Abe homotopy groups of homogeneous spaces with applications to topological phases of matter

by

Michael Hill

B.A., Butler University, 2011

M.S., Miami University, 2013

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

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Manhattan, Kansas

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Abstract

Describing physical systems can be complicated. In order to reduce the complexity of certain models, one will sometimes restrict attention to the so called order parameter field. The order parameter field can take many different forms depending on the physical system to be modeled. We can look at the topological characteristics of the order parameter space to model what has become known as topological phases of these systems.

The aspect of physical systems we will focus on here is topological excitations, also known as topological defects. These defects are often classified by the homotopy groups of the associated order parameter manifold M, with the n^{th} homotopy group of M, $\pi_n(M)$, describing the n-dimensional defects. These homotopy groups give all the possible defects that can occur in a given dimension, as well as how the defects can interact with each other. Furthermore, the action of the fundamental group of M on the homotopy group $\pi_n(M)$ tells how a one-dimensional defect acts on a n-dimensional defect. This information is given by the Abe homotopy group of the order parameter manifold.

We look to describe the Abe homotopy groups associated to some interesting physical cases and expand the results to a simplification of the computation of the Abe homotopy groups of all connected homogeneous spaces.

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Dedication

To my family for their support and encouragement.

Chapter 1

Introduction

Some fundamental physical models may be approximated near the local minima of the potential energy. In the presence of symmetry this leads to effective theories with fields taking values in an order parameter space given by the orbit of a group action which is a homogeneous space. Topological features of these models can describe phases of the system that are stable. This is the idea of a topological phase. One group [15] used something called the Abe homotopy groups in this context and computed a number of physically relevant Abe homotopy groups. There were four cases that the Abe homotopy groups were not completely understood and we wish to analyze those cases here.

In chapter 2, we describe three interesting physical systems: Liquid crystals, Bose-Einstein condensates, and liquid Helium. We begin in section 2.1, we review some background information on liquid crystals, describe the difference between a liquid and a crystal via density measures, describe the Landau-De Gennes Model of a uniaxial nematic liquid crystal, and the associated order parameter manifold that we are interested in analyzing. For section 2.2, we describe spin-1 and spin-2 Bose-Einstein Condensates by looking at the partial differential equation, energy functional, and Hamiltonian that describe them. After finding the symmetry group of the Hamiltonian, we determine the four possible stabilizer groups of the symmetry group. Two of these stabilizers correspond to order parameter manifolds for spin-1 Bose-Einstein condensates with the polar phase being of interest for

describing Abe homotopy groups. A brief description of the spin-2 uniaxial nematic phase we are also interested in studying is mentioned. The calculations are extremely similar to the spin-1 case and are omitted. In section 2.3 we describe the order parameter manifold for super cold ^3He. While the symmetry group for ^3He is known, there is no full analysis for all possible stabilizer groups. A short description for the A-phase of ^3He is given.

After a review of basic homotopy theory concepts in chapter 3, the definition of Abe homotopy is introduced and related to homotopy groups. The second and third Abe homotopy groups of the real protective plane are described with a computation for whether the action on each is trivial or nontrivial.

In chapter 4 the remaining three cases are computed fully answering the questions of [15]. The main result of the simplification for computations of Abe homotopy groups of a connected homogeneous spaces is given. This is done by looking at the *factorizing group*

$$F_M := H/((Z \cap H)H_0)$$

where $M = G/H$ is a connected homogeneous space represented as the quotient of a simply-connected Lie group and Z is the center of G. The action of $\pi_1(M)$ on $\pi_n(M)$ factors through the factorizing group.

In chapter 5, we review symmetric spaces and describe the action of the Abe homotopy group for $SU(n)/SO(n)$.

Chapter 2

Physical Systems

2.1 Liquid Crystals

2.1.1 Introduction

In this section we describe uniaxial nematic liquid crystals as an example of a physical system that has a model incorporating an order parameter. For the liquid crystal we will be looking at the orientation of the particles as the order parameter. Liquid crystals are so-called mesomorphic phases of matter. Mesomorphic phases of matter are those that are intermediate steps between the three traditional phases of gas, liquid, and solid. Liquid Crystals share physical properties with both liquids and solids. The solids in question are crystals. Crystals are defined to be a solid with a lattice structure, while liquids don't show any form of long range order. These traits can be explained by looking at something called the density-density correlation function and average particle density function [11]. We begin this chapter with a mathematical description of these phases of matter. After giving this description we describe one model for one phase that incorporates constraints from homotopy theory.

The materials that we consider are comprised of elementary units. These units may be collections of atoms in a fundamental cell of a crystal or molecules in a liquid. To understand the structure of the material, one selects a basic length scale ξ and computes the number

density of elementary units in a volume at length scale ξ. For large values of ξ this density is essentially constant. When the length scale becomes close to the size of the elementary unit the density will no longer be constant, but will depend on the location in space. We imagine that each elementary unit is associated with a unique point in space. This point might be the center of mass of the unit.

Consider the following model of a one-dimensional lattice with the integers representing the locations of the elementary units where we define the density measure to be $\rho_\xi(x) := \frac{\text{Number of integers in } B_\xi(x)}{2\xi}$ for different values of ξ.

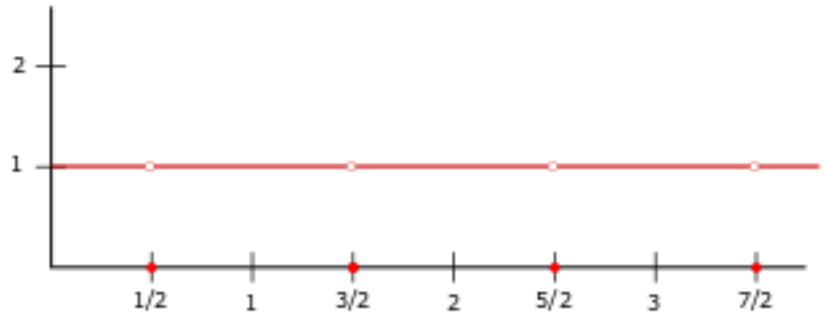

Figure 2.1: *The one-dimensional density measure* $\rho_{\frac{1}{2}}(x)$

For $\rho_{\frac{1}{2}}(x)$ the is only one integer in the ball of radius $\frac{1}{2}$ centered at any number except for when $x = \frac{1}{2} + k$ for $k \in \mathbb{Z}$. The next example of $\rho_{\frac{3}{2}}(x)$ has a value of $\frac{2}{3}$ when $x = \frac{1}{2} + k$ for $k \in \mathbb{Z}$, otherwise the value is 1. The third example of $\rho_{\frac{1}{4}}(x)$ is different than the first two. Here $\rho_{\frac{1}{4}}(x)$ takes the value of 2 on the intervals $(\frac{3+4k}{4}, \frac{5+4k}{4})$ for $k \in \mathbb{Z}$ and 0 otherwise.

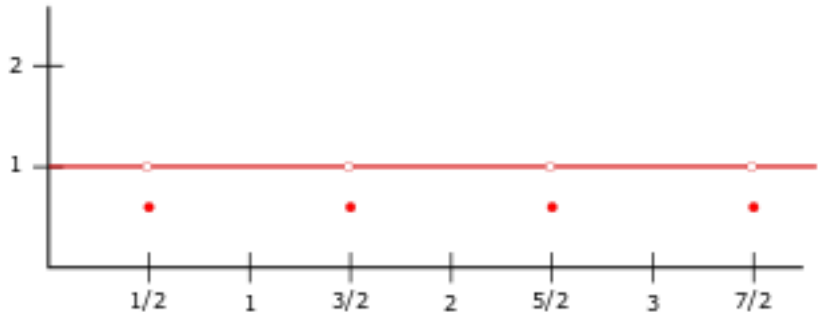

Figure 2.2: *The one-dimensional density measure* $\rho_{\frac{3}{2}}(x)$

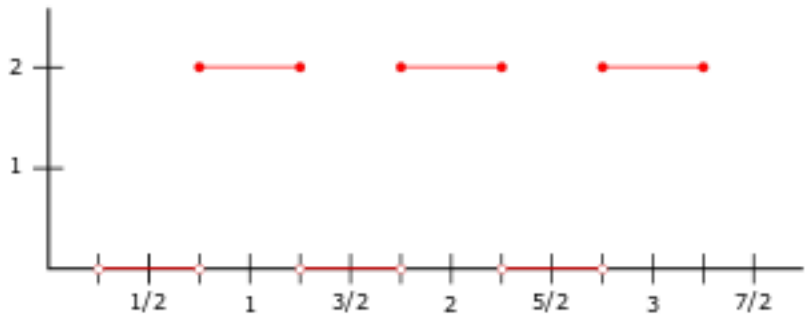

Figure 2.3: *The one-dimensional density measure* $\rho_{\frac{1}{4}}(x)$

The next example if for a two-dimensional density measure. We will build up to the full example by first looking at $\rho_{\frac{1}{4}}(x)$. Note that this function depends on $x, z \in \mathbb{R}/\mathbb{Z}$, so the contour diagram looks similar to the one dimensional case.

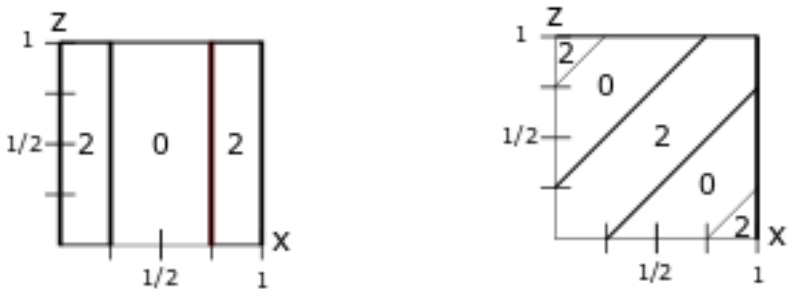

Figure 2.4: *The two-dimensional density measure* $\rho_{\frac{1}{4}}(x)$ *and* $\rho_{\frac{1}{4}}(x-z)$

For $\rho_{\frac{1}{4}}(x-z)$ we have the value of 2 between the lines $z = x - \frac{3}{4}$ and $z = x - \frac{1}{4}$. Combining the two we get the doubly periodic density measure $\rho_{\frac{1}{4}}(x)\rho_{\frac{1}{4}}(x-z)$.

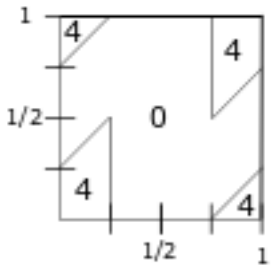

Figure 2.5: *The two-dimensional density measure* $\rho_{\frac{1}{4}}(x)\rho_{\frac{1}{4}}(x-z)$

Looking at the integral
$\frac{1}{2N}\int_{-N}^{N}\rho(x)\rho(x-z)dx = \int_0^1 \rho(x)\rho(x-z)dx = \int_0^1 \rho(x-y)\rho(x-y-z)dx$, since this function is independent of y and $\int_0^1 \rho(x-y)\rho(x-y-z)dx = \lim_{y\to\infty}\int_0^1 \rho(x-y)\rho(x-y-z)dx = F(z)$. This function is clearly periodic. Furthermore $\int_0^1 F(Z)dz = 1 = \bar{\rho}$, so we have the structure of a crystal.

The final example will be for a one-dimensional liquid. Let $\pi : [0,1) \to \mathbb{R}/\mathbb{Z}$ be a projection function. Pick $l : \mathbb{Z} \to [0,1)$ such that for all $\epsilon \in (0, \frac{1}{2})$, $x \in [0,1)$ to be the

random noise for the liquid. Then

$$\lim_{N\to\infty} \frac{\#((l^{-1}(\pi^{-1}(B_\epsilon(x)))) \cap [-N, N-2))}{2N} = 2\epsilon.$$

Let $\Lambda = \{n + l(n) | n \in \mathbb{Z}\}$ be the lattice structure and set $\rho_\xi^\Lambda = \frac{1}{2\xi}\#(B_\xi(x) \cap \Lambda)$ where we can break ξ into its integer and fractional part $\xi = \lfloor \xi \rfloor + \eta$ for $\eta \in [0, 1)$. Note that $\#B_\xi(x) \in [2\lfloor \xi \rfloor - 1, 2\lfloor \xi \rfloor + 2]$ so $\lim_{\xi\to\infty} \rho_\xi^\Lambda(x) = 1$.

With these examples in mind, when looking at a three-dimensional object, one suitable density measure would be

$$\rho_\xi(x) := \frac{\text{Number of units in } B_\xi(x)}{\frac{4}{3}\pi\xi^3}.$$

For large values of ξ this density will be nearly constant, representing the average number of units per volume $\overline{\rho}$. For small values of ξ this density function would be rough, bouncing between values of zero when no units are in the ball $B_\xi(x)$ and $\left(\frac{4}{3}\pi\xi^3\right)^{-1}$ when there is just one unit in the ball.

The density-density correlation function gives the probability to find a particle located at a point z in the given volume with the same characteristics as the particle at point x. The density-density correlation function is defined by

$$F(z) = \lim_{R\to\infty} \frac{1}{\frac{4}{3}\pi R^3} \int_{B_R(0)} \rho_\xi(x - z)\rho_\xi(x) d^3x.$$

Depending on the properties of this function $F(z)$, we can say a few things about the material it is describing. If $F(z)$ is constant, we say the system is amorphous and models a liquid or a gas. If $F(z)$ is not constant but triply periodic we say that the system is crystalline. When $F(z)$ is not constant nor triply periodic and does not have any rotational invariance, we say that the system is liquid crystal like. We can further describe the different phases of a liquid crystal by the periodic behavior that this function does have.

2.1.2 Phases of Liquid Crystals

Three dimensional liquid crystals are described by the dimension of positional ordering. Mathematically, this is the rank of the period lattice of the density-density correlation function. Columnar phases have a rank two group of translational symmetry. Smectic phases have a rank one group of translational symmetries. Nematic phases are systems containing no translational symmetries, provided the density-density correlation function is not rotationally symmetric. For most practical applications, X-ray diffraction is used to determine what phase a system is in.

Local Density	Phase
Triply Periodic	Solid Crystal
Doubly Periodic	Columanar Liquid Crystal
Singularity Perodic	Smectic Liquid Crystal
No Periodicity	Nematic Liquid Crystal

Table 2.1: *Phase based on local density without rotation*

Liquid crystals generally arise in materials having one of four structures outlined by De Gennes and Prost [11]. These structures are summarized here. All of these materials could be nematic, smectic, or columanar liquid crystals, although columanar liquid crystals tend to consist of disc like molecules.

Small elongated organic molecules, like para-azoxyanisole [20] and N-(p-methoxybenzylidene)-p-butylaniline are classic example of substances that form nematic liquid crystals. More modern examples include cholesterol esters, cyanobiphenyls, and cyanobicyclohexyl derivatives. Many of these examples have the property that they are thermotropic, meaning the system has its phase transitions governed by temperature.

The next type of molecules are long helical rods. For most of the folloing examples the rods are at least 10-15 times longer than they are wide. Synthetic polypeptides in suitable solvenents make up a large category of examples. There are also many biological examples including the Tobacco Mosaic Virus [6], and DNA in an electrolyte solution [7]. These examples are lyotropic, meaning that phase transitions in the systems are goverend by the

concentration of rods in the solution.

In addition to the rigid polymers that belong under the classification of long helical rods, main-chain polymers and side-chain polymers can also form the building blocks of liquid crystal systems. Side chain polymers are those that branch off the central main chain. They do not need to have the same structure or molecular composition as the main chain. There are many different examples of these types of liquid crystals due to the large number of combinations of materials that can make these types of polymers. Given the variable nature of this type of liquid crystal, they can be either a thermotropic or lytropic material.

