}{K} \le 1,$$

so that

$$\|fg\|_r \le K. \tag{1.9}$$

Since (1.9) holds for all $f, g \in \mathscr{M}$ with $\|f\|_p \le 1$ and $\|g\|_q \le 1$ and since $\|\cdot\|_r$ is homogeneous (of degree 1), (1.4) follows via a scaling argument. Finally, by sheer exhaustion of cases, it is easy to see that $K = 1$, whenever p and q are constant exponents. ∎

As an immediate application of Hölder's inequality, we can obtain a slightly more general version of a well-known embedding result. The set up is as follows: Assume p and r are two exponents with $r \le p$ a.e. Since $1 \le r \le p$, it is not difficult to see that $0 \le 1/r - 1/p \le 1 - 1/p \le 1$, whence

$$q := \frac{1}{\frac{1}{r} - \frac{1}{p}} = \frac{pr}{p-r}$$

is an exponent (i.e. $q \ge 1$). Assuming that $f \in L^p$ and $\mathbf{1}_X \in L^q$, then Hölder's inequality implies that $f = f \cdot \mathbf{1}_X \in L^r$, and in fact,

$$\|f\|_r \le K \|f\|_p \|\mathbf{1}_X\|_q .$$

This proves the following:

[1] Here is where we need that $r \ge 1$.

Proposition 1.7. *If* $1 \le r \le p$*, where* p *and* r *are exponents, and* $\mathbf{1}_X \in L^{\frac{pr}{p-r}}$*, then* L^p *continuously embeds in* L^r *with embedding norm at most* $K \left\| \mathbf{1}_X \right\|_{\frac{pr}{p-r}}$*.*

In the event that $\mu(X) < \infty$, then $\mathbf{1}_X \in L^q$, for all exponents q, whence $L^p \hookrightarrow L^r$, whenever $r \le p$, in this scenario.

1.4 Norm Conjugacy

The most common application of Hölder's inequality arises when the exponents p and q are stipulated to satisfy the extended real-valued equation:

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \tag{1.10}$$

for all $x \in X$. In this case, we say q is the conjugate exponent to p and is customarily written p' instead of q.

Having defined the conjugate exponent to p, we wish to define the conjugate norm to $\|\cdot\|_p$. This is the functional $\vert\kern-0.25ex\vert\kern-0.25ex\vert \cdot \vert\kern-0.25ex\vert\kern-0.25ex\vert_p$ defined on $\mathscr{M}$ given by

$$\vert\kern-0.25ex\vert\kern-0.25ex\vert f \vert\kern-0.25ex\vert\kern-0.25ex\vert_p := \sup_{\rho_{p'}(g) \le 1} \int_X |fg| \, d\mu.$$

The next proposition accounts for the variety of definitions for $\vert\kern-0.25ex\vert\kern-0.25ex\vert \cdot \vert\kern-0.25ex\vert\kern-0.25ex\vert_p$ used by different authors.

Proposition 1.8. *The following equalities hold:*

$$\begin{aligned} \vert\kern-0.25ex\vert\kern-0.25ex\vert f \vert\kern-0.25ex\vert\kern-0.25ex\vert_p &= \sup_{\|g\|_{p'} \le 1} \int_X |fg| \, d\mu = \sup_{\rho_{p'}(g) \le 1} \left| \int_X fg d\mu \right| = \sup_{\|g\|_{p'} \le 1} \left| \int_X fg d\mu \right| \\ &= \sup_{\rho_{p'}(g) \le 1, g \in \mathscr{S}} \int_X |fg| \, d\mu = \sup_{\|g\|_{p'} \le 1, g \in \mathscr{S}} \int_X |fg| \, d\mu \end{aligned}$$

where $\mathscr{S}$ *is the class of simple functions on* X*.*

Proof. The unit ball property implies that $\{\rho_{p'} \leq 1\} = \left\{ \|\cdot\|_{p'} \leq 1 \right\}$. This shows that the first and second, the third and fourth, and the fifth and sixth expressions are respectively equal. Moreover, it is clear that $\|\!|f|\!\|_p$ is at least as large as the third and fifth expressions. So, we need only show the reverse inequality holds in each of these cases. To this end, let $h \in L^{p'}$ with $\rho_{p'}(h) \leq 1$. Since $||h| \operatorname{sign} f| \leq |h|$, $\rho_{p'}(|h| \operatorname{sign} f) \leq \rho_{p'}(h) \leq 1$, whence

$$\int_X |fh|\, d\mu = \left| \int_X f\, (|h| \operatorname{sign} f)\, d\mu \right| \leq \sup_{\rho_{p'}(g) \leq 1} \left| \int_X fg d\mu \right|.$$

Taking the supremum over all such h proves the first desired inequality. On the other hand, we may find a sequence h_n of nonnegative simple functions that converge monotonically up to $|h|$. Since $|h_n| \leq |h|$, for each n, it follows, as above, that $\rho_{p'}(h_n) \leq 1$, for each n. Furthermore, the sequence $|fh_n|$ converges monotonically up to $|fh|$; and so, by the monotone convergence theorem,

$$\int_X |fh|\, d\mu = \lim_{n\to\infty} \int_X |fh_n|\, d\mu \leq \lim_{n\to\infty} \left[\sup_{\rho_{p'}(g)\leq 1, g\in\mathscr{S}} \int_X |fg|\, d\mu \right] = \sup_{\rho_{p'}(g)\leq 1, g\in\mathscr{S}} \int_X |fg|\, d\mu.$$

As above, taking the supremum over all such h yields the second desired inequality. ■

It is straight forward to verify that $\|\!|\cdot|\!\|_p$ is a semi-norm on the class of μ-measurable functions, f, for which $\|\!|f|\!\|_p < \infty$. For specificity, let us denote this class by CN^p. If μ is σ-finite, then this semi-norm is a full-fledged norm, provided we make the usual identification of functions that are equal μ a. e. This is not difficult to see at present, but we refrain from proving it at this point, since it is also an immediate consequence of the norm conjugate inequality below.

Hölder's inequality shows that L^p, as defined in the first section, continuously embeds in CN^p with embedding norm at most K, where $1 \leq K \leq 5$ is as in Theorem 1.6. We will soon see that the reverse inclusion also holds so that CN^p is nothing other than L^p. As hinted at a moment ago, the proof of this inclusion will require that the underlying measure be σ-finite.

Lemma 1.9. *Assume that μ is σ-finite. Then there exists a constant $K = K(p) \in [1,3]$ such that $\rho_p(f) \leq K = K(p)$, whenever $\|\|f\|\|_p \leq 1$. If p is constant, then $K = 1$.*

Proof. We first decompose F_p into two sets:

$$A := \{p = 1\}$$

$$B := \{1 < p < \infty\}.$$

Let us estimate on A. Put $g := \mathbf{1}_A$. Since $A = F_{p'}^c$, it follows that

$$\rho_{p'}(g) = \operatorname*{ess\,sup}_{F_{p'}^c} |g| \leq k(A) \leq 1.$$

Thus,

$$\int_A |f|^p = k(A) \int_X |fg| \leq k(A) \|\|f\|\|_p \leq k(A).$$

Now we estimate on B: Suppose additionally for the moment that $\int_B |f|^p < \infty$. Assume, for the sake of contradiction, that $\int_B |f|^p > 1$. Then it follows from Proposition 1.1(d) that there is a $t > 1$ so that $\int_B |f/t|^p = 1$. Put $g(x) = |f(x)/t|^{p(x)-1}\,\mathbf{1}_B(x)$. Then

$$\rho_{p'}(g) = \int_{F_{p'}} \left| \left| \frac{f(x)}{t} \right|^{p-1} \mathbf{1}_B(x) \right|^{p'} d\mu(x) + \operatorname*{ess\,sup}_{x \in F_{p'}^c} \left| \left| \frac{f(x)}{t} \right|^{p-1} \mathbf{1}_B(x) \right| = \int_B \left| \frac{f(x)}{t} \right|^p d\mu(x) = 1$$

Then

$$\|f\|_p \geq \int_X |f(x)g(x)|\, d\mu(x) = t \int_B \left|\frac{f(x)}{t}\right|^p d\mu(x) = t > 1.$$

This is a contradiction, whence $\int_B |f|^p \leq 1$, as desired.

