NUMERICAL METHODS FOR SOLVING LINEAR ILL-POSED PROBLEMS

by

SAPTO WAHYU INDRATNO

B.Sc., Bandung Institute of Technology, Indonesia, 1998
M.Sc., Twente University, The Netherlands, 2001
M.Sc., Kansas State University, USA, 2007

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2011

Abstract

A new method, the Dynamical Systems Method (DSM), justified recently, is applied to solving ill-conditioned linear algebraic system (ICLAS). The DSM gives a new approach to solving a wide class of ill-posed problems. In Chapter 1 a new iterative scheme for solving ICLAS is proposed. This iterative scheme is based on the DSM solution. An a posteriori stopping rules for the proposed method is justified. We also gives an a posteriori stopping rule for a modified iterative scheme developed in A.G.Ramm, JMAA,330 (2007),1338-1346, and proves convergence of the solution obtained by the iterative scheme. In Chapter 2 we give a convergence analysis of the following iterative scheme:

$$u_n^\delta = qu_{n-1}^\delta + (1-q)T_{a_n}^{-1}K^*f_\delta, \quad u_0^\delta = 0,$$

where $T := K^*K, \quad T_a := T + aI, \quad q \in (0,1), \quad a_n := \alpha_0 q^n,\ \alpha_0 > 0$, with finite-dimensional approximations of T and K^* for solving stably Fredholm integral equations of the first kind with noisy data. In Chapter 3 a new method for inverting the Laplace transform from the real axis is formulated. This method is based on a quadrature formula. We assume that the unknown function $f(t)$ is continuous with (known) compact support. An adaptive iterative method and an adaptive stopping rule, which yield the convergence of the approximate solution to $f(t)$, are proposed in this chapter.

NUMERICAL METHODS FOR SOLVING LINEAR ILL-POSED PROBLEMS

by

SAPTO WAHYU INDRATNO

B.Sc., Bandung Institute of Technology, Indonesia, 1998
M.Sc., Twente University, The Netherlands 2001
M.Sc., Kansas State University, USA 2007

A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2011

Approved by:

Major Professor
Alexander G. Ramm

Copyright

Abstract

A new method, the Dynamical Systems Method (DSM), justified recently, is applied to solving ill-conditioned linear algebraic system (ICLAS). The DSM gives a new approach to solving a wide class of ill-posed problems. In Chapter 1 a new iterative scheme for solving ICLAS is proposed. This iterative scheme is based on the DSM solution. An a posteriori stopping rules for the proposed method is justified. We also gives an a posteriori stopping rule for a modified iterative scheme developed in A.G.Ramm, JMAA,330 (2007),1338-1346, and proves convergence of the solution obtained by the iterative scheme. In Chapter 2 we give a convergence analysis of the following iterative scheme:

$$u_n^\delta = q u_{n-1}^\delta + (1-q) T_{a_n}^{-1} K^* f_\delta, \quad u_0^\delta = 0,$$

where $T := K^*K, \quad T_a := T + aI, \quad q \in (0,1), \quad a_n := \alpha_0 q^n,\ \alpha_0 > 0$, with finite-dimensional approximations of T and K^* for solving stably Fredholm integral equations of the first kind with noisy data. In Chapter 3 a new method for inverting the Laplace transform from the real axis is formulated. This method is based on a quadrature formula. We assume that the unknown function $f(t)$ is continuous with (known) compact support. An adaptive iterative method and an adaptive stopping rule, which yield the convergence of the approximate solution to $f(t)$, are proposed in this chapter.

Table of Contents

List of Figures

List of Tables

Acknowledgments

I would like to express my gratitude to Professor Alexander G. Ramm for being an outstanding advisor and excellent professor. His constant encouragement, support, and invaluable suggestions made this work successful. He has been everything that one could want in an advisor.

I am deeply indebted to my committee members Professor Diego Maldonado, Professor Marianne Korten and Professor Larry Weaver for their time and effort in reviewing this work.

I am also grateful to Anneke and Gus Van Der Hoeven for their help and valuable advice, and for being great friends throughout my study at Kansas State University.

I would like to thank all the staffs at department of Mathematics KSU for their tireless efforts. I am also very grateful to all my GTA friends at KSU.

Lastly, I am deeply and forever indebted to my wife Ida Widianingsih and my son Raditya A. Indratno for their love, support and encouragement throughout my entire life.

Dedication

I dedicate this thesis to my parents, **Maria Goretti Suwatni** and **Joseph Sutoyo**, who taught me the value of education. I am deeply indebted to them for their continued support and unwavering faith in me.

Preface

Consider the operator equation

$$Au = f, \tag{1}$$

where $A : X \to Y$ is an operator mapping a Banach space X into a Banach space Y. Problem (1) is called well-posed if A is bijective and A^{-1} is continuous. Then problem (1) is ill-posed if it is not well-posed. In[29] the following ill-posed problems that arise in many applications are given: stable numerical differential of noisy data, stable summation of the Fourier series and integrals with randomly perturbed coefficients, ill-conditioned linear algebraic systems, Fredholm and Volterra integral equations of the first kind, deconvolution problems, the Cauchy problem for Laplace's equation and the backwards heat equation. These problems can be reduced to equation (1). Therefore, it is important to develop a stable numerical method for solving ill-posed problem (1).

In applications the operator A is known and instead of the exact data f the noisy data f_δ are given, where $\|f - f_\delta\| \le \delta$ and δ is the noise level. It is natural to require that a numerical algorithm for solving problem (1) should have the following stability property: the less the noise level δ is, the closer approximation to y can be obtained.

Many methods have been developed for solving ill-posed problems stably. For example in[29,30] the following methods are discussed: variational regularization, quasisolutions, quasiinversion, iterative regularization method and the Dynamical Systems Method (DSM). In many papers the variational regularization is used for solving linear ill-posed problems $Au = f$. In this method one needs to minimize the functional

$$F(v) := \|Av - f_\delta\|^2 + \alpha\|v\|^2 = \inf, \tag{2}$$

where $\alpha > 0$ is a fixed parameter. It is proved in[23,29] that if $Ay = f$ and $y \perp N(A)$, where $N(A) := \{u \mid Au = 0\}$, then there exists a unique minimizer of (2) which is $u_{\alpha(\delta),\delta} =$

$(A^*A+\alpha I)^{-1}A^*f_\delta$, where I is the identity operator, and $\lim_{\delta\to 0}\|u_{\alpha(\delta),\delta}-y\|=0$ if $\delta\to 0$ and $\alpha(\delta)$ is chosen such that

$$\frac{\delta^2}{\alpha(\delta)}\to 0 \quad as\ \delta\to 0. \tag{3}$$

In many papers,[23,29,30], the parameter $\alpha(\delta)$ satisfying condition (3) is calculated by the discrepancy principle, i.e., the regularization parameter $\alpha(\delta)$ is obtained by solving the nonlinear equation:

$$V(\alpha):=\|Au_{\alpha,\delta}-f_\delta\|^2-\delta^2=0, \tag{4}$$

for α, where $u_{\alpha,\delta}=(A^*A+\alpha I)^{-1}A^*f_\delta$. The existence and uniqueness of the solution of equation (4) are proved in[29]. Numerically, one may use the Newton's method discussed in[23] to solve equation (4). The drawback of this method consists of the following: if the initial value of the regularization parameter α_0 is far from the solution of (4), Newton's method may fail to converge, and one needs to compute the derivative of the function $V(\alpha)$ which may be not easy.

The Dynamical Systems Method developed in[29,30] is a new general method for solving ill-posed problems. This method consists of finding an operator $\Phi(u,t)$ such that the Cauchy problem

$$\dot{u}=\Phi(u,t), \quad u(0)=u_0 \tag{5}$$

has the following properties:

$$\exists! u(t) \quad \forall t\geq 0, \quad \exists u(\infty), \quad and \quad Au(\infty)=f. \tag{6}$$

Some choices of the operator $\Phi(u,t)$ are given in[29,30]. For example when A is a linear operator, one may use the following $\Phi(u,t)$:

$$\Phi(u,t)=-u(t)+[A^*A+a(t)I]^{-1}A^*f, \tag{7}$$

where I is the identity operator, $a(t)>0$, $a(t)\searrow 0$ as $t\to\infty$. It is proved in[29] that Cauchy problem (5) with $\Phi(u,t)$ is defined in (7) yields properties (6). When the data f are noisy we replace the exact data f in (7) with the noisy data f_δ. In this case if t_δ is chosen such that

$$\frac{\delta^2}{a(t_\delta)} \to 0 \quad as \quad \delta \to 0, \quad \lim_{\delta\to 0} t_\delta = \infty, \tag{8}$$

then $\|u_\delta(t_\delta) - u\| \to 0$ as $\delta \to 0$, where

$$u_\delta(t_\delta) := u_0 e^{-t_\delta} + \int_0^{t_\delta} e^{-(t_\delta - s)}(A^*A + a(s)I)^{-1}A^* f_\delta ds. \tag{9}$$

Alternatively, one may use the following discrepancy-type principle for DSM developed in[28,33]:

$$\int_0^{t_\delta} e^{-(t_\delta - s)} a(s)\|Q^{-1}_{a(s)} f_\delta\| ds = C\delta, \quad C \in (1, 2], \tag{10}$$

where

$$Q_a := AA^* + aI. \tag{11}$$

In this thesis we develop iterative methods for solving linear ill-posed problems based on the DSM with $\Phi(u, t)$ defined in (7) which can be implemented easily numerically.

The main results of this thesis are:

(1) a new iterative scheme and discrepancy-type principle based on DSM for solving ill-conditioned linear algebraic systems stably (see Chapter 1),

(2) a modified iterative scheme developed in[32] (see Chapter 1),

(3) a new adaptive iterative scheme and adaptive discrepancy-type principle for solving stably Fredholm integral equations of the first kind (see Chapter 2), and

(4) a modified adaptive iterative scheme developed in Chapter 3 which is applied to inversion of the Laplace transform from the real axis (see Chapter 3).

The thesis is divided into three chapters. The first, second and third chapters are based on the published papers[16], [17] and[15], respectively. The thesis is organized as follows. In Chapter 1 a new iterative method and iterative discrepancy-type principle for solving linear ill-posed problem $Au = f$ are derived. This iterative method is based on the Dynamical

Systems Method (DSM) with $\Phi(u,t)$ defined in (7). The iterative discrepancy-type principle given in Section 2 is constructed from discrepancy-type principle for DSM (10). This iterative discrepancy-type principle is simpler than (4) or (10), since we do not need to solve the nonlinear equations (4) or (10). Another advantage of our iterative method is the following one: the initial regularization parameter α_0 can be chosen relatively large. Our method is new, since a numerical method relating the solution of Cauchy problem (9) and discrepancy-type principle (10) has not been developed in the literature, to our knowledge. The iterative scheme and the iterative discrepancy-type principle, derived in this Chapter are, respectively, of the form:

$$u_{n+1}^{\delta} = qu_n^{\delta} + (1-q)T_{a_{n+1}}^{-1}A^* f_\delta, \quad u_0 = 0, \quad q \in (0,1), \tag{12}$$

and

$$\begin{aligned} &\sum_{j=0}^{n_\delta - 1}(q^{n_\delta - j-1} - q^{n-j})a_{j+1}\|Q_{a_{j+1}}^{-1} f_\delta\| ds \le C\delta^{\varepsilon} \\ &< \sum_{j=0}^{n-1}(q^{n-j-1} - q^{n-j})a_{j+1}\|Q_{a_{j+1}}^{-1} f_\delta\|, \ 1 \le n < n_\delta, \end{aligned} \tag{13}$$

where $\|f_\delta - f\| \le \delta$, $a_n = a_0 q^n$,$a_0 > 0$, $T_a := A^*A + aI$, $Q_a := Q + aI$, $Q := AA^*$, I is the identity operator, $\epsilon \in (0,1)$ and $C \in (1,2)$. The convergence result for iterative method (12) is formulated in Theorem 1.2.10. We apply this iterative scheme to solve ill-conditioned linear algebraic system $Au = f$. In Section 3 we construct a modification of the following iterative scheme

$$u_{n+1}^{\delta} = aT_a^{-1}u_n^{\delta} + T_a^{-1}A^* f_\delta, \quad u_0 \perp N(A), \tag{14}$$

where $a > 0$ is a fixed parameter. This iterative scheme is developed in [32] and used for solving any solvable linear equation in a Hilbert space, including equations with unbounded, closed, densely defined linear operators. The numerical method for choosing the parameters a and n_δ, which yields the convergence result $\|u_{n_\delta} - y\| \to 0$ as $\delta \to 0$, has not been discussed in [32]. We modify iterative scheme (14) by replacing the fixed parameter a with the geometric series $a_n = a_0 q^n$, $q \in (0,1)$, $a_0 > 0$, so that the following stopping rule can be used:

$$\|AT^{-1}_{q^{n_\delta}} A^* f_\delta - f_\delta\| \leq C\delta^\varepsilon < \|AT^{-1}_{q^n} A^* f_\delta - f_\delta\|, \quad 1 \leq n < n_\delta,\ C > 1,\ \varepsilon \in (0,1). \tag{15}$$

The uniqueness and existence of the parameter n_δ satisfying inequalities (15) are proved in Lemma 1.2.5. We prove that iterative scheme (14) together with stopping rule (15) yield the convergence of iterative solution (14). This main result is formulated in Theorem 1.3.6.

In Chapter 1 an example of ill-conditioned linear algebraic system constructed by a projection method, is given. No method has been developed for choosing the number of basis functions needed in the projection method. In many papers, e.g.,[8,13,25], the number of basis functions is fixed and is large. If one chooses a large number of basis functions then the size of the linear system is also large. Therefore, if a large fixed matrix A is used in iterative scheme (12) then the computation time will be large when n_δ is large.

In Chapter 2 we develop an iterative method which allows us to choose the number of basis functions needed in the projection method. In this iterative scheme the number of basis functions may change in each iteration. Initially one may start with a small number of basis functions and at each iteration the number of the basis functions is increased only if some conditions hold (see pp.53-54) , so the computation time can be reduced. In Section 2 the adaptive iterative scheme is constructed for solving the linear operator equation $Ku = f$, where

$$(Ku)(x) := \int_a^b k(x,s)u(s)ds, \quad a < x < b, \tag{16}$$

$k(x,s)$ is a smooth kernel and $u \in L^2[a,b]$. This adaptive iterative scheme is constructed by finite-dimensional approximation of the operators $T := K^*K$ and K^*, where K^* is the adjoint operator of K. These approximations yield the following adaptive iterative scheme:

$$u^\delta_{n,m_n} = qu^\delta_{n-1,m_{n-1}} + (1-q)T^{-1}_{a_n,m_n} K^*_{m_n} f_\delta, \quad u^\delta_{0,m_0} = 0, \tag{17}$$

where $a_n := \alpha_0 q^n$, $\alpha_0 > 0$, $q \in (0,1)$, $T_{a,m} := T^{(m)} + aI$, $\|T^{(m)} - T\| \to 0$ as $m \to \infty$, K_m is a finite-dimensional approximation of the operator K^* and m is a parameter which measures the accuracy of the finite-dimensional approximations $T^{(m)}$ and K_m. Lemma 2.2.2 gives a rule for choosing the parameter m such that the finite-dimensional operator T_{a_n,m_n} is invertible. In Theorem 2.2.8 conditions (2.50)-(2.53) are used to prove the convergence of the iterative solution. The nontrivial task is to develop a stopping rule such that relation (2.53) holds. In Section 3 we consider the following adaptive stopping rule:

$$G_{n_\delta,m_{n_\delta}} \le C\delta^\varepsilon < G_{n,m_n}, \quad 1 \le n < n_\delta,\ C > 2, \quad \varepsilon \in (0,1), \tag{18}$$

where

$$\begin{aligned} &G_{n,m_n} = qG_{n-1,m_{n-1}} + (1-q)a_n\|Q^{-1}_{a_n,m_n} f_\delta\|, \\ &G_{0,m_0} = 0, \quad G_{1,m_1} \ge C\delta^\varepsilon, \quad a_n = qa_{n-1}, \quad a_0 = \alpha_0 = const > 0, \end{aligned} \tag{19}$$

$Q_{a,m} := Q^{(m)} + aI$, and $Q^{(m)}$ is the finite-dimensional approximation of the operator KK^*. Instead of using a fixed operator Q as in (13) we use the adaptive operator $Q^{(m_n)}$ which depends on the regularization parameter a_n. The existence and uniqueness of the parameter n_δ, satisfying (19), follows from Lemma 2.3.3 and definition (18). The convergence of the iterative method is formulated in Theorem 2.3.7. In Section 4 we give a simple example of finite-dimensional approximation operators $T^{(m)}$ and K^*_m.

In Chapter 3 we introduce a different approach to solving the Fredholm integral equations of the first kind described in Chapter 2. The advantage of this approach is: we only need a finite-dimensional approximation of the operator K^*K. Therefore, the rule of choosing the accuracy parameter m is much simpler than the one used in Chapter 2. In Section 2 an adaptive iterative scheme is constructed and applied to the inversion of the Laplace transform:

$$\mathcal{L}f(p) := \int_0^\infty e^{-ps} f(s)ds = F(p), \quad 0 < p < d < \infty, \tag{20}$$

where f is a real valued function in $X_{0,b}$, $X_{0,b}$ is defined in (3.2). A survey of the methods of the Laplace transform inversion has been given in[5]. In all of these methods the inversion of the Laplace transforms were taken from the complex axis. The methods mentioned in[2,5,10,18] do not include regularization techniques and therefore they can not be used in the case of noisy data. In[22] it is shown that the results of the inversion of the Laplace transform on the Mellin contour are more accurate than these of the inversion of the Laplace transform from the real axis. When $f(t)$ is a real valued function and $F(p)$ is known for all real and positive values of p the ill-posedness of Laplace transform inversion can be investigated by means of Mellin transform[1,26]. However, in practice $F(p)$ is known only at a finite set of points. Regularization methods where $F(p)$ is known at a finite set of points have been considered in[3,4,22,36,37]. In our method it is assumed that $f(t)$ is real-valued and $F(p)$ is known at a finite set points. The method, constructed in Section 2, is based on approximation of the kernel $\mathcal{L}^*\mathcal{L}$. The smoothness of the kernel allows one to use a simple quadrature formula: the compound Simpson's rule (see[6]). This approach yields the following approximation of the function $f(t)$:

$$f_\delta^{(m)}(t) = \sum_{j=0}^{m} c_j^{(m,\delta)} w_j^{(m)} e^{-p_j t}, \quad p_j = j\frac{d}{m}, \quad j = 0, \dots, m, \tag{21}$$

where $c_j^{(m,\delta)}$, $j = 1, 2, \dots, m$, are parameters obtained by the adaptive iterative scheme and $w_j^{(m)}$, $j = 1, 2, \dots, m$ are the weights of the compound Simpson's rule with m subintervals. In each iteration the number of basis functions used in (21) is obtained by rule (3.63). One can see that this rule is much simpler than the rule given in Theorem 2.2.8. The following iterative discrepancy-type principle is used as the stopping rule:

$$G_{n_\delta, m_{n_\delta}} \le C\delta^\varepsilon < G_{n,m_n}, \quad 1 \le n < n_\delta, \ C > \sqrt{d}, \ \varepsilon \in (0,1), \tag{22}$$

where

$$G_{n,m_n} = qG_{n-1,m_{n-1}} + (1-q)a_n \|c^{(m_n,\delta)}\|_{W^{m_n}}, \ G_{0,m_0} = 0, \tag{23}$$

where $a_n = a_0 q^n$, $a_0 > 0$, $q \in (0,1)$, $\|\cdot\|_{W^m}$ is defined in (3.7). The convergence of the iterative scheme, derived in Section 2 with stopping rule (22), is claimed in Theorem 3.2.17. The inversion method proposed in this Chapter is simpler than the methods given in[4,22,24,36], e.g., Fourier series expansion, regularized analytic continuation, eigenfunction expansion and Gauss-Laguerre quadrature method, since we only need the compound Simpson's quadrature in the discretization where the weights of the quadrature can be easily obtained exactly. Moreover, our representation of the approximation of the function $f(t)$ is new and uses only the weights of the compound Simpson's rule and the specific form of the Laplace Transform. The numerical results given in Section 3 show that our results are comparable with or better than the existing methods.

Chapter 1

Dynamical Systems Method for Solving Ill-conditioned Linear Algebraic Systems

1.1 Introduction

We consider a linear equation

$$Au = f, \tag{1.1}$$

where $A : \mathbb{R}^m \to \mathbb{R}^m$, and assume that equation (1.1) has a solution, possibly non-unique. According to Hadamard[30] p.9, problem (1.1) is called well-posed if the operator A is injective, surjective, and A^{-1} is continuous. Problem (1.1) is called ill-posed if it is not well-posed. Ill-conditioned linear algebraic systems arise as discretizations of ill-posed problems, such as Fredholm integral equations of the first kind,

$$\int_a^b k(x,t)u(t)dt = f(x),\ c \le x \le d, \tag{1.2}$$

where $k(x,t)$ is a smooth kernel. Therefore, it is of interest to develop a method for solving ill-conditioned linear algebraic systems stably. In this Chapter we give a method for solving linear algebraic system (1.1) with an ill-conditioned-matrix A. The matrix A is called ill-conditioned if $\kappa(A) >> 1$, where $\kappa(A) := ||A||||A^{-1}||$ is the condition number of A. If the null-space of A, $\mathcal{N}(A) := \{u : Au = 0\}$, is non-trivial, then $\kappa(A) = \infty$. Let $A = U\Sigma V^*$ be the singular value decomposition (SVD) of A, $UU^* = U^*U = I$, $VV^* = V^*V = I$, and

$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$, where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_m \ge 0$ are the singular values of A. Applying this SVD to the matrix A in (1.1), one gets

$$f = \sum_i \beta_i u_i \text{ and } y = \sum_{i, \sigma_i > 0} \frac{\beta_i}{\sigma_i} v_i, \tag{1.3}$$

where $\beta_i = \langle u_i, f \rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. The terms with small singular values σ_i in (1.3) cause instability of the solution, because the coefficients β_i are known with errors. This difficulty is essential when one deals with an ill-conditioned matrix A. Therefore a regularization is needed for solving ill-conditioned linear algebraic system (1.1). There are many methods to solve (1.1) stably: variational regularization, quasisolutions, iterative regularization (see e.g,[13],[23],[29],[30]). The method proposed in this Chapter is based on the Dynamical Systems Method (DSM) developed in[30] p.76. The DSM for solving equation (1.1) with, possibly, nonlinear operator A consists of solving the Cauchy problem

$$\dot{u}(t) = \Phi(t, u(t)), \quad u(0) = u_0; \; \dot{u}(t) := \frac{du}{dt}, \tag{1.4}$$

where $u_0 \in H$ is an arbitrary element of a Hilbert space H, and Φ is some nonlinearity, chosen so that the following three conditions hold: a) there exists a unique solution $u(t) \quad \forall t \ge 0$, b) there exists $u(\infty)$, and c) $Au(\infty) = f$.

In this Chapter we choose $\Phi(t, u(t)) = (A^*A + a(t)I)^{-1} f - u(t)$ and consider the following Cauchy problem:

$$\dot{u}_a(t) = -u_a(t) + [A^*A + a(t)I_m]^{-1} A^* f, \quad u_a(0) = u_0, \tag{1.5}$$

where

$$a(t) > 0, \text{ and } a(t) \searrow 0 \text{ as } t \to \infty, \tag{1.6}$$

A^* is the adjoint matrix and I_n is an $m \times m$ identity matrix. The initial element u_0 in (1.5) can be chosen arbitrarily in $N(A)^\perp$, where

$$\mathcal{N}(A) := \{u \mid Au = 0\}. \tag{1.7}$$

For example, one may take $u_0 = 0$ in (1.5) and then the unique solution to (1.5) with $u(0) = 0$ has the form

$$u(t) = \int_0^t e^{-(t-s)} T_{a(s)}^{-1} A^* f ds, \tag{1.8}$$

where $T := A^*A$, $T_a := T + aI$, I is the identity operator. In the case of noisy data we replace the exact data f with the noisy data f_δ in (1.8), i.e.,

$$u^\delta(t) = \int_0^{t_\delta} e^{-(t_\delta-s)} T_{a(s)}^{-1} A^* f_\delta ds, \tag{1.9}$$

where t_δ is the stopping time which will be discussed later. There are many ways to solve the Cauchy problem (1.5). For example, one may apply a family of Runge-Kutta methods for solving (1.5). Numerically, the Runge-Kutta methods require an appropriate stepsize to get an accurate and stable solution. Usually the stepsizes have to be chosen sufficiently small to get such a solution. The number of steps will increase when t_δ, the stopping time, increases, see[13]. Therefore the computation time will increase significantly. Since $\lim_{\delta\to 0} t_\delta = \infty$, as was proved in[30], the family of the Runge-Kutta method may be less efficient for solving the Cauchy problem (1.5) than the method, proposed in this Chapter. We give a simple iterative scheme, based on the DSM, which produces stable solution to equation (1.1). The novel points of this Chapter are iterative schemes (1.12) and (1.13) (see below), which are constructed on the basis of formulas (1.8) and (1.9), and a modification of the iterative scheme given in[32]. Our stopping rule for the iterative scheme (1.13) is given in (1.85) (see below). In[30] p.76 the function $a(t)$ is assumed to be a slowly decaying monotone function. In this thesis instead of using the slowly decaying continuous function $a(t)$ we use the following piecewise-constant function:

$$a^{(n)}(t) = \sum_{j=0}^{n-1} \alpha_0 q^{j+1} \chi_{(t_j,t_{j+1}]}(t), \quad q \in (0,1), \quad t_j = -j\ln(q),\ n \in \mathbb{N}, \tag{1.10}$$

where $\mathbb{N}$ is the set of positive integer, $t_0 = 0$, $\alpha_0 > 0$, and

$$\chi_{(t_j,t_{j+1}]}(t) = \begin{cases} 1, & t \in (t_j, t_{j+1}]; \\ 0, & \text{otherwise}. \end{cases} \tag{1.11}$$

The parameter α_0 in (1.10) is chosen so that assumption (1.17) (see below) holds. This assumption plays an important role in the proposed iterative scheme. Definition (1.10) allows one to write (1.8) in the form

$$u_{n+1} = qu_n + (1-q)T^{-1}_{\alpha_0 q^{n+1}} A^* f, \quad u_0 = 0. \tag{1.12}$$

A detailed derivation of the iterative scheme (1.12) is given in Section 2. When the data f are contaminated by some noise, we use f_δ in place of f in (1.8), and get the iterative scheme

$$u^\delta_{n+1} = qu^\delta_n + (1-q)T^{-1}_{\alpha_0 q^{n+1}} A^* f_\delta, \quad u^\delta_0 = 0. \tag{1.13}$$

We always assume that

$$\|f_\delta - f\| \le \delta, \tag{1.14}$$

where f_δ are the noisy data, which are known, while f is unknown, and δ is the level of noise. Here and throughout this Chapter the notation $\|z\|$ denotes the l^2-norm of the vector $z \in \mathbb{R}^m$. In this Chapter a discrepancy type principle (DP) is proposed to choose the stopping index of iteration (1.13). This DP is based on discrepancy principle for the DSM developed in[28,33], where the stopping time t_δ is obtained by solving the following nonlinear equation

$$\int_0^{t_\delta} e^{-(t_\delta - s)} a(s) \|Q^{-1}_{a(s)} f_\delta\| ds = C\delta, \quad C \in (1,2]. \tag{1.15}$$

It is a non-trivial task to obtain the stopping time t_δ satisfying (1.15). In this Chapter we propose a discrepancy type principle based on (1.15) which can be easily implemented numerically: iterative scheme (1.13) is stopped at the first integer n_δ satisfying the inequalities:

$$\begin{aligned} &\sum_{j=0}^{n_\delta - 1} (q^{n_\delta - j - 1} - q^{n-j}) \alpha_0 q^{j+1} \|Q^{-1}_{\alpha_0 q^{j+1}} f_\delta\| ds \le C\delta^\varepsilon \\ &< \sum_{j=0}^{n-1} (q^{n-j-1} - q^{n-j}) \alpha_0 q^{j+1} \|Q^{-1}_{\alpha_0 q^{j+1}} f_\delta\|, \ 1 \le n < n_\delta, \end{aligned} \tag{1.16}$$

and it is assumed that

$$(1-q)\alpha_0 q\|Q^{-1}_{\alpha_0 q} f_\delta\| \geq C\delta^\varepsilon, \quad C>1, \quad \varepsilon\in(0,1), \quad \alpha_0>0. \tag{1.17}$$

We prove in Section 2 that using discrepancy-type principle (1.16), one gets the convergence:

$$\lim_{\delta\to 0}\|u^\delta_{n_\delta}-y\|=0, \tag{1.18}$$

where u^δ_n is defined in (1.13). About other versions of discrepancy principles for DSM we refer the reader to[29],[27]. In this Chapter we assume that A is bounded. If the operator A is unbounded then f_δ may not belong to the domain of A^*. In this case the expression A^*f_δ is not defined. In[31],[32] and[34] solving (1.1) with unbounded operators is discussed. In these papers the unbounded operator A is assumed to be linear, closed, densely defined operator in a Hilbert space. Under these assumptions one may use the operator $A^*(AA^*+aI)^{-1}$ in place of $T_a^{-1}A^*$. This operator is defined for any f in the Hilbert space.
In[32] an iterative scheme with a constant regularization parameter is given:

$$u^\delta_{n+1}=aT_a^{-1}u^\delta_n+T_a^{-1}A^*f_\delta, \tag{1.19}$$

but the stopping rule, which produces a stable solution of equation (1.1) by this iterative scheme, has not been discussed in[32]. In this thesis the constant regularization parameter a in iterative scheme (1.19) is replaced with the geometric series $\{\alpha_0 q^n\}_{n=1}^\infty$, $\alpha_0>0,\ q\in(0,1)$, i.e.

$$u^\delta_{n+1}=\alpha_0 q^n T^{-1}_{\alpha_0 q^n}u^\delta_n+T^{-1}_{\alpha_0 q^n}A^*f_\delta. \tag{1.20}$$

Stopping rule (1.85) (see below) is used for this iterative scheme. Without loss of generality we use $\alpha_0=1$ in (1.20). The convergence analysis of this iterative scheme is presented in Section 3. In Section 4 some numerical experiments are given to illustrate the efficiency of the proposed methods.

1.2 Derivation of the proposed method

In this section we give a detailed derivation of iterative schemes (1.12) and (1.13). Let us denote by $y\in\mathbb{R}^m$ the unique minimal-norm solution of equation (1.1). Throughout this thesis we denote $T_{a(t)} := A^*A + a(t)I_m$, where I_m is the identity operator in $\mathbb{R}^m$, and $a(t)$ is given in (1.10).

