Maximum concurrent flow problems and p-modulus

by

Negar Orangi-Fard

M.S. University of Tabriz, Iran, 2011

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

Doctor of Philosophy

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Abstract

Maximum flow problems involve finding a feasible flow of maximum value through a single source-sink flow network. The flow must satisfy the restriction that the amount of flow into any node other than the source and sink node equals the amount of flow out of it. Moreover, the amount of flow across an edge cannot exceed the capacity of that edge.

Maximum multicommodity flow problems are a generalization of maximum flow problems that involve finding an optimal flow between multiple source and sink pairs. The maximum concurrent flow problem is a more complex and popular variation of the maximum multicommodity flow problem, where we are given a set of positive demands and the goal is finding an optimal flow between multiple source-sink pairs.

The p-modulus problem provides a general framework for quantifying the richness of a family of objects on a graph. Recent advances in the theory have led to several new interpretations of modulus. In the case of a single source and sink, it has been shown that the single source-sink maximum flow problem is dual to the 1-modulus problem. Similarly, it is known that 2-modulus is related to effective resistances and ∞-modulus is related to the shortest path problem.

Inspired by these properties of single source-sink networks, we show that the maximum concurrent flow problem can be embedded into a one-parameter family of p-modulus problems. Using the flexibility provided by the modulus framework, this allows us to introduce a family of generalizations of p-modulus problems. This connection to modulus provides a natural generalization of effective resistance and shortest path to networks with multiple source-sink pairs.

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Approved by:

Major Professor
Nathan Albin

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Acknowledgments

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Chapter 1

Introduction

A Graph is a mathematical representation of a network and it describes the relationship between lines and points. A graph consists of some points (nodes) and lines between them (edges). Network theory is the study of graphs as a representation of either symmetric relations or asymmetric relations between discrete objects. A network can be defined as a graph in which nodes and/or edges have attributes (e.g. names).

Network theory has applications in many disciplines including statistical physics, particle physics, computer science, electrical engineering, biology, economics, finance, operations research, climatology, ecology, public health, and sociology.

Some network problems involve finding an optimal way of doing something. Examples include network flow, shortest path problems, transport problems, transshipment problems, location problems, matching problems, assignment problems, packing problems, routing problems and critical path analysis. The focus of this dissertation is on maximum flow problems and, in particular, on a generalization called maximum concurrent flow.

Typically, maximum flow problems consist of a graph with capacities defined on the edges, and a source-target pair (s, t). The goal is to send the maximum amount of some commodity or substance from a given source node to a given sink node, while respecting the given capacities of each edge in the network.

Consider the network in Figure 1.1 where we want to send maximum flow from source

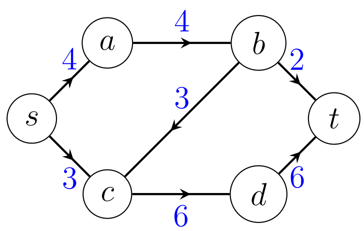

Figure 1.1: *Maximum flow example*

s to target t. The edge labels in the figure give the capacities of the corresponding edges. When considering a flow from s to t, it is often convenient to consider the various paths a unit of substance may follow in order to move from s to t. In this example, there are three possible paths that the commodity can follow. These paths can be represented as a sequence of the nodes they visit in order, as follows.

$$\gamma_1 = s \to a \to b \to t$$

$$\gamma_2 = s \to a \to b \to c \to d \to t$$

$$\gamma_3 = s \to c \to d \to t$$

The paths share the network resources and the sum of flows which can be send through certain edges should not exceed the resources on that edge. For example in this network paths γ_1 and γ_2 both use edge $s \to a$, so they share 4 units of the capacity available on this edge.

The maximum flow can be found by selecting the correct path flows. In this network we can send 2 units of flow γ_1 along path γ_1, 2 units of flow γ_2 along path γ_2, and finally 3 units of flow γ_3 along path γ_3. So the sum of all flows are 7 units. We can check that the capacity constraint of the edges are respected. For example γ_1 send 2 units of flow along edge $s \to a$, and γ_2 also send 2 units of flow along this edge, and the flows sum to 4 which is not exceeding the capacity of the edge $s \to a$.

Although it's not clear that this is the best possible flow, the max-flow min-cut theorem shows that this is the best possible flow and it maximizes the flow from source to target.

In computer science and optimization theory, the max-flow min-cut theorem states that the maximum flow through any network from a given source to a given sink is exactly the

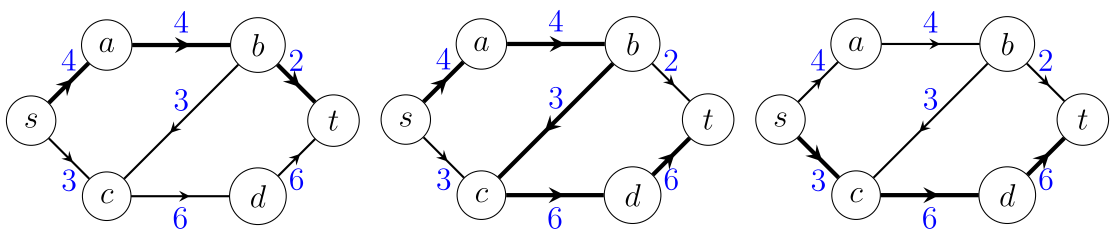

Figure 1.2: *γ_1=2, γ_2=2, γ_3=3, maximum flow* = 7

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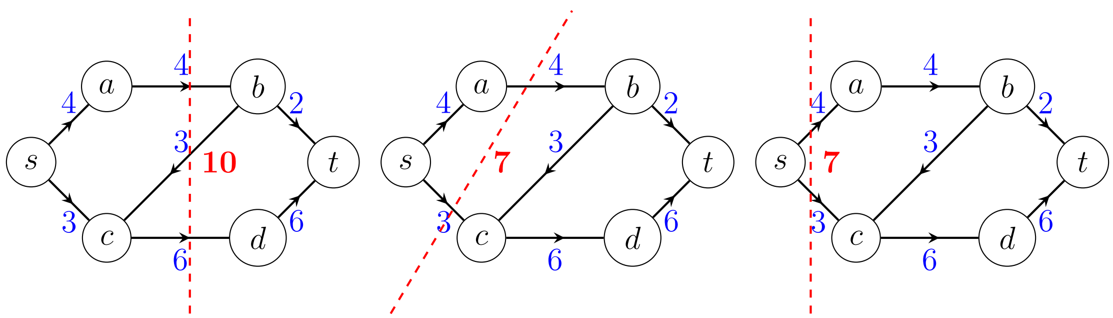

Figure 1.3: *Minimum cut example*

minimum sum of the edge weights that, if removed, would totally disconnect the source from the sink.

Some of the possible cuts for the above network is shown in Figure 1.3, and as it is shown in the graph the minimum cut in this network is 7. See Chapter 3.

Maximum concurrent flow problem is a popular version of maximum flow problem. It consists of a set of source-sink pairs, where we are given a set of positive demands for source-sink pairs. The edges have certain capacities, and the objective is to maximize the minimum ratio between the value of the ith flow and its corresponding demand. Flows share the given network resources and, hence must satisfy a joint capacity constraint.

In this dissertation, we study the dual theory of maximum concurrent flow problems, and make a connection with a general family of problems called p-modulus problems. Modulus is a powerful approach to the study of networks, and it provides a general framework for quantifying the richness of a family of objects. Modulus was originally introduced by Beurling and Ahlfors [1] in complex analysis. In recent years, the original continuum theory has been adapted to a theory of p-modulus on discrete graphs, exploiting its nature as a convex

	1 **pair**	k **pairs**
$p = 1$	maximum flow problem (minimum cut) $\sim$ 1-modulus problem	maximum concurrent flow problem $\sim$ **concurrent 1-modulus problem**
$p = 2$	effective resistance problem $\sim$ 2-modulus problem	**?** $\sim$ **concurrent 2-modulus problem**
$p = \infty$	shortest path problem $\sim$ ∞-modulus problem	**?** $\sim$ **concurrent ∞-modulus problem**

Table 1.1: *Connections between modulus and single- and multi-commodity flow problems. Items written in bold are introduced in this dissertation. The question marks indicate the questions that motivate this research.*

optimization problem. This has led to several new interpretations of p-modulus, including a probabilistic interpretation based on the concept of random objects on graphs. For certain values of the parameter p such as $1, 2$, and ∞, certain p-modulus problems recover classical quantities such as minimum cut, effective resistance, and shortest path.

Inspired by the fact that 1-modulus is the dual convex optimization problem of the single-source single-sink maximum flow problem (see Table 1), we have considered the dual to the concurrent maximum flow problem, which turns out to be a parameterized 1-modulus problem. We call this the concurrent p-modulus problem. Using the flexibility within the modulus framework, we define a class of generalized concurrent flow problems and analyze their properties.

This dissertation begins with a review of convex optimization problems, duality and optimality, then recalls that the classical maximum flow problem and its relationship to 1-modulus. This fact provides motivation to analyze the link between maximum concurrent flows and the theory of p-modulus. As we show, the maximum concurrent flow problem can indeed be embedded into a one-parameter family of p-modulus problems. In particular, when $p = 2$ and $p = \infty$ we obtain “multicommodity” generalizations of effective resistance and shortest paths. We also develop the probabilistic interpretation of these new problems and work out several examples.

The remainder of the thesis is organized as follows.

- In Chapter 2, we review the optimization problem and duality theory. Then we review the weak and strong duality and also necessary and efficient condition for optimality by recalling Karush-Kuhn-Tucker optimality condition.

- In Chapter 3, we review the definition of the graph, networks, flow and cuts, and maximum flow problems. We also recall the notion of p-modulus of families of objects and the fact that the dual of maximum flow is a modulus problem.

- In Chapter 4, we review the definition of concurrent maximum flow and present a new interpretation of its dual in the form of a parametrized modulus problem.

- In Chapter 5 we use the flexibility within the modulus framework to define a class of generalized concurrent flow problems and analyze their properties and present some examples to illustrate the theory thus far.

Chapter 2

Convex optimization

This chapter gives an overview of mathematical optimization, focusing on the special role of convex optimization. It reviews basic definitions and concepts of convex optimization, including duality and optimality conditions.

