Properties of p-modulus on radially symmetric infinite trees

by

Prem Raj Prasain

M.A., Tribhuvan University, 2009

M.S., Kansas State University, 2017

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
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Abstract

On a finite graph, the p-modulus of a family of paths takes the form of a standard convex optimization problem. The optimal value of this problem provides information about the underlying graph, and has been shown to be closely related to several important graph theoretic quantities including shortest path, effective resistance, and minimum cut. Some of these concepts have meaningful and interesting extensions to infinite graphs. For example, certain effective resistance calculations on infinite trees are known to be related to the transience or recurrence of random walks on these trees. The goal of this dissertation is to extend the theory of p-modulus to the case of infinite trees.

While many known results from finite graphs have analogs in the infinite tree setting, the infinite nature of the problem setting gives rise to a number of interesting and subtle properties. For example, while modulus problems on a finite graph involve optimization in a finite dimensional space with a finite number of constraints, the corresponding problem on an infinite tree requires optimization over a countably infinite number of decision variables with an uncountably infinite number of constraints. This makes the analysis both more interesting and more complex.

This dissertation first establishes the theory of p-modulus (specifically, the theory of p-modulus of a family of infinite paths) on an infinite rooted tree, then explores its interpretation and properties. One key result is the formulation of p-modulus in the infinite tree as a limit of p-modulus on truncated trees, with a formula given in terms of a series. The convergence of the series implies a positive modulus, while divergence implies that the modulus is zero. This distinction between trees with positive modulus and trees with zero modulus plays an important role in this dissertation. The value of modulus is related to the "bushiness" of the tree at infinity; trees which branch many times have larger modulus, while trees that do not branch much have smaller modulus. In fact, if a tree does not branch sufficiently, its modulus may be zero, a fact that is related to the recurrence of a random walk on the tree when the parameter p in the p-modulus equals 2.

This dissertation extends previous results on modulus for finite graphs by showing that 1-modulus is related to the minimum cut problem, 2-modulus is related to effective resistance, and ∞-modulus is related to shortest path.

Another key results of this dissertation is the existence of a critical p-value for certain trees. The value of p-modulus when p is above this critical value is zero, while the value of p-modulus when p is below the critical value is positive. This provides a sense of "dimension" for the tree. As one particular application, this dissertation also connects the value of this critical exponent to the transience or recurrence of a random walk.

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Major Professor
Dr. Nathan Albin

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Prem Raj Prasain

August 11, 2020

Preface

This dissertation is submitted for the Degree of Philosophy at Kansas State University. The research work described herein was conducted under the supervision of Professor Nathan Albin in the Department of Mathematics, Kansas State University between August 2014 to July 2020.

The purpose of this dissertation is to extend the concept of p-modulus in a finite graph to the infinite graph. This research is influenced by the research works of the modulus in a finite graph done by Supervisor Professor Nathan Albin along with his colleagues. The modulus properties in the infinite graphs are being studied. An especial focus is given on the p-modulus on a radially symmetric infinite tree. Another aspect of this dissertation is the analysis of the p-modulus problem in the terms of different parameters and established its properties, interpretation, and graphic measures in an infinite tree analog to the finite graph. Finally, this dissertation could be an interesting topic for the math scholars who like graph, the p-modulus, and random walk.

Prem Raj Prasain

August 11, 2020

Chapter 1

Introduction

In this chapter, we review several standard definitions from graph theory and review some of the theory of effective resistance and random walks on infinite trees.

1.1 Terminology

1.1.1 Graphs and trees

A *graph* is a tuple $G = (V, E)$ where V is the set of *vertices* or *nodes* of the graph and E is a set of unordered pairs of vertices, called *edges*. We denote the cardinalities of the edge set and vertex set by $|E|$ and $|V|$ respectively. A graph G is *finite* if both $|E|$ and $|V|$ are finite, otherwise G is *infinite*. A *finite walk* in a graph G is a sequence $\gamma = v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n$, where $v_0, v_1, v_2, \ldots, v_n \in V$ and $e_i = \{v_{i-1}, v_i\} \in E$ for $i = 1, 2, \ldots, n$. Such a walk is said to have *hop length* $\ell(\gamma) = n$. An *infinite walk* is similar, except the sequence does not terminate: $\gamma = v_0 e_1 v_1 e_2 v_2 \cdots$. In this case, we say that γ has length $\ell(\gamma) = +\infty$. A *finite (resp. infinite) path* is a finite (resp. infinite) walk in which $v_i \neq v_j$ for all $i \neq j$. A graph is called *simple* if it contains at most one edge between any two vertices and contains no self-loops (i.e., edges of the form $\{v, v\}$). For a simple graph, a walk can be uniquely determined from either its vertex sequence $v_0 v_1 v_2 \cdots$ or its edge sequence $e_1 e_2 e_3 \cdots$. A *cycle* is a finite walk $v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_0$ with the property that $v_0 e_1 v_1 e_2 \cdots v_{n-1}$ is a path.

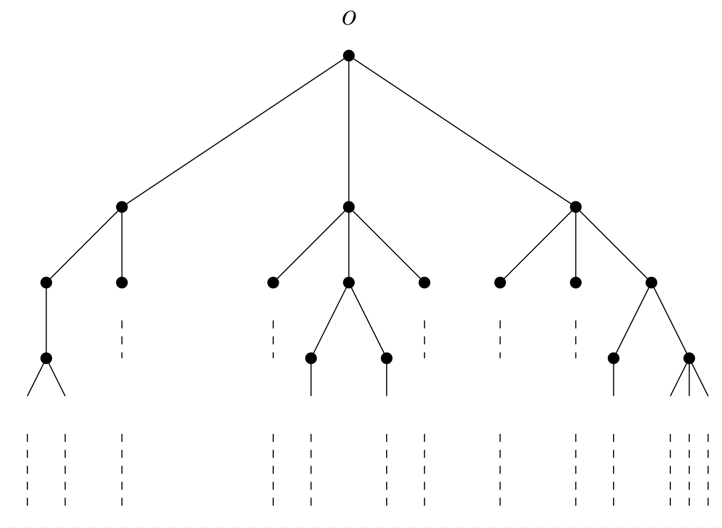


Figure 1.1: An infinite rooted tree $G = (G, E, o)$

Given two distinct vertices u and v in a graph, we denote by $\Gamma(u, v)$ the set of paths in the graph connecting u to v. A graph G is *connected* if it has at least one vertex and there is a path between every pair of vertices; that is, if $\Gamma(u, v) \neq \emptyset$ for all distinct pairs of vertices u and v. A *weighted graph* is a tuple $G = (V, E, \sigma)$ where the function $\sigma : E \to \mathbb{R}_{>0}$ which assigns a positive weight to each edge of G. The case $\sigma \equiv 1$ can be thought of as an unweighted graph and denoted by $G = (V, E)$. So, without loss of generality, we shall consider all graphs to be weighted.

A graph G is called a *tree* if it is connected and contains no cycles. Given any two distinct vertices u and v in a tree, the set $\Gamma(u, v)$ contains a single path, which we call γ_{uv}. A *rooted tree* is a tuple $G = (G, E, \sigma, o)$ where $G = (V, E, \sigma)$ is a tree and $o \in V$ is a specially identified vertex called the *root*.

1.1.2 Generations and descent

Every vertex, v, in a rooted tree has a *generation*, $\text{gen}(v)$ defined as

$$\text{gen}(v) = \begin{cases} 0 & \text{if } v = o, \\ \ell(\gamma_{ov}) & \text{if } v \neq o. \end{cases}$$

In words, $\text{gen}(v)$ is the number of "hops" required to move from the root o to the vertex v in the tree. The concept of generation can be extended to edges through the definition

$$\text{gen}(\{u, v\}) = \max\{\text{gen}(u), \text{gen}(v)\}.$$

In other words, the edges of generation n are those that connect vertices of generation $n-1$ to those of generation n.

Given two distinct vertices $u, v \in V$, we say that u is an *ancestor* of v (equivalently that v is a *descendent* of u if the path γ_{ov} passes through u. If additionally, $\{u, v\} \in E$, then u is called the *parent* of v and v is called a *child* of u. Each vertex other than the root has a unique parent, but vertices can have many children, or none at all. A vertex that has no children is called a *leaf*.

Similar definitions can be made for relationships among edges. Given two distinct edges $e, e' \in E$, we say that e is an *ancestor* of e' (equivalently that e' is a *descendent* of e) if every path from the root that contains e' also contains e. If additionally e and e' are adjacent, then e is called the *parent* of e' and e' is called a *child* of e.

Note that the edges in generation 1 do not have parents. Each other edge $e \in E$ has a unique parent edge denoted by $p(e)$. Each edges may have one or more child edges, denoted $c(e) \subset E$. The number of children of edge e is indicated by $C(e) = |c(e)|$. A tree is called locally finite if $C(e) < \infty$ for all $e \in E$. In this dissertation, we shall make the following definition. A tree is called a *proper infinite tree* if it is infinite, locally finite and $C(e) \geq 1$ for all $e \in E$. In words, a proper infinite tree is an infinite tree with no leaves and such that each vertex has only a finite number of children.

Special attention is paid in this dissertation to *radially symmetric trees*. These are trees for which $C(e)$ depends only on the generation of e. That is, $C(e) = C(\text{gen}(e))$.

By the *ball* B_n *of radius* n, we mean the set of edges which are within n hops from the root. That is,

$$B_n := \{e \in E : \text{gen}(e) \leq n\}$$

By the *shell of radius* n, we mean the set

$$S_n := B_n \setminus B_{n-1} = \{e \in E : \text{gen}(e) = n\}$$

Let $G = (V, E, o)$ be a proper tree and let $G' = (V', E', o)$ be a *subtree* with same root o. We say that G' is a truncation of tree G if

$$\forall e \in E \quad \text{either} \quad c(e) \cap E' = \emptyset \quad \text{or} \quad c(e) \cap E' = c(e)$$

In words, for each $e \in E$, G' either contains all or none of e's children in G.

1.1.3 Descending paths

In a rooted tree, we shall define a *descending path* to be a (finite or infinite) path $\gamma = e_1 e_2 e_3 \cdots$ starting at the root, o. For such a path, $e_i = p(e_{i+1})$ for $i = 1, 2, 3, \ldots$. The *length* of a descending path $\gamma \in \Gamma$ can be defined as

$$\ell(\gamma) := \sup_{e \in \gamma} \text{gen}(e),$$

which may either be some finite value, or $+\infty$. If γ is a finite, then $\ell(\gamma)$ coincides with the graph length of γ. In this dissertation, unless otherwise indicated, G shall be a proper infinite tree and we shall denote by Γ the family of descending paths on G. We shall also consider certain special subfamilies of Γ defined as

$$\Gamma_n := \{\gamma \in \Gamma : \ell(\gamma) = n\} \text{ when } n \geq 1 \quad \text{and} \quad \Gamma_\infty := \{\gamma \in \Gamma : \ell(\gamma) = +\infty\}$$

An important relationship among the Γ_n is the *extension property*: if $1 \leq m \leq n \leq \infty$ and if $\gamma \in \Gamma_n$ then there exists a unique $\gamma' \in \Gamma_m$ such that γ' is a sub-path of γ. We shall use the notation $\gamma' \preceq \gamma$ to indicate this sub-path relationship.

1.2 Effective resistance

In this section, we provide a heuristic way of thinking about effective resistance on an infinite graph. For a more mathematically precise treatment, see [8, 16].

On a finite tree, we can picture effective resistance by imagining the edges of the tree as electrical resistors with conductance (measured in mhos) given by the edge weights σ. The vertices represent the junctions at which these resistors are soldered together. Now we imagine connecting one terminal of a one-volt battery to the root node and the other terminal to all of the leaves. This will induce a certain amperage A to flow through the resistor network. The reciprocal, A^{-1}, of this amperage is called the effective resistance of the network.

On a finite tree, effective resistance can be found by working backward from the leaf nodes using the standard parallel and series rules for resistors. As an example, consider a perfect m-ary tree with depth n—that is, consider a finite tree such that all non-leaf nodes have exactly m children and all leaf nodes are at generation n. This is a special case of radially symmetric tree. Using the fact that $|S_k| = m^k$ for such a tree, it is a relatively straightforward exercise to find that the effective resistance, R_{eff}, is

$$R_{\text{eff}} = \sum_{k=1}^{n} \frac{1}{m^k} = \begin{cases} \frac{1-m^{-n}}{m-1} & \text{if } m \geq 2, \\ n & \text{if } m = 1. \end{cases} \tag{1.1}$$

This suggests a way of thinking about effective resistance on an infinite perfect m-ary tree. Imagine that we connect one terminal of a one-volt battery to the root of such a tree and we connect the other terminal to all nodes at generation $n \geq 1$. The effective resistance we would observe is given by (1.1). By making n arbitrarily large, we can observe the limiting behavior of the effective resistance

The general formula for the effective resistance is $R_{\text{eff}} = (m-1)^{-1}$ when $m \geq 2$. The case $m = 1$ is special. In this case, the effective resistance to the nth generation is n, yielding a sequence that diverges. We may think of the effective resistance of the infinite 1-ary tree as

$+\infty$.

1.3 Random walks

The connection between effective resistance and random walks in finite and infinite graph is established in [8, 13]. If we consider a random walk originating at the root of an infinite tree, then there are two fundamentally different types of behavior we might observe. The random walk may return infinitely often to the root, or it may return only finitely many times. The interesting thing about such random walks is that one or the other of these options has probability one. If the walk returns infinitely many times with probability one, then the walk is said to be *recurrent*. On the other hand, if the walk returns only finitely many times with probability one, the walk is said to be *transient*. In the electric network, transient and recurrent nature of a random walk expressed in term of voltage (see,[16, 18]). The connection to effective resistance is made through the following Theorem [8].

Theorem 1.3.1. *A random walk on an infinite tree is recurrent if and only if the effective resistance of the tree is infinite.*

A consequence of this theorem is that a random walk on an infinite perfect m-ary tree is recurrent if and only if $m = 1$.

1.4 Overview of this dissertation

The remainder of this dissertation is organized as follows.

- In Chapter 2, we review the theory of p-modulus on finite graphs and its connection to several important graph theoretic concepts including effective resistance.

- In Chapter 3, we expand upon the finite case to create a theoretical framework for modulus on the proper infinite tree and analogs properties of modulus on the infinite trees, especially in the radially symmetric infinite trees. We also introduce a parameter p which provides a sense of dimension on infinite radially symmetric trees.

- In Chapter 4, we develop an idea of the dimension of a an infinite rooted tree using p-modulus. The p-modulus as a parameter of p is either finite or zero, with a critical value of this parameter, p_c, marking the boundary between these two possibilities. In this chapter, we focus mainly on a critical value of p on 1-2 radially symmetric trees.

Chapter 2

Modulus on finite graphs

The theory of p-modulus came out of the theory of conformal modulus in complex analysis (see [1]). The study of quasiconformal maps extends the notion of p-modulus in more abstract metric measure spaces. Intuitively, p-modulus provides a method for quantifying the richness of a family of curves, in the sense that a family with many short curves will have a larger modulus than a family with fewer and longer curves. Recently, the p-modulus on the finite networks and its applications have been studied in [3, 4]. The p-modulus in the discrete setting have been studied in [12, 17].

Let $G = (V, E, \sigma)$ be a finite graph with vertex set V and edge set E and the weight function $\sigma : E \to \mathbb{R}_{>0}$. If $\sigma \equiv 1$, then graph G can be thought of as an unweighted graph.

The modulus framework is built on the abstract concept of an *object* on a graph. Examples of objects include paths, trees, cycles, forests, triangles, stars, etc. In modulus, one chooses a particular family of objects, Γ, to work with. Although the theory of discrete modulus has been developed for some infinite families of objects, this section considers only the case when Γ is finite.

Objects and edges are connected through a *usage function*, $\mathcal{N} : \Gamma \times E \to \mathbb{R}_{\geq 0}$, which assigns to each object $\gamma \in \Gamma$ and each edge $e \in E$ a value $\mathcal{N}(\gamma, e)$ indicating the degree to which γ "uses" e. The concept of usage is flexible. A common choice for objects that are subsets of edges (e.g., paths, trees, etc.) is the indicator function

$$\mathcal{N}(\gamma, e) = \mathbb{1}_\gamma(e) = \begin{cases} 1 & \text{if } e \in \gamma, \\ 0 & \text{if } e \notin \gamma. \end{cases} \tag{2.1}$$

The function $\mathcal{N}$ can be thought as a matrix $\mathcal{N} \in \mathbb{R}_{\geq 0}^{\Gamma \times E}$. This matrix is called the *usage matrix* for the family Γ.

A *density* on G is a function $\rho : E \to \mathbb{R}_{\geq 0}$. Similar to the usage matrix, it is often convenient to think of denisty ρ as a vector $\rho \in \mathbb{R}_{\geq 0}^E$. A density function can be thought as cost function on edge set. In particular, $\rho(e)$ is the *cost per unit usage of edge e*. For a density ρ and an object $\gamma \in \Gamma$, we define *ρ-length* of γ as

$$\ell_\rho(\gamma) := \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e),$$

which is simply the total usage cost that ρ assesses to γ. Thinking of $\mathcal{N}$ as a matrix and of ρ as a vector, we see that $\ell_\rho(\gamma) = (\mathcal{N}\rho)(\gamma)$. For a family Γ, the ρ-length of Γ is defined as

$$\ell_\rho(\Gamma) := \inf_{\gamma \in \Gamma} \ell_\rho(\gamma).$$

A density ρ is *admissible for* Γ if

$$\ell_\rho(\gamma) \geq 1 \qquad \forall \gamma \in \Gamma;$$

and we define the set $\mathrm{Adm}(\Gamma)$ to be set of all admissible densities. That is,

$$\mathrm{Adm}(\Gamma) = \{\rho \in \mathbb{R}_{\geq 0}^E : \ell_\rho(\Gamma) \geq 1\}.$$

Note that the set of admissible densities is the intersection of finitely many half-planes and, therefore, is a convex polytope.