In the event that it is unknown a priori that $\int_B |f|^p < \infty$, we consider the sequence of truncations given by

$$f_n(x) = \min\left\{n^{\frac{1}{p(x)}}, |f(x)|\right\}\mathbf{1}_{B_n}(x),$$

where the B_n are such that $B_n \subset B_{n+1} \subset B$ and $\bigcup_n B_n = B$ and $\mu(B_n) < \infty$, for each n. This is possible, by the (assumed) σ-finiteness of μ. Since $\int_B |f_n|^p \leq n\mu(E_n) < \infty$ and $\|f_n\|_p \leq \|f\|_p \leq 1$ (since $|f_n| \leq |f|$), the result above implies that

$$\int_B |f_n|^p \leq 1, \tag{1.11}$$

for each n. By construction, as $n \to \infty$, we have $|f_n|^p \nearrow |f|^p$ pointwise, whence

$$\int_B |f_n|^p \nearrow \int_B |f|^p. \tag{1.12}$$

Combining (1.11) with (1.12), we may deduce that $\int_B |f|^p \leq 1$ so that

$$\int_B |f|^p = k(B) \int_B |f|^p \leq k(B).$$

Finally, we estimate on F_p^c: Put $M := \operatorname{ess\,sup}_{F_p^c} |f|$. If $\mu(F_p^c) = 0$, there is nothing to prove. So, assume $\mu(F_p^c) > 0$. By the definition of essential supremum, given $0 < \epsilon < 1$, there exists a subset $E = E(\epsilon)$ of F_p^c such that $0 < \mu(E) < \infty$ (σ-finiteness used here) and $|f| \geq (1-\epsilon)M$, for all $x \in E$. Put $g = \frac{1}{\mu(E)}\mathbf{1}_E$. Since $E \subset F_p^c = \{p = \infty\} = \{p' = 1\}$, it follows that $\rho_{p'}(g) = 1$, whence

$$(1-\epsilon)M = \frac{1}{\mu(E)} \int_E (1-\epsilon)M \le \frac{1}{\mu(E)} \int_E |f| = \int_X |fg| \le |||f|||_p \le 1.$$

Letting $\epsilon \to 0$ shows that $M \le 1 = k(F_p^c)$.

In summary, we have shown that

$$|||f|||_p \le 1 \Rightarrow \rho_p(f) \le k(A) + k(B) + k(F_p^c) =: K.$$

Notice that $K = 1$ when exactly one of the three sets A, B or F_p^c has positive measure. In particular, this happens when p is constant. ■

Corollary 1.10. *$L^p = CS^p$, provided that μ is σ-finite. $L^p \subset CS^p$, even if μ is not σ-finite.*

Proof. As remarked above, Hölder's inequality implies that $L^p \subset CS^p$. On the other hand, if $f \in CS^p$, then the CS^p norm of $f / |||f|||_p$ is no greater than 1. Consequently, $\rho_p(f / |||f|||_p) < \infty$, by Lemma 1.9. (We are making the trivial assumption that $f \neq 0$.) Hence, $f \in L^p$, showing the opposite inclusion. ■

Corollary 1.11 (Norm conjugacy inequality)**.** *If μ is σ-finite, then there are constants $K_1 = K_1(p) \in [1/5, 1]$ and $K_2 = K_2(p) \in [1, 3]$ such that*

$$K_1 |||f|||_p \le \|f\|_p \le K_2 |||f|||_p .$$

The first inequality holds even if μ is not σ-finite. If p is constant, then $K_1 = K_2 = 1$ so that

$$\|f\|_p = |||f|||_p .$$

Proof. Through Corollary 1.10, we have established that $\|\|f\|\|_p$ is finite if and only if $\|f\|_p$ is also finite. (Note that the above chain of inequalities, properly interpreted, reflects this fact.) So, we may assume that both are finite. We have already argued that the first inequality is a consequence of Hölder's inequality. The constant $K_1 = 1/K$, where K is the constant from that theorem. To prove the second inequality, let K be as in Lemma 1.9. We now apply the same scaling technique that can be found as before: Since $f/\|\|f\|\|_p$ has CS^p norm no greater than 1, the convexity of ρ_p and Lemma 1.9 justify that

$$\rho_p\left(\frac{f}{K\|\|f\|\|_p}\right) \leq \frac{1}{K}\rho_p\left(\frac{f}{\|\|f\|\|_p}\right) \leq 1,$$

whence $f/\|\|f\|\|_p$ has L^p norm no greater than K. By homogeneity, $\|f\|_p \leq K\|\|f\|\|_p$. Taking $K_2 = K$ finishes the proof. ∎

Chapter 2

Advanced Topics

In this chapter, the intention is to demonstrate the uses of the basic tools that we have devised from the previous chapter in the settings of duality and Riesz-Thorin interpolation. In both instances, it will be shown that the now canonical proofs of these two results readily lend themselves to the variable setting. That said, the successful execution of these proofs, unsurprisingly, require some results to handle the nuances that arise due to the potentially variable exponent.

2.1 Duality

Given a normed vector space V, we denote by V' the normed vector space of bounded linear functionals from $V \to \mathbb{C}$ endowed with the usual operator norm. We wish to characterize $(L^p)'$. The characterization is motivated by norm conjugate inequality. More precisely, for $g \in L^{p'}$, we define the integral operator associated to g to be the operator $I(g) : L^p \to \mathbb{C}$

given by $I(g)(f) := \int_X fg d\mu$. Hölder's inequality ensures that $I(g)$ is a well-defined operator and that it is bounded. The linearity of the integral implies the linearity of $I(g)$, whence $I(g) \in (L^p)'$.

We have thus defined an operator $I : L^{p'} \to (L^p)'$. Again using the linearity of the integral, we find that I is linear. What's more, by Proposition 1.8, we have the identity

$$\|I(g)\|_{(L^p)'} = \|g\|_{p'} .$$

From this and the results from the previous section, it follows that I is an injective, bounded, linear operator from $L^{p'}$ into $(L^p)'$, whenever the underlying measure, μ, is σ-finite. If we tack on the additional hypothesis that p be a bounded exponent, then it actually turns out that this operator is an isomorphism.

Theorem 2.1. *Suppose μ is σ-finite and p is a bounded exponent. Then $I : L^{p'} \to (L^p)'$ is an isomorphism.*

Proof. Given the exposition preceding the statement of the theorem, it remains to prove that the I is surjective. Fix $T \in (L^p)'$.

We will first show the result with the added assumption that $\mu(X) < \infty$. Then $\mathbf{1}_E \in L^p$, for every measurable $E \subset X$. Thus, the set function $\nu(E) := T(\mathbf{1}_E)$ is well-defined on the same σ-algebra as μ. We wish to show that ν is a complex measure that is absolutely continuous with respect to μ. Observe that $\mu(E) = 0$ implies that $\mathbf{1}_E = 0$ μ a.e., and consequently $T(\mathbf{1}_E) = 0$, by linearity. In particular, $\nu(\varnothing) = 0$. Now, let $\{E_k\}$ be a pairwise disjoint sequence of measurable sets and put $E := \bigcup_k E_k$. Since

$$\rho_p\left(\mathbf{1}_E - \sum_{k=1}^{n} \mathbf{1}_{E_k}\right) = \rho_p\left(\sum_{k=n+1}^{\infty} \mathbf{1}_{E_k}\right) = \sum_{k=n+1}^{\infty} \mu(E_k) \to 0,$$

as $n \to \infty$ and since $p^+ < \infty$, we may conclude that $\sum_{k=1}^{n} \mathbf{1}_{E_k} \to \mathbf{1}_E$ in L^p, by Theorem 1.4. Hence,

$$\nu(E) = T(\mathbf{1}_E) = \sum_k T(\mathbf{1}_E) = \sum_k \nu(E_k),$$

by the continuity of T, completing the proof that ν is a complex measure absolutely continuous with respect to μ. By the theorem of Lebesgue, Radon and Nikodym, we may find a $g \in L^1$ such that

$$T(\mathbf{1}_E) = \nu(E) = \int_X \mathbf{1}_E g, \tag{2.1}$$

for every measurable $E \subset X$. By the linearity of T and the integral, (2.1) implies

$$T(f) = \int_X fg, \tag{2.2}$$

for every simple $f \in L^p$.

Let $f \in L^\infty$. Since $\mu(X) < \infty$, $L^\infty \subset L^p$. Also, since $p^+ < \infty$, simple functions are dense in L^p. Hence, by extracting a subsequence and truncating if needed, we may find a sequence, $\{f_n\}$, of simple functions in L^p that are uniformly bounded by $M := 2\,\text{ess sup}\,|f|$ and converge almost everywhere to f as well as in the L^p norm. Thus, we may invoke the continuity of T and the dominated convergence theorem to get

$$T(f) = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} \int f_n g = \int fg.$$

As $f \in L^\infty$ is arbitrary, we conclude that (2.2) holds on L^∞.