2.1 Definitions

A set $C \subseteq \mathbb{R}^n$ is an *affine set* if the line through any two distinct points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and $\theta \in \mathbb{R}$, we have

$$\theta x_1 + (1-\theta)x_2 \in C.$$

In other words, C contains the linear combination of any two points in C, provided the coefficients in the linear combination sum to one. This idea can be generalized to more than two points. We refer to a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$, where $\theta_1 + \cdots + \theta_k = 1$, as an affine combination of the points $x_1, \cdots, x_k$.

A set C is a *convex set* if the line segment between any two points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have

$$\theta x_1 + (1-\theta)x_2 \in C.$$

The *convex hull* of a set C, denoted $\operatorname{conv} C$, is the set of all convex combinations of points in C:

$$\operatorname{conv} C = \{\theta_1 x_1 + \cdots + \theta_k x_k |\ x_i \in C,\ \theta_i \geq 0, \quad i = 1, \cdots, k, \quad \theta_1 + \cdots + \theta_k = 1\}.$$

A set C is called a *cone*, or *non-negative homogeneous*, if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$. A set C is a *convex cone* if it is convex and a cone. A point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1, \cdots, \theta_k \geq 0$ is called a *conic combination* of $x_1, \cdots, x_k$.

A *hyperplane* is a set of the form

$$\{x |\ a^T x = b\},$$

where $a \in \mathbb{R}^n, a \neq 0$ and $b \in \mathbb{R}$, and it is affine set. Geometrically, a hyperplane can be interpreted as the set of points with a constant inner product to a given vector a, or as a hyperplane with normal vector parallel to a. A hyperplane divides $\mathbb{R}^n$ into two *halfspaces*. A *closed halfspace* is a set of the form

$$\{x |\ a^T x \leq b\},$$

where $a \neq 0$. Halfspaces are convex, but they are not affine. The set $\{x |\ a^T x < b\}$, which is the interior of the halfspace $\{x | a^T x \leq b\}$, is called an *open halfspace*.

A *polyhedron* is defined as the solution set of a finite number of linear equalities and inequalities:

$$\{x |\ a_j^T x \leq b_j,\ j = 1, \ldots m, \quad c_j^T x = d_j,\ j = 1, \ldots, p\}.$$

So a polyhedron is the intersection of a finite number of halfspaces and hyperplanes.

The *α-sublevel set* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\mathbf{C}_\alpha = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}.$$

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is an *affine function* if it is a sum of a linear function and a constant, i.e., if it has the form $f(x) = Ax + b$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *convex function* if for all $x, y \in \mathbb{R}^n$, and θ with $0 \leq \theta \leq 1$, we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y). \tag{2.1}$$

Geometrically, this inequality means that the line segment between $(x, f(x))$ and $(y, f(y))$, which is the chord from x to y, lies above the graph of f. A function f is strictly convex if strict inequality holds in (2.1) whenever $x \neq y$ and $0 < \theta < 1$. We say f is *concave function* if $-f$ is convex function.

A matrix $P \in \mathbb{R}^{n\times n}$ is *symmetric* if $P = P^T$, where P^T is the transpose of the matrix P. $P \in \mathbb{R}^{n\times n}$ is *symmetric positive definite* if it is symmetric and $x^T P x > 0$, for all $x \in \mathbb{R}^n_{>0}$. We use notation $\mathbb{S}^n_+$ for the set of all symmetric positive definite matrices.

2.2 Optimization problems

One important class of optimization problems has the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x), \\ \text{subject to} \quad & f_i(x) \leq 0, \qquad i = 1, \dots, m, \\ & h_i(x) = 0, \qquad i = 1, \dots, p. \end{aligned} \tag{2.2}$$

This defines the problem of finding the value of $x \in \mathbb{R}^n$ that minimizes a scalar function $f_0(x)$ among all x that satisfy the conditions $f_i(x) \leq 0$, $i = 1, \dots, m$, and $h_i(x) = 0$, $i = 1, \dots, p$. The vector x is called the *optimization variable* and the function $f_0 : \mathbb{R}^n \to \mathbb{R}$ is called the *objective function* or *cost function*. The inequalities $f_i(x) \leq 0$ are called *inequality constraints*, and the corresponding functions $f_i : \mathbb{R}^n \to \mathbb{R}$ are called the *inequality constraint functions*. The equations $h_i(x) = 0$ are called the *equality constraints*, and the functions $h_i : \mathbb{R}^n \to \mathbb{R}$ are the *equality constraint functions*. If there are no constraints (i.e., $m = p = 0$)

we say the problem (2.2) is unconstrained.

In some cases, one or more of the f_i or h_i may not be defined on all of $\mathbb{R}^n$. Traditionally, such cases are treated in one of two ways. Either one introduces the concept of *domain* to refer to the set of points for which a given function is defined, or one allows extended valued functions (i.e., functions whose value might be $\pm\infty$ at certain points). In this dissertation, the objective and constraint functions of interest can be defined on the whole space so, in order to avoid excess notation, we shall assume here that the f_i and h_i are defined throughout $\mathbb{R}^n$.

A point $x \in \mathbb{R}^n$ is *feasible* if it satisfies the constraints $f_i(x) \leq 0, i = 1, \ldots, m$, and $h_i(x) = 0, i = 1, \ldots, p$. The problem (2.2) is said to be feasible if there exists at least one feasible point, and *infeasible* otherwise. The set of all feasible points is called the *feasible set* or the *constraint set*.

The *optimal value* p^* of the problem (2.2) is defined as

$$p^* = \inf \left\{ f_0(x) \mid x \in \mathbb{R}^n,\ f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p \right\}.$$

If the problem is infeasible, then p^* is defined to be $+\infty$, since it is the infimum of the empty set. The problem (2.2) is said to be *unbounded below* if there are feasible points $\{x_k\}$ with $f_0(x_k) \to -\infty$ as $k \to \infty$. In this case, p^* is defined to be $-\infty$.

2.2.1 Convex optimization problem

A *convex optimization* problem has the standard form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m, \\ & a_i^T x = b_i, \qquad i = 1, \ldots, p, \end{aligned} \tag{2.3}$$

where $f_0, \ldots, f_m$ are convex functions and where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i = 1, \ldots, p$. Comparing (2.3) with the general standard form problem (2.2), the convex problem has the three following additional requirements:

- the objective function is convex,
- the inequality constraint functions are convex,
- the equality constraint functions $h_i(x) = a_i^T x - b_i$ are affine.

Such a problem is called convex both because the objective is a convex function and the feasible points form a convex set. (The intersection of m convex sublevel sets $\{x|f_i(x) \leq 0\}$ and p hyperplanes $\{x|a_i^T x = b_i\}$ is convex).

2.2.2 Concave maximization problems

We refer to

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m, \\ & a_i^T x = b_i, \qquad i = 1, \ldots, p, \end{aligned} \tag{2.4}$$

as a *concave optimization* problem if the objective function f_0 is concave, and the inequality constraint functions $f_1, \ldots, f_m$ are convex. This concave maximization problem is readily converted to a convex optimization problem by minimizing the convex objective function $-f_0$ subject to the same set of constraints.

2.2.3 Linear optimization problems

The optimization problem (2.3) is called a *linear program (LP)* when the objective and constraint functions are all affine. A general linear program has the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x + d \\ \text{subject to} \quad & Gx \leq h, \\ & Ax = b, \end{aligned} \tag{2.5}$$

where $c \in \mathbb{R}^n, d \in \mathbb{R}$, $G \in \mathbb{R}^{m\times n}, h \in \mathbb{R}^m, A \in \mathbb{R}^{p\times n}$ and $b \in \mathbb{R}^p$.

2.2.4 Quadratic optimization problems

The convex optimization problem (2.3) is called a *quadratic program (QP)* if the objective function is (convex) quadratic, and the constraint functions are affine. A quadratic program can be expressed in the form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & (1/2)x^T P x + q^T x + r \\ \text{subject to} \quad & Gx \leq h, \\ & Ax = b, \end{aligned} \tag{2.6}$$

where $P \in \mathbb{S}^n_+, q \in \mathbb{R}^n, r \in \mathbb{R}, G \in \mathbb{R}^{m\times n}, h \in \mathbb{R}^m, A \in \mathbb{R}^{p\times n}$ and $b \in \mathbb{R}^p$. In a quadratic program, we minimize a convex quadratic function over a polyhedron.

2.3 Duality

The concept of Lagrangian duality plays an important role in the theory of optimization. For every minimization problem of the form (2.2), there exists a corresponding maximization problem that provides valuable information about the original problem. In this section, we review the basic theory of duality.

2.3.1 Lagrangian

Consider the optimization problem (2.2) with domain $\mathbb{R}^n$, and denote the optimal value of this optimization problem by p^*.

Lagrangian duality takes the constraints in (2.2) into account by augmenting the objective function with a weighted sum of the constraint functions. We define the *Lagrangian* L :

$\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ associated with the problem (2.2) as

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x). \tag{2.7}$$

The value λ_i is called the *Lagrange dual variable* associated with the i^{th} inequality constraint $f_i(x) \leq 0$, and ν_i is the Lagrange dual variable associated with the i^{th} equality constraint $h_i(x) = 0$. The vectors λ and ν are called the *dual variables* or *Lagrange multiplier vectors* associated with the problem.

The Lagrangian is related to (2.2) by the fact that

$$\inf_{x \in \mathbb{R}^n} \sup_{\substack{\lambda \in \mathbb{R}^m_{\geq 0} \\ \nu \in \mathbb{R}^p}} L(x, \lambda, \nu) = p^*. \tag{2.8}$$

To verify this property, first assume that x is a feasible point for (2.2), i.e., $f_i(x) \leq 0$ and $h_i(x) = 0$ for all i. If $\lambda \in \mathbb{R}^m_{\geq 0}$ and $\nu \in \mathbb{R}^p$ then each of the terms $\lambda_i f_i(x)$ in L is non-positive, and each of the terms $\nu_i h_i(x)$ is zero. Thus,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \leq f_0(x).$$

Taking the supremum of both sides shows that

$$\sup_{\substack{\lambda \in \mathbb{R}^m_{\geq 0} \\ \nu \in \mathbb{R}^p}} L(x, \lambda, \nu) \leq f_0(x)$$

for all feasible x. In fact, since $L(x, 0, 0) = f_0(x)$, equality holds.