Lemma 1.2.1. *Let $g(x)$ be a continuous function on $(0, \infty)$, $c > 0$ and $q \in (0, 1)$ be constants. If*

$$\lim_{x \to 0^+} g(x) = g(0) := g_0, \tag{1.21}$$

then

$$\lim_{n \to \infty} \sum_{j=1}^{n-1} \left(q^{n-j-1} - q^{n-j}\right) g(cq^{j+1}) = g_0. \tag{1.22}$$

Proof. Let

$$\omega_j(n) := q^{n-j} - q^{n+1-j}, \quad \omega_j(n) > 0, \tag{1.23}$$

and

$$F_l(n) := \sum_{j=1}^{l-1} \omega_j(n) g(cq^j). \tag{1.24}$$

Then

$$|F_{n+1}(n) - g_0| \le |F_l(n)| + \left| \sum_{j=l}^{n} \omega_j(n) g(cq^j) - g_0 \right|.$$

Take $\epsilon > 0$ arbitrary small. For sufficiently large $l(\epsilon)$ one can choose $n(\epsilon)$, such that

$$|F_{l(\epsilon)}(n)| \le \frac{\epsilon}{2}, \ \forall n > n(\epsilon),$$

because $\lim_{n\to\infty} q^n = 0$. Fix $l = l(\epsilon)$ such that $|g(cq^j) - g_0| \le \frac{\epsilon}{2}$ for $j > l(\epsilon)$. This is possible because of (1.21). One has

$$|F_{l(\epsilon)}(n)| \le \frac{\epsilon}{2}, \ n > n(\epsilon)$$

and

$$\begin{aligned} \left| \sum_{j=l(\epsilon)}^{n} \omega_j(n) g(cq^j) - g_0 \right| &\le \sum_{j=l(\epsilon)}^{n} \omega_j(n) |g(cq^j) - g_0| + |\sum_{j=l(\epsilon)}^{n} \omega_j(n) - 1||g_0| \\ &\le \frac{\epsilon}{2} \sum_{j=l(\epsilon)}^{n} \omega_j(n) + q^{n-l(\epsilon)}|g_0| \\ &\le \frac{\epsilon}{2} + |g_0| q^{n-l(\epsilon)} \le \epsilon, \end{aligned}$$

if n is sufficiently large. Here we have used the relation

$$\sum_{j=l}^{n} \omega_j(n) = 1 - q^{n+1-l}.$$

Since $\epsilon > 0$ is arbitrarily small, Lemma 1.2.1 is proved.

□

Let us define

$$u_n := \int_0^{t_n} e^{-(t_n - s)} T^{-1}_{a^{(n)}(s)} A^* f ds, \quad t_n = -n \ln(q), \quad q \in (0,1). \tag{1.25}$$

Note that

$$\begin{aligned} u_n &= \int_0^{t_{n-1}} e^{-(t_n-s)} T^{-1}_{a^{(n)}(s)} A^* f ds + \int_{t_{n-1}}^{t_n} e^{-(t_n-s)} T^{-1}_{a^{(n)}(s)} A^* f ds \\ &= e^{-(t_n - t_{n-1})} \int_0^{t_{n-1}} e^{-(t_{n-1}-s)} T^{-1}_{a(s)} A^* f ds + \int_{t_{n-1}}^{t_n} e^{-(t_n-s)} T^{-1}_{a^{(n)}(s)} A^* f ds \\ &= e^{-(t_n - t_{n-1})} u_{n-1} + \int_{t_{n-1}}^{t_n} e^{-(t_n-s)} T^{-1}_{a^{(n)}(s)} A^* f ds. \end{aligned}$$

Using definition (1.10), one gets

$$\begin{aligned} u_n &= e^{-(t_n - t_{n-1})} u_{n-1} + [1 - e^{-(t_n - t_{n-1})}] T^{-1}_{\alpha_0 q^n} A^* f \\ &= \frac{q^n}{q^{n-1}} u_{n-1} + (1 - \frac{q^n}{q^{n-1}}) T^{-1}_{\alpha_0 q^n} A^* f. \end{aligned}$$

Therefore, (1.25) can be rewritten as iterative scheme (1.12).

Lemma 1.2.2. *Let u_n be defined in (1.12) and $Ay = f$. Then*

$$\|u_n - y\| \le q^n \|y\| + \sum_{j=0}^{n-1} \left(q^{n-j-1} - q^{n-j}\right) \alpha_0 q^{j+1} \|T^{-1}_{\alpha_0 q^{j+1}} y\|, \quad \forall n \ge 1, \tag{1.26}$$

and

$$\|u_n - y\| \to 0 \text{ as } n \to \infty. \tag{1.27}$$

Proof. By definitions (1.25) and (1.10) we obtain

$$u_n = \int_0^{t_n} e^{-(t_n-s)} T^{-1}_{a(s)} A^* f ds = \sum_{j=0}^{n-1} \left(\frac{q^n}{q^{j+1}} - \frac{q^n}{q^j} \right) T^{-1}_{\alpha_0 q^{j+1}} A^* f. \tag{1.28}$$

From (1.28) and the equation $Ay = f$, one gets:

$$\begin{aligned} u_n &= \sum_{j=0}^{n-1} \left(\frac{q^n}{q^{j+1}} - \frac{q^n}{q^j} \right) T^{-1}_{\alpha_0 q^{j+1}} A^* f \\ &= \sum_{j=0}^{n-1} \left(\frac{q^n}{q^{j+1}} - \frac{q^n}{q^j} \right) T^{-1}_{\alpha_0 q^{j+1}} (T_{\alpha_0 q^{j+1}} - \alpha_0 q^{j+1} I_m) y \\ &= \sum_{j=0}^{n-1} \left(q^{n-j-1} - q^{n-j} \right) y - \sum_{j=0}^{n-1} \left(q^{n-j-1} - q^{n-j} \right) \alpha_0 q^{j+1} T^{-1}_{\alpha_0 q^{j+1}} y \\ &= y - q^n y - \sum_{j=0}^{n-1} \left(q^{n-j-1} - q^{n-j} \right) \alpha_0 q^{j+1} T^{-1}_{\alpha_0 q^{j+1}} y. \end{aligned}$$

Thus, estimate (1.26) follows. To prove (1.27), we apply Lemma 1.2.1 with $g(a) := a\|T_a^{-1}y\|$. Since $y \perp \mathcal{N}(A)$, it follows from the spectral theorem that

$$\lim_{a\to 0} g^2(a) = \lim_{a\to 0} \int_0^\infty \frac{a^2}{(a+s)^2} d\langle E_s y, y\rangle = \|P_{\mathcal{N}(A)} y\|^2 = 0,$$

where E_s is the resolution of the identity corresponding to A^*A, and P is the orthogonal projector onto $\mathcal{N}(A)$. Thus, by Lemma 1.2.1, (1.27) follows. $\square$

Let us discuss iterative scheme (1.13). The following lemma gives the estimate of the difference of the solutions u_n^δ and u_n.

Lemma 1.2.3. *Let u_n and u_n^δ be defined in (1.12) and (1.13), respectively. Then*

$$\|u_n^\delta - u_n\| \le \frac{\sqrt{q}}{1 - q^{3/2}} w_n, \quad n \ge 0, \tag{1.29}$$

where $w_n := (1-q)\frac{\delta}{2\sqrt{q}\sqrt{\alpha_0 q^n}}$.

Proof. Let $H_n := \|u_n^\delta - u_n\|$. Then from the definitions of u_n^δ and u_n we get the estimate

$$H_{n+1} \le q\|u_n^\delta - u_n\| + (1-q)\|T^{-1}_{\alpha_0 q^{n+1}} A^*(f_\delta - f)\| \le qH_n + w_n. \tag{1.30}$$

Let us prove inequality (1.29) by induction. For $n = 0$ one has $u_0 = u_0^\delta = 0$, so (1.29) holds. For $n = 1$ one has $\|u_1^\delta - u_1\| \le (1-q)\frac{\delta}{2\sqrt{\alpha_0 q^2}}$, so (1.29) holds. If (1.29) holds for $n \le k$, then for $n = k+1$ one has

$$\begin{aligned} H_{k+1} &\le qH_k + w_k \le \left(\frac{q^{3/2}}{1-q^{3/2}} + 1\right) w_k = \frac{1}{1-q^{3/2}} w_k \\ &= \frac{1}{1-q^{3/2}} \frac{w_k}{w_{k+1}} w_{k+1} \le \frac{1}{1-q^{3/2}} \sqrt{q} w_{k+1}. \end{aligned} \tag{1.31}$$

Hence (1.29) is proved for $n \ge 0$. □

1.2.1 Stopping criterion

In this section we give a stopping rule for iterative scheme given in (1.13). Let $Q := AA^*$, $Q_a := Q + aI_m$, and

$$\begin{aligned} G_n &:= \int_0^{t_n} e^{-(t_n-s)} a(s) \|Q_{a(s)}^{-1} f_\delta\| ds \\ &= \sum_{j=0}^{n-1} (q^{n-j-1} - q^{n-j}) \alpha_0 q^{j+1} \|Q_{\alpha_0 q^{j+1}}^{-1} f_\delta\|, \quad n \ge 1, \end{aligned} \tag{1.32}$$

where $t_n = -n \ln q$, $q \in (0,1)$ and $\alpha_0 > 0$. Then stopping rule (1.16) can be rewritten as

$$G_{n_\delta} \le C\delta^\varepsilon < G_n, \quad 1 \le n < n_\delta, \quad \varepsilon \in (0,1), \quad C > 1, \quad G_1 > C\delta^\varepsilon. \tag{1.33}$$

Note that

$$\begin{aligned} G_{n+1} &= \sum_{j=0}^{n} (q^{n-j} - q^{n+1-j}) \alpha_0 q^{j+1} \|Q_{\alpha_0 q^{j+1}}^{-1} f_\delta\| \\ &= \sum_{j=0}^{n-1} (q^{n-j} - q^{n+1-j}) \alpha_0 q^{j+1} \|Q_{\alpha_0 q^{j+1}}^{-1} f_\delta\| + (1-q)\alpha_0 q^{n+1} \|Q_{\alpha_0 q^{n+1}}^{-1} f_\delta\| \\ &= qG_n + (1-q)\alpha_0 q^{n+1} \|Q_{\alpha_0 q^{n+1}}^{-1} f_\delta\|, \end{aligned}$$

so

$$G_n = qG_{n-1} + (1-q)\alpha_0 q^n \|Q_{\alpha_0 q^n}^{-1} f_\delta\|, \quad n \ge 1, \quad G_0 = 0. \tag{1.34}$$

Lemma 1.2.4. *Let G_n be defined in (1.34). Then*

$$G_n \le \frac{1}{1-\sqrt{q}} \sqrt{\alpha_0 q^n} \frac{\|y\|}{2} + \delta, \quad n \ge 1, \quad q \in (0,1). \tag{1.35}$$

Proof. Using the identity

$$-aQ_a^{-1} = AT_a^{-1}A^* - I_m,\ a > 0,\ T := A^*A,\ T_a := T + aI_m,$$

the estimates

$$a\|Q_a^{-1}\| \le 1,\ \|f_\delta - f\| \le \delta,$$

and

$$a\|AT_a^{-1}\| \le \frac{\sqrt{a}}{2},$$

where $Q := AA^*$, $Q_a := Q + aI_m$, we get

$$\begin{aligned}
G_n &= qG_{n-1} + (1-q)\|AT_{\alpha_0 q^n}^{-1}A^*f_\delta - f_\delta\| \\
&= qG_{n-1} + (1-q)\|AA^*Q_{\alpha_0 q^n}^{-1}f_\delta - f_\delta\| \\
&= qG_{n-1} + (1-q)\|(AA^* + \alpha_0 q^n I - \alpha_0 q^n I)Q_{q^n}^{-1}f_\delta - f_\delta\| \\
&= qG_{n-1} + (1-q)\alpha_0 q^n\|Q_{\alpha_0 q^n}^{-1}f_\delta\| \\
&= qG_{n-1} + (1-q)\alpha_0 q^n\|Q_{\alpha_0 q^n}^{-1}(f_\delta - f + f)\| \\
&\le qG_{n-1} + (1-q)\alpha_0 q^n\|Q_{\alpha_0 q^n}^{-1}(f_\delta - f)\| + (1-q)\alpha_0 q^n\|Q_{\alpha_0 q^n}^{-1}f\| \\
&\le qG_{n-1} + (1-q)\delta + (1-q)\|AT_{\alpha_0 q^n}^{-1}A^*f - f\| \\
&= qG_{n-1} + (1-q)\delta + (1-q)\|A(T_{\alpha_0 q^n}^{-1}A^*Ay - y)\| \\
&= qG_{n-1} + (1-q)\delta + (1-q)\|A(-\alpha_0 q^n T_{\alpha_0 q^n}^{-1}y)\| \\
&= qG_{n-1} + (1-q)\delta + (1-q)\alpha_0 q^n\|AT_{\alpha_0 q^n}^{-1}y\| \\
&\le qG_{n-1} + (1-q)\delta + (1-q)\alpha_0 q^n\frac{\|y\|}{2\sqrt{\alpha_0 q^n}} \\
&= qG_{n-1} + (1-q)\delta + (1-q)\sqrt{\alpha_0 q^n}\frac{\|y\|}{2} \\
&= qG_{n-1} + (1-q)\delta + \sqrt{q}\frac{\sqrt{\alpha_0 q^{n-1}}}{2}\|y\|.
\end{aligned} \tag{1.36}$$

Therefore,

$$G_n - \delta \le q(G_{n-1} - \delta) + \sqrt{q}\frac{\sqrt{\alpha_0 q^{n-1}}}{2}\|y\|, \quad n \ge 1,\ G_0 = 0. \tag{1.37}$$

Let us prove relation (1.35) by induction. From relation (1.37) we get

$$G_1 - \delta \le -q\delta + \frac{\sqrt{\alpha_0 q}}{2}\|y\| \le -q\delta + \frac{1}{1-\sqrt{q}}\frac{\sqrt{\alpha_0 q}}{2}\|y\| \le \frac{1}{1-\sqrt{q}}\frac{\sqrt{\alpha_0 q}}{2}\|y\|. \tag{1.38}$$

Thus, for $n = 1$ relation (1.35) holds. Suppose that

$$G_n - \delta \le \frac{1}{1-\sqrt{q}} \frac{\sqrt{\alpha_0 q^n}}{2} \|y\|, \quad 1 \le n \le k. \tag{1.39}$$

Then by inequalities (1.37) and (1.39) we obtain

$$\begin{aligned} G_{k+1} - \delta &\le q(G_k - \delta) + \sqrt{q} \frac{\sqrt{\alpha_0 q^k}}{2} \|y\| \\ &\le q \frac{1}{1-\sqrt{q}} \frac{\sqrt{\alpha_0 q^k}}{2} \|y\| + \sqrt{q} \frac{\sqrt{\alpha_0 q^k}}{2} \|y\| \\ &= \frac{\sqrt{q}}{1-\sqrt{q}} \frac{\sqrt{\alpha_0 q^k}}{2} \|y\| = \frac{\sqrt{q}}{1-\sqrt{q}} \frac{\sqrt{\alpha_0 q^k}}{2\sqrt{\alpha_0 q^{k+1}}} \sqrt{\alpha_0 q^{k+1}} \|y\| \\ &\le \frac{1}{1-\sqrt{q}} \frac{\sqrt{\alpha_0 q^{k+1}}}{2} \|y\|. \end{aligned} \tag{1.40}$$

Thus, relation (1.35) is proved. □

Lemma 1.2.5. *Let G_n be defined in (1.34), $q \in (0,1)$, and $\alpha_0 > 0$ be chosen such that $G_1 > C\delta^\varepsilon$, $\varepsilon \in (0,1)$, $C > 1$. Then there exists a unique integer n_c such that*

$$G_{n_c - 1} < G_{n_c} \text{ and } G_{n_c} > G_{n_c+1}, \quad n_c \ge 1. \tag{1.41}$$

Moreover,

$$G_{n+1} < G_n, \quad \forall n \ge n_c. \tag{1.42}$$

Proof. From Lemma 1.2.4 we have

$$G_n \le \frac{1}{1-\sqrt{q}} \sqrt{\alpha_0 q^n} \frac{\|y\|}{2} + \delta, \quad n \ge 1, \quad q \in (0,1).$$

Since $G_1 > C\delta^\varepsilon$ and $\limsup_{n\to\infty} G_n \le \delta < C\delta^\varepsilon$, it follows that there exists an integer $n_c \ge 1$ such that $G_{n_c-1} < G_{n_c}$ and $G_{n_c} > G_{n_c+1}$. Let us prove the monotonicity of G_n, for $n \ge n_c$. We have $G_{n_c+1} - G_{n_c} < 0$. Using definition (1.34), we get

$$\begin{aligned} G_{n_c+1} - G_{n_c} &= qG_{n_c} + (1-q)\alpha_0 q^{n_c+1} \|Q^{-1}_{\alpha_0 q^{n_c+1}} f_\delta\| - G_{n_c} \\ &= (1-q)\left(\alpha_0 q^{n_c+1} \|Q^{-1}_{\alpha_0 q^{n_c+1}} f_\delta\| - G_{n_c}\right) < 0. \end{aligned} \tag{1.43}$$

This implies

$$\alpha_0 q^{n_c+1}\|\alpha_0 Q^{-1}_{q^{n_c+1}} f_\delta\| - G_{n_c} < 0. \tag{1.44}$$

Note that

$$G_{n+1} - G_n = (1-q)(\alpha_0 q^{n+1}\|Q^{-1}_{\alpha_0 q^{n+1}} f_\delta\| - G_n).$$

Therefore, to prove the monotonicity of G_n for $n \geq n_c$, one needs to prove the inequality

$$\alpha_0 q^{n+1}\|Q^{-1}_{\alpha_0 q^{n+1}} f_\delta\| - G_n < 0, \quad \forall n \geq n_c.$$

This inequality is a consequence of the following lemma:

Lemma 1.2.6. *Let G_n be defined in (1.34), and (1.44) holds. Then*

$$\alpha_0 q^{n+1}\|Q^{-1}_{\alpha_0 q^{n+1}} f_\delta\| - G_n < 0, \quad \forall n \geq n_c. \tag{1.45}$$

Proof. Let us prove Lemma 1.2.6 by induction. Let

$$D_n := \alpha_0 q^{n+1}\|Q^{-1}_{\alpha_0 q^{n+1}} f_\delta\| - G_n$$

and

$$h(a) := a^2\|Q^{-1}_a f_\delta\|^2.$$

The function $h(a)$ is a monotonically growing function of a, $a > 0$. Indeed, by the spectral theorem, we get

$$\begin{aligned} h(a_1) = a_1^2\|Q^{-1}_{a_1} f_\delta\|^2 &= \int_0^\infty \frac{a_1^2}{(a_1+s)^2} d\langle F_s f_\delta, f_\delta\rangle \\ &\leq \int_0^\infty \frac{a_2^2}{(a_2+s)^2} d\langle F_s f_\delta, f_\delta\rangle = a_2^2\|Q^{-1}_{a_2} f_\delta\|^2 = h(a_2), \end{aligned} \tag{1.46}$$

where F_s is the resolution of the identity corresponding to $Q := AA^*$, because $\frac{a_1^2}{(a_1+s)^2} \leq \frac{a_2^2}{(a_2+s)^2}$ if $0 < a_1 < a_2$ and $s \geq 0$. By the assumption we have $D_{n_c} = \alpha_0 q^{n_c+1}\|Q^{-1}_{\alpha_0 q^{n_c+1}} f_\delta\| -$

$G_{n_c} < 0$. Thus, relation (1.45) holds for $n = n_c$. For $n = n_c + 1$ we get

$$\begin{aligned} D_{n_c+1} &= \alpha_0 q^{n_c+2} \|Q^{-1}_{\alpha_0 q^{n_c+2}} f_\delta\| - (1-q)\alpha_0 q^{n_c+1} \|Q^{-1}_{\alpha_0 q^{n_c+1}} f_\delta\| - qG_{n_c} \\ &= \sqrt{h(\alpha_0 q^{n_c+2})} - \sqrt{h(\alpha_0 q^{n_c+1})} + q\sqrt{h(\alpha_0 q^{n_c+1})} - qG_{n_c} \\ &= \sqrt{h(\alpha_0 q^{n_c+2})} - \sqrt{h(\alpha_0 q^{n_c+1})} + q(\sqrt{h(\alpha_0 q^{n_c+1})} - G_{n_c}) \\ &= \sqrt{h(\alpha_0 q^{n_c+2})} - \sqrt{h(\alpha_0 q^{n_c+1})} + qD_{n_c} \\ &= \sqrt{h(\alpha_0 q^{n_c+2})} - \sqrt{h(\alpha_0 q^{n_c+1})} + qD_{n_c} < 0. \end{aligned} \tag{1.47}$$

Here we have used the monotonicity of the function $h(a)$. Thus, relation (1.45) holds for $n = n_c + 1$. Suppose

$$D_n < 0, \quad n_c \le n \le n_c + k - 1.$$

This, together with the monotonically growth of the function $h(a) := a^2\|Q_q^{-1} f_\delta\|^2$, yields

$$\begin{aligned} D_{n_c+k} &= \alpha_0 q^{n_c+k+1} \|Q^{-1}_{\alpha_0 q^{n_c+k+1}} f_\delta\| - G_{n_c+k} \\ &= \sqrt{h(\alpha_0 q^{n_c+k+1})} - (1-q)\sqrt{h(\alpha_0 q^{n_c+k})} - qG_{n_c+k-1} \\ &= \sqrt{h(\alpha_0 q^{n_c+k+1})} - \sqrt{h(\alpha_0 q^{n_c+k})} + q(\sqrt{h(\alpha_0 q^{n_c+k})} - G_{n_c+k-1}) \\ &= \sqrt{h(\alpha_0 q^{n_c+k+1})} - \sqrt{h(\alpha_0 q^{n_c+k})} + qD_{n_c+k-1} \\ &= \sqrt{h(\alpha_0 q^{n_c+k+1})} - \sqrt{h(\alpha_0 q^{n_c+k})} + qD_{n_c+k-1} < 0. \end{aligned} \tag{1.48}$$

Thus, $D_n < 0, \; n \ge 1$. Lemma 1.2.6 is proved. □

Let us continue with the proof of Lemma 1.2.5. From relation (1.34) we have

$$\begin{aligned} G_{n+1} - G_n &= (q-1)G_n + (1-q)\alpha_0 q^{n+1} \|Q^{-1}_{\alpha_0 q^{n+1}} f_\delta\| \\ &= (1-q)\left(\alpha_0 q^{n+1} \|Q^{-1}_{\alpha_0 q^{n+1}} f_\delta\| - G_n\right). \end{aligned}$$

Using assumption (1.44) and applying Lemma 1.2.6, one gets

$$G_{n+1} - G_n < 0, \quad \forall n \ge n_c.$$

Let us prove that the integer n_c is unique. Suppose there exists another integer n_d such that $G_{n_d-1} < G_{n_d}$ and $G_{n_d} > G_{n_d+1}$. One may assume without loss of generality that $n_c < n_d$. Since $G_n > G_{n+1}, \; \forall n \ge n_c$, and $n_c < n_d$, it follows that $G_{n_d-1} > G_{n_d}$. This contradicts the assumption $G_{n_d-1} < G_{n_d}$. Thus, the integer n_c is unique. Lemma 1.2.5 is proved. □

Lemma 1.2.7. *Let G_n be defined in (1.34). If α_0 is chosen such that relations $G_1 > C\delta^\varepsilon$, $C > 1$, $\varepsilon \in (0,1)$, holds then there exists a unique n_δ satisfying inequality (1.33).*

Proof. Let us show that there exists an integer n_δ so that inequality (1.33) holds. Applying Lemma 1.2.4, one gets

$$\limsup_{n\to\infty} G_n \le \delta. \tag{1.49}$$

Since $G_1 > C\delta^\varepsilon$ and $\limsup_{n\to\infty} G_n \le \delta < C\delta^\varepsilon$, it follows that there exists an index n_δ satisfying stopping rule (1.33). The uniqueness of the index n_δ follows from the monotonicity of G_n, see Lemma 1.2.5. Thus, Lemma 1.2.7 is proved. □

Lemma 1.2.8. *Let $Ay = f$, $y \perp \mathcal{N}(A)$, and n_δ be chosen by rule (1.33). Then*

$$\lim_{\delta\to 0} q^{n_\delta} = 0, \quad q \in (0,1), \tag{1.50}$$

so

$$\lim_{\delta\to 0} n_\delta = \infty. \tag{1.51}$$

Proof. From rule (1.33) and relation (1.34) we have

$$\begin{aligned} qC\delta^\varepsilon + (1-q)\alpha_0 q^{n_\delta}\|Q^{-1}_{\alpha_0 q^{n_\delta}} f_\delta\| &< qG_{n_\delta-1} + (1-q)\alpha_0 q^{n_\delta}\|Q^{-1}_{\alpha_0 q^{n_\delta}} f_\delta\| \\ &= G_{n_\delta} \le C\delta^\varepsilon, \end{aligned} \tag{1.52}$$

so

$$(1-q)\alpha_0 q^{n_\delta}\|Q^{-1}_{\alpha_0 q^{n_\delta}} f_\delta\| \le (1-q)C\delta^\varepsilon. \tag{1.53}$$

Thus,

$$\alpha_0 q^{n_\delta}\|Q^{-1}_{\alpha_0 q^{n_\delta}} f_\delta\| < C\delta^\varepsilon. \tag{1.54}$$

Note that if $f \ne 0$ then there exists a $\lambda_0 > 0$ such that

$$F_{\lambda_0} f \ne 0, \quad \langle F_{\lambda_0} f, f\rangle := \xi > 0, \tag{1.55}$$

where ξ is a constant which does not depend on δ, and F_s is the resolution of the identity corresponding to the operator $Q := AA^*$. Let

$$h(\delta,\alpha) := \alpha^2 \|Q_\alpha^{-1} f_\delta\|^2.$$

For a fixed number $c_1 > 0$ we obtain

$$\begin{aligned} h(\delta, c_1) = c_1^2 \|Q_{c_1} f_\delta\|^2 = \int_0^\infty \frac{c_1^2}{(c_1+s)^2} d\langle F_s f_\delta, f_\delta\rangle \geq \int_0^{\lambda_0} \frac{c_1^2}{(c_1+s)^2} d\langle F_s f_\delta, f_\delta\rangle \\ \geq \frac{c_1^2}{(c_1+\lambda_0)^2} \int_0^{\lambda_0} d\langle F_s f_\delta, f_\delta\rangle = \frac{c_1^2 \|F_{\lambda_0} f_\delta\|^2}{(c_1+\lambda_0)^2}, \quad \delta > 0. \end{aligned} \tag{1.56}$$

Since F_{λ_0} is a continuous operator, and $\|f - f_\delta\| < \delta$, it follows from (1.55) that

$$\lim_{\delta\to 0} \langle F_{\lambda_0} f_\delta, f_\delta\rangle = \langle F_{\lambda_0} f, f\rangle > 0. \tag{1.57}$$

Therefore, for the fixed number $c_1 > 0$ we get

$$h(\delta, c_1) \geq c_2 > 0 \tag{1.58}$$

for all sufficiently small $\delta > 0$, where c_2 is a constant which does not depend on δ. For example one may take $c_2 = \frac{\xi}{2}$ provided that (1.55) holds. Let us derive from estimate (1.54) and the relation (1.58) that $q^{n_\delta} \to 0$ as $\delta \to 0$. From (1.54) we have

$$0 \leq h(\delta, \alpha_0 q^{n_\delta}) \leq (C\delta^\varepsilon)^2.$$

Therefore,

$$\lim_{\delta\to 0} h(\delta, \alpha_0 q^{n_\delta}) = 0. \tag{1.59}$$

Suppose $\lim_{\delta\to 0} q^{n_\delta} \neq 0$. Then there exists a subsequence $\delta_j \to 0$ such that

$$\alpha_0 q^{n_{\delta_j}} \geq c_1 > 0, \tag{1.60}$$

where c_1 is a constant. By (1.58) we get

$$h(\delta_j, \alpha_0 q^{n_{\delta_j}}) > c_2 > 0, \quad \delta_j \to 0 \text{ as } j \to \infty. \tag{1.61}$$

This contradicts relation (1.59). Thus, $\lim_{\delta\to 0} q^{n_\delta} = 0$. Lemma 1.2.8 is proved. □

Lemma 1.2.9. *Let n_δ be chosen by rule (1.33). Then*

$$\frac{\delta}{\sqrt{\alpha_0 q^{n_\delta}}} \to 0 \text{ as } \delta \to 0. \tag{1.62}$$

Proof. Relation (1.35), together with stopping rule (1.33), implies

$$C\delta^{\varepsilon} < G_{n_\delta -1} \le \frac{1}{1-\sqrt{q}}\frac{\sqrt{\alpha_0 q^{n_\delta -1}}}{2}\|y\| + \delta. \tag{1.63}$$

Then

$$\frac{1}{\sqrt{\alpha_0 q^{n_\delta -1}}} \le \frac{\|y\|}{2(1-\sqrt{q})\delta^{\varepsilon}(C-1)}, \quad \varepsilon \in (0,1). \tag{1.64}$$

This yields

$$\lim_{\delta\to 0}\frac{\delta}{\sqrt{\alpha_0 q^{n_\delta}}} = \lim_{\delta\to 0}\frac{\delta}{\sqrt{\alpha_0 q q^{n_\delta -1}}} \le \lim_{\delta\to 0}\frac{\delta^{1-\varepsilon}}{2\sqrt{q}(1-\sqrt{q})(C-1)}\|y\| = 0. \tag{1.65}$$

Lemma 1.2.9 is proved. □

Theorem 1.2.10. *Let $y \perp \mathcal{N}(A)$, $\|f_\delta - f\| \le \delta$, $\|f_\delta\| > C\delta^{\varepsilon}$, $C > 1$, $\varepsilon \in (0,1)$. Suppose n_δ is chosen by rule (1.33). Then*

$$\lim_{\delta\to 0}\|u_{n_\delta}^{\delta} - y\| = 0, \tag{1.66}$$

where u_n^δ is given in (1.13).

Proof. Using Lemma 1.2.2 and Lemma 1.2.3, we get the estimate

$$\begin{aligned}\|u_{n_\delta}^{\delta} - y\| &\le \|u_{n_\delta}^{\delta} - u_{n_\delta}\| + \|u_{n_\delta} - y\| \le \frac{\sqrt{q}}{1-q^{3/2}}(1-q)\frac{\delta}{2q\sqrt{\alpha_0 q^{n_\delta}}} + \|u_{n_\delta} - y\| \\ &:= I_1 + I_2,\end{aligned} \tag{1.67}$$

where $I_1 := \frac{\sqrt{q}}{1-q^{3/2}}(1-q)\frac{\delta}{2q\sqrt{\alpha_0 q^{n_\delta}}}$ and $I_2 := \|u_{n_\delta} - y\|$. Applying Lemma 1.2.9, one gets $\lim_{\delta\to 0} I_1 = 0$. Since $n_\delta \to \infty$ as $\delta \to 0$, it follows from Lemma 1.2.2 that $\lim_{\delta\to 0} I_2 = 0$. Thus, $\lim_{\delta\to 0}\|u_{n_\delta}^{\delta} - y\| = 0$. Theorem 1.2.10 is proved. □

The algorithm based on the proposed method can be stated as follows:

Step 1. Assume that (1.14) holds. Choose $C \in (1,2)$ and $\varepsilon \in (0.9,1)$. Fix $q \in (0,1)$, and choose $\alpha_0 > 0$ so that (1.17) holds. Set $n = 1$, and $u_0 = 0$.

Step 2. Use iterative scheme (1.13) to calculate u_n.

Step 3. Calculate G_n, where G_n is defined in (1.34).

Step 4. If $G_n \le C\delta^{\varepsilon}$ then stop the iteration, set $n_\delta = n$, and take $u_{n_\delta}^{\delta}$ as the approximate solution. Otherwise set $n = n + 1$, and go to Step 1.