Now, let $f \in L^p$ be simple with $\|f\|_p \le 1$. Since f is simple, $h := |f| \operatorname{sign} g \in L^\infty$ and $|h| \le |f|$, whence

$$\begin{aligned}
\int_X |fg| &= \left| \int_X hg \right| \qquad (2.3)\\
&= |T(h)| \\
&\le \|T\|_{L^p \to \mathbb{C}} \|h\|_p \\
&\le \|T\|_{L^p \to \mathbb{C}} \|f\|_p \\
&\le \|T\|_{L^p \to \mathbb{C}} .
\end{aligned}$$

Noting that $(p')' = p$ and taking the supremum in (2.3) over all such f, we may deduce that $\|g\|_{p'} \le \|T\|_{L^p \to \mathbb{C}}$. Thus, $g \in L^{p'}$.

Finally, we use the new found fact $g \in L^{p'}$ to conclude that (2.2) holds for all $f \in L^p$ as follows: Given any $f \in L^p$, choose a sequence, $\{f_n\}$, of simple functions in L^p that converge to f in the L^p norm. Then

$$\begin{aligned}
\left| Tf - \int fg \right| &= \lim_{n\to\infty} \left| Tf_n - \int fg \right| \\
&= \lim_{n\to\infty} \left| \int f_n g - \int fg \right| \\
&\le \lim_{n\to\infty} \int |f_n - f| \, |g| \\
&\le \lim_{n\to\infty} 5 \|f_n - f\|_p \|g\|_{p'} \\
&= 0,
\end{aligned}$$

whence $Tf = \int fg$, as desired.

We now replace the assumption that $\mu(X) < \infty$ with the assumption that μ is σ-finite. Choose a sequence, $\{X_k\}$, of measurable subsets of X such that $X_k \subset X_{k+1}$, $\mu(X_k) < \infty$

and $X = \bigcup_k X_k$. It is clear that that T restricts to a bounded linear operator on $L^p(X_k)$. By the work above, we may find $g_k \in L^{p'}(X_k)$ such that the restriction of T to $L^p(X_k)$ is given by integration over X_k against g. Extend each g_k to X by putting $g_k = 0$ outside of X_k. Since T is given by integration against g_k on X_k, it follows that $g_l = g_k$ μ a.e. on X_k, for every $l \geq k$. Consequently, the function $g = \lim_k g_k$ is well-defined and has the property that $g = g_k$ on X_k, for every k. Given $f \in L^p(X)$ with $\|f\|_p \leq 1$, the monotone convergence theorem implies

$$\begin{aligned}
\int_X |fg| &= \lim_{k\to\infty} \int_X |fg_k| \\
&= \lim_{k\to\infty} \left| \int_{X_k} (|f| \operatorname{sign} g_k)\, g_k \right| \\
&= \lim_{k\to\infty} |T\left(|f| \operatorname{sign} g_k\right)| \\
&\leq \lim_{k\to\infty} \|T\|_{L^p\to\mathbb{C}} \, \||f| \operatorname{sign} g_k\|_p \\
&\leq \|T\|_{L^p\to\mathbb{C}}\,,
\end{aligned}$$

whence $g \in L^{p'}$. Now that we know $g \in L^{p'}$, we will be able to argue that T is given by integration against g, thereby, completing the proof of this theorem. Given any $f \in L^p$, define $f_k := f\mathbf{1}_{X_k}$. Observe that $|f - f_k|^p \to 0$ everywhere and that $|f - f_k|^p \leq 2^{p^+} |f|^p \in L^1$. Consequently, $\rho_p(f - f_k) \to 0$, which, in turn, implies that $\|f - f_k\|_p \to 0$, since $p^+ < \infty$. Since $g \in L^{p'}$, its integral operator is continuous on L^p, by Hölder's inequality. This paired with the assumed continuity of T implies that

$$\begin{aligned}
T(f) &= \lim_{k\to\infty} T(f_k) = \lim_{k\to\infty} T|_{L^p(X_k)}(f_k|_{X_k}) = \lim_{k\to\infty} \int_{X_k} f_k|_{X_k} g_k = \lim_{k\to\infty} \int_X f_k g = \lim_{k\to\infty} I(g)(f_k) \\
&= I(g)(f).
\end{aligned}$$

$\blacksquare$

If in addition, p is (essentially) bounded away from one, then we have the obvious corollary:

Corollary 2.2. *Suppose μ is σ-finite and that $1 < \operatorname{ess\,inf}_X p \leq \operatorname{ess\,sup}_X p < \infty$, or equivalently, that p and p' are both bounded exponents. Then L^p is reflexive.*

Now that we effectively know how to identify bounded linear functional defined on L^p, we can revisit norm conjugacy.

Proposition 2.3. *Suppose μ is σ-finite and p is a bounded exponent. Then $f \in L^{p'}$ if and only if*

$$\sup \left\{ \left| \int_X f g d\mu \right| : \|g\|_p \leq 1, g \in \mathscr{S} \right\} \tag{2.4}$$

is finite. If $f \in L^{p'}$, then $\|f\|_{p'}$ is equal to (2.4).

Proof. For brevity, denote the supremum in (2.4) by A. By Proposition 1.8, we always have that $\|f\|_{p'} \geq A$, since the supremum for $\|f\|_{p'}$ is being taken over a larger set. So, if $f \in L^{p'}$, then $\|f\|_{p'} < \infty$, by Corllary 1.10, and therefore, $A < \infty$.

Let us assume, on the other hand, that $A < \infty$. Define $T = T_f : \mathscr{S} \cap L^{p'} \to \mathbb{C}$ by $Tg := \int fg$. Since p is bounded, $\mathscr{S} \cap L^p$ is dense in L^p. Moreover, by assumption, T is bounded on this dense subset. By the Hanh-Banach theorem, we may extend T to a bounded linear functional defined on all of L^p. By duality, our extension is given by integration against some $\tilde{f} \in L^{p'}$, whence $\int fg = \int \tilde{f} g$, for every $g \in \mathscr{S} \cap L^p$. This implies that $f = \tilde{f}$ almost everywhere so that $f \in L^{p'}$.

In any event, if we know that $f \in L^{p'}$, then we can show that $\|f\|_{p'} = A$ as follows: We have already pointed out that we always have $\|f\|_{p'} \geq A$. So, it remains to show the reverse inequality. To this end, let $g \in L^p$ with $\|g\|_p \leq 1$. Choose a sequence of simple functions, $\{g_n\}$, with $\|g_n\|_p \leq 1$, for all n, and such that $g_n \to g$ in L^p, as $n \to \infty$. Then

$$\left|\int fg\right| \leq \left|\int f(g - g_n)\right| + \left|\int fg_n\right| \leq C\,\|f\|_{p'}\,\|g - g_n\|_p + A. \tag{2.5}$$

Since $f \in L^{p'}$, the first term in sum on the right vanishes as $n \to \infty$. Thus, letting $n \to \infty$ first, then taking the supremum over all such g in (2.5) gives the desired inequality. ■

2.2 The Riesz-Thorin Interpolation Theorem

In this section, we will state and prove a version of the Riesz-Thorin interpolation theorem for variable Lebesgue spaces. As in the constant exponent setting, this theorem allows us determine the boundedness of a given operator for a certain range of L^p spaces into a corresponding range of L^q spaces, provided that we know that it is bounded on two "endpoint spaces". The major limitation of this particular interpolation theorem arises from the fact the intermediary exponents are defined so rigidly. More precisely, in practice, it often turns out that verifying that a certain operator, T, is bounded on the endpoint spaces boils down to determining whether or not the endpoint exponents have a certain property. Due to the rigidity of the way in which the intermediary exponents are defined, it frequently turns out that they also enjoy the same property as the endpoint exponents. As such, we could just as well have concluded that T is bounded on the intermediary space without having to appeal to the Riesz-Thorin theorem. Despite this limitation, the theorem is not without merit. Often times, it can provide a quick way to verify certain embeddings.

Before moving on to the statement of the theorem, we recall some notation: $p^- := \operatorname{ess\,inf}_X p$ and $p^+ := \operatorname{ess\,sup}_X p$. Having recalled this, let us state and prove the theorem.

Theorem 2.4. *Let (X, μ) and (Y, ν) be σ-finite, complete measure spaces. For $k = 0, 1$, assume that p_k and q_k are bounded exponents. Suppose that we have a linear operator, T, that carries $L^{p_0}(X) \cap L^{p_1}(X)$ into $\mathscr{M}(Y)$ and satisfies*

$$\|Tf\|_{q_k'} \leq M_k \|f\|_{p_k}, \tag{2.6}$$

for every $f \in \mathscr{S}(X) \cap L^{p_k}(X)$. For $z \in := \{z : 0 \leq Re\, z \leq 1\}$, define p_z and q_z by

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1} \quad \textit{and} \quad \frac{1}{q_z} = \frac{1-z}{q_0} + \frac{z}{q_1}.$$

Then, given any $\theta \in (0, 1)$, the inequality

$$\|Tf\|_{q_\theta'} \leq C M_0^{1-\theta} M_1^{\theta} \|f\|_{p_\theta} \tag{2.7}$$

holds, for every $f \in \mathscr{S}(X) \cap L^{p_\theta}(X)$, where $1 \leq C \leq 15$.