The final structure De Gennes and Prost mention is associated structures dealing with amphiphilic substances. Examples of these structures include some soap-water systems, as well as more complicated systems like those involving sodium dodecyl sulphate, decanol, hexane, and water. Unlike the previous building blocks, these systems can be either lyotropic or thermotropic depending on the materials involved.

2.1.3 Applications

Liquid crystals have many applications and are seen quite frequently in everyday life. One of the more common uses is in liquid crystal displays, like those used in calculators. For a monochromatic display like those pioneered by RCA in the 1970s [3], the display used a twisted nematic liquid crystal. There are many different compounds that can be used, but they all have to occur in a liquid crystal phase around room temperature, be made of rod like molecules, and respond to electrical currents. The displays consist of two glass panes with a polarizing film on the outside of each. These films are oriented perpendicular to each other so that very little light is able to pass through making the region appear dark. The interiors of the glass panes are etched with grooves parallel to their respective polarizing films. The grooves will align the rod like molecules of the liquid crystal in the same directions. The interior of the liquid crystal has the rods twisting in a helical fashion from one pane to the other. This twisting is a deformation to the nematic phase caused by the constraints of the container. This twisted nematic liquid crystal has the optical property of being able to

reorient the polarized light so that is passes through the second polarized film. Usually there is a mirrored surface behind the second pane of glass, allowing the light to reflect back out through the liquid crystal so that there is no image displayed.

Since the molecules are polarized, if the correct electrical current is introduced to the liquid crystal they will align to be perpendicular to the two panes of glass [4]. Any light that enters through the first pane of glass will be polarized, but the polarized light will not pass through the second pane of glass, since the liquid crystal is no longer in its twisted state. This will cause the area to appear darkened since very little light makes it to the mirrored surface to be reflected back out, which we see as a pixel appearing.

2.1.4 The Landau-De Gennes Model of a Uniaxial Nematic Liquid Crystal

Particles in a uniaxial nematic liquid crystal consist of rod-like molecules with one dimension longer than the other two. In order to describe these molecules, we start by putting an orthonormal frame of vectors (x, y, z) on the rod so that the longer dimension corresponds to the k vector. This leads to a rotational symmetry around a unit vector k coinciding with the longer dimension. Furthermore there is a mirror symmetry through the plane orthogonal to the vector k. Pick a length scale ξ and let $f(n, x)$ denote the probability density of finding the direction vector k of a molecule in $B_\xi(x)$ oriented in direction n. Here $f_x : S^2 \to [0, \infty)$ is given by $f_x(n) = f(n, x)$. For example, one could take the following model.

To model the rods we need more than the location of the center, we also need a direction. We can do this with a subset $\Lambda \subseteq \mathbb{R}^3$ representing the centers of the rods together with a function $k : \Lambda \to S^2$.

Thus $f_\xi(x, n) := \dfrac{2n+1}{\#(\Lambda \cap B_\xi(x)) \cdot 4\pi} \sum_{\lambda \in (\Lambda \cap B_\xi(x))} (n \cdot k(\lambda))^{2n}$

Lemma 1. *The first moment for the probability density $f_p(m)$, denoted $\bar{m}$ is zero. The second moment M is a symmetric (2,0)-tensor with trace 1.*

Proof. By definition, $\bar{m} = \int_{S^2} m f_p(m) d^2m$. Since $f_p(m)$ is an even function, by a change of

variables we have $\bar{m} = \int_{S^2} -mf_p(-m)d^2m = -\int_{S^2} mf_p(m)d^2m = -\bar{m}$. This implies that the integral is zero, so $\bar{m} = 0$

For the second moment, we have $M = \int_{S^2}(m \otimes m)f(m)d^2m$. This (2,0)-tensor is symmetric and semidefinite positive by definition. Instead of looking at the (2,0)-tensor, we will consider the (1,1)-tensor $M^* = \int_{S^2}(m \otimes m^*)f_p(m)d^2m$ obtained by the natural isomorphism between $\mathbb{R}^3$ and its dual $(\mathbb{R}^3)^*$. First look at the trace of the integrand. To compute $\mathrm{Tr}(m \otimes m^*)$ we can work in any coordinate system, so choose an orthonormal basis with m as one of the basis elements. Call the other two elements v_1 and v_2. Then $m \otimes m^*(m) = m$, and $m \otimes m^*(v_k) = 0$. Thus $\mathrm{Tr}(m \otimes m^*) = 1$. For the trace of M^*, we have $\mathrm{Tr}(M^*) = \int_{S^2} \mathrm{Tr}(m \otimes m^*)f_p(m)d^2m = \int_{S^2} f_p(m)d^2m = 1$. We note that since $\langle (m \otimes m^*)x, y\rangle = \langle (m \cdot x)m, y\rangle = (m \cdot x)(m \cdot y) = \langle x, (m \cdot y)m\rangle = \langle x, (m \otimes m^*)y\rangle$ we have a symmetric tensor. □

The order tensor we are concerned with is defined to be $Q := M - \frac{1}{3}I$ and it is a traceless tensor. Since Q is symmetric, it can be diagonalized. This order tensor is the order parameter of the model. Sample calculations of the second moment and order parameter for various distributions appear in Appendix A.

Remark. *The order parameter of a uniaxial nematic phase has exactly two distinct eigenvalues μ and λ.*

Definition 1. $Sym^2_{0,*} := \{Q | Q^* = Q,\ Q$ *has exactly one simple eigenvalue,* $Tr(Q) = 0\}$

An effective model of a a uniaxial nematic liquid crystal is given by the Landau-De Gennes Model. This model only depends upon the second moment of the probability density of the orientation of the molecules. In a general setting the Landau-De Gennes energy functional is

$$\mathcal{F}_{\mathrm{LG}}[Q] = \int_\Omega \frac{L}{2}|\nabla Q|^2(x) + f_B(Q(x))dx$$

where Q is the order parameter defined above, $L > 0$ is a material dependent elastic constant, $|\nabla Q|^2$ is the elastic energy density, and $f_B(Q)$ is a quartic polynomial in the Q-tensor components [17]. This work on describing phases of a liquid crystal as well as other forms

of matter like polymers lead to Pierre-Gilles de Gennes being awarded the Noble prize in physics in 1991 [2]. More specifically, we are only looking to describe the order parameter manifold for a uniaxial nematic liquid crystal in the following claim.

Claim. *Define* $N_1 := \{SP^{-1}DP | S = 1\}$ *where* $P \in O(3)$, S *is a scalar, and* D *is the diagonal matrix* $diag(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. *Then the order parameter manifold for a nematic uniaxial liquid crystal is* $N := Sym^2_{0,*} \cong \mathbb{R}^\times \times N_1$.

Proof. We have the decomposition $Q = P^{-1}\text{diag}(\mu, \lambda, \lambda)P$, where $\text{diag}(\mu, \lambda, \lambda)$ is a diagonal matrix with the given values. Since Q is traceless, $\mu + 2\lambda = 0$. Define $S = -3\lambda$ so that Q can be written as $Q = SP^{-1}\text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})P$.

Let $\vec{n}$ be an eigenvector with eigenvalue $\frac{2}{3}S$, and $(\vec{p}, \vec{\epsilon}, \vec{n}) \in O(3)$. Now $Q(\vec{n}) = \frac{2}{3}S\vec{n}$, $Q(\vec{p}) = -\frac{1}{3}S\vec{p}$, and $Q(\vec{q}) = -\frac{1}{3}S\vec{q}$. Set $Q' = S((n \otimes n^*) - \frac{1}{3}I)$ and check that $Q'(\vec{n}) = \frac{2}{3}S\vec{n}$, $Q'(\vec{p}) = -\frac{1}{3}S\vec{p}$, and $Q'(\vec{q}) = -\frac{1}{3}S\vec{q}$, so $Q' = Q = S((n \otimes n^*) - \frac{1}{3}I)$. This defines the order parameter manifold $N = \{SP^{-1}DP | P \in O(3)\}$. □

Proposition 1. *The stabilizer of the action of* $O(3)$ *on* N_1 *is* $\mathbb{Z}_2 \times O(2)$.

Proof. The action on the manifold $N_1 \times O(3) \to N_1$ is defined by $(Q, A) \mapsto A^{-1}QA$. So $N_1 \cong Q \cdot O(3)$ which implies that $N_1 \cong \text{stab}(Q) \backslash O(3)$ where $\text{stab}(Q)$ is the stabilizer of Q.

Next we want to compute the stabilizer of the action. First note that $Q = A^{-1}QA$ implies $QA = AQ$. Let n be an eigenvector of Q with eigenvalue $\frac{2}{3}S$, so that $Qn = \frac{2}{3}Sn$. Now $QAn = AQn = A\frac{2}{3}n = \frac{2}{3}An$, which makes An an eigenvector with eigenvalue $\frac{2}{3}$ and $An = \nu n$ with $\nu = \pm 1$ to preserve length.

In order to find the entries of $A = (a_{ij})_{1 \le i,j \le 3}$, we need to either look at the direct computations of QA compared to AQ or the eigenvector calculations. Let μ be the eigenvalue with eigenvector $(1, 0, 0)$. Then $(a_{11}, a_{21}, a_{31}) = A(1, 0, 0) = \mu(1, 0, 0)$, and we must have that a_{21} and a_{31} are zero. Since A is an orthogonal matrix, $|a_{11}| = 1$. Furthermore since $A \in O(3)$ we have that $A^T = A^{-1}$ so that a_{12} and a_{13} are also zero.

There are no other restrictions on the matrix A, so the remaining block of the matrix

can be thought of as a matrix in $O(2)$. This gives the elements of the stabilizer of the form

$$A = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \in \mathbb{Z}_2 \times O(2).$$

□

Finally, we show that $O(3)$ mod out by the stabilizer is isomorphic to a well known manifold whose homotopy groups will be discussed in later chapters.

Claim. *The map* $L : (\mathbb{Z}_2 \times O(2))\backslash O(3) \to \mathbb{RP}^2$ *defined by*

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \mapsto [c_{11} : c_{12} : c_{13}]$$

is an isomorphism.

Proof. First, this map is well defined since multiplying the matrix by an element of the stabilizer gives either $[c_{11} : c_{12} : c_{13}]$ or $[-c_{11} : -c_{12} : -c_{13}]$ which are the same as homogeneous coordinates. For surjectivity, given any point of $\mathbb{RP}^2$, we can make some scalar multiple of that point the first row of a matrix that represents an element of $(\mathbb{Z}_2 \times O(2))\backslash O(3)$. For injectivity, suppose that both matrices Q_1 and Q_2 map to $[c_{11} : c_{12} : c_{13}]$. This means that they are both of the form $\begin{bmatrix} \pm c_{11} & \pm c_{12} & \pm c_{13} \\ * & * & * \\ * & * & * \end{bmatrix}$, possibly after multiplication by some element of the stabilizer. Recall that since these matrices lie in $O(3)$, $Q_1^{-1} = Q_1^T$ and $Q_2^{-1} = Q_2^T$. Consider $Q_2 Q_1^T Q_1 = Q_2$. Looking at the multiplication for $Q_2 Q_1^T$, the entry in the first row and column is $\pm(c_{11})^2 \pm (c_{12})^2 \pm (c_{13})^2 = \pm 1$. This implies that $Q_2 Q_1^T$ is an element of the stabilizer. Therefore Q_1 and Q_2 belong to the same equivalence class.

Finally, L has a continuous inverse. The only property that is not immediately clear for the inverse is whether $(\mathbb{Z}_2 \times O(2))\backslash O(3)$ is Hausdorff. Since a topological group modulo a closed subgroup is Hausdorff, the inverse map will be continuous. □

This means that the order parameter manifold for the liquid crystal system in the uniaxial nematic phase is $\mathbb{R}^\times \times \mathbb{R}P^2$. This is homotopy equivalent to a disjoint union of two

copies of $\mathbb{R}P^2$. This essentially agrees with the result listed in [15].

2.2 Bose-Einstein Condensates

2.2.1 History

In this section we describe spin-1 polar Bose-Einstein Condensate (BEC) in terms of its order parameter manifold and the Abe homotopy groups that characterize what topological excitations can occur in the system. A BEC is a state of matter where atoms tend to behave like bosons at temperatures near absolute zero. A boson is a type of particle in Bose-Einstein statistics characterized by having an integer quantum spin number and identical bosons can occupy the same quantum state. The other type of particles are called fermions. These particles have half-integer quantum spin numbers and identical fermions can not occupy the same quantum state [9].

The concept of these condensates and Bose-Einstein statistics was developed by Satyendra Nath Bose and Albert Einstein around 1924 [1]. In 1995 Cornell and Wieman would create a BEC using rubidium. In the same year Ketterle would independently create a BEC out of sodium atoms. These three would later go on to share the Nobel prize for physics in 2001 for their work on BECs [1]. These condensates continue to be of great importance in research today [18].

In order to make similar calculations to those of a liquid crystal we need to look at the partial differential equation, energy functional, and Hamiltonian that describe a BEC.

2.2.2 Spin-1 Bose-Einstein Condensates

For a spin-1 BEC we have a 3-component complex wave function $\Psi = (\psi_1, \psi_0, \psi_{-1})^T$ whose dynamics are described by a generalized Gross-Pitaevskii equitation [13],

$$i\hbar\partial_t\psi = \frac{\delta\mathcal{E}[\psi]}{\delta\psi^*}, \text{where } \mathcal{E}[\psi] = \int H(\psi)dx.$$

The Hamiltonian is $H(\Psi) = \frac{\hbar^2}{2m}|\nabla\Psi|^2 + V(x)|\Psi|^2 + \frac{c_n}{2}|\Psi|^4 + \frac{c_s}{2}|\Psi^\dagger \mathbf{F}\Psi|^2$ with the terms representing kinetic energy, potential energy, spin-independent interaction between bosons, and spin-exchange interaction between bosons. The trapping potential, $V(x)$, that satisfies $V(x) \to \infty$ as $|x| \to \infty$, m is the mass of the bosons, $\Psi^\dagger = (\psi_1^*, \psi_0^*, \psi_{-1}^*)^T$ with ψ_k^* being the complex conjugate, and $\mathbf{F} = (F_x, F_y, F_z)$ are the spin-1 Pauli operators

$$F_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, F_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We use a version of these spin-1 Pauli matrices based on the quaternions. View the quaternions as a complex vector space with the complex numbers acting on the right. Recall that we can define the inner product on $\mathbb{C}$ for $a = x + iy$ and $b = v + iw$ as $\langle a, b\rangle = (x - iy)(v + iw)$ and the $\mathbb{C}$-inner product on $\mathbb{H}$ with viewing the complex numbers as $\mathbb{C} = \mathbb{R}[1, i]$ as $\mathrm{pr}_{\mathbb{C}}(q^*q) = \langle q, q\rangle$ The spin-1 representation that has a unitary basis of $\{1 \otimes 1, \frac{\sqrt{2}}{2}(1 \otimes j + j \otimes 1), j \otimes j\}$. Left multiplication by i, j, and k on these basis elements is computed below.

For multiplication by i we have,

$$\begin{aligned} i(1 \otimes 1) &= 1 \otimes i + i \otimes 1 \\ &= (1 \otimes 1)2i \\ i\frac{\sqrt{2}}{2}(1 \otimes j + j \otimes 1) &= \frac{\sqrt{2}}{2}(i \otimes j + 1 \otimes ij + jj \otimes 1 + j \otimes i) \\ &= \frac{\sqrt{2}}{2}((1 \otimes j)i + (1 \otimes j)(-i) + (j \otimes 1)(-i) + (j \otimes 1)(i) \\ &= 0 \\ i(j \otimes j) &= ij \otimes j + j \otimes ij \\ &= (j \otimes j)(-2i) \end{aligned}$$

which gives the matrix $2iF_z$.