1.3 Iterative scheme 2

In[32] the following iterative scheme for the exact data f is given:

$$u_{n+1} = aT_a^{-1}u_n + T_a^{-1}A^*f, \quad u_1 = u_1 \perp \mathcal{N}(A), \tag{1.68}$$

where a is a fixed positive constant. It is proved in[32] that iterative scheme (1.68) gives the relation

$$\lim_{n\to\infty} \|u_n - y\| = 0, \quad y \perp \mathcal{N}(A).$$

In the case of noisy data the exact data f in (1.68) is replaced with the noisy data f_δ, i.e.

$$u_{n+1}^\delta = aT_a^{-1}u_n^\delta + T_a^{-1}A^*f_\delta, \quad u_1 = u_1 \perp \mathcal{N}(A), \tag{1.69}$$

where $\|f_\delta - f\| \le \delta$ for sufficiently small $\delta > 0$. It is proved in[32] that there exist an integer n_δ such that

$$\lim_{\delta\to 0} \|u_{n_\delta}^\delta - y\| = 0, \tag{1.70}$$

where u_n^δ is the approximate solution corresponds to the noisy data. But a method of choosing the integer n_δ has not been discussed. In this section we modify iterative scheme (1.68) by replacing the constant parameter a in (1.68) with a geometric sequence $\{q^{n-1}\}_{n=1}^\infty$, i.e.

$$u_{n+1} = q^nT_{q^n}^{-1}u_n + T_{q^n}^{-1}A^*f, \quad u_1 = 0, \tag{1.71}$$

where $q \in (0,1)$. The initial approximation u_1 is chosen to be 0. In general one may choose an arbitrary initial approximation u_1 in the set $\mathcal{N}(A)^\perp$. If the data are noisy then the exact data f in (1.71) is replaced with the noisy data f_δ, and iterative scheme (1.69) is replaced with:

$$u_{n+1}^\delta = q^nT_{q^n}^{-1}u_n^\delta + T_{q^n}^{-1}A^*f_\delta, \quad u_1^\delta = 0. \tag{1.72}$$

We prove convergence of the solution obtained by iterative scheme (1.71) in Theorem 1.3.1 for arbitrary $q \in (0,1)$, i.e.

$$\lim_{n\to\infty} \|u_n - y\| = 0, \quad \forall q \in (0,1).$$

In the case of noisy data we use discrepancy-type principle (1.85) to obtain the integer n_δ such that

$$\lim_{\delta\to 0} \|u^\delta_{n_\delta} - y\| = 0. \tag{1.73}$$

We prove relation (1.73), for arbitrary $q \in (0,1)$, in Theorem 1.3.6.

Let us prove that the sequence u_n, defined by iterative scheme (1.71), converges to the minimal norm solution y of equation (1.1).

Theorem 1.3.1. *Consider iterative scheme* (1.71)*. Let $y \perp \mathcal{N}(A)$. Then*

$$\lim_{n\to\infty} \|u_n - y\| = 0. \tag{1.74}$$

Proof. Consider the identity

$$y = aT_a^{-1}y + T_a^{-1}A^*f, \quad Ay = f. \tag{1.75}$$

Let $w_n := u_n - y$ and $B_n := q^n T^{-1}_{q^n}$. Then $w_{n+1} = B_n w_n, \quad w_1 = y - u_1 = y$. One uses (1.75) and gets

$$\begin{aligned}
\|y-u_n\|^2 &= \|B_{n-1}B_{n-2}\dots B_1 w_1\|^2 = \|B_{n-1}B_{n-2}\dots B_1 y\|^2\\
&= \int_0^\infty \left(\frac{q^{n-1}}{q^{n-1}+s}\frac{q^{n-2}}{q^{n-2}+s}\cdots\frac{q}{q+s}\right)^2 d\langle E_s y, y\rangle\\
&= \int_0^\infty \left(\frac{q^{n-1}}{q^{n-1}+s}\right)^2\left(\frac{q^{n-2}}{q^{n-2}+s}\right)^2\cdots\left(\frac{q}{q+s}\right)^2 d\langle E_s y, y\rangle\\
&\le \int_0^\infty \frac{q^{2n}}{(q+s)^{2n}} d\langle E_s y, y\rangle,
\end{aligned} \tag{1.76}$$

where E_s is the resolution of the identity corresponding to the operator $T := A^*A$. Here we have used the identity (1.75) and the monotonicity of the function $\phi(x) := \frac{x^2}{(x+s)^2}, \ s \ge 0$. From estimate (1.76) we derive relation (1.74). Indeed, write

$$\int_0^\infty \frac{q^{2n}}{(q+s)^{2n}} d\langle E_s y, y\rangle = \int_0^b \frac{q^{2n}}{(q+s)^{2n}} d\langle E_s y, y\rangle + \int_b^\infty \frac{q^{2n}}{(q+s)^{2n}} d\langle E_s y, y\rangle, \tag{1.77}$$

where b is a sufficiently small number which will be chosen later. For any fixed $b > 0$ one has $\frac{q}{q+s} \le \frac{q}{q+b} < 1$ if $s \ge b$. Therefore, it follows that

$$\int_b^\infty \frac{q^{2n}}{(q+s)^{2n}} d\langle E_s y, y\rangle \to 0 \text{ as } n \to \infty. \tag{1.78}$$

On the other hand one has

$$\int_0^b \frac{q^{2n}}{(q+s)^{2n}} d\langle E_s y, y\rangle \leq \int_0^b d\langle E_s y, y\rangle. \tag{1.79}$$

Since $y \perp \mathcal{N}(A)$, one has $\lim_{b\to 0}\int_0^b d\langle E_s y, y\rangle = 0$. Therefore, given an arbitrary number $\epsilon > 0$ one can choose $b(\epsilon)$ such that

$$\int_0^{b(\epsilon)} \frac{q^{2n}}{(q+s)^{2n}} d\langle E_s y, y\rangle < \frac{\epsilon}{2}. \tag{1.80}$$

Using this $b(\epsilon)$, one chooses sufficiently large $n(\epsilon)$ such that

$$\int_{b(\epsilon)}^{\infty} \frac{q^{2n}}{(q+s)^{2n}} d\langle E_s y, y\rangle < \frac{\epsilon}{2}, \quad \forall n > n(\epsilon). \tag{1.81}$$

Since $\epsilon > 0$ is arbitrary, Theorem 1.3.1 is proved. $\square$

As we mentioned before if the exact data f are contaminated by some noise then iterative scheme (1.72) is used, where $\|f_\delta - f\| \leq \delta$. Note that

$$\|u_{n+1}^\delta - u_{n+1}\| \leq q^n \|T_{q^n}^{-1}(u_n^\delta - u_n)\| + \frac{\delta}{2\sqrt{q^n}} \leq \|u_n^\delta - u_n\| + \frac{\delta}{2\sqrt{q^n}}. \tag{1.82}$$

To prove the convergence of the solution obtained by iterative scheme (1.72), we need the following lemmas:

Lemma 1.3.2. *Let u_n and u_n^δ be defined in (1.71) and (1.72), respectively. Then*

$$\|u_n^\delta - u_n\| \leq \frac{\sqrt{q}}{1-\sqrt{q}} \frac{\delta}{2\sqrt{q^n}}, \quad n \geq 1. \tag{1.83}$$

Proof. Let us prove relation (1.83) by induction. For $n = 1$ one has $u_1^\delta - u_1 = 0$. Thus, for $n = 1$ the relation holds. Suppose

$$\|u_l^\delta - u_l\| \leq \frac{\sqrt{q}}{1-\sqrt{q}} \frac{\delta}{2\sqrt{q^l}}, \quad 1 \leq l \leq k. \tag{1.84}$$

Then from (1.82) and (1.84) we have

$$\begin{aligned}\|u_{k+1}^\delta - u_{k+1}\| &\leq \|u_k^\delta - u_k\| + \frac{\delta}{2\sqrt{q^k}} \leq \frac{\sqrt{q}}{1-\sqrt{q}} \frac{\delta}{2\sqrt{q^k}} + \frac{\delta}{2\sqrt{q^k}} \\ &\leq \sqrt{q} \frac{\delta}{(1-\sqrt{q})2\sqrt{q^{k+1}}}.\end{aligned}$$

Thus,

$$\|u_n^\delta - u_n\| \leq \sqrt{q}\frac{\delta}{(1-\sqrt{q})2\sqrt{q^n}}, \quad n \geq 1.$$

Lemma 1.3.2 is proved. $\square$

Let us formulate our stopping rule: the iteration in iterative scheme (1.72) is stopped at the first integer n_δ satisfying

$$\|AT_{q^{n_\delta}}^{-1}A^*f_\delta - f_\delta\| \leq C\delta^\varepsilon < \|AT_{q^n}^{-1}A^*f_\delta - f_\delta\|, \quad 1 \leq n < n_\delta,\ C > 1,\ \varepsilon \in (0,1), \tag{1.85}$$

and it is assumed that $\|f_\delta\| > C\delta^\varepsilon$.

Lemma 1.3.3. *Let u_n^δ be defined in (1.72), and $W_n := \|AT_{q^n}^{-1}A^*f_\delta - f_\delta\|$. Then*

$$W_{n+1} \leq W_n, \quad n \geq 1. \tag{1.86}$$

Proof. Note that

$$W_n = \|AA^*Q_{q^n}^{-1}f_\delta - f_\delta\| = \|(Q_{q^n} - q^nI_m)Q_{q^n}^{-1}f_\delta - f_\delta\| = \|q^nQ_{q^n}^{-1}f_\delta\|, \tag{1.87}$$

where $Q := AA^*$, and $Q_a := Q + aI_m$. Using the spectral theorem, one gets

$$W_{n+1}^2 = \int_0^\infty \frac{q^{2(n+1)}}{(q^{n+1}+s)^2}d\langle F_sf_\delta, f_\delta\rangle \leq \int_0^\infty \frac{q^{2n}}{(q^n+s)^2}d\langle F_sf_\delta, f_\delta\rangle = W_n^2, \tag{1.88}$$

where F_s is the resolution of the identity corresponding to the operator $Q := AA^*$. Here we have used the monotonicity of the function $g(x) = \frac{x^2}{(x+s)^2}, \quad s \geq 0$. Thus,

$$W_{n+1} \leq W_n, \quad n \geq 1. \tag{1.89}$$

Lemma 1.3.3 is proved. $\square$

Lemma 1.3.4. *Let u_n^δ be defined in (1.72), and $\|f_\delta\| > C\delta^\varepsilon$, $\varepsilon \in (0,1)$, $C > 1$. Then there exists a unique index n_δ such that inequality (1.85) holds.*

Proof. Let $e_n := AT_{q^n}^{-1}A^*f_\delta - f_\delta$. Then

$$e_n = q^n Q_{q^n}^{-1} f_\delta, \tag{1.90}$$

where $Q_a := AA^* + aI$. Therefore,

$$\begin{aligned} \|e_n\| &\le \|q^n Q_{q^n}^{-1}(f_\delta - f)\| + \|q^n Q_{q^n}^{-1} f\| \\ &\le \|f_\delta - f\| + \|q^n Q_{q^n}^{-1} Ay\| \le \delta + \frac{\sqrt{q^n}}{2}\|y\|, \end{aligned} \tag{1.91}$$

where the estimate $\|Q_a^{-1}A\| = \|AT_a^{-1}\| \le \frac{1}{2\sqrt{a}}$ was used. Thus,

$$\limsup_{n\to\infty} \|e_n\| \le \delta.$$

This shows that the integer n_δ, satisfying (1.85), exists. The uniqueness of n_δ follows from its definition. □

Lemma 1.3.5. *Let u_n^δ be defined in (1.72). If n_δ is chosen by rule (1.85) then*

$$\lim_{\delta\to 0} \frac{\delta}{\sqrt{q^{n_\delta}}} = 0. \tag{1.92}$$

Proof. From (1.91) we have

$$\|AT_{q^{n-1}}^{-1}A^*f_\delta - f_\delta\| \le \delta + \frac{\sqrt{q^{n-1}}}{2}\|e_1\|, \tag{1.93}$$

where $e_1 := u_1 - y = -y$. It follows from stopping rule (1.85) and estimate (1.93) that

$$C\delta^\varepsilon \le \|AT_{q^{n_\delta-1}}^{-1}A^*f_\delta - f_\delta\| \le \frac{\sqrt{q^{n_\delta-1}}}{2}\|e_1\| + \delta. \tag{1.94}$$

Therefore,

$$(C-1)\,\delta^\varepsilon \le \frac{\sqrt{q^{n_\delta-1}}}{2}\|e_1\|, \tag{1.95}$$

and so

$$\frac{1}{\sqrt{q^{n_\delta-1}}} \le \frac{\|e_1\|}{2(C-1)\delta^\varepsilon}, \quad \varepsilon \in (0,1). \tag{1.96}$$

This implies

$$\frac{\delta}{\sqrt{q^{n_\delta}}} = \frac{\delta}{\sqrt{qq^{n_\delta-1}}} \le \frac{\|e_1\|\delta}{2q^{1/2}(C-1)\delta^\varepsilon} = \frac{\|e_1\|}{2q^{1/2}(C-1)}\delta^{1-\varepsilon}. \tag{1.97}$$

Thus, $\frac{\delta}{\sqrt{q^{n_\delta}}} \to 0$ as $\delta \to 0$. Lemma 1.3.5 is proved. □

The proof of convergence of the solution obtained by iterative scheme (1.72) is given in the following theorem:

Theorem 1.3.6. *Let u_n^δ be defined in (1.72), $y \perp \mathcal{N}(A)$, $\|f_\delta\| > C\delta^\varepsilon$, $\varepsilon \in (0,1)$, $C > 1$, $\quad q \in (0,1)$. If n_δ is chosen by rule (1.85), then*

$$\|u_n^\delta - y\| \to 0 \text{ as } \delta \to 0. \tag{1.98}$$

Proof. From Lemma 1.3.2 we get the following estimate:

$$\|u_{n_\delta}^\delta - y\| \leq \|u_{n_\delta}^\delta - u_{n_\delta}\| + \|u_{n_\delta} - y\| \leq \frac{\sqrt{q}}{1-\sqrt{q}} \frac{\delta}{2\sqrt{q^{n_\delta}}} + \|u_{n_\delta} - y\| := I_1 + I_2, \tag{1.99}$$

where $I_1 := \frac{\sqrt{q}}{1-\sqrt{q}} \frac{\delta}{2\sqrt{q^{n_\delta}}}$ and $I_2 := \|u_{n_\delta} - y\|$. By Lemma 1.3.5 one gets $I_1 \to 0$ as $\delta \to 0$. To prove $\lim_{\delta\to 0} I_2 = 0$ one needs the relation $\lim_{\delta\to 0} n_\delta = \infty$. This relation is a consequence of the following lemma:

Lemma 1.3.7. *If n_δ is chosen by rule (1.85), then*

$$q^{n_\delta} \to 0 \text{ as } \delta \to 0, \tag{1.100}$$

so

$$\lim_{\delta\to 0} n_\delta = \infty. \tag{1.101}$$

Proof. Note that

$$\begin{aligned} AT_a^{-1}A^* f_\delta - f_\delta &= AA^*Q_a^{-1} f_\delta - f_\delta = (AA^* + aI_m - aI_m)Q_a^{-1} f_\delta - f_\delta \\ &= f_\delta - aQ_a^{-1} f_\delta - f_\delta = -aQ_a^{-1} f_\delta, \end{aligned}$$

where $a > 0$, $Q := AA^*$ and $Q_a := Q + aI$. From stopping rule (1.85) we have $0 \leq \|AT_{q^{n_\delta}}^{-1} A^* f_\delta - f_\delta\| \leq C\delta^\varepsilon$. Thus,

$$\lim_{\delta\to 0} \|AT_{q^{n_\delta}}^{-1} A^* f_\delta - f_\delta\| = \lim_{\delta\to 0} \|q^{n_\delta} Q_{q^{n_\delta}}^{-1} f_\delta\| = 0. \tag{1.102}$$

Using an argument given in the proof of Lemma 1.2.8, (see formulas (1.54)-(1.61) in which $\alpha_0 = 1$), one gets $\lim_{\delta\to 0} q^{n_\delta} = 0$, so $\lim_{\delta\to 0} n_\delta = \infty$. Lemma 1.3.7 is proved. □

Lemma 1.3.7 and Theorem 1.3.1 imply $I_2 \to 0$ as $\delta \to 0$. Thus, Theorem 1.3.6 is proved. □

1.4 Numerical experiments

In all the experiments we measure the accuracy of the approximate solutions using the relative error:

$$\text{Rel.Err} = \frac{\|u^\delta_{n_\delta} - y\|}{\|y\|},$$

where $\|.\|$ is the Euclidean norm in $\mathbb{R}^n$. The exact data are perturbed by some noises so that

$$\|f_\delta - f\| \leq \delta,$$

where

$$f_\delta = f + \delta \frac{e}{\|e\|},$$

δ is the noise level, and $e \in \mathbb{R}^n$ is the noise taken from the Gaussian distribution with mean 0 and standard deviation 1. The MATLAB routine called "randn" with seed 15 is used to generate the vector e. The iterative schemes (1.13) and (1.72) will be denoted by IS_1 and IS_2, respectively. In the iterative scheme IS_1, for fixed $q \in (0, 1)$, one needs to choose a sufficiently large $\alpha_0 > 0$ so that inequality (1.17) hold, for example one may choose $\alpha_0 \geq 1$. The number of iterations of IS_1 and IS_2 are denoted by $Iter_1$ and $Iter_2$, respectively. We compare the results obtained by the proposed methods with the results obtained by using the variational regularization method (VR). In VR we use the Newton method for solving the equation for regularization parameter. In[23] the nonlinear equation

$$\|Au_{VR}(a) - f_\delta\|^2 = (C\delta)^2, \ C = 1.01, \tag{1.103}$$

where $u_{VR}(a) := T_a^{-1} A^* f_\delta$, is solved by the Newton's method. In this thesis the initial value of the regularization parameter is taken to be $\alpha_0 = \frac{\alpha_0}{2^{k_\delta}}$, where k_δ is the first integer such that the Newton's method for solving (1.103) converges. We stop the iteration of the Newton's method at the first integer n_δ satisfying the inequality $|\|AT_{a_n}^{-1}A^* f_\delta - f_\delta\|^2 - (C\delta)^2| \leq 10^{-3}(C\delta)^2, \quad a_0 := \alpha_0$. The number of iterations needed to complete a convergent Newton's method is denoted by $Iter_{VR}$.

1.4.1 Ill-conditioned linear algebraic systems

Table 1.1: *Condition number of some Hilbert matrices*

n	$\kappa(A) = \|A\|\|A^{-1}\|$
10	1.915×10^{13}
20	1.483×10^{28}
70	8.808×10^{105}
100	1.262×10^{150}
200	1.446×10^{303}

Consider the following system:

$$H^{(m)}u = f, \tag{1.104}$$

where

$$H_{ij}^{(m)} = \frac{1}{i+j+1}, \quad i, j = 1, 2, ..., m,$$

is a Hilbert matrix of dimension m. The system (1.104) is an example of a severely ill-posed problem if $m > 10$, because the condition number of the Hilbert matrix is increasing exponentially as m grows, see Table (1.1). The minimal eigenvalues of Hilbert matrix of dimension m can be obtained using the following formula

$$\lambda_{\min}(H^{(m)}) = 2^{15/4}\pi^{3/2}\sqrt{m}(\sqrt{2}+1)^{-(4m+4)}(1+o(1)). \tag{1.105}$$

This formula is proved in[20]. Since $\kappa(H^{(m)}) = \frac{\lambda_{max}(H^{(m)})}{\lambda_{min}(H^{(m)})}$, it follows from (1.105) that the condition number grows as $O(\frac{e^{3.5255m}}{\sqrt{m}})$. The following exact solution is used to test the proposed methods:

$$y \in \mathbb{R}^m, \text{ where } y_k = \sqrt{.5k}, \; k = 1, 2, \ldots, m.$$

The Hilbert matrix of dimension $m = 200$ is used in the experiments. This matrix has condition number of order 10^{303}, so it is a severely ill-conditioned matrix. In Table 2 one can see that the number of iterations of the iterative scheme IS_1 and IS_2 increases as the value of q increases. The relative errors start to increase at $q = .125$. By these

Table 1.2: *Hilbert matrix problem: the number of iterations and the relative errors with respect to the parameter q ($\alpha_0 = 1, \quad \delta = 10^{-2}$).*

q	IS_1		IS_2	
	REl.Err	$Iter_1$	REl.Err	$Iter_2$.
.5	0.031	24	0.032	23
.25	0.031	13	0.032	13
.125	0.032	9	0.032	9

observations, we suggest to choose the parameter q in the interval $(.125, .5)$. In Table 3 the results of the experiments with various values of δ are presented. Here the parameter ε

Table 1.3: *ICLAS with Hilbert matrix: the relative errors and the number of iterations*

δ	IS_1		IS_2		VR	
	REl.Err	$Iter_1$	REl.Err	$Iter_2$	REl.Err	$Iter_{VR}$.
5%	0.038	11	0.043	11	0.055	13
3%	0.037	12	0.034	12	0.045	14
1%	0.031	13	0.032	13	0.034	15

was .99. The geometric sequence $\{.25^{n-1}\}_{n=1}^{\infty}$ was used in the iterative schemes IS_1 and IS_2. The parameter C in (1.16) and (1.85) were 1.01. The parameter k_δ in the variational regularization method was 1. One can see that the relative errors of IS_1 and IS_2 are smaller than these for the VR. The relative error decreases as the noise level decreases which can be seen on the same table. This shows that the proposed method produces stable solutions.

1.4.2 Fredholm integral equations of the first kind (FIEFK)

Here we consider two Fredholm integral equations :

a)

$$f(s) = \int_{-3}^{3} k(t-s)u(t)dt, \tag{1.106}$$

where

$$k(z) = \begin{cases} 1 + \cos(\frac{\pi}{3}z), & |z| < 3; \\ 0, & |z| \geq 3, \end{cases} \tag{1.107}$$

and

$$f(z) = \begin{cases} (6+z)\left[1 - \frac{1}{2}\cos(\frac{\pi}{3}z)\right] - \frac{9}{2\pi}\sin(\frac{\pi z}{3}), & |z| \leq 6; \\ 0, & |z| > 6. \end{cases} \tag{1.108}$$

b)

$$f(s) = \int_0^1 k(s,t)u(t)dt, s \in (0,1), \tag{1.109}$$

where

$$k(s,t) = \begin{cases} s(t-1), & s < t; \\ t(s-1), & s \geq t, \end{cases} \tag{1.110}$$

and

$$f(s) = (s^3 - s)/6. \tag{1.111}$$

The problem $a)$ is discussed in[25] where the solution to this problem is $u(x) = k(x)$. The second problem is taken from[7] where the solution is $u(x) = x$. The Galerkin's method is used to discretized the integrals (1.106) and (1.109). For the basis functions we use the following orthonormal box functions

$$\phi_i(s) := \begin{cases} \sqrt{\frac{m}{c_1}}, & [s_{i-1}, s_i]\ ; \\ 0, & \text{otherwise}, \end{cases} \tag{1.112}$$

and

$$\psi_i(t) := \begin{cases} \sqrt{\frac{m}{c_2}}, & [t_{i-1}, t_i]\ ; \\ 0, & \text{otherwise}, \end{cases} \tag{1.113}$$

where $s_i = d_1 + i\frac{d_2}{m}$, $t_i = d_3 + i\frac{d_4}{m}$, $i = 0, 1, 2, \ldots, m$. In the problem $a)$ the parameters c_1, c_2, d_1, d_2, d_3 and d_4 are set to 12, 6, -6, 12, -3 and 6, respectively. In the second problem we use $d_1 = d_3 = 0$ and $c_1 = c_2 = d_2 = d_4 = 1$. Here we approximate the solution $u(t)$ by $\tilde{u} = \sum_{j=1}^{m} c_j \psi_j(t)$. Therefore solving problem (1.106) is reduced to solving the linear algebraic system

$$A\tilde{c} = f,\ \tilde{c}, f \in \mathbb{R}^m, \tag{1.114}$$

where in problem $a)$

$$A_{ij} = \int_{-3}^{3}\int_{-6}^{6} k(t-s)\phi_i(s)\psi_j(t)dsdt$$

and $f_i = \int_{-6}^{6} f(s)\phi_i(s)ds,\ i,j = 1,2,\ldots,m$, and in problem $b)$

$$A_{ij} = \int_{0}^{1}\int_{0}^{1} k(s,t)\phi_i(s)\psi_j(t)dsdt$$

and $f_i = \int_0^1 f(s)\phi_i(s)ds$, and $\tilde{c}_j = c_j\ i,j = 1,2,\ldots,m$.

Table 1.4: *Problem a): the number of iterations and the relative errors with respect to the parameter q* $(\alpha_0 = 2, \quad \delta = 10^{-2})$.

q	IS_1		IS_2	
	REl.Err	$Iter_1$	REl.Err	$Iter_2$.
.5	0.008	12	0.007	11
.25	0.009	7	0.007	7
.125	0.009	5	0.008	5

Table 1.5: *Problem a): the relative errors and the number of iterations*

δ	IS_1		IS_2		VR	
	REl.Err	$Iter_1$	REl.Err	$Iter_2$	REl.Err	$Iter_{VR}$.
5%	0.018	6	0.014	6	0.016	11
3%	0.013	6	0.011	6	0.013	12
1%	0.009	7	0.007	7	0.008	15

The parameter $m = 600$ is used in problem $a)$. In this case the condition number of the matrix A with $m = 600$ is 3.427×10^9, so it is an ill-conditioned matrix. Here the parameter C in IS_1 and IS_2 are 2 and 1.01, respectively. For problem $b)$ the parameter m is 200. In this case the condition number of the matrix A is 4.863×10^4. The parameter C is 1.01 in the both iterative schemes IS_1 and IS_2. In Tables 4 and 6 we give the relation between the parameter q and the number of iterations and the relative errors of the iterative schemes IS_1 and IS_2. The closer the parameter q to 1, the larger number of iterations we get, and

Table 1.6: *Problem b): the number of iterations and the relative errors with respect to the parameter q* ($\alpha_0 = 4, \quad \delta = 10^{-2}$).

q	IS_1		IS_2	
	REl.Err	$Iter_1$	REl.Err	$Iter_2$.
.5	0.428	17	0.446	15
.25	0.421	9	0.436	9
.125	0.439	6	0.416	7

Table 1.7: *Problem b): the relative errors and the number of iterations*

δ	IS_1		IS_2		VR	
	REl.Err	$Iter_1$	REl.Err	$Iter_2$	REl.Err	$Iter_{VR}$.
5%	0.618	7	0.621	7	0.627	12
3%	0.541	8	0.559	8	0.584	13
1%	0.421	9	0.436	9	0.457	13

the closer the parameter q to 0, the smaller the number of iterations we get. But the relative error starts to increase if the parameter q is chosen too small. Based on the numerical results given in Tables 4 and 5, we suggest to choose the parameter q in the interval $(0.125, 0.5)$. In the iterative schemes IS_1 and IS_2 we use the geometric sequence $\{2 \times .25^{n-1}\}_{n=1}^{\infty}$ for problem a). The geometric series $\{4 \times .25^{n-1}\}_{n=1}^{\infty}$ is used in problem b). In the variational regularization method we use $\alpha_0 = 2$ and $\alpha_0 = 4$ as the initial regularization parameter of the Newton's method in problem a) and b), respectively. Since the Newton's method for solving (1.103) is locally convergent, in problem b) we need to choose a smaller regularization parameter α_0 than for IS_1 and IS_2 methods. Here $k_\delta = 8$ was used. The numerical results on Table 5 show that the solutions produced by the proposed iterative schemes are stable. In problem a) the relative errors of the iterative scheme IS_2, are smaller than these for the iterative scheme IS_1 and than these for the variational regularization, VR. In Table 7 the relative errors produced by the three methods for solving problem b) are presented. The relative error of IS_1 is smaller than the one for the other two methods.

1.5 Conclusion

We have demonstrated that the proposed iterative schemes can be used for solving ill-conditioned linear algebraic systems stably. The advantage of the iterative scheme (1.13) compared with iterative scheme (1.72) is the following: one applies the operator T_a^{-1} only once at each iteration. Note that the difficulty of using the Newton's method is in choosing the initial value for the regularization parameter, since the Newton's method for solving equation (1.103) converges only locally. In solving (1.103) by the Newton's method one often has to choose an initial regularization parameter a_0 sufficiently close to the root of equation (1.103) as shown in problem b) in Section 4.2. In our iterative schemes the initial regularization parameter can be chosen in the interval $[1, 4]$ which is larger than the initial regularization parameter used in the variational regularization method. In the iterative scheme IS_1 we modified the discrepancy-type principle

$$\int_0^t e^{-(t-s)} a(s) \|Q_{a(s)}^{-1}\| ds = C\delta, \ C \in (1, 2),$$

given in[28,33], by using (1.10) to get discrepancy-type principle (1.33), which can be easily implemented numerically. In Section 3 we used the geometric series $\{\alpha_0 q^n\}_{n=1}^{\infty}$ in place of the constant regularization parameter a in the iterative scheme

$$u_{n+1} = aT_a^{-1}u_n + T_a^{-1}A^* f_\delta$$

developed in[32]. This geometric series of the regularization parameter allows one to use the a posteriori stopping rule given in (1.85). We proved that this stopping rule produces stable approximation of the minimal norm solution of equation (1.1). In all the experiments stopping rules (1.33) and (1.85) produce stable approximations to the minimal norm solution of equation (1.1). It is of interest to develop a method for choosing the parameter q in the proposed methods which gives sufficiently small relative error and small number of iterations.

