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Dynamical Systems Method (DSM) for solving equations with monotone operators without smoothness assumptions on $F'(u)$

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Abstract

A version of the Dynamical Systems Method (DSM) for solving ill-posed nonlinear equations $F(u) = f$ with monotone operators F in a Hilbert space is studied in this paper under less restrictive assumptions on the nonlinear operators F than the assumptions used earlier. A new method of proof of the basic results is used. An *a posteriori* stopping rule, based on a discrepancy-type principle, is proposed and justified mathematically under weaker assumptions on the nonlinear operator F, than the assumptions used earlier.

Keywords. Dynamical systems method (DSM), nonlinear operator equations, monotone operators, discrepancy principle.

MSC: 65R30; 47J05; 47J06; 47J35.

1 Introduction

In this paper we study a version of the Dynamical Systems Method (DSM) (see [18]) for solving the equation

$$F(u) = f, \tag{1}$$

where F is a nonlinear Fréchet differentiable monotone operator in a real Hilbert space H, and equation (1) is assumed solvable. Monotonicity means that

$$\langle F(u) - F(v), u - v\rangle \geq 0, \quad \forall u, v \in H. \tag{2}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product in H. It is known (see, e.g., [18]), that the set $\mathcal{N} := \{u : F(u) = f\}$ is closed and convex if F is monotone and continuous. A closed and convex set in a Hilbert space has a unique minimal-norm element. This element in $\mathcal{N}$ we denote y, $F(y) = f$. We assumed in earlier works that $F'(u)$ is locally Lipschitz. This assumption is considerably weakened in this work: we assume now only the continuity of $F'(u)$. Since F is monotone, one has $F'(u) \geq 0$, so $||[F'(u) + a(t)I]^{-1}|| \leq \frac{1}{a(t)}$ if $a(t) > 0$.

The local and global existence and uniqueness of the solution to (3) were proved under these weak assumptions in [13]. This proof is not included in the paper. The emphasis in this paper is on the new methods and ideas for proving the basic result of the paper, namely, Theorem 7.

Assume that f is not known but f_δ, the noisy data, are known, and $\|f_\delta - f\| \le \delta$. If $F'(u)$ is not boundedly invertible, then solving for u, given noisy data f_δ, is often (but not always) an ill-posed problem. When F is a linear bounded operator many methods for stable solution of (1) were proposed (see [14], [15], [26], [7], [18] and references therein). However, when F is nonlinear then the theory is less complete.

The DSM for solving equation (1) was studied extensively in [18]–[25], [9]-[11], where also numerical examples, illustrating efficiency of the algorithms, based on the DSM methods, were given. In [18] the following version of the DSM for solving equation (1) was studied:

$$\dot{u}_\delta = -\big(F'(u_\delta) + a(t)I\big)^{-1}\big(F(u_\delta) + a(t)u_\delta - f_\delta\big), \quad u_\delta(0) = u_0. \tag{3}$$

Here F *is a monotone operator, and* $a(t) > 0$ *is a continuous function, defined for all* $t \ge 0$*, strictly monotonically decaying,* $\lim_{t\to\infty} a(t) = 0$. These assumptions on $a(t)$ hold throughout the paper and are not repeated. Additional assumptions on $a(t)$ will appear in Theorem 7. Convergence of the above DSM was proved in [18] for any initial value u_0 with an *a priori* choice of stopping time t_δ, provided that $a(t)$ is suitably chosen. In this paper an a posteriori choice of t_δ is formulated and justified rigorously.

The theory of monotone operators is presented in many books, e.g., in [3], [17], [28]. Many of the results of the theory of monotone operators, used in this paper, can be found in [18]. In [16] methods for solving well-posed nonlinear equations in a finite-dimensional space are discussed.

Methods for solving equation (1) with monotone operators are quite important in many applications. It is proved in [18] that solving any solvable linear operator equation $Au = f$ with a closed densely defined linear operator A can be reduced to solving equation (1) with a monotone operator. Equations (1) with monotone operators arise often when the physical system is dissipative. In the earlier papers and in monograph [18] it was assumed that F is locally twice Fréchet differentiable, and a nonlinear differential inequality ([18], p.97) was used in a study of the behavior of the solution to the DSM (3). The smoothness assumptions on F are weakened in this paper, the method of our proofs is new, and, as a result, the proofs are shorter and simpler than the earlier ones. The assumptions on the "regularizing function" $a(t)$ are also weakened.