On the other hand, if x is not feasible, then at least one of the $f_i(x)$ is positive or at least one of the $h_i(x)$ is nonzero. In either case, one can see that by making the corresponding λ_i or ν_i arbitrarily large in magnitude we have

$$\sup_{\substack{\lambda \in \mathbb{R}^m_{\geq 0} \\ \nu \in \mathbb{R}^p}} L(x, \lambda, \nu) = +\infty.$$

Combining the two cases shows that

$$\sup_{\substack{\lambda \in \mathbb{R}^m_{\geq 0} \\ \nu \in \mathbb{R}^p}} L(x, \lambda, \nu) = \begin{cases} f_0(x) & \text{if } x \text{ is feasible,} \\ +\infty & \text{if } x \text{ is not feasible.} \end{cases} \tag{2.9}$$

Equation (2.8) then follows.

2.3.2 The Lagrange dual problem

The max-min inequality allows us to interchange the order of the infimum and supremum in (2.8).

$$p^* = \inf_{x \in \mathbb{R}^n} \sup_{\substack{\lambda \in \mathbb{R}^m_{\geq 0} \\ \nu \in \mathbb{R}^p}} L(x, \lambda, \nu) \geq \sup_{\substack{\lambda \in \mathbb{R}^m_{\geq 0} \\ \nu \in \mathbb{R}^p}} \inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu). \tag{2.10}$$

In light of the right-hand side of this inequality, we define the *Lagrange dual objective function* $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as

$$g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu) = \inf_{x \in \mathbb{R}^n} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right), \tag{2.11}$$

for $\lambda \in \mathbb{R}^m, \nu \in \mathbb{R}^p$. If for a given λ and ν, $x \mapsto L(x, \lambda, \nu)$ is unbounded below, then we define $g(\lambda, \nu) = -\infty$.

Inequality (2.10) shows that for each pair (λ, ν) with $\lambda \geq 0$, the Lagrange dual function gives us a lower bound on the optimal value p^* of the optimization problem (2.2). The best lower bound that can be obtained in this way is itself the solution to an optimization problem:

$$\begin{aligned} \underset{\lambda, \nu}{\text{maximize}} \quad & g(\lambda, \nu) \\ \text{subject to} \quad & \lambda \geq 0. \end{aligned} \tag{2.12}$$

This problem is called the *Lagrange dual problem* associated with the *primal problem* (2.2). A pair (λ, ν) where $\lambda \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$ is called *dual feasible*. A pair (λ^*, ν^*) is called *dual optimal* or *optimal Lagrange multipliers* if it is optimal for the problem (2.12).

The dual function $g(\lambda, \nu)$ is the pointwise infimum of a family of affine functions of (λ, ν), so it is concave. (This is true even if the primal problem (2.2) is not convex.) Since the objective to be maximized in (2.12) is concave and the constraint $\lambda \geq 0$ are convex, the Lagrange dual problem (2.12) is a convex optimization problem.

2.3.3 Weak duality

Recall that the dual function $g(\lambda, \nu)$ gives a lower bound on the primal optimal value, i.e. $g(\lambda, \nu) \leq p^*$ for any $\lambda \geq 0$ and ν. If we denote the optimal value of the Lagrange dual problem (2.12) by d^*, then d^* is the best such lower bound on p^*. In particular, we get

$$d^* \leq p^*. \tag{2.13}$$

This inequality is called *weak duality.* The gap, $p^* - d^*$, is called the *optimal duality gap* of the original problem.

The bound (2.13) can be used to find a lower bound on optimal value of the problem which is hard to solve. Since the dual problem is always convex and in many cases can be solved efficiently to find d^*.

2.3.4 Strong duality

If the optimal duality gap is zero, then we say *strong duality* holds:

$$d^* = p^*. \tag{2.14}$$

Weak duality always holds, but strong duality does not hold in general. If the primal problem (2.2) is convex, i.e. the objective function and all of the inequality constraints are convex and the equality constraints are affine, then, as we shall see, we often have strong duality. Once convexity of the primal problem is established, there are many sufficient conditions called *constraint qualifications* that imply strong duality.

2.3.5 Slater's condition

One of the simplest constraint qualifications is *Slater's condition.* Consider the original optimization problem

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m, \\ & h_i(x) = 0, & i = 1, \ldots, p. \end{array}$$

The problem is said to satisfy *Slater's condition* if there exists a *strictly feasible* point in $\mathbb{R}^n$, i.e. there exists an $x \in \mathbb{R}^n$ such that

$$f_i(x) < 0,\ i = 1, \ldots, m, \qquad \text{and} \qquad h_i(x) = 0,\ i = 1, \ldots, p.$$

Theorem 2.3.1 (Slater's Theorem)**.** *If Slater's condition holds for a convex optimization problem, then strong duality also holds, i.e.* $p^* = d^*$*.*

For a proof, see [29].

Remark 2.3.2. Slater's theorem can be refined as follows. If the first k constraint functions $f_1, \ldots, f_k$ are affine, then strong duality holds provided the following weaker condition holds: There exists an $x \in \mathbb{R}^n$ with

$$f_i(x) \leq 0,\ i = 1, \ldots, k, \quad \text{and} \quad f_i(x) < 0,\ i = k+1, \ldots, m, \quad \text{and} \quad h_i(x) = 0,\ i = 1, \ldots, p.$$

So, the affine constraints do not need to hold with strict inequality.

Both Slater's condition and its refinement above not only imply strong duality for convex problems, but they also imply that the dual optimal value is attained as long as $d^* > -\infty$, i.e., there exists a dual feasible (λ^*, ν^*) with

$$g(\lambda^*, \nu^*) = d^* = p^*.$$

2.4 KKT optimality conditions

This section studies the Karush-Kuhn-Tucker (KKT) necessary and sufficient conditions for the optimality of optimization problems.

2.4.1 Saddle points of the Lagrangian

A point $(x^*, \lambda^*, \nu^*) \in \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^p$ is called a *saddle point* for the Lagrangian (2.7) if it satisfies

$$L(x^*, \lambda, \nu) \leq L(x^*, \lambda^*, \nu^*) \leq L(x, \lambda^*, \nu^*),$$

for every $(x, \lambda, \nu) \in \mathbb{R}^n \times \mathbb{R}^m_{\geq 0} \times \mathbb{R}^p$. In other words, x^* minimizes $L(x, \lambda^*, \nu^*)$ over x and (λ^*, ν^*) maximizes $L(x^*, \lambda, \nu)$ over (λ, ν). So, an equivalent formulation of the saddle point property is

$$L(x^*, \lambda^*, \nu^*) = \inf_{x} L(x, \lambda^*, \nu^*), \qquad L(x^*, \lambda^*, \nu^*) = \sup_{\substack{\lambda \geq 0 \\ \nu}} L(x^*, \lambda, \nu). \tag{2.15}$$

Saddle points of the Lagrangian are important because of the following theorems.

Theorem 2.4.1. *If (x^*, λ^*, ν^*) is a saddle point of the Lagrangian, then x^* is optimal for the primal problem, (λ^*, ν^*) is optimal for the dual problem, and the duality gap is zero.*

Theorem 2.4.2. *If strong duality holds and if x^*, (λ^*, ν^*) are primal and dual optimal values respectively, then (x^*, λ^*, ν^*) is a saddle point for the Lagrangian.*

Theorems 2.4.1 and 2.4.2 are proved at the end of Section 2.4.3.

The KKT conditions provide necessary conditions for a saddle point to exist. As described shortly, under some additional assumptions the KKT conditions are also sufficient.

2.4.2 Primal and dual feasibility

The two simplest KKT conditions to describe are primal and dual feasibility. These are simply the observation that x^* must satisfy the constraints of the primal problem while (λ^*, ν^*) must satisfy the constraints of the dual problem. That is,

$$\begin{aligned} f_i(x^*) &\leq 0, \qquad i = 1, \ldots, m, \\ h_i(x^*) &= 0, \qquad i = 1, \ldots, p, \\ \lambda_i^* &\geq 0, \qquad i = 1, \ldots, m. \end{aligned} \tag{2.16}$$

2.4.3 Complementary slackness

The next condition connects the inequality constraints to their corresponding dual variables.

Proposition 2.4.3. *Suppose that (x^*, λ^*, ν^*) is a saddle point of the Lagrangian. Then*

$$\lambda_i^* f_i(x^*) = 0, \qquad i = 1, \ldots, m.$$

Proof. From the assumptions,

$$\begin{aligned} f_0(x^*) &= g(\lambda^*, \nu^*) \\ &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \right) \\ &\leq f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \\ &\leq f_0(x^*). \end{aligned} \tag{2.17}$$

To see the last inequality, note that, since x^* is primal feasible, $f_i(x^*) \leq 0$ and $h_i(x^*) = 0$ for all i. Moreover, since (λ^*, ν^*) is dual feasible, $\lambda_i^* \geq 0$ for all i. Thus, each term $\lambda_i^* f_i(x^*) \leq 0$ and each term $\nu_i^* h_i(x^*) = 0$. We conclude that the two inequalities in (2.17) hold as equality, and therefore that

$$\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0.$$

Since each term in this sum is non-positive, each must be zero.

□

The complementary slackness condition can also be expressed as either of the following implications.

$$\begin{aligned} \lambda_i^* > 0 \quad &\Rightarrow \quad f_i(x^*) = 0, \\ f_i(x^*) < 0 \quad &\Rightarrow \quad \lambda_i^* = 0. \end{aligned}$$

When $f_i(x^*) = 0$ for some i, we say that the i^{th} inequality constraint is *active*. In words, it means that if the i^{th} optimal Lagrange multiplier is positive, then the i^{th} inequality constraint is active at the corresponding primal optimal point.

Now we are ready to prove Theorems 2.4.1 and 2.4.2.