Chapter 2

An iterative method for solving Fredholm integral equations of the first kind

2.1 Introduction

We consider a linear operator

$$(Ku)(x) := \int_a^b k(x,z)u(z)dz = f(x), \quad a \le x \le b, \tag{2.1}$$

where $K : L^2[a,b] \to L^2[a,b]$ is a linear compact operator. We assume that $k(x,z)$ is a smooth function on $[a,b]\times[a,b]$. Since K is compact, the problem of solving equation (2.1) is ill-posed. Some applications of the Fredholm integral equations of the first kind can be found in[19],[29],[30]. There are many methods for solving equation (2.1): variational regularization, quasi-solution, iterative regularization, the Dynamical Systems Method (DSM). A detailed description of these methods can be found in[23],[29],[30]. In this Chapter we propose an iterative scheme for solving equation (2.1) based on the DSM. We refer the reader to[29] and[30] for a detailed discussion of the DSM. When we are trying to solve (2.1) numerically, we need to carry out all the computations with finite-dimensional approximation K_m of the operator K, $\lim_{m\to\infty}\|K_m - K\| = 0$. One approximates a solution to (2.1) by a linear combination of basis functions $v_m(x) := \sum_{i=1}^m \zeta_j^{(m)}\phi_j(x)$, where $\zeta_j^{(m)}$ are constants, and $\phi_i(x)$ are orthonormal basis functions in $L^2[0,1]$. Here the constants $\zeta_j^{(m)}$ can be obtained by solving the ill-conditioned linear algebraic system:

$$\sum_{j=1}^{m}(K_m)_{ij}\zeta_j^{(m)} = g_i, \quad i = 1, 2, \dots, m, \tag{2.2}$$

where $(K_m)_{ij} := \int_a^b \int_a^b k(x,s)\phi_j(s)ds\overline{\phi_i(x)}dx$, $1 \le i, j \le m$, and $g_i := \int_a^b f(x)\overline{\phi_i(x)}dx$. In applications, the exact data f may not be available, but noisy data f_δ, $\|f_\delta - f\| \le \delta$, are available. Therefore, one needs a regularization method to solve stably equation (2.2) with the noisy data $g_i^\delta := \int_a^b f_\delta(x)\overline{\phi_i(x)}dx$ in place of g_i. In the variational regularization (VR) method for a fixed regularization parameter $a > 0$ one obtains the coefficients $\zeta_j^{(\delta,m)}$ by solving the linear algebraic system:

$$a\zeta_i^{(\delta,m)} + \sum_{j=1}^{m}(K_m^*K_m)_{ij}\zeta_j^{(\delta,m)} = g_i^\delta, \quad i = 1, 2, \dots, m, \tag{2.3}$$

where

$$(K_m^*K_m)_{ij} := \int_a^b \int_a^b \overline{k(s,x)\phi_i(x)} \int_a^b k(s,z)\phi_j(z)dzdsdx,$$

$\|f - f_\delta\| \le \delta$, and $\overline{k(s,x)}$ is the complex conjugate of $k(s,x)$. In the VR method one has to choose the regularization parameter a. In[23] the Newton's method is used to obtain the parameter a which solves the following nonlinear equation:

$$F(a) := \|K_m\zeta_m - g^\delta\|^2 = (C\delta)^2, \quad C \ge 1, \tag{2.4}$$

where $\zeta_m = (aI + K_m^*K_m)^{-1}K_m^*g^\delta$, and K_m^* is the adjoint of the operator K_m. In Chapter 1 the following iterative scheme for obtaining the coefficients $\zeta_j^{(m)}$ is studied:

$$\zeta_{n,m}^\delta = q\zeta_{n-1,m}^\delta + (1-q)T_{a_n,m}^{-1}K_m^*g^\delta, \quad d_0^\delta = 0, \quad a_n := \alpha_0 q^n, \tag{2.5}$$

where $\alpha_0 > 0$, $q \in (0,1)$,

$$T_{a,m} := T^{(m)} + aI, \quad T^{(m)} := K_m^*K_m, \quad a > 0, \tag{2.6}$$

and I is the identity operator. Iterative scheme (2.5) is derived from a DSM solution of equation (2.1) obtained in[29] p.44. In iterative scheme (2.5) adaptive regularization parameters a_n are used. A discrepancy-type principle for DSM is used to define the stopping rule for the iteration processes.

The value of the parameter m in (2.4) and (2.5) is fixed at each iteration, and is usually large. The method for choosing the parameter m has not been discussed in Chapter 1. In this Chapter we choose the parameter m as a function of the regularization parameter a_n, and approximate the operator $T := K^*K$ (respectively K^*) by a finite-rank operator $T^{(m)}$ (respectively K^*_m):

$$\lim_{m\to\infty} \|T^{(m)} - T\| = 0. \tag{2.7}$$

Condition (2.7) can be satisfied by approximating the kernel $g(x, z)$ of T,

$$g(x, z) := \int_a^b \overline{k(s, x)}k(s, z)ds, \tag{2.8}$$

with the degenerate kernel

$$g_m(x, z) := \sum_{i=1}^{m} w_i\overline{k(s_i, x)}k(s_i, z), \tag{2.9}$$

where $\{s_i\}_{i=1}^m$ are the collocation points, and w_i,$1 \le i \le m$, are the quadrature weights. Quadrature formulas (2.9) can be found in[6]. Let K^*_m be a finite-dimensional approximation of K^* such that

$$\lim_{m\to\infty} \|K^* - K^*_m\| = 0. \tag{2.10}$$

One may choose $K^*_m = P_mK^*$, where P_m is a sequence of orthogonal projection operators on $L^2[a, b]$ such that $P_mx \to x$ as $m \to \infty$, $\forall x \in L^2[a, b]$. We propose the following iterative scheme:

$$u^\delta_{n,m_n} = qu^\delta_{n-1,m_{n-1}} + (1 - q)T^{-1}_{a_n,m_n}K^*_{m_n}f_\delta, \quad u^\delta_{0,m_0} = 0, \tag{2.11}$$

where $a_n := \alpha_0q^n$, $\alpha_0 > 0$, $q \in (0, 1)$, $\|f_\delta - f\| \le \delta$, $T_{a,m}$ is defined in (2.6) with $T^{(m)}$ satisfying condition (2.7), K^*_m is chosen so that condition(2.10) holds, and m_n in (2.11) is a parameter which measures the accuracy of the finite-dimensional approximations $T^{(m_n)}$ and $K^*_{m_n}$ at the $n-$th iteration. We propose a rule for choosing the parameters m_n so that m_n depend on the parameters a_n. This rule yields a non-decreasing sequence m_n. Since m_n is a non-decreasing sequence, we may start to compute $T^{-1}_{a_n,m_n}K^*_{m_n}f_\delta$ using a small size linear algebraic system

$$T_{a_n,m_n}g^\delta = K^*_{m_n}f_\delta, \tag{2.12}$$

and increase the value of m_n only if $G_{n,m_n} > C\delta^\varepsilon$, $C > 2$, $\varepsilon \in (0,1)$, where G_{n,m_n} is defined below, in (2.74). Parameters m_n may take large values for $n \le n_\delta$, where n_δ is defined below, in (2.73). The choice of the parameters m_i, $i = 1, 2, \ldots$, in (2.11), which guarantees convergence of the iterative process (2.11), is given in Section 2. We prove in Section 3 that the discrepancy-type principle, proposed in Chapter 1, with $T^{(m)}$ and K_m^* in place of T and K^* respectively, guarantees the convergence of the approximate solution u_{n,m_n}^δ to the minimal norm solution of equation (2.1). Throughout this Chapter we assume that

$$y \perp \mathcal{N}(K), \tag{2.13}$$

and

$$Ky = f, \tag{2.14}$$

where $\mathcal{N}(K)$ is the nullspace of K.

Throughout this Chapter we denote by K_m^* the operator approximating K^*, and define

$$T_a := T + aI, \quad T := K^*K, \tag{2.15}$$

where $a = const > 0$ and I is the identity operator.

The main result of this Chapter is Theorem 2.3.7 in Section 3.

2.2 Convergence of the iterative scheme

In this section we derive sufficient conditions on the parameters m_i, $i = 1, 2, \ldots$, for the iterative process (2.11) to converge to the minimal-norm solution y. The estimates of the following Lemma are known (see, e.g.,[30]), so their proofs are omitted.

Lemma 2.2.1. *One has:*

$$\|T_a^{-1}\| \le \frac{1}{a} \tag{2.16}$$

and

$$\|T_a^{-1}K^*\| \le \frac{1}{2\sqrt{a}}, \tag{2.17}$$

for any positive constant a.

While T_a is boundedly invertible for every $a > 0$, $T_{a,m}$ may be not invertible. The following lemma provides sufficient conditions for $T_{a,m}$ to be boundedly invertible.

Lemma 2.2.2. *Suppose that*

$$\|T - T^{(m)}\| < \epsilon a, \quad a = const > 0, \tag{2.18}$$

where $\epsilon \in (0, 1/2]$*. Then the following estimates hold*

$$\|T_{a,m}^{-1}\| \le \frac{2}{a}, \tag{2.19}$$

$$\|T_{a,m}^{-1} K^*\| \le \frac{1}{\sqrt{a}} \tag{2.20}$$

and

$$\|T_{a,m}^{-1} K^* K\| \le 2. \tag{2.21}$$

Proof. Write

$$T_{a,m} = T_a \left[I + T_a^{-1}(T^{(m)} - T) \right]. \tag{2.22}$$

It follows from (2.18) and (2.16) that

$$\|T_a^{-1}(T^{(m)} - T)\| \le \|T_a^{-1}\| \|(T^{(m)} - T)\| \le \epsilon < 1. \tag{2.23}$$

Therefore the operator $I + T_a^{-1}(T^{(m)} - T)$ is boundedly invertible. Since T_a is invertible, it follows from (2.22) and (2.23) that $T_{a,m}$ is invertible and

$$T_{a,m}^{-1} = \left[I + T_a^{-1}(T^{(m)} - T) \right]^{-1} T_a^{-1}. \tag{2.24}$$

Let us estimate the norm $\|T_{a,m}^{-1}\|$. We have $0 < \epsilon \le 1/2$, so

$$\left\| \left[I + T_a^{-1}(T^{(m)} - T) \right]^{-1} \right\| \le \frac{1}{1 - \|T_a^{-1}(T^{(m)} - T)\|} \le \frac{1}{1 - \epsilon} \le 2. \tag{2.25}$$

This, together with (2.16) and (2.24), yields

$$\|T_{a,m}^{-1}\| \le \frac{2}{a}. \tag{2.26}$$

Thus, estimate (2.19) is proved. To prove estimate (2.20), write

$$T_{a,m}^{-1}K^* = \left[I + T_a^{-1}(T^{(m)} - T)\right]^{-1} T_a^{-1}K^*.$$

Using estimates (2.25) and (2.17), one gets

$$\|T_{a,m}^{-1}K^*\| \le \frac{1}{\sqrt{a}}$$

which proves estimate (2.20). Let us derive estimate (2.21). One has:

$$T_{a,m}^{-1}K^*K = \left[I + T_a^{-1}(T^{(m)} - T)\right]^{-1} T_a^{-1}K^*K.$$

Using the estimates $\|T_a^{-1}T\| \le 1$ and (2.25), one obtains

$$\|T_{a,m}^{-1}T\| \le \frac{1}{1-\epsilon} \le 2.$$

Lemma 2.2.2 is proved. $\square$

Lemma 2.2.3. *Let $g(x)$ be a continuous function on $(0,\infty)$, $c > 0$ and $q \in (0,1)$ be constants. If*

$$\lim_{x\to 0^+} g(x) = g(0) := g_0, \tag{2.27}$$

then

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(q^{n-j-1} - q^{n-j}\right) g(cq^{j+1}) = g_0. \tag{2.28}$$

Proof. Let

$$w_j^{(n)} := q^{n-j} - q^{n+1-j}, \quad w_j^{(n)} > 0, \tag{2.29}$$

and

$$F_l(n) := \sum_{j=1}^{l-1} w_j^{(n)} g(cq^j). \tag{2.30}$$

Then

$$|F_{n+1}(n) - g_0| \le |F_l(n)| + \left|\sum_{j=l}^{n} w_j^{(n)} g(cq^j) - g_0\right|.$$

Take $\epsilon > 0$ arbitrary small. For sufficiently large $l(\epsilon)$ one can choose $n(\epsilon)$, such that

$$|F_{l(\epsilon)}(n)| \le \frac{\epsilon}{2},\ \forall n > n(\epsilon),$$

because $\lim_{n\to\infty} q^n = 0$. Fix $l = l(\epsilon)$ such that $|g(cq^j) - g_0| \le \frac{\epsilon}{2}$ for $j > l(\epsilon)$. This is possible because of (2.27). One has

$$|F_{l(\epsilon)}(n)| \le \frac{\epsilon}{2}, \ n > n(\epsilon)$$

and

$$\begin{aligned}\left|\sum_{j=l(\epsilon)}^{n} w_j^{(n)} g(cq^j) - g_0\right| &\le \sum_{j=l(\epsilon)}^{n} w_j^{(n)} |g(cq^j) - g_0| + |\sum_{j=l(\epsilon)}^{n} w_j^{(n)} - 1||g_0| \\ &\le \frac{\epsilon}{2} \sum_{j=l(\epsilon)}^{n} w_j^{(n)} + q^{n-l(\epsilon)}|g_0| \\ &\le \frac{\epsilon}{2} + |g_0| q^{n-l(\epsilon)} \le \epsilon,\end{aligned}$$

if n is sufficiently large. Here we have used the relation

$$\sum_{j=l}^{n} w_j^{(n)} = 1 - q^{n+1-l}.$$

Since $\epsilon > 0$ is arbitrarily small, relation (2.28) follows.

Lemma 2.2.3 is proved. □

Lemma 2.2.4. *Let*

$$u_n = qu_{n-1} + (1-q)T_{a_n}^{-1} K^* f, \quad u_0 = 0, \quad a_n := \alpha_0 q^n, \quad q \in (0,1). \tag{2.31}$$

Then

$$\|u_n - y\| \le q^n \|y\| + \sum_{j=0}^{n-1} \left(q^{n-j-1} - q^{n-j}\right) a_{j+1} \|T_{a_{j+1}}^{-1} y\|, \quad \forall n \ge 1, \tag{2.32}$$

and

$$\|u_n - y\| \to 0 \ \text{as } n \to \infty. \tag{2.33}$$

Proof. By induction, we obtain

$$u_n = \sum_{j=0}^{n-1} w_j^{(n)} T_{a_{j+1}}^{-1} K^* f, \tag{2.34}$$

where $w_j^{(n)} = q^{n-j-1} - q^{n-j}$. This, together with the identities $Ky = f$,

$$T_a^{-1} K^* K = T_a^{-1}(K^* K + aI - aI) = I - aT_a^{-1} \tag{2.35}$$

and

$$\sum_{j=0}^{n} w_j^{(n)} = 1 - q^n, \tag{2.36}$$

yield

$$\begin{aligned} u_n &= \sum_{j=0}^{n-1} w_j^{(n)} T_{a_{j+1}}^{-1} (T_{a_{j+1}} - a_{j+1} I) y \\ &= \sum_{j=0}^{n-1} w_j^{(n)} y - \sum_{j=0}^{n-1} w_j^{(n)} a_{j+1} T_{a_{j+1}}^{-1} y \\ &= y - q^n y - \sum_{j=0}^{n-1} w_j^{(n)} a_{j+1} T_{a_{j+1}}^{-1} y. \end{aligned}$$

Thus, estimate (2.32) follows. To prove (2.33), we apply Lemma 2.2.3 with $g(a) := a\|T_a^{-1} y\|$. Since $y \perp \mathcal{N}(K)$, it follows from the spectral theorem that

$$\lim_{a\to 0} g^2(a) = \lim_{a\to 0} \int_0^\infty \frac{a^2}{(a+s)^2} d\langle E_s y, y\rangle = \|P_{\mathcal{N}(K)} y\|^2 = 0,$$

where E_s is the resolution of the identity corresponding to K^*K, and P is the orthogonal projector onto $\mathcal{N}(K)$. Thus, by Lemma 2.2.3, (2.33) follows.

Lemma 1.2.2 is proved. □

Lemma 2.2.5. *Let u_n and $a_n = \alpha_0 q^n$, $\alpha_0 > 0$, $q \in (0,1)$ be defined in (2.31), $T_{a,m}$ be defined in (2.6), m_i be chosen so that*

$$\|T - T^{(m_i)}\| \le \frac{a_i}{2}, \quad 1 \le i \le n, \tag{2.37}$$

and

$$u_{n,m_n} = q u_{n-1,m_{n-1}} + (1-q) T_{a_n,m_n}^{-1} K_{m_n}^* f, \quad u_{0,m_0} = 0. \tag{2.38}$$

Then

$$\begin{aligned} \|u_{n,m_n} - u_n\| \le q^n\|y\| + \|y - u_n\| + 2\sum_{j=0}^{n-1} w_{j+1}^{(n)} \frac{\|(K_{m_{j+1}}^* K - T^{(m_{j+1})})y\|}{a_{j+1}} \\ + 2\sum_{j=0}^{n-1} w_{j+1}^{(n)} a_{j+1} \|T_{a_{j+1}}^{-1} y\|, \end{aligned} \tag{2.39}$$

where $w_j^{(n)}$ are defined in (2.29).

Proof. One has $w_i^{(n)} > 0$, $0 < q < 1$, and

$$\sum_{j=0}^{n-1} w_{j+1}^{(n)} = 1 - q^n \to 1, \quad as \quad n \to \infty.$$

Therefore one may use $w_{j+1}^{(n)}$ for large n as quadrature weights. To prove inequality (2.39), the following lemma is needed:

Lemma 2.2.6. *Let u_{n,m_n} be defined in (2.38). Then*

$$u_{n,m_n} = \sum_{j=0}^{n-1} w_{j+1}^{(n)} T_{a_{j+1},m_{j+1}}^{-1} K_{m_{j+1}}^* f, \quad n > 0, \tag{2.40}$$

where w_j are defined in (2.29).

Proof. Let us prove equation (2.40) by induction. For $n = 1$ we get

$$\begin{aligned} u_{1,m_1} &= qu_0 + (1-q)T_{a_1,m_1}^{-1} K_{m_1}^* f = (1-q)T_{a_1,m_1}^{-1} K_{m_1}^* f \\ &= w_1^{(1)} T_{a_1,m_1}^{-1} K_{m_1}^* f, \end{aligned}$$

so equation (2.40) holds. Suppose equation (2.40) holds for $1 \le n \le k$. Then

$$\begin{aligned} u_{k+1,m_{k+1}} &= qu_{k,m_k} + (1-q)T_{a_{k+1},m_{k+1}}^{-1} K_{m_{k+1}}^* f \\ &= q\sum_{j=0}^{k-1} w_{j+1}^{(k)} T_{a_{j+1},m_{j+1}}^{-1} K_{m_{j+1}}^* f + (1-q)T_{a_{k+1},m_{k+1}}^{-1} K_{m_{k+1}}^* f \\ &= \sum_{j=0}^{k-1} w_{j+1}^{(k+1)} T_{a_{j+1},m_{j+1}}^{-1} K_{m_{j+1}}^* f + w_{k+1}^{(k+1)} T_{a_{k+1},m_{k+1}}^{-1} K_{m_{k+1}}^* f \\ &= \sum_{j=0}^{k} w_{j+1}^{(k+1)} T_{a_{j+1},m_{j+1}}^{-1} K_{m_{j+1}}^* f. \end{aligned} \tag{2.41}$$

Here we have used the identities $qw_j^{(n)} = w_j^{(n+1)}$ and $1 - q = w_j^{(j)}$. Equation (2.40) is proved. □

By Lemma 2.2.6, one gets:

$$\begin{aligned} u_{n,m_n} - u_n &= \sum_{j=0}^{n-1} w_{j+1}^{(n)} T_{a_{j+1},m_{j+1}}^{-1} K_{m_{j+1}}^* Ky - u_n \\ &= \sum_{j=0}^{n-1} w_{j+1}^{(n)} T_{a_{j+1},m_{j+1}}^{-1} (K_{m_{j+1}}^* K - T^{(m_{j+1})} + T^{(m_{j+1})})y - u_n \\ &:= I_1 + I_2, \end{aligned}$$

where

$$I_1 := \sum_{j=0}^{n-1} w_{j+1}^{(n)} T_{a_{j+1},m_{j+1}}^{-1} (K_{m_{j+1}}^* K - T^{(m_{j+1})} + T^{(m_{j+1})}) y,$$

and

$$I_2 := -u_n.$$

We get

$$\begin{aligned}
I_1 &= \sum_{j=0}^{n-1} w_{j+1}^{(n)} \left[T_{a_{j+1},m_{j+1}}^{-1} (K_{m_{j+1}}^* K - T^{(m_{j+1})}) y + T_{a_{j+1},m_{j+1}}^{-1} T^{(m_{j+1})} y \right] \\
&= \sum_{j=0}^{n-1} w_{j+1}^{(n)} \left[T_{a_{j+1},m_{j+1}}^{-1} (K_{m_{j+1}}^* K - T^{(m_{j+1})}) y + y - a_{j+1} T_{a_{j+1},m_{j+1}}^{-1} y \right] \\
&= \sum_{j=0}^{n-1} w_{j+1}^{(n)} \left[T_{a_{j+1},m_{j+1}}^{-1} (K_{m_{j+1}}^* K - T^{(m_{j+1})}) y - a_{j+1} T_{a_{j+1},m_{j+1}}^{-1} y \right] \\
&+ y - q^n y \\
&= \sum_{j=0}^{n-1} w_{j+1}^{(n)} T_{a_{j+1},m_{j+1}}^{-1} (K_{m_{j+1}}^* K - T^{(m_{j+1})}) y \\
&- \sum_{j=0}^{n-1} w_{j+1}^{(n)} a_{j+1} (T_{a_{j+1},m_{j+1}}^{-1} - T_{a_{j+1}}^{-1} + T_{a_{j+1}}^{-1}) y + y - q^n y \\
&= y - q^n y + \sum_{j=0}^{n-1} w_{j+1}^{(n)} T_{a_{j+1},m_{j+1}}^{-1} (K_{m_{j+1}}^* K - T^{(m_{j+1})}) y \\
&- \sum_{j=0}^{n-1} w_{j+1}^{(n)} \left[a_{j+1} T_{a_{j+1},m_{j+1}}^{-1} (T - T^{(m_{j+1})}) T_{a_{j+1}}^{-1} y + a_{j+1} T_{a_{j+1}}^{-1} y \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_1 + I_2 = y - u_n - q^n y &+ \sum_{j=0}^{n-1} w_{j+1}^{(n)} T_{a_{j+1},m_{j+1}}^{-1} (K_{m_{j+1}}^* K - T^{(m_{j+1})}) y \\
&- \sum_{j=0}^{n-1} w_{j+1}^{(n)} \left[a_{j+1} T_{a_{j+1}}^{-1} + a_{j+1} T_{a_{j+1},m_{j+1}}^{-1} (T - T^{(m_{j+1})}) T_{a_{j+1}}^{-1} \right] y.
\end{aligned} \tag{2.42}$$

Applying the estimates $\|T^{(m_i)} - T\| \le \frac{a_i}{2}$ and $\|T^{-1}_{a_i,m_i}\| \le \frac{2}{a_i}$ in (2.43), one gets

$$\begin{aligned}
\|u_{n,m} - u_n\| &\le q^n\|y\| + \|y-u_n\| + \sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|T^{-1}_{a_{j+1}} y\| \\
&+ \sum_{j=0}^{n-1} w^{(n)}_{j+1} \|T^{-1}_{a_{j+1},m_{j+1}}(K^*_{m_{j+1}}K - T^{(m_{j+1})})y\| \\
&+ \sum_{j=0}^{n-1} w^{(n)}_{j+1} \|a_{j+1}T^{-1}_{a_{j+1},m_{j+1}}(T - T^{(m_{j+1})})T^{-1}_{a_{j+1}} y\| \\
&\le q^n\|y\| + \|y-u_n\| + \sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|T^{-1}_{a_{j+1}} y\| \\
&+ \sum_{j=0}^{n-1} w^{(n)}_{j+1} \|T^{-1}_{a_{j+1},m_{j+1}}\|\|(K^*_{m_{j+1}}K - T^{(m_{j+1})})y\| \qquad (2.43)\\
&+ \sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|T^{-1}_{a_{j+1},m_{j+1}}\|\|T - T^{(m_{j+1})}\|\|T^{-1}_{a_{j+1}} y\| \\
&\le q^n\|y\| + \|y-u_n\| + \sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|T^{-1}_{a_{j+1}} y\| \\
&+ \sum_{j=0}^{n-1} w^{(n)}_{j+1} \frac{2}{a_{j+1}}\|(K^*_{m_{j+1}}K - T^{(m_{j+1})})y\| \\
&+ \sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|T^{-1}_{a_{j+1}} y\|.
\end{aligned}$$

Lemma 2.2.5 is proved. □

Lemma 2.2.7. *Under the assumptions of Lemma 2.2.5 if*

$$\|K^*_{m_n} - K^*\| \le \frac{\sqrt{a_n}}{2} \tag{2.44}$$

then

$$\|u_{n,m_n} - u^\delta_{n,m_n}\| \le \frac{\sqrt{q}}{1-q^{3/2}} \frac{2\delta}{\sqrt{q}\sqrt{a_n}}. \tag{2.45}$$

Proof. We have

$$\begin{aligned}
u_{n,m_n} - u^\delta_{n,m_n} &= q(u_{n-1,m_{n-1}} - u^\delta_{n-1,m_{n-1}}) + (1-q)T^{-1}_{a_n,m_n}K^*_{m_n}(f - f_\delta) \\
&= q(u_{n-1,m_{n-1}} - u^\delta_{n-1,m_{n-1}}) + (1-q)T^{-1}_{a_n,m_n}(K^*_{m_n} - K^*)(f - f_\delta) \qquad (2.46)\\
&+ (1-q)T^{-1}_{a_n,m_n}K^*(f - f_\delta).
\end{aligned}$$

Since $\|f - f_\delta\| \le \delta$, $\|T^{-1}_{a_n,m_n} K^*\| \le \frac{1}{\sqrt{a_n}}$ and $\|K^*_{m_n} - K^*\| \le \frac{\sqrt{a_n}}{2}$, it follows that

$$\|u_{n,m_n} - u^\delta_{n,m_n}\| \le q\|u_{n-1,m_{n-1}} - u^\delta_{n-1,m_{n-1}}\| + 2\frac{\delta}{\sqrt{a_n}}. \tag{2.47}$$

Let us prove estimate (2.45) by induction. Define $H_n := \|u_{n,m_n} - u^\delta_{n,m_n}\|$ and $h_n := 2\frac{\delta}{\sqrt{q}\sqrt{a_n}}$. For $n = 0$ we get $H_0 = 0 < \frac{\sqrt{q}}{1-q^{3/2}} h_0$. Thus (2.45) holds. Suppose estimate (2.45) holds for $0 \le n \le k$. Then

$$\begin{aligned} H_{k+1} &\le qH_k + h_k \le q\frac{\sqrt{q}}{1-q^{3/2}} h_k + h_k = \left(q\frac{\sqrt{q}}{1-q^{3/2}} + 1 \right) h_k \\ &= \frac{1}{1-q^{3/2}} \frac{h_k}{h_{k+1}} h_{k+1} \le \frac{\sqrt{q}}{1-q^{3/2}} h_{k+1}. \end{aligned} \tag{2.48}$$

Here we have used the relation

$$\frac{h_k}{h_{k+1}} = \frac{2\frac{\delta}{\sqrt{q}\sqrt{a_k}}}{2\frac{\delta}{\sqrt{q}\sqrt{a_{k+1}}}} = \frac{\sqrt{a_{k+1}}}{\sqrt{a_k}} = \frac{\sqrt{qa_k}}{\sqrt{a_k}} = \sqrt{q}. \tag{2.49}$$

Lemma 2.2.7 is proved. □

The following theorem gives the convergence of the iterative scheme (2.11).

Theorem 2.2.8. *Let u^δ_{n,m_n} be defined in (2.11), m_i be chosen so that*

$$\|T - T^{(m_i)}\| \le a_i/2, \tag{2.50}$$

$$\|T^{(m_i)} - K^*_{m_i} K\| \le a_i^2, \tag{2.51}$$

$$\|K^*_{m_i} - K^*\| \le \sqrt{a_i}/2, \tag{2.52}$$

and n_δ satisfies the following relations:

$$\lim_{\delta\to 0} n_\delta = \infty, \quad \lim_{\delta\to 0} \frac{\delta}{\sqrt{a_{n_\delta}}} = 0. \tag{2.53}$$

Then

$$\lim_{\delta\to 0} \|u^\delta_{n_\delta,m_{n_\delta}} - y\| = 0. \tag{2.54}$$

Proof. We have

$$\|y - u^\delta_{n,m_n}\| \leq \|y - u_n\| + \|u_n - u_{n,m_n}\| + \|u_{n,m_n} - u^\delta_{n,m_n}\|. \tag{2.55}$$

From (2.39) and estimate (2.51) we get

$$\|u_{n,m_n} - u_n\| \leq q^n\|y\| + \|y - u_n\| + 2\sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|y\| + 2\sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|T^{-1}_{a_{j+1}} y\|. \tag{2.56}$$

This, together with Lemma 2.2.7, implies

$$\|y - u^\delta_{n,m_n}\| \leq 2\left(J(n) + \frac{\delta}{(1-q^{3/2})\sqrt{a_n}}\right), \tag{2.57}$$

where

$$J(n) := \frac{q^n}{2}\|y\| + \|y - u_n\| + \sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|y\| + \sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|T^{-1}_{a_{j+1}} y\|, \tag{2.58}$$

and $w^{(n)}_j$ are defined in (2.29). Since $y \perp \mathcal{N}(A)$, it follows that

$$\lim_{a\to 0} a^2\|T^{-1}_a y\|^2 = \int_0^\infty \frac{a^2}{(a+s)^2} d\langle E_s y, y\rangle = \|P_{\mathcal{N}(K)} y\|^2 = 0,$$

where E_s is the resolution of the identity of the selfadjoint operator T, and $P_{\mathcal{N}(K)}$ is the orthogonal projector onto the nullspace $\mathcal{N}(K)$. Applying Lemma 2.2.3 with $g(a) := a\|T^{-1}_a y\|$, one gets

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|T^{-1}_{a_{j+1}} y\| = 0. \tag{2.59}$$

Similarly, letting $g(a) := a\|y\|$ in Lemma 2.2.3, we get

$$\lim_{n\to\infty} 2\sum_{j=0}^{n-1} w^{(n)}_{j+1} a_{j+1}\|y\| = 0. \tag{2.60}$$

Relations (2.59) and (2.60), together with Lemma 1.2.2, imply

$$\lim_{n\to\infty} J(n) = 0. \tag{2.61}$$

If we stop the iteration at $n = n_\delta$ such that assumptions (2.53) hold then $\lim_{\delta\to 0} J(n_\delta) = 0$ and $\lim_{\delta\to 0} \frac{\delta}{\sqrt{a_{n_\delta}}} = 0$. Therefore, relation (2.54) is proved. This proves Theorem 2.2.8. □

2.3 A discrepancy-type principle for DSM

In this section we propose an adaptive stopping rule for the iterative scheme (2.11). Throughout this section the parameters $m_i,\ i = 1, 2, \dots,$ are chosen so that conditions (2.50)-(2.52) hold,

$$\|Q - Q^{(m_i)}\| \leq \epsilon a_i, \quad \epsilon \in (0, 1/2], \quad a_i = \alpha_0 q^i, \quad \alpha_0 = const > 0, \tag{2.62}$$

where

$$Q := KK^*, \tag{2.63}$$

and $Q^{(m)}$ is a finite-dimensional approximation of Q. One may satisfy condition (2.62) by approximating the kernel $q(x, s)$ of Q,

$$q(x, s) = \int_a^b k(x, z)\overline{k(s, z)}dz, \tag{2.64}$$

with

$$q_m(x, s) = \sum_{i=1}^{m} \gamma_i k(x, z_i)\overline{k(s, z_i)}, \tag{2.65}$$

where $\gamma_i,\ i = 1, 2, \dots, m,$ are some quadrature weights and z_i are the collocation points.

Lemma 2.3.1.

$$\|Q_a^{-1}\| \leq \frac{1}{a} \tag{2.66}$$

and

$$\|Q_a^{-1}K\| \leq \frac{1}{2\sqrt{a}}, \tag{2.67}$$

for any positive constant a.

Proof. Since $Q = Q^* \geq 0$, one uses the spectral theorem and gets:

$$\|Q_a^{-1}\| = \sup_{s>0} \frac{1}{s + a} \leq \frac{1}{a}.$$

Inequality (2.67) follows from the identity

$$Q_a^{-1}K = KT_a^{-1}, \quad T := K^*K, \quad T_a := T + aI, \tag{2.68}$$

and the estimate

$$\|KT_a^{-1}\| = \|UT^{1/2}T_a^{-1}\| \le \|T^{1/2}T_a^{-1}\| = \sup_{s\ge 0}\frac{s^{1/2}}{a+s} \le \frac{1}{2\sqrt{a}}, \tag{2.69}$$

where the polar decomposition was used: $K = UT^{1/2}$, U is a partial isometry, $\|U\| = 1$. Lemma 2.3.1 is proved. $\square$

Lemma 2.3.2. *Suppose m is chosen so that*

$$\|Q - Q^{(m)}\| \le \epsilon a, \quad \epsilon \in (0, 1/2], \quad a > 0. \tag{2.70}$$

Then the following estimates hold:

$$\|Q_{a,m}^{-1}\| \le \frac{2}{a}, \tag{2.71}$$

$$\|Q_{a,m}^{-1}K\| \le \frac{1}{\sqrt{a}}. \tag{2.72}$$

Proof of Lemma 2.3.2 is similar to the proof of Lemma 2.2.2 and is omitted.