Now, given a real parameter $1 \leq p \leq \infty$, the *p-energy* of a density ρ is defined to be

$$\mathcal{E}_{p,\sigma}(\rho) = \begin{cases} \sum\limits_{e \in E} \sigma(e)\rho(e)^p & \text{if } 1 \leq p < \infty \\ \max\limits_{e \in E} \sigma(e)\rho(e) & \text{if } p = \infty \end{cases}$$

Note that

$$\lim_{p \to \infty} (\mathcal{E}_{p,\sigma^p}(p))^{\frac{1}{p}} = \max_{e \in E} \sigma(e)\rho(e) = \mathcal{E}_{\infty,\sigma}(\rho).$$

Definition 2.0.1. *Given a graph $G = (V, E, \sigma)$ with non-empty family of objects Γ and exponent $1 \leq p \leq \infty$, the p-modulus of Γ is defined as*

$$\text{Mod}_{p,\sigma}(\Gamma) := \inf_{\rho \in \text{Adm}(\Gamma)} \mathcal{E}_{p,\sigma}(\rho). \tag{2.2}$$

Equivalently, p-modulus corresponds the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \mathcal{E}_{p,\sigma}(\rho) \\ \text{subject to} \quad & \ell_\rho(\gamma) \geq 1 \qquad \forall\, \gamma \in \Gamma, \\ & \rho(e) \geq 0 \qquad \forall\, e \in E \end{aligned} \tag{2.3}$$

where each object $\gamma \in \Gamma$ determines one inequality constraint. The admissible set $\text{Adm}(\Gamma)$ is defined by a set of linear inequalities. Each row of $\mathcal{N}$ corresponds to an object $\gamma \in \Gamma$. Since the family of objects Γ may be infinite, $\mathcal{N}$ may have infinitely many rows. However, under certain conditions (e.g., that $\mathcal{N}$ is integer-valued), the set of constraints in the above optimization problem (2.3) can be assumed to be finite by removing all but an *essential subfamily* of objects from Γ (see [4]). For instance, if Γ is the set of all walks between two distinct vertices with $\mathcal{N}(\gamma, e)$ defined to be the number of times γ traverses e, then the infinite family of walks can be replaced by the finite family of simple paths connecting the two vertices.

In order to avoid trivial cases, we assume that $\text{Adm}(\Gamma)$ is nonempty. One way to guarantee this is to require that no row of $\mathcal{N}$ is identically zero, and that

$$\mathcal{N}_{\min} := \inf\{\mathcal{N}(\gamma, e) : \mathcal{N}(\gamma, e) > 0\} > 0.$$

In other words, the positive entries of $\mathcal{N}$ are bounded away from zero. Under these assumptions, the density $\rho \equiv \mathcal{N}_{\min}^{-1}$ is admissible.

A density $\rho^* \in \mathrm{Adm}(\Gamma)$ is called *extremal* if $\mathcal{E}_{p,\sigma}(\rho^*) = \mathrm{Mod}_{p,\sigma}(\Gamma)$. On finite graphs, the existence of an extremal density follows from the facts that $\mathrm{Adm}(\Gamma)$ is closed. When $1 < p < \infty$, uniqueness follows from the strict convexity of ρ-energy $\mathcal{E}_{p,\sigma}(\rho)$ [2, 4]. Moreover, when $\mathcal{N}_{\min}$ is positive, the extremal density is bounded [3].

Lemma 2.0.1. *Given Γ and $1 \leq p \leq \infty$, the extremal density ρ^* exists and unique for $1 < p < \infty$.*

The basic properties of modulus of non-empty families of walks is discussed in [4] for the case $p = 2$, $\sigma \equiv 1$. In general, the following proposition describes the basic properties of p-modulus of a non-trivial family of objects on a finite and weighted graph G [3, 6].

Proposition 2.0.1. *Let $G = (V, E, \sigma)$ be a simple finite graph with edge-weights $\sigma \in \mathbb{R}^E_{>0}$ and Γ be a non-trivial family of objects on G Then, for $1 \leq p \leq \infty$, the following hold:*

1. ***Monotonicity in** Γ: Suppose Γ and Γ' are families of objects on G such that $\Gamma \subset \Gamma'$. Then,*

$$\mathrm{Mod}_{p,\sigma}(\Gamma) \leq \mathrm{Mod}_{p,\sigma}(\Gamma') \tag{2.4}$$

2. ***Monotonicity in** p: For $1 < p \leq q < \infty$ and Γ is family of object on G. Then*

$$\mathrm{Mod}_{q,\sigma}(\Gamma) \leq \mathrm{Mod}_{p,\sigma}(\Gamma) \tag{2.5}$$

3. ***Monotonocity in** σ: Let $\sigma \leq \sigma'$ and Γ be the family of objects on G. Then for $1 \leq p \leq \infty$,*

$$\mathrm{Mod}_{q,\sigma}(\Gamma) \leq \mathrm{Mod}_{p,\sigma'}(\Gamma) \tag{2.6}$$

4. ***Countable Subadditivity***: *For $1 \leq p < \infty$ and let $\{\Gamma_i\}_{i=1}^{\infty}$ be a sequence of families of objects on G. Then*

$$\mathrm{Mod}_{p,\sigma}(\bigcup_{i=1}^{\infty}\Gamma_i) \leq \sum_{i=1}^{\infty}\mathrm{Mod}_{p,\sigma}(\Gamma_i) \tag{2.7}$$

The extremal density of the modulus problem is continuous in the exponent $p > 1$ and is closely related to the gradient of the modulus with respect to the edge weights σ. The modulus problem $\mathrm{Mod}_{p,\sigma}(\Gamma)$ is *Lipschitz continuous* and *concave* in σ (see [2]).

The concept of p-modulus for a connecting family of paths is explicitly connected with the graph theoretic concepts of *shortest path*, *effective conductance* and *min-cut*. Let $G = (V, E, \sigma)$ and two distinct vertices s and t in V be given. Let $\Gamma(s,t)$ be the *connecting family* of all the paths in G that start at s and end at t. By shortest path we mean a path in G connecting s to t which has the shortest length (in term of hops). The following Theorem proved in [2] gives the connection between the p-modulus of $\Gamma(s,t)$ and the shortest path between s and t.

Theorem 2.0.1. *Let $\Gamma = \Gamma(s,t)$ be a family of connecting paths in G, then*

$$\mathrm{Mod}_{\infty,1}(\Gamma) = \frac{1}{\ell(\Gamma)}$$

where $\ell(\Gamma) = \inf_{\gamma \in \Gamma} \ell(\gamma)$ is the shortest path between s and t.

The modulus problem is also closely related to the *Max-Flow Min-Cut Theorem* connected to Ford and Fulkerson's orignal work [10]. The maximum flow problem of Ford and Fulkerson is equivalent to a dual formulation of the 1-*modulus* of $\Gamma(s,t)$ [2].

A *cut* in a graph G is subset $C \subset E$ whose removal disconnects the graph G. A subset C is a cut for $\Gamma(s,t)$ (also called an *st*-cut) if $\gamma \cap C \neq \emptyset$ for every $\gamma \in \Gamma(s,t)$. For a weighted graph G, the size of a cut C is defined to be $\sigma(C) := \sum_{e \in C} \sigma(e)$. The connection between this and modulus is established in the following theorem (see, [2, 3]).

Theorem 2.0.2. *Let $\Gamma = \Gamma(s,t)$ be family of connecting path in weighted graph $G = (V, E, \sigma)$. Then,*

$$\mathrm{Mod}_{1,\sigma}(\Gamma) = \min\{\sigma(C) : C \textit{ is a cut of } \Gamma\}$$

An undirected graph $G = (E, V, \sigma)$ can be considered as an *electrical network* with edge *conductance* given by the weights σ (see [8]). The *effective conductance* $\mathcal{C}_{\text{eff}}(s,t)$ is the reciprocal of *effective resistance* between two distinct vertices s and t in the graph G. The following theorem shows that the extremal density for the modulus of a connecting family of walks can be related to a generalized *voltage potential* in a network of nonlinear resistors (see [2, 9]).

Theorem 2.0.3. *Let $G = (V, E, \sigma)$ be an undirected graph, let $\Gamma = \Gamma(s,t)$ be the connecting family of walks between two distinct vertices s and t in V, and let $1 < p < \infty$. Let ρ^* be the extremal density for (2.2). Then there exists a vertex potential $\phi^* : V \to \mathbb{R}$ such that $\phi^*(s) = 0$, $\phi^*(t) = 1$, and*

$$\rho^*(\{x,y\}) = |\phi^*(x) - \phi^*(y)| \qquad \forall \{x,y\} \in E.$$

Moreover, this ϕ^* solves the optimization problem

$$\begin{aligned} &\textit{minimize} \quad \sum_{\{x,y\}\in E} \sigma(\{x,y\})|\phi(x) - \phi(y)|^p \\ &\textit{subject to} \quad \phi(s) = 0 \quad \phi(t) = 1. \end{aligned}$$

When $p = 2$, it follows that $\text{Mod}_{2,\sigma}(\Gamma) = \mathcal{C}_{\mathit{eff}}(s,t)$.

Consider an example which helps us to understand how p-modulus generalizes three fundamental quantities in graph theory: shortest path length, effective conductance, and min-cut. Consider the simple undirected and unweighted graph which contains k parallel simple paths of ℓ hops connecting node s to t (see Figure 2.1). Let $\Gamma = \Gamma(s,t)$ be the connecting family of paths between s and t. It is straightforward to show that the choice $\rho^* = \frac{1}{\ell}$ is extremal for all $1 \le p \le \infty$. Thus, for $1 \le p < \infty$, modulus of Γ is

$$\text{Mod}_{p,1}(\Gamma) = \mathcal{E}_{p,1}(\rho^*) = k\ell \left(\frac{1}{\ell}\right)^p = \frac{k}{\ell^{p-1}}$$

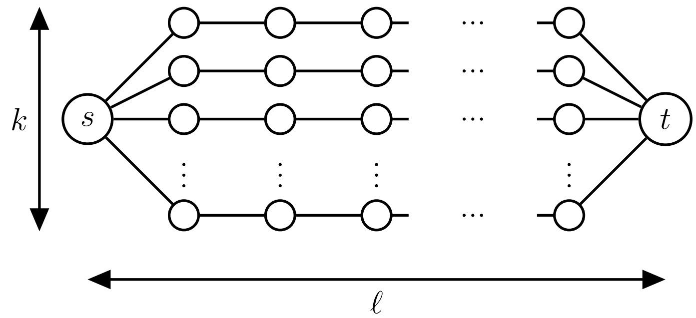


Figure 2.1: A graph consisting of k simple walks connecting s to t

and for $p = \infty$,

$$\mathrm{Mod}_\infty(\Gamma) = \mathcal{E}_{\infty,1}(\rho^*) = \frac{1}{\ell}.$$

We can build up several interesting observations from this example. In particular, the p-modulus is determined by the number of walks k, the length of a walk ℓ, and p. For $1 < p < \infty$, the modulus becomes large if Γ contains many, short walks, and small if Γ contains few, long walks. When $p \approx 1$, the modulus is more sensitive to the number of parallel walks k and less sensitive with their length ℓ. When p is very large, the modulus depends more on the length of the walks ℓ and less on the number k. In particular, when $p = 1$, the modulus does not depend on ℓ: a large family of walks has large modulus regardless of their lengths. On the other hand, when $p = \infty$ the modulus depends only on the length of the shortest walk and not at all on the number of walks. When $p = 2$, the modulus is k/ℓ, which is the effective conductance of the graph considered as a resistor network between s and t with unit resistors placed on each edge. Moreover, $\mathrm{Mod}_{p,1}(\Gamma)$ is continuous, monotone decreasing parameter p, and $\lim\limits_{p\to\infty} \mathrm{Mod}_{p,1}(\Gamma)^{1/p} = \mathrm{Mod}_{\infty,1}(\Gamma)$.

2.1 Langrangian duality and its probability interpretation

In this section, we review the formulation of the Lagrangian dual of p-modulus problem and the existence of optimal dual variable and strong duality. The Lagrangian dual is used in the probability interpretation of p-modulus problem.

Returning to (2.3), we can see that the p-modulus problem is a convex optimization problem. The *p-energy* $\mathcal{E}_{p,\sigma}(\rho)$ of the density ρ is a convex function of ρ and $\mathrm{Adm}(\Gamma)$ is a convex set. If Γ is finite, moreover, then since $\mathrm{Adm}(\Gamma)$ is defined by a finite set of inequalities, making (2.3) a standard ordinary convex program (see [4]). Even if Γ is infinite, it is sometimes possible to reduce the set of inequalities to a finite set. The following lemma proved in [4] shows that a family of walks on a finite graph can be replaced by a finite family of walks with the same admissible set. Such a finite subfamily is called an *essential subfamily* for the modulus problem.

Lemma 2.1.1. *Let Γ be a given family of walks on a finite graph. There exists a finite subfamily $\Gamma^* \subseteq \Gamma$ such that $\mathrm{Adm}(\Gamma^*) = \mathrm{Adm}(\Gamma)$.*

Using standard techniques from convex optimization, the Lagrangian for the problem (2.3) can be written as

$$\mathcal{L}_{p,\sigma}(\rho, \lambda, \mu) = \mathcal{E}_{p,\sigma}(\rho) + \sum_{\gamma\in\Gamma} \lambda(\gamma) \left(1 - \sum_{e\in E} \mathcal{N}(\gamma, e)\rho(e)\right) - \sum_{e\in E} \mu(e)\rho(e) \tag{2.8}$$

where $\lambda : \Gamma \to \mathbb{R}_{\geq 0}$ and $\mu : E \to \mathbb{R}_{\geq 0}$ are the Lagrange dual variables associated with the admissibility constraints. The Lagrangian dual and derivation of Lagrangian dual problem of optimization problem (2.3) is studied extensively in [2, 4].

For $1 < p < \infty$, the Lagrangian is differentiable. Moreover, it can be shown that the inequality constraints of the form $\rho(e) \geq 0$ do not need to be enforced explicitly, so the dual variable μ is not needed. The Lagrange dual optimization problem in this case is derived as

$$\begin{aligned} \text{maximize} \quad & \sum_{\gamma\in\Gamma} \lambda(\gamma) - (p-1) \sum_{e\in E} \sigma(e) \left(\frac{1}{p\sigma(e)} \sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)\lambda(\gamma)\right)^{\frac{p}{p-1}} \\ \text{subject to} \quad & \lambda(\gamma) \geq 0 \quad \forall\gamma \in \Gamma, \end{aligned} \tag{2.9}$$

Moreover, it can be shown using *Slater's condition* (see [7]) that *strong duaity* holds, meaning that the minimum value in (2.3) is equal to the maximum value in (2.9) (see [2]). This duality can be useful in proving bounds on modulus. Any feasible density ρ yields an upper bound on modulus while any non-negative λ yields a lower bound. The extremal density ρ^* can be obtained from the extremal dual variable λ^*, which is typically not unique. However, the extremal density ρ^* is unique.

For $p = 1$, the 1-modulus optimization problem (2.3) can be replaced by the following equivalent problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{\gamma\in\Gamma} \sigma(e)\rho(e) \\ \text{subject to} \quad & \sum_{e\in E} \mathcal{N}(\gamma, e)\rho(e) \geq 1 \quad \forall\gamma \in \Gamma \\ & \rho(e) \geq 0 \quad \forall e \in E \end{aligned} \tag{2.10}$$

Its Lagrangian dual problem can be written as

$$\begin{aligned} \text{maximize} \quad & \sum_{\gamma\in\Gamma} \lambda(\gamma) \\ \text{subject to} \quad & \sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)\lambda(\gamma) \leq \sigma(e) \quad \forall\, \gamma \in \Gamma \\ & \lambda(\gamma) \geq 0 \quad \forall\, \gamma \in \Gamma \end{aligned} \tag{2.11}$$

In the case of $p = \infty$, the optimization problem (2.3) can be stated as the following equivalent problem (see [7])

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & \sum_{e\in E} \mathcal{N}(\gamma, e)\rho(e) \geq 1 \quad \forall\gamma \in \Gamma \\ & 0 \leq \rho(e) \leq t \quad \forall e \in E \end{aligned} \tag{2.12}$$

and its dual problem is formulated as

$$\begin{aligned} \text{maximize} \quad & \sum_{\gamma\in\Gamma} \lambda(\gamma) \\ \text{subject to} \quad & \sum_{\gamma\in\Gamma} \mathcal{N}(\gamma,e)\lambda(\gamma) \leq \eta(e) \quad \forall\, e\in E \\ & \sum_{e\in E} \eta(e) = 1 \\ & \eta(e) \geq 0 \quad \forall\, e\in E \\ & \lambda(\gamma) \geq 0 \quad \forall\, \gamma\in\Gamma \end{aligned} \tag{2.13}$$

where $\eta : E \to \mathbb{R}_{\geq 0}$ is a dual variable defined on the edge set E. The Lagrangian dual (2.9) is interpreted in a probabilistic setting in [3]. For this, $\mathcal{P}(\Gamma)$ is defined to be the set of probability mass function (pmfs) on the set Γ. That is, $\mathcal{P}(\Gamma)$ contains the set of vectors $\mu \in \mathbb{R}^{\Gamma}_{\geq 0}$ with the property that $\mu^T\mathbf{1} = 1$. Given such a μ, a random variable $\underline{\gamma}$ is defined with the distribution given by μ: $\mathbb{P}_\mu(\underline{\gamma} = \gamma) = \mu(\gamma)$. The value $\mathcal{N}(\underline{\gamma}, e)$ is a random variable for each given edge $e \in E$, and its expectation is denoted by $\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]$. The following theorem states the probabilistic interpretation of the Lagrangian dual.

Theorem 2.1.1. *Let $G = (V, E, \sigma)$ be a weighted finite graph and let Γ be a non-trivial finite family of objects on G withe usage matrix $\mathcal{N}$. Then, for any $1 < p < \infty$, letting $q = p/(p-1)$ be the conjugate exponent to p, we have*

$$\mathrm{Mod}_{p,\sigma}(\Gamma)^{-\frac{1}{p}} = \left(\min_{\mu\in\mathcal{P}(\Gamma)} \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]^q \right)^{\frac{1}{q}} \tag{2.14}$$

Moreover, any optimal measure μ^*, must satisfy

$$\mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, e)] = \frac{\sigma(e)\rho^*(e)^{\frac{p}{q}}}{\mathrm{Mod}_{p,\sigma}(\Gamma)}$$

where ρ^* is the unique extremal density for $\mathrm{Mod}_{p,\sigma}(\Gamma)$.

