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On the relation between the $S-$matrix and the spectrum of the interior Laplacian

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Abstract

The main results of this paper are:

1) a proof that a necessary condition for 1 to be an eigenvalue of the S-matrix is real analyticity of the boundary of the obstacle,

2) a short proof of the conclusion stating that if 1 is an eigenvalue of the S-matrix, then k^2 is an eigenvalue of the Laplacian of the interior problem, and that in this case there exists a solution to the interior Dirichlet problem for the Laplacian, which admits an analytic continuation to the whole space R^3 as an entire function.

1. Introduction and Statement of the Result

We consider below the obstacle scattering problem in R^3, but the argument and the results remain valid in R^n, $n \geq 2$.

Let the obstacle $D \subset R^3$ be a bounded domain with a Lipschitz boundary S. Denote by $D' = R^3 \setminus D$ the exterior domain and by N, the unit normal to S, pointing into D'. Let $k > 0$ be the wave number, and S^2 be the unit sphere in R^3. The scattering matrix $\mathcal{S} = \mathcal{S}(k) = I - \frac{k}{2\pi i}A$ for the obstacle scattering problem is a unitary operator in $L^2(S^2)$, I is the identity operator and A is an integral operator in $L^2(S^2)$, whose kernel $A(\beta, \alpha, k)$ is the scattering amplitude, which is defined in formula (5) below. The operator $\mathcal{S}$ has an eigenvalue 1 if and only if equation $Aw = 0$ has a non-trivial solution. The eigenvalues of $\mathcal{S}$ have 1 as an accumulation point, they all have absolute values equal to 1 since $\mathcal{S}$ is unitary.

The following conjecture, (the Doron-Smilansky (DS) conjecture) is known:

DS conjecture: A number $k^2 > 0$ is a Dirichlet eigenvalue of the Laplacian in a bounded domain D if and only if the corresponding $\mathcal{S}$-matrix for the scattering problem by the obstacle D has an eigenvalue 1.

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This conjecture is discussed in [1]-[3], and in [2] a counterexample to this conjecture is mentioned.

From the definition of the $\mathcal{S}$-matrix it follows that 1 is its eigenvalue if and only if 0 is an eigenvalue of A, that is, equation (12) (see below) has a non-trivial solution.

We prove (see Theorem 2) that if equation (12) has a non-trivial solution, then the boundary S of D is an analytic set. Since generically S is not an analytic set, it follows that the DS conjecture is incorrect. Our result gives a necessary condition for 1 to be an eigenvalue of the $\mathcal{S}$-matrix. This condition is necessary but not sufficient for 1 to be an eigenvalue of the $\mathcal{S}$-matrix (and, therefore, not sufficient for the DS conjecture to hold for the domain D).

In [2] it is proved that if $D \subset R^2$ is a bounded domain with a sufficiently smooth boundary S, and if 1 is a Dirichlet eigenvalue of $\mathcal{S}$, then k^2 is a Dirichlet eigenvalue of the Laplacian in D. An open problem, stated in [2], is to give a proof of such a statement for $D \subset R^n$ with $n > 2$. This is done in our paper by a method different from the one in [2]. Our proof is short and simple.

Let S^2_j, $j = 1, 2$, be arbitrary small fixed open subsets of S^2, and the boundary conditions on S be either the Dirichlet, or the Neumann, or the Robin conditions.

The following theorem is proved in [5], p.85:

Theorem (Ramm) *The knowledge of* $A(\beta, \alpha, k)$, $\forall \alpha \in S^2_1$, $\forall \beta \in S^2_2$, *and for a fixed* $k > 0$, *determines* S *and the boundary conditions on* S *uniquely.*

It follows from this result that the knowledge of the $S-$ matrix $\mathcal{S}(k)$ at a fixed $k > 0$ determines the boundary S of the obstacle and the boundary condition on S uniquely.

Therefore, the discrete spectrum of the Laplacian in D, corresponding to this boundary condition, is determined uniquely by the knowledge of $\mathcal{S}(k)$ at a fixed $k > 0$.

This conclusion establishes a relation between the $S-$matrix and the spectrum of the Laplacian in D.

Let us now formulate the obstacle scattering problem, introduce basic notions, and formulate our results.