Proof. We present and adaptation of the now standard proof for the constant exponent case. This, for example, can be found in [7]. Fix $\theta \in (0, 1)$ and let $f \in \mathscr{S}(X) \cap L^{p_\theta}(X)$. We will break the proof of the theorem into steps:

Step 1: Make reductions. Because T is linear, we may assume that $f \neq 0$, since the desired inequality holds for $f = 0$. Since a constant multiple of a simple function is still a simple function, the linearity of T, the homogenity of the norm and a scaling argument allows us to further assume that $\|f\|_{p_\theta} = 1$ and show

$$\|Tf\|_{q'_\theta} \leq CM_0^{1-\theta} M_1^{\theta}. \tag{2.8}$$

Furthermore, by Proposition 2.3, to prove (2.8), it suffices to prove that

$$\left| \int_Y Tf(y)g(y)d\nu(y) \right| \leq KM_0^{1-\theta} M_1^{\theta}, \tag{2.9}$$

for every $g \in \mathscr{S}(Y) \cap L^{q_\theta}(Y)$, with $\|g\|_{q_\theta} \leq 1$. Note that K is the Hölder inequality constant and C is product of K and the norm conjugate exponent. This is why $1 \leq C \leq 15$.

Step 2: Set notation and definitions. Under the above assumptions, we may write

$$f = \sum_{j=1}^{m} a_j e^{i\alpha_j} \mathbf{1}_{A_j}$$

and

$$g = \sum_{k=1}^{n} b_k e^{i\beta_k} \mathbf{1}_{B_k},$$

where the $a_j, b_k > 0$, $\alpha_j, \beta_k \in \mathbb{R}$, $\mu(A_j), \mu(B_k) < \infty$, and the A_j and B_k are, respectively, pairwise disjoint. Now, we define

$$f_z(x) := \sum_{j=1}^{m} a_j^{\frac{p_\theta(x)}{p_z(x)}} e^{i\alpha_j} \mathbf{1}_{A_j(x)}$$

and

$$g_z(y) := \sum_{k=1}^{n} b_k^{\frac{q_\theta(y)}{q_z(y)}} e^{i\beta_k} \mathbf{1}_{B_k(y)}.$$

Finally, we put

$$F(z) := \int_Y Tf_z(y) g_z(y) d\nu(y).$$

Step 3: Argue that F is continuous and bounded on $\mathbb{S}$ and analytic on $\mathrm{int}(\mathbb{S})$. Note first, by construction, $p_\theta(x) \in [1, \infty]$. Now, we have that $(1-\theta)p_1(x) + \theta p_0(x) \geq 1$, since $p_0, p_1 \geq 1$ and $\theta \in (0,1)$. Consequently, for a.e. $x \in X$,

$$p_\theta(x) = \frac{p_0(x)p_1(x)}{(1-\theta)p_1(x) + \theta p_0(x)} \leq p_0(x)p_1(x) \leq p_0^+ p_1^+ < \infty.$$

So, we, in fact, have $p_\theta(x) \in [1, p_0^+ p_1^+]$. Note also that

$$-1 < \frac{1}{p_1^-} - 1 \leq \frac{1}{p_1(x)} - \frac{1}{p_0(x)} \leq 1 - \frac{1}{p_0^-(x)} < 1,$$

for a.e. $x \in X$. Hence, regarding

$$\frac{p_\theta(x)}{p_z(x)} = p_\theta(x)\left[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right] z + \frac{p_\theta(x)}{p_0(x)}.$$

as a linear polynomial in z, it follows that the linear coefficient maps $X \to [-p_0^+ p_1^+, p_0^+ p_1^+]$ while the constant coefficient maps $X \to [0, p_0^+ p_1^+]$, since $p_0 \geq 1$. In a symmetric manner, we deduce that, if we regard

$$\frac{q_\theta(x)}{q_z(x)} = q_\theta(x)\left[\frac{1}{q_1(x)} - \frac{1}{q_0(x)}\right] z + \frac{q_\theta(x)}{q_0(x)}$$

as a polynomial in z, then the linear coefficient maps $Y \to [-q_0^+ q_1^+, q_0^+ q_1^+]$ while the constant coefficient maps $Y \to [0, q_0^+ q_1^+]$. Since we may rewrite F as

$$F(z) = \sum_{j=1}^{m}\sum_{k=1}^{n} \int_Y T\left[a_j^{\frac{p_\theta(\cdot)}{p_z(\cdot)}} \mathbf{1}_{A_j}(\cdot)\right](y) b_k^{\frac{q_\theta(y)}{q_z(y)}} \mathbf{1}_{B_k}(y) d\nu(y),$$

Lemma 2.6, which is stated and proved below, ensures that F is the finite sum of functions that are continuous and bounded on $\mathbb{S}$ and analytic on $\mathrm{int}(\mathbb{S})$, and therefore, is itself such a function.

Step 4: Obtain bounds for F on the $\partial\mathbb{S}$. Assume that $\operatorname{Re} z = 0$ and write $z = it$, with $t \in \mathbb{R}$. Since the A_j are pairwise disjoint and the $a_j > 0$, we have

$$\begin{aligned}
\rho_{p_0}(f_z) &= \int_X \left| \sum_{j=1}^m a_j^{\frac{p_\theta(x)}{p_z(x)}} e^{i\alpha_j} \mathbf{1}_{A_j}(x) \right|^{p_0(x)} d\mu(x) \\
&= \int_X \left| \sum_{j=1}^m a_j^{p_\theta(x)\left[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right]it + \frac{p_\theta(x)}{p_0(x)}} e^{i\alpha_j} \mathbf{1}_{A_j}(x) \right|^{p_0(x)} d\mu(x) \\
&= \int_X \sum_{j=1}^m \left| a_j^{p_\theta(x)\left[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right]it + \frac{p_\theta(x)}{p_0(x)}} e^{i\alpha_j} \mathbf{1}_{A_j}(x) \right|^{p_0(x)} d\mu(x) \\
&= \int_X \sum_{j=1}^m a_j^{p_\theta(x)} \mathbf{1}_{A_j}(x) d\mu(x) \\
&= \int_X \left| \sum_{j=1}^m a_j e^{i\alpha_j} \mathbf{1}_{A_j}(x) \right|^{p_\theta(x)} d\mu(x) \\
&= \rho_{p_\theta}(f) \\
&\le 1,
\end{aligned}$$

since $\|f\|_{p_\theta} = 1$. In brief, for z with $\operatorname{Re} z = 0$, we have $\rho_{p_0}(f_z) \le 1$, and therefore, $\|f_z\|_{p_0} \le 1$. Making the obvious substitutions, the same argument shows that $\|g_z\|_{q_0} \le 1$, for z with $\operatorname{Re} z = 0$. Thus, for z with $\operatorname{Re} z = 0$, Hölder's inequality gives

$$\begin{aligned}
|F(z)| &\le \left| \int_Y Tf_z(y) g_z(y) d\nu(y) \right| \qquad (2.10) \\
&\le K \|Tf_z\|_{q_0'} \|g_z\|_{q_0} \\
&\le K M_0 \|f_z\|_{p_0} \|g_z\|_{q_0} \\
&\le K M_0.
\end{aligned}$$

The above argument readily lends itself to the alternate situation in which $\operatorname{Re} z = 1$. In this case, we find that $\|f_z\|_{p_1}$ and $\|g_z\|_{q_1}$ are both no greater than 1, and as such, we may apply Hölder's inequality to ultimately get

$$|F(z)| \leq KM_1, \tag{2.11}$$

for z with $\operatorname{Re} z = 1$.

Step 5: Invoke Hadamard's three lines lemma. Using Hadamard's three lines lemma with (2.10) and (2.11) as our respective boundary estimates, we find that

$$|F(z)| \leq (KM_0)^{1-\tau}(KM_1)^{\tau} = KM_0^{1-\tau}M_1^{\tau}, \tag{2.12}$$

for all $\tau \in [0,1]$ and z such that $\operatorname{Re} z = \tau$. In particular, choosing $\tau = z = \theta$ in (2.12), we get exactly (2.9), since $f_\theta = f$ and $g_\theta = g$. ■

Corollary 2.5. *Under the same hypothesis as Theorem 2.4, T has a unique continuous extension to all of $L^{p_\theta}(X)$.*

Proof. Since p_θ is a bounded exponent, $\mathscr{S}(X) \cap L^{p_\theta}(X)$ is a dense linear subspace of $L^{p_\theta}(X)$, whence the corollary follows from the general theory of Banach spaces. ■

It should also be remarked that the Riesz-Thorin interpolation theorem has even more generalized version in the Orlicz space setting in which the operator is further allowed to be sublinear. The proof of this result is highly technical and requires quite a bit of Orlicz space machinery, and therefore, is not presented in detail here. The necessary details can be found in [8].