Proof of Theorem 2.4.1. If a saddle point of the Lagrangian exists, then

$$\begin{aligned} p^* &= \inf_x \sup_{\substack{\lambda \geq 0 \\ \nu}} L(x, \lambda, \nu) && \text{by (2.8)} \\ &\leq \sup_{\substack{\lambda \geq 0 \\ \nu}} L(x^*, \lambda, \nu) \\ &= \inf_x L(x, \lambda^*, \nu^*) && \text{by (2.15)} \\ &\leq \sup_{\substack{\lambda \geq 0 \\ \nu}} \inf_x L(x, \lambda, \nu) = d^*. \end{aligned}$$

Combining this with (2.13) we get $p^* = d^*$ and all inequalities above hold as equality. In particular,

$$p^* = \sup_{\substack{\lambda \geq 0 \\ \nu}} L(x^*, \lambda, \nu) \qquad \text{and} \qquad d^* = \inf_x L(x, \lambda^*, \nu^*). \tag{2.18}$$

Now we need to show that x^* and (λ^*, ν^*) are optimal points for their respective problems.

First we show that x^* is an optimal point for the primal problem, i.e. $p^* = f_0(x^*)$. The point (x^*, λ^*, ν^*) is a saddle point, so we get

$$\begin{aligned}
p^* &= \sup_{\substack{\lambda \geq 0 \\ \nu}} L(x^*, \lambda, \nu) = L(x^*, \lambda^*, \nu^*), && \text{by (2.15) and (2.18)} \\
&= f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* h_i(x^*), && \text{by (2.7)} \\
&= f_0(x^*), && \text{by Proposition 2.4.3}
\end{aligned}$$

which means that x^* is a optimal point for the primal problem.

Next we show that (λ^*, ν^*) is a dual optimal point, i.e. $d^* = g(\lambda^*, \nu^*)$. By assumption, the point (x^*, λ^*, ν^*) is a saddle point, so, by (2.18), we get

$$d^* = \inf_x L(x, \lambda^*, \nu^*) = g(\lambda^*, \nu^*).$$

Therefore (λ^*, ν^*) is an optimal point for the dual problem. □

Proof of Theorem 2.4.2. Since x^* is a primal optimal point, (2.9) implies that

$$p^* = f_0(x^*) = \sup_{\substack{\lambda \geq 0 \\ \nu}} L(x^*, \lambda, \nu).$$

Also, since (λ^*, ν^*) is optimal for the dual problem,

$$d^* = g(\lambda^*, \nu^*) = \inf_x L(x, \lambda^*, \nu^*).$$

Since the strong duality $p^* = d^*$ holds by assumption, it follows that

$$\sup_{\substack{\lambda \geq 0 \\ \nu}} L(x^*, \lambda, \nu) = \inf_x L(x, \lambda^*, \nu^*).$$

So,

$$L(x^*, \lambda^*, \nu^*) \leq \sup_{\substack{\lambda \geq 0 \\ \nu}} L(x^*, \lambda, \nu) = \inf_x L(x, \lambda^*, \nu^*) \leq L(x^*, \lambda^*, \nu^*),$$

which implies (2.15). ∎

2.4.4 Stationarity

The stationarity condition is easiest to describe when $f_i, i = 0, \dots, m$, and $h_i, i = 1, \dots, p$ are all differentiable. Although stationarity conditions exist for the non-differentiable case, this is not needed for the results of this dissertation.

Proposition 2.4.4. *Assume that all f_i and h_i are differentiable and that (x^*, λ^*, ν^*) is a saddle point of the Lagrangian. Then,*

$$\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0. \tag{2.19}$$

Proof. Since the variable x in (2.15) is unconstrained, the gradient of L with respect to x must vanish at a saddle point, which gives (2.19). ∎

Combining (2.16) with Propositions 2.4.3 and 2.4.4 yields the following set of necessary conditions on a saddle point of the Lagrangian when the objective and all constraints are differentiable.

$$\begin{aligned}
&f_i(x^*) \le 0, \qquad i = 1, \dots, m, \\
&h_i(x^*) = 0, \qquad i = 1, \dots, p, \\
&\lambda_i^* \ge 0, \qquad i = 1, \dots, m, \\
&\lambda_i^* f_i(x^*) = 0, \qquad i = 1, \dots, m, \\
&\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0.
\end{aligned} \tag{2.20}$$

These set of condition are called the *Karush-Kuhn-Tucker (KKT)* conditions.

It is possible to solve the KKT conditions analytically in few special cases. More generally, many algorithms for convex optimization can be interpreted as, methods for solving the KKT conditions.

2.4.5 KKT conditions for special cases

The KKT conditions play an important role in optimization. In this section we study the KKT conditions for some important types of optimization problems.

KKT conditions in convex optimization problems

The optimization problem (2.3) is a convex optimization problem, where the objective and all inequality functions are convex and all equality functions are affine functions.

For convex optimization problems, the KKT conditions also provide sufficient conditions; there are two senses in which this can be understood.

Theorem 2.4.5. *For a convex optimization problem* (2.3) *with differentiable objective and constraints, a point* (x^*, λ^*, ν^*) *is a saddle point of the Lagrangian if and only if it satisfies the KKT conditions.*

Proof. Necessity of the KKT conditions has already been established, so it remains to show sufficiency. By assumption, the function

$$x \mapsto L(x, \lambda^*, \nu^*) = f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* (a_i^T x - b_i)$$

is convex, so the stationarity condition implies that x^* is a global minimizer of this function, establishing the first equality in (2.15).

Using this together with weak duality and the complementary slackness condition shows that

$$\begin{aligned} p^* \geq d^* \geq g(\lambda^*, \nu^*) = \inf_x L(x, \lambda^*, \nu^*) &= L(x^*, \lambda^*, \nu^*) \\ &= f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} \nu_i^* (a_i^T x^* - b_i) \\ &= f_0(x^*) \geq p^*, \end{aligned}$$

which establishes that x^* is primal optimal, (λ^*, ν^*) is dual feasible, and the duality gap is zero. Thus (x^*, λ^*, ν^*) is a saddle point of the Lagrangian. □

Theorem 2.4.6. *Suppose that strong duality holds for a convex optimization problem* (2.3) *with differentiable objective and constraints and that the dual optimum is attained. Then x^* is optimal for the primal problem if and only if there exists a dual feasible point (λ^*, ν^*) that, together with x^*, satisfies the KKT conditions.*

Proof. If a triple (x^*, λ^*, ν^*) satisfying the KKT conditions can be found, then Theorem 2.4.5 shows that x^* is primal optimal. On the other hand, if an optimal x^* exists, then the assumptions together with Theorem 2.4.2 imply that (x^*, λ^*, ν^*) is a saddle point. □

Theorem 2.4.6 is most useful when combined with a constraint qualification like Slater's condition. The following corollary shows an example of this.

Corollary 2.4.7. *Consider a convex optimization problem* (2.3) *with differentiable objective and constraints. Suppose that Slater's condition is satisfied and that $d^* > -\infty$. Then x^* is optimal for the primal problem if and only if there exists (λ^*, ν^*) so that (x^*, λ^*, ν^*) satisfies the KKT conditions.*

KKT conditions in linear optimization problems

The Lagrangian for the linear optimization problem in (2.5) is:

$$\begin{aligned}
L(x, \lambda, \nu) &= c^T x + d + \sum_{i=1}^{m} \lambda_i (g_i^T x - h_i) + \sum_{i=1}^{p} \nu_i (a_i^T x - b_i) \\
&= c^T x + d + \sum_{i=1}^{m} \lambda_i (g_i^T x) - \sum_{i=1}^{m} \lambda_i h_i + \sum_{i=1}^{p} \nu_i (a_i^T x) - \sum_{i=1}^{p} \nu_i b_i \\
&= c^T x + d + (\lambda^T G) x - \lambda^T h + \nu^T A x - \nu^T b \\
&= x^T \left(c + G^T \lambda + A^T \nu \right) + d - \lambda^T h - \nu^T b,
\end{aligned}$$

with Lagrange dual function

$$
\begin{aligned}
g(\lambda, \nu) &= \inf_x L(x, \lambda, \nu) \\
&= \begin{cases} d - \lambda^T h - \nu^T b & \text{if} \quad c + G^T \lambda + A^T \nu = 0, \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

So the dual problem takes the form

$$
\begin{aligned}
\underset{\lambda, \nu}{\text{maximize}} \qquad & d - \lambda^T h - \nu^T b, \\
\text{subject to} \qquad & c + G^T \lambda + A^T \nu = 0, \\
& \lambda \in \mathbb{R}^m_{\geq 0}, \\
& \nu \in \mathbb{R}^p_{\geq 0}.
\end{aligned}
$$

Let x^* be an optimal point for the primal problem and (λ^*, ν^*) an optimal point for its dual problem, where the the primal problem satisfies in Slater's condition.

$$
\begin{aligned}
\inf_x L(x, \lambda^*, \nu^*) &= \inf_x \left(c^T x + d + \sum_{i=1}^{m} {\lambda_i^*}^T (g_i x - h_i) + \sum_{i=1}^{p} {\nu_i^*}^T (a_i x - b_i) \right) \\
&= c^T x^* + d + \sum_{i=1}^{m} {\lambda_i^*}^T (g_i x^* - h_i) + \sum_{i=1}^{p} {\nu_i^*}^T (a_i x^* - b_i) \\
&= L(x^*, \lambda^*, \nu^*).
\end{aligned}
$$

Since x^* minimizes the Lagrangian, its gradient with respect to x vanishes at x^*:

$$
\nabla L(x^*, \lambda^*, \nu^*) = c^T \mathbb{1} + \sum_{i=1}^{m} {\lambda_i^*}^T g_i + \sum_{i=1}^{p} {\nu_i^*}^T a_i = 0.
$$

So, the KKT conditions for linear optimization problem are:

$$
\begin{aligned}
&g_i x^* \leq h_i, \qquad i = 1, \ldots, m, \\
&a_i x^* = b_i, \qquad i = 1, \ldots, p, \\
&\lambda_i^* \geq 0, \qquad i = 1, \ldots, m, \\
&\lambda_i^* (g_i^T x^* - h_i) = 0, \qquad i = 1, \ldots, m, \\
&c^T \mathbb{1} + \sum_{i=1}^{m} {\lambda_i^*}^T g_i + \sum_{i=1}^{p} {\nu_i^*}^T a_i = 0.
\end{aligned}
$$