These formulas simplify in the particular case that $p = 2$, $\sigma \equiv 1$, and Γ is a collection of subsets of E, so that $\mathcal{N}$ is a (0,1)-matrix defined as $\mathcal{N}(\gamma, e) = \mathbb{1}_\gamma(e)$. In this case, this duality relation is expressed as

$$\text{Mod}_2(\Gamma)^{-1} = \min_{\mu \in \mathcal{P}(\Gamma)} \mathbb{E}_\mu |\underline{\gamma} \cap \underline{\gamma}'|$$

where $\underline{\gamma}$ and $\underline{\gamma}'$ are two identically independent random variables chosen according to the pmf μ and $|\underline{\gamma} \cap \underline{\gamma}'|$ is their *overlap*. In other words, computing the 2-modulus in this setting is equivalent to finding a pmf that minimizes the expected overlap of two iid Γ-valued random variables.

2.2 Blocking duality and p-modulus

In this section, we will review the *Fulkerson blocker* or *blocker*, *Fulkerson Theorem*, and blocking duality for p-modulus and its connection to the probabilistic interpretation. We first focus on defining the blocking polyhedron to $\text{Adm}(\Gamma)$.

A set D is a *convex* if the line segment connecting any two points in D lies within the set D, that is, for any $x_1, x_2 \in D$, we have $tx_1 + (1-t)x_2 \in D$, for any $0 \leq t \leq 1$. Let $\mathcal{D}$ be the set of all closed convex sets $D \subset \mathbb{R}^E_{\geq 0}$ that are *recessive*, in the sense that $D + \mathbb{R}^E_{\geq 0} = D$. It is assumed that $\emptyset \subsetneq D \subsetneq \mathbb{R}^E_{\geq 0}$ to avoid trivial cases. For each $D \in \mathcal{D}$, the blocker or blocking polyhedron associated with the set D is defined as

$$\text{BL}(D) = \{\eta \in \mathbb{R}^E_{\geq 0} : \eta^T \rho \geq 1 \quad \forall\, \rho \in D\}.$$

A point $x \in D \in \mathcal{D}$ is an *extreme point* of D if x is not contained in the relative interior of any line segment in D. In other words, if $x = tx_1 + (1-t)x_2$ for some $x_1, x_2 \in D$ and some $t \in (0,1)$, then $x = x_1 = x_2$. The set of extreme points of D is denoted by $\text{ext}(D)$. A *convex hull* of the set D, denoted by $\text{co}(D)$, is the set of all convex combination, that is,

$$\text{co}(D) = \{t_1 x_1 + \cdots + t_n x_n :\ x_i \in D,\ t_i \geq 0, i = 1, 2, \cdots, n,\ t_1 + \cdots + t_n = 1\}$$

Note that the convex hull $\mathrm{co}(D)$ is the smallest convex set that contains the set D. The *dominant* of a set $D \subseteq \mathbb{R}^E_{\geq 0}$ is the recessive closed convex set

$$\mathrm{Dom}(D) = \mathrm{co}(D) + \mathbb{R}^E_{\geq 0}.$$

Note that if Γ is a finite non-trivial family of objects on a graph G, the admissible set $\mathrm{Adm}(\Gamma)$ is determined by finitely many inequalities and, thus, has finitely many extreme points. These extreme points play a special role in the theory of blocker duality.

Definition 2.2.1. *Let $G = (V, E)$ be a finite graph and Γ be a finite non-trivial family of objects on G. The family of extreme points, that is,*

$$\hat{\Gamma} = \mathrm{ext}\,(\mathrm{Adm}(\Gamma)) = \{\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_s\}$$

is the Fulkerson blocker of Γ

In order to make sense of this definition, we observe that an object $\gamma \in \Gamma$ can be associated with its edge usage vector $\mathcal{N}(\gamma, \cdot)$. Similarly, any vector in $\mathbb{R}^E_{\geq 0}$ can be thought of as an object on G. Thus, we may always think of a family as a set of vectors in $\mathbb{R}^E_{\geq 0}$. In that sense, we may interpret each of the vectors $\hat{\gamma}_i$ in the Definition 2.2.1 as the usage vector for some object, which we also call $\hat{\gamma}_i$ through a slight abuse of notation. The following Fulkerson Theorem [11] states that $\mathrm{Adm}(\Gamma)$ is determined by the dominant of the set of blockers and the admissible set for the blocking family $\hat{\Gamma}$ is determined by the dominant of the set Γ (see, [5]).

Theorem 2.2.1. *Let $G = (V, E)$ be a graph and let Γ be a non-trivial finite family of objects on G. Let $\hat{\Gamma}$ be the Fulkerson blocker of Γ. Then*

1. $\mathrm{Adm}(\Gamma) = \mathrm{Dom}(\hat{\Gamma}) = \mathrm{BL}(\mathrm{Adm}(\hat{\Gamma}))$;
2. $\mathrm{Adm}(\hat{\Gamma}) = \mathrm{Dom}(\Gamma) = \mathrm{BL}(\mathrm{Adm}(\Gamma))$;
3. $\hat{\hat{\Gamma}} \subset \Gamma$

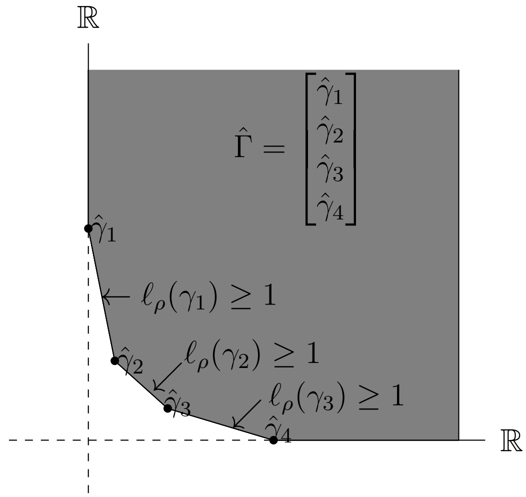


Figure 2.2: Adm(Γ) is determined by a finite family of fulkerson blockers

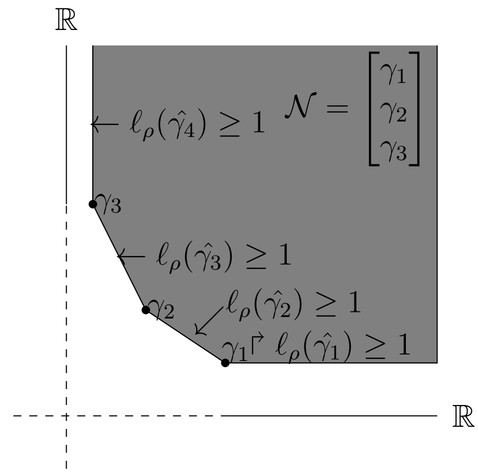


Figure 2.3: Adm($\hat{\Gamma}$) determined by a finite family of objects

As an example, in Figure (2.2), the extreme points or "vertices" of the shaded region which is Adm(Γ) of a finite family of objects are Fulkerson blockers for Γ. Similarly, in Figure (2.3), the extreme points or "vertices" of shaded region which is Adm($\hat{\Gamma}$) of a finite family of some of objects. The Fulkerson blocker $\hat{\Gamma}$ being a family of objects, it is natural to define the modulus on it. The following theorem shows how the modulus of a family of objects and it's Fulkerson blocker are related (see [3, Theorem 3.7]).

Theorem 2.2.2. *Let $G = (V, E, \sigma)$ be a graph and let Γ be a non-trivial family of objects on G with usage matrix $\mathcal{N}$. Let the exponent $1 < p < \infty$ be given with $q = (p-1)/p$ its conjugate exponent. Then*

$$\mathrm{Mod}_{p,\sigma}(\Gamma)^{\frac{1}{p}} \, \mathrm{Mod}_{p,\hat{\sigma}}(\hat{\Gamma})^{\frac{1}{q}} = 1 \tag{2.15}$$

where $\hat{\sigma} \in \mathbb{R}^E_{>0}$ is dual weight defined as $\widehat{\sigma(e)} = \sigma(e)^{-\frac{q}{p}}$ for all $e \in E$.
Moreover, the optimal $\rho^* \in \mathrm{Adm}(\Gamma)$ and $\eta^* \in \mathrm{Adm}(\hat{\Gamma})$ are unique and are related as follows:

$$\rho^*(e) = \frac{\hat{\sigma}(e)\eta^*(e)^{q-1}}{\mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})} \qquad \forall\, e \in E \tag{2.16}$$

In the case for $p = 2$, the relation (2.15) becomes

$$\mathrm{Mod}_{2,\sigma}(\Gamma)\,\mathrm{Mod}_{2,\sigma^{-1}}(\hat{\Gamma}) = 1, \tag{2.17}$$

and (2.16) becomes

$$\sigma(e)\rho^*(e) = \mathrm{Mod}_{2,\sigma}(\Gamma)\eta^*(e) \quad \forall e \in E.$$

A similar theorem holds when $p = 1$ or $p = \infty$ (see [3, Theorem 3.10]).

Theorem 2.2.3. *Under the assumptions of Theorem 2.2.2,*

$$\mathrm{Mod}_{1,\sigma}(\Gamma)\,\mathrm{Mod}_{\infty,\sigma^{-1}}(\hat{\Gamma}) = 1.$$

As an example of blocking duality, consider a family $\Gamma = \Gamma(s,t)$ consisting of all paths connecting two distinct vertices $s, t \in V$, with edge usages defined as in (2.1). In this case, (2.11) can be seen to be the standard path formulation of the maximum flow problem; we wish to assign certain non-negative flow amounts along each st-path in such a way that the total flow is maximized subject to the constraint that the amount of flow allowed along an edge $e \in E$ is no larger than $\sigma(e)$. The blocking family, $\hat{\Gamma}$, is known to be the family of minimal st-cuts, again with usage defined as in (2.1). In this case, Theorem 2.2.3 is equivalent to the max-flow min-cut theorem. For some more general families, Γ, the theorem provides a generalization of the max-flow min-cut theorem (see [5, Proposition 4.1]).

For a planar graph, G, this duality has an interesting interpretation. Every minimal st-cut corresponds to an $\hat{s}\hat{t}$-path in a dual graph, $\hat{G}$ (see Figure 2.4). Each edge of $\hat{G}$ (blue dashed edges in the figure) corresponds to the edge of G (black edges in the figure) that it crosses, and receives a weight $\hat{\sigma}$ depending on the weight σ of the corresponding edge and on p. Thus, the modulus of a connecting family in a planar graph is dual to the modulus of a corresponding connecting family in a related dual graph (see [9] for details when $p = 2$).

The probabilistic interpretation of p-modulus described in Section 2.1 is also closely connected to Fulkerson blocking duality. The following theorem proved in [4] shows that

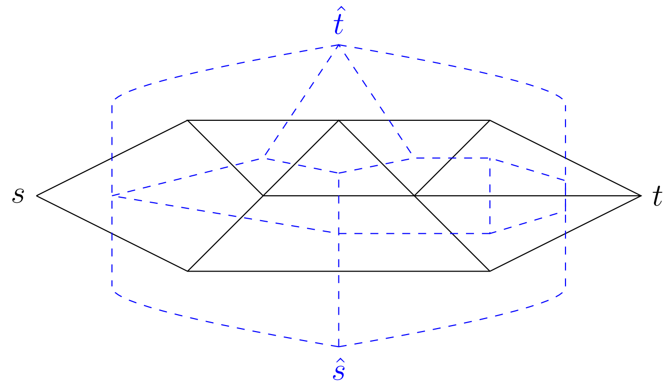


Figure 2.4: The family $\Gamma(s,t)$ of simple path connecting s to t and family $\hat{\Gamma}(\hat{s},\hat{t})$ of st-cuts for $\Gamma(s,t)$

optimal dual density, η^*, for the q-modulus of the Fulkerson blocker is exactly the optimal expected usage of that edge in the probabilistic interpretation of p-modulus.

Theorem 2.2.4. *Let $G = (V, E, \sigma)$ be a graph and Γ a finite family of objects on G with associated usage matrix N. For a given $1 < p < \infty$, let μ^* be an optimal* pmf *for the minimization problem in* (2.14) *and let η^* be optimal for* $\mathrm{Mod}_{q,\hat{\sigma}}(\hat{\Gamma})$. *Then,*

$$\eta^*(e) = \mathbb{E}_{\mu^*}[\mathcal{N}(\underline{\gamma}, e)].$$

Chapter 3

Modulus on radially symmetric infinite trees

Now we turn our attention to p-modulus on infinite trees. In particular, we make the following assumptions on G.

- G is a proper infinite tree.
- G is radially symmetric in the sense that $C(e) = C(\text{gen}(e))$—that is, the number of children of a given edge depends only on the generation of that edge.
- The edge weights σ are radially symmetric in the sense that $\sigma(e) = \sigma(\text{gen}(e))$—that is, the weight of an edge depends only on the generation of that edge.

Let $G = (E, V, \sigma, o)$ be a tree satisfying the above assumptions and let Γ, Γ_n and Γ_∞ be the families of descending paths described in Section 1.1.3. Our goal is to define and compute the p-modulus, $\text{Mod}_{p,\sigma}(\Gamma_\infty)$. Typically, we treat the cases $1 < p < \infty$, $p = 1$ and $p = \infty$ separately. In the discussion, we routinely use several facts about an exponent $p \in (1, \infty)$ and its corresponding Hölder conjugate exponent q:

$$q = \frac{p}{p-1}, \quad p = \frac{q}{q-1}, \quad \frac{p}{q} = p - 1 = \frac{1}{q-1}, \quad \frac{q}{p} = q - 1 = \frac{1}{p-1}.$$

3.1 Defining modulus

The definition of modulus in (2.2) extends readily to the infinite setting; we only need to take a little care when dealing with the infinite sums. Note that both $\mathcal{E}_{p,\sigma}(\rho)$ and $\ell_\rho(\gamma)$ in the formula are sums over the countably infinite set E and that their terms are non-negative. Thus, in both cases, the partial sums are monotone and each sum either converges to a finite value or diverges. In the latter case, we understand the value of the sum to be $+\infty$. With that understanding, we can define $\text{Mod}_{p,\sigma}(\Gamma_\infty)$ to be the value of

$$\begin{aligned} \text{minimize} \quad & \mathcal{E}_{p,\sigma}(\rho) \\ \text{subject to} \quad & \ell_\rho(\gamma) \geq 1 \qquad \forall\, \gamma \in \Gamma_\infty, \\ & \rho(e) \geq 0 \qquad \forall\, e \in E. \end{aligned} \tag{3.1}$$

The following lemma shows an upper bound for the modulus of a proper infinite tree.

Lemma 3.1.1. *For all $p \in [1, \infty]$,*

$$\text{Mod}_{p,\sigma}(\Gamma_\infty) \leq \inf_n \sigma(n)|S_n|.$$

Proof. Let n be any generation and define

$$\rho(e) = \begin{cases} 1 & \text{if } e \in S_n, \\ 0 & \text{if } e \notin S_n. \end{cases}$$

Since every $\gamma \in \Gamma_\infty$ contains an edge in S_n, this density is admissible and so its energy provides an upper bound on the modulus. Taking the infimum of energy over all generations establishes the lemma. □

3.2 Modulus as a limit

Thinking back to the discussion of effective resistance in Section 1.2, it is natural to wonder how $\text{Mod}_{p,\sigma}(\Gamma_\infty)$ is related to $\text{Mod}_{p,\sigma}(\Gamma_n)$. The goal of this section is to prove the following theorem.

Theorem 3.2.1. *For any* $p \in (1, \infty)$,

$$\lim_{n\to\infty} \text{Mod}_{p,\sigma}(\Gamma_n) = \text{Mod}_{p,\sigma}(\Gamma_\infty).$$

We begin by establishing some helpful lemmas.

Lemma 3.2.1. *Let* $\rho \in \mathbb{R}^E_{\geq 0}$. *Then*

$$\lim_{n\to\infty} \ell_\rho(\Gamma_n) = \ell_\rho(\Gamma_\infty).$$

Proof. Note that $\{\ell_\rho(\Gamma_n)\}$ is a monotone sequence, so either the limit exists as a finite number or the sequence diverges to $+\infty$, in which case we say that the limit is $+\infty$.

Let $A_0 = \{\gamma_n\}$ be a sequence of paths with $\gamma_n \in \Gamma_n$ such that $\ell_\rho(\gamma_n) = \ell_\rho(\Gamma_n)$. By the assumption of local finiteness, there must be an edge $e_1 \in S_1$ that intersects infinitely many paths in A_0. Let $A_1 \subseteq A_0$ be the infinite subsequence produced by removing all paths that do not pass through e_1. Again, there must be some edge $e_2 \in S_2$ that intersects infinitely many paths in A_1. Let $A_2 \subseteq A_1$ be the infinite subsequence produced by removing all paths that do not use this edge. Repeating this procedure, we may produce an infinite sequence of edges $\{e_1, e_2, e_3, \ldots\}$ and a chain of infinite subsets $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ with the property that every path in A_n begins with the subpath $e_1 e_2 \cdots e_n$ and every path in A_n is a shortest-length path in some Γ_m with $m \geq n$.

Now consider the infinite path $\gamma = e_1 e_2 e_3 \cdots$. For any $n \geq 1$, there exists an $m \geq n$ and a $\gamma' \in A_n \cap \Gamma_m$ with the property that $\ell_\rho(\gamma') = \ell_\rho(\Gamma_m)$. So,

$$\ell_\rho(\gamma \cap B_n) = \ell_\rho(\gamma' \cap B_n) \leq \ell_\rho(\gamma') = \ell_\rho(\Gamma_m).$$

Since $\ell_\rho(\gamma \cap B_n) \to \ell_\rho(\gamma)$ as $n \to \infty$, this shows that

$$\lim_{n\to\infty} \ell_\rho(\Gamma_n) \geq \ell_\rho(\gamma) \geq \ell_\rho(\Gamma_\infty).$$

On the other hand, if $\gamma \in \Gamma_\infty$ then for any $n \geq 1$,

$$\ell_\rho(\Gamma_n) \leq \ell_\rho(\gamma \cap B_n) \leq \ell_\rho(\gamma).$$

Taking the infimum over $\gamma \in \Gamma_\infty$ establishes the opposite inequality. □

Lemma 3.2.2. *If* $1 \leq m \leq n \leq \infty$, *then* $\mathrm{Adm}(\Gamma_m) \subseteq \mathrm{Adm}(\Gamma_n)$.