Also, it worth mentioning that the Riesz-Thorin theorem is often presented along side the Marcinkiewicz interpolation theorem. At this point, it is unknown whether or not this latter interpolation theorem has a generalization to the variable setting. The major complication halting progress with the Marcinkiewicz theorem is that it seems intimately tied to the Cavalieri's principle, and therefore, distribution functions. These complications and related ideas are presented [3].

We end this section by stating and prove the following somewhat technical lemma that was required for the third step of the proof for the Riesz-Thorin theorem.

Lemma 2.6. *Let p_1 and p_2 be bounded exponents. Suppose $T : L^{p_1}(X_1, \mu_1) \to L^{p_2'}(X_2, \mu_2)$ is linear and continuous. For $k = 1, 2$, suppose we are given positive real numbers a_k, B_k and M_k, measurable sets A_k with $\mu_k(A_k) < \infty$ and measurable functions $m_k, b_k : X_k \to \mathbb{R}$ that satisfy $-M_k \leq m_k(x_k) \leq M_k$ and $0 \leq b_k(x_k) \leq B_k$, for a.e. $x_k \in X_k$. For $z \in \mathbb{S}$, define*

$$F(z) := \int_{X_2} T\left[a_1^{m_1(\cdot)z+b_1(\cdot)} \mathbf{1}_{A_1}(\cdot)\right](x_2) a_2^{m_2(x_2)z+b_2(x_2)} \mathbf{1}_{A_2}(x_2) d\mu_2(x_2).$$

Then F is continuous and bounded on $\mathbb{S}$ and analytic on $\mathrm{int}(\mathbb{S})$.

Proof. To make the reading slightly less cluttered, we introduce the notation

$$\alpha_k(x_k, z) := a_k^{m_k(x_k)z+b_k(x_k)} \mathbf{1}_{A_k}(x_k)$$

and

$$Q_k(x_k, z, w) := \frac{\alpha_k(x_k, z) - \alpha_k(x_k, w)}{z - w} - \alpha_k(x_k, z) m_k(x_k) \log a_k$$

Note that under this notation, we may rewrite F as

$$F(z) = \int_{X_2} T[\alpha_1(\cdot, z)](x_2) \alpha_2(x_2, z) d\mu_2(x_2).$$

We now make some preliminary estimates, the first of which is basic. For a.e. $x_k \in X_k$,

$$\begin{aligned} |\alpha_k(x_k, z)| &= \left| a_k^{m_k(x_k)\operatorname{Re} z + b_k(x_k)} a_k^{i m_k(x_k)\operatorname{Im} z} \mathbf{1}_{A_k}(x_k) \right| \\ &= a_k^{m_k(x_k)\operatorname{Re} z + b_k(x_k)} \mathbf{1}_{A_k}(x_k) \\ &\le C_k \mathbf{1}_{A_k}(x_k), \end{aligned} \tag{2.13}$$

where $C_k := \max_{t \in [-M_k, M_k + B_k]} a_k^t$. From (2.13), we get

$$\begin{aligned} |Q_k(x_k, z, w)| &= |\alpha_k(x_k, z)| \left| \frac{a_k^{m_k(x_k)(w-z)} - 1}{w - z} - m_k(x_k)\log a_k \right| \\ &= |\alpha_k(x_k, z)| \left| \frac{e^{[m_k(x_k)\log a_k](w-z)} - 1}{w - z} - m_k(x_k)\log a_k \right| \\ &= |\alpha_k(x_k, z)| \left| \frac{\left[\sum_{j=0}^{\infty} \frac{([m_k(x_k)\log a_k](w-z))^j}{j!}\right] - 1}{w - z} - m_k(x_k)\log a_k \right| \\ &= |\alpha_k(x_k, z)| \left| \sum_{j=2}^{\infty} \frac{[m_k(x_k)\log a_k]^j (w-z)^{j-1}}{j!} \right| \\ &\le C_k \mathbf{1}_{A_k}(x_k) \sum_{j=2}^{\infty} \frac{(M_k |\log a_k|)^j |w-z|^{j-1}}{j!} \\ &= C_k M_k^2 |\log a_k|^2 |z-w| \mathbf{1}_{A_k}(x_k) \sum_{j=0}^{\infty} \frac{(M_k |\log a_k| |z-w|)^j}{(j+2)!} \\ &\le C_k M_k^2 |\log a_k|^2 |z-w| e^{M_k |\log a_k| |z-w|} \mathbf{1}_{A_k}(x_k), \end{aligned} \tag{2.14}$$

for a.e. $x_k \in X_k$. Now, for $|z-w|$ sufficiently small, the expression $C_k M_k^2 |\log a_k|^2 |z-w| e^{M_k |\log a_k| |z-w|}$ is no greater then 1. Thus,

$$\begin{aligned} \rho_{p_k}(Q_k(\cdot, z, w)) &:= \int_{X_k} |Q_k(x_k, z, w)|^{p_k(x_k)} \, d\mu_k(x_k) \\ &\le C_k M_k^2 |\log a_k|^2 |z-w| e^{M_k |\log a_k| |z-w|} \mu_k(A_k) \\ &\to 0, \end{aligned} \tag{2.15}$$

as $w \to z$, since $\mu(A_k) < \infty$. Since $p_k^+ < \infty$, it follows that

$$\lim_{\mathrm{int}(\mathbb{S}) \ni \mathrm{w} \to \mathrm{z}} \|Q_k(\cdot, z, w)\|_{p_k} = 0. \tag{2.16}$$

Note also that (2.16) immediately gives

$$\lim_{\mathrm{int}(\mathbb{S}) \ni \mathrm{w} \to \mathrm{z}} \|\alpha_k(\cdot, z) - \alpha_k(\cdot, w)\|_{p_k} = 0, \tag{2.17}$$

since

$$\|\alpha_k(\cdot, z) - \alpha_k(\cdot, w)\|_{p_k} \le |z - w| \left(\|Q_k(\cdot, z, w)\|_{p_k} + \|\alpha_k(\cdot, z) m_k(\cdot) \log a_k\|_{p_k} \right).$$

We can now proceed with proving the analyticity of F. We decompose the defining difference quotient as the sum of two integrals:

$$\begin{aligned}
\frac{F(z) - F(w)}{z - w} &= \frac{\int_{X_2} T[\alpha_1(\cdot, z)](x_2)\alpha_2(x_2, z) d\mu_2(x_2) - \int_{X_2} T[\alpha_1(\cdot, w)](x_2)\alpha_2(x_2, w) d\mu_2(x_2)}{z - w} \\
&= \int_{X_2} \frac{T[\alpha_1(\cdot, z)](x_2)\alpha_2(x_2, z) - T[\alpha_1(\cdot, w)](x_2)\alpha_2(x_2, z)}{z - w} d\mu_2(x_2) \\
&\quad + \int_{X_2} \frac{T[\alpha_1(\cdot, w)](x_2)\alpha_2(x_2, z) - T[\alpha_1(\cdot, w)](x_2)\alpha_2(x_2, w)}{z - w} d\mu_2(x_2) \\
&= \underbrace{\int_{X_2} T\left[\frac{\alpha_1(\cdot, z) - \alpha_1(\cdot, w)}{z - w}\right](x_2)\alpha_2(x_2, z) d\mu_2(x_2)}_{I} \\
&\quad + \underbrace{\int_{X_2} T\left[\alpha_1(\cdot, w)\right](x_2) \left[\frac{\alpha_2(x_2, z) - \alpha_2(x_2, w)}{z - w}\right] d\mu_2(x_2)}_{J}.
\end{aligned}$$

Put

$$I' := \int_{X_2} T\left[\alpha_1(\cdot, z) m_1(\cdot) \log a_1\right](x_2)\alpha_2(x_2, z) d\mu(x_2)$$

and

$$J' := \int_{X_2} T\left[\alpha_1(\cdot, z)\right](x_2)\alpha_2(x_2, z) m_2(x_2) \log a_2 d\mu_2(x_2).$$