KKT conditions in quadratic optimization problems

The Lagrangian for the quadratic optimization problem in (2.6) is:

$$
\begin{aligned}
L(x, \lambda, \nu) &= \frac{1}{2} x^T P x + q^T x + r + \lambda^T (Gx - h) + \nu^T (Ax - b) \\
&= \frac{1}{2} x^T P x + q^T x + r + \lambda^T G x - \lambda^T h + \nu^T A x - \nu^T b \\
&= \frac{1}{2} x^T P x + (G^T \lambda + A^T \nu + q)^T x - \lambda^T h + \nu^T b + r.
\end{aligned}
$$

Since x^* minimizes the Lagrangian, therefore $\nabla_{x^*} L = 0$, so

$$
\nabla L = P^T x^* + G^T \lambda + A^T \nu + q = 0,
$$

and $x^* = -P^{-1}(G^T \lambda + A^T \nu + q)$, and

$$
\begin{aligned}
g(\lambda, \nu) &= \inf_x L(x, \lambda, \nu) \\
&= \frac{1}{2} {x^*}^T P x^* + (G^T \lambda + A^T \nu + q)^T x^* - \lambda^T h + \nu^T b + r \\
&= \frac{-1}{2} P^{-1} (G^T \lambda + A^T \nu + q)^2 - \lambda^T h + \nu^T b + r.
\end{aligned}
$$

So the dual problem will be

$$\begin{aligned}
\underset{\lambda,\nu}{\text{minimize}} \quad & \frac{1}{2}P^{-1}(G^T\lambda + A^T\nu + q)^2 + \lambda^T h - \nu^T b - r \\
\text{subject to} \quad & \lambda \in \mathbb{R}^m_{\geq 0,} \\
& \nu \in \mathbb{R}^p_{\geq 0}.
\end{aligned}$$

Finally the KKT conditions for quadratic optimization problem are:

$$\begin{aligned}
& g_i x^* \leq h_i, \qquad i = 1, \ldots, m, \\
& a_i x^* = b_i, \qquad i = 1, \ldots, p, \\
& \lambda_i^* \geq 0, \qquad i = 1, \ldots, m, \\
& \lambda_i^*(g_i^T x^* - h_i) = 0, \qquad i = 1, \ldots, m, \\
& \sum_{i=1}^{n} p_{ij}^T x_i + \sum_{i=1}^{m} g_{ij}^T \lambda_i^* + \sum_{i=1}^{p} a_{ij}^T \nu^* + q_j = 0, \qquad j = 1, \ldots, n.
\end{aligned}$$

Chapter 3

Flows and modulus

In this chapter, we will start with reviewing some basic definitions and properties of discrete graphs. We will also review maximum flow and minimum cut problems. Next, we will present modulus of families of objects on graphs and its connection to maximum flow problems.

3.1 Graph preliminaries

In this dissertation, the notation $G = (V, E)$ represents a *graph*, where V is a set whose elements are called vertices, and E is a set of elements called edges. A graph may either be *undirected*, in which case the edges are unorderdered pairs of vertices, or *directed*, in which case the edges are ordered pairs of vertices. The graph G is *simple* if it contains no edges connecting a vertex to itself and at most one undirected edge between any two distinct vertices. The graph is *finite* if the vertex set has cardinality $|V| = N \in \mathbb{N}$. Although the theory developed in this dissertation can be extended to other cases, for simplicity of presentation in what follows we shall assume that G is finite, undirected and simple. Two vertices u and v are *neighbors* if and only if $\{u, v\} \in E$. This relationship is indicated by $u \sim v$. The number of neighbors of a vertex x is the *degree* of x, indicated by $deg(x)$.

Given an integer $n \geq 1$, a string $\gamma = x_0 e_1 x_1 e_2 x_2 \cdots e_n x_n$ with $x_i \in V$ for $i = 0, \ldots, n$ and $e_k = \{x_{k-1}, x_k\} \in E$, for $k = 1, \ldots, n$, is called a *walk* with n hops from x_0 to x_n. For simplicity, when a graph is simple we will just list the vertices visited by the walk, and write $\gamma = x_1 \cdots x_n$. A walk that does not revisit any vertex is called a *simple walk*, or *path*. The graph G is *connected* if, for any two distinct vertices $x, y \in V$, G contains a walk from x to y. A walk $\gamma = x_0 e_1 x_1 e_2 x_2 \cdots e_n x_n$ such that $x_0 e_1 x_1 e_2 x_2 \cdots e_{n-1} x_{n-1}$ is a path and $x_n = x_0$ where $n \geq 2$ is called a *cycle*.

Given graph G and two nodes $s \neq t \in V$, the *connecting family* $\Gamma(s,t)$ is the family of all paths in G that start at s and end at t:

$$\Gamma(s,t) := \{\gamma : \gamma \text{ is a path } s \rightsquigarrow t\}.$$

Connecting families of finite graphs are always finite families of paths.

3.2 Flows and cuts on the network

Let $G = (V, E, \sigma)$ be an undirected network with finite vertex set V and edge set E. To every edge $e \in E$ there corresponds a weight $0 < \sigma(e) < \infty$ which we will think of as a "capacity."

Definition 3.2.1. Given a pair of distinct vertices $s, t \in V$, an *st-flow* can be defined as a function $f : V \times V \to \mathbb{R}$ with the following properties:

- Flow f is restricted to edges of the graph:

$$f(u,v) = 0, \quad \text{if} \quad \{u,v\} \notin E,$$

- Flow f is anti-symmetric:

$$f(u,v) = -f(v,u), \quad \forall u, v \in V,$$

  which gives a sense of direction to the flow; if there is a positive flow from u to v, then there is a corresponding negative flow from v to u.

- Flow f satisfies the divergence-free condition:

$$\operatorname{div}_f(u) := \sum_{v \in V} f(u,v) = 0, \qquad \forall u \in V \setminus \{s,t\},$$

divergence-free condition states that the flow substance is conserved at all nodes other than the source and sink.

Remark 3.2.2. Define the *adjacency matrix* as $A(u,v) = \mathbb{1}_E(\{u,v\})$. Sums over neighbors can be extended to sums over all vertices V using the following:

$$\operatorname{div}_f(u) := \sum_{v \sim u} f(u,v) = \sum_{v \in V} A(u,v) f(u,v).$$

Define the *value of a flow* from s to t to be

$$\operatorname{Val}(f) := \operatorname{div}_f(s). \tag{3.1}$$

Note that the divergence-free condition and anti-symmetry lead to

$$\operatorname{Val}(f) + \operatorname{div}_f(t) = \operatorname{div}_f(s) + \operatorname{div}_f(t) = \sum_{u \in V} \operatorname{div}_f(u) = \sum_{u \in V} \sum_{v \in V} A(u,v) f(u,v) = 0.$$

So $\operatorname{div}_f(t) = -\operatorname{Val}(f)$.

The flow f from s to t is *feasible* if:

$$|f(u,v)| \leq \sigma(u,v), \quad \text{for every} \quad \{u,v\} \in E,$$

and write $f \in \mathcal{F}$.

$$F^* := \max_{f \in \mathcal{F}} \operatorname{Val}(f)$$

A feasible flow f^* such that $\operatorname{Val}(f^*) = F^*$ is called a maximum flow.

A *cut* $C \subset V$ is a partition of the vertices of the graph into two disjoint subsets. Any cut determines a *cut-set* or *edge-boundary*, the set of edges that have one endpoint in each subset of the partition

$$\partial C := \{e = \{x, y\} \in E : x \in C, \text{and } y \notin C\}.$$

An *st-cut* $C \subset V$ is a cut $C \subset V$ with $s \in C$ but $t \notin C$.

The capacity of a cut C is

$$\sigma(C) := \sum_{e \in \partial C} \sigma(e).$$

An important relationship between *st*-paths and *st*-cuts is described by the following theorem.

Theorem 3.2.3. *If C is an st-cut then $\gamma \cap \partial C \neq \emptyset$ for all $\gamma \in \Gamma(s, t)$. Conversely, if $E' \subseteq E$ has the property that $\gamma \cap E' \neq \emptyset$ for all $\gamma \in \Gamma(s, t)$, then there exists an st-cut C such that $\partial C \subseteq E'$.*

Theorem 3.2.4. *(Max-flow min-cut Theorem [11]) Let $\mathcal{C}$ be the set of all cuts for flows from s to t. Then*

$$F^* = \min_{C \in \mathcal{C}} \sigma(C).$$

3.3 Maximum flow problem

The maximum flow problem is defined as the maximum amount of flow that a network would allow to flow from source to sink. This problem was first formulated in 1954 by T. E. Harris and F. S. Ross as a simplified model of railway traffic flow [19]. The maximum flow problems involve finding a feasible flow through a single-source, single-sink flow network where the amount of flow being sent from a source node s to a sink node t is maximized. In 1955, L. R. Ford and D. R. Fulkerson created the first known algorithm, the FordFulkerson algorithm. [11] [12].

Ford and Fulkerson assumed the capacities of the edges of the problem to be constant. This problem can be easily interpreted in the evacuation context as evacuating as many as possible people from a danger zone (source node) into a safe zone (sink node).

Over the years, various improved algorithms for the maximum flow problem were introduced, notably the shortest augmenting path algorithm of Edmonds and Karp [9] and independently Dinitz [7], the push-relabel algorithm of Goldberg and Tarjan [15, 16], and the binary blocking flow algorithm of Goldberg and Rao [14].

The goal in the maximum flow problem is to find the maximum value of all *st*-flows satisfying a capacity constraint on every edge, which can be written as follows:

$$|f(u,v)| = |f(v,u)| \leq \sigma(e) \quad \text{for all } e = \{u,v\} \in E. \tag{3.2}$$

For a given capacity constraints the *st*-flow is called *feasible* if in addition to the requirements in Definition 3.2.1, the capacity constraint (3.2) also holds.

The maximum flow problem can be written as an optimization problem as follows.

$$\begin{aligned} \underset{f}{\text{maximize}} \qquad & \mathrm{Val}(f) \\ \text{subject to} \qquad & f \text{ is a feasible } st\text{-flow.} \end{aligned} \tag{3.3}$$

For any distinct nodes $s, t \in V$, let $\Gamma = \Gamma(s,t)$ be the family of all paths in G connecting s to t, and let C represent the family of all cycles. To each $\gamma \in \Gamma(s,t)$, we can assign the unit *path flow* $f(\gamma)$ defined by

$$f(\gamma)(u,v) = \begin{cases} 1 & \text{if } \gamma \text{ crosses from } u \text{ to } v, \\ -1 & \text{if } \gamma \text{ crosses from } v \text{ to } u, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $f(\gamma)$ represents a unit flow from s to t that follows γ. In a similar way, one can define a unit *cycle flow* $f(c)$ for each cycle $c \in C$.