We propose the following *stopping rule*:

Choose n_δ so that the following inequalities hold

$$G_{n_\delta, m_{n_\delta}} \le C\delta^\varepsilon < G_{n,m_n}, \quad 1 \le n < n_\delta, \ C > 2, \quad \varepsilon \in (0,1), \tag{2.73}$$

where

$$\begin{aligned} G_{n,m_n} &= qG_{n-1,m_{n-1}} + (1-q)a_n\|Q_{a_n,m_n}^{-1}f_\delta\|, \\ G_{0,m_0} &= 0, \quad G_{1,m_1} \ge C\delta^\varepsilon, \quad a_n = qa_{n-1}, \quad a_0 = \alpha_0 = const > 0, \end{aligned} \tag{2.74}$$

and

$$Q_{a,m} := Q^{(m)} + aI. \tag{2.75}$$

The discrepancy-type principle (2.73) is derived from the following discrepancy principle for DSM proposed in[28,33]:

$$\int_0^{t_\delta} e^{-(t_\delta - s)}a(s)\|Q_{a(s)}^{-1}f_\delta\|ds = C\delta, \quad C > 1, \tag{2.76}$$

where t_δ is the stopping time, and we assume that

$$a(t) > 0, \quad a(t) \searrow 0.$$

The derivation of the stopping rule (2.73) with $Q^{(m)} = Q$ is given in Chapter 1. Let us prove that there exists an integer n_δ such that inequalities (2.73) hold. To prove the existence of such an integer, we derive some properties of the sequence G_{n,m_n} defined in (2.74). Using Lemma 2.3.2, the relation $Ky = f$, and the assumption $\|f_\delta - f\| \le \delta$, we get

$$\begin{aligned} a_n\|Q^{-1}_{a_n,m_n} f_\delta\| &\le a_n\|Q^{-1}_{a_n,m_n}(f_\delta - f)\| + a_n\|Q^{-1}_{a_n,m_n} f\| \\ &\le 2\delta + 2\sqrt{a_n}\|y\|, \end{aligned} \tag{2.77}$$

where estimates (2.71) and (2.72) were used. This, together with (2.74), yield

$$G_{n,m_n} \le qG_{n-1,m_{n-1}} + (1-q)2\delta + (1-q)2\sqrt{a_n}\|y\|, \tag{2.78}$$

so

$$G_{n,m_n} - 2\delta \le q(G_{n-1,m_{n-1}} - 2\delta) + (1-q)2\sqrt{q}\sqrt{a_{n-1}}\|y\|, \tag{2.79}$$

where the relation $a_n = qa_{n-1}$, $a_0 = \alpha_0 = const > 0$, was used. Define

$$\Psi_n := G_{n,m_n} - 2\delta, \tag{2.80}$$

where $G_{n,m}$ is defined in (2.74), and let

$$\psi_n := (1-q)2\sqrt{a_n}\|y\|. \tag{2.81}$$

Then

$$\Psi_n \le q\Psi_{n-1} + \sqrt{q}\psi_{n-1}. \tag{2.82}$$

Lemma 2.3.3. *If* (2.80) *and* (2.81) *hold, then*

$$\Psi_n \le \frac{1}{1-\sqrt{q}}\psi_n, \quad n \ge 0. \tag{2.83}$$

Proof. Let us prove this lemma by induction. For $n = 0$ we get

$$\Psi_0 = -2\delta \le \frac{1}{1-\sqrt{q}}\psi_0.$$

Suppose estimate (2.83) is true for $0 \le n \le k$. Then

$$\begin{aligned}\Psi_{k+1} &\le q\Psi_k + \sqrt{q}\psi_k \le \frac{q}{1-\sqrt{q}}\psi_k + \sqrt{q}\psi_k = \frac{\sqrt{q}}{1-\sqrt{q}}\psi_k \\ &= \frac{\sqrt{q}}{1-\sqrt{q}}\frac{\psi_k}{\psi_{k+1}}\psi_{k+1} \le \frac{\sqrt{q}}{1-\sqrt{q}}\frac{1}{\sqrt{q}}\psi_{k+1} = \frac{1}{1-\sqrt{q}}\psi_{k+1}.\end{aligned} \tag{2.84}$$

Here we have used the relation

$$\frac{\psi_k}{\psi_{k+1}} = \frac{(1-q)2\sqrt{a_k}\|y\|}{(1-q)2\sqrt{a_{k+1}}\|y\|} = \frac{\sqrt{a_k}}{\sqrt{a_{k+1}}} = \frac{\sqrt{a_k}}{\sqrt{qa_k}} = \frac{1}{\sqrt{q}}. \tag{2.85}$$

Thus, Lemma 2.3.3 is proved. □

By definitions (2.80), (2.81), and Lemma 2.3.3, we get the estimate

$$G_{n,m_n} \le 2\delta + \frac{1}{1-\sqrt{q}}(1-q)2\sqrt{a_n}\|y\|, \quad n \ge 0, \tag{2.86}$$

so

$$\limsup_{n\to\infty} G_{n,m_n} \le 2\delta \tag{2.87}$$

because $\lim_{n\to\infty} a_n = 0$.

Since $G_{1,m_1} \ge C\delta^{\varepsilon}$, $C > 2$, $\varepsilon \in (0,1)$ and $\limsup_{n\to\infty} G_{n,m_n} \le 2\delta$, it follows that there exists an integer n_δ such that inequalities (2.73) hold. The uniqueness of the integer n_δ follows from its definition.

Lemma 2.3.4. *If n_δ is chosen by the rule (2.73), then*

$$\frac{\delta}{\sqrt{a_{n_\delta}}} \to 0 \text{ as } \delta \to 0. \tag{2.88}$$

Proof. From the stopping rule (2.73) and estimate (2.86) we get

$$C\delta^{\varepsilon} < G_{n_\delta-1,m_{n_\delta-1}} \le 2\delta + \frac{1}{1-\sqrt{q}}(1-q)2\sqrt{a_{n_\delta-1}}\|y\|. \tag{2.89}$$

This implies

$$\frac{1}{\sqrt{a_{n_\delta-1}}} \le \frac{1}{(1-\sqrt{q})(C-2)\delta^\varepsilon}(1-q)2\|y\|, \tag{2.90}$$

so

$$\frac{\delta}{\sqrt{a_{n_\delta}}} \le \frac{\delta^{1-\varepsilon}}{\sqrt{q}(1-\sqrt{q})(C-2)}(1-q)2\|y\| \to 0 \text{ as } \delta \to 0. \tag{2.91}$$

Lemma 2.3.4 is proved. □

Lemma 2.3.5. *If n_δ is chosen by the rule (2.73), then*

$$\lim_{\delta\to 0} n_\delta = \infty. \tag{2.92}$$

Proof. From the stopping rule (2.73) we get

$$\begin{aligned} &qC\delta^\varepsilon + (1-q)a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| < qG_{n_\delta-1,m_{n_\delta-1}} + (1-q)a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| \\ &= G_{n_\delta,m_{n_\delta}} < C\delta^\varepsilon. \end{aligned} \tag{2.93}$$

This implies

$$0 \le a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| < C\delta^\varepsilon \to 0 \text{ as } \delta \to 0. \tag{2.94}$$

Note that

$$\begin{aligned} 0 &\le a_{n_\delta}\|Q^{-1}_{a_{n_\delta}} f_\delta\| \le a_{n_\delta}\|(Q^{-1}_{a_{n_\delta}} - Q^{-1}_{a_{n_\delta},m_{n_\delta}})f_\delta\| + a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| \\ &= a_{n_\delta}\|Q^{-1}_{a_{n_\delta}}(Q_{a_{n_\delta},m_{n_\delta}} - Q_{a_{n_\delta}})Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| + a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| \\ &= a_{n_\delta}\|Q^{-1}_{a_{n_\delta}}(Q^{(m_{n_\delta})} - Q)Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| + a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| \\ &\le a_{n_\delta}\|Q^{-1}_{n_\delta}\|\|Q^{(m_{n_\delta})} - Q\|\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| + a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| \\ &\le a_{n_\delta}\frac{2}{a_{n_\delta}}\epsilon a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| + a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\| \\ &\le 2a_{n_\delta}\|Q^{-1}_{a_{n_\delta},m_{n_\delta}} f_\delta\|, \end{aligned} \tag{2.95}$$

where estimates (2.62), (2.71) and $0 < \epsilon < \frac{1}{2}$ were used. This, together with (2.94), yield

$$\lim_{\delta\to 0} a_{n_\delta}\|Q^{-1}_{a_{n_\delta}} f_\delta\| = 0. \tag{2.96}$$

To prove relation (2.92) the following lemma is needed:

Lemma 2.3.6. *Suppose condition $\|f - f_\delta\| \leq \delta$ and relation (2.96) hold. Then*

$$\lim_{\delta\to 0} a_{n_\delta} = 0. \tag{2.97}$$

Proof. If $f \neq 0$ then there exists a $\lambda_0 > 0$ such that

$$F_{\lambda_0} f \neq 0, \quad \langle F_{\lambda_0} f, f\rangle := \xi > 0, \tag{2.98}$$

where ξ is a constant which does not depend on δ, and F_s is the resolution of the identity corresponding to the operator $Q := KK^*$. Let

$$h(\delta, \alpha) := \alpha^2 \|Q_\alpha^{-1} f_\delta\|^2, \quad Q := KK^*, \ Q_a := aI + Q.$$

For a fixed number $c_1 > 0$ we obtain

$$\begin{aligned} h(\delta, c_1) = c_1^2 \|Q_{c_1} f_\delta\|^2 = \int_0^\infty \frac{c_1^2}{(c_1+s)^2} d\langle F_s f_\delta, f_\delta\rangle \geq \int_0^{\lambda_0} \frac{c_1^2}{(c_1+s)^2} d\langle F_s f_\delta, f_\delta\rangle \\ \geq \frac{c_1^2}{(c_1+\lambda_0)^2} \int_0^{\lambda_0} d\langle F_s f_\delta, f_\delta\rangle = \frac{c_1^2 \|F_{\lambda_0} f_\delta\|^2}{(c_1+\lambda_0)^2}, \quad \delta > 0. \end{aligned} \tag{2.99}$$

Since F_{λ_0} is a continuous operator, and $\|f - f_\delta\| < \delta$, it follows from (2.98) that

$$\lim_{\delta\to 0} \langle F_{\lambda_0} f_\delta, f_\delta\rangle = \langle F_{\lambda_0} f, f\rangle > 0. \tag{2.100}$$

Therefore, for the fixed number $c_1 > 0$ we get

$$h(\delta, c_1) \geq c_2 > 0 \tag{2.101}$$

for all sufficiently small $\delta > 0$, where c_2 is a constant which does not depend on δ. For example one may take $c_2 = \frac{\xi}{2}$ provided that (2.98) holds. It follows from relation (2.96) that

$$\lim_{\delta\to 0} h(\delta, a_{n_\delta}) = 0. \tag{2.102}$$

Suppose $\lim_{\delta\to 0} a_{n_\delta} \neq 0$. Then there exists a subsequence $\delta_j \to 0$ such that

$$\alpha_0 a_{n_{\delta_j}} \geq c_1 > 0, \tag{2.103}$$

where c_1 is a constant. By (2.101) we get

$$h(\delta_j, a_{n_{\delta_j}}) > c_2 > 0, \quad \delta_j \to 0 \text{ as } j \to \infty. \tag{2.104}$$

This contradicts relation (2.102). Thus, $\lim_{\delta\to 0} a_{n_\delta} = 0$.

Lemma 2.3.6 is proved. □

Applying Lemma 2.3.6 with $a_{n_\delta} = \alpha_0 q^{n_\delta}$, $q \in (0,1)$, $\alpha_0 > 0$, one gets relation (2.92).

Lemma 2.3.5 is proved. □

We formulate the main result of this chapter in the following theorem:

Theorem 2.3.7. *Suppose m_i are chosen so that conditions (2.50)-(2.52) and (2.62) hold, and n_δ is chosen by rule (2.73). Then*

$$\lim_{\delta\to 0} \|u^\delta_{n_\delta, m_{n_\delta}} - y\| = 0. \tag{2.105}$$

Proof. From (2.57) we get the estimate

$$\|y - u^\delta_{n_\delta, m_{n_\delta}}\| \le 2\left(J(n_\delta) + \frac{\delta}{(1-q^{3/2})\sqrt{a_{n_\delta}}}\right), \tag{2.106}$$

where $J(n)$ is defined in (2.58). It is proved in Theorem 2.2.8 that $\lim_{n\to\infty} J(n) = 0$. By Lemma 2.3.5, one gets $n_\delta \to \infty$ as $\delta \to 0$, so $\lim_{\delta\to 0} J(n_\delta) = 0$. From Lemma 2.3.4 we get $\lim_{\delta\to 0} \frac{\delta}{\sqrt{a_{n_\delta}}} = 0$. Thus,

$$\lim_{\delta\to 0} \|y - u^\delta_{n_\delta, m_{n_\delta}}\| = 0.$$

Theorem 2.3.7 is proved. □

2.4 Numerical experiments

Consider the following Fredholm integral equation:

$$Ku(s) := \int_0^1 e^{-st} u(t)dt = f(s), \quad s \in [0,1]. \tag{2.107}$$

The function $u(t) = t$ is the solution to equation (2.107) corresponding to $f(s) = \frac{1-(s+1)e^{-s}}{s^2}$. We perturb the exact data $f(s)$ by a random noise δ, $\delta > 0$, and get the noisy data

$f_\delta(s) = f(s) + \delta$. The compound Simpson's rule (see[6]) with the step size $\frac{1}{2^m}$ is used to approximate the kernel $g(x, z)$, defined in (2.8). This yields

$$T^{(m)}u := \sum_{j=1}^{2^m+1} \beta_j^{(m)} k(s_j, x) \int_0^1 k(s_j, z)u(z)dz,$$

where $k(s, t) := e^{-st}$, $\beta_j^{(m)}$ are the compound Simpson's quadrature weights: $\beta_1^{(m)} = \beta_{2^m+1}^{(m)} = \frac{1/3}{2^m}$, and for $j = 2, 3, \dots, 2^m$

$$\beta_j^{(m)} = \begin{cases} \frac{4/3}{2^m}, & j \text{ is even;} \\ \frac{2/3}{2^m}, & \text{otherwise,} \end{cases} \tag{2.108}$$

and s_j are the collocation points: $s_j = \frac{j-1}{2^m}$, $j = 1, 2, \dots, 2^m + 1$.

Let

$$\gamma_m := \|(T - T^{(m)})u\|,$$

$$h(s, x, z) := k(s, x)k(s, z)$$

and

$$c_1 := \frac{1}{180} \max_{x,z\in[0,1]} \max_{s\in[0,1]} \left| \frac{\partial^4 h(s, x, z)}{\partial s^4} \right| = \frac{16}{180}. \tag{2.109}$$

Then

$$\begin{aligned} \gamma_m^2 &= \int_0^1 \left| \int_0^1 \left(\int_0^1 h(s, x, z)ds - \sum_{j=1}^{2^m+1} \beta_j^{(m)} h(s_j, x, z) \right) u(z)dz \right|^2 dx \\ &\le \int_0^1 \left| \int_0^1 \frac{c_1}{2^{4m}} u(z)dz \right|^2 dx \le \left(\frac{c_1}{2^{4m}} \right)^2 \|u\|^2. \end{aligned} \tag{2.110}$$

The upper bound c_1 for the error of the compound Simpson's quadrature can be found in[6]. Thus,

$$\|T - T^{(m)}\| \le \frac{c_1}{2^{4m}} \to 0 \text{ as } m \to \infty.$$

Similarly, we approximate the kernel $q(x, s)$ defined in (2.64) by the Simpson's rule with the step size $\frac{1}{2^m}$ and get

$$\|Q - Q^{(m)}\| \le \frac{c_1}{2^{4m}} \to 0 \text{ as } m \to \infty. \tag{2.111}$$

Let us partition the interval $[0,1]$ into $2^m 180$, $m>0$, equisized subintervals D_j, where $D_j=[d_{j-1},d_j)$, $j=1,2,\dots,2^m$. Then $|d_j-d_{j-1}|=\frac{1}{2^m 180}$, $j=1,2,\dots,2^m$, and using the Taylor expansion of e^{st} about $s=d_{j-1}$, one gets

$$\begin{aligned}
&|e^{-st}-e^{-d_{j-1}t}[1-t(s-d_{j-1})]| \le \sum_{l=2}^{\infty}\frac{(s-d_j)^l}{l!} \le (s-d_{j-1})^2\sum_{j=0}^{\infty}(s-d_{j-1})^j \\
&\le \frac{1}{2^{2m}180^2}\sum_{j=0}^{\infty}\left(\frac{1}{2^m 180}\right)^j = \frac{1}{2^{2m}180^2}\frac{2^m 180}{2^m 180-1} \\
&= \frac{1}{2^m 180(2^m 180-1)} \le \frac{1}{2^{2m}180}, \quad \forall s\in D_j,\ t\in[0,1].
\end{aligned} \tag{2.112}$$

This allows us to define

$$K_m^* u(t)=\sum_{j=1}^{2^m}\int_{D_j} e^{-d_{j-1}t}[1-t(s-d_{j-1})]u(s)ds. \tag{2.113}$$

This, together with condition (2.112), yields

$$\begin{aligned}
\|(K^*-K_m^*)u\|^2 &= \int_0^1\left|\sum_{j=1}^{2^m}\int_{D_j}\left(e^{-st}-e^{-d_{j-1}t}[1-t(s-d_{j-1})]\right)u(t)dt\right|^2 ds \\
&\le \frac{1}{2^{2m}180^2}\int_0^1\left|\sum_{j=1}^{2^m}\int_{D_j}|u(t)|dt\right|^2 ds \le \frac{1}{2^{4m}180^2}\|u\|^2.
\end{aligned} \tag{2.114}$$

Thus,

$$\|K^*-K_m^*\| \le \frac{1}{2^{2m}180} \to 0 \quad as\ m\to\infty. \tag{2.115}$$

Moreover

$$\begin{aligned}
\|(T^{(m)}-K_m^*K)u\| &\le \|(T^{(m)}-T)u\|+\|(T-K_m^*K)u\| \\
&\le \frac{c_1}{2^{4m}}\|u\|+\|K^*-K_m^*\|\|Ku\| \\
&\le \frac{16}{2^{4m}180}\|u\|+\frac{1}{2^{2m}180}\|u\| \le \frac{17}{2^{2m}180}\|u\|.
\end{aligned} \tag{2.116}$$

Here we have used the constant $c_1=16/180$ and the estimate $|k(s,t)|\le \max_{s,t\in[0,1]}|e^{-st}|=1$. Thus,

$$\|T^{(m)}-K_m^*K\| \le \frac{17}{2^{2m}180}. \tag{2.117}$$

To satisfy condition (2.50) the parameter m_i may be chosen by solving the equation

$$\frac{c_1}{2^{4m_i}} = \frac{a_i}{2}. \tag{2.118}$$

To get m_i satisfying condition (2.51), one solves the equation

$$\frac{17}{2^{2m_i}180} = \eta a_i^2, \tag{2.119}$$

where $\eta = const \geq 10$. Here we have used the estimate $\|T^{(m_i)} - K_{m_i}^* K\| \leq \eta a_i^2$ instead of estimate (2.51). This estimate will not change our main results. The reason of using the constant $\eta \geq 10$ than of 1 in (2.119) is to control the decaying rate of the parameter a_i^2 so that the growth rate of the parameter m_i in (2.119) can be made as slow as we wish. To obtain the parameter m_i satisfying condition (2.52), one solves

$$\frac{c_1}{2^{2m_i}} = \frac{\sqrt{a_i}}{2}. \tag{2.120}$$

Hence to satisfy all the conditions in Theorem 2.3.7, one may choose m_i such that

$$m_i := \max\left\{ \left\lceil \frac{\ln(2c_1/a_i)}{4\ln 2} \right\rceil, \left\lceil \frac{\ln(\frac{17}{180(\eta a_i^2)})}{2\ln 2} \right\rceil, \left\lceil \frac{\ln(2c_1/\sqrt{a_i})}{2\ln 2} \right\rceil \right\}, \tag{2.121}$$

where $\lceil x \rceil$ is the smallest integer not less than x, c_1 is defined in (2.109), $a_i = \alpha_0 q^i, \quad \alpha_0 > 0, \quad q \in (0,1)$. In all the experiments the parameter η in (2.121) is equal to 10 which is sufficient for the given problem. To obtain the approximate solution to problem (2.107), we consider a finite-dimensional approximate solution

$$u_{n,m_n}^{\delta}(x) := P_m u(x) = \sum_{j=1}^{2^m} \zeta_j^{(m_n,\delta)} \Phi_j(x), \tag{2.122}$$

$P_m : L^2[0,1] \to L_m$,

$$L_m = \text{span}\{\Phi_1, \Phi_2, \ldots, \Phi_{2^m}\}, \tag{2.123}$$

where $\{\Phi_i\}$ are the Haar basis functions (see[35]): $\Phi_1(x) = 1 \ \forall x \in [0,1]$, and for $j = 2^{l-1} + p,\ l = 1, 2, \ldots, m,\ p = 1, 2, \ldots, 2^{l-1}$

$$\Phi_j(x) = \begin{cases} 2^{(l-1)/2}, & x \in [\frac{p-1}{2^{l-1}}, \frac{p-1/2}{2^{l-1}}); \\ -2^{(l-1)/2}, & x \in [\frac{p-1/2}{2^{l-1}}, \frac{p}{2^{l-1}}); \\ 0, & \text{otherwise}. \end{cases} \tag{2.124}$$

Let us formulate an algorithm for obtaining the approximate solution to (2.107) using iterative scheme (2.11), where the discrepancy-type principle for DSM defined in Section 3 is used as the stopping rule.

(1) Given data: K, f_δ, δ;

(2) initialization : $\alpha_0 > 0$, $\eta \geq 10$, $q \in (0,1)$, $C > 2$, $u^\delta_{0,m_0} = 0$, $G_0 = 0$, $n = 1$;

(3) iterate, starting with $n = 1$, and stop until the condition (2.133) below holds,

   (a) $a_n = \alpha_0 q^n$,

   (b) choose $m_n = \max\left\{\left\lceil \frac{\ln(2c_1/a_n)}{4\ln 2}\right\rceil, \left\lceil \frac{\ln(17/(180\eta a_n^2))}{2\ln 2}\right\rceil, \left\lceil \frac{\ln(2c_1/\sqrt{a_n})}{2\ln 2}\right\rceil\right\}$, where c_1 is defined in (2.109), and a_n are defined in (a),

   (c) construct the vectors v^δ and g^δ:

$$v_i^\delta := \langle K^*_{m_n} f_\delta, \Phi_i\rangle, \quad i = 1, 2, \dots, 2^{m_n}, \tag{2.125}$$

$$g_i^\delta = \langle f_\delta, \Phi_i\rangle \quad i = 1, 2, \dots, 2^{m_n}, \tag{2.126}$$

   (d) construct the matrices A_{m_n} and B_{m_n}:

$$\begin{aligned}(A_{m_n})_{ij} :&= \sum_{l=1}^{2^{m_n}+1} \beta_l^{(m_n)} \langle k(s_l,\cdot), \Phi_i\rangle\langle k(s_l,\cdot)\Phi_j\rangle,\\ &i,j = 1,2,3,\dots,2^{m_n},\end{aligned} \tag{2.127}$$

$$\begin{aligned}(B_{m_n})_{ij} :&= \sum_{l=1}^{2^{m_n}+1} \eta_l^{(m_n)} \langle k(\cdot,s_l), \Phi_i\rangle\langle k(\cdot,s_l)\Phi_j\rangle,\\ &i,j = 1,2,3,\dots,2^{m_n},\end{aligned} \tag{2.128}$$

   where $\beta_i^{(m_n)}$ and $\eta_l^{(m_n)}$ are the quadrature weights and s_l are the collocation points,

   (e) solve the following two linear algebraic systems:

$$(a_n I + A_{m_n})\zeta^{(m_n,\delta)} = v^\delta, \tag{2.129}$$

where $(\zeta^{(m_n,\delta)})_i = \zeta_i^{(m_n,\delta)}$ and

$$(a_n I + B_{m_n})\gamma^{(m_n,\delta)} = g^\delta, \tag{2.130}$$

where $(\gamma^{(m_n,\delta)})_i = \gamma_i^{(m_n,\delta)}$,

(f) update the coefficient $\langle \zeta^{(m_n,\delta)}, \Phi_i \rangle$ of the approximate solution $u_{n,m_n}(x)$ in (2.122) by the iterative formula:

$$u^\delta_{n,m_n}(x) = q u^\delta_{n-1,m_{n-1}}(x) + (1-q)\sum_{j=1}^{2^{m_n}} \zeta_j^{(m_n,\delta)} \Phi_j(x), \tag{2.131}$$

where

$$u^\delta_{0,m_0}(x) = 0, \tag{2.132}$$

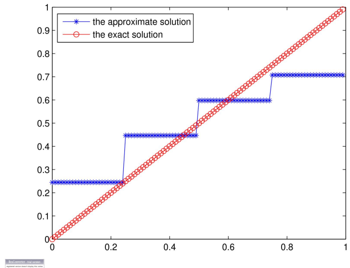
until

$$G_{n,m_n} = qG_{n-1,m_{n-1}} + a_n \|\gamma^{(m_n,\delta)}\| \le C\delta^\varepsilon. \tag{2.133}$$

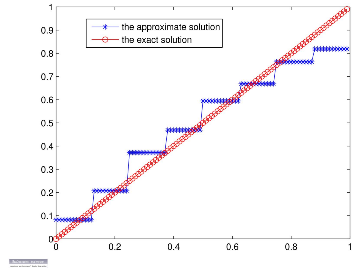
Since K is a selfadjoint operator, the matrix B_{m_n} in step (d) is equal to the matrix A_{m_n}. We measure the accuracy of the approximate solution $u^\delta_{m_{n_\delta}}$ by the following average error formula:

$$Avg := \frac{\sum_{j=1}^{100} |u(t_j) - u^\delta_{m_{n_\delta}}(t_j)|}{100}, \quad t_1 = 0, \quad t_j = 0.01j, \ j = 2, 3, \dots, 99, \tag{2.134}$$

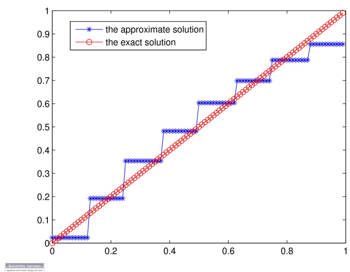
where $u(t)$ is the exact solution to problem (2.107). In all the experiments we use $\alpha_0 = 1$, $q = 0.25$, $C = 2.01$ and $\varepsilon = 0.99$. The linear algebraic systems (2.129) and (2.130) are solved using MATLAB. The levels of noise: 5%, 1%, and .05% are used in the experiments. For the level of noise 5% the stopping condition is satisfied at $m_{n_\delta} = 2$. The resulting average error is 0.1095. When the noise level δ is decreased to the level of noise 1%, we get the average error $Avg = 0.0513$, so the accuracy of the approximate solution is improved. The parameter m_{n_δ} for this level of noise is 3, so one needs to solve a larger linear algebraic system to get such accuracy. When the noise is .5% the average error is improved without increasing the value of the parameter m_n. In this level of noise we get $Avg = 0.0452$. The



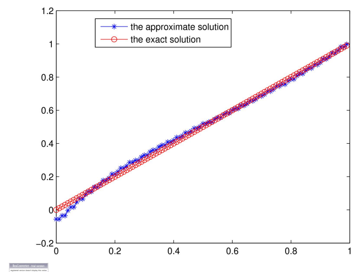
$\delta = 5\%$, $m_{n_\delta} = 2$



$\delta = 1\%$, $m_{n_\delta} = 3$



$\delta = .5\%$, $m_{n_\delta} = 3$



$\delta = .05\%$, $m_{n_\delta} = 5$

Figure 2.1: *Reconstruction of the exact solution $u(t) = t$ using the proposed iterative scheme*

value of the parameter m_n increases to 4 as the level of noise δ decreases to 0.05%. The average error is improved to 0.0250. Figure 1 shows the reconstructions with the proposed iterative scheme for the noise levels: 5%, 1%, 0.5% and 0.05%.

We compare the results of the proposed iterative scheme with the iterative scheme proposed in Chapter 2:

$$u_n^\delta = qu_{n-1}^\delta + (1-q)T_{a_n}^{-1}K^* f_\delta, \quad u_0 = 0, \quad a_n = \alpha_0 q^n,\ \alpha_0 > 0. \tag{2.135}$$

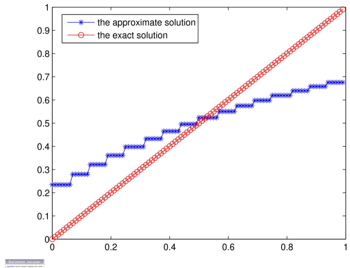
In this iterative scheme we need to solve the following equation:

$$(a_n I + A)z = A^* f_\delta, \tag{2.136}$$

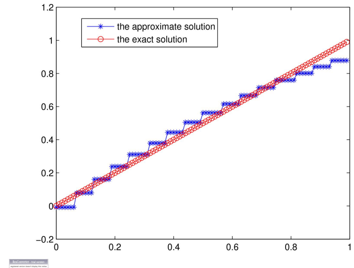
where

$$(A)_{i,j} := \int_0^1 \Phi_i(s) \int_0^1 e^{-st}\Phi_j(t)dtds, \quad i,j = 1,2,\dots,2^m, \tag{2.137}$$

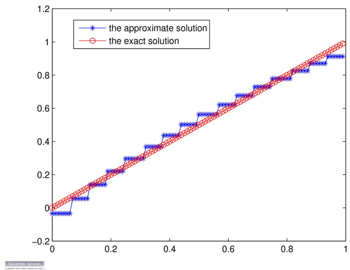
$$(f_\delta)_i := \int_0^1 f_\delta(s)\Phi_i(s)ds, \quad i = 1,2,\dots,2^m, \tag{2.138}$$



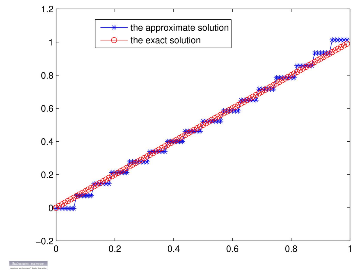
$\delta = 5\%,\ m = 4$



$\delta = 1\%,\ m = 4$



$\delta = .5\%,\ m = 4$



$\delta = .05\%,\ m = 4$

Figure 2.2: *Reconstruction of the exact solution* $u(t) = t$ *using iterative scheme* (2.135)

and $\Phi_i(x)$ are the Haar basis functions. In all the experiments the value of the parameter m in (2.137) and (2.138) is 4, so the size of the matrix A in (2.136) is fixed to 16×16 at each iteration.