Proof. Let $\rho \in \mathrm{Adm}(\Gamma_m)$ and let $\gamma \in \Gamma_n$. By the extension property, there is sub-path $\gamma' \in \Gamma_m$ such that $\gamma' \preceq \gamma$, that is, γ is an extension of γ'. Thus, since $\rho \geq 0$,

$$\ell_\rho(\gamma) = \sum_{e\in\gamma} \rho(e) \geq \sum_{e\in\gamma'} \rho(e) = \ell_\rho(\gamma') \geq 1.$$

Thus, $\rho \in \mathrm{Adm}(\Gamma_n)$. □

It is natural to expect the density ρ to share the symmetry of the tree. We shall call ρ *radially symmetric* if $\rho(e) = \rho(\mathrm{gen}(e))$—that is, if $\rho(e)$ depends only on the generation of e.

Lemma 3.2.3. *Suppose* $\rho \in \mathrm{Adm}(\Gamma_n)$ *for some* $1 \leq n \leq \infty$ *and that* ρ *is not radially symmetric. Then there exists a* $\rho' \in \mathrm{Adm}(\Gamma_n)$ *such that*

$$\mathcal{E}_{p,\sigma}(\rho') < \mathcal{E}_{p,\sigma}(\rho).$$

Proof. Let $m \leq n$ be the earliest generation where ρ fails to be radially symmetric and let $e, e' \in S_m$ be such that $\rho(e) \neq \rho(e')$. Define $\tilde{\rho}$ from ρ by swapping the values on e and its children with the corresponding values on e' and its children. This density is still admissible and has the same energy as ρ. Now define $\rho' = (\rho + \tilde{\rho})/2$. As an average of two admissible densities, ρ' is also admissible. Moreover, by the strict convexity of the energy, it follows that

$$\mathcal{E}_{p,\sigma}(\rho') < \frac{1}{2}\mathcal{E}_{p,\sigma}(\tilde{\rho}) + \frac{1}{2}\mathcal{E}_{p,\sigma}(\rho) = \mathcal{E}_{p,\sigma}(\rho).$$

□

Lemma 3.2.4. *When computing modulus on a radially symmetric tree, it is sufficient to consider radially symmetric densities.*

Proof. This is a consequence of the previous lemma. □

In the case of radially symmetric tree, the Lemma 3.2.1 is more easily established.

Lemma 3.2.5. *Let ρ be a radially symmetric density on G. Then*

$$\lim_{n\to\infty} \ell_\rho(\Gamma_n) = \ell_\rho(\Gamma_\infty). \tag{3.2}$$

Proof. To see that the limit exists (if we include $+\infty$ as a possible limit), denote by ρ_k the value of ρ on S_k. For any $1 \le n < \infty$,

$$\ell_\rho(\Gamma_n) = \sum_{k=1}^{n} \rho_k. \tag{3.3}$$

Moreover, if γ is any path in Γ_∞, then

$$\ell_\rho(\Gamma_\infty) = \ell_\rho(\gamma) = \sum_{k=1}^{\infty} \rho_k.$$

Taking the limit in (3.3) and substituting shows (3.2). □

Now we are ready to prove the main theorem of this section.

Proof of Theorem 3.2.1. Lemma 3.2.2 implies that the sequence $\{\mathrm{Mod}_{p,\sigma}(\Gamma_n)\}_{n=1}^{\infty}$ is a non-increasing sequence of non-negative numbers and, therefore, approaches a limit. Now let $\epsilon > 0$ be arbitrary and let $\rho \in \mathrm{Adm}(\Gamma_\infty)$ be radially symmetric and have the property that

$$\mathcal{E}_{p,\sigma}(\rho) \le \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) + \epsilon.$$

Since $\ell_\rho(\Gamma_\infty) \geq 1$ by assumption, Lemma 3.2.1 implies that $\ell_\rho(\Gamma_n) > 0$ for sufficiently large n. For these n, it is straightforward to check that $\rho/\ell_\rho(\Gamma_n) \in \mathrm{Adm}(\Gamma_n)$.

It follows that for sufficiently large n,

$$\mathrm{Mod}_{p,\sigma}(\Gamma_n) \leq \mathcal{E}_{p,\sigma}\left(\frac{\rho}{\ell_\rho(\Gamma_n)}\right) = \sum_{e\in E}\sigma(e)\left(\frac{\rho(e)}{\ell_\rho(\Gamma_n)}\right)^p = \frac{\mathcal{E}_{p,\sigma}(\rho)}{\ell_\rho(\Gamma_n)^p}.$$

Taking a limit as $n \to \infty$ shows that $\lim_{n\to\infty} \mathrm{Mod}_{p,\sigma}(\Gamma_n) \leq \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) + \epsilon$. Since $\epsilon > 0$ was arbitrary, this implies that $\lim_{n\to\infty} \mathrm{Mod}_{p,\sigma}(\Gamma_n) \leq \mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$. On the other hand, Lemma 3.2.2 implies the opposite inequality, thus establishing equality. □

3.3 Computing modulus

Now we are ready to write the formula for p-modulus of an infinite, radially symmetric tree.

Modulus for $1 < p < \infty$. We begin by considering the modulus of a truncated family Γ_n with $1 \leq n < \infty$. This family has a unique optimal density, ρ_n which is radially symmetric by Lemma 3.2.3. Denote by σ_k and $\rho_{n,k}$ the values of σ and ρ_n on S_k respectively. Then ρ_n is optimal for the problem

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{n} |S_k|\sigma_k\rho_{n,k}^p, \\ \text{subject to} \quad & \sum_{k=1}^{n} \rho_{n,k} \geq 1. \end{aligned}$$

By introducing a Lagrange multiplier λ for the constraint, one can show that ρ_n satisfies

$$\rho_{n,k} = \lambda\left(\sigma_k|S_k|\right)^{-\frac{q}{p}}.$$

Solving for λ to satisfy the constraint shows that

$$\lambda = \left(\sum_{k=1}^{n} (\sigma_k|S_k|)^{-\frac{q}{p}}\right)^{-1}, \qquad \text{so} \qquad \rho_{n,k} = \frac{(\sigma_k|S_k|)^{-\frac{q}{p}}}{\sum\limits_{\ell=1}^{n} (\sigma_\ell|S_\ell|)^{-\frac{q}{p}}}.$$

From this, we find that

$$\mathrm{Mod}_{p,\sigma}(\Gamma_n) = \left(\sum_{k=1}^{n} (\sigma_k|S_k|)^{-\frac{q}{p}}\right)^{-\frac{p}{q}}.$$

Taking the limit, we find a formula for the modulus of the infinite tree:

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) = \left(\sum_{k=1}^{\infty} (\sigma_k|S_k|)^{-\frac{q}{p}}\right)^{-\frac{p}{q}}, \tag{3.4}$$

with the understanding that $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) = 0$ if the sum diverges.

As an example, consider the case of an unweighted perfect m-ary tree with $p = q = 2$. In this case, we have $|S_k| = m^k$, so

$$\sum_{k=1}^{\infty} (\sigma_k|S_k|)^{-\frac{q}{p}} = \sum_{k=1}^{\infty} \frac{1}{m^k} = \begin{cases} \frac{1}{m-1} & \text{if } m \geq 2, \\ +\infty & \text{if } m = 1 \end{cases}$$

and, therefore,

$$\mathrm{Mod}_{2,\sigma}(\Gamma_\infty) = \begin{cases} m-1 & \text{if } m \geq 2, \\ 0 & \text{if } m = 1. \end{cases}$$

As we might expect, modulus in this case is the reciprocal of effective resistance, if we understand $1/+\infty = 0$.

Duality for p-modulus. For the $p = 1$ and $p = \infty$ cases, it is helpful to establish lower bounds on modulus by establishing a dual problem. Applying classical Lagrangian duality to the p-modulus problem is difficult in part due to the fact that there are uncountably many inequality constraints. In a sense, the following construction serves as a replacement to the classical dual by providing a family of lower bounds for p-modulus. The largest such lower bound can be characterized as a type of dual problem and, at least in the case of radially symmetric trees, the lower bound is exact.

We begin by defining

$$\Lambda := \left\{ \eta \in \mathbb{R}^E_{\geq 0} : \sum_{e \in S_1} \eta(e) = 1, \sum_{e' \in c(e)} \eta(e') = \eta(e) \, \forall e \in E \right\}.$$

The connection between Λ and modulus comes from the following theorem.

Theorem 3.3.1. *Let $\rho \in \mathbb{R}^E_{\geq 0}$. Then $\rho \in \mathrm{Adm}(\Gamma_\infty)$ if and only if*

$$\sum_{e \in E} \rho(e)\eta(e) \geq 1 \qquad \text{for all } \eta \in \Lambda. \tag{3.5}$$

Before proving Theorem 3.3.1, we establish a few important properties of Λ. First, a simple calculation shows that the set Λ is convex subset of $\mathbb{R}^E_{\geq 0}$. For this, let $\theta \in [0, 1]$ and $\eta_1, \eta_2 \in \Lambda$. Let $\eta = \theta\eta_1 + (1 - \theta)\eta_2$ then clearly $\eta \geq 0$. For any $e \in S_1$

$$\sum_{e \in S_1} (\theta\eta_1 + (1 - \theta)\eta_2)(e) = \theta \sum_{e \in S_1} \eta_1(e) + (1 - \theta) \sum_{e \in S_1} \eta_2(e) = 1$$

and

$$\begin{aligned} \sum_{e' \in c(e)} \eta(e') = \sum_{e' \in c(e)} (\theta\eta_1 + (1 - \theta)\eta_2)(e') &= \theta \sum_{e' \in c(e)} \eta_1(e') + (1 - \theta) \sum_{e' \in c(e)} \eta_2(e') \\ &= \theta\eta_1(e) + (1 - \theta)\eta_2(e) \\ &= \eta(e) \end{aligned}$$

This shows that $\eta \in \Lambda$ and hence it is a convex subset of $\mathbb{R}^E_{\geq 0}$.

Another consequence of the definition is that if $\eta \in \Lambda$, then η assigns unit mass to all shells in the tree.

Lemma 3.3.1. *Let $\eta \in \Lambda$ and let $n \geq 1$. Then*

$$\eta(S_n) = \sum_{e \in S_n} \eta(e) = 1.$$

Proof. The case $n = 1$ is a direct consequence of the definition of Λ. Proceeding by induction, assume the equality is true for some S_n. Then

$$\eta(S_{n+1}) = \sum_{e \in S_n} \sum_{e' \in c(e)} \eta(e') = \sum_{e \in S_n} \eta(e) = 1.$$

□

The following lemma provides a useful interpretation of the partial sums for the series on the left-hand side of (3.5).

Lemma 3.3.2. *Let $\rho \in \mathbb{R}^E_{\geq 0}$ and let $\eta \in \Lambda$. Then, for any $n \geq 1$*

$$\sum_{e \in B_n} \rho(e)\eta(e) = \sum_{e \in S_n} \ell_\rho(\gamma_e)\eta(e),$$

where γ_e is the path descending from the root and terminating after passing through the edge e.

Proof. The equality is trivially true when $n = 1$. Suppose it is true for some $n \geq 1$, then

$$\begin{aligned}
\sum_{e \in B_{n+1}} \rho(e)\eta(e) &= \sum_{e \in B_n} \rho(e)\eta(e) + \sum_{e \in S_{n+1}} \rho(e)\eta(e) \\
&= \sum_{e \in S_n} \ell_\rho(\gamma_e)\eta(e) + \sum_{e \in S_n} \sum_{e' \in c(e)} \rho(e')\eta(e') \\
&= \sum_{e \in S_n} \ell_\rho(\gamma_e) \sum_{e' \in c(e)} \eta(e') + \sum_{e \in S_n} \sum_{e' \in c(e)} \rho(e')\eta(e') \\
&= \sum_{e \in S_n} \sum_{e' \in c(e)} \left(\ell_\rho(\gamma_e) + \rho(e')\right) \eta(e') \\
&= \sum_{e \in S_n} \sum_{e' \in c(e)} \ell_\rho(\gamma_{e'})\eta(e') \\
&= \sum_{e \in S_{n+1}} \ell_\rho(\gamma_e)\eta(e).
\end{aligned}$$

□

Now we are ready to prove the characterization of $\mathrm{Adm}(\Gamma_\infty)$ given in Theorem 3.3.1.

Proof of Theorem 3.3.1. Suppose $\rho \in \mathrm{Adm}(\Gamma_\infty)$. Then for any $\eta \in \Lambda$, Lemma 3.3.2 implies that

$$\sum_{e \in E} \rho(e)\eta(e) = \lim_{n \to \infty} \sum_{e \in B_n} \rho(e)\eta(e) = \lim_{n \to \infty} \sum_{e \in S_n} \ell_\rho(\gamma_e)\eta(e).$$

Since $\gamma_e \in \Gamma_n$ for $e \in S_n$, it follows that $\ell_\rho(\gamma_e) \geq \ell_\rho(\Gamma_n)$. Thus, by Lemma 3.3.1, the sum on the right is a convex combination of values, each of which is bounded below by $\ell_\rho(\Gamma_n)$. Thus, by Lemma 3.2.1,

$$\sum_{e \in E} \rho(e)\eta(e) \geq \lim_{n \to \infty} \ell_\rho(\Gamma_n) = \ell_\rho(\Gamma_\infty) \geq 1.$$

On the other hand, suppose $\rho \notin \mathrm{Adm}(\Gamma_\infty)$. Then, there exists $\gamma \in \Gamma_\infty$ such that $\ell_\rho(\gamma) < 1$. Define η to be the indicator function of γ: $\eta = \mathbf{1}_\gamma$. Note that $\eta \in \Lambda$. Moreover,

$$\sum_{e \in E} \rho(e)\eta(e) = \sum_{e \in \gamma} \rho(e) = \ell_\rho(\gamma) < 1.$$

□

Theorem 3.3.2. *Let $\eta \in \Lambda$, then*

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) \geq \begin{cases} \left(\sum_{e\in E} \sigma(e)^{-\frac{q}{p}}\eta(e)^q\right)^{-\frac{p}{q}} & \text{if } p \in (1,\infty), \\ \left(\sup_{e\in E} \sigma(e)^{-1}\eta(e)\right)^{-1} & \text{if } p = 1, \\ \left(\sum_{e\in E} \sigma(e)^{-1}\eta(e)\right)^{-1} & \text{if } p = \infty. \end{cases}$$

Proof. Let $\rho \in \mathrm{Adm}(\Gamma_\infty)$. First, consider the case $p \in (1,\infty)$. By Theorem 3.3.1 and Hölder's inequality, we have that

$$\begin{aligned} 1 &\leq \sum_{e\in E} \rho(e)\eta(e) = \sum_{e\in E} \sigma(e)^{\frac{1}{p}}\rho(e)\sigma(e)^{-\frac{1}{p}}\eta(e) \\ &\leq \left(\sum_{e\in E} \sigma(e)\rho(e)^p\right)^{\frac{1}{p}} \left(\sum_{e\in E} \sigma(e)^{-\frac{q}{p}}\eta(e)^q\right)^{\frac{1}{q}}, \end{aligned}$$

from which it follows that

$$\mathcal{E}_{p,\sigma}(\rho) \geq \left(\sum_{e\in E} \sigma(e)^{-\frac{q}{p}}\eta(e)^q\right)^{-\frac{p}{q}}.$$

For $p = 1$,

$$1 \leq \sum_{e\in E} \rho(e)\eta(e) = \sum_{e\in E} \sigma(e)\rho(e)\left(\sigma(e)^{-1}\eta(e)\right) \leq \left(\sum_{e\in E} \sigma(e)\rho(e)\right)\left(\sup_{e\in E} \sigma(e)^{-1}\eta(e)\right)$$

And for $p = \infty$,

$$1 \leq \sum_{e\in E} \sigma(e)\rho(e)\left(\sigma(e)^{-1}\eta(e)\right) \leq \left(\sup_{e\in E} \sigma(e)\rho(e)\right)\left(\sum_{e\in E} \sigma(e)^{-1}\eta(e)\right)$$

Taking the infimum over all admissible densities yields the desired inequality.

□

From the Theorem 3.3.2, we have two immediate important consequences.

Corollary 3.3.1. *Suppose* $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ *for* $1 < p < \infty$*, then*

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) \geq \left(\inf_{\eta\in\Lambda} \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \eta(e)^q \right)^{-\frac{p}{q}}$$

Moreover, the equality is attained for a radially symmetric infinite tree and the optimal density is

$$\rho_k^* = \frac{\left(\sigma_k \eta_k^{-1}\right)^{-\frac{q}{p}}}{\sum_{\ell=1}^{\infty} \left(\sigma_k \eta_k^{-1}\right)^{-\frac{q}{p}}}$$

Proof. By the Theorem 3.3.2, for any $\eta \in \Lambda$

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) \geq \left(\sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \eta(e)^q \right)^{-\frac{p}{q}}$$

Taking the supremum of the right-hand side over $\eta \in \Lambda$ and using the monotonicity of the function $x \mapsto x^{-\frac{p}{q}}$, we have

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) \geq \left(\inf_{\eta\in\Lambda} \sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \eta(e)^q \right)^{-\frac{p}{q}}.$$

Now, for a radially symmetric tree, choose $\eta(e) = |S_k|^{-1}$ for all $e \in S_k$ for all k. It is straightforward to verify that $\eta \in \Lambda$. We can write dual energy as

$$\sum_{e\in E} \sigma(e)^{-\frac{q}{p}} \eta(e)^q = \sum_{k=1}^{\infty} \sigma_k^{-\frac{q}{p}} |S_k|^{1-q} = \sum_{k=1}^{\infty} \left(\sigma_k |S_k|\right)^{-\frac{q}{p}}.$$

Using (3.4), the equality holds and the optimal density is

$$\rho_k^* = \frac{\left(\sigma_k \eta_k^{-1}\right)^{-\frac{q}{p}}}{\sum_{\ell=1}^{\infty} \left(\sigma_\ell \eta_\ell^{-1}\right)^{-\frac{q}{p}}}$$

□

Corollary 3.3.2. *Let $G = (V, E, \sigma, o)$ be a radially symmetric infinite tree. Then*

$$\mathrm{Mod}_{1,\sigma}(\Gamma_\infty) \geq \inf_{k\geq 1} \sigma(S_k).$$

Proof. This is a consequence of Theorem 3.3.2 with the choice of η defined as $|S_k|^{-1}$ on each shell S_k. □

Modulus for $p = 1$. By (3.1), the 1-*modulus* of a family of descending paths in a radially symmetric tree is defined as

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} \sigma(e)\rho(e) \\ \text{subject to} \quad & \ell_\rho(\gamma) \geq 1 \qquad \forall\, \gamma \in \Gamma_\infty, \\ & \rho(e) \geq 0 \qquad \forall\, e \in E. \end{aligned}$$

In this case, the p-energy is not strictly convex, so the proof used when $1 < p < \infty$, which relies on the existence of unique optimal densities, does not apply. For a finite graph, 1-modulus of a connecting family of two distinct vertices is equivalent to the minimum cut problem (see, [2]). Analogously, the 1-modulus in the radially symmetric tree is the infimum over cuts for descending paths.