The scattering solution $u(x, \alpha, k)$ is the solution to the following scattering problem:

$$Lu := (\nabla^2 + k^2)u = 0 \text{ in } D', \tag{1}$$

$$u \mid_S = 0, \tag{2}$$

$$u = u_0 + v, \qquad u_0 := e^{ik\alpha\cdot x}, \tag{3}$$

$$\frac{\partial v}{\partial r} - ikr = o\left(\frac{1}{r}\right), \qquad r := |x| \to \infty. \tag{4}$$

Here $\alpha \in S^2$ is the incident direction, i.e., the direction of the incident plane wave u_0, v is the scattered field which satisfies the radiation condition (4). This condition implies that

$$v := v(x, \alpha, k) = A(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \beta := \frac{x}{r}. \tag{5}$$

The function $A := A(\beta, \alpha, k)$ is called the scattering amplitude. Let us denote by $A : L^2(S^2) \to L^2(S^2)$ the operator

$$Aw := \int_{S^2} A(\beta, \alpha, k) w(\alpha) d\alpha. \tag{6}$$

It is well known (see [5]), that problem $(1)-(4)$ has a unique solution $u(x, \alpha, k)$,

$$A(\beta, \alpha, k) = -\frac{1}{4\pi} \int_S e^{-ik\beta\cdot s} u_N(s, \alpha, k) ds, \tag{7}$$

where $u_N(s, \alpha, k)$ is the normal derivative of the scattering solution $u(x, \alpha, k)$ on S, and the following relation holds:

$$u(x, \alpha, k) = e^{ik\alpha\cdot x} - \int_S g(x, s, k) u_N(s, \alpha, k) ds. \tag{8}$$

Here G, the resolvent kernel of the Dirichlet Laplacian in the exterior domain D', satisfies the following equation:

$$G(x, y, k) = g(x, y, k) - \int_S g(x, s, k) G_N(s, y, k) ds, \tag{9}$$

where

$$g(x, y, k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \tag{10}$$

The function G solves the boundary value problem:

$$LG = -\delta(x-y) \,\text{in}\, D', \qquad G \mid_S = 0, \tag{11}$$

and satisfies the radiation condition (4).

Let σ denote the set of the eigenvalues of the Dirichlet Laplacian in D. This set is discrete.

It is proved in [5], pp.52-57, that:

a) The function $A(\beta, \alpha, k)$ admits a meromorphic continuation as a function of k from the ray $(0, \infty)$ to the whole complex k-plane,

b) The scattering amplitude $A(\beta, \alpha, k)$ is analytic in the region $Imk \geq 0$ (if $D \subset R^{2n}$ then $k = 0$ is a logarithmic branch point),

c) $A(\beta, \alpha, k)$ has infinitely many poles on the imaginary axis in the region $Imk < 0$,

d) As a function of α and β, the scattering amplitude $A(\beta, \alpha, k)$ admits analytic continuation from $S^2 \times S^2$ to the set $M \times M$, where $M := \{\Theta : \Theta \in \mathbb{C}^3,\ \Theta \cdot \Theta = 1\}$, $\Theta \cdot \omega := \sum_{j=1}^{3} \Theta_j\, \omega_j$. The set M is a non-compact algebraic variety in $\mathbb{C}^3$.

Let us now state our basic results:

Theorem 1. *If $\mathcal{S}(k)$ has an eigenvalue* 1*, that is, the equation*

$$Aw = \int_{S^2} A(\beta, \alpha, k) w(\alpha) d\alpha = 0 \tag{12}$$

has a non-trivial solution w, then $k^2 \in \sigma$, and there is a solution to the problem $(\nabla^2 + k^2)W = 0$ in D, $W|_S = 0$, which can be extended from D to R^3 as a bounded entire function of x.

Theorem 2. *If equation (12) has a non-trivial solution, then the boundary S is an analytic set.*

An analytic set is a set of zeros of (a finite collection of) analytic functions. One can find definition and properties of analytic sets in [4], Section 1.4. If S is an analytic set, then S is piecewise real analytic surface. Generically, S is not piecewise real analytic surface. Therefore, it follows from Theorem 2 that the DS conjecture is incorrect.