In Table 1 we compare the results of the proposed iterative scheme with of iterative scheme (2.135). Here the proposed iterative and iterative scheme (2.135) are denoted by It_1 and It_2, respectively. For the levels of noise 5%, 1%, 0.5% the CPU time of iterative scheme (2.135) are larger than of these for the proposed iterative scheme, since at each iteration of iterative scheme (2.135) one needs to solve linear algebraic system (2.136) with the matrix A of the size 16×16 while in the proposed iterative scheme one only needs to use smaller sizes of the matrix A at each iteration. In general the average errors of the proposed iterative scheme are comparable to of these for iterative scheme (2.135).

Table 2.1: *Fixed vs adaptive iterative scheme*

δ	It_1			It_2		
	Avg	m_{n_δ}	CPU time (seconds)	Avg	m	CPUtime (seconds)
5%	0.1095	2	0.1563	0.1346	4	0.5313
1%	0.0513	3	0.2188	0.0339	4	0.5313
0.5%	0.0452	3	0.2344	0.0300	4	0.5469
0.05%	0.0250	5	0.8281	0.0206	4	0.5313

2.5 Conclusion

A stopping rule with the parameters m_n depending on the regularization parameters a_n is proposed. The m_n is an increasing sequence of the regularization parameter a_n. This allows one to start by solving a small size linear algebraic system (2.129), and one increases the size of the linear algebraic system only if $G_n > C\delta^{\varepsilon}$. In the numerical example it is demonstrated that a simple quadrature method, compound Simpson's quadrature, can be used for approximating the kernel $g(x,z)$, defined in (2.8). Our method yields convergence of the approximate solution u^{δ}_{n,m_δ} to the minimal norm solution of (2.1). Numerical experiments show that all the average errors of the proposed method are comparable to of these for iterative scheme (2.135). Our numerical experiments demonstrate that the adaptive choice of the parameter m_n is more efficient, in the following sense: the value of the parameters m_n of the proposed iterative scheme at the noise levels 5%,1% and 0.5% are smaller than of the parameter m, used in the iterative scheme (2.135). Therefore the computational time of the proposed method at these levels of noise is smaller than the computational time for the iterative scheme (2.135). The adaptive choice of the parameters m_n may give a large size of the matrix A_{m_n} in (2.129), since m_n is a non-decreasing sequence depending on the geometric sequence a_n, so the CPU time increases as the value of the parameter m_n increases. In the iterative scheme (2.135) the size of the matrix A in (2.136) is fixed at each iteration, so the CPU time depends on the number of iterations. The drawback of using a fixed size $2^m \times 2^m$ of the matrix A in (2.136) at each iteration is: the solution u^{δ}_n, defined by formula (2.135), where $n = n(\delta)$ is found by the stopping rule (2.73) with $m_n = m \ \forall n$, may approximate the minimal norm solution on the finite-dimensional space $L_m = \text{span}\{\Phi_1, \Phi_2, \dots, \Phi_{2^m}\}$ not accurately, so that for some levels of the noise the exact solution to problem (2.107) will not be well approximated by any function from L_m. From Table 1 one can see that the number of basis functions used for an approximation of the minimal norm solution with the accuracy 0.1095 by the iterative scheme with the adaptive choice of m_n is four times smaller than the number of these functions used in the iterative scheme with a fixed m, while the accuracy is 0.1095 in It_1 and 0.1346 in It_2 (see line 1 in Table 1).

Chapter 3

Inversion of the Laplace transform from the real axis using an adaptive iterative method

3.1 Introduction

Consider the Laplace transform :

$$\mathcal{L}f(p) := \int_0^\infty e^{-pt}f(t)dt = F(p), \quad \mathrm{Re}p > 0, \tag{3.1}$$

where $\mathcal{L} : X_{0,b} \to L^2[0,\infty)$,

$$X_{0,b} := \{f \in L^2[0,\infty) \mid \mathrm{supp}f \subset [0,b)\}, \quad b > 0. \tag{3.2}$$

We assume in (3.2) that f has compact support. This is not a restriction practically. Indeed, if $\lim_{t\to\infty} f(t) = 0$, then $|f(t)| < \delta$ for $t > t_\delta$, where $\delta > 0$ is an arbitrary small number. Therefore, one may assume that $\mathrm{supp}f \subset [0, t_\delta]$, and treat the values of f for $t > t_\delta$ as noise. One may also note that if $f \in L^1(0,\infty)$, then

$$F(p) := \int_0^\infty f(t)e^{-pt}dt = \int_0^b f(t)e^{-pt}dt + \int_b^\infty f(t)e^{-pt}dt := F_1(p) + F_2(p),$$

and $|F_2(p)| \le e^{-bp}\delta$, where $\int_b^\infty |f(t)|dt \le \delta$. Therefore, the contribution of the "tail" $f_b(t)$ of f,

$$f_b(t) := \begin{cases} 0, & t < b, \\ f(t), & t \ge b, \end{cases}$$

can be considered as noise if $b > 0$ is large and $\delta > 0$ is small. We assume in (3.2) that $f \in L^2[0,\infty)$. One may also assume that $f \in L^1[0,\infty)$, or that $|f(t)| \le c_1 e^{c_2 t}$, where c_1, c_2 are positive constants. If the last assumption holds, then one may define the function $g(t) := f(t)e^{-(c_2+1)t}$. Then $g(t) \in L^1[0,\infty)$, and its Laplace transform $G(p) = F(p + c_2 + 1)$ is known on the interval $[c_2+1, c_2+1+b]$ of real axis if the Laplace transform $F(p)$ of $f(t)$ is known on the interval $[0, b]$. Therefore, our inversion methods are applicable to these more general classes of functions f as well.

The operator $\mathcal{L} : X_{0,b} \to L^2[0,\infty)$ is compact. Therefore, the inversion of the Laplace transform (3.1) is an ill-posed problem (see[23],[29]). Since the problem is ill-posed, a regularization method is needed to obtain a stable inversion of the Laplace transform. There are many methods to solve equation (3.1) stably: variational regularization, quasisolutions, iterative regularization (see e.g,[23],[29],[30]). In this Chapter we propose an adaptive iterative method based on the Dynamical Systems Method (DSM) developed in[29],[30]. Some methods have been developed earlier for the inversion of the Laplace transform (see[2],[5],[10],[18]). In many papers the data $F(p)$ are assumed exact and given on the complex axis. In[22] it is shown that the results of the inversion of the Laplace transform from the complex axis are more accurate than these of the inversion of the Laplace transform from the real axis. The reason is the ill-posedness of the Laplace transform inversion from the real axis. A survey regarding the methods of the Laplace transform inversion has been given in[5]. There are several types of the Laplace inversion method compared in[5]. The inversion formula for the Laplace transform is well known:

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(p)e^{pt}dp, \quad \sigma > 0, \tag{3.3}$$

is used in some of these methods, and then $f(t)$ is computed by some quadrature formulas, and many of these formulas can be found in[6] and[21]. Moreover, the ill-posedness of the Laplace transform inversion is not discussed in all the methods compared in[5]. The approximate $f(t)$, obtained by these methods when the data are noisy, may differ significantly from $f(t)$. There are some papers in which the inversion of the Laplace transform from the real axis was studied (see[24],[1],[4],[9],[12],[22],[26],[36],[37]). In[1] and[26] a method based on the Mellin transform is developed. In this method the Mellin transform of the data $F(p)$ is calculated first and then inverted for $f(t)$. In[4] a Fourier series method for the inversion of Laplace transform from the real axis is developed. The drawback of this method comes from the ill-conditioning of the discretized problem. It is shown in[4] that if one uses some basis functions in $X_{0,b}$, the problem becomes extremely ill-conditioned if the number m of the basis functions exceeds 20. In[12] a reproducing kernel method is used for the inversion of the Laplace transform. In the numerical experiments in[12] the authors use double and multiple precision methods to obtain high accuracy inversion of the Laplace transform. The usage of the multiple precision increases the computation time significantly which is observed in[12], so this method may be not efficient in practice. A detailed description of the multiple precision technique can be found in[11] and[14]. Moreover, the Laplace transform inversion with perturbed data is not discussed in[12]. In[37] the authors develop an inversion formula, based on the eigenfunction expansion for the Laplace transform. The difficulties with this method are: a) the inversion formula is not applicable when the data are noisy, b) even for exact data the inversion formula is not suitable for numerical implementation.

The Laplace transform as an operator from C_{0k} into L^2, where

$$C_{0k} = \{f(t) \in C[0,+\infty) \mid \mathrm{supp} f \subset [0,k)\},\ k = const > 0,\ L^2 := L^2[0,\infty),$$

is considered in[9]. The finite difference method is used in[9] to discretize the problem, where the size of the linear algebraic system obtained by this method is fixed at each iteration, so the computation time increases if one uses large linear algebraic systems. The method of choosing the size of the linear algebraic system is not given in[9]. Moreover, the inversion of the Laplace transform when the data $F(p)$ is given only on a finite interval $[0,d]$, $d > 0$, is not discussed in[9].

The novel points of this Chapter are:

1) the representation of the approximation solution (3.73) of the function $f(t)$ which depends only on the kernel of the Laplace transform,

2) the adaptive iterative scheme (3.76) and adaptive stopping rule (3.87), which generate the regularization parameter, the discrete data $F_\delta(p)$ and the number of terms in (3.73), needed for obtaining an approximation of the unknown function $f(t)$.

We study the inversion problem using the pair of spaces $(X_{0,b}, L^2[0,d])$, where $X_{0,b}$ is defined in (3.2), develop an inversion method, which can be easily implemented numerically, and demonstrate in the numerical experiments that our method yields the results comparable in accuracy with the results, presented in the literature, e.g., with the double precision results given in [12].

The smoothness of the kernel allows one to use the compound Simpson's rule in approximating the Laplace transform. Our approach yields a representation (3.73) of the approximate inversion of the Laplace transform. The number of terms in approximation (3.73) and the regularization parameter are generated automatically by the proposed adaptive iterative method. Our iterative method is based on the iterative method proposed in Chapter 2. The adaptive stopping rule we propose here is based on the discrepancy-type principle, established in [28,33]. This stopping rule yields convergence of the approximation (3.73) to $f(t)$ when the noise level $\delta \to 0$.

A detailed derivation of our inversion method is given in Section 2. In Section 3 some results of the numerical experiments are reported. These results demonstrate the efficiency and stability of the proposed method.

3.2 Description of the method

Let $f \in X_{0,b}$. Then equation (3.1) can be written as:

$$(\mathcal{L}f)(p) := \int_0^b e^{-pt} f(t)dt = F(p), \quad 0 \leq p. \tag{3.4}$$

Let us assume that the data $F(p)$, the Laplace transform of f, are known only for $0 \leq p \leq d < \infty$. Consider the mapping $\mathcal{L}_m : L^2[0,b] \to \mathbb{R}^{m+1}$, where

$$(\mathcal{L}_m f)_i := \int_0^b e^{-p_i t} f(t)dt = F(p_i), \quad i = 0, 1, 2, \ldots, m, \tag{3.5}$$

$$p_i := ih, \quad i = 0, 1, 2, \ldots, m, \ h := \frac{d}{m}, \tag{3.6}$$

and m is an even number which will be chosen later. Then the unknown function $f(t)$ can be obtained from a finite-dimensional operator equation (3.5). Let

$$\langle u, v\rangle_{W^m} := \sum_{j=0}^{m} w_j^{(m)} u_j v_j \quad \textit{and} \quad \|u\|_{W^m} := \langle u, u\rangle_{W^m} \tag{3.7}$$

be the inner product and norm in $\mathbb{R}^{m+1}$, respectively, where $w_j^{(m)}$ are the weights of the compound Simpson's rule (see[6] p.58), i.e.,

$$w_j^{(m)} := \begin{cases} h/3, & j = 0, m; \\ 4h/3, & j = 2l - 1, \ l = 1, 2, \ldots, m/2; \\ 2h/3, & j = 2l, \ l = 1, 2, \ldots, (m-2)/2, \end{cases} \qquad h = \frac{d}{m}, \tag{3.8}$$

where m is an even number. Then

$$\begin{aligned} \langle \mathcal{L}_m g, v\rangle_{W^m} &= \sum_{j=0}^{m} w_j^{(m)} \int_0^b e^{-p_j t} g(t) dt v_j \\ &= \int_0^b g(t) \sum_{j=0}^{m} w_j^{(m)} e^{-p_j t} v_j dt = \langle g, \mathcal{L}_m^* v\rangle_{X_{0,b}}, \end{aligned} \tag{3.9}$$

where

$$\mathcal{L}_m^* v = \sum_{j=0}^{m} w_j^{(m)} e^{-p_j t} v_j, \quad v := \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^{m+1}. \tag{3.10}$$

and

$$\langle g, h\rangle_{X_{0,b}} := \int_0^b g(t) h(t) dt. \tag{3.11}$$

It follows from (3.5) and (3.10) that

$$(\mathcal{L}_m^* \mathcal{L}_m g)(t) = \sum_{j=0}^{m} w_j^{(m)} e^{-p_j t} \int_0^b e^{-p_j z} g(z) dz := (T^{(m)} g)(t), \tag{3.12}$$

and

$$\mathcal{L}_m \mathcal{L}_m^* v = \begin{pmatrix} \int_0^b e^{-p_0 t} \sum_{j=0}^{m} w_j^{(m)} e^{-p_j t} v_j dt \\ \int_0^b e^{-p_1 t} \sum_{j=0}^{m} w_j^{(m)} e^{-p_j t} v_j dt \\ \vdots \\ \int_0^b e^{-p_m t} \sum_{j=0}^{m} w_j^{(m)} e^{-p_j t} v_j dt \end{pmatrix} := Q^{(m)} v, \tag{3.13}$$

where

$$(Q^{(m)})_{ij} := w_j^{(m)} \int_0^b e^{-(p_i+p_j)t} dt = w_j^{(m)} \frac{1-e^{-b(p_i+p_j)}}{p_i+p_j}, \quad i,j = 0,1,2,\dots,m. \tag{3.14}$$

Lemma 3.2.1. *Let $w_j^{(m)}$ be defined in (3.8). Then*

$$\sum_{j=0}^{m} w_j^{(m)} = d, \tag{3.15}$$

for any even number m.

Proof. From definition (3.8) one gets

$$\begin{aligned}\sum_{j=0}^{m} w_j^{(m)} &= w_0^{(m)} + w_m^{(m)} + \sum_{j=1}^{m/2} w_{2j-1}^{(m)} + \sum_{j=1}^{(m-2)/2} w_{2j}^{(m)} \\ &= \frac{2h}{3} + \sum_{j=1}^{m/2} \frac{4h}{3} + \sum_{j=1}^{(m-2)/2} \frac{2h}{3} \\ &= \frac{2h}{3} + \frac{2hm}{3} + \frac{h(m-2)}{3} = hm = \frac{d}{m} m = d.\end{aligned} \tag{3.16}$$

Lemma 3.2.1 is proved. □

Lemma 3.2.2. *The matrix $Q^{(m)}$, defined in (3.14), is positive semidefinite and self-adjoint in $\mathbb{R}^{m+1}$ with respect to the inner product (3.7).*

Proof. Let

$$(H_m)_{ij} := \int_0^b e^{-(p_i+p_j)t} dt = \frac{1-e^{-b(p_i+p_j)}}{p_i+p_j}, \tag{3.17}$$

and

$$(D_m)_{ij} = \begin{cases} w_i^{(m)}, & i = j; \\ 0, & \text{otherwise}, \end{cases} \tag{3.18}$$

$w_j^{(m)}$ are defined in (3.8). Then $\langle D_m H_m D_m u, v \rangle_{\mathbb{R}^{m+1}} = \langle u, D_m H_m D_m v \rangle_{\mathbb{R}^{m+1}}$, where

$$\langle u, v \rangle_{\mathbb{R}^{m+1}} := \sum_{j=0}^{m} u_j v_j, \quad u, v \in \mathbb{R}^{m+1}. \tag{3.19}$$

We have

$$\begin{aligned}\langle Q^{(m)}u, v\rangle_{W^m} &= \sum_{j=0}^{m} w_j^{(m)}(Q^{(m)}u)_j v_j = \sum_{j=0}^{m}(D_m H_m D_m u)_j v_j \\ &= \langle D_m H_m D_m u, v\rangle_{\mathbb{R}^{m+1}} = \langle u, D_m H_m D_m v\rangle_{\mathbb{R}^{m+1}} \\ &= \sum_{j=0}^{m} u_j (D_m H_m D_m v)_j = \sum_{j=0}^{m} u_j w_j^{(m)} (H_m D_m v)_j \\ &= \langle u, Q^{(m)}v\rangle_{W^m}.\end{aligned} \tag{3.20}$$

Thus, $Q^{(m)}$ is self-adjoint with respect to inner product (3.7). We have

$$\begin{aligned}(H_m)_{ij} &= \int_0^b e^{-(p_i+p_j)t}dt = \int_0^b e^{-p_i t}e^{-p_j t}dt \\ &= \langle \phi_i, \phi_j\rangle_{X_{0,b}}, \quad \phi_i(t) := e^{-p_i t},\end{aligned} \tag{3.21}$$

where $\langle \cdot, \cdot\rangle_{X_{0,b}}$ is defined in (3.11). This shows that H_m is a Gram matrix. Therefore,

$$\langle H_m u, u\rangle_{\mathbb{R}^{m+1}} \geq 0, \;\; \forall u \in \mathbb{R}^{m+1}. \tag{3.22}$$

This implies

$$\langle Q^{(m)}u, u\rangle_{W^m} = \langle Q^{(m)}u, D_m u\rangle_{\mathbb{R}^{m+1}} = \langle H_m D_m u, D_m u\rangle_{\mathbb{R}^{m+1}} \geq 0. \tag{3.23}$$

Thus, $Q^{(m)}$ is a positive semidefinite and self-adjoint matrix with respect to the inner product (3.7). □

Lemma 3.2.3. *Let $T^{(m)}$ be defined in (3.12). Then $T^{(m)}$ is self-adjoint and positive semidefinite operator in $X_{0,b}$ with respect to inner product (3.11).*

Proof. From definition (3.12) and inner product (3.11) we get

$$\begin{aligned}\langle T^{(m)}g, h\rangle_{X_{0,b}} &= \int_0^b \sum_{j=0}^{m} w_j^{(m)} e^{-p_j t} \int_0^b e^{-p_j z} g(z)dz h(t)dt \\ &= \int_0^b g(z) \sum_{j=0}^{m} w_j^{(m)} e^{-p_j z} \int_0^b e^{-p_j t} h(t)dtdz \\ &= \langle g, T^{(m)}h\rangle_{X_{0,b}}.\end{aligned} \tag{3.24}$$

Thus, $T^{(m)}$ is a self-adjoint operator with respect to inner product (3.11). Let us prove that $T^{(m)}$ is positive semidefinite. Using (3.12), (3.8), (3.7) and (3.11), one gets

$$\begin{aligned}
\langle T^{(m)}g, g\rangle_{X_{0,b}} &= \int_0^b \sum_{j=0}^m w_j^{(m)} e^{-p_j t} \int_0^b e^{-p_j z} g(z)dz g(t)dt \\
&= \sum_{j=0}^m w_j^{(m)} \int_0^b e^{-p_j z} g(z)dz \int_0^b e^{-p_j t} g(t)dt \\
&= \sum_{j=0}^m w_j^{(m)} \left(\int_0^b e^{-p_j z} g(z)dz\right)^2 \geq 0.
\end{aligned} \tag{3.25}$$

Lemma 3.2.3 is proved. $\square$

From (3.10) we get Range$[\mathcal{L}_m^*] = span\{w_j^{(m)} k(p_j, \cdot, 0)\}_{j=0}^m$, where

$$k(p, t, z) := e^{-p(t+z)}. \tag{3.26}$$

Let us approximate the unknown $f(t)$ as follows:

$$f(t) \approx \sum_{j=0}^m c_j^{(m)} w_j^{(m)} e^{-p_j t} = T_{a,m}^{-1} \mathcal{L}_m^* F^{(m)} := f_m(t), \tag{3.27}$$

where p_j are defined in (3.6), $T_{a,m}$ is defined in (3.34), and $c_j^{(m)}$ are constants obtained by solving the linear algebraic system:

$$(aI + Q^{(m)})c^{(m)} = F^{(m)}, \tag{3.28}$$

where $Q^{(m)}$ is defined in (3.13),

$$c^{(m)} := \begin{pmatrix} c_0^{(m)} \\ c_1^{(m)} \\ \vdots \\ c_m^{(m)} \end{pmatrix} \quad and \quad F^{(m)} := \begin{pmatrix} F(p_0) \\ F(p_1) \\ \vdots \\ F(p_m) \end{pmatrix}. \tag{3.29}$$

To prove the convergence of the approximate solution $f(t)$, we use the following estimates, which are proved in[30], so their proofs are omitted.

Lemma 3.2.4. *Let $T^{(m)}$ and $Q^{(m)}$ be defined in (3.12) and (3.13), respectively. Then, for $a > 0$, the following estimates hold:*

$$\|Q_{a,m}^{-1}\mathcal{L}_m\| \le \frac{1}{2\sqrt{a}}, \tag{3.30}$$

$$a\|Q_{a,m}^{-1}\| \le 1, \tag{3.31}$$

$$\|T_{a,m}^{-1}\| \le \frac{1}{a}, \tag{3.32}$$

$$\|T_{a,m}^{-1}\mathcal{L}_m^*\| \le \frac{1}{2\sqrt{a}}, \tag{3.33}$$

where

$$Q_{a,m} := Q^{(m)} + aI \quad T_{a,m} := T^{(m)} + aI, \tag{3.34}$$

I is the identity operator and $a = const > 0$.

Estimates (3.30) and (3.31) are used in proving inequality (3.92), while estimates (3.32) and (3.33) are used in the proof of lemmas 2.9 and 2.10, respectively.

Let us formulate an iterative method for obtaining the approximation solution of $f(t)$ with the exact data $F(p)$. Consider the following iterative scheme

$$u_n(t) = qu_{n-1}(t) + (1-q)T_{a_n}^{-1}\mathcal{L}^*F, \quad u_0(t) = 0, \tag{3.35}$$

where $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$, i.e.,

$$(\mathcal{L}^*g)(t) = \int_0^d e^{-pt}g(p)dp, \tag{3.36}$$

$$\begin{aligned}(Tf)(t) := (\mathcal{L}^*\mathcal{L}f)(t) &= \int_0^b \int_0^d k(p,t,z)dpf(z)dz \\ &= \int_0^b \frac{f(z)}{t+z}\left(1 - e^{-d(t+z)}\right)dz,\end{aligned} \tag{3.37}$$

$k(p,t,z)$ is defined in (3.26),

$$T_a := aI + T, \quad a > 0, \tag{3.38}$$

$$a_n := qa_{n-1}, \quad a_0 > 0, \quad q \in (0,1). \tag{3.39}$$

Lemma 3.2.5. *Let T_a be defined in (3.38), $\mathcal{L}f = F$, and $f \perp \mathcal{N}(\mathcal{L})$, where $\mathcal{N}(\mathcal{L})$ is the null space of $\mathcal{L}$. Then*

$$a\|T_a^{-1}f\| \to 0 \quad \text{as } a \to 0. \tag{3.40}$$

Proof. Since $f \perp \mathcal{N}(\mathcal{L})$, it follows from the spectral theorem that

$$\lim_{a\to 0} a^2\|T_a^{-1}f\|^2 = \lim_{a\to 0}\int_0^\infty \frac{a^2}{(a+s)^2} d\langle E_s f, f\rangle = \|P_{\mathcal{N}(\mathcal{L})}f\|^2 = 0,$$

where E_s is the resolution of the identity corresponding to $\mathcal{L}^*\mathcal{L}$, and P is the orthogonal projector onto $\mathcal{N}(\mathcal{L})$.

Lemma 3.2.5 is proved. □

Theorem 3.2.6. *Let $\mathcal{L}f = F$, and u_n be defined in (3.35) Then*

$$\lim_{n\to\infty} \|f - u_n\| = 0. \tag{3.41}$$

Proof. By induction we get

$$u_n = \sum_{j=0}^{n-1} \omega_j^{(n)} T_{a_{j+1}}^{-1}\mathcal{L}^* F, \tag{3.42}$$

where T_a is defined in (3.38), and

$$\omega_j^{(n)} := q^{n-j-1} - q^{n-j}. \tag{3.43}$$

Using the identities

$$\mathcal{L}f = F, \tag{3.44}$$

$$T_a^{-1}\mathcal{L}^*\mathcal{L} = T_a^{-1}(T + aI - aI) = I - aT_a^{-1} \tag{3.45}$$

and

$$\sum_{j=0}^{n-1} \omega_j^{(n)} = 1 - q^n, \tag{3.46}$$

we get

$$\begin{aligned} f - u_n &= f - \sum_{j=0}^{n-1}\omega_j^{(n)} f + \sum_{j=0}^{n-1}\omega_j^{(n)} a_{j+1} T_{a_{j+1}}^{-1} f \\ &= q^n f + \sum_{j=0}^{n-1}\omega_j^{(n)} a_{j+1} T_{a_{j+1}}^{-1} f. \end{aligned} \tag{3.47}$$

Therefore,

$$\|f-u_n\| \leq q^n\|f\| + \sum_{j=0}^{n-1} \omega_j^{(n)} a_{j+1} \|T_{a_{j+1}}^{-1} f\|. \tag{3.48}$$

To prove relation (3.41) the following lemma is needed:

Lemma 3.2.7. *Let $g(x)$ be a continuous function on $(0,\infty)$, $c>0$ and $q\in(0,1)$ be constants. If*

$$\lim_{x\to 0^+} g(x) = g(0) := g_0, \tag{3.49}$$

then

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \left(q^{n-j-1} - q^{n-j}\right) g(cq^{j+1}) = g_0. \tag{3.50}$$

Proof. Let

$$F_l(n) := \sum_{j=1}^{l-1} \omega_j^{(n)} g(cq^{j+1}), \tag{3.51}$$

where $\omega_j^{(n)}$ are defined in (3.43). Then

$$|F_{n+1}(n) - g_0| \leq |F_l(n)| + \left| \sum_{j=l}^{n} \omega_j^{(n)} g(cq^{j+1}) - g_0 \right|.$$

Take $\epsilon>0$ arbitrarily small. For sufficiently large fixed $l(\epsilon)$ one can choose $n(\epsilon) > l(\epsilon)$, such that

$$|F_{l(\epsilon)}(n)| \leq \frac{\epsilon}{2},\ \forall n > n(\epsilon),$$

because $\lim_{n\to\infty} q^n = 0$. Fix $l = l(\epsilon)$ such that $|g(cq^j) - g_0| \leq \frac{\epsilon}{2}$ for $j > l(\epsilon)$. This is possible because of (3.49). One has

$$|F_{l(\epsilon)}(n)| \leq \frac{\epsilon}{2},\ n > n(\epsilon) > l(\epsilon)$$

and

$$\begin{aligned}\left| \sum_{j=l(\epsilon)}^{n} \omega_j^{(n)} g(cq^{j+1}) - g_0 \right| &\leq \sum_{j=l(\epsilon)}^{n} \omega_j^{(n)} |g(cq^{j+1}) - g_0| + |\sum_{j=l(\epsilon)}^{n} \omega_j^{(n)} - 1||g_0| \\ &\leq \frac{\epsilon}{2} \sum_{j=l(\epsilon)}^{n} \omega_j^{(n)} + q^{n-l(\epsilon)}|g_0| \\ &\leq \frac{\epsilon}{2} + |g_0|q^{n-l(\epsilon)} \leq \epsilon,\end{aligned}$$

if $n(\epsilon)$ is sufficiently large. Here we have used the relation

$$\sum_{j=l}^{n} \omega_j^{(n)} = 1 - q^{n-l}.$$

Since $\epsilon > 0$ is arbitrarily small, relation (3.50) follows.

Lemma 1.2.1 is proved. $\square$

Lemma 3.2.5 together with Lemma 1.2.1 with $g(a) = a\|T_a^{-1}f\|$ yield

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \omega_j^{(n)} a_{j+1} \|T_{a_{j+1}}^{-1} f\| = 0. \tag{3.52}$$

This together with estimate (3.48) and condition $q \in (0,1)$ yield relation (3.41).

Theorem 3.2.6 is proved. $\square$

Lemma 3.2.8. *Let T and $T^{(m)}$ be defined in (3.37) and (3.12), respectively. Then*

$$\|T - T^{(m)}\| \le \frac{(2bd)^5}{540\sqrt{10}m^4}. \tag{3.53}$$

Proof. From definitions (3.37) and (3.12) we get

$$\begin{aligned} |(T - T^{(m)})f(t)| &\le \int_0^b \left| \int_0^d k(p,t,z)dp - \sum_{j=0}^{m} w_j^{(m)} k(p_j,t,z) \right| |f(z)|dz \\ &\le \int_0^b \left| \frac{d^5}{180m^4} \max_{p\in[0,d]} (t+z)^4 e^{-p(t+z)} \right| |f(z)|dz \\ &= \int_0^b \frac{d^5}{180m^4}(t+z)^4|f(z)|dz \le \frac{d^5}{180m^4} \left(\int_0^b (t+z)^8 dz \right)^{1/2} \|f\|_{X_{0,b}} \\ &= \frac{d^5}{180m^4} \left[\frac{(t+b)^9 - t^9}{9} \right]^{1/2} \|f\|_{X_{0,b}}, \end{aligned} \tag{3.54}$$

where the following upper bound for the error of the compound Simpson's rule was used (see[6] p.58): for $f \in C^{(4)}[x_0, x_{2l}]$, $x_0 < x_{2l}$,

$$\left| \int_{x_0}^{x_{2l}} f(x)dx - \frac{h}{3}\left[f_0 + 4\sum_{j=1}^{l} f_{2(j-1)} + 2\sum_{j=1}^{l-1} f_{2j} + f_{x_{2l}} \right] \right| \le R_l, \tag{3.55}$$

where

$$f_j := f(x_j), \quad x_j = x_0 + jh,\ j = 0,1,2,\ldots,2l,\ h = \frac{x_{2l} - x_0}{2l}, \tag{3.56}$$

and

$$R_l = \frac{(x_{2l} - x_0)^5}{180(2l)^4} |f^{(4)}(\xi)|, \quad x_0 < \xi < x_{2l}. \tag{3.57}$$

This implies

$$\|(T - T^{(m)})f\|_{X_{0,b}} \le \frac{d^5}{540m^4} \left[\frac{(2b)^{10} - 2b^{10}}{10}\right]^{1/2} \|f\|_{X_{0,b}} \le \frac{(2bd)^5}{540\sqrt{10}m^4} \|f\|_{X_{0,b}}, \tag{3.58}$$

so estimate (3.53) is obtained.