In this paper we propose and justify a stopping rule for solving ill-posed equation (1) based on a discrepancy principle (DP) for the DSM (3). The main result of this paper is Theorem 7 in which a DP is formulated, the existence of the stopping time t_δ is proved, and the convergence of the DSM (3) with the proposed DP is justified under some natural assumptions for a wide class of nonlinear equations with monotone operators.

Our result is novel because the convergence of the DSM is justified under less restrictive assumptions on F than in [18], [12], where twice Fréchet differentiability was assumed and the DP was not established for problem (3). Moreover, the rate of decay of the function $a(t)$ as $t \to \infty$ can be arbitrary in the power scale, while in [18] $a(t)$ was often assumed to satisfy the condition $\int_0^\infty a(t)dt = \infty$ which implies the decay in the power scale not faster than $O(\frac{1}{t})$ as $t \to \infty$.

These new theoretical results are useful practically. The auxiliary results in our paper are borrowed from [8] and their proofs are omitted.

A few remarks about the history of the method (3) may be useful for the reader. Probably the first paper in which a continuous analog of the Newton's method was proposed for solving well-posed operator equation (1) was the paper [4]. Method (3) has been studied in the literature earlier by several authors, (see [1], [18], and references therein), usually under the assumption that $F'(u)$ satisfies a Lipschitz condition. Iterative versions of the method (3) were also studied (see, e.g., in [1], [6], [18]), and in some of the cited papers by the authors, also under some smoothness assumptions on $F'(u)$. In [5] iterative methods of Gauss-Newton type are studied under the assumption that $F'(u)$ satisfies a Lipschitz condition. The discrepancy principle for linear ill-posed problems was proposed by V.Morozov (see, e.g., [15]). We mention paper [27] and book [2].

To the authors' knowledge it is for the first time a justification of the convergence of the method (3) is proved in this paper under the minimal assumption of the continuity of $F'(u)$. The method of the proof is novel and can be used can be used in a study of other problems. The justification of the discrepancy principle for stable solution of (1) with noisy data by the method (3) is also given under the minimal assumption of the continuity of $F'(u)$.

2 Auxiliary results

Let us consider the following equation

$$F(V_{\delta,a}) + aV_{\delta,a} - f_\delta = 0, \qquad a > 0, \tag{4}$$

where $a = const$. It is known (see, e.g., [18]) that equation (4) with monotone continuous operator F has a unique solution for any $f_\delta \in H$.

Let us recall the following result from [18, p.112]:

Lemma 1 *Assume that equation* (1) *is solvable, y is its minimal-norm solution, and F is monotone and continuous. Then*

$$\lim_{a\to 0} \|V_{0,a} - y\| = 0,$$

where $V_{0,a}$ solves (4) *with $\delta = 0$.*

Lemma 2 (Lemma 3, [8]) *If* (2) *holds and F is continuous, then $\|V_{\delta,a}\| = O(\frac{1}{a})$ as $a \to \infty$, and*

$$\lim_{a\to\infty} \|F(V_{\delta,a}) - f_\delta\| = \|F(0) - f_\delta\|. \tag{5}$$

Let $a = a(t)$, $0 < a(t) \searrow 0$, and assume $a \in C^1[0,\infty)$. Then the solution $V_\delta(t) := V_{\delta,a(t)}$ of (4) is a function of t. From the triangle inequality one gets

$$\|F(V_\delta(0)) - f_\delta\| \geq \|F(0) - f_\delta\| - \|F(V_\delta(0)) - F(0)\|.$$

From Lemma 2 it follows that for large $a(0)$ one has

$$\|F(V_\delta(0)) - F(0)\| \leq M_1\|V_\delta(0)\| = O\left(\frac{1}{a(0)}\right), \quad M_1 = \max_{||u||\leq||V_\delta(0)||} ||F'(u)||.$$

Therefore, if $\|F(0) - f_\delta\| > C\delta$, then $\|F(V_\delta(0)) - f_\delta\| \geq (C-\epsilon)\delta$, where $\epsilon > 0$ is arbitrarily small, for sufficiently large $a(0) > 0$.