For multiplication by j we have,

$$\begin{aligned}
j(1\otimes 1) &= 1\otimes j + j\otimes 1 \\
&= \frac{\sqrt{2}}{2}(1\otimes j + j\otimes 1)(\sqrt{2}) \\
j\frac{\sqrt{2}}{2}(1\otimes j + j\otimes 1) &= \frac{\sqrt{2}}{2}(j\otimes j + 1\otimes jj + jj\otimes 1 + j\otimes j) \\
&= \frac{\sqrt{2}}{2}((j\otimes j) + (1\otimes 1)(-i) + (1\otimes 1)(-i) + (j\otimes j)) \\
&= (j\otimes j)(\sqrt{2}) + (1\otimes 1)(-\sqrt{2}) \\
j(j\otimes j) &= jj\otimes j + j\otimes jj = -1\otimes j + j\otimes -1 \\
&= \frac{\sqrt{2}}{2}(1\otimes j + j\otimes 1)(-\sqrt{2})
\end{aligned}$$

which gives the matrix $-2iF_y$.

For multiplication by k we have,

$$\begin{aligned}
k(1\otimes 1) &= 1\otimes k + k\otimes 1 = 1\otimes ij + ij\otimes 1 \\
&= 1\otimes -ji + -ji\otimes 1 = (1\otimes j + j\otimes 1)(-i) \\
&= \frac{\sqrt{2}}{2}(1\otimes j + j\otimes 1)(-i\sqrt{2}) \\
k\frac{\sqrt{2}}{2}(1\otimes j + j\otimes 1) &= \frac{\sqrt{2}}{2}(k\otimes j + 1\otimes kj + kj\otimes 1 + j\otimes k) \\
&= \frac{\sqrt{2}}{2}(ij\otimes j + 1\otimes ijj + ijj\otimes 1 + j\otimes ij) \\
&= \frac{\sqrt{2}}{2}((j\otimes j)(-i) + (1\otimes 1)(-i) + (1\otimes 1)(-i) + (j\otimes j)(-i)) \\
&= [(1\otimes 1) + (j\otimes j)](-\sqrt{2}i) \\
k(j\otimes j) &= kj\otimes j + j\otimes kj = ijj\otimes j + j\otimes ijj \\
&= -i\otimes j + j\otimes -i \\
&= \frac{\sqrt{2}}{2}(1\otimes j + j\otimes 1)(-i\sqrt{2})
\end{aligned}$$

which gives the matrix $-2iF_x$.

In order to define the order parameter manifold for the spin-1 case, we first define the group and algebra actions. The group action $G : Sp(1) \times \mathrm{Sym}_2\mathbb{H} \to \mathrm{Sym}_2\mathbb{H}$ is defined by $q(x \otimes y + y \otimes x) = qx \otimes qy + qy \otimes qx$.

Another action on this group is right multiplication by complex numbers. The group action commutes with this multiplication of complex numbers allowing us to define the action $C : \mathrm{Sym}_2\mathbb{H} \times S^1 \to \mathrm{Sym}_2\mathbb{H}$ given by $(x \otimes y + y \otimes x)\alpha = x\alpha \otimes y + x \otimes y\alpha + y \otimes x\alpha + y\alpha \otimes x$.

Looking at the Hamiltonian, we want to see that it has the symmetry group $G = \mathrm{Sp}(1) \times S^1$. We can go through term by term to check that G is in fact the symmetry group. To see the S^1 symmetry consider the change $\Psi \to \Psi\lambda$ where $\lambda \in S^1$. Then we have $\frac{1}{4}|\bar{\lambda}\Psi^\dagger \mathbf{F} \Psi\lambda|^2 = \frac{1}{4}|\bar{\lambda}\Psi^\dagger i\Psi\lambda|^2 + \frac{1}{4}|\bar{\lambda}\Psi^\dagger j\Psi\lambda|^2 + \frac{1}{4}|\bar{\lambda}\Psi^\dagger k\Psi\lambda|^2$ for one of the terms. The first summand, ignoring the constant term $\frac{1}{4}$, is

$$\begin{aligned}
|\bar{\lambda}\Psi^\dagger i\Psi\lambda|^2 &= \mathrm{Re}((\bar{\lambda}\Psi^\dagger i\Psi\lambda)^*(\bar{\lambda}\Psi^\dagger i\Psi\lambda)) \\
&= \mathrm{Re}(\bar{\lambda}\Psi^\dagger(-i)\Psi\lambda\bar{\lambda}\Psi^\dagger i\Psi\lambda) \\
&= \mathrm{Re}(\bar{\lambda}\Psi^\dagger(-i)\Psi\Psi^\dagger i\Psi\lambda) \\
&= \mathrm{Re}(-\Psi^\dagger i\Psi\Psi^\dagger i\Psi) = |\Psi^\dagger i\Psi|^2
\end{aligned}$$

with the other two summands having similar calculations.

To see the Sp(1) symmetry consider the changes $\Psi \to e^{i\theta}\Psi$ and $\Psi \to e^{j\theta}\Psi$. These two elementary changes form a basis for any quaternionic change. We have $\frac{1}{4}|\Psi^\dagger e^{-i\theta}\mathbf{F}e^{i\theta}\Psi|^2 = \frac{1}{4}|\Psi^\dagger e^{-i\theta}ie^{i\theta}\Psi|^2 + \frac{1}{4}|\Psi^\dagger e^{-i\theta}je^{i\theta}\Psi|^2 + \frac{1}{4}|\Psi^\dagger e^{-i\theta}ie^{i\theta}\Psi|^2$, so the first summand

is preserved. For the other summands, again ignoring the constant term, we have

$$\begin{aligned}
|\Psi^\dagger e^{-i\theta} j e^{i\theta}\Psi|^2 &= |\Psi^\dagger j e^{2i\theta}\Psi|^2 \\
&= |\Psi^\dagger j\cos(2\theta)\Psi - \Psi^\dagger k\sin(2\theta)\Psi|^2 \\
&= \cos^2(2\theta)|\Psi^\dagger j\Psi|^2 + \sin^2(2\theta)|\Psi^\dagger k\Psi|^2 \\
&\quad - \cos(2\theta)\sin(2\theta)\left(\langle\Psi^\dagger j\Psi, \Psi^\dagger k\Psi\rangle + \langle\Psi^\dagger k\Psi, \Psi^\dagger j\Psi\rangle\right),
\end{aligned}$$

and

$$\begin{aligned}
|\Psi^\dagger e^{-i\theta} k e^{i\theta}\Psi|^2 &= |\Psi^\dagger k e^{2i\theta}\Psi|^2 \\
&= |\Psi^\dagger k\cos(2\theta)\Psi + \Psi^\dagger j\sin(2\theta)\Psi|^2 \\
&= \cos^2(2\theta)|\Psi^\dagger k\Psi|^2 + \sin^2(2\theta)|\Psi^\dagger j\Psi|^2 \\
&\quad + \cos(2\theta)\sin(2\theta)\left(\langle\Psi^\dagger j\Psi, \Psi^\dagger k\Psi\rangle + \langle\Psi^\dagger k\Psi, \Psi^\dagger j\Psi\rangle\right).
\end{aligned}$$

Thus,

$$|\Psi^\dagger e^{-i\theta} j e^{i\theta}\Psi|^2 + |\Psi^\dagger e^{-i\theta} k e^{i\theta}\Psi|^2 = |\Psi^\dagger j\Psi|^2 + |\Psi^\dagger k\Psi|^2.$$

The analogous results will occur for the change $\Psi \to e^{j\theta}\Psi$. Since Sp(1) is generated by the set containing elements of the form $e^{i\theta}$ and elements of the form $e^{j\theta}$, we see that $|\Psi^\dagger \mathbf{F}\Psi|^2$ is invariant under the Sp(1) action.

Next consider the change $\Psi \to \Psi\lambda$ for the term $|\Psi|^2$.

$$|\Psi\lambda|^2 = \mathrm{Re}(\bar{\lambda}\Psi^\dagger\Psi\lambda) = \mathrm{Re}(\Psi^\dagger\Psi) = |\Psi|^2$$

For the changes $\Psi \to e^{i\theta}\Psi$ and $\Psi \to e^{j\theta}\Psi$ both calculations are similar:

$$|e^{i\theta}\Psi|^2 = \mathrm{Re}(\Psi^\dagger e^{-i\theta} e^{i\theta}\Psi) = \mathrm{Re}(\Psi^\dagger\Psi) = |\Psi|^2.$$

The remaining terms in the Hamiltonian $|\Psi|^4$ and $|\nabla\Psi|^2$ follow from the computations for

$|\Psi|^2$ since squaring or taking the derivative will not change the symmetry.

Before moving on to determining the stabilizers, it is worth noting that all symmetric 2-tensors are decomposable. Indeed, to write $2a(1 \otimes 1) + 2b(j \otimes j) + c(1 \otimes j + j \otimes 1)$ as a decomposable symmetric 2-tensor let

$$\begin{aligned}&(1z_1 + jz_2) \otimes (1w_1 + jw_2) + (w_1 + jw_2) \otimes (1z_1 + jz_2)\\ =&2(1 \otimes 1)z_1w_1 + 2(j \otimes j)z_2w_2 + (1 \otimes j + j \otimes 1)(z_1w_2 + z_2w_1)\end{aligned}$$

be a decomposable element. Set $a = z_1w_1$, $b = z_2w_2$, and $c = z_1w_2 + z_2w_1$. Then we can solve for $w_1 \frac{a}{z_1}$, $w_2 = \frac{b}{z_2}$, and $c = \frac{bz_1}{z_2} + \frac{az_2}{z_1}$. If we write $x = \frac{z_1}{z_2}$, then the equation for c becomes $c = bx + \frac{a}{x}$, which is a quadratic equation in x that is solvable when $b \neq 0$.

If $b = 0$, then $x = \frac{a}{c}$. This leads to two more cases of either $a = 0$ or $a \neq 0$, both which have solutions.

Proposition 2. *Up to conjugacy, the four possible stabilizer groups of $G = Sp(1) \times S^1$ acting on $Sym^2\mathbb{H}$ are $Sp(1) \times S^1$, $Pin(2)$, S^1, and $\mathbb{Z}_2$.*

Proof. Looking at $x \otimes y + y \otimes x$, we begin with the case when $x = 0$. Here we have that $x \otimes y + y \otimes x = 0$ so the stabilizer is the entire group.

If $x \neq 0$, then we can write x in polar form as $x = re^{\theta v}$ where r is a positive real number, v is a purely imaginary quaternion, and $\theta \in [0, 2\pi)$. Take

$$\begin{aligned}e^{-\theta v}(x \otimes y + y \otimes x) &= e^{-\theta v}x \otimes e^{-\theta v}y + e^{-\theta v}y \otimes e^{-\theta v}x\\ &= r \otimes e^{-\theta v}y + e^{-\theta v}y \otimes r\end{aligned}$$

thus there is an element in the orbit in which x is a positive real, so we may assume that x is a real number. Note that the stabilizer of an element is equal to the stabilizer of a real multiple of that element,
$(q, \alpha)(rx \otimes y + y \otimes rx) = r(q, \alpha)(x \otimes y + y \otimes x)$, so without loss of generality we can now take $x = 1$.

Consider $1 \otimes (z + jw) + (z + jw) \otimes 1 = (1 \otimes 1)(2z) + (1 \otimes j + j \otimes 1)w$. If $|z| = 0$ then using the S^1 symmetry and real scale we may assume that $w = 1$ and have $(q, \alpha)(1 \otimes j + j \otimes 1) = q \otimes qj\alpha + qj \otimes q\alpha$. We want this quantity to be $1 \otimes j + j \otimes 1$, so consider the equivalent representatives $q\beta \otimes qj\alpha\beta^{-1} + qj\gamma \otimes q\alpha\gamma^{-1}$. One possibility is $q\beta = 1$, $qj\gamma = j$, and $qj\alpha\beta^{-1} = j$. This leads to $q = \beta^{-1} = \gamma$ and $\alpha = 1$ which is one family of solutions. The other possibility is $q\beta = j$, $qj\gamma = 1$, and $qj\alpha\beta^{-1} = 1$ which leads to $q = j\beta^{-1} = -j\gamma$ and $\alpha = -1$. If we take q to be complex and set $\beta = q^{-1}$ then $1 \otimes qj\alpha q^{-1} = 1 \otimes j\alpha$, so $(q, \alpha) = (q, 1)$. Together these imply that the stabilizer is $\{(\lambda, 1), (j\lambda, -1)\}$ which is isomorphic to $S^1 \cup jS^1 = \mathrm{Pin}(2)$.

The next case is when $x = 1$ and $z \neq 0$, so up to the S^1-symmetry and scaling we may assume $z = 1$, $(q, \alpha)(1 \otimes (1 + jw) + (1 + jw) \otimes 1) = q \otimes q(1 + jw)\alpha + q(1 + jw) \otimes q\alpha$. If $w = 0$ then we would need $q \in \mathbb{C}$ with $q^2\alpha = 1$ and this has a unique solution for any unit complex q implying a stabilizer of S^1.

For the last case when $w \neq 0$, we either have $q \otimes q(1 + jw)\alpha = 1 \otimes (1 + jw)$ or $q \otimes q(1 + jw)\alpha = (1 + jw) \otimes 1$. This first case implies that q is complex, giving $\alpha = 1$ and $q = \pm 1$ with a stabilizer of $\mathbb{Z}_2$. The second case implies that there is a complex β so that $q(1 + jw)\alpha\beta = 1$ and $q\beta^{-1} = 1 + jw$, giving $q = (1 + jw)\beta$ and

$$1 = (1 + jw)\beta(1 + j)\alpha\beta = \alpha(\beta^2 - |w|^2) + jw\alpha(\beta^2 + 1)\,.$$

This is impossible. □

Using these stabilizers, we can determine the possible order parameter manifolds. In the first case when $x = 0$ the stabilizer was the entire group, which means the order parameter manifold is $M = G/G = \text{point}$. When $x \otimes y + y \otimes x$ is a complex multiple of $1 \otimes 1$ the order parameter manifold is $M = (Sp(1) \times S^1)/S^1$. These two order parameter manifolds don't correspond to physical state of the spin-1 Bose Einstein Condensate.

In the last case of $1 \otimes u + u \otimes 1$ with $u \in \mathbb{H} - (\mathfrak{sp}_1 \cup 0)$, the order parameter manifold is $M = (Sp(1) \times S^1)/\mathbb{Z}_2$. This corresponds to the order parameter manifold of $SO(3)$, which

is the ferromagnetic phase[21].

For the case of $x \in (0, \infty)$ and $y \in j\mathbb{C}$, the order parameter manifold is $M = (\mathrm{Sp}(1) \times S^1)/\mathrm{Pin}(2)$. This corresponds to the order parameter manifold of $(S^1 \times S^2)/\mathbb{Z}_2$, which is the anti-ferromagnetic phase, also known as the polar phase[21]. We are interested in this phase and determining its Abe homotopy groups.

2.2.3 Spin-2 Bose-Einstein Condensates

There is also a spin-2 phase that we are interested in determining its Abe homotopy groups. As there are many more phases for the spin-2 Bose Einstein Condensate, we will briefly outline the particular case we are interested in.

For a spin-2 BEC we have a 5-component complex wave function $\Psi = (\psi_2, \psi_1, \psi_0, \psi_{-1}, \psi_{-2})^T$ whose dynamics are described by a generalized Gross-Pitaevskii equitation [13],

$$i\hbar\partial_t\psi = \frac{\delta\mathcal{E}[\psi]}{\delta\psi^*}, \text{where } \mathcal{E}[\psi] = \int H(\psi)dx.$$

The Hamiltonian for the spin-2 case is $H(\Psi) = \frac{\hbar^2}{2m}|\nabla\Psi|^2 + V(x)|\Psi|^2 + \frac{c_1}{2}|\Psi|^4 + \frac{c_2}{2}|\Psi^\dagger\mathbf{F}\Psi|^2 + \frac{c_3}{2}|A_{00}^2|$ with the terms representing kinetic energy, potential energy, spin-independent interaction between bosons, spin-exchange interaction between bosons, and the amplitude of the spin-singlet pair. The trapping potential, $V(x)$, that satisfies $V(x) \to \infty$ as $|x| \to \infty$, m is the mass of the bosons, $\Psi^\dagger = (\psi_2^*, \psi_1^*, \psi_0^*, \psi_{-1}^*, \psi_{-2}^*)^T$ with ψ_k^* being the complex conjugate, and $\mathbf{F} = (F_x, F_y, F_z)$ are the spin-2 Pauli operators.