As a corresponding definition of a cut set for the infinite tree, we shall call a subset of edges $C \subseteq E$ a cut for Γ_∞ if $|\gamma \cap C| > 0$ for every $\gamma \in \Gamma_\infty$.

Theorem 3.3.3. *Let $G = (V, E, \sigma, o)$ be a radially symmetric tree. Then*

$$\mathrm{Mod}_{1,\sigma}(\Gamma_\infty) = \inf\{\sigma(C) : C \textit{ is a cut of } \Gamma_\infty\}.$$

Proof. Let $C \subseteq E$ be a cut set for Γ_∞. Define a density ρ on E as follows

$$\rho(e) = \begin{cases} 1 & \text{if } e \in C, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\rho \in \mathrm{Adm}(\Gamma_\infty)$ since every descending path crosses at least one edge of C. Moreover,

$$\mathrm{Mod}_{1,\sigma}(\Gamma_\infty) \le \mathcal{E}_{1,\sigma}(\rho) = \sum_{e\in E} \sigma(e)\rho(e) = \sum_{e\in C} \sigma(e) = \sigma(C).$$

Taking the infimum shows that

$$\mathrm{Mod}_{1,\sigma}(\Gamma_\infty) \le \inf\{\sigma(C) : C \text{ is a cut of } \Gamma_\infty\}.$$

On the other hand, the shell S_k is a cut for the family of descending paths for all k. So, as a consequence of Corollary 3.3.2, we have

$$\mathrm{Mod}_{1,\sigma}(\Gamma_\infty) \ge \inf_{k\ge 1} \sigma(S_k) \ge \inf\{\sigma(C) : C \text{ is a cut for } \Gamma_\infty\}.$$

□

Modulus for $p = \infty$. By (3.1), the ∞-*modulus* of a family of descending paths in a proper infinite tree defined as

$$\begin{aligned} \text{minimize} \quad & \sup_{e\in E} \sigma(e)\rho(e) \\ \text{subject to} \quad & \ell_\rho(\gamma) \ge 1 \qquad \forall\, \gamma \in \Gamma_\infty, \\ & \rho(e) \ge 0 \qquad \forall\, e \in E. \end{aligned}$$

As with the 1-energy, the ∞-energy is not strictly convex, meaning that an optimal density (if it exists) need not be unique. To establish context, we recall that in the case of unweighted finite graph, ∞-modulus of a connecting family of two distinct vertices is equal to the reciprocal shortest *hop length* (see, [2]). Analogously, the ∞-modulus in the radially symmetric tree is the reciprocal of the weighted length of the family of descending paths.

Theorem 3.3.4. *Let $G = (V, E, \sigma, o)$ be a radially symmetric tree. Then*

$$\mathrm{Mod}_{\infty,\sigma}(\Gamma_\infty) = \begin{cases} \frac{1}{\ell_{\sigma^{-1}}(\Gamma_\infty)} & \text{if } \ell_{\sigma^{-1}}(\Gamma_\infty) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\eta \in \Lambda$. From Lemma 3.3.2 and 3.2.1, it follows that

$$\sum_{e\in E} \sigma(e)^{-1}\eta(e) = \lim_{n\to\infty} \sum_{e\in B_n} \sigma(e)^{-1}\eta(e) = \lim_{n\to\infty} \sum_{e\in S_n} \ell_{\sigma^{-1}}(\gamma_e)\eta(e) = \lim_{n\to\infty} \ell_{\sigma^{-1}}(\Gamma_n) = \ell_{\sigma^{-1}}(\Gamma_\infty).$$

First, suppose that $\ell_{\sigma^{-1}}(\Gamma_\infty) < 0$. Then, Theorem 3.3.2 shows

$$\mathrm{Mod}_{\infty,\sigma}(\Gamma_\infty) \geq \left(\sum_{e\in E} \sigma(e)^{-1}\eta(e)\right)^{-1} \geq (\ell_{\sigma^{-1}}(\Gamma_\infty))^{-1} \tag{3.6}$$

Define the density

$$\rho(e) = \sigma(e)^{-1}(\ell_{\sigma^{-1}}(\Gamma_\infty))^{-1}$$

for all $e \in E$. The density ρ is admissible since for $\gamma \in \Gamma_\infty$,

$$\ell_\rho(\gamma) = \sum_{e\in\gamma} \rho(e) = \frac{1}{\ell_{\sigma^{-1}}(\Gamma_\infty)} \sum_{e\in\gamma} \sigma(e)^{-1} = 1.$$

Now,

$$\sigma(e)\rho(e) = \frac{1}{\ell_{\sigma^{-1}}(\Gamma_\infty)}.$$

and

$$\sup_{e\in E} \sigma(e)\rho(e) = \frac{1}{\ell_{\sigma^{-1}}(\Gamma_\infty)}$$

This shows that

$$\mathrm{Mod}_{\infty,\sigma}(\Gamma_\infty) \leq \sup_{e\in E} \sigma(e)\rho(e) = \frac{1}{\ell_{\sigma^{-1}}(\Gamma_\infty)}. \tag{3.7}$$

Hence, by (3.6) and (3.7)

$$\mathrm{Mod}_{\infty,\sigma}(\Gamma_\infty) = \frac{1}{\ell_{\sigma^{-1}}(\Gamma_\infty)}.$$

On the other hand, suppose $\ell_{\sigma^{-1}}(\Gamma_\infty) = \infty$. Let $\epsilon > 0$ and consider the density $\rho(e) = \epsilon\sigma(e)^{-1}$. Since

$$\ell_\rho(\Gamma_\infty) = \epsilon\ell_{\sigma^{-1}}(\Gamma_\infty) = \infty,$$

this density is admissible. Thus,

$$\mathrm{Mod}_{\infty,\sigma}(\Gamma_\infty) \leq \mathcal{E}_{\infty,\sigma}(\rho) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the modulus is zero. □

A consequence of Theorem 3.3.4 is that the ∞-modulus in the unweighted radially symmetric tree is zero. Indeed, setting $\sigma \equiv 1$ for all $e \in E$, gives $\ell_{\sigma^{-1}}(\Gamma_\infty) = \infty$.

3.4 Optimal densities

When $1 \leq n < \infty$, the standard theory of p-modulus shows that there exists a unique optimal density $\rho^* \in \mathrm{Adm}(\Gamma_n)$ satisfying $\mathcal{E}_{p,\sigma}(\rho) = \mathrm{Mod}_{p,\sigma}(\Gamma_n)$ (see, e.g., [4]). It is natural to wonder, then, whether or not an optimal density exists for $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$.

First, we consider an example demonstrating that an optimal density need not exist. Consider the infinite 1-ary tree with $\sigma \equiv 1$ and assume $1 < p < \infty$. It is a straightforward exercise to show that the unique optimal density ρ_n^* on for $\mathrm{Mod}_{p,\sigma}(\Gamma_n)$ takes the value $\frac{1}{n}$ on all edges of B_n and zero elsewhere. Thus,

$$\mathrm{Mod}_{p,\sigma}(\Gamma_n) = \frac{n}{n^p} = n^{-\frac{p}{q}}.$$

By Theorem 3.2.1, then,

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) = 0.$$

The only density satisfying $\mathcal{E}_{p,\sigma}(\rho) = 0$ is the zero density, but this density is clearly not admissible for Γ_∞. Thus, we know of at least one case in which an optimal density does not exist. In fact, the existence or nonexistence of an optimal density is closely related to the convergence or divergence of the infinite sum in (3.4).

Theorem 3.4.1. *An optimal density for* $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$ *exists if and only if* $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$.

Proof. As already discussed, if the modulus is zero, then no optimal density can exist. Suppose, on the other hand, that the modulus is positive. Then the series

$$\sum_{k=1}^{\infty} (\sigma_k |S_k|)^{-\frac{q}{p}}$$

converges.

Define a radially symmetric density ρ so that on S_k it takes the value

$$\rho_k := \frac{(\sigma_k|S_k|)^{-\frac{q}{p}}}{\sum\limits_{\ell=1}^{\infty} (\sigma_\ell|S_\ell|)^{-\frac{q}{p}}}. \tag{3.8}$$

Note that ρ is admissible, since $\sum\limits_{k=1}^{\infty} \rho_k = 1$. Moreover,

$$\begin{aligned} \mathcal{E}_{p,\sigma}(\rho) &= \sum_{k=1}^{\infty} \frac{\sigma_k|S_k|\,(\sigma_k|S_k|)^{-q}}{\left(\sum\limits_{\ell=1}^{\infty}(\sigma_\ell|S_\ell|)^{-\frac{q}{p}}\right)^{p}} = \frac{\sum\limits_{k=1}^{\infty}(\sigma_k|S_k|)^{-\frac{q}{p}}}{\left(\sum\limits_{\ell=1}^{\infty}(\sigma_\ell|S_\ell|)^{-\frac{q}{p}}\right)^{p}} \\ &= \left(\sum_{k=1}^{\infty}(\sigma_k|S_k|)^{-\frac{q}{p}}\right)^{-\frac{p}{q}} = \mathrm{Mod}_{p,\sigma}(\Gamma_\infty), \end{aligned} \tag{3.9}$$

showing that ρ is optimal. □

The result can be summarized as follows. If the quantities $\sigma_k|S_k|$ grow sufficiently fast, then $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ and an optimal density exists. Otherwise, $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) = 0$ and there is no optimal density.

An interesting consequence of this theorem is that the p-modulus of a radially symmetric tree can either be zero or positive for a given $p > 1$. In Chapter 4, we revisit this fact and show the existence of a critical value for the parameter p which determines the boundary between positive and zero modulus for a given tree.

3.5 Dependence on p and σ

In the present section, we will explore the dependence of *p-modulus* on the parameters p and σ. For context, we recall that if Γ is a family of connecting paths between two vertices in a finite graph, then the function $p \mapsto \mathrm{Mod}_{p,\sigma}(\Gamma)$ is continuous and nonincreasing. The p-modulus as a function of the weight parameter σ is concave and Lipschitz continuous (see, [2, 5]). In this section, we extend these properties to infinite radially symmetric trees

3.5.1 Dependence on p

In the previous section, we formulated $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$ as the reciprocal of an infinite sum. Therefore, $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$ is either zero or some positive finite value. In this section, we assume that for a given weight function $\sigma : E \to (0,\infty)$, the corresponding $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$. The following property is the extension of the theory on finite graphs.

Theorem 3.5.1. *Suppose* $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ *for all* p *in a neighborhood of* $p_0 \in (1,\infty)$*, then the function* $p \mapsto \mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$ *is continuous at* p_0*.*

Proof. Suppose the modulus is positive in a closed interval $[p_0-\epsilon, p_0+\epsilon] \subset (1,\infty)$ with $\epsilon > 0$ and let p belong to this interval. The fact that $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ implies that the infinite sum converges. Let q_0 and q be the Hölder conjugate exponents of p_0 and p respectively. The limit test implies that the terms $\sigma_k|S_k|$ diverge to $+\infty$ as $k \to \infty$. Thus, there exists a constant $c > 0$ such that $\sigma_k|S_k| \geq c$ for all k. From (3.4), we have

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)^{-\frac{1}{p-1}} = \sum_{k=1}^{\infty} (\sigma_k|S_k|)^{-\frac{1}{p-1}} = c^{-\frac{1}{p-1}} \sum_{k=1}^{\infty} \left(\frac{\sigma_k|S_k|}{c}\right)^{-\frac{1}{p-1}}. \tag{3.10}$$

In order to pass to the limit as $p \to p_0$, we verify the conditions of Tannery's theorem [15].

Since $\sigma_k|S_k|/c \geq 1$ for all k, we have that

$$\left(\frac{\sigma_k|S_k|}{c}\right)^{-\frac{1}{p-1}} \leq \left(\frac{\sigma_k|S_k|}{c}\right)^{-\frac{1}{p_0+\epsilon-1}}.$$

By assumption,

$$\sum_{k=1}^{\infty}\left(\frac{\sigma_k|S_k|}{c}\right)^{-\frac{1}{p_0+\epsilon-1}} = c^{\frac{1}{p_0+\epsilon-1}}\operatorname{Mod}_{p_0+\epsilon,\sigma}(\Gamma_\infty)^{-\frac{1}{p_0+\epsilon-1}}.$$

Applying Tannery's theorem, we may pass to the limit in (3.10), obtaining

$$\lim_{p\to p_0}\operatorname{Mod}_{p,\sigma}(\Gamma_\infty)^{-\frac{1}{p-1}} = c^{-\frac{1}{p_0-1}}\sum_{k=1}^{\infty}\left(\frac{\sigma_k|S_k|}{c}\right)^{-\frac{1}{p_0-1}} = \operatorname{Mod}_{p_0,\sigma}(\Gamma_\infty)^{-\frac{1}{p_0-1}}.$$

Using the positivity of modulus near p_0 and the continuity of the logarithm, it follows that

$$\lim_{p\to p_0}\left(-\frac{1}{p-1}\log\operatorname{Mod}_{p,\sigma}(\Gamma_\infty)\right) = -\frac{1}{p_0-1}\log\operatorname{Mod}_{p_0,\sigma}(\Gamma_\infty),$$

from which it follows that

$$\lim_{p\to p_0}\operatorname{Mod}_{p,\sigma}(\Gamma_\infty) = \operatorname{Mod}_{p_0,\sigma}(\Gamma_\infty).$$

□

The following lemma extends the property of monotonicity in parameter p in an infinite graph as well.

Lemma 3.5.1. *Let $1 < p \le p' < \infty$ and $\operatorname{Mod}_{p,\sigma}(\Gamma_\infty) > 0$. Then*

$$\operatorname{Mod}_{p',\sigma}(\Gamma_\infty) \le \operatorname{Mod}_{p,\sigma}(\Gamma_\infty)$$

Proof. Let $1 < p \le p' < \infty$ and let q and q' be their respective conjugate exponents. Then

$$\sum_{k=1}^{\infty}(\sigma_k|S_k|)^{-\frac{q}{p}} = \sum_{k=1}^{\infty}(\sigma_k|S_k|)^{1-q} \le \sum_{k=1}^{\infty}(\sigma_k|S_k|)^{1-q'} = \sum_{k=1}^{\infty}(\sigma_k|S_k|)^{-\frac{q'}{p'}}.$$

which gives

$$\left(\sum_{k=1}^{\infty}(\sigma_k|S_k|)^{-\frac{q'}{p'}}\right)^{-\frac{p}{q}} \le \left(\sum_{k=1}^{\infty}(\sigma_k|S_k|)^{-\frac{q}{p}}\right)^{-\frac{p}{q}}.$$

This completes the proof. □

Theorem 3.5.2. *Suppose* $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ *for all* p *in a neighborhood of* $p_0 \in (1, \infty)$ *and let* $e \in E$*, then the function* $p \mapsto \rho_p^*(e)$*, is continuous at* p_0*, where* ρ_p^* *is the unique optimal density for* p*-modulus.*

Proof. Suppose that $e \in S_k$. The formula in (3.8) shows that

$$\rho_p^*(e) = \rho_{p,k}^* := \left(\frac{\sigma_k |S_k|}{\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)}\right)^{-\frac{1}{p-1}}.$$

Continuity follows from Theorem 3.5.1 and the fact that the function $p \mapsto x^{-\frac{1}{p-1}}$ is continuous on $(1, \infty)$ for positive x. □

For the finite graph, the ∞-*modulus* of the connecting family of two distinct vertices is the reciprocal of the shortest path between two vertices (see [2]). The following theorem generalize the ∞-modulus on a infinite tree that ∞-*modulus* of the family of descending paths is zero which the reciprocal of infinite length descending paths.

Theorem 3.5.3. *In the limit,*

$$\lim_{p\to\infty} \mathrm{Mod}_{p,1}(\Gamma_\infty)^{\frac{1}{p}} = 0.$$

Proof. Let $\epsilon > 0$ and choose $n > 2/\epsilon$. Define the density ρ to take the value $1/n$ on B_n and 0 elsewhere. Since every path in Γ_∞ must use n edges from B_n, this density is admissible, so for any $1 \le p < \infty$,

$$\mathrm{Mod}_{p,1}(\Gamma_\infty)^{\frac{1}{p}} \le \mathcal{E}_{p,1}(\rho)^{\frac{1}{p}} = \frac{|B_n|^{\frac{1}{p}}}{n}.$$

By choosing p sufficiently large, we can ensure that the numerator does not exceed 2, which shows that

$$0 \le \mathrm{Mod}_{p,1}(\Gamma_\infty)^{\frac{1}{p}} \le \frac{2}{n} < \epsilon$$

for all p sufficiently large. □

3.5.2 Dependence on σ

Now we consider the dependence of modulus on the edge weights σ, establishing some analogs to the finite graph case.

Theorem 3.5.4. *Let $1 < p < \infty$ and suppose $\text{Mod}_{p,\sigma}(\Gamma_\infty) > 0$. Then the function $\sigma \mapsto \text{Mod}_{p,\sigma}(\Gamma_\infty)$ is differentiable with respect to each σ_ℓ. Moreover,*

$$\frac{\partial}{\partial \sigma_\ell} \text{Mod}_{p,\sigma}(\Gamma_\infty) = |S_\ell| \left(\frac{\sigma_\ell |S_\ell|}{\text{Mod}_{p,\sigma}(\Gamma_\infty)} \right)^{-\frac{p}{p-1}} = |S_\ell| (\rho^*_\ell)^p,$$

where ρ^* is the unique optimal density for $\text{Mod}_{p,\sigma}(\Gamma_\infty)$.