Below the words decreasing and increasing mean strictly decreasing and strictly increasing.

Lemma 3 (Lemma 2, [8]) *Assume* $\|F(0) - f_\delta\| > 0$. *Let* $0 < a(t) \searrow 0$, *and* F *be monotone. Denote*

$$\phi(t) := \|F(V_\delta(t)) - f_\delta\|, \quad \psi(t) := \|V_\delta(t)\|,$$

where $V_\delta(t)$ *solves* (4) *with* $a = a(t)$. *Then* $\phi(t)$ *is decreasing, and* $\psi(t)$ *is increasing.*

Lemma 4 (cf. Lemma 4, [8]) *Assume* $0 < a(t) \searrow 0$. *Then the following inequality holds*

$$\lim_{t\to\infty} \|F(V_\delta(t)) - f_\delta\| \leq \delta. \tag{6}$$

Remark 5 Let $V := V_\delta(t)|_{\delta=0}$, so $F(V) + a(t)V - f = 0$. Let y be the minimal-norm solution to the equation $F(u) = f$. We claim that

$$\|V_\delta - V\| \leq \frac{\delta}{a}. \tag{7}$$

Indeed, from (4) one gets

$$F(V_\delta) - F(V) + a(V_\delta - V) = f_\delta - f.$$

Multiply this equality by $V_\delta - V$ and use (2) to obtain

$$\begin{aligned}\delta\|V_\delta - V\| &\geq \langle f_\delta - f, V_\delta - V\rangle \\ &= \langle F(V_\delta) - F(V) + a(V_\delta - V), V_\delta - V\rangle \\ &\geq a\|V_\delta - V\|^2.\end{aligned}$$

This implies (7).

Similarly, from the equation

$$F(V) + aV - F(y) = 0$$

one can derive that

$$\|V\| \leq \|y\|. \tag{8}$$

From (7) and (8), one gets the following estimate:

$$\|V_\delta\| \leq \|V\| + \frac{\delta}{a} \leq \|y\| + \frac{\delta}{a}. \tag{9}$$

Lemma 6 *Let $a(t)$ satisfy* (16)*. Then one has*

$$e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot a(s)|\|V_\delta(s)\|ds \le \frac{1}{2}a(t)\|V_\delta(t)\|, \qquad t\ge 0. \tag{10}$$

Proof. Let us check that

$$e^{\frac{t}{2}}|\dot a(t)| \le \frac{d}{dt}\left(\frac{1}{2}a(t)e^{\frac{t}{2}}\right), \qquad t>0. \tag{11}$$

One has

$$\frac{d}{dt}\left(\frac{1}{2}a(t)e^{\frac{t}{2}}\right) = \frac{a(t)e^{\frac{t}{2}}}{4} + \frac{\dot a(t)e^{\frac{t}{2}}}{2} = \frac{a(t)e^{\frac{t}{2}}}{4} - \frac{|\dot a(t)|e^{\frac{t}{2}}}{2}. \tag{12}$$

Thus, inequality (11) is equivalent to

$$\frac{3}{2}|\dot a(t)| \le \frac{1}{4}a(t), \qquad \forall t>0. \tag{13}$$

Inequality (13) holds because by our assumptions the function $a(t)$ satisfies (16). Integrating both sides of (11) from 0 to t, one gets

$$\int_0^t e^{\frac{s}{2}}|\dot a(s)|ds \le \frac{1}{2}a(t)e^{\frac{t}{2}} - \frac{1}{2}a(0)e^0 < \frac{1}{2}a(t)e^{\frac{t}{2}}, \qquad t\ge 0. \tag{14}$$

Multiplying (14) by $e^{-\frac{t}{2}}\|V_\delta(t)\|$, and using the fact that $\|V_\delta(t)\|$ is increasing (see Lemma 3), one gets (10). Lemma 6 is proved. □

3 Main result

Denote

$$A := F'(u_\delta(t)), \quad A_a := A + aI,$$

where I is the identity operator, and $u_\delta(t)$ solves the following Cauchy problem:

$$\dot u_\delta = -A^{-1}_{a(t)}[F(u_\delta)+a(t)u_\delta - f_\delta], \quad u_\delta(0)=u_0, \tag{15}$$

where $u_0\in H$. Assume

$$0<a(t)\searrow 0, \quad \frac{1}{6}\ge \frac{|\dot a(t)|}{a(t)}\searrow 0, \qquad t\ge 0. \tag{16}$$

Assume that equation (1) has a solution, possibly nonunique, and y is the minimal-norm solution to this equation. Let f be unknown but f_δ be given, $\|f_\delta - f\|\le\delta$.