For the spin-2 system, we can make similar computations with the basis

$\{1 \otimes 1 \otimes 1 \otimes 1,$

$\frac{1}{2}j \otimes 1 \otimes 1 \otimes 1 + 1 \otimes j \otimes 1 \otimes 1 + 1 \otimes 1 \otimes j \otimes 1 + 1 \otimes 1 \otimes 1 \otimes j,$

$\frac{1}{\sqrt{6}}j \otimes j \otimes 1 \otimes 1 + 1 \otimes 1 \otimes j \otimes j + j \otimes 1 \otimes j \otimes 1 + 1 \otimes j \otimes 1 \otimes j + 1 \otimes j \otimes j \otimes 1 + j \otimes 1 \otimes 1 \otimes j,$

$\frac{1}{2}1 \otimes j \otimes j \otimes j + j \otimes 1 \otimes j \otimes j + j \otimes j \otimes 1 \otimes j + j \otimes j \otimes j \otimes 1,$

$j \otimes j \otimes j \otimes j\}.$

This gives the matrix $2iF_z = \begin{pmatrix} 4i & 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 0 & -4i \end{pmatrix}$ when doing multiplication by i, $-2iF_y = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$ for multiplication by j, and $-2iF_x = \begin{pmatrix} 0 & -2i & 0 & 0 & 0 \\ -2i & 0 & -\sqrt{6}i & 0 & 0 \\ 0 & -2i & 0 & -2i & 0 \\ 0 & 0 & -\sqrt{6}i & 0 & -2i \\ 0 & 0 & 0 & -2i & 0 \end{pmatrix}$ for multiplication by k.

Like with the spin-1 case, the spin-2 Hamilitonian has a symmetry group of $G = \mathrm{Sp}(1) \times S^1$.

The phase we are interested in for the spin-2 case is the uniaxial nematic phase. The stabilizer for this phase is Pin(2). We can determine this in a similar but longer computation compared to the spin-1 case. We want $(q, \alpha)(1 \otimes 1 \otimes j \otimes j + 1 \otimes j \otimes 1 \otimes j + 1 \otimes j \otimes j \otimes 1 + j \otimes 1 \otimes j \otimes 1 + j \otimes j \otimes 1 \otimes 1 + j \otimes 1 \otimes 1 \otimes j) = 1 \otimes 1 \otimes j \otimes j + 1 \otimes j \otimes 1 \otimes j + 1 \otimes j \otimes j \otimes 1 + j \otimes 1 \otimes j \otimes 1 + j \otimes j \otimes 1 \otimes 1 + j \otimes 1 \otimes 1 \otimes j$. Multiplying each term by a complex scalar like in the proof of Proposition 2 and analyzing the results gives a group isomorphic to Pin(2). Therefore we have the order parameter manifold for the spin-2 uniaxial nemtaic phase is $M = (\mathrm{Sp}(1) \times S^1)/\mathrm{Pin}(2)$, which is isomorphic to $S^1 \times \mathbb{R}P^2$[15].

2.3 Liquid Helium

One of the physical models we are interested in analyzing is for super-cooled liquid Helium. Below the critical temperature of 2.7 mK, ^3He will become a superfluid. There are three experimentally known phases of ^3He in this state[16], the A, A_1, and B phases. The mathematics for determining all possible phases is extremely difficult and no full analysis has been done. The theoretically known phases are planar, polar, bipolar, axial, α, β, and γ. The phase we are interested in is the A phase which has been found experimentally, but the

relevant Abe homotopy group was previously unknown.

The phases of ^3He can be found by looking at the collective action given by [14]

$$A_{coll}[A^*, A] = -\frac{i}{2}\text{Tr log}\begin{pmatrix} i\partial_t - \xi(-i\bigtriangledown) & i\tilde{\bigtriangledown}_i\sigma_a A_{ai} \\ i\tilde{\bigtriangledown}_i\sigma_a A^*_{ai} & i\partial_t + \xi(i\bigtriangledown_i) \end{pmatrix} - \frac{1}{3}\int d^4x A^*_{ai}(x)A_{ai}(x)$$

and calculating its local minima. The symmetry of this action field is determined by the phase, spin, and orbital rotations giving a symmetry group of $SU(2) \times SU(2) \times S^1$. In order to describe the possible phases of ^3He, we need to see what happens to the field A_{ai} after small oscillations. Below the critical temperature where ^3He is a superfluid, these oscillations will move A_{ai} to a new minimum and hence a new phase. This classification of finding new minimums after oscillations is the mathematically difficult part of the analysis.

The formula that characterizes the A phase is[14] $A_{ai} = \Delta_A d_a(\phi^{(1)} + i\phi^{(2)})$, where $d, \phi^{(1)}, \phi^{(2)}$ are arbitrary real unit vectors with $\phi^{(1)} \perp \phi^{(2)}$. Since there are no restrictions on d, in the order parameter manifold it is represents by a sphere S^2. Since $\phi^{(1)} \perp \phi^{(2)}$, we can take their cross product to get a third vector $\phi^{(3)}$ that will describe an orthonormal frame $(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) \in \text{SO}(3)$. Finally multiplying both d and $(\phi^{(1)} + i\phi^{(2)})$ by -1 will not affect the actions, so we will mod out by $\mathbb{Z}_2$ in the final space. This give the order parameter space for ^3He-A as $(S^2 \times \text{SO}(3))/\mathbb{Z}_2$.

Chapter 3

Homotopy Theory

3.1 Homotopy Groups

Topological excitations are textures and singularities that are stable under arbitrary continuous transformations of the order parameter manifold. The excitations are often classified by homotopy groups, but there are some instances where different types of excitations interact with each other, so more information is needed to have a complete classification of the excitations. This provides motivation for study of the so called Abe homotopy groups.

The study of homotopy groups is common in algebraic topology. There are many well established books covering homotopy groups in detail [12; 19]. For the computations we will do for Abe homotopy groups, the following lemmas for homotopy groups will be useful.

Lemma 2. *If X and Y are pointed topological spaces, then*

$$\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0).$$

For a proof of this result, see [12] page 343.

Lemma 3. *For a based space (X, x_0) with subspace A containing the base point, the sequence*

$$\ldots \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \ldots$$

is exact. Here i_* and j_* are induced inclusion maps, and ∂ is a boundary map defined by restricting the map $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to $(S^{n-1}, s_0) \to (A, x_0)$.

For a proof of this result, see [12] page 344.

3.2 Abe Homotopy

Abe defined a generalization of the standard homotopy groups based on cylinders [5]. These groups are based on maps of the form

$$f : ([0,1] \times S^{n-1}, 0 \times S^{n-1} \cup 1 \times S^{n-1}) \to (X, x_0).$$

Here the subspaces $0 \times S^{n-1}$ and $1 \times S^{n-1}$ which can be thought of as the top and bottom of an n-dimensional cylinder respectively, are mapped onto the base point x_0. The set of maps defined above is denoted $K_n(X, x_0)$, and the groups of homotopy equivalence classes of elements of $K_n(X, x_0)$ are the Abe homotopy groups, which are denoted $\kappa_n(X, x_0)$. The group structure is derived from concatenation of cylinders given by

$$f * g(t, v) = \begin{cases} f(2t, v), \text{ for } t \in [0, 1/2], \\ g(2t-1, v), \text{ for } t \in [1/2, 1]. \end{cases}$$

If one lets $v_0 \in S^{n-1}$ represent a basepoint, there is a natural projection

$$p : \kappa_n(X, x_0) \to \pi_1(X, x_0),$$

given by $p([f]) = [f \circ \iota_{v_0}]$ where $\iota_{v_0} : I \to [0,1] \times S^{n-1}$ is the standart inclusion. One also has a natural inclusion

$$j : \pi_1(X, x_0) \to \kappa_n(X, x_0),$$

given by $j([\gamma]) = [\gamma \circ q]$ where $q : [0,1] \times S^{n-1} \to [0,1]$ is the projection. The main result about the Abe homotopy groups [5] is

Theorem 1. *The kernel of $p : \kappa_n(X, x_0) \to \pi_1(X, x_0)$ is isomorphic to $\pi_n(X, x_0)$, and we have split exact sequence*

$$0 \to \pi_n(X, x_0) \to \kappa_n(X, x_0) \to \pi_1(X, x_0) \to 0.$$

It follows that we can view the Abe homotopy group as the following semi-direct product $\kappa_n(X, x_0) = \pi_1(X, x_0) \ltimes \pi_n(X, x_0)$. In this model the right action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is given by $R_\gamma(\alpha) = \gamma^{-1} * \alpha * \gamma$, where $*$ is the concetanition operation defined above. In a more typical model of $\pi_n(X, x_0)$ based on maps $(D^n, \partial D^n) \to (X, x_0)$ this aciton is given by

$$R_\gamma(\alpha) = \begin{cases} \alpha(2v), \text{ for } |v| \leq 1/2, \\ \gamma(2|v| - 1), \text{ for } |v| \geq 1/2. \end{cases}$$

Definition 2. *Let H and K be groups and let φ be a homomorphism from K into $Aut(H)$. Let $\cdot$ denote the right action of K on H determined by φ. Then the semi-direct product $K \ltimes H$ is the set of ordered pairs (k, h) with the following multiplication: $(k_1, h_1)(k_2, h_2) = (k_1 k_2, h_1 h_2 \cdot k_1)$.*

Another way to define a semi-direct product structure on a group G is by saying we have subgroups H, K of G that satisfy the following conditions:

(1) H is a normal subegroup of G

(2) $H \cap K = \{1\}$

(3) $HK = G$

The fact that we were concerned with the split short exact sequence in the proof of the Abe homotopy group being isomorphic to a semi-direct product comes from the following proposition.

Proposition 3. *Let $1 \to H \xrightarrow{f} G \xrightarrow{g} K \to 1$ be a short exact sequence. Suppose there is a splitting homomorphism $s : K \to G$ such that $gs = 1_K$. Then $G \cong K \ltimes H$.*

Proof. We need to show that G satisfies the conditions of a semi-direct product. Note that

$H \cong \text{im}(f)$ and $K \cong \text{im}(s)$. Since $H \cong \text{im}(f) = \ker(g)$ it is a normal subgroup. Suppose that $x \in H \cap K$. Then there exists $h \in H$ and $k \in K$ such that $x = f(h) = s(k)$. Then $k = gs(k) = gf(h) = 1$ since gf is the trivial homomorphism. Suppose $x \in G$. Then we can write $x = xzz^{-1}$ where $z = sg(x)$. Applying g to both sides of $z = sg(x)$ we have that $g(z) = gsg(x)$, but since gs is the identity function on K $g(z) = g(x)$. Applying g to xz^{-1} we have $g(xz^{-1}) = g(x)g(z^{-1}) = g(x)g(z)^{-1} = g(x)g(x)^{-1} = 1$ so $xz^{-1} \in \ker(g) \cong \text{im}(f) \cong H$. Sing $z \in \text{im}(s) \cong K$ we have $HK = G$. □

Given $\gamma \in \pi_1(X, x_0)$ and $\alpha \in \pi_n(X, x_0)$ the semi-direct product $\pi_1(X, x_0) \ltimes \pi_n(X, x_0)$ is defined by $(\gamma_1, \alpha_1) * (\gamma_2, \alpha_2) = (\gamma_1 * \gamma_2, R_{\gamma_2}(\alpha_1) * \alpha_2)$, where $R_\gamma(\alpha) = \gamma^{-1} * \alpha * \gamma$ is the product between elements of the first and n^{th} homotopy groups. Here $R_\gamma(\alpha)$ is an element of $\pi_n(M, \phi_0)$ since the map first does the inverse of the loop γ, then the sphere α before the loop γ, which is homotopic to a sphere.

The action $R_\gamma(\alpha)$ is important in physical models because it models how topological charge interacts with a vortex. The topological charge may change while the Abe conjugacey classes generated by $R_\gamma(\alpha)$ will be constant. This means is will be physically relevant to know if the action on $\kappa_n(X, x_0)$ is trivial or not.

Chapter 4

Physical Results

4.1 Introduction

In this chapter we review the results for the uniaxial nematic liquid crystal and analyze the remaining three groups in question from the Kobayashi et al. paper [15]. In section 2.1, we described how to obtain the order parameter manifold for a liquid crystal in the uniaxial nematic phase. In chapter 3 we obtained the second and third Abe Homotopy groups based on the order parameter manifold. Now we would like to describe the second and third Abe homotopy groups for the remaining three cases from that paper.

4.2 Uniaxial Nematic Liquid Crystal

We can calculate the Abe homotopy groups of a space if we know the homotopy groups of that space. When we discussed the order parameter manifold of an uniaxal nematic liquid crystal, we were able to calculate the order parameter space to be homotopy equivalent to $\mathbb{R}P^2$. In order to compute the relevant Abe homotopy group we need the first three homotopy groups of $\mathbb{R}P^2$.

For these calculations we will assume the homotopy groups $\pi_k(S^n)$ for $k \leq n$ and all the homotopy groups of S^1. The first non-obvious homotopy group to compute is $\pi_3(S^2)$. One

important fact is that fibrations induce long exact sequences of homotopy group. We can compute $\pi_3(S^2)$ by looking at the Hopf fibration $S^1 \to S^3 \to S^2$. The portion of the long exact sequence we need is

$$... \to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1) \to ...$$

which gives an isomorphism $\mathbb{Z} \cong \pi_3(S^3) \cong \pi_3(S^2)$, since $\pi_k(S^1)$ is trivial for $k > 1$. The fibration we will use to calculate the relevant homotopy groups of $\mathbb{R}P^2$ is $\mathbb{Z}_2 \to S^2 \to \mathbb{R}P^2$. The three relevant portions of the long exact sequence are

$$\begin{aligned}
\cdots \to &\pi_3(\mathbb{Z}_2) \to \pi_3(S^2) \to \pi_3(\mathbb{R}P^2) \to \pi_2(\mathbb{Z}_2) \to \cdots \\
\cdots \to &\pi_2(\mathbb{Z}_2) \to \pi_2(S^2) \to \pi_2(\mathbb{R}P^2) \to \pi_1(\mathbb{Z}_2) \to \cdots \\
\cdots \to &\pi_1(S^2) \to \pi_1(\mathbb{R}P^2) \to \pi_0(\mathbb{Z}_2) \to \pi_0(S^2) \to \cdots .
\end{aligned}$$

In each of the three parts the first and last homotopy groups are trivial which leads to the following three isomorphisms $\mathbb{Z} \cong \pi_3(S^2) \cong \pi_3(\mathbb{R}P^2)$, $\mathbb{Z} \cong \pi_2(S^2) \cong \pi_2(\mathbb{R}P^2)$, and $\pi_1(\mathbb{R}P^2) \cong \pi_0(\mathbb{Z}_2) \cong \mathbb{Z}_2$.

This means that the Abe homotopy groups of $\mathbb{R}P^2$ are $\kappa_2(\mathbb{R}P^2) \cong \mathbb{Z}_2 \ltimes \mathbb{Z}$ and $\kappa_3(\mathbb{R}P^2) \cong \mathbb{Z}_2 \ltimes \mathbb{Z}$. This doesn't give a full picture of the Abe homotopy groups, since it is possible to have some influence of vortices on either π_2 or π_3. We need to describe the action $\gamma(\alpha)$ to have a full picture. The main focus will be to calculate whether this action is trivial or nontrivial.