Proof. Suppose $\text{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ and let ℓ be given. Note that

$$\sum_{k=1}^{\infty} (\sigma_k |S_k|)^{-\frac{q}{p}} = (\sigma_\ell |S_\ell|)^{-\frac{q}{p}} + \sum_{k \neq \ell} (\sigma_k |S_k|)^{-\frac{q}{p}}.$$

Since the second term on the right is independent of σ_ℓ, it follows that

$$\begin{aligned} \frac{\partial}{\partial \sigma_\ell} \left(\text{Mod}_{p,\sigma}(\Gamma_\infty)^{-\frac{1}{p-1}} \right) &= \frac{\partial}{\partial \sigma_\ell} \left((\sigma_\ell |S_\ell|)^{-\frac{1}{p-1}} \right) \\ &= -\frac{1}{p-1} |S_\ell| \left(\sigma_\ell |S_\ell| \right)^{-\frac{p}{p-1}}, \end{aligned}$$

and so

$$\begin{aligned} \frac{\partial}{\partial \sigma_\ell} \text{Mod}_{p,\sigma}(\Gamma_\infty) &= \frac{\partial}{\partial \sigma_\ell} \left[\left(\text{Mod}_{p,\sigma}(\Gamma_\infty)^{-\frac{1}{p-1}} \right)^{-(p-1)} \right] \\ &= -(p-1) \left(\text{Mod}_{p,\sigma}(\Gamma_\infty)^{-\frac{1}{p-1}} \right)^{-p} \frac{\partial}{\partial \sigma_\ell} \text{Mod}_{p,\sigma}(\Gamma_\infty)^{-\frac{1}{p-1}} \\ &= |S_\ell| \left(\frac{\sigma_\ell |S_\ell|}{\text{Mod}_{p,\sigma}(\Gamma_\infty)} \right)^{-\frac{p}{p-1}} = |S_\ell| (\rho^*_\ell)^p. \end{aligned}$$

□

Theorem 3.5.5. *Let $1 < p < \infty$ and suppose $\text{Mod}_{p,\sigma}(\Gamma_\infty) > 0$. The function $\sigma \mapsto \rho^*_\sigma(e)$ is differentiable with respect to each σ_ℓ for each e.*

Proof. Using (3.8) and (3.9), the optimal density on generation k can be expressed as

$$\rho_k^* = \frac{(\sigma_k|S_k|)^{\frac{1}{1-p}}}{\operatorname{Mod}_{p,\sigma}(\Gamma_\infty)^{\frac{1}{1-p}}}. \tag{3.11}$$

Since the modulus is assumed to be positive, Theorem 3.5.4 implies that ρ_k^* is a ratio of functions both of which are differentiable in σ_ℓ.

Now, rewrite (3.11) as

$$\rho_k^{*(1-p)} \operatorname{Mod}_{p,\sigma}(\Gamma_\infty) = \sigma_k|S_k|.$$

Differentiating both sides with respect to σ_ℓ yields

$$(1-p)\rho_k^{*-p} \operatorname{Mod}_{p,\sigma}(\Gamma_\infty)\frac{\partial \rho_k^*}{\partial \sigma_\ell} + \rho_k^{*(1-p)}\frac{\partial}{\partial \sigma_\ell} \operatorname{Mod}_{p,\sigma}(\Gamma_\infty) = \delta_{k\ell}|S_k|$$
$$(1-p)\rho_k^{*-p} \operatorname{Mod}_{p,\sigma}(\Gamma_\infty)\frac{\partial \rho_k^*}{\partial \sigma_\ell} + \rho_k^{*(1-p)}|S_\ell|\rho_\ell^{*p} = \delta_{k\ell}|S_k|.$$

So,

$$\frac{\partial \rho_k^*}{\partial \sigma_\ell} = \frac{\delta_{k\ell}|S_k|\rho_k^{*p} - |S_\ell|\rho_k^*\rho_\ell^{*p}}{(1-p)\operatorname{Mod}_{p,\sigma}(\Gamma_\infty)}.$$

□

Theorem 3.5.6. *Let* $1 < p < \infty$ *and suppose* $\operatorname{Mod}_{p,\sigma}(\Gamma_\infty) > 0$. *The function* $\sigma \mapsto \operatorname{Mod}_{p,\sigma}(\Gamma_\infty)$ *is concave.*

For the proof of this theorem, we prove the two lemmas first. The following lemma shows that if the σ is scaled by some positive number then p-modulus of also scaled by same amount.

Lemma 3.5.2. *Suppose* $\operatorname{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ *for some* $1 < p < \infty$ *and let* $\theta > 0$. *Then*

$$\operatorname{Mod}_{p,\theta\sigma}(\Gamma_\infty) = \theta \operatorname{Mod}_{p,\sigma}(\Gamma_\infty)$$

Proof. The proof is a straightforward calculation using (3.4):

$$\mathrm{Mod}_{p,\theta\sigma}(\Gamma_\infty) = \left[\sum_{k=1}^{\infty}(\theta\sigma_k|S_k|)^{\frac{1}{1-p}}\right]^{1-p} = \theta\left[\sum_{k=1}^{\infty}(\sigma_k|S_k|)^{\frac{1}{1-p}}\right]^{1-p} = \theta\,\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$$

□

Lemma 3.5.3. *Let* $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ *for* $1 < p < \infty$. *Let* $\sigma^1, \sigma^2 : E \to \mathbb{R}_{>0}$. *Then*

$$\mathrm{Mod}_{p,\sigma^1+\sigma^2}(\Gamma_\infty) \geq \mathrm{Mod}_{p,\sigma_1}(\Gamma_\infty) + \mathrm{Mod}_{p,\sigma_2}(\Gamma_\infty).$$

Proof. Let $\sigma = \sigma^1 + \sigma^2$ and ρ^* be the optimal density for σ. Let ρ^1 and ρ^2 be the optimal densities for σ^1 and σ^2 respectively. Now for $1 < p < \infty$, by (3.4)

$$\begin{aligned}
\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) &= \mathrm{Mod}_{p,\sigma^1+\sigma^2}(\Gamma_\infty) \\
&= \sum_{k=1}^{\infty}(\sigma_k^1 + \sigma_k^2)\rho^{*p}|S_k| \\
&= \sum_{k=1}^{\infty}\sigma_k^1\rho^{*p}|S_k| + \sum_{k=1}^{\infty}\sigma_k^2\rho^{*p}|S_k| \\
&\geq \sum_{k=1}^{\infty}\sigma_k^1(\rho_k^1)^p|S_k| + \sum_{k=1}^{\infty}\sigma_k^2(\rho_k^2)^p|S_k| \\
&= \mathrm{Mod}_{p,\sigma^1}(\Gamma_\infty) + \mathrm{Mod}_{p,\sigma^2}(\Gamma_\infty).
\end{aligned}$$

□

The proof of Theorem 3.5.6 follows from Lemmas 3.5.2 and 3.5.3.

Proof of Theorem 3.5.6. Suppose that $\mathrm{Mod}_{p,\sigma^1}(\Gamma_\infty) > 0$ and $\mathrm{Mod}_{p,\sigma^2}(\Gamma\infty) > 0$ and let $\theta \in [0,1]$. Then

$$\begin{aligned}
\mathrm{Mod}_{p,\theta\sigma^1+(1-\theta)\sigma^2}(\Gamma_\infty) &\geq \mathrm{Mod}_{p,\theta\sigma^1}(\Gamma_\infty) + \mathrm{Mod}_{(p,1-\theta)\sigma^2}(\Gamma_\infty) \\
&= \theta\,\mathrm{Mod}_{p,\sigma^1}(\Gamma_\infty) + (1-\theta)\,\mathrm{Mod}_{p,\sigma^2}(\Gamma_\infty).
\end{aligned}$$

□

In the infinite graph case, there are several possible analogs to the Lipschitz continuity property on finite graphs, each depending on restricting the weights σ to some class of weights. Here we study one particular class. We say $\sigma \in \mathbb{R}^E_{>0}$ is uniformly elliptic if there exists a constant $\alpha_1 > 0$ such that $\sigma \geq \alpha_1$ and say that σ is *bounded* if there exists a constant $\alpha_2 > 0$ such that $\sigma \leq \alpha_2$. Just as the assumption of bounded uniform ellipticity plays an important role in PDEs, the corresponding concept allows us to prove strong theorems for p-modulus on infinite trees.

In the general case, a positive p-modulus for a weighted radially symmetric trees does not imply a positive p-modulus on the unweighted version of the tree and vice versa. The details with examples is discussed in Chapter 4. The following result is a consequence of Lemma 3.5.4 which states that positivity of p-modulus on a weighted tree is equivalent to positivity of p-modulus on the unweighted tree if the weights are bounded and uniformly elliptic.

Lemma 3.5.4. *Suppose σ is a radially symmetric weight that is bounded and uniformly elliptic with $0 < \alpha_1 \leq \sigma \leq \alpha_2$. Then*

$$\alpha_1 \operatorname{Mod}_{p,1}(\Gamma_\infty) \leq \operatorname{Mod}_{p,\sigma}(\Gamma_\infty) \leq \alpha_2 \operatorname{Mod}_{p,1}(\Gamma_\infty). \tag{3.12}$$

Proof. This is a consequence of the assumptions on σ and the monotonicity of the functions $x \mapsto x^{-q/p}$ and $x \mapsto x^{-p/q}$. □

Corollary 3.5.1. *Suppose σ is a radially symmetric weight that is bounded and uniformly elliptic, then $\operatorname{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ if and only if $\operatorname{Mod}_{p,1}(\Gamma_\infty) > 0$.*

Theorem 3.5.7. *Let $\alpha_1, \alpha_2 > 0$ and let Σ be the set of all bounded elliptic weights of the form*

$$\Sigma = \left\{\sigma \in \mathbb{R}^E_{>0} : \alpha_1 \leq \sigma \leq \alpha_2\right\}.$$

Suppose $\mathrm{Mod}_{p,1}(\Gamma_\infty) > 0$ *for some* $1 < p < \infty$. *Then the function* $F : \sigma \to \mathbb{R}$ *defined as* $F(\sigma) = \mathrm{Mod}_{p,\sigma}(\Gamma)$ *is Lipschitz continuous on* Σ *with respect to the* ∞*-norn.*

Proof. By Corollary 3.5.1 and Theorem 3.4.1 there exist optimal ρ^* and $\hat{\rho}^*$ for $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$ and $\mathrm{Mod}_{p,\hat{\sigma}}(\Gamma_\infty)$ respectively. Note that, for any $N \geq 1$,

$$\begin{aligned}
\sum_{k=1}^{N} \hat{\sigma}_k |S_k| (\rho_k^*)^p &= \sum_{k=1}^{N} \sigma_k |S_k| (\rho_k^*)^p + \sum_{k=1}^{N} (\hat{\sigma}_k - \sigma_k) |S_k| (\rho_k^*)^p \\
&\leq \sum_{k=1}^{N} \sigma_k |S_k| (\rho_k^*)^p + \sum_{k=1}^{N} \frac{|\hat{\sigma}_k - \sigma_k|}{\sigma_k} \sigma_k |S_k| (\rho_k^*)^p \\
&\leq \sum_{k=1}^{\infty} \sigma_k |S_k| (\rho_k^*)^p + \frac{1}{\alpha_1} \sup_k |\hat{\sigma}_k - \sigma_k| \sum_{k=1}^{\infty} \sigma_k |S_k| (\rho_k^*)^p \\
&\leq \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) + \frac{1}{\alpha_1} \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) \|\hat{\sigma} - \sigma\|_\infty \\
&\leq \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) + \frac{\alpha_2}{\alpha_1} \mathrm{Mod}_{p,1}(\Gamma_\infty) \|\hat{\sigma} - \sigma\|_\infty.
\end{aligned}$$

Taking the limit as $N \to \infty$ shows that

$$\mathrm{Mod}_{p,\hat{\sigma}}(\Gamma_\infty) - \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) \leq \frac{\alpha_2}{\alpha_1} \mathrm{Mod}_{p,1}(\Gamma_\infty) \|\hat{\sigma} - \sigma\|_\infty.$$

Repeating the same argument with σ and $\hat{\sigma}$ interchanged establishes the theorem. □

The final two theorems of this section give two additional useful consequences of assuming bounded uniformly elliptic weights.

Theorem 3.5.8. *Suppose* σ *is a radially symmetric weight that is bounded and uniformly elliptic, then* $\mathrm{Mod}_{1,\sigma}(\Gamma_\infty) > 0$.

Proof. This is a consequence of Corollary 3.5.1 and the following lemma. □

For finite graphs 1-*modulus* of connecting family of walks is the min-cut (see, Theorem 2.0.2). In the case of an unweighted proper tree, each shell S_k is a cut for the family of descending paths. As the following lemma shows, the first shell, that is edges incident on the root is a minimum cut for the family of descending paths.

Lemma 3.5.5. *The* 1*-modulus satisfies*

$$\mathrm{Mod}_{1,1}(\Gamma_\infty) = |S_1|.$$

Proof. First, note that the density ρ_1, which assigns the value 1 to every edge in S_1 and 0 to every other edge, is admissible. So

$$\mathrm{Mod}_{1,1}(\Gamma_\infty) \leq \mathcal{E}_{1,1}(\rho_1) = |S_1|.$$

On the other hand, suppose ρ is admissible. Let $\{e_1, e_2, e_3, \ldots, e_r\}$ be an enumeration of S_1 with $r = |S_1|$, and let γ_i be an arbitrary path in Γ_∞ that passes through e_i for $i = 1, 2, 3, \ldots, r$. Note that the paths $\{\gamma_i\}$ are pairwise disjoint. Thus,

$$\mathcal{E}_{1,1}(\rho) = \sum_{e\in E} \rho(e) \geq \sum_{e\in \bigcup\limits_{i=1}^{r} \gamma_i} \rho(e) = \sum_{i=1}^{r} \sum_{e\in\gamma_i} \rho(e) \geq r = |S_1|.$$

Since this is true from an arbitrary admissible ρ, it follows that $\mathrm{Mod}_{1,1}(\Gamma_\infty) \geq |S_1|$. Since the opposite inequality was already established, equality holds. □

Theorem 3.5.9. *Suppose σ is a radially symmetric weight that is bounded and uniformly elliptic, then*

$$\lim_{p\to\infty} \mathrm{Mod}_{p,\sigma}(\Gamma_\infty)^{\frac{1}{p}} = 0 = \mathrm{Mod}_{\infty,\sigma}(\Gamma_\infty).$$

Proof. That the limit is zero is a consequence of Theorem 3.5.3 and Lemma 3.5.4. □

3.6 Random walks and p-modulus on radially symmetric infinite trees

There is a well-known and established relationship between an electrical network and a random walk on a graph (see, for example, [8, 13]). In this section, we review the connection between a random walk and modulus in an infinite tree, focusing on radially symmetric trees. We begin by reviewing the connection between random walks and effective resistance.

As discussed in the Section 1.2, imagine the tree as an infinite resistor network with resistor conductances given by σ on each edge. Suppose we connect a one-volt battery between the root o and all the nodes in the shell S_n so that the voltage at o is 1 and at S_n is 0. Then by Kirchoff's Law and Ohm's law, current will flow along the edges and establish certain voltage potentials at the other vertices in the graph. The voltage at any vertex v has an interpretation in the context of a weighted random walk. In a weighted random walk, the probability of moving from v to one of its neighbors is proportional to the value of σ on the edge connecting the two. It can be shown that the voltage at v is equal to the probability that a weighted random walk starting from v visits the root o before it visits any node in S_n. Taking the limit as $N \to \infty$ the limiting voltages are all 1 if and only if the limiting probabilities are all 1, which is the same thing as saying that on the infinite tree, the probability of visiting the root from any vertex is 1. That is, the random walk is recurrent. Otherwise the random walk is transient (see [8]).

To aid in understanding effective conductance (or effective resistance) on an proper infinite rooted tree, consider each ball B_n as a finite subgraph of the infinite tree G. Then the sequence $\{B_n\}_{n=1}^{\infty}$ satisfies $B_n \subseteq B_{n+1}$ and $G = \bigcup\limits_n B_n$. For any $n \geq 1$, let $B_n^c = G \setminus B_n$. If $\mathbb{P}(o \to B_n^c)$ is the probability of a random walk staring from o reaching B_n^c before returning to o, then the limit $\lim\limits_{n\to\infty} \mathbb{P}(o \to B_n^c)$ is the probability of never returning to o in the infinite graph G. This is called the *escape probability* from o. The escape probability is positive if and only if the random walk on G is transient. The effective conductance on the infinite tree is closely related to this escape probability (see, [8]). Let $C_{\text{eff}}(o, B_n^c)$ be the effective conductance between o and the vertices in B_n^c. Then the effective conductance of G is defined as the limit $C_{\text{eff}}(G) = \lim_{n\to\infty} C_{\text{eff}}(o, B_n^c)$. The following theorem [8] shows the relation between effective conductance and random walks in an infinite graph.

Theorem 3.6.1. *A random walk on an infinite connected graph is transient if and only if the effective conductance from any of its vertices to infinity is positive.*

By Theorem 2.0.3, the effective conductance $C_{\text{eff}}(o, B_n^c)$ is $\text{Mod}_{2,\sigma}(\Gamma_n)$, the modulus of the family of descending paths from the root on the truncated (finite) rooted tree. In the case of the infinite tree, letting $n \to \infty$ and using Theorem 3.2.1 shows that the effective conductance on the infinite tree is $\text{Mod}_{2,\sigma}(\Gamma_\infty)$. So, the following corollary relates the modulus of Γ_∞ in an infinite tree with a random walk on this tree.

Corollary 3.6.1. *A random walk in an infinite tree G is transient if and only if* $\text{Mod}_{2,\sigma}(\Gamma_\infty) > 0$.

An infinite unweighted tree can be thought as electrical network with unit resistances (see [8, 18]). A random walk is transient or recurrent depending on how the number of children in each generation grows. By the (3.4) of the Chapter 3, we have a direct consequence which can be also seen in [13].

Corollary 3.6.2. *The random walk on an unweighted infinite tree is transient if and only if* $\sum_{k=1}^{\infty} \frac{1}{|S_k|} < \infty$.

From this corollary, we can conclude that a random walk in a radially symmetric tree whose edges each have more than one child is always transient.