We claim that $I \to I'$ and $J \to J'$, as $w \to z$. To see that $I \to I'$, we use the linearity of the integral and T along with Hölder's inequality to obtain

$$\begin{aligned} |I - I'| &= \left| \int_{X_2} T\left[\frac{\alpha_1(\cdot, z) - \alpha_1(\cdot, w)}{z - w}\right](x_2)\alpha_2(x_2, z)d\mu_2(x_2) \right. \\ &\qquad \left. - \int_{X_2} T\left[\alpha_1(\cdot, z)m_1(\cdot)\log a_1\right](x_2)\alpha_2(x_2, z)d\mu(x_2) \right| \\ &= \left| \int_{X_2} T\left[Q_1(\cdot, z, w)\right](x_2)\alpha_2(x_2, z)d\mu_2(x_2) \right| \\ &\leq K \left\| T\left[Q_1(\cdot, z, w)\right](\cdot) \right\|_{p_2'} \left\| \alpha_2(\cdot, z) \right\|_{p_2} \\ &\leq K \left\| T \right\|_{L^{p_1} \to L^{p_2'}} \left\| Q_1(\cdot, z, w) \right\|_{p_1} \left\| \alpha_2(\cdot, z) \right\|_{p_2}, \end{aligned}$$

where K is the constant appearing in Hölder's inequality. The desired conclusion now follows by (2.16). The computation for J in relation to J' is slightly more tedious, but no more difficult. First, we mimic what we did earlier and break $J - J'$ into two sub-integrals:

$$\begin{aligned} J - J' &= \int_{X_2} T\left[\alpha_1(\cdot, w)\right](x_2)\left[\frac{\alpha_2(x_2, z) - \alpha_2(x_2, w)}{z - w}\right]d\mu_2(x_2) \\ &\qquad - \int_{X_2} T\left[\alpha_1(\cdot, z)\right](x_2)\alpha_2(x_2, z)m_2(x_2)\log a_2 d\mu_2(x_2) \\ &= \int_{X_2} T\left[\alpha_1(\cdot, w)\right](x_2)\left[\frac{\alpha_2(x_2, z) - \alpha_2(x_2, w)}{z - w} - \alpha_2(x_2, z)m_2(x_2)\log a_2\right]d\mu_2(x_2) \\ &\qquad + \int_{X_2} \left(T\left[(\alpha_1(\cdot, w)\right](x_2) - T\left[\alpha_1(\cdot, z)\right](x_2)\right)\alpha_2(x_2, z)m_2(x_2)\log a_2 d\mu_2(x_2) \\ &= \underbrace{\int_{X_2} T\left[\alpha_1(\cdot, w)\right](x_2)Q_2(x_2, z, w)d\mu_2(x_2)}_{S_1} \\ &\qquad + \underbrace{\int_{X_2} T\left[\alpha_1(\cdot, w) - \alpha_1(\cdot, z)\right](x_2)\alpha_2(x_2, z)m_2(x_2)\log a_2 d\mu_2(x_2)}_{S_2}. \end{aligned}$$

Next, we estimate each of these individually:

$$\begin{aligned}|S_1| &\leq K \left\|T\left[\alpha_1(\cdot,w)\right](\cdot)\right\|_{p_2'} \left\|Q_2(\cdot,z,w)\right\|_{p_2} \\ &\leq K \left\|T\right\|_{L^{p_1}\to L^{p_2'}} \left\|\alpha_1(\cdot,w)\right\|_{p_1} \left\|Q_2(\cdot,z,w)\right\|_{p_2} \\ &\leq KC_1 \left\|T\right\|_{L^{p_1}\to L^{p_2'}} \left\|\mathbf{1}_{A_1}\right\|_{p_1} \left\|Q_2(\cdot,z,w)\right\|_{p_2},\end{aligned} \tag{2.18}$$

where we have used Hölder's inequality, the boundedness of T and (2.13), respectively. Following analogous arguments, we have

$$|S_2| \leq K \left\|T\right\|_{L^{p_1}\to L^{p_2'}} \left\|\alpha_1(\cdot,w)-\alpha_1(\cdot,z)\right\|_{p_1} \left\|\alpha_2(\cdot,z)m_2(\cdot)\log a_2\right\|_{p_2}. \tag{2.19}$$

Now, the expression on the far right of (2.18) tends to 0, as $w \to z$, by (2.16). On the other hand, the expression on the far right of (2.19) tends to 0, as $w \to z$, by (2.17). Consequently, $J \to J'$, as $w \to z$. Hence, we have shown F is analytic on $\operatorname{int}(\mathbb{S})$; and moreover, the derivative F' is given by $I' + J'$.

We now move on to the continuity of F on the entire strip $\mathbb{S}$. This poses no additional difficulty, for we simply use the same techniques as above:

$$\begin{aligned}|F(z)-F(w)| &= \left| \int_{X_2} T\left[\alpha_1(\cdot,z)-\alpha_1(\cdot,w)\right](x_2)\alpha_2(x_2,z)d\mu_2(x_2) \right. \\ &\qquad \left. + \int_{X_2} T\left[\alpha_1(\cdot,w)\right](x_2)\left[\alpha_2(x_2,z)-\alpha_2(x_2,w)\right]d\mu_2(x_2) \right| \\ &\leq K \left\|T\right\|_{L^{p_1}\to L^{p_2'}} \left\|\alpha_1(\cdot,z)-\alpha_1(\cdot,w)\right\|_{p_1} \left\|\alpha_2(\cdot,z)\right\|_{p_2} \\ &\qquad + KC_1 \left\|T\right\|_{L^{p_1}\to L^{p_2'}} \left\|\mathbf{1}_{A_1}\right\|_{p_1} \left\|\alpha_1(\cdot,z)-\alpha_2(\cdot,z)\right\|_{p_2}.\end{aligned}$$

As $w \to z$ in $\mathbb{S}$, both terms in this final sum tend to zero by (2.18), proving the (uniform) continuity of F in $\mathbb{S}$.

Finally, we argue that F is bounded in $\mathbb{S}$. Having laid the ground work above, this is quite straight forward:

$$\begin{aligned}
|F(z)| &\leq K \left\| T\left[\alpha_1(\cdot, z)\right](\cdot) \right\|_{p_2'} \left\| \alpha_2(\cdot, z) \right\|_{p_2} \\
&\leq K \left\| T \right\|_{L^{p_1} \to L^{p_2'}} \left\| \alpha_1(\cdot, z) \right\|_{p_1} \left\| \alpha_2(\cdot, z) \right\|_{p_2} \\
&\leq K C_1 C_2 \left\| T \right\|_{L^{p_1} \to L^{p_2'}} \left\| \mathbf{1}_{A_1} \right\|_{p_1} \left\| \mathbf{1}_{A_2} \right\|_{p_2} \\
&< \infty.
\end{aligned}$$

■

Chapter 3

The Boundedness of a Certain Square Function on $L^p(\mathbb{R}^n)$

3.1 Motivation

A fundamental strategy that surfaces frequently in mathematics is the ability to ascertain properties of functions by investigating their image under certain operators. While this theme is present in virtually all branches of mathematics in one guise or another, it is at the forefront of Fourier and, more generally, Harmonic analysis. In fact, one customarily begins their study of these subjects with the examination of the Fourier transform (resp. series). In doing so, one is invariably lead to a pantheon of results that relate a given function to its transformed image; e.g. the Riemann-Lebesgue lemma and the Plancherel identity. The success of analyzing functions via their Fourier transform (resp. series) has lead mathematicians to mimic this scheme. This brings us to the subject matter of this final chapter: square functions.

Since their first appearance nearly a century ago, square functions have earned some recognition among prominent mathematicians and have established themselves as an indispensable tool in the realm of function theory. To somewhat oversimplify their construction, the basic idea is as follows: Partition the domain of definition of a given function. On each piece of the partition, transform the function via a "local squaring transformation". Amalgamate these pieces (via integration or summation). And finally, take the square root of the amalgamation.

As alluded to above, the desire is be able to estimate each of the local transformations and hope that the interactions of these local estimates are amenable enough so that, upon piecing them together, our created operator is bounded, or in an ideal situation, has norm comparable to the transformed function. To provide a more concrete example, we briefly recount the groundbreaking work of Littlewood and Paley in the 1930's.

The setup for their celebrated result is as follows: Given a sequence of Fourier coefficients $\{a_n\}_{n=0}^{\infty}$, consider the formally defined function $f(\theta) := \sum_n a_n e^{in\theta}$. Put $\Delta_0 f(\theta) := a_0$ and

$$\Delta_k f(\theta) := \sum_{2^{k-1} \leq n < 2^k} a_n e^{in\theta},$$

for $k \geq 1$. The Δ_k are simply the dyadic truncations of the Fourier series. The associated square function of this dyadic decomposition is

$$Sf(\theta) := \left(\sum_{k=0}^{\infty} |\Delta_k f(\theta)|^2 \right)^{\frac{1}{2}}.$$

Here, the $|\Delta_k f|^2$ is our "local squaring transformation". With this setup, the Littlewood-Paley result is that $\|Sf\|_p \simeq \|f\|_p$, for $1 < p < \infty$, and typifies the ideal scenario for square functions. Thus, if one wishes to estimate the norm of a given function, f, and is allowed some flexibility with these estimates, it suffices to estimate the image, Sf. This can be advantageous precisely because estimating Sf effectively boils down to estimating the localized dyadic truncations. For a more comprehensive history of square functions and survey of some related results, the reader is referred to [11].