The family of *st*-flows has a natural vector space structure, with addition and scalar multiplication of functions defined as usual, and the value function is a linear functional on this vector space. Any st-flow f has a decomposition in the form

$$f = \sum_{\gamma \in \Gamma(s,t)} x(\gamma) f(\gamma) + \sum_{c \in C} x(c) f(c),$$

where the $x(\gamma)$ and $x(c)$ are real numbers.

This structure allows every flow to be decomposed into a combination of *path flows* and *cycle flows* (flows whose nonzero values are restricted to the edges of a path or a cycle of the graph respectively). In the context of the maximum flow problem (3.3), cycles may be omitted because they use capacity while contributing nothing to the value of the flow (see Lemma 4.1.1), so for maximum-valued flow problem, we may ignore flows that contain cycles of positive flow. For similar reasons, negative path flows $x(\gamma) < 0$ may be removed from consideration. This observation leads to the *path formulation* of the maximum flow problem, summarized here.

Since for any γ, $\mathrm{Val}(f(\gamma)) = 1$, the value of such an f can be expressed as

$$\mathrm{Val}(f) = \sum_{\gamma \in \Gamma(s,t)} x(\gamma) \, \mathrm{Val}(f(\gamma)) = \sum_{\gamma \in \Gamma(s,t)} x(\gamma),$$

leading to the standard path formulation of the max flow problem:

$$\begin{aligned} \underset{x(\gamma)}{\text{maximize}} \quad & \sum_{\gamma \in \Gamma(s,t)} x(\gamma) \\ \text{subject to} \quad & \sum_{\gamma \in \Gamma(s,t): e \in \gamma} x(\gamma) \le \sigma(e), \quad \forall e \in E, \\ & x(\gamma) \ge 0, \quad \forall \gamma \in \Gamma(s,t), \end{aligned} \tag{3.4}$$

where the notation $e \in \gamma$ means that the path γ traverses the edge e in either direction. This form is convenient because the flow conditions in Definition 3.2.1 are satisfied automatically; one only needs to take care of the capacity constraint (3.2) which, as indicated, requires that the sum of the flow strengths $x(\gamma)$ along each path using a particular edge must not exceed the capacity of that edge.

3.4 Modulus of families of objects

The theory of modulus of families of curves in the plane was originally introduced by A. Beurling and L. V. Ahlfors [1] to solve famous open questions in function theory. The theory has been extended over the years to families of curves in $\mathbb{R}^n$ and to abstract metric spaces as well.

R. J. Duffin [8] developed the related notion of extremal length on graphs, mostly in the context of planar effective resistance problems. More recently, O. Schramm [30] used a notion of modulus on graphs to prove a striking uniformization theorem with squares. See also J. W. Cannon [6] for the relation of discrete modulus with the classical Riemann mapping theorem, and P. Haïssinsky [18] for a nice introduction to modulus on graphs.

3.4.1 Modulus in the continuum

In the classical setting of modulus, one considers a family of curves Γ in a domain Ω in the plane. A *density* in this setting is a measurable function $\rho : \Omega \to [0, \infty)$. The density ρ is said to be *admissible* for the family Γ if

$$\int_\gamma \rho \, ds \geq 1, \quad \text{for all } \gamma \in \Gamma.$$

The collection of all admissible densities is called the admissible set, $\mathrm{Adm}(\Gamma)$.

The 2-modulus of the family Γ is defined as

$$\mathrm{Mod}_2(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_\Omega \rho^2 dA.$$

More generally, the p-modulus is defined as

$$\mathrm{Mod}_p(\Gamma) := \inf_{\rho \in \mathrm{Adm}(\Gamma)} \int_\Omega \rho^p dA.$$

The p-modulus framework provides a method for quantifying the richness of a family of curves. Families with many short curves will have a larger modulus than families with fewer and longer curves. The parameter p in the modulus tends to favor the "many curves" aspect when p is close to 1 and the "short curves" aspect as p becomes large.

In this research we will study the discrete version of this theory on graphs using its nature as a convex optimization problem.

3.4.2 Families of objects

This section, reviews the basic framework of modulus on graphs. It considers families Γ of objects γ on G, such as families of walks, cuts, trees, etc., and begins with the general theory, which can be applied to any of these families of objects. To simplify the discussion, we assume that Γ is finite.

A *usage function,* $\mathcal{N}$, defined on $\Gamma \times E$ assigns to each object $\gamma \in \Gamma$ and each edge $e \in E$ a non-negative value $\mathcal{N}(\gamma, e)$ representing the amount by which the object γ "uses" the edge e.

The definition of usage can be tailored to the family of objects and particular application. For instance, if Γ is a family of walks, we can define $\mathcal{N}(\gamma, e)$ as the number of times γ traverses e. If γ is a path, a common choice is

$$\mathcal{N}(\gamma, e) := \mathbb{1}_\gamma(e) = \begin{cases} 1 & \text{if } e \in \gamma, \\ 0 & \text{if } e \notin \gamma. \end{cases} \tag{3.5}$$

Since we are working with finite families of objects on finite graphs, it is often convenient to think of $\mathcal{N}$ as a matrix in $\mathbb{R}_{\geq 0}^{\Gamma \times E}$. Each row of the matrix, $\mathcal{N}(\gamma, \cdot) \in \mathbb{R}_{\geq 0}^{E}$, corresponds to a particular object $\gamma \in \Gamma$ and is referred to as the *usage vector* for that object. It is also often convenient to associate each $\gamma \in \Gamma$ with its associated usage vector. In this way, we may think of any family Γ as a subset of $\mathbb{R}_{\geq 0}^{E}$.

Example 3.4.1. Here are three main examples of possible object families and their corresponding usage matrices.

- A walk $\gamma = x_0 e_1 x_2 e_2 \cdots e_n x_n$ is associated to the function $\mathcal{N}(\gamma, e)$= number times γ traverses e. In this case $\mathcal{N}(\gamma, \cdot)^T \in \mathbb{Z}^E_{\geq 0}$. Note that two distinct walks could have the same usage function $\mathcal{N}(\gamma, \cdot)$.

- A set $T \subset E$, such as a spanning tree or a simple cycle, is associated to the indicator $\mathcal{N}(T, e) = \mathbb{1}_T(e)$, which is equal to 1 if $e \in T$, and equal to 0 otherwise. In this case $\mathcal{N}(T, \cdot)^T \in \{0, 1\}^E$.

- A flow f can be given the usage function $\mathcal{N}(f, e) := |f(e)|$. Then, in this case, $\mathcal{N}(\gamma, \cdot)^T \in \mathbb{R}^E_{\geq 0}$.

3.4.3 Admissible densities

Given a density ρ and an object $\gamma \in \Gamma$, the *total usage cost*, or *ρ-length* of γ is defined as

$$\ell_\rho(\gamma) := \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) = (\mathcal{N}\rho)(\gamma).$$

Given a family of objects Γ, a density $\rho \in \mathbb{R}^E_{\geq 0}$ is *admissible for the family* Γ, if

$$\ell_\rho(\gamma) \geq 1, \quad \forall \gamma \in \Gamma.$$

Equivalently, ρ is admissible for Γ if

$$\ell_\rho(\Gamma) := \inf_{\gamma \in \Gamma} \ell_\rho(\gamma) \geq 1.$$

In matrix notation, we want

$$\mathcal{N}\rho \geq 1,$$

where $\mathcal{N}$ is the $\Gamma \times E$ usage matrix and the inequality is understood elementwise.

The set of admissible densities

$$\mathrm{Adm}(\Gamma) := \{\rho \in \mathbb{R}^E_{\geq 0} : \mathcal{N}\rho \geq 1\} = \bigcap_{\gamma \in \Gamma} \left\{ \rho \in \mathbb{R}^E_{\geq 0} : \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) \geq 1 \right\} \tag{3.6}$$

is an intersection of closed half-spaces, and hence is a closed convex set in $\mathbb{R}^E$.

Example 3.4.2. Consider the unweighted graph G with two edges $e_1 = \{a, b\}$ and $e_2 = \{b, c\}$ in series. In this case $\mathbb{R}^E_{\geq 0} = \mathbb{R}^2_{\geq 0}$ is the first quadrant in the plane. Let $\Gamma = \Gamma(a, c)$ be the connecting family of all walks that start at a and end at c. Every walk $\gamma \in \Gamma$ uses edge e_1 a number of times, n, and edge e_2 a number of times, m, where $n, m \geq 1$ are both odd. Moreover, every pair of odd integers (n, m) is the usage vector for some walk $\gamma \in \Gamma$. The corresponding line

$$n\rho_1 + m\rho_2 = 1$$

intersects the coordinate axes at $(\frac{1}{n}, 0)$ and $(0, \frac{1}{m})$. We can see by inspection that in this case

$$\mathrm{Adm}(\Gamma) = \{\rho \in \mathbb{R}^2_{\geq 0} : \rho_1 + \rho_2 \geq 1\}, \tag{3.7}$$

since the inequality with $(n, m) = (1, 1)$ implies all of the other inequalities.