Theorem 7 *Let $a(t)$ satisfy* (16)*. Let $C>0$ and $\zeta\in(0,1]$ be constants such that $C\delta^\zeta>\delta$. Assume that $F: H\to H$ is a Fréchet differentiable monotone operator, and u_0 is an element of H, satisfying the following inequalities*

$$\|F(u_0)+a(0)u_0 - f_\delta\| \le \frac{1}{4}a(0)\|V_\delta(0)\|, \qquad \|F(u_0)-f_\delta\| > C\delta^\zeta, \tag{17}$$

where $V_\delta(t) := V_{\delta,a(t)}$ *solves* (4) *with* $a = a(t)$*. Then the solution* $u_\delta(t)$ *to problem* (15) *exists on an interval* $[0, T_\delta]$*,* $\lim_{\delta\to 0} T_\delta = \infty$*, and there exists a unique* t_δ*,* $t_\delta \in (0, T_\delta)$*, such that* $\lim_{\delta\to 0} t_\delta = \infty$ *and*

$$\|F(u_\delta(t_\delta)) - f_\delta\| = C\delta^\zeta, \quad \|F(u_\delta(t)) - f_\delta\| > C\delta^\zeta, \quad \forall t \in [0, t_\delta). \tag{18}$$

If $\zeta \in (0,1)$ *and* t_δ *satisfies* (18)*, then*

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0. \tag{19}$$

Remark 8 In Theorem 7 the existence of t_δ satisfying (18) is guaranteed for any $\zeta \in (0,1]$. However, we prove relation (19) for $\zeta \in (0,1)$. If $\zeta = 1$ it is possible to prove that $u_\delta(t_\delta)$ converges to a solution to (1), but it is not known whether this solution is the minimal-norm solution of (1) if (1) has more than one solution.

Further results on the choices of ζ require extra assumptions on F and y. Since the minimal-norm solution y satisfies the relation $\|F(y) - f_\delta\| = \|f - f_\delta\| \le \delta$, it is natural to choose $C > 0$ and $\zeta \in (0,1)$ so that $C\delta^\zeta$ be close to δ.

One can choose u_0 satisfying the first inequality in (17). Indeed, if u_0 approximates $V_\delta(0)$, the solution to equation (4), with a small error, then the first inequality in (17) is satisfied. The first inequality in (17) is a sufficient condition for (40), i.e.,

$$e^{-\frac{t}{2}}\|F(u_0) + a(0)u_0 - f_\delta\| \le \frac{1}{4}a(t)\|V_\delta(t)\|, \quad t \ge 0, \tag{20}$$

to hold. In our proof inequality (20) is used at $t = t_\delta$. The stopping time t_δ is often sufficiently large for the quantity $e^{\frac{t_\delta}{2}}a(t_\delta)$ to be large. This follows from the fact that $\lim_{t\to\infty} e^{\frac{t}{2}}a(t) = \infty$ (see (44) below). In this case inequality (20) with $t = t_\delta$ is satisfied for a wide range of u_0.

The second inequality in (17) is a natural assumption because if this inequality does not hold and $\|u_0\|$ is not "too large", then u_0 can be considered as an approximate solution to (1).

Proof. [Proof of Theorem 7] The uniqueness of t_δ follows from (18). Indeed, if t_δ and $\tau_\delta > t_\delta$ both satisfy (18), then the second inequality in (18) does not hold on the interval $[0, \tau_\delta)$.