In general, a transient random walk in an unweighted proper infinite tree does not imply that a weighted random walk will also be transient. However, in the case of bounded and uniformly elliptic weights, this connection can be made. Note that the two p-moduli are equivalent in this case (see Theoerem 3.5.4). In particular case when $p = 2$, we have

$$\alpha_1 \text{Mod}_{2,1}(\Gamma_\infty) \le \text{Mod}_{2,\sigma}(\Gamma_\infty) \le \alpha_2 \text{Mod}_{2,1}(\Gamma_\infty).$$

This shows that a random walk in a weighted radially symmetric tree with bounded uniformly elliptic weights is transient if and only if the random walk in the corresponding unweighted tree is transient.

3.7 Application of modulus

In this section, we discuss an application of p-modulus to define metrics on the family Γ_∞ of descending paths in an infinite tree. For this, we briefly review metrics derived from p-modulus on a finite graph. For a given connected finite graph $G = (V, E, \sigma)$, and distinct vertices s and t, we consider the connecting family $\Gamma = \Gamma(s,t)$ of walks between s and t. The reciprocals of modulus

$$\mathrm{Mod}_{\infty,1}(\Gamma(s,t))^{-1}, \quad \mathrm{Mod}_{2,\sigma}(\Gamma(s,t))^{-1}, \quad \text{and} \quad \mathrm{Mod}_{1,\sigma}(\Gamma(s,t))^{-1},$$

(considered as functions of s and t) are equivalent to well-known graph metrics. The case $\mathrm{Mod}_{\infty,\sigma}(\Gamma(s,t))^{-1}$ produces the shortest path metric. The case $\mathrm{Mod}_{1,\sigma}(\Gamma(s,t))^{-1}$ is equivalent to the reciprocal of the min-cut (st-cut), which is an ultra-metric [5]. As discussed earlier, $\mathrm{Mod}_{2,\sigma}(\Gamma(s,t))^{-1}$, is the effective resistance between s and t with edge conductances σ. The fact that effective resistance is a metric has been proved using current flow, and commute times [14]. The following theorem provides another proof that effective resistance is a metric on a finite graph [5].

Theorem 3.7.1. *Suppose $G = (V, E, \sigma)$ is a weighted, connected, and simple graph. Let $\Gamma = \Gamma(s,t)$ be the connecting family of all walks from s and t. Let $1 < p < \infty$ and let $q = p/(p-1)$ be the conjugate exponent of p. Then*

$$d_p(s,t) := \begin{cases} \mathrm{Mod}_{p,\sigma}(\Gamma)^{-q/p} & \text{if } s \neq t, \\ 0 & \text{if } s = t \end{cases}$$

is a metric on G.

In particular when $p = 2$, $d_2(s,t) = \text{Mod}_{2,\sigma}(\Gamma(s,t))^{-1}$, which is the well-known effective resistance metric.

In the case of a proper infinite tree there is a natural metric on the *boundary* of the tree, which is equivalent to the family of all descending paths from the root o (see [13]). As an example, for a homogeneous infinite tree of degree $b \geq 3$, a function $d : \Gamma_\infty \times \Gamma_\infty \to \mathbb{R}$ defined as

$$d(\gamma, \gamma') = b^{-n} \tag{3.13}$$

where

$$n = n(\gamma, \gamma') = \max\{\text{gen}(e) : e \in \gamma \cap \gamma'\}.$$

is a metric (see [18]). In fact, the function d is a metric if b is replaced by any number greater than 1.

Now we use p-modulus to measure the distance between two distinct infinite descending paths in a radially symmetric infinite tree. Recall that the p-modulus of a family of descending paths is either finite or zero. In the case when p-modulus $\text{Mod}_{p,\sigma}(\Gamma_\infty) > 1$ then a function d defined in (3.13) with b replaced by $\text{Mod}_{p,\sigma}(\Gamma_\infty)$ is a metric. For example, in an unweighted radially symmetric tree with $|S_k| = 3^k$ for each k, the 2-modulus of the family of descending paths given by (3.4) is

$$\begin{aligned}
\text{Mod}_{2,1}(\Gamma) &= \left(\sum_{k=1}^{\infty} |S_k|^{-1}\right)^{-1} \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{3^k}\right)^{-1} \\
&= 2
\end{aligned}$$

Replacing b in (3.13) by 2, the function d is metric and hence the distance between two infinite paths $\gamma, \gamma' \in \Gamma_\infty$ is

$$d(\gamma, \gamma') = 2^{-n}.$$

On the other hand, consider the case when $0 < \text{Mod}_{p,\sigma}(\Gamma_\infty) \leq 1$ for some radially symmetric infinite tree $G = (V, E, \sigma, o)$. For each edge e, let Γ_e be the subfamily of descending paths crossing e. That is, $\Gamma_e = \{\gamma \in \Gamma : e \in \gamma\}$. For any $k \geq 1$, the family Γ_∞ is the finite union of the disjoint subfamilies Γ_e with $e \in S_k$. That is,

$$\Gamma_\infty = \bigcup_{e \in S_k} \Gamma_e.$$

Let $\mathrm{Mod}_{p,\sigma}(\Gamma_e)$ be the p-modulus the family Γ_e. Then by Proposition 2.0.1

$$0 < \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) \leq \sum_{e \in S_k} \mathrm{Mod}_{p,\sigma}(\Gamma_e) \tag{3.14}$$

since $\Gamma_e \subseteq \Gamma_\infty$ and $0 < \mathrm{Mod}_{p,\sigma}(\Gamma_e) \leq \mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$ for any $e \in S_k$. In fact, this relation can also be found from (3.4). Suppose that $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ for some $1 < p < \infty$ and define the function $d : \Gamma_\infty \times \Gamma_\infty \to \mathbb{R}$ as follows. For any $\gamma_1, \gamma_2 \in \Gamma_\infty$,

$$d_p(\gamma_1, \gamma_2) = \begin{cases} 0 & \text{if } \gamma_1 = \gamma_2, \\ \mathrm{Mod}_{p,\sigma}(\Gamma_e) & \text{if } 0 < |\gamma_1 \cap \gamma_2| < \infty \\ \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) & \text{otherwise} \end{cases} \tag{3.15}$$

where e in the second case refers to edge of largest generation shared by the the paths γ_1 and γ_2. Note that in all cases, $d_p(\gamma_1, \gamma_2) \leq \mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$. The following lemma shows that the function d_p is a metric on Γ_∞.

Lemma 3.7.1. *Suppose $0 < \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) \leq 1$ for $1 < p < \infty$. Then the function defined in (3.15) is a metric on Γ_∞.*

Proof. Suppose that γ_1, γ_2 and γ_3 are in Γ_∞. Then

1. $d_p(\gamma_1, \gamma_2) \geq 0$ by definition

2. $d_p(\gamma_1, \gamma_2) = 0 \Leftrightarrow \mathrm{Mod}_{p,\sigma}(\Gamma_e) = 0 \Leftrightarrow \gamma_1 = \gamma_2$.

3. Symmetry: $d_p(\gamma_1, \gamma_2) = \mathrm{Mod}_{p,\sigma}(\Gamma_e) = d_p(\gamma_2, \gamma_1)$.

4. Triangle Inequality: For any γ_1, γ_2, and $\gamma_3 \in \Gamma_\infty$, we need to show that

$$d_p(\gamma_1, \gamma_2) \leq d_p(\gamma_1, \gamma_3) + d_p(\gamma_2, \gamma_3) \tag{3.16}$$

The inequality is straightforward if two or more of the paths are equal. If $\gamma_1 \cap \gamma_2 = \emptyset$, then at least one of the intersections $\gamma_1 \cap \gamma_3$ and $\gamma_2 \cap \gamma_3$ is also empty and the inequality is satisfied. If $\gamma_1 \cap \gamma_3 = \emptyset$, then $d_p(\gamma_1, \gamma_2) \leq \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) = d_p(\gamma_1, \gamma_3)$ and the inequality is satisfied, and similarly if $\gamma_2 \cap \gamma_3 = \emptyset$.

Finally, consider the case that $\gamma_1 \cap \gamma_2 \cap \gamma_3 \neq \emptyset$. We might have the following two subcases:

(a) Suppose $\gamma_1 \cap \gamma_2 = \gamma_1 \cap \gamma_3 = \gamma_2 \cap \gamma_3$. Let e be edge of largest generation shared by each of γ_1, γ_2, and γ_3. Then

$$d_p(\gamma_1, \gamma_2) = d_p(\gamma_1, \gamma_3) = d_p(\gamma_2, \gamma_3) = \mathrm{Mod}_{p,\sigma}(\Gamma_e)$$

and triangle inequality (3.16) easily satisfied.

(b) Otherwise, without loss of generality, we may assume that $|\gamma_1 \cap \gamma_3| > |\gamma_2 \cap \gamma_3| > 0$. Let e be the last edge where γ_1 and γ_2 meet and let e' be the last edge where γ_1 and γ_3 meet. Note that either e' must be a descendent of e or e must be a descendent of e'. In the first case,

$$d_p(\gamma_1, \gamma_2) = d_p(\gamma_2, \gamma_3) = \mathrm{Mod}_{p,\sigma}(\Gamma_e),$$

which implies the triangle inequality. In the second case

$$d_p(\gamma_1, \gamma_3) = d_p(\gamma_2, \gamma_3) = \mathrm{Mod}_{p,\sigma}(\Gamma_{e'}) \geq \mathrm{Mod}_{p,\sigma}(\Gamma_{e'}),$$

and again the triangle inequality holds.

□

Note that the metric in (3.15) is also an *ultra-metric* on Γ_∞ since it satisfies the stronger inequality

$$d_p(\gamma_1, \gamma_2) \leq \max\{d_p(\gamma_1, \gamma_3), d_p(\gamma_2, \gamma_3)\}.$$

3.8 Modulus as a measure of dimension

For unweighted trees ($\sigma \equiv 1$), we can think of modulus as providing a sense of dimension as follows. Let $1 < p_1 < p_2 < \infty$ and let q_1 and q_2 be their respective Hölder conjugate exponents. Then

$$\sum_{k=1}^{\infty} |S_k|^{-\frac{q_1}{p_1}} = \sum_{k=1}^{\infty} |S_k|^{1-q_1} \leq \sum_{k=1}^{\infty} |S_k|^{1-q_2} = \sum_{k=1}^{\infty} |S_k|^{-\frac{q_2}{p_2}}.$$

Thus, if $\text{Mod}_{p_2,1}(\Gamma_\infty)$ is positive, then so is $\text{Mod}_{p_1,1}(\Gamma_\infty)$. Note that for given $p > 1$ if $\text{Mod}_{p,1}(\Gamma_\infty)$ is positive then by the Corollary 3.3.2 $\text{Mod}_{1,1}(\Gamma_\infty)$ is bounded below by the cut weight and the Theorem 3.5.3 the modulus is zero as p approaches to infinity. Therefore, for $\sigma \equiv 1$ we can consider the quantity

$$p_c = \sup\{p > 1 : \text{Mod}_{p,1}(\Gamma_\infty) > 0\}$$

as a measure of how "bushy" the tree is. At one extreme we have the 1-ary tree for which $p_c = 1$. On the other hand, we have the binary tree for which $p_c = +\infty$. More generally, all radially symmetric trees for which each vertex has at least two children have $p_c = +\infty$. In the next section, we will study more closely trees for which $p_c \in (1, \infty)$ and we extend this concept to weighted infinite trees.

Chapter 4

Modulus dimension of 1-2 trees

In this chapter we examine the existence of a critical exponent for a special class of radially symmetric trees that we call 1-2 trees. Assuming the tree is unweighted (i.e., that $\sigma \equiv 1$), the formula given in (3.4) of Chapter 3 shows that the *p-modulus* $(1 < p < \infty)$ is always positive if each edge of a shell has at least two children, that is, if $C(e) \geq 2$ for all $e \in E$. On the other hand, if $C(e) = 1$ for all $e \in E$, then the *p-modulus* is zero. Moreover, the *p-modulus* in a radially symmetric tree is either finite or zero since it depends on the given sequence of children in each generation. Interesting cases occur in radially symmetric trees when $C(e) = 1$ for many but not all edges. In this chapter, we focus on radially symmetric trees with $C(e) \in \{1, 2\}$, which we call 1-2 trees. As we shall see, the *p-modulus* of such a tree is finite or zero depending on whether the parameter p is larger than or smaller than a certain critical value.

By Theorem 3.2.1, for $1 < p < \infty$, the *p-modulus* of the family of descending paths is the limit of *p-modulus* on the truncated 1-2 trees. That is,

$$\lim_{n \to \infty} \mathrm{Mod}_{p,1}(\Gamma_n) = \mathrm{Mod}_{p,1}(\Gamma_\infty).$$

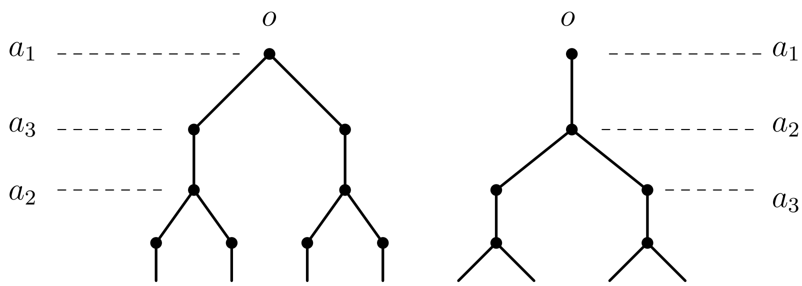


Figure 4.1: A graph of 1-2 tree

Formula (3.4) on an unweighted 1-2 tree can be written as

$$\mathrm{Mod}_{p,1}(\Gamma_\infty) = \left(\sum_{k=1}^{\infty} |S_k|^{-\frac{q}{p}} \right)^{-\frac{p}{q}}. \tag{4.1}$$

In the case $C(e) \equiv 1$, this implies that the modulus is zero for any $p \in (1, \infty)$. When $C(e) \equiv 2$,

$$\mathrm{Mod}_{p,1}(\Gamma_\infty) = \left(\sum_{k=1}^{\infty} (2^k)^{-\frac{q}{p}} \right)^{-\frac{p}{q}} = \left(\frac{2^{-\frac{q}{p}}}{1 - 2^{-\frac{q}{p}}} \right)^{-\frac{p}{q}} = 2 \left(1 - 2^{-\frac{q}{p}}\right)^{\frac{p}{q}},$$

which is positive for any $p \in (1, \infty)$. For general 1-2 trees, it turns out that there exists a critical value for the parameter p at which point $\mathrm{Mod}_{p,1}(\Gamma_\infty)$ switches between positive and zero.

The situation is very interesting when the convergence of the series depends on the parameter p. Consider a sequence $\{a_i : a_i \in \{1, 2\}, i = 1, 2, \cdots\}$. This sequence produces a 1-2 tree (see Figure 4.1) where the number of children of each vertex in generation $i - 1$ is given by a_i. Note that, if the sequence contains only finitely many 2s, then the modulus is 0 for all $1 < p < \infty$, so the interesting cases correspond to sequences with infinitely many 2s.

We first consider the case that $a_1 = 2$ (that is, the root has two children). Let $K = \{1, k_1, k_2, k_3, \cdots\}$ be the sequence of indices where a_i takes the value 2. More specifically, $k_0 = 1 < k_1 < k_2 < k_3 < \cdots$ and $a_k = 2$ if and only if $k \in K$. Define a skip sequence of indices $c_i = k_i - k_{i-1}$ for $i = 1, 2, \cdots$, then (4.1) takes the form

$$\text{Mod}_{p,1}(\Gamma_\infty) = \left[\sum_{k=1}^{\infty} \frac{c_k}{2^{\frac{k}{p-1}}} \right]^{1-p}. \tag{4.2}$$

For example, if

$$\{a_i\} = \{2, 1, 1, 1, 2, 1, 2, 2, \ldots\}$$

then the skip sequence is $\{4, 2, 1, \ldots\}$. Notice that

$$|S_1| = |S_2| = |S_3| = |S_4| = 2,\ |S_5| = |S_6| = 4,\ |S_7| = 8,\ |S_8| = 16.$$

Thus, the first few terms in (4.1) are

$$\begin{aligned} \text{Mod}_{p,1}(\Gamma_\infty) &= \left(2^{-\frac{q}{p}} + 2^{-\frac{q}{p}} + 2^{-\frac{q}{p}} + 2^{-\frac{q}{p}} + 4^{-\frac{q}{p}} + 4^{-\frac{q}{p}} + 8^{-\frac{q}{p}} + 16^{-\frac{q}{p}} + \cdots\right)^{-\frac{p}{q}} \\ &= \left(4(2^{-\frac{q}{p}}) + 2(4^{-\frac{q}{p}}) + 1(8^{-\frac{q}{p}}) + \cdots\right)^{-\frac{p}{q}} \\ &= \left(\frac{c_1}{2^{\frac{1}{p-1}}} + \frac{c_2}{2^{\frac{2}{p-1}}} + \frac{c_3}{2^{\frac{3}{p-1}}} + \cdots\right)^{-\frac{p}{q}}. \end{aligned}$$

In the case $a_1 = 1$, suppose that 2 appears for the first time at the generation m then we define $K = \{m, k_1, k_2, k_3, \ldots\}$ and the formula becomes

$$\text{Mod}_{p,1}(\Gamma_\infty) = \left[m + \sum_{k=1}^{\infty} \frac{c_k}{2^{\frac{k}{p-1}}} \right]^{1-p}.$$

Thus, the *p-modulus* is either finite or zero depending on whether the series on the right-hand side of (4.2) converges or diverges. The convergence of the series depends on the sequence $\{c_i\}$ and the parameter p. However, when $c_i = 1$ (or $c_i = \infty$) for all i, that is, $a_i = 2$ (or $a_i = 1$) for all i, then the series in the right hand side converges (or diverges) for all $1 < p < \infty$. As a more interesting example, suppose that $c_i = 2^i$ then

$$\sum_{k=1}^{\infty} \frac{c_k}{2^{\frac{k}{p-1}}} = \sum_{k=1}^{\infty} 2^{(\frac{p-2}{p-1})k}$$

diverges for $p \geq 2$ and converges for $1 < p < 2$. Thus, the modulus in this case is positive if and only if $1 \leq p < 2$. In general, if $\text{Mod}_{p,1}(\Gamma_\infty)$ is positive for some p then we are interested in the critical value of the parameter p which demarcates the boundary between zero and finite modulus.