Lemma 3.2.8 is proved. □

Lemma 3.2.9. *Let* $0 < a < a_0$*,*

$$m = \kappa \left(\frac{a_0}{a}\right)^{1/4}, \quad \kappa > 0. \tag{3.59}$$

Then

$$\|T - T^{(m)}\| \le \frac{(2bd)^5}{540\sqrt{10}a_0\kappa^4} a, \tag{3.60}$$

where T *and* $T^{(m)}$ *are defined in* (3.37) *and* (3.12)*, respectively.*

Proof. Inequality (3.60) follows from estimate (3.53) and formula (3.59). □

Lemma 3.2.9 leads to an adaptive iterative scheme:

$$u_{n,m_n}(t) = qu_{n-1,m_{n-1}} + (1-q)T^{-1}_{a_n,m_n}\mathcal{L}^*_{m_n}F^{(m_n)}, \quad u_{0,m_0}(t) = 0, \tag{3.61}$$

where $q \in (0,1)$, a_n are defined in (3.39), $T_{a,m}$ is defined in (3.34), $A_m\mathcal{L}$ is defined in (3.5), and

$$F^{(m)} := \begin{pmatrix} F(p_0) \\ F(p_1) \\ \cdots \\ F(p_m) \end{pmatrix} \in \mathbb{R}^{m+1}, \tag{3.62}$$

p_j are defined in (3.6). In the iterative scheme (3.61) we have used the finite-dimensional operator $T^{(m)}$ approximating the operator T. Convergence of the iterative scheme (3.61) to the solution f of the equation $\mathcal{L}f = F$ is established in the following lemma:

Lemma 3.2.10. *Let $\mathcal{L}f = F$ and u_{n,m_n} be defined in (3.61). If m_n are chosen by the rule*

$$m_n = \left\lceil \left[\kappa \left(\frac{a_0}{a_n} \right)^{1/4} \right] \right\rceil, \quad a_n = qa_{n-1}, \ q \in (0,1), \ \kappa, a_0 > 0, \tag{3.63}$$

where $\lceil [x] \rceil$ is the smallest even number not less than x, then

$$\lim_{n\to\infty} \|f - u_{n,m_n}\| = 0. \tag{3.64}$$

Proof. Consider the estimate

$$\|f - u_{n,m_n}\| \le \|f - u_n\| + \|u_n - u_{n,m_n}\| := I_1(n) + I_2(n), \tag{3.65}$$

where $I_1(n) := \|f - u_n\|$ and $I_2(n) := \|u_n - u_{n,m_n}\|$. By Theorem 3.2.6, we get $I_1(n) \to 0$ as $n \to \infty$. Let us prove that $\lim_{n\to\infty} I_2(n) = 0$. Let $U_n := u_n - u_{n,m_n}$. Then, from definitions (3.35) and (3.61), we get

$$U_n = qU_{n-1} + (1-q)\left(T_{a_n}^{-1}\mathcal{L}^*F - T_{a_n,m_n}^{-1}\mathcal{L}_{m_n}^*F^{(m_n)}\right), \quad U_0 = 0. \tag{3.66}$$

By induction we obtain

$$U_n = \sum_{j=0}^{n-1} \omega_j^{(n)} \left(T_{a_{j+1}}^{-1}\mathcal{L}^*F - T_{a_{j+1},m_{j+1}}^{-1}(\mathcal{L}_{m_{j+1}})^*F^{(m_{j+1})} \right), \tag{3.67}$$

where ω_j are defined in (3.43). Using the identities $\mathcal{L}f = F$, $\mathcal{L}_m f = F^{(m)}$,

$$T_a^{-1}T = T_a^{-1}(T + aI - aI) = I - aT_a^{-1}, \tag{3.68}$$

$$T_{a,m}^{-1}T^{(m)} = T_{a,m}^{-1}(T^{(m)} + aI - aI) = I - aT_{a,m}^{-1}, \tag{3.69}$$

$$T_{a,m}^{-1} - T_a^{-1} = T_{a,m}^{-1}(T - T^{(m)})T_a^{-1}, \tag{3.70}$$

one gets

$$\begin{aligned} U_n &= \sum_{j=0}^{n-1} \omega_j^{(n)} a_{j+1} \left(T_{a_{j+1},m_{j+1}}^{-1} - T_{a_{j+1}}^{-1} \right) f \\ &= \sum_{j=0}^{n-1} \omega_j^{(n)} a_{j+1} T_{a_{j+1},m_{j+1}}^{-1} \left(T - T^{(m_{j+1})} \right) T_{a_{j+1}}^{-1} f. \end{aligned} \tag{3.71}$$

This together with the rule (3.63), estimate (3.32) and Lemma 3.2.8 yield

$$\begin{aligned}\|U_n\| &\leq \sum_{j=0}^{n-1} \omega_j^{(n)} a_{j+1} \|T^{-1}_{a_{j+1},m_{j+1}}\| \|T - T^{(m_{j+1})}\| \|T^{-1}_{a_{j+1}} f\| \\ &\leq \frac{(2bd)^5}{540\sqrt{10}a_0\kappa^4} \sum_{j=0}^{n-1} \omega_j^{(n)} a_{j+1} \|T^{-1}_{a_{j+1}} f\|.\end{aligned} \tag{3.72}$$

Applying Lemma 3.2.5 and Lemma 1.2.1 with $g(a) = a\|T_a^{-1} f\|$, we obtain $\lim_{n\to\infty} \|U_n\| = 0$. Lemma 3.2.10 is proved. □

3.2.1 Noisy data

When the data $F(p)$ are noisy, the approximate solution (3.27) is written as

$$f_m^\delta(t) = \sum_{j=0}^{m} w_j^{(m)} c_j^{(m,\delta)} e^{-p_j t} = T^{-1}_{a,m} \mathcal{L}_m^* F_\delta^{(m)}, \tag{3.73}$$

where the coefficients $c_j^{(m,\delta)}$ are obtained by solving the following linear algebraic system:

$$Q_{a,m} c^{(m,\delta)} = F_\delta^{(m)}, \tag{3.74}$$

$Q_{a,m}$ is defined in (3.34),

$$c^{(m,\delta)} := \begin{pmatrix} c_0^{(m,\delta)} \\ c_1^{(m,\delta)} \\ \dots \\ c_m^{(m,\delta)} \end{pmatrix}, \quad F_\delta^{(m)} := \begin{pmatrix} F_\delta(p_0) \\ F_\delta(p_1) \\ \dots \\ F_\delta(p_m) \end{pmatrix}, \tag{3.75}$$

$w_j^{(m)}$ are defined in (3.8), and p_j are defined in (3.6).

To get the approximation solution of the function $f(t)$ with the noisy data $F_\delta(p)$, we consider the following iterative scheme:

$$u^\delta_{n,m_n} = q u^\delta_{n-1,m_{n-1}} + (1-q) T^{-1}_{a_n,m_n} \mathcal{L}^*_{m_n} F_\delta^{(m_n)}, \quad u^\delta_{0,m_0} = 0, \tag{3.76}$$

where $T_{a,m}$ is defined in (3.34), a_n are defined in (3.39), $q \in (0,1)$, $F_\delta^{(m)}$ is defined in (3.75), and m_n are chosen by the rule (3.63). Let us assume that

$$F_\delta(p_j) = F(p_j) + \delta_j, \quad 0 < |\delta_j| \leq \delta, \quad j = 0, 1, 2, \dots, m, \tag{3.77}$$

where δ_j are random quantities generated from some statistical distributions, e.g., the uniform distribution on the interval $[-\delta, \delta]$, and δ is the noise level of the data $F(p)$. It follows from assumption (3.77), definition (3.8), Lemma 3.2.1 and the inner product (3.7) that

$$\|F_\delta^{(m)} - F^{(m)}\|_{W^m}^2 = \sum_{j=0}^{m} w_j^{(m)} \delta_j^2 \le \delta^2 \sum_{j=0}^{m} w_j^{(m)} = \delta^2 d. \tag{3.78}$$

Lemma 3.2.11. *Let u_{n,m_n} and u_{n,m_n}^δ be defined in (3.61) and (3.76), respectively. Then*

$$\|u_{n,m_n} - u_{n,m_n}^\delta\| \le \frac{\sqrt{d}\delta}{2\sqrt{a_n}}(1-q^n), \quad q \in (0,1), \tag{3.79}$$

where a_n are defined in (3.39).

Proof. Let $U_n^\delta := u_{n,m_n} - u_{n,m_n}^\delta$. Then, from definitions (3.61) and (3.76),

$$U_n^\delta = qU_{n-1}^\delta + (1-q)T_{a_n,m_n}^{-1}\mathcal{L}_{m_n}^*(F^{(m_n)} - F_\delta^{(m_n)}), \quad U_0^\delta = 0. \tag{3.80}$$

By induction we obtain

$$U_n^\delta = \sum_{j=0}^{n-1} \omega_j^{(n)} T_{a_{j+1},m_{j+1}}^{-1} (\mathcal{L}_{m_{j+1}})^* (F^{(m_{j+1})} - F_\delta^{(m_{j+1})}), \tag{3.81}$$

where $\omega_j^{(n)}$ are defined in (3.43). Using estimates (3.78) and inequality (3.33), one gets

$$\|U_n^\delta\| \le \sqrt{d} \sum_{j=0}^{n-1} \omega_j^{(n)} \frac{\delta}{2\sqrt{a_{j+1}}} \le \frac{\sqrt{d}\delta}{2\sqrt{a_n}} \sum_{j=0}^{m} \omega_j^{(n)} = \frac{\sqrt{d}\delta}{2\sqrt{a_n}}(1-q^n), \tag{3.82}$$

where ω_j are defined in (3.43).

Lemma 3.2.11 is proved. □

Theorem 3.2.12. *Suppose that conditions of Lemma 3.2.10 hold, and n_δ satisfies the following conditions:*

$$\lim_{\delta \to 0} n_\delta = \infty, \quad \lim_{\delta \to 0} \frac{\delta}{\sqrt{a_{n_\delta}}} = 0. \tag{3.83}$$

Then

$$\lim_{\delta \to 0} \|f - u_{n_\delta, m_{n_\delta}}^\delta\| = 0. \tag{3.84}$$

Proof. Consider the estimate:

$$\|f - u^{\delta}_{n_\delta,m_{n_\delta}}\| \le \|f - u_{n_\delta,m_{n_\delta}}\| + \|u_{n_\delta,m_{n_\delta}} - u^{\delta}_{n_\delta,m_{n_\delta}}\|. \tag{3.85}$$

This together with Lemma 3.2.11 yield

$$\|f - u^{\delta}_{n_\delta,m_{n_\delta}}\| \le \|f - u_{n_\delta,m_{n_\delta}}\| + \frac{\sqrt{d}\delta}{2\sqrt{a_{n_\delta}}}(1-q^n). \tag{3.86}$$

Applying relations (3.83) in estimate (3.86), one gets relation (3.84).

Theorem 3.2.12 is proved. □

In the following subsection we propose a stopping rule which implies relations (3.83).

3.2.2 Stopping rule

In this subsection a stopping rule which yields relations (3.83) in Theorem 3.2.12 is given. We propose the stopping rule

$$G_{n_\delta,m_{n_\delta}} \le C\delta^{\varepsilon} < G_{n,m_n}, \quad 1 \le n < n_\delta,\ C > \sqrt{d},\ \varepsilon \in (0,1), \tag{3.87}$$

where

$$G_{n,m_n} = qG_{n-1,m_{n-1}} + (1-q)\|\mathcal{L}_{m_n} z^{(m_n,\delta)} - F^{(m_n)}_{\delta}\|_{W^{m_n}},\ G_{0,m_0} = 0, \tag{3.88}$$

$\|\cdot\|_{W^m}$ is defined in (3.7),

$$z^{(m,\delta)} := \sum_{j=0}^{m} c^{(m,\delta)}_j w^{(m)}_j e^{-p_j t}, \tag{3.89}$$

$w^{(m)}_j$ and p_j are defined in (3.8) and (3.6), respectively, and $c^{(m,\delta)}_j$ are obtained by solving linear algebraic system (3.74).

We observe that

$$\begin{aligned}
\mathcal{L}_{m_n} z^{(m_n,\delta)} - F^{(m_n)}_{\delta} &= Q^{(m_n)} c^{(m_n,\delta)} - F^{(m_n)}_{\delta} \\
&= Q^{(m_n)}(a_n I + Q^{(m_n)})^{-1} F^{(m_n)}_{\delta} - F^{(m_n)}_{\delta} \\
&= (Q^{(m_n)} + a_n I - a_n I)(a_n I + Q^{(m_n)})^{-1} F^{(m_n)}_{\delta} - F^{(m_n)}_{\delta} \\
&= -a_n(a_n I + Q^{(m_n)})^{-1} F^{(m_n)}_{\delta} = -a_n c^{(m_n,\delta)}.
\end{aligned} \tag{3.90}$$

Thus, the sequence (3.88) can be written in the following form

$$G_{n,m_n} = qG_{n-1,m_{n-1}} + (1-q)a_n\|c^{(m_n,\delta)}\|_{W^{m_n}},\ G_{0,m_0} = 0, \tag{3.91}$$

where $\|\cdot\|_{W^m}$ is defined in (3.7), and $c^{(m,\delta)}$ solves the linear algebraic system (3.74).

It follows from estimates (3.78), (3.30) and (3.31) that

$$\begin{aligned}
a_n\|c^{(m_n,\delta)}\|_{W^{m_n}} &= a_n\|(a_nI + Q^{(m_n)})^{-1}F_\delta^{(m_n)}\|_{W^{m_n}} \\
&\le a_n\|(a_nI + Q^{(m_n)})^{-1}(F_\delta^{(m_n)} - F^{(m_n)})\|_{W^{m_n}} \\
&+ a_n\|(a_nI + Q^{(m_n)})^{-1}F^{(m_n)}\|_{W^{m_n}} \\
&\le \|F_\delta^{(m_n)} - F^{(m_n)}\|_{W^{m_n}} \\
&+ a_n\|(a_nI + Q^{(m_n)})^{-1}\mathcal{L}_{m_n}f\|_{W^{m_n}} \\
&\le \delta\sqrt{d} + \sqrt{a_n}\|f\|_{X_{0,b}}.
\end{aligned} \tag{3.92}$$

This together with (3.91) yield

$$G_{n,m_n} \le qG_{n-1,m_{n-1}} + (1-q)\left(\delta\sqrt{d} + \sqrt{a_n}\|f\|_{X_{0,b}}\right), \tag{3.93}$$

or

$$G_{n,m_n} - \delta\sqrt{d} \le q(G_{n-1,m_{n-1}} - \delta\sqrt{d}) + (1-q)\sqrt{a_n}\|f\|_{X_{0,b}}. \tag{3.94}$$

Lemma 3.2.13. *The sequence* (3.91) *satisfies the following estimate:*

$$G_{n,m_n} - \delta\sqrt{d} \le \frac{(1-q)\sqrt{a_n}\|f\|_{X_{0,b}}}{1-\sqrt{q}}, \tag{3.95}$$

where a_n are defined in (3.39).

Proof. Define

$$\Psi_n := G_{n,m_n} - \delta\sqrt{d} \tag{3.96}$$

and

$$\psi_n := (1-q)\sqrt{a_n}\|f\|_{X_{0,b}}. \tag{3.97}$$

Then estimate (3.94) can be rewritten as

$$\Psi_n \le q\Psi_{n-1} + \sqrt{q}\psi_{n-1}, \tag{3.98}$$

where the relation $a_n = qa_{n-1}$ was used. Let us prove estimate (3.95) by induction. For $n = 0$ we get

$$\Psi_0 = -\delta\sqrt{d} \leq \frac{(1-q)\sqrt{a_0}\|f\|_{X_{0,b}}}{1-\sqrt{q}}. \tag{3.99}$$

Suppose estimate (3.95) is true for $0 \leq n \leq k$. Then

$$\begin{aligned}
\Psi_{k+1} &\leq q\Psi_k + \sqrt{q}\psi_k \leq \frac{q}{1-\sqrt{q}}\psi_k + \sqrt{q}\psi_k \\
&= \frac{\sqrt{q}}{1-\sqrt{q}}\psi_k = \frac{\sqrt{q}}{1-\sqrt{q}}\frac{\psi_k}{\psi_{k+1}}\psi_{k+1} \\
&= \frac{\sqrt{q}}{1-\sqrt{q}}\frac{\sqrt{a_k}}{\sqrt{a_{k+1}}}\psi_{k+1} = \frac{1}{1-\sqrt{q}}\psi_{k+1},
\end{aligned} \tag{3.100}$$

where the relation $a_{k+1} = qa_k$ was used.

Lemma 3.2.13 is proved. □

Lemma 3.2.14. *Suppose*

$$G_{1,m_1} > \delta\sqrt{d}, \tag{3.101}$$

where G_{n,m_n} are defined in (3.91). Then there exist a unique integer n_δ, satisfying the stopping rule (3.87) with $C > \sqrt{d}$.

Proof. From Lemma 3.2.13 we get the estimate

$$G_{n,m_n} \leq \delta\sqrt{d} + \frac{(1-q)\sqrt{a_n}\|f\|_{X_{0,b}}}{1-\sqrt{q}}, \tag{3.102}$$

where a_n are defined in (3.39). Therefore,

$$\limsup_{n\to\infty} G_{n,m_n} \leq \delta\sqrt{d}, \tag{3.103}$$

where the relation $\lim_{n\to\infty} a_n = 0$ was used. This together with condition (3.101) yield the existence of the integer n_δ. The uniqueness of the integer n_δ follows from its definition.

Lemma 3.2.14 is proved. □

Lemma 3.2.15. *Suppose conditions of Lemma 3.2.14 hold and n_δ is chosen by the rule (3.87). Then*

$$\lim_{\delta\to 0}\frac{\delta}{\sqrt{a_{n_\delta}}} = 0. \tag{3.104}$$

Proof. From the stopping rule (3.87) and estimate (3.102) we get

$$C\delta^{\varepsilon} \le G_{n_\delta-1,m_{n_\delta-1}} \le \delta\sqrt{d} + \frac{(1-q)\sqrt{a_{n_\delta-1}}\|f\|_{X_{0,b}}}{1-\sqrt{q}}, \tag{3.105}$$

where $C > \sqrt{d}$, $\varepsilon \in (0,1)$. This implies

$$\frac{\delta(C\delta^{\varepsilon-1}-\sqrt{d})}{\sqrt{a_{n_\delta-1}}} \le \frac{(1-q)\|f\|_{X_{0,b}}}{1-\sqrt{q}}, \tag{3.106}$$

so, for $\varepsilon \in (0,1)$, and $a_{n_\delta} = q a_{n_\delta-1}$, one gets

$$\lim_{\delta\to 0}\frac{\delta}{\sqrt{a_{n_\delta}}} = \lim_{\delta\to 0}\frac{\delta}{\sqrt{q}\sqrt{a_{n_\delta-1}}} \le \lim_{\delta\to 0}\frac{(1-q)\delta^{1-\varepsilon}\|f\|_{X_{0,b}}}{(\sqrt{q}-q)(C-\delta^{1-\varepsilon}\sqrt{d})} = 0. \tag{3.107}$$

Lemma 3.2.15 is proved. □

Lemma 3.2.16. *Consider the stopping rule (3.87), where the parameters m_n are chosen by rule (3.63). If n_δ is chosen by the rule (3.87) then*

$$\lim_{\delta\to 0} n_\delta = \infty. \tag{3.108}$$

Proof. From the stopping rule (3.87) with the sequence G_n defined in (3.91) one gets

$$\begin{aligned} qC\delta^{\varepsilon} + (1-q)a_{n_\delta}\|c^{(m_{n_\delta},\delta)}\|_{W^{m_{n_\delta}}} &\le qG_{n_\delta-1,m_{n_\delta-1}} \\ + (1-q)a_{n_\delta}\|c^{(m_{n_\delta},\delta)}\|_{W^{m_{n_\delta}}} &= G_{n_\delta,m_{n_\delta}} < C\delta^{\varepsilon}, \end{aligned} \tag{3.109}$$

where $c^{(m,\delta)}$ is obtained by solving linear algebraic system (3.74). This implies

$$0 < a_{n_\delta}\|c^{(m_{n_\delta},\delta)}\|_{W^{m_{n_\delta}}} \le C\delta^{\varepsilon}. \tag{3.110}$$

Thus,

$$\lim_{\delta\to 0} a_{n_\delta}\|c^{(m_{n_\delta},\delta)}\|_{W^{m_{n_\delta}}} = 0. \tag{3.111}$$

If $F^{(m)} \ne 0$, then there exists a $\lambda_0^{(m)} > 0$ such that

$$E^{(m)}_{\lambda_0^{(m)}}F^{(m)} \ne 0, \quad \langle E^{(m)}_{\lambda_0}F^{(m)}, F^{(m)}\rangle_{W^m} := \xi^{(m)} > 0, \tag{3.112}$$

where $E_s^{(m)}$ is the resolution of the identity corresponding to the operator $Q^{(m)} := \mathcal{L}_m\mathcal{L}_m^*$. Let

$$h_m(\delta,\alpha) := \alpha^2\|Q^{-1}_{m,\alpha}F^{(m)}_\delta\|^2_{W^m}, \quad Q_{m,a} := aI + Q^{(m)}.$$

For a fixed number $a > 0$ we obtain

$$
\begin{aligned}
h_m(\delta, a) &= a^2 \|Q_{m,a}^{-1} F_\delta^{(m)}\|_{W^m}^2 \\
&= \int_0^\infty \frac{a^2}{(a+s)^2} d\langle E_s^{(m)} F_\delta^{(m)}, F_\delta^{(m)} \rangle_{W^m} \\
&\geq \int_0^{\lambda_0^{(m)}} \frac{a^2}{(a+s)^2} d\langle E_s^{(m)} F_\delta^{(m)}, F_\delta^{(m)} \rangle_{W^m} \\
&\geq \frac{a^2}{(a+\lambda_0)^2} \int_0^{\lambda_0^{(m)}} d\langle E_s^{(m)} F_\delta^{(m)}, F_\delta^{(m)} \rangle_{W^m} \\
&= \frac{a^2 \|E_{\lambda_0^{(m)}}^{(m)} F_\delta^{(m)}\|_{W^m}^2}{(a+\lambda_0^{(m)})^2}.
\end{aligned}
\tag{3.113}
$$

Since $E_{\lambda_0}^{(m)}$ is a continuous operator, and $\|F^{(m)} - F_\delta^{(m)}\|_{W^m} < \sqrt{d}\delta$, it follows from (3.112) that

$$
\lim_{\delta \to 0} \langle E_{\lambda_0}^{(m)} F_\delta^{(m)}, F_\delta^{(m)} \rangle_{W^m} = \langle E_{\lambda_0}^{(m)} F^{(m)}, F^{(m)} \rangle_{W^m} > 0. \tag{3.114}
$$

Therefore, for the fixed number $a > 0$ we get

$$
h_m(\delta, a) \geq c_2 > 0 \tag{3.115}
$$

for all sufficiently small $\delta > 0$, where c_2 is a constant which does not depend on δ. Suppose $\lim_{\delta \to 0} a_{n_\delta} \neq 0$. Then there exists a subsequence $\delta_j \to 0$ as $j \to \infty$, such that

$$
a_{n_{\delta_j}} \geq c_1 > 0, \tag{3.116}
$$

and

$$
0 < m_{n_{\delta_j}} = \left\lceil [\kappa (a_0/a_{n_{\delta_j}})^{1/4}] \right\rceil \leq \left\lceil [\kappa (a_0/c_1)^{1/4}] \right\rceil := c_3 < \infty, \quad \kappa, a_0 > 0, \tag{3.117}
$$

where the rule (3.63) was used to obtain the parameters $m_{n_{\delta_j}}$. This together with (3.112) and (3.115) yield

$$
\begin{aligned}
\lim_{j \to \infty} h_{m_{n_{\delta_j}}}(\delta_j, a_{n_{\delta_j}}) &\geq \lim_{j \to \infty} \frac{a_{n_{\delta_j}}^2 \|E_{\lambda_0^{(m_{n_{\delta_j}})}}^{(m_{n_{\delta_j}})} F_{\delta_j}^{(m_{n_{\delta_j}})}\|_{W^{m_{n_{\delta_j}}}}^2}{(a_{n_{\delta_j}} + \lambda_0^{(m_{n_{\delta_j}})})^2} \\
&\geq \liminf_{j \to \infty} \frac{c_1^2 \|E_{\lambda_0^{(m_{n_{\delta_j}})}}^{(m_{n_{\delta_j}})} F^{(m_{n_{\delta_j}})}\|_{W^{m_{n_{\delta_j}}}}^2}{(c_1 + \lambda_0^{(m_{n_{\delta_j}})})^2} > 0.
\end{aligned}
\tag{3.118}
$$

This contradicts relation (3.111). Thus, $\lim_{\delta\to 0} a_{n_\delta} = \lim_{\delta\to 0} a_0 q^{n_\delta} = 0$, i.e., $\lim_{\delta\to 0} n_\delta = \infty$. Lemma 3.2.16 is proved. $\square$

It follows from Lemma 3.2.15 and Lemma 3.2.16 that the stopping rule (3.87) yields the relations (3.83). We have proved the following theorem:

Theorem 3.2.17. *Suppose all the assumptions of Theorem 3.2.12 hold, m_n are chosen by the rule (3.63), n_δ is chosen by the rule (3.87) and $G_{1,m_1} > C\delta$, where G_{n,m_n} are defined in (3.91), then*

$$\lim_{\delta\to 0} \|f - u^\delta_{n_\delta, m_{n_\delta}}\| = 0. \tag{3.119}$$

3.2.3 The algorithm

Let us formulate the algorithm for obtaining the approximate solution f_m^δ:

(1) The data $F_\delta(p)$ on the interval $[0, d]$, $d > 0$, the support of the function $f(t)$, and the noise level δ;

(2) initialization : choose the parameters $\kappa > 0$, $a_0 > 0$, $q \in (0, 1)$, $\varepsilon \in (0, 1)$, $C > \sqrt{d}$, and set $u^\delta_{0,m_0} = 0$, $G_0 = 0$, $n = 1$;

(3) iterate, starting with $n = 1$, and stop when condition (2.133) (see below) holds,

  (a) $a_n = a_0 q^n$,

  (b) choose m_n by the rule (3.63),

  (c) construct the vector $F_\delta^{(m_n)}$:

$$(F_\delta^{(m_n)})_l = F_\delta(p_l), \quad p_l = lh,\ \ h = d/m_n,\ \ l = 0, 1, \dots, m, \tag{3.120}$$

  (d) construct the matrices H_{m_n} and D_{m_n}:

$$(H_{m_n})_{ij} := \int_0^b e^{-(p_i+p_j)t} dt = \frac{1 - e^{-b(p_i+p_j)}}{p_i + p_j}, \quad i, j = 1, 2, 3, \dots, m_n \tag{3.121}$$

$$(D_{m_n})_{ij} = \begin{cases} w_i^{(m_n)}, & i = j; \\ 0, & \text{otherwise}, \end{cases} \tag{3.122}$$

where $w_j^{(m)}$ are defined in (3.8),

(e) solve the following linear algebraic system:

$$(a_n I + H_{m_n} D_{m_n}) c^{(m_n,\delta)} = F_\delta^{(m_n)}, \tag{3.123}$$

where $(c^{(m_n,\delta)})_i = c_i^{(m_n,\delta)}$,

(f) update the coefficient $c_j^{(m_n,\delta)}$ of the approximate solution $u^\delta_{n,m_n}(t)$ defined in (3.73) by the iterative formula:

$$u^\delta_{n,m_n}(t) = q u^\delta_{n-1,m_{n-1}}(t) + (1-q) \sum_{j=1}^{m_n} c^{(m_n,\delta)} w_j^{(m_n)} e^{-p_j t}, \tag{3.124}$$

where

$$u^\delta_{0,m_0}(t) = 0. \tag{3.125}$$

Stop when for the first time the inequality

$$G_{n,m_n} = q G_{n-1,m_{n-1}} + a_n \|c^{(m_n,\delta)}\|_{W^{m_n}} \le C\delta^\varepsilon \tag{3.126}$$

holds, and get the approximation $f^\delta(t) = u^\delta_{n_\delta,m_{n_\delta}}(t)$ of the function $f(t)$ by formula (3.124).

3.3 Numerical experiments

3.3.1 The parameters κ, a_0, d

From definition (3.39) and the rule (3.63) we conclude that $m_n \to \infty$ as $a_n \to 0$. Therefore, one needs to control the value of the parameter m_n so that it will not grow too fast as a_n decreases. The role of the parameter κ in (3.63) is to control the value of the parameter m_n so that the value of the parameter m_n will not be too large. Since for sufficiently small noise level δ, namely $\delta \in (10^{-16}, 10^{-6}]$, the regularization parameter a_{n_δ}, obtained by the stopping rule (3.87), is at most $O(10^{-9})$, we suggest to choose κ in the interval $(0, 1]$.

For the noise level $\delta \in (10^{-6}, 10^{-2}]$ one can choose $\kappa \in (1, 3]$. To reduce the number of iterations we suggest to choose the geometric sequence $a_n = a_0\delta^{\alpha n}$, where $a_0 \in [0.1, 0.2]$ and $\alpha \in [0.5, 0.9]$. One may assume without loss of generality that $b = 1$, because a scaling transformation reduces the integral over $(0, b)$ to the integral over $(0, 1)$. We have assumed that the data $F(p)$ are defined on the interval $J := [0, d]$. In the case the interval $J = [d_1, d]$, $0 < d_1 < d$, the constant d in estimates (3.60), (3.78), (3.79), (3.82), (3.94), (3.95), and (3.102) are replaced with the constant $d - d_1$. If $b = 1$, i.e., $f(t) = 0$ for $t > 1$, then one has to take d not too large. Indeed, if $f(t) = 0$ for $t > 1$, then an integration by parts yields: $F(p) = [f(0) - e^{-p}f(1)]/p + O(1/p^2)$, $p \to \infty$. If the data are noisy, and the noise level is δ, then the data becomes indistinguishable from noise for $p = O(1/\delta)$. Therefore it is useless to keep the data $F_\delta(p)$ for $d > O(1/\delta)$. In practice one may get a satisfactory accuracy of inversion by the method, proposed in Section 2, when one uses the data with $d \in [1, 20]$ when $\delta \leq 10^{-2}$. In all the numerical examples we have used $d = 5$. Given the interval $[0, d]$, the proposed method generates automatically the discrete data $F_\delta(p_j)$, $j = 0, 1, 2, \ldots, m$, over the interval $[0, d]$ which are needed to get the approximation of the function $f(t)$.

3.3.2 Experiments

To test the proposed method we consider some examples proposed in[24],[1],[2],[3],[4],[5],[10],[12],[22], and[37]. To illustrate the numerical stability of the proposed method with respect to the noise, we use the noisy data $F_\delta(p)$ with various noise levels $\delta = 10^{-2}$, $\delta = 10^{-4}$ and $\delta = 10^{-6}$. The random quantities δ_j in (3.77) are obtained from the uniform probability density function over the interval $[-\delta, \delta]$. In examples 1-12 we choose the value of the parameters as follows: $a_n = 0.1q^n$, $q = \delta^{1/2}$ and $d = 5$. The parameter $\kappa = 1$ is used for the noise levels $\delta = 10^{-2}$ and $\delta = 10^{-4}$. When $\delta = 10^{-6}$ we choose $\kappa = 0.3$ so that the value of the parameters m_n are not very large, namely $m_n \leq 300$. Therefore, the computation time for solving linear algebraic system (3.123) can be reduced significantly. We assume that the support of the function $f(t)$ is in the interval $[0, b]$ with $b = 10$. In the stopping rule (3.87) the following parameters are used: $C = \sqrt{d} + 0.01$, $\varepsilon = 0.99$. In example 13 the function $f(t) = e^{-t}$ is used to test the applicability of the proposed method to functions without compact support. The results are given in Table 13 and Figure 13.

For a comparison with the exact solutions we use the mean absolute error:

$$MAE := \left[\frac{\sum_{j=1}^{100}(f(t_i) - f^{\delta}_{m_{n_\delta}}(t_i))^2}{100}\right]^{1/2}, \; t_j = 0.01 + 0.1(j-1), \; j = 1, \dots, 100, \quad (3.127)$$

where $f(t)$ is the exact solution and $f^{\delta}_{m_{n_\delta}}(t)$ is the approximate solution. The computation time (CPU time) for obtaining the approximation of $f(t)$, the number of iterations (Iter.), and the parameters m_{n_δ} and a_{n_δ} generated by the proposed method are given in each experiment (see Tables 1-12). All the calculations are done in double precision generated by MATLAB.