3.2 Terminology and Notation

To streamline the exposition to follow, we will set forth some notation to be used for the remainder of the text.

We will be using the familiar calculus evaluation notion $f(x)|_{x=a}^{b}$ which, of course, is simply a less space consuming way of writing $f(b) - f(a)$. Along side this notation, however, we introduce the following:

$$f(x)\{_{x=a}^{b} := f(a) + f(b).$$

For us, the primary advantage of these two evaluation notations is their ability to properly bookkeep certain sums. This will become evident in the proof of Theorem 3.1. Another nice property that these evaluations have is that they behave nicely when dealing with products of functions of distinct variables. More precisely, suppose $f(x_1, ..., x_n) = \prod_{k=1}^{n} f_k(x_k)$. Then

$$f(x_1, ..., x_n)|_{x_j=a_j}^{b_j} = [f(b_k) - f(a_k)] \prod_{k \neq j} f_k(x_k)$$

and

$$f(x_1, ..., x_n)\{_{x_j=a_j}^{b_j} = [f(a_k) + f(b_k)] \prod_{k \neq j} f_k(x_k).$$

These observations justify some computations that arise in the following section.

Moving on, let $\mathbb{O}^n$ denote the n-dimensional first "octant". That is,

$$\mathbb{O}^n := (\mathbb{R}_+)^n = \{s = (s_1, ..., s_n) \in \mathbb{R}^n : s_k > 0, k = 1, ..., n\}.$$

$\mathbb{O}^n$ has a natural group structure being the n-fold product of the Abelian group $\mathbb{R}_+$. More precisely, the multiplication in $\mathbb{O}^n$ that we will be using is $st := (s_1t_1, ..., s_nt_n)$ and inverses will be given by $s^{-1} = 1/s := (1/s_1, ..., 1/s_n)$. With this group structure, we let $\mathbb{O}^n$ act on $\mathbb{R}^n$ in an obvious way: i.e. $sx := (s_1x_1, ..., s_nx_n)$. Finally, we will be using a group homomorphic functional $\pi : \mathbb{O}^n \to \mathbb{R}_+$, which is given by $\pi(s) := 1/(s_1 \cdots s_n)$. Note that π is a product of n functions each of which is just a function of an individual s_k, $k = 1, ..., n$. Thus, it behaves nicely respect with the evaluations at the outset of the section in the way described earlier.

With this structure and notation, we now let $\mathbb{O}^n$ act on (complex vector-valued) functions, f, defined on $\mathbb{R}^n$ as follows:

$$f_s(x) := \pi(s) f\left(\frac{x}{s}\right) = \frac{1}{s_1 \cdots s_n} f\left(\frac{x_1}{s_1}, ..., \frac{x_n}{s_n}\right).$$

We call f_s the s-dilation of f. Note that we will often be regarding $f_s(x)$ as a function defined on $\mathbb{R}^n \times \mathbb{R}^n$ in the natural way: $(x, s) \mapsto f_s(x)$.

All of this notation was set so that we could define, and later manipulate, a type of square function succinctly: Given $K : \mathbb{R}^n \to \mathbb{R}$, we define the square function associated to K to be the operator

$$S_K f(x) := \sqrt{\int_{\mathbb{O}^n} |(K_s * f)(x)|^2 \, \pi(s) ds.}$$

Here we are letting K and f be any measurable functions for which the right hand side makes sense. This will work, if, for example, K is smooth, $f \geq 0$ and both have compact support.

The function K will be called the kernel of our square function. As it turns out, we will not be concerned with the square function associated to K. Rather, we will focus on the square function associated to a certain transformation of K. This transformation is a differential operator, being a composition of multiple differential operators, all of the same form. The individual operators will be given by

$$D_j K(x) := x_j \partial_j K(x) + K(x).$$

The composite operator is then

$$DK(x) := D_n \cdots D_1 K(x).$$

Although it is not truly essential to the discussion, it is worth noting that, with a some computation and induction, one has the nice formula

$$DK(x) = (-1)^n \sum_{j=0}^{n} \sum_{\substack{\alpha \in \{0,1\}^n \\ |\alpha| = j}} x^\alpha \partial_\alpha K(x).$$

The construction of this operator is motivated by the fact that it behaves nicely with respect to the dilation involved in the formulation of our square functions. To see this, assume that $K \in C^\infty$ and observe that

$$
\begin{aligned}
\frac{\partial}{\partial s_j}\left[K_s(x)\right] &= \frac{\partial}{\partial s_j}\left[\pi(s) K\left(\frac{x}{s}\right)\right] \\
&= \frac{1}{s_1 \cdots s_{j-1} s_{j+1} \cdots s_n} \frac{\partial}{\partial s_j}\left[\frac{1}{s_j} K\left(\frac{x_1}{s_1}, \ldots, \frac{x_n}{s_n}\right)\right] \\
&= \frac{1}{s_1 \cdots s_{j-1} s_{j+1} \cdots s_n}\left(\frac{1}{s_j}\left[\partial_j K\left(\frac{x_1}{s_1}, \ldots, \frac{x_n}{s_n}\right)\right]\left(-\frac{x_j}{s_j^2}\right)\right. \\
&\qquad \left. + \left[K\left(\frac{x_1}{s_1}, \ldots, \frac{x_n}{s_n}\right)\right]\left(-\frac{1}{s_j^2}\right)\right) \\
&= -\pi(s)\left[\frac{x_j}{s_j} \cdot \partial_j K\left(\frac{x}{s}\right) + K\left(\frac{x}{s}\right)\right] \frac{1}{s_j} \cdot \\
&= -(D_j K)_s(x) \frac{1}{s_j}.
\end{aligned}
$$

Our final expression is also smooth, since K is. Thus, we may iterate this computation, starting with $j = 1$ and proceeding incrementally one step at a time. More precisely, at the second iteration, we replace $K_s(x)$ with $-(D_1 K)_s(x)\frac{1}{s_1}$ to get

$$
\begin{aligned}
\frac{\partial^2}{\partial s_2 \partial s_1}\left[K_s(x)\right] &= \frac{\partial}{\partial s_2}\left[-(D_1 K)_s(x) \frac{1}{s_1}\right] \\
&= (-1) \frac{\partial}{\partial s_2}\left[(D_1 K)_s(x)\right] \frac{1}{s_1} \\
&= (-1)^2 (D_2 D_1 K)_s(x) \frac{1}{s_2 s_1}.
\end{aligned}
$$

Continuing on in the fashion, we ultimately obtain the identity

$$
\frac{\partial^n}{\partial s_n \cdots \partial s_1}\left[K_s(x)\right] = (-1)^n \left(DK\right)_s(x) \pi(s). \tag{3.1}
$$

This identity paired with some well executed integration techniques will be the cornerstones to proving the unboundedness of S_{DK} on variable L^p, for a nontrivial class of functions K.

3.3 A Counter Example

As mentioned, we will eventually see that for a large class of functions, K, the transformed kernel, DK, will not admit a bounded square function from variable $L^p(\mathbb{R}^n)$ into itself, where $n \geq 1$. The nuance of this result is twofold in the sense that if p is constant or if $n = 1$, then there are examples for which our constructed transformation will be bounded from $L^p(\mathbb{R}^n)$ into itself. Our null result ultimately hinges on the following:

Theorem 3.1. *Suppose $K \in C^\infty(\mathbb{R}^n)$ is real-valued, does not vanish at the origin and is such that $DK \in L^1(\mathbb{R}^n)$. Then there exists a constant $C = C(K, n) > 0$ such that*

$$\frac{C}{|R|}\int_R f(y)dy \leq S_{DK}f(x), \tag{3.2}$$

for every rectangle R, every $x \in R$ and $f \geq 0$ supported in R; or equivalently,

$$\left[\frac{C}{|R|}\int_R f(y)dy\right]\mathbf{1}_R \leq S_{DK}f, \tag{3.2$'$}$$

for every $f \geq 0$ supported in R.