Let us prove the existence of t_δ. From (15), one obtains:

$$\frac{d}{dt}\big(F(u_\delta) + au_\delta - f_\delta\big) = A_a\dot{u}_\delta + \dot{a}u_\delta = -\big(F(u_\delta) + au_\delta - f_\delta\big) + \dot{a}u_\delta.$$

This and (4) imply:

$$\frac{d}{dt}\big[F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta)\big] = -\big[F(u_\delta) - F(V_\delta) + a(u_\delta - V_\delta)\big] + \dot{a}u_\delta. \tag{21}$$

Denote

$$v := v(t) := F(u_\delta(t)) - F(V_\delta(t)) + a(t)(u_\delta(t) - V_\delta(t)), \qquad h := h(t) := \|v(t)\|.$$

Multiply (21) by v and get

$$h\dot{h} = -h^2 + \langle v, \dot{a}(u_\delta - V_\delta)\rangle + \dot{a}\langle v, V_\delta\rangle \leq -h^2 + h|\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|h\|V_\delta\|. \tag{22}$$

This implies

$$\dot{h} \leq -h + |\dot{a}|\|u_\delta - V_\delta\| + |\dot{a}|\|V_\delta\|. \tag{23}$$

Since $\langle F(u_\delta) - F(V_\delta), u_\delta - V_\delta\rangle \geq 0$, one obtains from two equations

$$\langle v, u_\delta - V_\delta\rangle = \langle F(u_\delta) - F(V_\delta) + a(t)(u_\delta - V_\delta), u_\delta - V_\delta\rangle,$$

and

$$\langle v, F(u_\delta) - F(V_\delta)\rangle = \|F(u_\delta) - F(V_\delta)\|^2 + a(t)\langle u_\delta - V_\delta, F(u_\delta) - F(V_\delta)\rangle,$$

the following two inequalities:

$$a\|u_\delta - V_\delta\|^2 \leq \langle v, u_\delta - V_\delta\rangle \leq \|u_\delta - V_\delta\|h, \tag{24}$$

and

$$\|F(u_\delta) - F(V_\delta)\|^2 \leq \langle v, F(u_\delta) - F(V_\delta)\rangle \leq h\|F(u_\delta) - F(V_\delta)\|. \tag{25}$$

Inequalities (24) and (25) imply:

$$a\|u_\delta - V_\delta\| \leq h, \quad \|F(u_\delta) - F(V_\delta)\| \leq h. \tag{26}$$

Inequalities (23) and (26) imply

$$\dot{h} \leq -h\Big(1 - \frac{|\dot{a}|}{a}\Big) + |\dot{a}|\|V_\delta\|. \tag{27}$$

By the assumption, $1 - \frac{|\dot{a}|}{a} \geq \frac{1}{2}$, so inequality (27) implies

$$\dot{h} \leq -\frac{1}{2}h + |\dot{a}|\|V_\delta\|. \tag{28}$$

Inequality (28) implies

$$h(t) \leq h(0)e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot{a}(s)|\|V_\delta(s)\|ds. \tag{29}$$

From (29) and (26), one gets

$$\|F(u_\delta(t)) - F(V_\delta(t))\| \leq h(0)e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot{a}|\|V_\delta\|ds. \tag{30}$$

Hence, using the triangle inequality and (30), one gets:

$$\|F(u_\delta(t)) - f_\delta\| \leq \|F(V_\delta(t)) - f_\delta\| + h(0)e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot{a}|\|V_\delta\|ds. \tag{31}$$

Since $a(s)\|V_\delta(s)\| = \|F(V_\delta(s)) - f_\delta\|$ is decreasing, by Lemma 3, one obtains

$$\lim_{t\to\infty} e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} |\dot{a}| \|V_\delta\| ds \le \lim_{t\to\infty} e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} \frac{|\dot{a}(s)|}{a(s)} a(0)\|V_\delta(0)\| ds. \tag{32}$$

The assumption $\lim_{t\to\infty} \frac{|\dot{a}(t)|}{a(t)} = 0$ implies

$$\lim_{t\to\infty} \frac{\int_0^t e^{\frac{s}{2}} \frac{|\dot{a}(s)|}{a(s)} a(0)\|V_\delta(0)\| ds}{e^{\frac{t}{2}}} = 0. \tag{33}$$

Indeed, if $I := \int_0^\infty e^{\frac{s}{2}} \frac{|\dot{a}(s)|}{a(s)} a(0)\|V_\delta(0)\| ds < \infty$ then (33) is obvious. If $I = \infty$, then (33) follows from L'Hospital's rule.