The following two corollaries come as result of the Theorem 3.5.3 and by the Lemma 3.5.1.

Corollary 4.0.1. *In a* 1-2 *radially symmetric tree,*

$$\lim_{p\to\infty} \text{Mod}_{p,1}(\Gamma_\infty)^{\frac{1}{p}} = 0. \tag{4.3}$$

Corollary 4.0.2. *In a* 1-2 *radially symmetric tree, suppose that* $1 < p \leq p' < \infty$ *and that* $\text{Mod}_{p,1}(\Gamma_\infty) > 0$. *Then*

$$\text{Mod}_{p',1}(\Gamma_\infty) \leq \text{Mod}_{p,1}(\Gamma_\infty) \tag{4.4}$$

In particular, for an unweighted 1-2 radially symmetric tree, the 1-*modulus* is bound above by the number of children of the root. This result comes as corollary from the Lemma 3.5.5 since S_1 is minimum cut of Γ_∞ in the radially symmetric tree.

Corollary 4.0.3. *Let* Γ_∞ *be the family of descending paths in* 1-2 *radially symmetric tree. Then*

$$\text{Mod}_{1,1}(\Gamma_\infty) = |S_1|$$

Thus, we know that *p-modulus* is always finite for $p = 1$ and it decays as p approaches infinity by Corollary 4.0.2. This suggests that there may be a value for the parameter p such that *p-modulus* becomes zero for for all larger values of p. We define the critical exponent for the *p-modulus* in a 1-2 radially symmetric tree as

$$p_c = \sup\{p > 1 : \text{Mod}_{p,1}(\Gamma_\infty) > 0\} = \inf\{p > 1 : \text{Mod}_{p,1}(\Gamma_\infty) = 0\},$$

with the understanding that p_c may be either 1 (if $\text{Mod}_{p,1}(\Gamma_\infty) = 0$ for all $p > 1$) or $+\infty$ (if $\text{Mod}_{p,1}(\Gamma_\infty) > 0$ for all $p > 1$).

The following theorem shows that 1-2 trees can have nontrivial critical p_c (that is $1 < p_c < \infty$).

Theorem 4.0.1. *For any $1 < r < \infty$, there exists an unweighted 1-2 tree with critical exponent $p_c = r$.*

Proof. For $k = 1, 2, \ldots$, define

$$c_k = \left\lceil 2^{\frac{k}{r-1}} \right\rceil .$$

Since each $c_k \geq 1$, the sequence $\{c_k\}$ is a skip sequence for some 1-2 tree. Moreover,

$$\left(2^{\frac{1}{r-1}-\frac{1}{p-1}}\right)^k \leq \frac{c_k}{2^{\frac{k}{p-1}}} \leq \left(2^{\frac{1}{r-1}-\frac{1}{p-1}}\right)^k + \left(2^{-\frac{1}{p-1}}\right)^k \qquad \text{for all } k.$$

Comparison with the geometric series shows that the infinite series converges if and only if $1 < p < r$. □

For an example, take a 1-2 tree in the following manner: $c_{k+1} = 4c_k$, then by the above formula, *p-modulus* is

for $p = 2$,

$$\mathrm{Mod}_{2,1}(\Gamma_\infty) = \left[\sum_{k=1}^{\infty} \frac{c_k}{2^k}\right]^{-1} = \left[c_1 \sum_{k=1}^{\infty} \frac{4^k}{2^k}\right]^{-1} = 0$$

for $p = \frac{5}{4}$,

$$\mathrm{Mod}_{\frac{5}{4},1}(\Gamma_\infty) = \left[\sum_{k=1}^{\infty} \frac{c_k}{2^{4k}}\right]^{-1} = \left[c_1 \sum_{k=1}^{\infty} \frac{4^k}{4^{2k}}\right]^{-1} > 0$$

Recall Theorem 2.0.3 and Section 3.6 which connect 2-*modulus* of the family of descending paths in an infinite tree to its effective conductance and to the transience or recurrence of a random walk. In particular, a random walk is recurrent (or transient) in a relatively *skinny* (or *bushy*) unweighted radially symmetric trees. The critical exponent p_c tells us about the bushiness or skinniness of the 1-2 tree.

Theorem 4.0.2. *Let p_c be the critical exponent for* $\mathrm{Mod}_{p,1}(\Gamma_\infty)$. *Then*

1. *If $p_c > 2$ then the random walk is transient.*

2. If $p_c < 2$ then the random walk is recurrent.

Proof. For 1), suppose the random walk is recurrent, then by 3.6.1 $\text{Mod}_{2,1}(\Gamma_\infty) = 0$, so $p_c \leq 2$. For 2), suppose the random walk is transient, then by 3.6.1 $\text{Mod}_{2,1}(\Gamma_\infty) > 0$, so $p_c \geq 2$. □

Note that the case $p_c = 2$ cannot be decided in this way since there is currently no general theory to determine whether $\text{Mod}_{p_c,1}(\Gamma_\infty)$ is positive or zero. As an example, take the skip sequence $c_k = 2^k$ for all k, then by the Theorem 4.0.1 the critical exponent is $p_c = 2$ and the $\text{Mod}_{2,1}(\Gamma_\infty) = 0$. On other hand pick a skip sequence as

$$c_k = \left\lceil \frac{2^k}{k^2} \right\rceil \geq 1 \quad \text{for } k = 1, 2, 3, \ldots.$$

Then

$$\sum_{k=1}^{\infty} \frac{c_k}{2^{\frac{k}{p-1}}} = \sum_{k=1}^{\infty} \frac{\left\lceil \frac{2^k}{k^2} \right\rceil}{2^{\frac{k}{p-1}}} \geq \sum_{k=1}^{\infty} \frac{\frac{2^k}{k^2}}{2^{\frac{k}{p-1}}} = \sum_{k=1}^{\infty} \frac{2^{\left(\frac{p-2}{p-1}\right)k}}{k^2}$$

By the ratio test, the series in the right side diverges if $p > 2$

$$\lim_{k\to\infty} \frac{2^{\left(\frac{p-2}{p-1}\right)(k+1)}}{(k+1)^2} \cdot \frac{k^2}{2^{\left(\frac{p-2}{p-1}\right)k}} = 2^{\frac{p-2}{p-1}}$$

This shows that $\text{Mod}_{p,1}(\Gamma_\infty) = 0$ for $p > 2$. Therefore, the critical exponent satisfies $p_c \leq 2$ but at $p = 2$, we have

$$\text{Mod}_{2,1}(\Gamma_\infty)^{-1} = \sum_{k=1}^{\infty} \frac{c_k}{2^k} = \sum_{k=1}^{\infty} \frac{\left\lceil \frac{2^k}{k^2} \right\rceil}{2^k} \leq \sum_{k=1}^{\infty} \frac{\frac{2^k}{k^2} + 1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{2^k}.$$

Both series on the right converge absolutely so by comparison test, we have $\text{Mod}_{2,1}(\Gamma_\infty) > 0$ and, therefore $p_c = 2$.

We could extend this idea of critical exponent in 1-n an infinite tree whose number of children at each generation either 1 or n. A 1-n infinite tree which is bushy at the bottom and skinny at the top has zero p-modulus (see the right tree in Figure 4.2) and hence $p_c = 1$. On the other way, a infinite tree that is skinny at the bottom and bushy at the top has

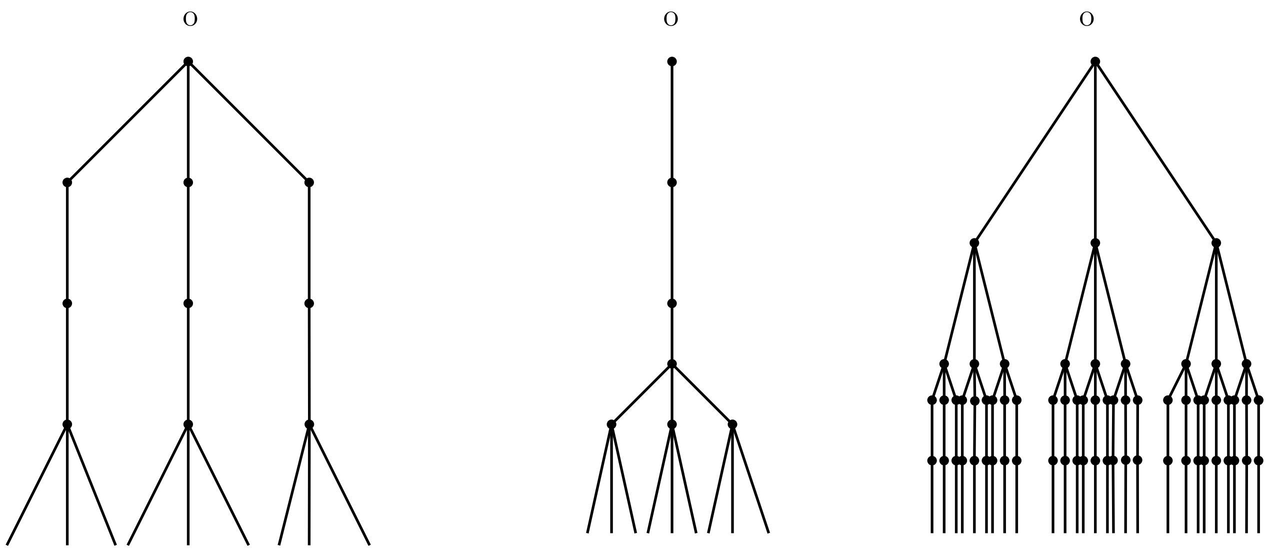


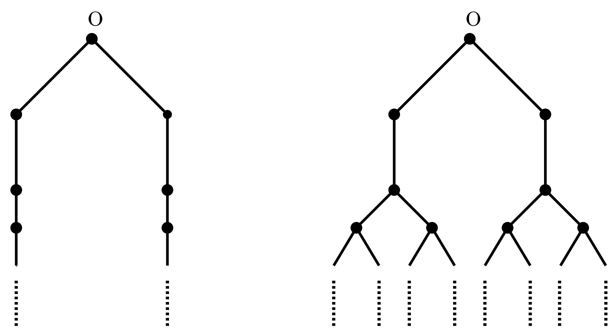
Figure 4.2: The graphs of 1-n tree: bushy-skinny, bottom-skinny and bottom-bushy from left to right.

a positive p-modulus (see middle graph in Figure 4.2 where the bottom means the root). Hence the $p_c = +\infty$. This shows that an infinite tree which has bushy and skinny generation must have such critical value p_c such that for $p < p_c$, the p-modulus is positive or for $p > p_c$, the p-modulus is zero. Also we can extend the Theorem 3.5.6 for 1-4 radially symmetric tree.

Next, in a weighted infinite rooted 1-2 radially symmetric tree the positive (or zero) *p-modulus* may not imply a tree is dense (or thin) in the sense of the number of children. Consider the path tree that is, $S_k = 1$ for all k, then the formula for *p-modulus* is expressed in

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) = \left(\sum_{k=1}^{\infty} \sigma_k^{-\frac{q}{p}} \right)^{-\frac{p}{q}} \tag{4.5}$$

In this case, given $p > 1$, the weight σ determines whether an infinite tree has the positive or zero modulus. For an example, let $\sigma_k = 2^k$, then the series on the right hand side of 4.5 for $1 < p < 2$ converges and diverges for any $p \geq 2$. Hence $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ is positive for $1 < p < 2$ and $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0$ is zero for $p \geq 2$. Consider another example of a dense



$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0,\ \sigma_k = 2^k$ $\qquad$ $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) = 0,\ \sigma_k = 2^{-k}$

Figure 4.3: The weighted infinite thin tree and dense tree with positive modulus and zero modulus from the left to right.

infinite tree, $|S_k| = 2^k$ for $k = 1, 2, \cdots$ then the *p-modulus* is given by

$$\mathrm{Mod}_{p,\sigma}(\Gamma_\infty) = \left(\sum_{k=1}^{\infty} (\sigma_k 2^k)^{-\frac{q}{p}} \right)^{-\frac{p}{q}} \tag{4.6}$$

If $\sigma_k = 2^{-k}$ for $k = 1, 2, \cdots$, then the *p-modulus* zero for any $p > 1$. This shows that *p-modulus* on a dense 1-2 infinite tree has modulus zero as the weight decay proportional to the number of children of each shell. Hence, we define the critical exponent for weighted 1-2 radially symmetric tree.

$$p_c^\sigma = \sup\{p > 1 : \mathrm{Mod}_{p,\sigma}(\Gamma_\infty) > 0\}$$

In the case of $\sigma \equiv 1$, we have the critical exponent p_c for the unweighted 1-2 radially symmetric tree. The following theorem characterize the nature of a random walk in the weighted 1-2 radially symmetric tree. The proof is similar arguments of the Theorem 4.0.2.

Theorem 4.0.3. *Let p_c be the critical exponent for $\mathrm{Mod}_{p,\sigma}(\Gamma_\infty)$. Then*

1. *If $p_c^\sigma > 2$ then the random walk is transient.*

2. *If $p_c^\sigma < 2$ then the random walk is recurrent.*

In the case, the weight is bounded and uniformly elliptic, then it is more easier to get critical exponent of 1-2 radially symmetric infinite tree. Recall the relation (3.12) where the weights are bounded and uniformly elliptic the *p-moduli* $\text{Mod}_{p,\sigma}(\Gamma_\infty)$ and $\text{Mod}_{p,1}(\Gamma_\infty)$ are equivalent. That is,

$$\alpha_1 \text{Mod}_{p,1}(\Gamma_\infty) \leq \text{Mod}_{p,\sigma}(\Gamma_\infty) \leq \alpha_2 \text{Mod}_{p,1}(\Gamma_\infty)$$

Theorem 4.0.4. *For $0 < p < \infty$ and σ be radially symmetric, bounded, and uniformly elliptic. Then*

$$p_c^\sigma = p_c$$

.

Proof. If $\text{Mod}_{p,1}(\Gamma_\infty) = 0$ then $\text{Mod}_{p,\sigma}(\Gamma_\infty) = 0$. Hence $p_c^\sigma = p_c = 1$. Let $\text{Mod}_{p,1}(\Gamma_\infty) > 0$, then by first inequality we have $p_c \leq p_c^\sigma$ and by the second inequality, we have, $p_c^\sigma \leq p_c$. $\square$

The following corollary is a direct consequence of the Theorem 4.0.4 and it relates the critical value of parameter p with a random walk in unweighted and weighted infinite tree.

Corollary 4.0.4. *Let σ be a bounded and uniformly elliptic weight and let $G = (G, V, \sigma, o)$ be a 1-2 radially symmetric tree. Then a random walk is transient (recurrent) if and only if $p_c^\sigma > 2$ ($p_c^\sigma < 2$).*

Chapter 5

Conclusion

In this dissertation, we have extended the concept of p-modulus on finite graphs to infinite graphs. The main focus has been given to the p-modulus of a family of descending paths on radially symmetric infinite trees, its interpretation, and properties.

In the Chapter 1 we formally introduced the terminology, basic definitions, and assumptions that are used in the rest of the chapters. Definitions of graph, descending paths, family of descending paths, length, infinite tree, descendants and generation, ball, shell, and random walks are the key points in this chapter.

In the Chapter 2, we have reviewed the p-modulus on the finite graph. The basics properties are studied in Proposition 2.0.1 in terms of parameter p, the family of objects Γ, and weight σ. We have established the connection between p-modulus and graphic quantities: shortest path, min-cut, and effective resistance for the finite graph. Special focus has been given to the 1-modulus which is the min-cut (see Theorem 2.0.2) and the 2-modulus which the effective conductance (see Theorem 2.0.3) and later we have established the extension of these theorems to infinite trees in Chapter 3. The dual formulation of the p-modulus problem is established in equation (2.9) using standard convex optimization theory and special cases for $p = 1$ and $p = \infty$ have been stated separately (see (2.11), (2.13)). The Lagrangian dual for the p-modulus problem provides the probability interpretation of modulus (see Theorem 2.1.1) and how the p-modulus is related to q-modulus of its family of blockers in the Theorem 2.15 and its probability interpretation in the Theorem 2.2.4.

In Chapter 3, we introduced the p-modulus in the infinite tree as a convex optimization problem with uncountably many constraints and established p-modulus as the limit of the p-modulus of truncated trees (see Theorem 3.2.1). More specifically, we derived the formula (3.4) for computing the p-modulus in the radially symmetric tree in terms of a series whose convergence implies positive modulus and divergence implies zero modulus. As an application, for $p = 2$, the convergence of the series is related to transience or recurrence of a random walk in the infinite tree (see, Corollary 3.6.1). We derived a dual formulation of p-modulus in the radially symmetric tree. Using the dual formulation, we proved that the 1-modulus in a radially symmetric tree is the infimum of cuts of the family of descending paths (see Theorem 3.3.3) and ∞-modulus is zero in a limit sense as p approaches to infinity (see Theorem 3.3.3 and Theorem 3.5.3). The dual formula of the p-modulus problem provides a lower bound established in Theorem 3.3.2. Some other properties of p-modulus and its dependence on different parameters have been studied in Section 3.5.

In Chapter 4 we focused on a special class of 1-2 radially symmetric infinite tree. The monotone property of the parameter p in 1-2 infinite tree shows the existence of a critical parameter p_c which identifies the boundary between zero and positive p-modulus. In particular, such a critical parameter measures the dimension of the infinite tree in the sense of how "bushy" the tree is. Also, it is related to the transience or recurrence of a random walk in the tree (see, Theorem 4.0.2). This chapter also extends the concept of the critical exponent to weighted radially symmetric infinite trees.

Finally, we conclude our dissertation with fundamental questions of p-modulus problems on proper infinite trees. In the Theorem 3.2.1, the p-modulus of a family of descending paths in a proper infinite tree is the limit of p-modulus in the truncated tree. Positive p-modulus implies that the existence of an optimal density. In general (non-symmetric) trees, we expect that the optimal density exists and is the limit of the optimal densities for p-modulus in the truncated trees as long as the limiting density is nowhere zero. A math scholar can ask multiple questions if p-modulus is positive. Does a limit of the sequence of the optimal density for p-modulus in the truncated tree exist? If the limit exists, does it converges point-wise or uniformly? More generally, in which topology does the sequence converge. Is the limit density unique? The next question might be more intriguing. What could be a standard Lagrangian of p-modulus in the proper infinite tree and what could be the probability interpretation and dual modulus in the proper tree?

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