First is a theorem from [15] that works for a specific type of manifold. Namely we are dealing with ${}^{S^n}\!/_K$ where K is a discrete subgroups of $SO(n+1)$ which freely acts on S^n.

Theorem 2. *For any $\gamma \in \pi_1(S^n/K, x_0)$ and any $\alpha \in \pi_n(S^n/K, x_0)$ with $n \geq 2$ the action is given by* $R_\gamma(\alpha) = \begin{cases} \alpha^{-1} & \text{if } n \text{ is even and } {}^{O(n)}\!/_{SO(n)} \subset K \\ \alpha & \text{otherwise} \end{cases}$.

The non-trivial case is a narrow since K must contain the subgroup $\mathbb{Z}_2 \cong {}^{O(n)}\!/_{SO(n)}$,

which can be thought of as the matrices $\pm$I that are contained in $\mathrm{SO}(n+1)$ as $\pm\mathrm{I} \oplus 1$ and n must be even. Fortunately this does cover the liquid crystal system that we discussed in Section 2.1. In that case we have that $\mathbb{R}P^2 \cong {}^{S^2}\!/_{\mathbb{Z}_2}$, so the dimension of the sphere is even and $K = \mathbb{Z}_2$ meaning that in this liquid crystal system with $\alpha \in \pi_2(\mathbb{R}P^2)$ and $\gamma \in \pi_1(\mathbb{R}P^2)$, we have $R_\gamma(\alpha) = \alpha^{-1}$. This means that $\kappa_2(\mathbb{R}P^2)$ is the non-trivial semi-direct product $\mathbb{Z}_2 \ltimes \mathbb{Z}$. This theorem don't apply to the third Abe homotopy group for the liquid crystal system, $\kappa_3(\mathbb{R}P^2)$. In order to see what $\kappa_3(\mathbb{R}P^2)$ and its action will be, we need to analyze $R_\gamma(\alpha)$ for γ, the generator of $\pi_1(\mathbb{R}P^2)$ and $\alpha \in \pi_3(\mathbb{R}P^2)$.

When looking at the map $f : S^3 \to \mathbb{R}P^2$ representing the generator $\alpha \in \pi_3(\mathbb{R}P^2)$, we identify S^3 with the unit quaternions and $\mathbb{R}P^2$ with the purely imaginary unit quaternions with antipodes identified. To describe this map we will look at it as the composition $f = h \circ \sigma \circ g$. The first map is $g : [-1,1] \times S^2 \to S^3$ given by $g(t,v) = e^{\pi t v}$. This collapses the bottom of the cylinder $\{0\} \times S^2$ to $1 \in S^3$. The second map is the hopf map $\sigma : S^3 \to S^2$ given by $\sigma(q) = q^{-1} i q$. This is the Hopf fibration. Finally we have the covering projection $h : S^2 \to \mathbb{R}P^2$ given by $h(x) = [x]$. We can now define f as a composition $f = h \circ \sigma \circ g$. Notice that the family of maps $\sigma_s(q) = (e^{\pi k s/2} q)^{-1} i e^{\pi k s/2} q$ is a (free) homotopy between σ and $\sigma'(q) = -q^{-1} i q$. This implies that the generator is also represented by $f' = h \circ \sigma' \circ g$. One easily checks that $f(\{0,1\} \times S^2) = f'(\{0,1\} \times S^2) = [i]$ and $f(I \times \{i\}) = f'(I \times \{i\}) = [i]$, so both maps really do represent elements of $\pi_3(\mathbb{R}P^2)$.

In order to get useful information about the product $\gamma^{-1} * \alpha * \gamma$ in the Abe homotopy group, let $\gamma \in \pi_1(\mathbb{R}P^2)$ be a generator. To be specific, choose $\gamma(t) = h \circ \sigma(e^{\pi k(1-t)/2})$. (Here we will use the same letter to denote a representative of a homotopy class and the homotopy class.) Notice that $\sigma(e^{\pi k(1-t)/2})$ is a path form $-i$ to i in S^2. The function $\hat{\gamma}(t,v) = \gamma(t)$ represents an element of the Abe homotopy group that maps to γ. Of course, $\gamma^{-1}(t) = h \circ \sigma(e^{\pi k t/2}) = \hat{\gamma}^{-1}(t,v)$. The product of these three elements of the Abe homotopy group is just the concatenation of the cylinders. We now describe a homotopy rel $\partial I \times S^2$ between this representative of $\hat{\gamma}^{-1} * \alpha * \hat{\gamma}$ and a representative of α. This homotopy will be defined on each cylinder, and we will check that the portions agree on the ends of the cylinders.

Set $\hat{\gamma}_s(t,v) = h \circ \sigma(e^{\pi k s(1-t)/2})$, so $\hat{\gamma}_s^{-1}(t,v) = h \circ \sigma(e^{\pi k s t/2})$. Also set $f_s = h \circ \sigma_s \circ g$. The desired homotopy is just $\hat{\gamma}_s^{-1} * f_s * \hat{\gamma}_s$. Now $\hat{\gamma}_s^{-1}(1,v) = h \circ \sigma(e^{\pi k s/2})$, and $f_s(0,v) = h \circ \sigma_s(1) = h \circ \sigma(e^{\pi k s/2})$. Similarly, $\hat{\gamma}_s(0,v) = h \circ \sigma(e^{\pi k s/2})$, and $f_s(1,v) = h \circ \sigma_s(-1) = h \circ \sigma(e^{\pi k s/2})$, so the concatenation $\hat{\gamma}_s^{-1} * f_s * \hat{\gamma}_s$ is well-defined. Furthermore, $\hat{\gamma}_s^{-1}(0,v) = h(i)$, and $\hat{\gamma}_s(1,v) = h(i)$ so the concatenation is an equivalence between two elements of the Abe homotopy group. At $s = 1$ it is just $\hat{\gamma}^{-1} * \alpha * \hat{\gamma}$. At $\hat{\gamma}_0(t,v) = h(i)$, so $\hat{\gamma}_0^{-1} * f_0 * \hat{\gamma}_0$ is a representative of α. This shows that the action is trivial.

4.3 Order Parameter Manifolds

Kobayashi et. al. [15] identified several physically interesting order parameter manifolds and computed the relevant Abe homotopy groups in most cases. There were four interesting Abe homotopy groups that they did not compute. We now compute those groups.

In addition to the order parameter manifold for a uniaxial nematic liquid crystal that we discussed previously, the other cases are stated in the following definition.

Definition 3 (Order parameter manifolds)**.** *We have*
$X_{unLC} := \mathbb{R}^\times \times \mathbb{R}P^2$ *with the action of reflection. (This is the order parameter manifold for uniaxial nematic liquid crystal.)*

$X_{s=1,pBEC} := (S^1 \times S^2)/\mathbb{Z}_2$ *with the* $\mathbb{Z}_2$ *acting diagonally by reflection on* S^1 *and antipodally on* S^2*. (This is the order parameter manifold for spin-1, polar Bose Einstein condensates.)*

$X_{s=2,unBEC} := S^1 \times \mathbb{R}P^2$*. (This is the order parameter manifold for spin-2, uniaxial nematic Bose Einstein condensates)*

$X_{^3He} := (SO^3 \times S^2)/\mathbb{Z}_2$ *with the* $\mathbb{Z}_2$ *acting diagonally by 1/2 rotation on* $SO(3)$ *and antipodally on* S^2*. ((This is the order parameter manifold for dipole-free* ^{3}*Helium-A)*

For these four spaces we have the following propositions describing the second and thrid Abe homotopy groups.

Proposition 4 (Kobayashi et. al.)**.** *The second Abe homotopy groups with a non-trivial action by π_1 are:*

$\kappa_2(\mathbb{R}P^2) = \mathbb{Z} \rtimes \mathbb{Z}_2$

$\kappa_2(X_{s=1,pBEC}) = \mathbb{Z} \rtimes (\mathbb{Z} \times \mathbb{Z}_2)$

$\kappa_2(X_{s=2,unBEC}) = \mathbb{Z} \rtimes (\mathbb{Z} \times \mathbb{Z}_2)$

$\kappa_2(X_{^3He}) = \mathbb{Z} \rtimes \mathbb{Z}_4$

Proposition 5. *The action of π_1 on the third homotopy group of each of these four order parameter manifolds is trivial. Thus, the third Abe homotopy groups are:*

$\kappa_3(\mathbb{R}P^2) = \mathbb{Z} \times \mathbb{Z}_2$

$\kappa_3(X_{s=1,pBEC}) = \mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}_2)$

$\kappa_3(X_{s=2,unBEC}) = \mathbb{Z} \times (\mathbb{Z} \times \mathbb{Z}_2)$

$\kappa_3(X_{^3He}) = (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z}_4$

Proof of Proposition 5. We will reduce each case to the computation of the action of π_1 on $\pi_n(S^n/K)$ and the action of π_1 on $\pi_3(\mathbb{R}P^2)$. At the same time we will recover the proof of Proposition 4. This reduction will be apparent in each case after we identify the universal cover. The first action follows from the following theorem.

Theorem 3 (Kobayashi et. al.)**.** *Let $K < O(n+1)$ be discrete. Then $\pi_1(S^n/K) \cong K$ and $g \in K$ acts on $\alpha \in \pi_n(S^n/K)$ via $g \cdot \alpha = (\det g)\alpha$.*

This theorem follows from a direct computation via differential forms.

Remark. *Note that in this theorem Kobayashi et. al. are viewing the group structure as additive, while I have be viewing the group structure as multiplicative throughout this paper.*

The actions for the manifold $\mathbb{R}P^2$ were computed in section 4.2. It showed that the action is trivial on the third Abe homotopy group of $\mathbb{R}P^2$.

When looking at the spin-1 polar BEC, we have the order parameter manifold of $(U(1)\times S^2)/\mathbb{Z}_2$, which is isomorphic to $(S^1\times S^2)/\mathbb{Z}_2$. Note that the action of $\mathbb{Z}_2$ is a diagonal action that is reflection on S^1 and antipodal on S^2.

In order to analyze the homotopy groups of this space we will look at the universal cover $\mathbb{R}\times S^2$, and the group $\Gamma = \mathbb{Z}\rtimes\mathbb{Z}_2$. The product in Γ is $(n_1,\epsilon_1)(n_2,\epsilon_2) = (n_1+\epsilon_1 n_2, \epsilon_1\epsilon_2)$. We define an action $f:\Gamma\times\mathbb{R}\times S^2\to\mathbb{R}\times S^2$ by $f_{n,\epsilon}(t,x) = (\epsilon t+n, \epsilon x)$. If we were to compose two of these functions together, we would see the semidirect product structure of Γ in the first component.

$$f_{n_1,\epsilon_1}f_{n_2,\epsilon_2}(t,x) = (\epsilon_1\epsilon_2 t + n_1 + \epsilon_1 n_2, \epsilon_1\epsilon_2 x)$$

Lemma 4. *The function $\Psi:(\mathbb{R}\times S^2)/\Gamma\to(S^1\times S^2)/\mathbb{Z}_2$ given by $\Psi([t,x]_\Gamma) = [(e^{2\pi it},x)]_{\mathbb{Z}_2}$ is an isomorphism.*

Proof. The function is well defined. For $\Psi([\epsilon t+n,\epsilon x]_\Gamma) = [(e^{2\pi i(\epsilon t+n)},\epsilon x)]_{\mathbb{Z}_2} = [(e^{2\pi i\epsilon t},\epsilon x)]_{\mathbb{Z}_2}$ we have two cases. When $\epsilon=1$, $[(e^{2\pi it},x)]_{\mathbb{Z}_2} = \Psi([t,x]_\Gamma)$ and when $\epsilon=-1$, $[(e^{-2\pi it},-x)]_{\mathbb{Z}_2} = [(e^{2\pi it},x)]_{\mathbb{Z}_2} = \Psi([t,x]_\Gamma)$ so the function is well defined. The function is clearly surjective. For injectivity, we look at $[(e^{2\pi it_1},x_1)]_{\mathbb{Z}_2} = [(e^{2\pi it_2},x_2)]_{\mathbb{Z}_2}$ and consider when $x_1=\epsilon x_2$. When $\epsilon=1$, we have $x_1=x_2$ and $t_1=t_2+n$ for some $n\in\mathbb{Z}$ which means that (t_1,x_1) and (t_2,x_2) are in the same equivalence class of $(\mathbb{R}\times S^2)/\Gamma$. When $\epsilon=-1$, we have $x_1=-x_2$ and $t_1=-t_2+n$ for some $n\in\mathbb{Z}$. Again, these are in the same equivalence class of $(\mathbb{R}\times S^2)/\Gamma$ and the function is injective. □

It follows that $\widetilde{X}=\mathbb{R}\times S^2$ really is the universal cover of the space $X=(S^1\times S^2)/\mathbb{Z}_2$. We know that $\pi_2(X)=\pi_2(\widetilde{X})$. We would like to see how the generators of Γ act on $\widetilde{X}$. The two natural generators of Γ are $(1,1)$ and $(0,-1)$. The first component can be viewed as a deck transformation of $\mathbb{R}$ acting as a translation. The second component represents either the identity map or the antipodal map on S^2.

For the action of $\pi_1(X)\cong\mathbb{Z}_2$ on $\pi_2(\widetilde{X})$, we see that the generator of π_1 acts trivially

on $(1,1)$. The action of the generator of π_1 on the the antipodal map on S^2, i.e., $(0,-1)$ is non-trivial as given Theorem 3. This means that $\kappa_2(X_{s=1,pBEC})$ is as stated in Proposition 4.

We would like to do the same analysis for π_3. On $\pi_3(\widetilde{X})$, $(1,1)$ acts trivially on the homotopy groups as one can see by the homotopy corresponding to $(s,1)$. Note that the generator of $\pi_3(X)$ is just the map $(0,f)$ where f is the generator of $\pi_3(\mathbb{R}P^2)$ described above. The action of $(0,-1)$ on $(0,f)$ corresponds exactly to the action of -1 on f and we see that the action is trivial. This shows that $\kappa_3(X_{s=1,pBEC})$ is as it was specified in Proposition 5.

The next case to analyze is for the spin-2 uniaxial nematic BEC. This physical system has an order parameter manifold of $U(1)\times(S^2/\mathbb{Z}_2)$ which is isomorphic to $X=S^1\times\mathbb{R}P^2$ and has universal cover $\widetilde{X}=\mathbb{R}\times S^2$. Here the $\mathbb{Z}_2$ action is the antipodial map. Note that $\pi_1(U(1))$ is giving translations of $\mathbb{R}$ component of the universal cover $\widetilde{X}$, which won't change the results on those homotopy groups. We are left to analyze the action on a manifold isomorphic to $\mathbb{R}P^2$. Again by the theorem of Kobayashi et. al., the action of $\pi_1(X)$ on $\pi_2(X)$ will be nontrivial. For the action of $\pi_1(X)$ on $\pi_3(X)$, the homotopy obtained in the previous case by looking $\gamma(\alpha)$ will apply to this case and shows that the action will be trivial.

The final case to look at is for ${}^3He-A$ in a dipole-free phase. It has an order parameter manifold of $(S^2\times SO(3))/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ action is antipodal on the S^2 component and antipodal on the $SO(3)$ component. Note that the order parameter manifold is isomorphic to $X=(S^2\times\mathbb{R}P^3)/\mathbb{Z}_2$ which has a universal cover of $\widetilde{X}=S^2\times S^3$.

In order to analyze the action of the fundamental group on higher homotopy groups, we first have the following isomorphism of groups.