3.4.4 The p-energy

For $1 \leq p < \infty$, the *p-energy* of a density $\rho \in \mathbb{R}^E_{\geq 0}$ is defined as

$$\mathcal{E}_{p,\sigma}(\rho) := \sum_{e \in E} \sigma(e)\rho(e)^p, \tag{3.8}$$

and when $p = \infty$, the *∞-energy* is defined as

$$\mathcal{E}_{\infty,\sigma}(\rho) := \max_{e \in E} \sigma(e)\rho(e). \tag{3.9}$$

Lemma 3.4.3. *Suppose that $s : (1, \infty) \to (0, \infty)$ is a function with the property that*

$s(p)/p \to r$ as $p \to \infty$, for some $r \in [0, \infty)$. Then for any density $\rho \in \mathbb{R}^E_{\geq 0}$

$$\lim_{p\to\infty} \mathcal{E}_{p,\sigma^{s(p)}}(\rho)^{1/p} = \mathcal{E}_{\infty,\sigma^r}(\rho).$$

Proof. Let $p \in (1, \infty)$ and let $s = s(p)$. Choose $e_0 \in E$ with the property that $\mathcal{E}_{\infty,\sigma^{s/p}}(\rho) = \sigma(e_0)^{s/p}\rho(e_0)$. Then

$$\begin{aligned}
\mathcal{E}_{\infty,\sigma^{s/p}}(\rho) &= \sigma(e_0)^{s/p}\rho(e_0) \\
&\leq \left(\sum_{e\in E} \sigma(e)^s \rho(e)^p\right)^{1/p} \\
&= \mathcal{E}_{p,\sigma^s}(\rho)^{1/p} \\
&= \mathcal{E}_{\infty,\sigma^{s/p}}(\rho) \left(\sum_{e\in E} \left(\frac{\sigma(e)^{s/p}\rho(e)}{\sigma(e_0)^{s/p}\rho(e_0)}\right)^p\right)^{1/p} \\
&\leq \mathcal{E}_{\infty,\sigma^{s/p}}(\rho)|E|^{1/p}.
\end{aligned}$$

The limit formula follows by continuity of $\mathcal{E}_{\infty,\sigma}(\rho)$ in σ. $\square$

In particular, the lemma implies the following two limits.

$$\lim_{p\to\infty} \mathcal{E}_{p,\sigma}(\rho)^{1/p} = \mathcal{E}_{\infty,1}(\rho) \qquad \text{and} \qquad \lim_{p\to\infty} \mathcal{E}_{p,\sigma^p}(\rho)^{1/p} = \mathcal{E}_{\infty,\sigma}(\rho).$$

3.4.5 Definition of p-modulus

Given a graph $G = (V, E, \sigma)$, an exponent $1 \leq p \leq \infty$, and a family Γ, the *p-modulus of* Γ is defined as

$$\mathrm{Mod}_{p,\sigma}(\Gamma) := \inf_{\rho\in \mathrm{Adm}(\Gamma)} \mathcal{E}_{p,\sigma}(\rho). \tag{3.10}$$

Written in more standard convex optimization notation, p-modulus is the value of the problem

$$\begin{aligned} \underset{\rho}{\text{minimize}} \quad & \mathcal{E}_{p,\sigma}(\rho) \\ \text{subject to} \quad & \mathcal{N}\rho \geq 1, \\ & \rho \in \mathbb{R}^E_{\geq 0}. \end{aligned} \tag{3.11}$$

where each object γ adds one constraint to be satisfied.

3.4.6 Some properties of p-modulus

Assumption 3.4.4. In what follows, we make the following assumptions on the family Γ of objects on $G = (V, E, \sigma)$.

1. **Non-emptyness:** $\Gamma \neq \emptyset$.
2. **Non-triviality:** Every object $\gamma \in \Gamma$ has non-trivial usage $\mathcal{N}(\gamma, \cdot) \not\equiv 0$.
3. **Discreteness:** Every edge used by an object should be used a definite amount:

$$\mathcal{N}_{\min} := \inf_{\gamma \in \Gamma} \min_{e \in E : \mathcal{N}(\gamma, e) \neq 0} \mathcal{N}(\gamma, e) > 0.$$

Proposition 3.4.5. *Let Γ be a family of objects as in Assumption 3.4.4, then*

1. *For $1 \leq p \leq \infty$ a minimizer ρ^* for (3.11) always exist. These minimizers are called extremal densities.*
2. *Extremal densities satisfy*

$$0 \leq \rho^* \leq \mathcal{N}_{\min}^{-1}.$$

3. *For $1 < p < \infty$, the extremal density ρ^* is unique.*

Proof. To prove that a minimizer ρ^* always exist in 1, we use the non-triviality assumption. We can choose $R > 0$ large enough so that

$$K := \mathrm{Adm}(\Gamma) \cap \{\rho \in \mathbb{R}^E_{\geq 0} \mid \mathcal{E}_{p,\sigma}(\rho) \leq R\} \neq \emptyset.$$

K is compact, since it is a closed and bounded set and $\mathbb{R}^E$ has finite dimension. The p-norms are continuous and therefore there always exists a minimizer.

For 2, suppose ρ^* is an extremal density for $\mathrm{Mod}_{p,\sigma}(\Gamma)$ and assume that $\rho^*(e_0) > \mathcal{N}_{\min}^{-1}$ on some edge $e_0 \in E$. Define a new density $\rho(e)$ as

$$\rho(e) := \begin{cases} \rho^*(e) & e \neq e_0, \\ \mathcal{N}_{\min}^{-1} & e = e_0. \end{cases}$$

We claim that ρ is admissible for Γ. If an object γ does not use e_0, then $\ell_\rho(\gamma) = \ell_{\rho^*}(\gamma) \geq 1$. Otherwise, if an object γ uses e_0 then,

$$\ell_\rho(\gamma) = \sum_{e \neq e_0} \mathcal{N}(\gamma, e)\rho(e) + \mathcal{N}(\gamma, e_o)\mathcal{N}_{\min}^{-1} \geq 1.$$

However,

$$\begin{aligned} \mathcal{E}_{p,\sigma}(\rho) &= \sum_{e \neq e_0} \sigma(e)\rho(e)^p + \sigma(e_0)\mathcal{N}_{\min}^{-p} \\ &< \sum_{e \neq e_0} \sigma(e)\rho(e)^p + \sigma(e_0)(\rho^*)^p = \mathcal{E}_{p,\sigma}(\rho^*), \end{aligned}$$

which is contradiction since ρ^* is an extremal density.

Finally to prove 3, assume that we have two extremal densities ρ_1^* and ρ_2^*, where $\rho_1^* \neq \rho_2^*$. The set $\mathrm{Adm}(\Gamma)$ is convex, so

$$\rho_0 := \frac{\rho_1^* + \rho_2^*}{2},$$

is also admissible. However, for $\rho \in (1, \infty)$ the function $\mathcal{E}_{p,\sigma}(\rho)$ is strictly convex, so

$$\mathcal{E}_{p,\sigma}(\rho_0) < \frac{\mathcal{E}_{p,\sigma}(\rho_1^*) + \mathcal{E}_{p,\sigma}(\rho_2^*)}{2} = \mathrm{Mod}_{p,\sigma}(\Gamma).$$

which is contradiction, so $\rho_1^* = \rho_2^*$. □

Proposition 3.4.6 (Basic properties of of modulus). *Let Γ be a family of walks in graph G. The following properties hold*

1. ***Constant walks:*** *If Γ contains a constant walk (walk with zero hops), then* $\mathrm{Mod}(\Gamma) = \infty$.
2. ***Empty family:*** *If* $\Gamma = \emptyset$, *then* $\mathrm{Mod}(\Gamma) = 0$.
3. ***Monotonicity:*** *If* $\Gamma_1 \subset \Gamma_2$, *then* $\mathrm{Mod}(\Gamma_1) \leq \mathrm{Mod}(\Gamma_2)$.
4. ***Countable subadditivity:*** $\mathrm{Mod}(\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} \mathrm{Mod}(\Gamma_i)$.

Proof. Refer to [5]. □

3.5 Connections between p-modulus and max flow

Let $\mathcal{N}$ be the usage matrix in (3.5). The standard path formulation of the maximum flow problem in (3.4) can be rewritten as the path formulation of the maximum flow problem:

$$\begin{aligned} \underset{x(\gamma)}{\text{maximize}} \quad & \sum_{\gamma \in \Gamma} x(\gamma) \\ \text{subject to} \quad & \sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e) x(\gamma) \leq \sigma(e), \quad \forall e \in E, \\ & x(\gamma) \in \mathbb{R}^{\Gamma}_{\geq 0}. \end{aligned} \tag{3.12}$$

Define the vectors

$$x = [x(\gamma)], \quad \forall \gamma \in \Gamma,$$
$$\sigma = [\sigma(e)], \quad \forall e \in E,$$
$$\mathbb{1} = [\mathbb{1}_\Gamma],$$

where

$$\mathbb{1}_\Gamma = \begin{cases} 1 & \gamma \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the objective function $\sum_{\gamma\in\Gamma} x(\gamma)$, and constraint $\sum_{\gamma} \mathcal{N}(\gamma, e)x(\gamma)$ in (3.12) can be written as

$$\sum_{\gamma\in\Gamma} x(\gamma) = \mathbb{1}^T x,$$
$$\sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)x(\gamma) = (\mathcal{N}^T x)_e, \quad \forall e \in E.$$

So, the matrix form of the problem (3.12) will be

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & -\mathbb{1}^T x \\ \text{subject to} \quad & \mathcal{N}^T x \leq \sigma, \\ & x \geq 0. \end{aligned} \tag{3.13}$$

Now, consider the dual to the maximum flow problem (3.13). We start with the Lagrangian

$$\begin{aligned} L(x, \rho, \lambda) &= -\mathbb{1}^T x + \rho^T(\mathcal{N}^T x - \sigma) - \lambda^T x \\ &= x^T(\mathbb{1} + \mathcal{N}\rho - \lambda) - \rho^T \sigma, \end{aligned}$$

with

$$\rho = [\rho(e)], \quad \forall e \in E,$$
$$\lambda = [\lambda_\gamma], \quad \gamma \in \Gamma,$$

as dual variables, and next find the Lagrangian dual function $g(\rho, \lambda)$:

$$\begin{aligned} g(\rho,\lambda) &= \inf_x L(x,\rho,\lambda) \\ &= \begin{cases} -\rho^T\sigma & -\mathbb{1} + \mathcal{N}\rho - \lambda = 0, \\ -\infty & \text{else.} \end{cases} \end{aligned}$$

So, the dual problem for (3.13) takes the form:

$$\begin{aligned} \underset{\rho}{\text{minimize}} \quad & \sigma^T\rho \\ \text{subject to} \quad & \mathcal{N}\rho \geq \mathbb{1}, \\ & \rho \in \mathbb{R}^E_{\geq 0}. \end{aligned} \tag{3.14}$$