Proof. Fix a rectangle $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$ in $\mathbb{R}^n$. Since $K(0) \neq 0$, we may choose $\epsilon > 0$ so small so that $(-1)^n K(0) - \epsilon$ and $(-1)^n K(0) + \epsilon$ are nonzero and have the same sign. By continuity, choose $\delta > 0$ so that $\max_k |z_k| \leq \delta$ implies $|K(z) - K(0)| < 3^{-n}\epsilon$. Put

$l_k = (b_k - a_k)/\delta$ and $l = (l_1, ..., l_n)$. Then, for $x, y \in R$ observe that

$$\begin{aligned}
&\left| [K_s(x-y) - K(0)\pi(s)] \Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n} \right| \qquad (3.3)\\
&= \left| \left[K(s^{-1}(x-y)) - K(0) \right] \pi(s) \Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n} \right| \\
&\le \left| K(s^{-1}(x-y)) - K(0) \right| \pi(s) \Big\{_{s_1=l_1}^{2l_1} \cdots \Big\{_{s_n=l_n}^{2l_n} \\
&= \sum_{t \in \{1,2\}^n} \left| K((tl)^{-1}(x-y)) - K(0) \right| \pi(tl) \\
&< 3^{-n} \epsilon \sum_{t \in \{1,2\}^n} \pi(tl) \\
&= 3^{-n} \epsilon \pi(s) \Big\{_{s_1=l_1}^{2l_1} \cdots \Big\{_{s_n=l_n}^{2l_n} \\
&= 3^{-n} \epsilon \prod_{k=1}^{n} \frac{3}{2l_k} \\
&= \epsilon \frac{\delta^n}{2^n} \frac{1}{|R|}.
\end{aligned}$$

Also note that, for $x, y \in R$, we have

$$\begin{aligned}
K_s(x-y)\Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n} &= (K(0)\pi(s) + [K_s(x-y) - K(0)\pi(s)]) \Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n} \qquad (3.4)\\
&= K(0) \left[\pi(s)\Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n} \right] + [K_s(x-y) - K(0)\pi(s)] \Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n} \\
&= K(0) \left[\prod_{k=1}^{n} \left(s_k^{-1} \Big|_{s_k=l_k}^{2l_k} \right) \right] + [K_s(x-y) - K(0)\pi(s)] \Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n} \\
&= K(0) \prod_{k=1}^{n} \left[-(2l_k)^{-1} \right] + [K_s(x-y) - K(0)\pi(s)] \Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n}. \\
&= (-1)^n K(0) \frac{\delta^n}{2^n} \frac{1}{|R|} + [K_s(x-y) - K(0)\pi(s)] \Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n}.
\end{aligned}$$

Combining (3.3) and (3.4), we get

$$[(-1)^n K(0) - \epsilon] \frac{\delta^n}{2^n} \frac{1}{|R|} \le K_s(x-y)\Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n} \le [(-1)^n K(0) + \epsilon] \frac{\delta^n}{2^n} \frac{1}{|R|}. \qquad (3.5)$$

Consequently, either

$$\frac{C}{|R|} \leq -K_s(x-y)\Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n}, \qquad (x, y \in R) \tag{3.6}$$

or

$$\frac{C}{|R|} \leq K_s(x-y)\Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n}, \qquad (x, y \in R) \tag{3.7}$$

where

$$C := \frac{\delta^n}{2^n} \max\left\{-\left[(-1)^n K(0) + \epsilon\right], (-1)^n K(0) - \epsilon\right\}.$$

Note that $C > 0$, by the choice of ϵ, and depends only on K and n. In particular, C does not depend on f or R.

Without loss of generality, we may assume that we are in the case of (3.6). Then, for $x \in R$, $\mathrm{supp} f \subset R$ and $f \geq 0$, we have

$$\begin{aligned}
\frac{C}{|R|} \int_R f(y)dy &\leq \int_R - \left[K_s(x-y)\Big|_{s_1=l_1}^{2l_1} \cdots \Big|_{s_n=l_n}^{2l_n}\right] f(y)dy \\
&= \int_{\mathbb{R}^n} - \left(\int_{l_1}^{2l_1} \cdots \int_{l_n}^{2l_n} \frac{\partial^n}{\partial s_1 \cdots \partial s_n} \left[K_s(x-y)\right] ds_n \cdots ds_1\right) f(y)dy \\
&= \int_{\mathbb{R}^n} - \left(\int_{l_1}^{2l_1} \cdots \int_{l_n}^{2l_n} (-1)^n (DK)_s(x-y)\pi(s) ds_n \cdots ds_1\right) f(y)dy \\
&= \int_{l_1}^{2l_1} \cdots \int_{l_n}^{2l_n} (-1)^{n+1} (DK)_s * f(x)\pi(s) ds_n \cdots ds_1 \\
&= \int_{l_1}^{2l_1} \cdots \int_{l_n}^{2l_n} \sqrt{\pi(s)} \left[(-1)^{n+1}(DK)_s * f(x)\sqrt{\pi(s)}\right] ds_n \cdots ds_1 \\
&\leq \left(\int_{l_1}^{2l_1} \cdots \int_{l_n}^{2l_n} \pi(s) ds_n \cdots ds_1\right)^{1/2} S_{DK} f(x) \\
&\leq (\log 2)^{\frac{n}{2}} S_{DK} f(x).
\end{aligned}$$

Dividing through by $(\log 2)^{\frac{n}{2}}$, gives (3.2). Note that the resulting constant still only depends on K and n. ■

Corollary 3.2. *If S_{DK} is bounded on $L^p(\mathbb{R}^n)$, then there exists a constant $C = C(K,n)$ such that*

$$\frac{1}{|R|}\left\|\mathbf{1}_R\right\|_p \left\|\mathbf{1}_R\right\|_{p'} \leq C, \tag{3.8}$$

for any rectangle $R \subset \mathbb{R}^n$.

Proof. Fix a rectangle R. Let $f \geq 0$ be supported in R with $\|f\|_p \leq 1$. By the homogeneity and monotonicity of the norm, Theorem 3.1, and the assumed boundedness of S_{DK} on L^p, we have

$$\begin{aligned} \frac{1}{|R|}\left[\int_{\mathbb{R}^n} f(y)\mathbf{1}_R(y)dy\right]\left\|\mathbf{1}_R\right\|_p &= \left\|\left[\frac{1}{|R|}\int_R f(y)dy\right]\mathbf{1}_R\right\|_p \\ &\leq \left\|CS_{DK}f\right\|_p \\ &\leq C\left\|f\right\|_p \\ &\leq C. \end{aligned} \tag{3.9}$$

By norm conjugacy (for positive functions), the supremum of $\int_{\mathbb{R}^n} f(y)\mathbf{1}_R(y)dy$ over all such f is, up to a factor of five, $\left\|\mathbf{1}_R\right\|_{p'}$. Thus, taking the supremum in (3.9) over all such f yields (3.8). ■

Corollary 3.3. *Suppose p is a bounded exponent and $n \geq 2$. If S_{DK} is bounded on $L^p(\mathbb{R}^n)$, then p must be a constant exponent.*

Proof. By Corollary 3.2, the estimate (3.8) holds for every rectangle in $\mathbb{R}^n$. Since $n \geq 2$, it must be that p is constant. See the discussion on pages 2009 to 2010 of [5] for more details. ■

In view of these results, we have shown that there is no hope for our constructed square functions to behave nicely with respect to variable Lebesgue spaces of higher (than linear) dimensional Euclidean space (equipped with Lebesgue measure). This is the case no matter how nice our underlying kernel is, as long as it still satisfies the hypothesis of Theorem 3.1. Needless to say, this would be of little interest if our square functions always behaved poorly; i.e. if they behaved poorly even in the constant exponent case. As it turns out, with a little more structure on the underlying kernel, we can ensure that the square function associated to the transformed kernel is bounded on L^p, provided that the exponent is constant. Let us illustrate.

For this illustration, we will assume K is a smooth, product kernel supported in $[-1,1]^n$ and does not vanish at the origin. More precisely, we assume that

$$K(x_1, ..., x_n) = k_1(x_1) \cdots k_n(x_n),$$

where each $k_j \in C_c^\infty(\mathbb{R}, \mathbb{R})$ is supported in $(-1,1)$ and $k_j(0) \neq 0$. Since K is a product kernel, it readily follows that

$$DK(x) = D_n \cdots D_1 \left[k_1(x_1) \cdots k_n(x_n)\right] = \prod_{j=1}^{n} \left[x_j k_j'(x_j) + k_j(x_j)\right].$$

From this, we see that the transformed kernel, DK, is also a smooth, product kernel supported in $[-1,1]^n$. Put $\psi_j(x_j) := x_j k_j'(x_j) + k_j(x_j)$ and observe that $\psi_j(x_j) = \frac{d}{dx_j}\left[x_j k_j(x_j)\right]$. From this, we may deduce both that ψ_j is not identically zero, for otherwise, k_j would not be smooth at the origin. We may also deduce from $\psi_j(x_j) = \frac{d}{dx_j}[x_j k_j(x_j)]$ that each ψ_j has mean value zero. In summary, DK is a non-trivial, smooth, product kernel supported $[-1,1]^n$ such that each factor has mean value zero. By the work of R. Fefferman, et al, (see [4]) it follows that the operator S_{DK} is bounded on $L^p(\mathbb{R}^n)$, for every constant $p \in (1,\infty)$. This, of course, is to be held in contrast with the fact that S_{DK} is not bounded on any non-constant variable Lebesgue space, by the results above.

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