Lemma 5. *The map given by* $f:(S^3\times Sp(1))/\mathbb{Z}_4\to(S^2\times SO(3))/\mathbb{Z}_2$ *given by* $f([x,q]_{\mathbb{Z}_4})=[x,[q]_\pm]_{\mathbb{Z}_2}$ *is an isomorphism.*

Recall that the $\mathbb{Z}_4$ action on $S^2\times Sp(1)$ is $(x,q)[n]\mapsto(xe^{\pi in},qe^{(\pi in)/2})$.

Proof. The function is well defined since
$f([xe^{\pi in},qe^{(\pi in)/2}]_{\mathbb{Z}_4})=[xe^{\pi in},[qe^{(\pi in)/2}]_\pm]_{\mathbb{Z}_2}=[x,[q]_\pm]_{\mathbb{Z}_2}=f([x,q]_{\mathbb{Z}_4})$. Here the middle

equality can be seen in two cases. If n is even, then x does not change and $e^{(\pi i n)/2} = \pm 1$ so the second component is the same up to equivalence. If n is odd, then there is a $\mathbb{Z}_2$ difference in each component. The function is clearly surjective. For injectivity, consider $f([x_1, q_1]_{\mathbb{Z}_4}) = f([x_2, q_2]_{\mathbb{Z}_4})$. Then $[x_1, [q_1]_{\pm}]_{\mathbb{Z}_2} = [x_2, [q_2]_{\pm}]_{\mathbb{Z}_2}$, so there exists $n \in \mathbb{Z}_4$ such that $x_1 = x_2 e^{\pi i n}$ and $q_1 = \pm q_2 e^{(\pi i n)/2}$. Therefore $x_1 x_2^{-1} = e^{\pi i n}$ and $q_1 q_2^{-1} = \pm e^{(\pi i n)/2}$ making the function injective. □

Note that $\pi_3((S^2 \times Sp(1))/\mathbb{Z}_4) \cong \pi_3(S^2) \times \pi_3(Sp(1)) \cong \mathbb{Z} \times \mathbb{Z}$. We can view the two generators of this group as $\eta \times pt$ and $pt \times j$. Once again we are using the hopf map $\eta : S^3 \to S^2$ given by $\eta(q) = qiq^*$ and defining the map of the generator as $\eta \times pt : S^3 \to S^2 \times Sp(1)$ given by $q \mapsto (qiq^*, 1)$. Similarly, the other generating map is $pt \times j : S^3 \to S^2 \times Sp(1)$ is given by $q \mapsto (i, q)$.

For $x \in S^2$ the action by an element $n \in \mathbb{Z}_4$ is given by $x[n] = xe^{\pi i n}$ which is well defined.

Now we need to look at the action $q[n] = qe^{(\pi i n)/2}$ We claim that $q \mapsto qe^{(\pi i n)/2}$, which can be seen by writing $e^{(\pi i n)/2} = \cos((\pi i)/2) + i \sin((\pi i)/2)$ and viewing $\mathbb{H} \cong \mathbb{R}^4$ with basis $\{1, i, j, k\}$. Then the action $[n]$ can be view as the matrix insert here, which has determinant 1. If follows from [15] that the action is trivial. Since the action was trivial on S^3, it will also be the trivial action on $Sp(1)$.

4.4 General Results for Homogeneous Spaces

Now that we have answered the four cases, we will generalize the results to homogeneous spaces. A homogeneous space M is a topological space with a transitive group action given by a Lie group G. The stabilizer of an element $x \in G$ is $G_x = \{g \in G | gx = x\}$.

Lemma 6. *The function $F : G/G_x \to M$ given by $F([g]) = gx$ is an isomorphism.*

Proof. The function is well defined. Suppose that $[g_1] = [g_2]$. Then there is a $\hat{g} \in G_x$ such that $g_1 \hat{g} = g_2$. Then $F([g_2]) = g_2 x = g_1 \hat{g} x = g_1 x = F([g_1])$. The function is also surjective since the group action is transitive. For any $y \in M$ there exists a $g \in G$ such that $gx = y$.

Finally the map is injective. Suppose that $F([g_1]) = F([g_2])$, then $g_1 x = g_2 x$ which implies that $g_2^{-1} g_1 x = x$. Therefore $g_2^{-1} g_1 \in G_x$ and $[g_1] = [g_2]$. □

Remark. *The choice of x for the stabilizer was arbitrary. Assume that $\bar{g} \in G_x$ and $gx = y$. Then $g\bar{g}g^{-1}y = g\bar{g}x = gx = y$, so $g\bar{g}g^{-1} \in G_y$. Therefore the stabilizers for any element y are a group of conjungates of the elements in the stabilizer of x.*

Lemma 7. *For a group G, the connected component of the identity G_0 is a normal subgroup.*

Proof. To show that $G_0 < G$, choose $x \in G_0$, let $\gamma_x : [0,1] \to G_0$ be a path with $\gamma_x(0) = e$ and $\gamma_x(1) = x$. Let $a, b \in G_0$ and $\gamma_{ab} : [0,1] \to G_0$ be the path given by $\gamma_{ab}(t) = \gamma_a(t)\gamma_b(t)$. It follows that G_0 is closed since $\gamma_{ab}(0) = e$ and $\gamma_{ab}(1) = ab$. Similarly,
$\gamma_{a^{-1}} = (\gamma_a(t))^{-1}$ is also a path in G_0.

To show that $G_0 \lhd G$, let $x \in G_0$ with γ_x defined as above and $\gamma_g : [0,1] \to G$ be a path with $\gamma_g(0) = e$ and $\gamma_g(1) = g$ for $g \in G$. Then
$gxg^{-1} = \gamma_{gxg^{-1}}(1) = \gamma_g(1)\gamma_x(1)(\gamma_g(1))^{-1} = \gamma_x(1) \in G_0$ □

The following lemmas gives a series of isomorphisms for connected homogeneous spaces M. The goal is to analyze how the group $\pi_1(M)$ acts on higher homotopy groups. We have seen above that a homogeneous space can be represented by a quotient of a Lie group G by the stabilizer of an arbitrary element of the group. Note that if M is connected, the Lie group G need not be. This first isomorphism shows that G can be chosen so that it is connected.

Proposition 6. *If $M = G/H$ is a connected homogeneous space then there exists a group G' that is connected and $H' \subseteq G'$ such that $G/H \cong G'/H'$.*

Proof. We will show the map $R : G_0/(H \cap G_0) \to G/H$ given by $R([a]_{H \cap G_0}) = [a]_H$ is an isomorphism. First, this map is well defined. Suppose that $[b]_{H \cap G_0} = [b]_{H \cap G_0}$, then there exists $d \in H \cap G_0$ such that $cd = b$. Then $R([b]_{H \cap G_0}) = [b]_H = [cd]_H = [c]_H = R([c]_{H \cap G_0})$. For injectivity, suppose that $R([a]_{H \cap G_0}) = R([b]_{H \cap G_0})$. Then $[a]_H = [b]_H$ so there exists $h \in H$ such that $ah = b$. Note that $a^{-1}b = h \in G_0$ and thus $[a]_{H \cap G_0} = [b]_{H \cap G_0}$. For

surjectivity, consider maps $p : G \to G/H$ given by projection, $i : \{0\} \rightarrowtail [0,1]$ given by inclusion, $\gamma : [0,1] \to G/H$ with $\gamma(0) = [e]$ and $\gamma(1) = [a]_H$, and the constant map $\{0\} \to G$. The map p is a serre fibration so there exists a lift of the map γ, say δ. Clearly $\delta(1) \in G_0$ and it maps to a. □

Now that G can be picked to be connected, we want to show that it can be further simplified so that G may be taken to be simply-connected.

Proposition 7. *Let $M = G/H$ be a homogeneous space with G connected, then there exists a group G' such that $\pi_1(G') = 1$ and there is a subgroup $H' \subseteq G'$ so that $G/H \cong G'/H'$.*

Proof. We claim that $G' = \widetilde{G}$, the universal cover with covering map $p : \widetilde{G} \to G$ and $H' = p^{-1}(H)$. Consider the map $S : G'/H' \to G/H$ given by $S([a]_{H'}) = [p(a)]_H$ which we claim is an isomorphism. We see that the map S is well defined. Indeed, suppose that $[\gamma]_{H'} = [\delta]_{H'}$. Then there exists $\epsilon \in H'$ such that $\gamma\epsilon = \delta$, and thus
$S([\delta]_{H'}) = [\delta(1)]_H = [\gamma(1)\epsilon(1)]_H = [\gamma(1)]_H = S([\gamma]_{H'})$. For injectivity, suppose that $S([\delta]_{H'}) = S([\gamma]_{H'})$, then $[\delta(1)]_H = [\gamma(1)]_H$ so that there exists $\epsilon \in H'$ such that $\gamma(1)\epsilon = \delta(1)$. Let $\epsilon(t)$ be any path starting at the identity and ending at ϵ, then $\gamma(1)\epsilon(1) = \delta(1)$. Thus $[\delta]_{H'} = [\gamma\epsilon]_{H'} = [\gamma]_{H'}$. For surjectivity, let $[\gamma(1)]_H \in G/H$. Consider $m \in p^{-1}(\gamma(1))$ and let μ be the path from the identity to m. Then $S([\mu]_{H'}) = [\mu(1)]_H = [\gamma(1)]_H$. □

With G now simply connected, we can easily determine the first homotopy group of M.

Lemma 8. *If $M = G/H$ and G is simply connected, then $\pi_1(M) = H/H_0$. If we only assume that G is connected, we obtain $\pi_1(G/H) = p^{-1}(H)/p^{-1}(H)_0$.*

Proof. Consider the homotopy exact sequence induced by $H \to G \to G/H$.

$$... \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to ...$$

Since $\pi_1(G)$ and $\pi_0(G)$ are trivial, $\pi_1(G/H) \cong \pi_0(H) \cong H/H_0$. The more general case follows by applying Proposition 7 giving $G/H \cong \widetilde{G}/p^{-1}(H)$. □

Now that we can compute $\pi_1(M)$, the next few propositions describe the action when both G and H are connected.

Proposition 8. *If G is connected and $p : \widetilde{G} \to G$ is the universal cover, then $p^{-1}(e) \subset Z$ where Z is the center of $\widetilde{G}$.*

Proof. We use the description of the universal cover as equivalence classes of paths. Let $f, g \in \widetilde{G}$ with $f = [z_t]$ and $g = [x_s]$ such that $z_0 = \widetilde{e} = x_0$ and $e = p(f) = z_1$. We will show that $gfg^{-1} = f$. To do so, we just define an appropriate homotopy. Define $H_s(t) = x_{st} z_t x_{st}^{-1}$. when $s = 0$ $H_0(t) = x_0 z_t x_0^{-1} = z_t$ which is a representative of f, and when $s = 1$ $H_1(t) = x_t z_t x_t^{-1}$ which is a representative of gfg^{-1}. We now show that this is homotopy rel boundary. Indeed, when $t = 0$ $H_s(0) = x_0 z_0 x_0^{-1} = e$, when $t = 1$ $H_s(1) = x_s z_1 x_s^{-1} = e$. Thus, $f \in p^{-1}(e)$ is in the center. □

Proposition 9. *The group $p^{-1}(H)/p^{-1}(H)_0$ is isomorphic to $(Z \cap p^{-1}(H)) \,/\, (Z \cap p^{-1}(H)_0)$. In particular it is abelian.*

Proof. Define $\psi : (Z \cap p^{-1}(H)) \,/\, (Z \cap p^{-1}(H)_0) \to p^{-1}(H)/p^{-1}(H)_0$ by
$\psi([w]_{Z \cap p^{-1}(H)_0}) = [w]_{p^{-1}(H)_0}$.

The function is well defined. Suppose $[w]_{(Z \cap (p^{-1}(H)_0))} = [y]_{(Z \cap (p^{-1}(H)_0))}$ then there exists $x \in (Z \cap (p^{-1}(H)_0))$ such that $wx = y$. Then
$\psi([y]_{(Z \cap (p^{-1}(H)_0))}) = \psi([wx]_{(Z \cap (p^{-1}(H)_0))}) = [wx]_{p^{-1}(H)_0} = [w]_{p^{-1}(H)_0} = \psi([w]_{(Z \cap (p^{-1}(H)_0))})$. For surjectivity, let $y \in p^{-1}(H)$. Then $p(y) \in H$, and there exists $h : [0,1] \to H$ with $h_0 = p(y)$ and $h_1 = 1$ which is true since H is connected. There is a lift of this to $\widetilde{G}$ starting at $y = \widetilde{h}_1$. Let $\widetilde{h} : [0,1] \to \widetilde{G}$ be such that $p(\widetilde{h}_t) = h_t \subset H$ and $p(\widetilde{h}_1) = h_1$. Then $p(\widetilde{h}_1) \in Z$. For injectivity, suppose that $\psi([w]_{(Z \cap (p^{-1}(H)_0))}) = \psi([y]_{(Z \cap (p^{-1}(H)_0))})$, so there exists $v \in p^{-1}(H)_0$ such that $p(w^{-1}y) = v$. Then $w^{-1}y \in (Z \cap (p^{-1}(H)_0))$ and $[w]_{(Z \cap (p^{-1}(H)_0))} = [ww^{-1}y]_{(Z \cap (p^{-1}(H)_0))} = [y]_{(Z \cap (p^{-1}(H)_0))}$.

It follows that the group is abelian since the elements of the center commute with everything.

□

In trying to analyze the action of $\pi_1(M)$ on $\pi_n(M)$, one important case is when the action is trivial. We say that the space M is n-simple if $\pi_1(M)$ acts trivially on $\pi_n(M)$ and if M is n-simple for all n, then it is simple. The next proposition will show that $M = G/H$ is simple when both G and H are connected.

Recall that we have a left action so that for each $h \in \widetilde{G}$, the map $L_h : \widetilde{G}/p^{-1}(H)_0 \to \widetilde{G}/p^{-1}(H)_0$ is given by $L_h([g]) = [hg]$. Using this we get the following natural isomorphism:

$$\tau : \left(Z \cap p^{-1}(H)\right) / \left(Z \cap p^{-1}(H)_0\right) \to \mathrm{Deck}(\widetilde{G}/p^{-1}(H)_0),$$

where $\tau_{[z]} = L_z$. This is well-defined since z is in the center.

Proposition 10. *If G and H are connected then G/H is simple. Note that G need not be simply-connected.*

Proof. By Proposition 9 we have that $\pi_1(G/H)$ is abelian and isomorphic to $(Z \cap p^{-1}(H))/(Z \cap p^{-1}(H)_0)$. Consider $[z] \in (Z \cap p^{-1}(H)) / (Z \cap p^{-1}(H)_0)$ as a representative of the fundamental group. Let

$$\alpha : (D^n, \partial D^n) \to (\widetilde{G}/p^{-1}(H)_0, [1])$$

represent an element of $\pi_n(\widetilde{G}/p^{-1}(H)_0) \cong \pi_n(G/H)$. Given $z \in \widetilde{G}$ let γ^z be a path with $\gamma^z(0) = 1$ and $\gamma^z(1) = z$. The product of $[z]$ and $[\alpha]$ is represented by

$$R_z(\alpha) = \begin{cases} z^{-1}\alpha(2v), \text{ for } |v| \leq 1/2, \\ \gamma^z(1-2|v|)^{-1}, \text{ for } |v| \geq 1/2. \end{cases}$$

Now define a homotopy between α and $R_z(\alpha)$ by

$$H_s(v) = \begin{cases} \gamma^z(s)^{-1}\alpha(\frac{2v}{2-s}), \text{ for } |v| \in [0, 1-\frac{1}{2}s], \\ \gamma^z(\frac{s-2}{s}(|v|-1))^{-1}, \text{ for } |v| \in [1-\frac{1}{2}s, 1]. \end{cases}$$

□

Now that we have seen that the action will be trivial when H is connected, we would like to describe the behavior of the action in a broader setting. For this we will assume that $M = G/H$ is a connected homogeneous space and $\pi_1(G)$ is trivial. Recall that we have $\pi_1(M) \cong H/H_0 \cong (Z \cap H)/(Z \cap H_0)$. There is a large subgroup of the fundamental group that acts trivially on the higher homotopy groups. Thus, one can just consider the action of a certain quotient of the fundamental group on the higher homotopy groups.