It follows from (31)–(33) and Lemma 4 that

$$\lim_{t\to\infty} \|F(u_\delta(t)) - f_\delta\| \le \lim_{t\to\infty} \|F(V_\delta(t)) - f_\delta\| + \lim_{t\to\infty} e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} |\dot{a}| \|V_\delta\| ds \le \delta. \tag{34}$$

The assumption $\|F(u_0) - f_\delta\| > C\delta^\zeta > \delta$ and inequality (34) imply the existence of a $t_\delta > 0$ such that (18) holds because $\|F(u_\delta(t)) - f_\delta\|$ is a continuous function of t.

We claim that

$$\lim_{\delta\to 0} t_\delta = \infty. \tag{35}$$

Let us prove (35). From the triangle inequality and (30) one gets

$$\begin{aligned} \|F(u_\delta(t)) - f_\delta\| &\ge \|F(V_\delta(t)) - f_\delta\| - \|F(V_\delta(t)) - F(u_\delta(t))\| \\ &\ge a(t)\|V_\delta(t)\| - h(0)e^{-\frac{t}{2}} - e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} |\dot{a}| \|V_\delta\| ds. \end{aligned} \tag{36}$$

Recall that $a(t)$ satisfies (16) by our assumptions. From (16) and Lemma 6 one obtains

$$\frac{1}{2} a(t)\|V_\delta(t)\| \ge e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} |\dot{a}| \|V_\delta(s)\| ds. \tag{37}$$

From (17) we have

$$h(0)e^{-\frac{t}{2}} \le \frac{1}{4} a(0)\|V_\delta(0)\| e^{-\frac{t}{2}}, \qquad t \ge 0. \tag{38}$$

It follows from (16) that

$$e^{-\frac{t}{2}} a(0) \le a(t). \tag{39}$$

Specifically, inequality (39) is obviously true for $t = 0$, and

$$\left(a(t)e^{\frac{t}{2}}\right)'_t = a(t)e^{\frac{t}{2}} \left(\frac{1}{2} - \frac{|\dot{a}(t)|}{a(t)}\right) > 0,$$

by (16). Therefore, one gets from (39) and (38) the following inequality:

$$e^{-\frac{t}{2}} h(0) \le \frac{1}{4} a(t)\|V_\delta(0)\| \le \frac{1}{4} a(t)\|V_\delta(t)\|, \quad t \ge 0, \tag{40}$$

Here, we have used the inequality $\|V_\delta(t')\| \le \|V_\delta(t)\|$ for $t' < t$, established in Lemma 3 in Section 2. From (18) and (36)–(40), one gets

$$C\delta^\zeta = \|F(u_\delta(t_\delta)) - f_\delta\| \ge \frac{1}{4}a(t_\delta)\|V_\delta(t_\delta)\|. \tag{41}$$

From (7) and the triangle inequality one derives

$$a(t)\|V(t)\| \le a(t)\|V(t) - V_\delta(t)\| + a(t)\|V_\delta(t)\| \le \delta + a(t)\|V_\delta(t)\|, \qquad \forall t \ge 0. \tag{42}$$

It follows from (41) and (42) that

$$0 \le \lim_{\delta\to 0} a(t_\delta)\|V(t_\delta)\| \le \lim_{\delta\to 0}\big(\delta + 4C\delta^\zeta\big) = 0. \tag{43}$$

Since $\|V(t)\|$ increases (see Lemma 3), the above formula implies $\lim_{\delta\to 0} a(t_\delta) = 0$. Since $0 < a(t) \searrow 0$, it follows that $\lim_{\delta\to 0} t_\delta = \infty$, i.e., (35) holds.

Let us prove that

$$\lim_{t\to\infty} e^{\frac{t}{2}} a(t) = \infty. \tag{44}$$

We claim that, for sufficiently large $t > 0$, the following inequality holds

$$\frac{t}{2} > \ln\frac{1}{a^2(t)}. \tag{45}$$

By L'Hospital's rule and (16), one obtains

$$\lim_{t\to\infty} \frac{t}{2\ln\frac{1}{a^2(t)}} = \lim_{t\to\infty} \frac{1}{2a^2(t)\frac{-2\dot a(t)}{a^3(t)}} = \lim_{t\to\infty} \frac{a(t)}{4|\dot a(t)|} = \infty. \tag{46}$$

This implies that (44) holds for $t > 0$ sufficiently large. From (45) one concludes

$$\lim_{t\to\infty} e^{\frac{t}{2}} a(t) \ge \lim_{t\to\infty} e^{\ln\frac{1}{a^2(t)}} a(t) = \lim_{t\to\infty} \frac{1}{a(t)} = \infty. \tag{47}$$

Thus, relation (44) is proved.