In Section 2 Theorems 1 and 2 are proved. In the proofs, the following result of the author is used:

Lemma 1. ([5], p.46) *One has*

$$G(x, y, k) = \frac{e^{ik|y|}}{4\pi|y|} u(x, \alpha, k)[1 + o(1)], \quad |y| \to \infty, \quad \frac{y}{|y|} = -\alpha, \tag{13}$$

where $u(x, \alpha, k)$ is the scattering solution, i.e., the solution to (1) - (4).

Lemma 1 yields formula (8) as a consequence of (9), while formula (9) is obtained by Green's formula. Formula (7) follows from (8).

2. Proofs.

Proof of Theorem 1. *Let us prove that if $w \not\equiv 0$ solves (12) then $k^2 \in \sigma$.*

Assume that equation (12) has a non-trivial solution w. Multiply (7) by $w = w(\alpha)$ and integrate over S^2 with respect to α. The result is

$$\int_S e^{-ik\beta\cdot s} p(s) ds = 0, \qquad p(s) := \int_{S^2} u_N(s, \alpha, k) w(\alpha) d\alpha. \tag{14}$$

Let us prove that $p(s) \not\equiv 0$.

Indeed, if

$$p(s) = \int_{S^2} u_N(s, \alpha, k) w(\alpha) d\alpha = 0 \qquad \forall s \in S, \tag{15}$$

then the function $w(\alpha) = 0$ because the set $\{u_N(s, \alpha, k)\}_{\forall \alpha \in S^2}$ is total (dense) in $L^2(S)$ for any fixed $k > 0$ ([5], p.162).

Let us continue the proof of Theorem 1 and prove that $k^2 \in \sigma$ if equation (12) has a non-trivial solution.

Equation (14) and Lemma 1 imply that

$$\nu(x) := \int_S \frac{e^{ik|x-s|}}{4\pi|x-s|} p(s) ds = 0 \quad \text{in } D'. \tag{16}$$

Indeed, this ν solves equation (1), satisfies the radiation condition (4), and (14) implies

$$\nu(x) = o\Big(\frac{1}{|x|}\Big), \qquad |x| \to \infty. \tag{17}$$

Relation (17) and Lemma 1 in [5], p.25, imply that

$$\nu(x) = 0 \quad \text{in } D'. \tag{18}$$

Therefore, by the jump formula for the normal derivative of the single layer potential (16) ([5], p.14), one gets

$$\frac{\partial \nu}{\partial N_+} = p(s) \not\equiv 0, \tag{19}$$

where $\frac{\partial}{\partial N_+}$ denotes the limiting value on S of the normal derivative from inside of D.

This implies that $k^2 \in \sigma$.

Indeed, $\nu(x)$ solves the equation

$$(\nabla^2 + k^2)\nu = 0 \quad \text{in } D', \tag{20}$$

and satisfies the boundary condition

$$\nu|_S = 0, \tag{21}$$

due to (18) and the continuity of ν across S. Finally, $\nu \not\equiv 0$ in D because of (19).

The last statement of Theorem 1, namely, the existence of the solution to problem (20)-(21) which can be analytically continued to the whole space R^3 as an entire function of x, is proved as follows.

The reciprocity relation $A(\beta, \alpha, k) = A(-\alpha, -\beta, k)$ (see [5], p.53) and equation (12) imply:

$$0 = \int_{S^2} A(\beta, \alpha, k) w(\alpha) d\alpha = -\frac{1}{4\pi} \int_S \Big(\int_{S^2} e^{ik\alpha\cdot s} w(\alpha) d\alpha \Big) u_N(s, -\beta) ds \qquad \forall \beta \in S^2. \tag{22}$$

Since the set $\{u_N(s, \alpha, k)\}_{\forall \alpha \in S^2}$ is total (dense) in $L^2(S)$ for any fixed $k > 0$ ([5], p.162), relation (22) implies

$$\int_{S^2} e^{ik\alpha\cdot s} w(\alpha) d\alpha = 0 \quad \forall s \in S. \tag{23}$$

Therefore, the function

$$W(x) := \int_{S^2} e^{ik\alpha\cdot x} w(\alpha) d\alpha, \qquad x \in R^3, \tag{24}$$

satisfies all the requirements mentioned in the last statement of Theorem 1.