Definition 4. *If $M = G/H$ is a connected homogeneous space represented as the quotient of a simply-connected Lie group, the* factorizing group *is*

$$F_M := H/((Z \cap H)H_0).$$

Theorem 4. *For any connected homogeneous space M represented as G/H with G simply-connected, the action of $\pi_1(M)$ on $\pi_n(M)$ factors through the factorizing group $F_M := H/((Z \cap H)H_0)$, where Z is the center of G.*

We start by finding an equivalent form for $M = G/H$ and F_M.

Lemma 9. *If N is a normal subgroup of G and a subgroup of $H < G$, then*

$$G/H \cong (G/N)/(H/N).$$

In particular,

1. $G/H \cong (G/H_0)/(H/H_0)$,
2. $G/((Z \cap H) \cong (G/H_0)/\left((Z \cap H)/H_0\right)$,
3. $H/((Z \cap H)H_0)) \cong (H/H_0)/((Z \cap H)H_0)/H_0)$, *and*
4. $G/((Z \cap H)H_0) \cong (G/(Z \cap H))/(((Z \cap H)H_0)/(Z \cap H))$.

Proof. Consider the function $\Phi : G/H \to (G/N)/(H/N)$ given by $\Phi([f]_H) = [[f]_H]_{H/N}$. The function is well defined. Suppose that $[g]_H = [f]_H$, then there exists $h \in H$ such that $gh = f$ so that $\Phi([f]_H) = [[f]_H]_{H/N} = [[gh]_H]_{H/N} = [[g]_H]_{H/N} = \Phi([g]_H)$. The function is clearly surjective. For injectivity, suppose that $\Phi([g_1]_H) = \Phi([g_2]_H)$. Then $[[g_1]_H]_{H/N} = [[g_2]_H]_{H/N}$ so there exists $h \in H$ such that $g_2 h = g_1$. It follows that $g_1 g_2^{-1} = h$ so the function is injective.

In particular, we see that $(Z \cap H)H_0$ is a subgroup of H. Let $z, w \in (Z \cap H)$ and $\gamma_1, \delta_1 \in H_0$. Here we are representing elements in the identity component H_0 as endpoints of paths $\gamma : I \to H_0$ with $\gamma_0 = e$ and γ_1 as the desired element. Then $z\gamma_1 w\delta_1 = zw\gamma_1\delta_1$ since $w \in Z$ and thus $((Z \cap H)H_0)$ is a subgroup of H. In Lemma 7 we saw that H_0 is a normal subgroup of H.Next $(Z \cap H)$ is normal in G. Let $z \in (Z \cap H)$ and $g \in G$. Then $g^{-1}zg = g^{-1}gz = z \in (Z \cap H)$ due to $z \in Z$.

□

Remark. *The space G/H is covered by $G/((Z \cap H)H_0)$ so for $n > 1$,*

$$\pi_n(G/H) \cong \pi_n(G/((Z \cap H)H_0)).$$

Since $(Z \cap H)H_0$ is connected, By Proposition 10 we see that $G/((Z \cap H)H_0)$ is simple and $\pi_1(G/((Z \cap H)H_0))$ acts trivially on $\pi_n(G/H)$.

It follows from this remark that the action of $\pi_1(M)$ on $\pi_n(M)$ will factor through $F_M = H/((Z \cap H)H_0) \cong (H/H_0)/(((Z \cap H)H_0)/H_0)$. This completes the proof of Theorem 4.

Chapter 5

Symmetric Spaces

While not all of the examples with liquid crystals and Bose Einstein Condensates had order parameter manifolds that were symmetric spaces, having a categorization of the Abe homotopy groups of symmetric spaces could be applied to other physical models.

Definition 5. *For $p \in M$ a Riemannian metric $g(\cdot,\cdot)_p$ on M is a covariant symmetric 2-tensor such that $g(v,v) > 0$ for all $v \in T_pM$.*

Definition 6. *A Riemannian manifold is a smooth manifold equipped with a Riemannian metric.*

Definition 7. *A symmetric space is a Riemannian manifold (M,g) such that for every point $p \in M$ there exists an isometry σ_p of (M,g) called an involution that satisfies $\sigma_p(p) = p$ and $d\sigma_p = -id_{T_pM}$.*

Remark. *A symmetric space can also be defined as a manifold M with differentiable multiplication $\mu : M \times M \to M$, written as $\mu(x,y) = x \cdot y$ that satisfies the following properties:*
(1) $x \cdot x = x$
(2) $x \cdot (x \cdot y) = y$
(3) $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$
(4) Every x has a neighborhood U such that $x \cdot y = y$ implies $x = y$ for all $y \in U$

There are many examples of classes of symmetric spaces, with the simplest being the direct product of two symmetric spaces with componentwise multiplication. Some other fundamental examples include:

Lie groups

For a Lie group G with $x, y \in G$, multiplication is given by $x \cdot y = xy^{-1}x$.

Spheres

Let (x, y) be a nonsingular symmetric bilinear form on $\mathbb{R}^n$. Define the sphere of radius r as the set $M_r = \{x \in \mathbb{R}^n | (x, x) = r\}$ and multiplication of two points as $x \cdot y = 2\frac{(x,y)}{(x,x)}x - y$, which is reflection of y through x.

Grassmannians

Let $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ and let $\mathbb{K}^n$ have the hermitian scalar product $(x, y) = \sum \bar{x}_i y_i$. Let $M = M(n, k)$ be the set of all linear subspaces of $\mathbb{K}^n$ and $M_q = M(q, n, k)$ be the set of subspaces of dimension q. Note that $M = M_0 \cup M_1 \cup M_2 \cup \ldots M_n$.

For $V \in M$ we have the decomposition $\mathbb{K}^n = V \oplus V^\perp$. Let S_V be the reflection in V which for a vector $x = x_1 + x_2$ in $\mathbb{K}^n$, $S_V(x) = x_1 - x_2$.

Definition 8. *Let $\gamma : [a, b] \to M$ be a curve segment and $Y : [a, b] \to TM$ be a vector field along γ. Y is called parallel along γ if $\dot{Y} + \nabla_{\dot{\Gamma}(Y)} = 0$.*

Definition 9. *The holonomy group at a point $p \in M$ is the group of all linear transformations of T_pM obtained from parallel transport along piecewise smooth curves starting and ending at p.*

Definition 10. *Riemannian manifold is called irreducible if the holomony group acts irreducibly on the tangent space.*

Remark. *Any simply connected Riemannian symmetric space is a product of irreducible ones.*

Definition 11. *A division algebra is a ring in which every nonzero element has an inverse, but multiplication is not necessarily commutative.*

Theorem (Hurwitz)**.** *The only division composition algebras over* $\mathbb{R}$ *up to isomorphism are* $\mathbb{R}$*,* $\mathbb{C}$*,* $\mathbb{H}$*, and* $\mathbb{O}$*.*

Using the language of division algebras, we can describe the seven infinite families of symmetric spaces [10].

Grassmannians $\{\mathbb{K}^p \subset \mathbb{K}^n\} = G_p(\mathbb{K}^n)$ for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

$\mathbb{R}$-structures on $\mathbb{C}^n$ given by $\{\mathbb{R}^n \subset \mathbb{C}^n\} = \mathrm{U}_n/\mathrm{SO}_n$.

$\mathbb{C}$-structures on $\mathbb{H}^n$ given by $\{\mathbb{C}^n \subset \mathbb{H}^n\} = \mathrm{Sp}_n/\mathrm{U}_n$.

$\mathbb{C}$-structures on $\mathbb{R}^{2n}$ given by $\{\mathbb{R}^{2n} \cong \mathbb{C}^n\} = \mathrm{SO}_{2n}/\mathrm{U}_n$.

$\mathbb{H}$-structures on $\mathbb{C}^{2n}$ given by $\{\mathbb{C}^{2n} \cong \mathbb{H}^n\} = \mathrm{U}_{2n}/\mathrm{Sp}_n$.

A symmetric space $M = G/H$ is the quotient of a connected Lie group G by a subgroup H which is the invariant group of an involution of G. Due to Cartan we have a complete classification of symmetric spaces. Here is a table of the seven irreducible infinite families and their labels.

Label	G	H
AI	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$
AII	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$
AIII	$\mathrm{SU}(p+q)$	$\mathrm{SU}(p) \times \mathrm{SU}(q)$
BDI	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p) \times \mathrm{SO}(q)$
DIII	$\mathrm{SO}(2n)$	$\mathrm{U}(n)$
CI	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$
CII	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p) \times \mathrm{Sp}(q)$

Table 5.1: *Irreducible Infinite Families of Symmetric Spaces*

In order to describe the homotopy groups of symmetric spaces we first need to look at the classical Lie groups.

5.1 Classical Groups

The four groups that will be used in the quotients of symmetric spaces are the unitary group $\mathrm{U}(n)$, special unitary group $\mathrm{SU}(n)$, special orthogonal group $\mathrm{SO}(n)$, and the symplectic

group $\mathrm{Sp}(n)$. The orthogonal group $\mathrm{O}(n)$ will also appear in some calculations. In order to say anything meaningful about the Abe homotopy groups of the symmetric spaces, we need to consider the fundamental groups as well as the stable range of the homotopy groups. The stable range for a homotopy group of a series of spaces X_n is a value N such that $\pi_k(X_n) \cong \pi_k(X_{n+1})$ for all $n \geq N$.

By looking at some basic fibrations it is possible to know where the stable ranges will occur without knowing the exact homotopy groups. First consider the fibration
$\mathrm{SO}(n) \to \mathrm{O}(n) \to \mathbb{Z}_2$. Looking at the long exact sequence associated to this fibration we have

$$\cdots \to \pi_{k+1}(\mathbb{Z}_2) \to \pi_k(\mathrm{SO}(n)) \to \pi_k(\mathrm{O}(n)) \to \pi_k(\mathbb{Z}_2) \to \cdots$$

. We know that $\mathbb{Z}_2$ has trivial homotopy groups for $k \geq 1$, so we have that
$\pi_k(\mathrm{SO}(n)) \cong \pi_k(\mathrm{O}(n))$ for $k \geq 1$. This fact allows us to find the stable ranges for $\mathrm{SO}(n)$ by using the fibration $\mathrm{SO}(n) \to \mathrm{SO}(n+1) \to S^n$ and the associated long exact sequence

$$\cdots \to \pi_{k+1}(S^n) \to \pi_k(\mathrm{SO}(n)) \to \pi_k(\mathrm{SO}(n+1)) \to \pi_k(S^n) \to \cdots$$

where since $\pi_k(S^n)$ is trivial if $n \geq k$, we have the isomorphism $\pi_k(\mathrm{SO}(n)) \cong \pi_k(\mathrm{SO}(n+1))$ when $n > k+1$. Therefore $\pi_k(\mathrm{O}(n)) \cong \pi_k(\mathrm{O}(n+1))$ when $n > k+1$.

For the unitary group we can look at the fibration $\mathrm{U}(n) \to \mathrm{U}(n+1) \to S^{2n+1}$. We can see that in the long exact sequence $\pi_k(S^{2n+1})$ is trivial if $2n+1 > k$, so $\pi_k(\mathrm{U}(n)) \cong \pi_k(\mathrm{U}(n+1))$ for $2n > k$.

For the special unitary group, we can use the fibration $\mathrm{SU}(n) \to \mathrm{U}(n) \to S^1$ to get the isomorphism $\pi_k(\mathrm{SU}(n)) \cong \pi_k(\mathrm{U}(n))$ for $k > 2$ regardless of the value of n. Therefore $\pi_k(\mathrm{SU}(n)) \cong \pi_k(\mathrm{SU}(n+1))$ for $2n > k \geq 2$.

For the symplectic group, consider the fibration $\mathrm{Sp}(n) \to \mathrm{Sp}(n+1) \to S^{4n+3}$ to see that $\pi_k(\mathrm{Sp}(n)) \cong \pi_k(\mathrm{Sp}(n+1))$ for $4n+2 > k$.

Using Morse theory Bott was able to compute all of the homotopy groups of these classical groups in the stable range [8]. The homotopy groups of $\mathrm{O}(n)$, $\mathrm{SO}(n)$, and $\mathrm{Sp}(n)$ are eight-

fold periodic and U(n), Su(n) are two-fold periodic we have the following table of homotopy groups in the stable range.

Group	π_{8p}	π_{8p+1}	π_{8p+2}	π_{8p+3}	π_{8p+4}	π_{8p+5}	π_{8p+6}	π_{8p+7}
O(n)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
SO(n)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
U(n)	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
SU(n)	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
Sp(n)	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Example

The action of π_1 is trivial if n is odd and $\mathbb{Z}_2$ is n is even for $SU(n)/SO(n)$.

Let $G = SU(n) = \{B \subset Mat_{n\times n}(\mathbb{C}) | B^*B = I\}$. Note that $Z(G) = \mathbb{Z}_n = \{\omega I | \omega^n = 1\}$. Let σ be an involution of $SU(n)$, $\sigma \in Aut(SU(n))$ such that $\sigma(A) = \bar{A}$. Define K to be the fixed set of σ, $K = Fix(\sigma) = \{A | \sigma(A) = A\} = SO(n)$.

The normalizer of K is $N(K) = \{B | BAB^* \in K \text{ for all } A \in K\}$. Notice that $\bar{B}\bar{A}\bar{B}^* = BAB^*$ and therefore $B^*\bar{B}A = AB^*\bar{B}$ since BAB^* is a real valued matrix. For ease of notation, set $C := B^*\bar{B} = B^{-1}\sigma(B)$.

Lemma 10. *For $B \in N(K)$, set $C = Ad_\sigma(B) = B^{-1}\sigma(B)$. Then $C \in Z(SU(n))$.*

Proof. First note that for a matrix $A \in SO(n)$, A will commute with C. The matrix A is defined to be an elementary rotation matrix $A = \begin{bmatrix} 1 & 0 & \dots & & & \dots & 0 \\ 0 & \ddots & & & & & \\ \vdots & & \cos\theta & & & -\sin\theta & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & \sin\theta & & & \cos\theta & \\ 0 & & & & & & 1 \end{bmatrix}$ where all the entries on the diagonal are one except for $(a_{ii}) = \cos\theta$ and $(a_{kk}) = \cos\theta$ with all other entries being zero except for $(a_{ik}) = -\sin\theta$, and $(a_{ki}) = \sin\theta$ with $i < k$. We want to show that the matrix C is a diagonal matrix whose entries are all the same. In order to do so, we

have that any eigenvector v of C has only two nonzero entries. For example $C\begin{bmatrix}1\\i\\0\\\vdots\\0\end{bmatrix} = \lambda_1\begin{bmatrix}1\\i\\0\\\vdots\\0\end{bmatrix}$, $C\begin{bmatrix}1\\0\\i\\\vdots\\0\end{bmatrix} = \lambda_2\begin{bmatrix}1\\0\\i\\\vdots\\0\end{bmatrix}$, and $C\begin{bmatrix}1\\-i\\0\\\vdots\\0\end{bmatrix} = \lambda_n\begin{bmatrix}1\\-i\\0\\\vdots\\0\end{bmatrix}$. Since λ_i is an arbitrary number, C is a diagonal matrix. Since A commutes with C, we have that $ACv = CAv = C\lambda v = \lambda Cv$. Consider the matrix $A_{\frac{\pi}{2}} = \begin{bmatrix}1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1\end{bmatrix}$. Then $\lambda_2\begin{bmatrix}1\\i\\0\\\vdots\\0\end{bmatrix} = \lambda_2 A_{\frac{\pi}{2}}\begin{bmatrix}1\\0\\i\\\vdots\\0\end{bmatrix} = A_{\frac{\pi}{2}}\lambda_2\begin{bmatrix}1\\0\\i\\\vdots\\0\end{bmatrix} = A_{\frac{\pi}{2}}C\begin{bmatrix}1\\0\\i\\\vdots\\0\end{bmatrix} = CA_{\frac{\pi}{2}}\begin{bmatrix}1\\0\\i\\\vdots\\0\end{bmatrix} = C\begin{bmatrix}1\\i\\0\\\vdots\\0\end{bmatrix} = \lambda_1\begin{bmatrix}1\\i\\0\\\vdots\\0\end{bmatrix}$ and therefore $\lambda_1 = \lambda_2$. Since we can choose different matrices A to switch two rows, all the eigenvalues are the same. This implies that $C = \lambda I$. For any $D \in SU(n)$ we have $\lambda I D(\lambda I)^{-1}D^{-1} = 1$ and therefore $C \in Z(SU(n))$. $\square$

Now define the adjoint map $Ad_\sigma : N(K) \to Z(G)$ given by $Ad_\sigma(B) = B^{-1}\sigma(B)$. This map is a group homomorphism,

$$\begin{aligned} Ad_\sigma(BC) &= (BC)^{-1}\sigma(BC) \\ &= C^{-1}B^{-1}\sigma(B)\sigma(C) \\ &= B^{-1}\sigma(B)C^{-1}\sigma(C) \\ &= Ad_\sigma(B)Ad_\sigma(C). \end{aligned}$$

For this map we have that $\mathrm{Ker}(Ad_\sigma) = K$ and by the first homomorphism theorem, $N(K)/K = \mathrm{im}(\mathrm{Ad}_\sigma)$. This will be a easier form to compute.