- ***Example 1. (see[12])***

$$f_1(t) = \begin{cases} 1, & 1/2 \le t \le 3/2, \\ 0, & otherwise, \end{cases} \qquad F_1(p) = \begin{cases} 1, & p = 0, \\ \frac{e^{-p/2} - e^{-3p/2}}{p}, & p > 0. \end{cases}$$

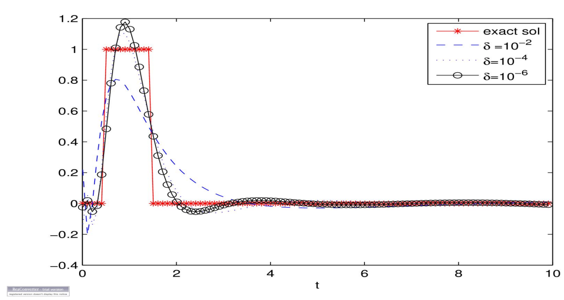


Figure 3.1: *Example 1: the stability of the approximate solution*

The reconstruction of the exact solution for different values of the noise level δ is shown in Figure 1. When the noise level $\delta = 10^{-6}$, our result is comparable with the double precision results shown in[12]. The proposed method is stable with respect to the noise δ as shown in Table 1.

Table 3.1: *Example 1.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time(second)	a_{n_δ}
1.00×10^{-2}	9.62×10^{-2}	30	3	3.13×10^{-2}	2.00×10^{-3}
1.00×10^{-4}	5.99×10^{-2}	32	4	6.25×10^{-2}	2.00×10^{-7}
1.00×10^{-6}	4.74×10^{-2}	54	5	3.28×10^{-1}	2.00×10^{-10}

- ***Example 2. (see[4], [12])***

$$f_2(t) = \begin{cases} 1/2, & t = 1, \\ 1, & 1 < t < 10, \\ 0, & \text{elsewhere}, \end{cases} \qquad F_2(p) = \begin{cases} 9, & p = 0, \\ \frac{e^{-p} - e^{-10p}}{p}, & p > 0. \end{cases}$$

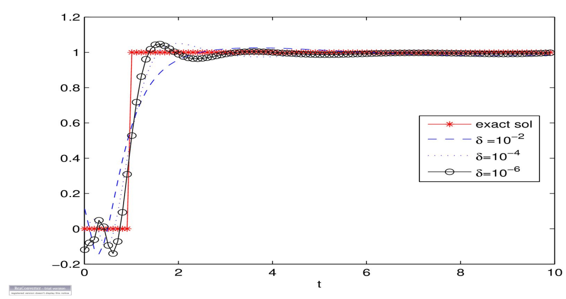


Figure 3.2: *Example 2: the stability of the approximate solution*

Table 3.2: *Example 2.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	1.09×10^{-1}	30	2	3.13×10^{-2}	2.00×10^{-3}
1.00×10^{-4}	8.47×10^{-2}	32	3	6.25×10^{-2}	2.00×10^{-6}
1.00×10^{-6}	7.41×10^{-2}	54	5	4.38×10^{-1}	2.00×10^{-12}

The reconstruction of the function $f_2(t)$ is plotted in Figure 2. In[12] a high accuracy result is given by means of the multiple precision. But, as reported in[12], to get such high accuracy results, it takes 7 hours. From Table 2 and Figure 2 we can see that the proposed method yields stable solution with respect to the noise level δ. The reconstruction of the exact solution obtained by the proposed method is better than the reconstruction shown in[4]. The result is comparable with the double precision results given in[12]. For $\delta = 10^{-6}$ and $\kappa = 0.3$ the value of the parameter m_{n_δ} is bounded by the constant 54.

- ***Example 3. (see[24], [1], [4], [5], [37])***

$$f_3(t) = \begin{cases} te^{-t}, & 0 \le t < 10, \\ 0, & \text{otherwise}, \end{cases} \qquad F_3(p) = \frac{1 - e^{-(p+1)10}}{(p+1)^2} - \frac{10e^{-(p+1)10}}{p+1}.$$

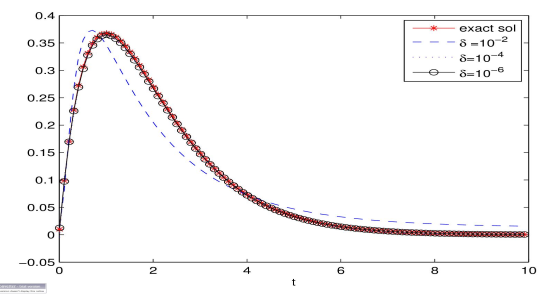


Figure 3.3: *Example 3: the stability of the approximate solution*

Table 3.3: *Example 3.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	2.42×10^{-2}	30	2	3.13×10^{-2}	2.00×10^{-3}
1.00×10^{-4}	1.08×10^{-3}	30	3	3.13×10^{-2}	2.00×10^{-6}
1.00×10^{-6}	4.02×10^{-4}	30	4	4.69×10^{-2}	2.00×10^{-9}

We get an excellent agreement between the approximate solution and the exact solution when the noise level $\delta = 10^{-4}$ and 10^{-6} as shown in Figure 3. The results obtained by the proposed method are better than the results given in[4]. The mean absolute error MAE decreases as the noise level decreases which shows the stability of the proposed method. Our results are more stable with respect to the noise δ than the results presented in[37]. The value of the parameter m_{n_δ} is bounded by the constant 30 when the noise level $\delta = 10^{-6}$ and $\kappa = 0.3$.

- ***Example 4. (see[4], [12])***

$$f_4(t) = \begin{cases} 1 - e^{-0.5t}, & 0 \le t < 10, \\ 0, & \text{elsewhere.} \end{cases}$$

$$F_4(p) = \begin{cases} 8 + 2e^{-5}, & p = 0, \\ \frac{1-e^{-10p}}{p} - \frac{1-e^{-(p+1/2)10}}{p+0.5}, & p > 0. \end{cases}$$

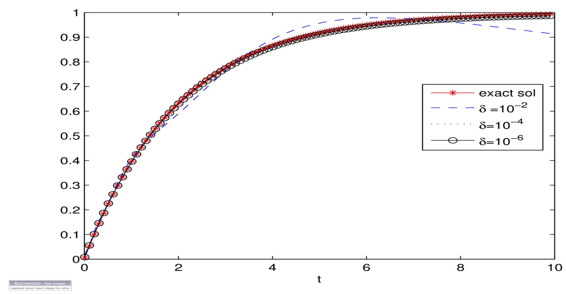


Figure 3.4: *Example 4: the stability of the approximate solution*

As in our example 3 when the noise $\delta = 10^{-4}$ and 10^{-6} are used, we get a satisfactory agreement between the approximate solution and the exact solution. Table 4 gives the results of the stability of the proposed method with respect to the noise level δ. Moreover, the reconstruction of the function $f_4(t)$ obtained by the proposed method is better than the reconstruction of $f_4(t)$ shown in[4], and is comparable with the double precision reconstruction obtained in[12].

Table 3.4: *Example 4.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	1.59×10^{-2}	30	2	3.13×10^{-2}	2.00×10^{-3}
1.00×10^{-4}	8.26×10^{-4}	30	3	9.400×10^{-2}	2.00×10^{-6}
1.00×10^{-6}	1.24×10^{-4}	30	4	1.250×10^{-1}	2.00×10^{-9}

In this example when $\delta = 10^{-6}$ and $\kappa = 0.3$ the value of the parameter m_{n_δ} is bounded by the constant 109 as shown in Table 4.

- ***Example 5. (see[2],[4],[10])***

$$f_5(t) = 2/\sqrt{3}e^{-t/2}\sin(t\sqrt{3}/2)$$
$$F_5(p) = \frac{1-\cos(10\sqrt{3}/2)e^{-10(p+0.5)}}{[(p+0.5)^2+3/4]} - \frac{2(p+0.5)e^{-10(p+0.5)}\sin(10\sqrt{3}/2)}{\sqrt{3}[(p+0.5)^2+3/4]}.$$

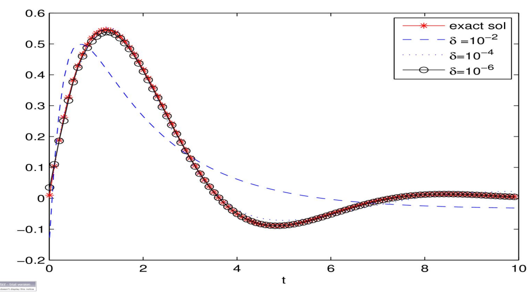


Figure 3.5: *Example 5: the stability of the approximate solution*

Table 3.5: *Example 5.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	4.26×10^{-2}	30	3	6.300×10^{-2}	2.00×10^{-3}
1.00×10^{-4}	1.25×10^{-2}	30	3	9.38×10^{-2}	2.00×10^{-6}
1.00×10^{-6}	1.86×10^{-3}	54	4	3.13×10^{-2}	2.00×10^{-9}

This is an example of the damped sine function. In[2] and[10] the knowledge of the exact data $F(p)$ in the complex plane is required to get the approximate solution. Here we only use the knowledge of the discrete perturbed data $F_\delta(p_j)$, $j = 0, 1, 2, \dots, m$, and get a satisfactory result which is comparable with the results given in[2] and[10] when the level noise $\delta = 10^{-6}$. The reconstruction of the exact solution $f_5(t)$ obtained by our method is better than this of the method given in[4]. Moreover, our method yields stable solution with respect to the noise level δ as shown in Figure 5 and Table 5 show. In this example when $\kappa = 0.3$ the value of the parameter m_{n_δ} is bounded by 54 for the noise level $\delta = 10^{-6}$ (see Table 5).

- ***Example 6. (see[12])***

$$f_6(t) = \begin{cases} t, & 0 \le t < 1, \\ 3/2 - t/2, & 1 \le t < 3, \\ 0, & \text{elsewhere.} \end{cases}$$

$$F_6(p) = \begin{cases} 3/2, & p = 0, \\ \frac{1-e^{-p}(1+p)}{p^2} + \frac{e^{-3p}+e^{2p}(2p-1)}{2p^2}, & p > 0. \end{cases}$$

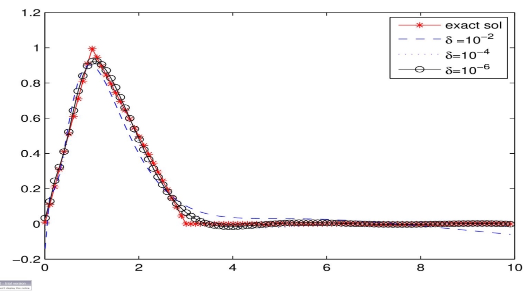


Figure 3.6: *Example 6: the stability of the approximate solution*

Table 3.6: *Example 6.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	4.19×10^{-2}	30	2	4.700×10^{-2}	2.00×10^{-3}
1.00×10^{-4}	1.64×10^{-2}	32	3	9.38×10^{-2}	2.00×10^{-6}
1.00×10^{-6}	1.22×10^{-2}	54	4	3.13×10^{-2}	2.00×10^{-9}

Example 6 represents a class of piecewise continuous functions. From Figure 6 the value of the exact solution at the points where the function is not differentiable can not be well approximated for the given levels of noise by the proposed method. When the noise level $\delta = 10^{-6}$, our result is comparable with the results given in[12]. Table 6 reports the stability of the proposed method with respect to the noise δ. It is shown in Table 6 that the value of the parameter m generated by the proposed adaptive stopping rule is bounded by the constant 54 for the noise level $\delta = 10^{-6}$ and $\kappa = 0.3$ which gives a relatively small computation time.

- ***Example 7. (see[12])***

$$f_7(t) = \begin{cases} -te^{-t} - e^{-t} + 1, & 0 \le t < 1, \\ 1 - 2e^{-1}, & 1 \le t < 10, \\ 0, & \text{elsewhere,} \end{cases}$$

$$F_7(p) = \begin{cases} 3/e - 1 + 9(1 - 2/e), & p = 0, \\ e^{-1-p}\frac{e^{1+p} - e(1+p)^2 + p(3+2p)}{p(p+1)^2} + (e-2)e^{-1-p-10p}\frac{e^{10p} - e^p}{p}, & p > 0. \end{cases}$$

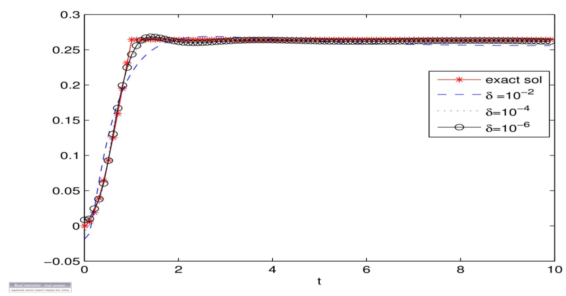


Figure 3.7: *Example 7: the stability of the approximate solution*

Table 3.7: *Example 7.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	1.52×10^{-2}	30	2	4.600×10^{-2}	2.00×10^{-3}
1.00×10^{-4}	2.60×10^{-3}	30	3	9.38×10^{-2}	2.00×10^{-6}
1.00×10^{-6}	2.02×10^{-3}	30	4	3.13×10^{-2}	2.00×10^{-9}

When the noise level $\delta = 10^{-4}$ and $\delta = 10^{-6}$, we get numerical results which are comparable with the double precision results given in [12]. Figure 7 and Table 7 show the stability of the proposed method for decreasing δ.

- ***Example 8. (see [3], [4])***

$$f_8(t) = \begin{cases} 4t^2e^{-2t}, & 0 \le t < 10, \\ 0, & \text{elsewhere.} \end{cases}$$

$$F_8(p) = \frac{8 + 4e^{-10(2+p)}[-2 - 20(2+p) - 100(2-p)^2]}{(2+p)^3}.$$

The results of this example are similar to the results of Example 3. The exact solution can be well reconstructed by the approximate solution obtained by our method at

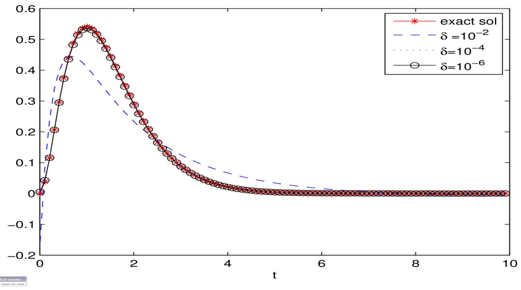


Figure 3.8: *Example 8: the stability of the approximate solution*

the levels noise $\delta = 10^{-4}$ and $\delta = 10^{-6}$ (see Figure 8). Table 8 shows that the MAE decreases as the noise level decreases which shows the stability of the proposed method with respect to the noise. In all the levels of noise δ the computation time of the proposed method in obtaining the approximate solution are relatively small. We get better reconstruction results than the results shown in[4]. Our results are comparable with the results given in[3].

Table 3.8: *Example 8.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	2.74×10^{-2}	30	2	1.100×10^{-2}	2.00×10^{-3}
1.00×10^{-4}	3.58×10^{-3}	30	3	3.13×10^{-2}	2.00×10^{-6}
1.00×10^{-6}	5.04×10^{-4}	30	4	4.69×10^{-2}	2.00×10^{-9}

- ***Example 9. (see[24])***

$$f_9(t) = \begin{cases} 5 - t, & 0 \le t < 5, \\ 0, & \text{elsewhere}, \end{cases}$$

$$F_9(p) = \begin{cases} 25/2, & p = 0, \\ \frac{e^{-5p} + 5p - 1}{p^2}, & p > 0. \end{cases}$$

As in Example 6 the error of the approximate solution at the point where the function is not differentiable dominates the error of the approximation. The reconstruction of the exact solution can be seen in Figure 9. The detailed results are presented in Table 9. When the double precision is used, we get comparable results with the results

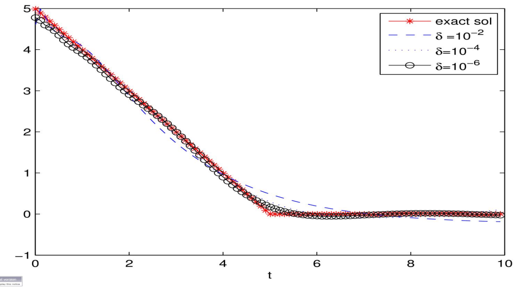


Figure 3.9: *Example 9: the stability of the approximate solution*

shown in[24].

Table 3.9: *Example 9.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	2.07×10^{-1}	30	3	6.25×10^{-2}	2.00×10^{-6}
1.00×10^{-4}	7.14×10^{-2}	32	4	3.44×10^{-1}	2.00×10^{-9}
1.00×10^{-6}	2.56×10^{-2}	54	5	3.75×10^{-1}	2.00×10^{-12}

- ***Example 10. (see[5])***

$$f_{10}(t) = \begin{cases} t, & 0 \le t < 10, \\ 0, & \text{elsewhere}, \end{cases}$$

$$F_{10}(p) = \begin{cases} 50, & p = 0, \\ \frac{1-e^{-10p}}{p^2} - \frac{10e^{-10p}}{p}, & p > 0. \end{cases}.$$

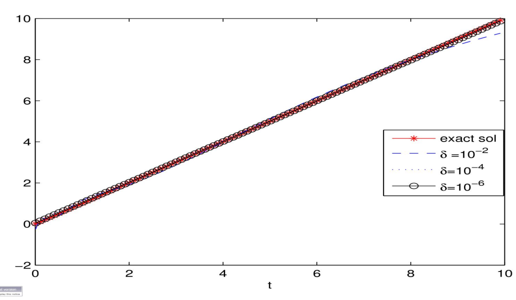


Figure 3.10: *Example 10: the stability of the approximate solution*

Table 10 shows the stability of the solution obtained by our method with respect to the noise level δ. We get an excellent agreement between the exact solution and the

Table 3.10: *Example 10.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	2.09×10^{-1}	30	3	3.13×10^{-2}	2.00×10^{-6}
1.00×10^{-4}	1.35×10^{-2}	32	4	9.38×10^{-2}	2.00×10^{-9}
1.00×10^{-6}	3.00×10^{-3}	54	4	2.66×10^{-1}	2.00×10^{-9}

approximate solution for all the noise levels δ as shown in Figure 10.

- ***Example 11. (see[5],[22])***

$$f_{11}(t) = \begin{cases} \sin(t), & 0 \le t < 10, \\ 0, & \text{elsewhere}, \end{cases}$$

$$F_{11}(p) = \frac{1 - e^{-10p}(p\sin(10) + \cos(10))}{1 + p^2}.$$

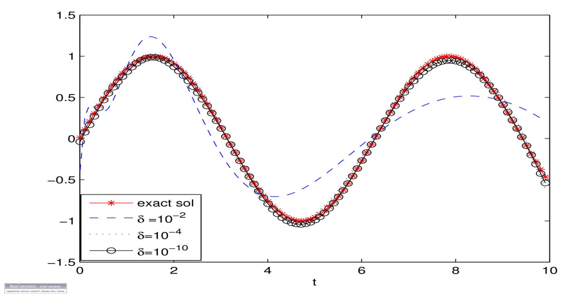


Figure 3.11: *Example 11: the stability of the approximate solution*

Here the function $f_{11}(t)$ represents the class of periodic functions. It is mentioned in[22] that oscillating function can be found with acceptable accuracy only for relatively small values of t. In this example the best approximation is obtained when the noise level $\delta = 10^{-6}$ which is comparable with the results given in[5] and[22]. The reconstruction of the function $f_{11}(t)$ for various levels of the noise δ are given in Figure 11. The stability of the proposed method with respect to the noise δ is shown in Table 11. In this example the parameter m_{n_δ} is bounded by the constant 54 when the noise level $\delta = 10^{-6}$ and $\kappa = 0.3$.

Table 3.11: *Example 11.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	2.47×10^{-1}	30	3	9.38×10^{-2}	2.00×10^{-6}
1.00×10^{-4}	4.91×10^{-2}	32	4	2.50×10^{-1}	2.00×10^{-9}
1.00×10^{-6}	2.46×10^{-2}	54	5	4.38×10^{-1}	2.00×10^{-12}

- ***Example 12. (see[3], [5])***

$$
\begin{aligned}
f_{12}(t) &= \begin{cases} t\cos(t), & 0 \le t < 10, \\ 0, & \text{elsewhere}, \end{cases} \\
F_{12}(p) &= \frac{(p^2-1) - e^{-10p}(-1+p^2+10p+10p^3)\cos(10)}{(1+p^2)^2} \\
&\quad + \frac{e^{-10p}(2p+10+10p^2)\sin(10)}{(1+p^2)^2}.
\end{aligned}
$$

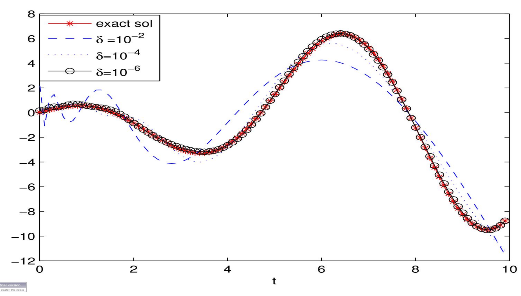


Figure 3.12: *Example 12: the stability of the approximate solution*

Here we take an increasing function which oscillates as the variable t increases over the interval $[0, 10)$. A poor approximation is obtained when the noise level $\delta = 10^{-2}$. Figure 12 shows that the exact solution can be approximated very well when the noise level $\delta = 10^{-6}$. The results of our method are comparable with these of the methods given in[3] and[5]. The stability of our method with respect to the noise level is shown in Table 12.

- ***Example 13.***

$$
f_{13}(t) = e^{-t}, \quad F_{13}(p) = \frac{1}{1+p}.
$$

Table 3.12: *Example 12.*

δ	MAE	m_{n_δ}	$Iter.$	CPU time (seconds)	a_{n_δ}
1.00×10^{-2}	1.37×10^{0}	96	3	9.38×10^{-2}	2.00×10^{-6}
1.00×10^{-4}	5.98×10^{-1}	100	4	2.66×10^{-1}	2.00×10^{-9}
1.00×10^{-6}	2.24×10^{-1}	300	5	3.44×10^{-1}	2.00×10^{-12}

Here the support of $f_{13}(t)$ is not compact. From the Laplace transform formula one gets

$$\begin{aligned} F_{13}(p) &= \int_0^\infty e^{-t} e^{-pt} dt = \int_0^b e^{-(1+p)t} dt + \int_b^\infty e^{-(1+p)t} dt \\ &= \int_0^b f_{13}(t) e^{-pt} dt + \frac{e^{-(1+p)b}}{1+p} := I_1 + I_2, \end{aligned}$$

where $\delta(b) := e^{-b}$. Therefore, I_2 can be considered as noise of the data $F_{13}(p)$, i.e.,

$$F_{13}^{\delta}(p) := F_{13}(p) - \delta(b), \tag{3.128}$$

where $\delta(b) := e^{-b}$. In this example the following parameters are used: $d = 2$, $\kappa = 10^{-1}$ for $\delta = e^{-5}$ and $\kappa = 10^{-5}$ for $\delta = 10^{-8}$, 10^{-20} and 10^{-30}. Table 13 shows that the error decreases as the parameter b increases. The approximate solution obtained by the proposed method converges to the function $f_{13}(t)$ as b increases (see Figure 13).

Table 3.13: *Example 13.*

b	MAE	m_δ	Iter	CPU time (seconds)
5	1.487×10^{-2}	2	4	3.125×10^{-2}
8	2.183×10^{-4}	2	4	3.125×10^{-2}
20	4.517×10^{-9}	2	4	3.125×10^{-2}
30	1.205×10^{-13}	2	4	3.125×10^{-2}

3.4 Conclusion

We have tested the proposed algorithm on the wide class of examples considered in the literature. Using the rule (3.63) and the stopping rule (3.87), the number of terms in rep-

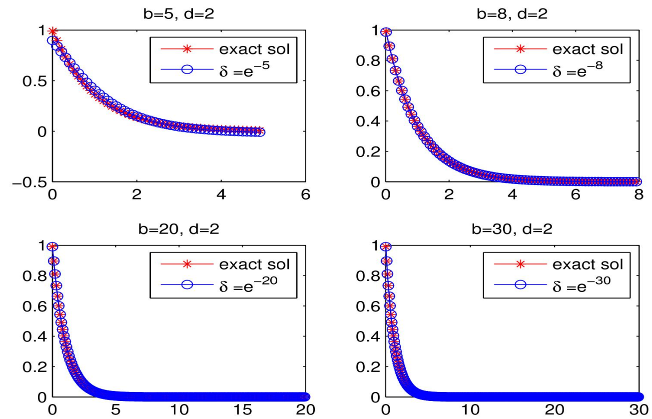


Figure 3.13: *Example 13: the stability of the approximate solution*

resentation (3.73), the discrete data $F_\delta(p_j)$, $j = 0, 1, 2, \dots, m$, and regularization parameter a_{n_δ}, which are used in computing the approximation $f_m^\delta(t)$ (see (3.73)) of the unknown function $f(t)$, are obtained automatically. Our numerical experiments show that the computation time (CPU time) for approximating the function $f(t)$ is small, namely CPU time ≤ 1 seconds, and the proposed iterative scheme and the proposed adaptive stopping rule yield stable solution with respect to the noise level δ. The proposed method also works for f without compact support as shown in Example 13. Moreover, in the proposed method we only use a simple representation (3.73) which is based on the kernel of the Laplace transform integral, so it can be easily implemented numerically.

Bibliography

[1] R. G. Airapetyan and A. G. Ramm. Numerical inversion of the Lapace transform from the real axis. *Jour. Math. Anal. Appl.*, 248:572–587, 2000.

[2] K. S. Crump. Numerical inversion of Laplace transforms using a Fourier series approximation. *Journal of the association for computing machinery*, 23(1):89–96, 1976.

[3] S. Cuomo, L. D'Amore, A. Murli, and M. Rizzardi. Computation of the inverse Laplace transform based on a collocation method which uses only real values. *Journal of Computational and Applied Mathematics*, 198:98–115, 2007.

[4] L. D'Amore and A. Murli. Reguarization of a Fourier series method for the Laplace transform inversion with real data. *Inverse Probems*, 18:1185–1205, 2002.

[5] B. Davies and B. Martin. Numerical inversion of the Laplace transform: a survey and comparison of methods. *J. of Comp. Phys.*, 22:1–32, 1979.

[6] P. J. Davis and P. Rabinowitz. *Methods of numerical integration.* Academic Press, INC., London, 1984.

[7] L. M. Delves and J. L. Mohamed. *Computational Methods for Integral Equations.* Cambridge University Press, New York, 1985.

[8] A. M. Denisov and A. S. Krylov. Numerical solution of integral equations of the first kind. *Computational mathematics and modeling*, 1:137–142, 1990.

[9] C. W. Dong. A regularization method for the numerical inversion of the Laplace transform. *SIAM J. Numer. Anal.*, 30(3):759–773, 1993.

[10] H. Dubner and J. Abate. Numerical inversion of Laplace transforms by relating them to the finite Fourier cosine transform. *Journal of the Association for computing machinery*, 15(1):115–123, 1968.

[11] H. Fujiwara. A multiple precision arithmetic software. http://www-an.acs.i.kyoto-u.ac.jp/fujiwara/exflib.

[12] H. Fujiwara, T. Matsura, S. Saitoh, and Y. Sawano. Numerical real inversion of the Laplace transform by using a high-accurate numerical method (preprint).

[13] N. S. Hoang and A. G. Ramm. Solving ill-conditioned linear algebraic systems by the Dynamical Systems Method. *Inverse Problems in Sci. and Engineering*, 16:617–630, 2008.

[14] K. M Howell. Multiple precision arithmetic techniques. *The computer journal*, 9(4):383–387, 1967.

[15] S. W. Indratno and A. G. Ramm. Inversion of the laplace transform with real data using an adaptive iterative method. *Internat. Jour. Math. Math. Sci. (IJMMS), 2009, Article 898195, 38 pages, doi:10.1155/2009/898195.*

[16] S. W. Indratno and A. G. Ramm. Dynamical systems method for solving ill-conditioned linear algebraic systems. *Int. J. of Computing Science and Mathematics*, 2,N4:308–333, 2009.

[17] S. W. Indratno and A. G. Ramm. An iterative method for solving fredholm integral equations of the first kind. *Int. J. of Computing Science and Mathematics*, 2,N4:354–379, 2009.

[18] P. Iseger. Numerical transform inversion using Gaussian quadrature. *Probability in engineering and informational science*, 20:1–44, 2006.

[19] V. Ivanov, V. Vasin, and V. Tanana. *Theory of linear inverse and ill-posed problems and its applications*. VSP, Utrecht, 2002.

[20] G. A. Kalyabin. An asymptotic formula for the minimal eigenvalues of Hilbert type matrices. *Functional analysis and its applications*, 35, No.1:67–70, 2001.

[21] V. I. Krylov and N. Skoblya. *Reference book on numerical inversion of the Laplace transform*. Nauka i technika, Minsk, 1968.

[22] V. V. Kryzhniy. Numerical inversion of the Lapace transform: analysis via reguarized analytic continuation. *Inverse Problems*, 22:579–597, 2006.

[23] V. Morozov. *Methods of solving incorrectly posed problems*. Springer-Verlang, New York, 1984.

[24] A. Murli, S. Cuomo, L. D'Amore, and Galleti. Numerical regularization of a real inversion formula based on the Laplace transform's eigenfunction expansion of the inverse function. *Inverse Problems*, 23:713–731, 2007.

[25] D. L. Phillips. A technique for the numerical solution of certain integral equations of the first kind. *J. Assoc. Comput. Mach.*, 9:84–97, 1962.

[26] A. G. Ramm. Inversion of the Laplace transform. *Inverse Problems*, 2:55–59, 1986.

[27] A. G. Ramm. A new discrepancy principle. *Journ. Math. Anal. Appl.*, 310:342–345, 2005.

[28] A. G. Ramm. Discrepancy principle for DSM, I. *Comm. Nonlin. Sci. and Numer. Simulation*, 1:95–101, 2005.

[29] A. G. Ramm. *Inverse problems*. Springer, New York, 2005.

[30] A. G. Ramm. *Dynamical Systems Method for solving operator equations*. Elsevier, Amsterdam, 2007.

[31] A. G. Ramm. Ill-posed problems with unbounded operators. *Journ. Math. Anal. Appl.*, 325:490–495, 2007.

[32] A. G. Ramm. Iterative solution of linear equations with unbounded operators. *Journ. Math. Anal. Appl.*, 330:1338–1346, 2007.

[33] A. G. Ramm. Discrepancy principle for DSM II. *Comm. Nonlin. Sci. and Numer. Simulation*, 13:1256–1263, 2008.

[34] A. G. Ramm. On unbounded operators and applications. *Appl. Math. Lett.*, 21:377–382, 2008.

[35] I. M. Sobol. *Multidimensional Quadrature Formulas and Haar Functions.* Nauka, Moscow, 1969.

[36] J. Varah. Pitfalls in the numerical solution of linear ill posed problems. *SIAM J. Stat. Comput.*, 4:164–176, 1983.

[37] J. G. Whirter and E. R. Pike. Laplace transform and other similar Fredholm integral equations of the first kind. *J. Phys. A: Math. Gen*, 11:1729–1745, 1978.