From (31), (37), (40) and (9) one gets

$$C\delta^\zeta \le a(t_\delta)\|V_\delta(t_\delta)\|\big(1 + \frac{1}{2} + \frac{1}{4}\big) \le \frac{7}{4}\Big(a(t_\delta)\|y\| + \delta\Big). \tag{48}$$

This and the relation $\lim_{\delta\to 0}\frac{\delta}{\delta^\zeta} = 0$, for a fixed $\zeta \in (0,1)$, imply

$$\lim_{\delta\to 0} \frac{\delta^\zeta}{a(t_\delta)} \le \frac{7\|y\|}{4C} < \frac{2\|y\|}{C}. \tag{49}$$

It follows from inequality (29) and the first inequality in (26) that

$$a(t)\|u_\delta(t) - V_\delta(t)\| \le h(0)e^{-\frac{t}{2}} + e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}|\dot a(s)|\|V_\delta(s)\|ds. \tag{50}$$

From (49) and the first inequality in (9) one gets, for sufficiently small δ, the following inequality

$$\|V_\delta(t)\| \le \|y\| + \frac{\delta}{a(t)} < \|y\| + \frac{2\|y\|}{C}, \qquad 0 \le t \le t_\delta. \tag{51}$$

Therefore,

$$\lim_{\delta\to 0} \frac{\int_0^{t_\delta} e^{\frac{s}{2}}|\dot{a}(s)|\|V_\delta(s)\|ds}{e^{\frac{t_\delta}{2}}a(t_\delta)} \le \big(\|y\| + \frac{2\|y\|}{C}\big) \lim_{\delta\to 0} \frac{\int_0^{t_\delta} e^{\frac{s}{2}}|\dot{a}(s)|ds}{e^{\frac{t_\delta}{2}}a(t_\delta)}. \tag{52}$$

Let us show that

$$\lim_{\delta\to 0} \frac{\int_0^{t_\delta} e^{\frac{s}{2}}|\dot{a}(s)|ds}{e^{\frac{t_\delta}{2}}a(t_\delta)} = 0. \tag{53}$$

The denominator of (53) tends to ∞ as $\delta \to 0$ by (44). Thus, if the numerator of (53) is bounded then (53) holds. Otherwise, relation (35) and L'Hospital's rule yield

$$\lim_{\delta\to 0} \frac{\int_0^{t_\delta} e^{\frac{s}{2}}|\dot{a}(s)|ds}{e^{\frac{t_\delta}{2}}a(t_\delta)} = \lim_{t\to\infty} \frac{e^{\frac{t}{2}}|\dot{a}(t)|}{\frac{1}{2}e^{\frac{t}{2}}a(t) - e^{\frac{t}{2}}|\dot{a}(t)|} = 0. \tag{54}$$

It follows from (52) and (53) that

$$\lim_{\delta\to 0} \frac{\int_0^{t_\delta} e^{\frac{s}{2}}|\dot{a}(s)|\|V_\delta(s)\|ds}{e^{\frac{t_\delta}{2}}a(t_\delta)} = 0. \tag{55}$$

From (55), (50), and (35), one gets

$$0 \le \lim_{\delta\to 0} \|u_\delta(t_\delta) - V_\delta(t_\delta)\| = \lim_{\delta\to 0} \frac{h(t_\delta)}{a(t_\delta)} = 0. \tag{56}$$

It is now easy to finish the proof of Theorem 7.

From the triangle inequality and inequality (7) one obtains

$$\begin{aligned} \|u_\delta(t_\delta) - y\| &\le \|u_\delta(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - y\| \\ &\le \|u_\delta(t_\delta) - V_\delta(t_\delta)\| + \frac{\delta}{a(t_\delta)} + \|V(t_\delta) - y\|, \end{aligned} \tag{57}$$

where $V(t_\delta) = V_{0,a(t_\delta)}$ (see equation (4)). From (56), (35), inequality (57), and Lemma 1 one obtains (19). Theorem 7 is proved. $\Box$

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