Thus, Theorem 1 is proved. $\square$

Remark 1. A similar argument yields the following result:

If $\sigma_{\mathcal{N}}$ is the set of the eigenvalues of the Neumann Laplacian, and $A_{\mathcal{N}}(\beta, \alpha, k)$ is the scattering amplitude, corresponding to the plane wave scattering by the obstacle D on the boundary of which the Neumann boundary condition holds, then if equation (12), with $A_{\mathcal{N}}$ in place of A, has a non-trivial solution, then $k^2 \in \sigma_{\mathcal{N}}$.

Remark 2. If $k^2 \in \sigma$, then any non-trivial solution to (20)-(21) can be written in the form (16) with $p(s)$ defined in (19), and the boundary condition (18) holds. Taking $|x| \to \infty$, $\frac{x}{|x|} = \beta$, in (16) and using (18), one obtains

$$\int_S e^{-ik\beta\cdot s} p(s) ds = 0 \qquad \forall \beta \in S^2, \qquad p(s) \not\equiv 0. \tag{25}$$

Thus, if $k^2 \in \sigma$, then equation (25) has a non-trivial solution $p(s)$.

Proof of Theorem 2. Suppose equation (12) has a solution $\eta \in L^2(S^2)$, $\eta \not\equiv 0$. Then

$$\int_S ds u_N(s, \alpha) \int_{S^2} e^{-ik\beta\cdot s} \eta(\beta) d\beta = 0 \qquad \forall \alpha \in S^2. \tag{26}$$

Since the set $\{u_N(s, \alpha)\}_{\forall \alpha \in S^2}$ is total in $L^2(S)$, one concludes from (26) that

$$\psi(s) := \int_{S^2} e^{-ik\beta\cdot s} \eta(\beta) d\beta = 0 \qquad \forall s \in S, \tag{27}$$

where

$$\psi(x) := \int_{S^2} e^{-ik\beta\cdot x} \eta(\beta) d\beta.$$

The function $\psi(x)$ is an entire function of x, that is, an analytic function of $x \in \mathbb{C}^3$. It vanishes on S, so S is *an analytic set* (see [2] for the definition and properties of analytic sets). Generically, the boundary S is not an analytic set.

Thus, Theorem 2 is proved. $\square$

Remark 3. If one uses the reciprocity relation $A(\beta, \alpha, k) = A(-\alpha, -\beta, k)$, then one concludes that zero is an eigenvalue of A if either

$$\int_S e^{-ik\beta\cdot s} \int_{S^2} u_N(s, \alpha, k) w(\alpha) d\alpha = 0 \qquad \forall \beta \epsilon S^2, \qquad w \not\equiv 0, \tag{28}$$

or

$$\int_S \Big(\int_{S^2} e^{ik\alpha\cdot s} w(\alpha) d\alpha \Big) u_N(s, -\beta) ds = 0 \qquad \forall \beta \in S^2, \qquad w \not\equiv 0. \tag{29}$$

The last relation implies equation (28) (with $\beta = -\alpha$ and $\eta(\beta) = w(\alpha)$).

Let us denote $T_k p := \int_S g(s, t, k) p(t) dt$ and $U := U(x, k) := \int_S g(x, t, k) p(t) dt$, so $U|_S = T_k p$.

Remark 4. *The operator T_k^{-1} has simple poles at the points $k^2 = k_j^2$, where $k_j^2 \in \sigma$.*

Remark 4 shows that the knowledge of the set of poles of the operator T_k^{-1} allows one to find the spectrum of the interior Dirichlet Laplacian in D.

Proof of Remark 4. Consider the equation $T_k p = f$. Then

$$U(x) = \int_S g(x,t,k)p(t)dt$$

solves the problem

$$(\nabla^2 + k^2)U = 0 \text{ in } D, \qquad U|_S = f. \tag{30}$$

Let

$$(\nabla^2 + k^2)\Gamma = -\delta(x-y) \text{ in } D, \qquad \Gamma\ |_S = 0.$$

Then Green's formula yields the following representation of the solution to problem (30):

$$U(x) = -\int_S f(t)\Gamma_{N_t}(t,x,k)dt, \quad x \in D, \qquad k^2 \neq k_j^2. \tag{31}$$

Since $\Gamma(x,y,k) = \sum_{j=1}^{\infty} \frac{\phi_j(x)\overline{\phi_j(y)}}{k^2 - k_j^2}$ has a simple pole at $k^2 = k_j^2$, the claim is proved. Here ϕ_j are the normalized eigenfunctions of the Dirichlet Laplacian in D.

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