USEFUL METHODS FOR THE DISTRIBUTIONS OF PRODUCTS OF RANDOM VARIABLES

by

ROBERT DWIGHT BOWSER

B. S., Kansas State University, 1970

613-8301

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Statistics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1973

Ray A. Waller
Major Professor

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I. INTRODUCTION

In the field of statistics it is often of interest to examine the distribution of a random variable which is some combination of other random variables with known distributions. Many papers in the literature are devoted to finding the distribution of sums of random variables, while very few are concerned with the products or quotients of random variables. This paper is a consolidation and review of the literature in the area of products of random variables and their distributions.

Some of the more important techniques which have been used to find the distribution function of a product or quotient of two or more random variables are, what we call, the density transform technique, the distribution function technique, the characteristic function technique, and the Mellin transform technique. We first examine each of these techniques in a completely general setting in Section II, which we call "Methods and Mathematics." That section is included to review, or introduce, the techniques in the most general terms, so that when we use them to derive distribution functions and probability density functions for products of random variables in Section III, entitled "Products of Random Variables", the basic concepts will be well in mind. Although not a new technique for statisticians, Epstein (1948), the Mellin transform has not been incorporated into textbooks and therefore is receiving the most emphasis.

We should note at this time that approximation methods do exist, in contrast to the exact methods mentioned above. Only two such methods are briefly discussed here, as our main concern is a study of the exact methods. First, suppose X_i, i=1,2,3,...is a collection of independent identically distributed positive valued random variables to be multiplied "n" at a time. If we can assume that the random variable formed by taking the logarithm of the X_i has finite mean and variance, then we can apply the Central Limit Theorem to this new set of random variables. Now, since the logarithm of the product is equal to the sum of the logarithms of the components, the Central Limit Theorem states that the logarithm of the product approaches a Normal distribution as n increases under quite general conditions. This idea is further expounded in both Shellard (1952) and Broadbent (1956). Second, Arojan (1947) has given a limiting distribution for the product of two normals, not necessarily independent, for the case when the coefficients of variation $(\frac{\sigma_x}{\mu_x}, \frac{\sigma_y}{\mu_y})$ both approach infinity. No more is made of either of these methods.

II. Methods and Mathematics

In this section we first provide a short review of transformation techniques similar to those found in most intermediate and advanced textbooks in statistics (e.g. Kendall and Stuart, 1969; Fisz, 1963; Hogg and Craig, 1970). We mention only the general characteristics here, since the conditions which must be met for the application of each method are easily accessible. This section concludes with a short study of the Mellin transform which provides background information in the use of that technique. Throughout this section and the remainder of this paper we restrict our attention to continuous distributions with tractable probability density functions. We do this solely for ease of notation, as insertion of the Lebesgue integral in place of the Riemann integral allows the techniques to be used on all types of distributions with certain regularity conditions.

Within this section we make several assumptions. For all techniques we assume that the random variable Y is a function of the random vector $\underline{X}$, $Y = f(\underline{X})$. It is also convenient to assume that the vector ($\underline{X} = [X_1, X_2, \ldots, X_n]$) has the probability density function $g(\underline{x})$ and distribution function $G(\underline{x})$. Further assumptions will be added when necessary.

In order to treat the density transform technique in all its generality, we begin the discussion of this technique with some additional assumptions (Hogg and Craig, 1970). Let us first assume that $Y = f(\underline{X})$, $Z_2 = f_2(\underline{X})$, $Z_3 = f_3(\underline{X})$, $\ldots$, $Z_n = f_n(\underline{X})$ is a transformation from the n-dimensional space where $g(\underline{x}) > 0$, onto a subset of the $[Y, Z_2, Z_3, \ldots, Z_n]$ space. In addition, let us assume that the set $A = \{\underline{x} | g(\underline{x}) > 0\}$ can be