Recall that $\det(C) = 1$ since $C \in SU(n)$ and thus $\lambda^N = 1$, making λ a n^{th} root of unity.

Recall from the definition of $Z(G)$ that $\omega^N = 1$ and set $N = \omega^p = \omega^{2x} = \omega^{yN}$. Assume that $N \equiv 1(mod\ 2)$ and $Ad_\sigma(B) = NI$, then

$$\begin{aligned} Ad_\sigma(\omega^x B) &= \omega^{-1}B^{-1}\omega^{-1}\sigma(B) \\ &= \omega^{-2x}Ad_\sigma(B) \\ &= \omega^{-2x}NI \\ &= I \end{aligned}$$

so that $\omega^x B = A \in K$. We can write $B = \omega^{-x}A$. This implies that $N(K) \subseteq ZK$.

Let $NI \in Z$ and $A \in K$, then

$$\begin{aligned} Ad_\sigma(NA) &= N^{-1}A^{-1}N^{-1}\sigma(A) \\ &= N^{-2}A^{-1}\sigma(A) \\ &= N^{-2}I \in Z \end{aligned}$$

which implies that $ZK \subseteq N(K)$.

Note $Ad_\sigma(ZK) = Z^2 := \{\omega^{2x}I | \omega^{2x} \in Z\}$, $B^*B = 1$, $det(B) = 1$, and $Ad_\sigma(B) = e^{\frac{-i\pi}{k}}I$.

If $M \in Z$ then $M = e^{\frac{-i\pi}{k}}I$. For $n = 2l$, $M = Ad(e^{\frac{-i\pi l}{k}}I)$, for $n = 2l-1$, $M = Ad(e^{\frac{-i\pi l}{k}}IB)$. This implies that $Ad_\sigma : N(K) \to Z$ is a surjective function, therefore $N(K)/K \cong Z$.

$Ad_\sigma(ZK) = Z^2$ so $(ZK)/Z \cong Z^2$ and $N(K) = ZK \cup BZK$.
Thus $N(K)/(ZK) \cong (ZK \cup BZK)/(ZK) \cong \mathbb{Z}_2$ and $N(K)/(ZK))_0 = ZK$.

It follows that the factorizing group $F_M = K/((Z \cap K)K_0)$ will be trivial for $SU(n)/SO(n)$ when n is odd and $\mathbb{Z}_2$ when n is even.

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Appendix A

Moment Calculations

In this appendix we compute the second moment of a probability density $S^2 \to \mathbb{R}$ that factors through $\mathbb{R}P^2 \to \mathbb{R}$. As mentioned before in Chapter 2 no useful information can be taken from the first moment since the probability density $f : S^2 \to \mathbb{R}_+$ is an even function, but the second moment does help us to understand the order parameter manifold.
For $m = (x, y, z)$ the second moment is defined to be $M := \int_{S^2}(m \otimes m)f(m)d^2m$. By identifying $\mathbb{R}$ with $\mathbb{R}^*$ in the standard way, we can instead look at $M^* := \int_{S^2}(m \otimes m^*)f(m)d^2m$ which will give the equivalent information about the trace and eigenvalues.

We look at the case where the probability density function is the uniform distribution, which is clearly even and compute each entry in the matrix

$$M^* = \int_{S^2} \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix} d^2m.$$

Due to symmetry it is only necessary to compute 6 integrals. We use spherical coordinates $x = r\sin\varphi\cos\theta$, $y = r\sin\varphi\sin\theta$, and $z = r\cos\varphi$ on S^2, and set $r = 1$. The normalizing constant for the calculations will be $\frac{1}{4\pi}$, since $\int_{S^2} 1 \; d^2m = 1$.

Starting with the off-diagonal terms, we have the following computations.

$$\frac{1}{4\pi}\int_{S^2} xyd^2m = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \sin^3\varphi \sin\theta\cos\theta d\theta d\varphi = 0$$

$$\frac{1}{4\pi}\int_{S^2} xzd^2m = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin^2\varphi\cos\varphi\cos\theta d\varphi d\theta = 0$$

$$\frac{1}{4\pi}\int_{S^2} yzd^2m = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin^2\varphi\cos\varphi\sin\theta d\varphi d\theta = 0$$

This means that M^* is a diagonal matrix. Looking at the diagonal terms we have

$$\frac{1}{4\pi}\int_{S^2} x^2d^2m = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin^3\varphi\cos^2\theta d\varphi d\theta = \frac{1}{3}$$

$$\frac{1}{4\pi}\int_{S^2} y^2d^2m = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin^3\varphi\sin^2\theta d\varphi d\theta = \frac{1}{3}$$

$$\frac{1}{4\pi}\int_{S^2} z^2d^2m = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\varphi\cos^2\theta d\varphi d\theta = \frac{1}{3}$$

This gives us $M^* = diag(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ which we would expect using the uniform probability density. Furthermore it has trace 1, which means M will have trace 1 as well.

For a slightly more complicated tensor, consider the probability density $f(x, y, z) = ax^2 + by^2$ for $a, b \in \mathbb{R}$. Now for this probability density, the normalizing constant will be $\dfrac{3}{4\pi(a+b)}$, since $\dfrac{3}{4\pi(a+b)}\int_{S^2} f(x,y,z)\ d^2m = 1$.
Looking at similar computation to the uniform probability density we have

$$\frac{3}{4\pi(a+b)}\int_{S^2} xyf(x,y,z)d^2m = 0$$

$$\frac{3}{4\pi(a+b)}\int_{S^2} yzf(x,y,z)d^2m = 0$$

$$\frac{3}{4\pi(a+b)}\int_{S^2} xzf(x,y,z)d^2m = 0$$

so again we will end up with a diagonal matrix. The other three terms are

$$\frac{3}{4\pi(a+b)}\int_{S^2} x^2 f(x,y,z)d^2m = \frac{3}{4\pi(a+b)}\int_0^{2\pi}\int_0^{\pi} a\sin^5\varphi\cos^4\theta + b\sin^5\varphi\sin^2\theta\cos^2\theta d\varphi d\theta = \frac{3a+b}{5(a+b)}$$

$$\frac{3}{4\pi(a+b)}\int_{S^2} y^2 f(x,y,z)d^2m = \frac{3}{4\pi(a+b)}\int_0^{2\pi}\int_0^{\pi} a\sin^5\varphi\sin^2\theta\cos^2\theta + b\sin^5\varphi\sin^4\theta d\varphi d\theta = \frac{a+3b}{5(a+b)}$$

$$\frac{3}{4\pi(a+b)}\int_{S^2} z^2 f(x,y,z)d^2m = \frac{3}{4\pi(a+b)}\int_0^{2\pi}\int_0^{\pi} a\sin^3\varphi\cos^2\varphi\cos^2\theta + b\sin^3\varphi\cos^2\varphi\sin^2\theta d\varphi d\theta = \frac{1}{5}$$

and so we end up with the matrix $M^* = diag(\frac{3a+b}{5(a+b)}, \frac{a+3b}{5(a+b)}, \frac{1}{5})$ which will have trace 1 as long as one of a or b is a nonzero number. Recall that the order tensor is defined to be $Q := M - \frac{1}{3}I$, which will be a traceless tensor in either of the above cases. The other condition that was specific to liquid crystals in the uniaxial nematic phase was the tensor had two distinct eigenvalues. The only example that would fit this criterion would be if $a = b$. In other words the symmetry of the molecules affects the possible probability densities.

Another example is the function $\hat{f}(x,y,z) = z^{2n} = \cos^{2n}(\varphi)$. Here we have the integral $\int_{S^2}\hat{f}d^2m = \int_0^{2\pi}\int_0^{\pi}\cos^{2n}(\varphi)\sin(\varphi)d\varphi d\theta = \frac{4\pi}{2n+1}$, so our normalizing constant is $\frac{2n+1}{4\pi}$. We define $\bar{f}(\varphi,\theta) = \frac{2n+1}{4\pi}\cos^{2n}(\varphi)$ as our normalized probability density.

Looking at the off diagonal terms we have

$$\int_{S^2} xy\bar{f}d^2m = \int_0^{2\pi}\int_0^{\pi}\sin\theta\cos\theta\sin^2\varphi\cos^{2n}\varphi d\varphi d\theta = 0$$

$$\int_{S^2} yz\bar{f}d^2m = \int_0^{2\pi}\int_0^{\pi}\cos\theta\cos^{2n+1}\varphi\sin\varphi d\varphi d\theta = 0$$

$$\int_{S^2} xz\bar{f}d^2m = \int_0^{2\pi}\int_0^{\pi}\sin\theta\sin\varphi\cos^{2n+1}\varphi d\varphi d\theta = 0$$

which leads to a diagonal matrix.

Looking at the diagonal components, we have

$$\frac{2n+1}{4\pi}\int_{S^2} z^2\bar{f}d^2m = \frac{2m+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\varphi\cos^{2n+2}\varphi d\varphi d\theta = \frac{2m+1}{4\pi}\int_0^{2\pi}\frac{2}{2n+3}d\theta = \frac{2m+1}{2m+3}$$

$$\frac{2m+1}{4\pi}\int_{S^2} x^2\bar{f}d^2m = \frac{2m+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin^3\varphi\cos^{2n}\varphi\cos^2\theta d\varphi d\theta = \frac{2m+1}{4}\int_0^{\pi}\sin^3\varphi\cos^{2n}\varphi d\varphi$$

$$\frac{2m+1}{4\pi}\int_{S^2} y^2\bar{f}d^2m = \frac{2m+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin^3\varphi\cos^{2n}\varphi\sin^2\theta d\varphi d\theta = \frac{2m+1}{4}\int_0^{\pi}\sin^3\varphi\cos^{2n}\varphi d\varphi$$

There is no need to calculate the last two integrals, since $\int_{S^2} x^2\bar{f}d^2m = \int_{S^2} y^2\bar{f}d^2m$ and the trace is one, we must have that $\frac{2m+1}{4\pi}\int_{S^2} x^2\bar{f}d^2m = \frac{1}{2n+3}$

Appendix B

Unitary Groups

The calculations for the stable homotopy groups of the unitary groups was first done by Bott [8] and we will recreate some of those results here. Since S^1 is isomorphic to $U(1)$ we will assume those homotopy groups.

Theorem 5. *Let $k < 2n$, then if $k = 2l$, $\pi_{2l}(U(n)) = 0$ and if $k = 2l+1$, $\pi_{2l+1}(U(n)) = \mathbb{Z}$.*

Proof. We will start with some base cases for small values of k.
When $k = 1$, $\pi_1(U(1)) = \mathbb{Z}$. Using the fibration, $\pi_1(U(n)) \cong \pi_1(U(n+1)) \cong \mathbb{Z}$ for $n \geq 1$.
When $k = 2$, first consider the portion of the long exact sequence
$0 = \pi_2(U(1)) \to \pi_2(U(2)) \to \pi_2(S^3) = 0$ to see that $\pi_2(U(2)) = 0$. Using this result in the piece of the long exact sequence $\pi_3(S^{2n+1}) \to \pi_2(U(n)) \to \pi_2(U(n+1)) \to \pi_2(S^{2n+1})$ results in the homotopy groups $\pi_2(U(n)) = 0$ for all n.

When $k = 3$, $0 = \pi_3(U(1)) \to \pi_3(U(2)) \to \pi_3(S^3) \to \pi_2(U(1)) = 0$ so $\pi_3(U(2)) = \mathbb{Z}$. This means that $\pi_3(U(n)) = \mathbb{Z}$ in the stable range of $n \geq 2$.

The base cases for induction can be taken as all of the previous computations of homotopy groups from the fibrations of $U(n) \to U(n+1) \to S^{2n+1}$. We proceed by induction on l. Assume that the statement is true for all values less than $l + 1$. We have the following long

exact sequence,

$$\pi_{2l}(U(2n)) \to \pi_{2l}(\frac{U(2n)}{U(n)\times U(n)}) \to \pi_{2l-1}(U(n)\times U(n)) \to \pi_{2l-1}(U(2n)) \to \pi_{2l-1}(\frac{U(2n)}{U(n)\times U(n)}) \to \pi_{2l-2}(U(n)\times U(n))$$

Looking at the first four terms under our assumptions, we have

$$0 \to \pi_{2l}(\frac{U(2n)}{U(n)\times U(n)}) \to \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}.$$

For the group $U(2n)$, we have the diagonal matrix $I_n \oplus -I_n$ as the base point for the homotopy class. When trying to find the stabilizer of this matrix by the action of conjugation, we need to find g such that $g(I_n \oplus -I_n)g^{-1} = I_n \oplus -I_n$. The only type of matrix will do this is a block matrix of unitary groups. Therefore $\text{stab}(U(2n)) = U(n)\times U(n)$. Note that this is the stabilizer of the entire path, not just the point. This means the map ${}^{U(2n)}\!/_{U(n)\times U(n)} \to U(2n)^\nu$ is given by $[g] \mapsto g\alpha g^{-1}$. Here $U(2n)^\nu$ is the set of all geodesics of minimal length that start and end at the base point $I_n \oplus -I_n$ and are contained in the trivial homotopy class. This means that the map $\pi_{2l}(\frac{U(2n)}{U(n)\times U(n)}) \to \pi_{2l-1}(U(n)\times U(n))$ has kernel zero and image $\mathbb{Z}$, so $\pi_{2l}(\frac{U(2n)}{U(n)\times U(n)}) = \mathbb{Z}$. Using Bott's isomorphism for homotopy groups of symmetric spaces and the set of all minimal length geodesics between two points in a particular homotopy class $\pi_k(M^\nu) = \pi_{k+1}(M)$, we have that $\mathbb{Z} = \pi_{2l}(\frac{U(2n)}{U(n)\times U(n)}) \cong \pi_{2l+1}(U(2n))$.
Similarly, looking at the last 4 terms in the long exact sequence, we have

$$\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z} \to \pi_{2l-1}(\frac{U(2n)}{U(n)\times U(n)}) \to 0.$$

This time $im(\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}) = \mathbb{Z}$ for the first map, so that $0 \cong \pi_{2l-1}(\frac{U(2n)}{U(n)\times U(n)}) \cong \pi_{2l}(U(2n))$.

□