We use the following equality to relate this problem to the 1-modulus problem

$$(\mathcal{N}\rho)_\gamma = \sum_e \mathcal{N}(\gamma, e)\rho(e) = \ell_\rho(\gamma), \quad \gamma \in \Gamma,$$

and rewrite the equivalent form of the problem (3.14) as its modulus form:

$$\begin{aligned} \underset{\rho}{\text{minimize}} \quad & \sigma^T\rho \\ \text{subject to} \quad & \sum_{e\in E} \mathcal{N}(\gamma, e)\rho \geq 1, \quad \gamma \in \Gamma, \\ & \rho \in \mathbb{R}^E_{\geq 0}. \end{aligned} \tag{3.15}$$

3.6 Special cases

The p-modulus, $\mathrm{Mod}_{p,\sigma}(\Gamma)$ problem is

$$\begin{aligned} \underset{\rho}{\text{minimize}} \quad & \mathcal{E}_{p,\sigma}(\rho) \\ \text{subject to} \quad & \sum_{e\in E} \mathcal{N}(\gamma,e)\rho(e) \geq 1, \quad \gamma \in \Gamma, \\ & \rho \in \mathbb{R}^E_{\geq 0}. \end{aligned} \tag{3.16}$$

When $\Gamma = \Gamma(s,t)$, p-modulus is related to certain classical graph-theoretic quantities [3]. In particular,

- $\mathrm{Mod}_{1,\sigma}(\Gamma(s,t))$ is the value of the maximum st-flow, where the σ are treated as capacities. By the max-flow min-cut theorem, $\mathrm{Mod}_{1,\sigma}(\Gamma(s,t))$ is also equal to the value of the min cut for s and t,
- $\mathrm{Mod}_{2,\sigma}(\Gamma(s,t))^{-1} = \mathrm{R}_{\text{eff}}(s,t)$ is the effective resistance in the network between s and t, where the σ are treated as conductances, and
- $\mathrm{Mod}_{\infty,\sigma}(\Gamma(s,t))^{-1} = \mathrm{dist}_{\sigma^{-1}}(s,t)$ is the usual graph distance measured with respect to edge weights σ^{-1}.

3.7 Duality and the probabilistic interpretation

Let $\mathcal{P} = \mathcal{P}(\Gamma)$ represent the set of probability mass functions (pmfs) on the set Γ,

$$\mathcal{P}(\Gamma) = \left\{ \mu \in \mathbb{R}^\Gamma_{\geq 0} : \sum_{\gamma\in\Gamma} \mu(\gamma) = 1 \right\},$$

in other words $\mathcal{P}$ is the set of $\mu \in \mathbb{R}^\Gamma_{\geq 0}$ with the property that $\mu^T \mathbb{1} = 1$. From such a μ we can define $\underline{\gamma}$ a random object in Γ sampled with a given probability μ:

$$\mathbb{P}_\mu(\underline{\gamma} = \gamma) := \mu(\gamma).$$

For an edge $e \in E$, the value $\mathcal{N}(\underline{\gamma}, e)$ is a random variable with expectation

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e)\mu(\gamma).$$

In the particular case that $\mathcal{N}$ is defined as in (3.5), this expectation is actually an *edge usage probability*:

$$\mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)] = \mathbb{P}_\mu(e \in \underline{\gamma}).$$

The probabilistic interpretation of the dual for (3.11) can be stated as follows [4, Remark 5.5].

$$\left(\mathrm{Mod}_{p,\sigma}(\Gamma)\right)^{-\frac{q}{p}} = \min_{\mu \in \mathcal{P}(\Gamma)} \sum_{e \in E} \sigma(e)^{-\frac{q}{p}} \mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)]^q, \tag{3.17}$$

where $q = \frac{p}{p-1}$.

When $p = 2$, and Γ is a path family with $\mathcal{N}$ given in (3.5), the formula simplifies to

$$\begin{aligned} \mathrm{Mod}_{2,\sigma}(\Gamma)^{-1} &= \min_{\mu \in \mathcal{P}(\Gamma)} \sum_{e \in E} \sigma(e)^{-1} \mathbb{E}_\mu(\mathcal{N}(\underline{\gamma}, e))^2 \\ &= \min_{\mu \in \mathcal{P}(\Gamma)} \sum_{e \in E} \sigma(e)^{-1} \mathbb{P}_\mu(e \in \underline{\gamma})^2. \end{aligned} \tag{3.18}$$

Remark 3.7.1. In general, (3.17) admits infinitely many minimizing pmfs μ^*. However, when $1 < p < \infty$, all such minimizers give the same expected edge usages

$$\eta^*(e) := \mathbb{E}_\mu[\mathcal{N}(\underline{\gamma}, e)].$$

Indeed, $\mu \in \mathcal{P}$ is optimal for (3.17) if and only if its expected edge usages are given by this unique η^*.

Chapter 4

Maximum concurrent flow and modulus

The extension of *st*-flow problems to the case of multiple source-sink pairs, called *multicommodity flow problems* has been discussed by many researchers [2, 11, 12, 21, 23, 24], but the first work dealing with multicommodity flow problems is due to L. R. Ford, D.R. Fulkerson and W. S. Jewell [11, 12, 22]. Even though the underlying idea is common in these various works, the treatment of multicommodity flow tends to differ in some details such as the way in which capacity constraints, or the objective function, are formulated.

In some of these problems, there may be capacities only on the total flow on an edge, some cases have individual capacities for each commodity on each edge, and some have both types of capacities. Typically, multicommodity flow problems consist of a graph with capacities defined on the edges, and a set of source-sink pairs $\{(s_i, t_i), i = 1, \ldots, k\}$. The task is to find flows $f_1, \ldots, f_k$, where f_i is a flow of the i-th commodity from s_i to t_i, so that some objective function of the value of these flows is optimized. The simplest multicommodity flow problem is the *maximum multicommodity flow problem*. In this case the object is to maximize the sum of the flow values. K. Onaga [20] established a necessary and sufficient condition for the existence of a feasible multicommodity flow configuration on a capacity-constrained undirected network when the locations of source and sink as well as the total flow value are given for each commodity. A generalization of this problem is the *maximum weight multicommodity flow problem* in which we are given weights and we want to maximize the weighted sum of the flows. The *maximum concurrent flow problem* is a more complex and popular variation of the maximum multicommodity flow problem, where we are given a set of k positive demands $d_1, \cdots, d_k$. F. Shahrokhi and D. W. Matula [31] developed the first fully polynomial-time approximation scheme for the maximum concurrent flow problems with uniform edge capacities. They introduced the idea of using an exponential length function to control edge congestion. The maximum concurrent flow problem is also interesting in the context of approximation algorithms, due to the fact that it is the dual of the LP relaxation of an NP-hard sparsest cut problem [25].

A line of research based on Lagrangian relaxation and linear programming decomposition techniques has decreased the running times for multicommodity flow problems [17, 24, 27, 28].

N. E. Young [32] described a randomized algorithm which departs from this general theme. At every step his algorithm solves a shortest path problem, instead of a maximum single commodity flow problem in earlier algorithms.

N. Grag and J. Könemann [13] followed a similar approach. They managed to provide an elegant framework for solving multicommodity flow problems that yields a simple analysis of the correctness of the obtained algorithm. While Young's procedure pushes a unit flow at each step along a shortest path, Grag and Könemann introduced a new but similar procedure which pushes enough flow so as to saturate the minimum capacity edge of the path. Their results generalize to fractional packing problems.

L. K. Fleisher [10] developed significantly faster algorithms for various multicommodity flow problems. She noticed that we can use an approximate shortest path in order to avoid recalculations of shortest path for commodities with a common source. Her results for concurrent flow problems were later improved by G. Karakostas [23], who was able to reduce the dependence of the computational complexity on the number of source-sink pairs, k.

A. Madry [26] combined the work of Grag and Könemann, and Fleischer with ideas from dynamic graph algorithms to obtain faster $(1-\epsilon)$-approximation schemes for various versions of the multicommodity flow problem.

This thesis studies the maximum concurrent flow problem where we are given a set of k positive demands $d_1, \cdots, d_k$, and are asked to find a multicommodity flow that is feasible and that routes at least a fixed proportion zd_i of each demand between each source-sink pair, with the goal of maximizing the common factor of proportionality z. For simplicity, in this dissertation we will assume that the demands add up to one. (See Remark 5.1.1.) It will sometimes be beneficial to additionally assume that the edge capacities are all equal to one. We will call this the unit capacity version of the problem.

4.1 Maximum concurrent flow problems

In the multicommodity maximum flow problem, the single source-sink pair (s, t) is replaced with a set of sources and sinks $D = \{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}$. The goal is to choose a set of flows, $f_1, f_2, \ldots, f_k$, where each f_i is an s_it_i-flow. These flows represent the transfer of k different commodities among the k sources and sinks within the network. The flows are considered to share the network's resources and, therefore, are given a *joint capacity constraint* of the form

$$\sum_{i=1}^{k} |f_i(e)| \leq \sigma(e), \qquad \forall e \in E.$$

The concurrent maximum flow problem is a type of multiobjective optimization problem. We would like to simultaneously maximize the values of all flows. However, since the flows share network resources, some balance among these objectives is needed. For the concurrent maximum flow problem, this balance is achieved by maximizing the objective

$$\min_{i=1,\ldots,k} \frac{\mathrm{Val}(f_i)}{d_i},$$

where the d_i are positive numbers, typically referred to as *demands*. By introducing an auxiliary variable z to replace the minimum, the maximum concurrent flow problem can be written in the form

$$\begin{aligned}
\underset{f_i,z}{\text{maximize}} \quad & z \\
\text{subject to} \quad & f_i \text{ an } s_it_i\text{-flow}, \quad \text{for } i = 1,\dots,k, \\
& \mathrm{Val}(f_i) \geq zd_i, \quad \text{for } i = 1,\dots,k, \\
& \sum_{i=1}^{k} |f_i(e)| \leq \sigma(e), \quad \text{for all } e \in E.
\end{aligned}$$

Using the definitions of st-flow in Definition 3.2.1 and value of a flow $\mathrm{Val}(f_i)$ in in (3.1), the concurrent problem can be written equivalently as

$$\begin{aligned}
\underset{f_i,z}{\text{maximize}} \quad & z \\
\text{subject to} \quad & 1) \sum_{u \sim v} f_i(u,v) = 0, \quad \forall i = 1,\dots,k,\ \forall v \in V \setminus \{s_i,t_i\}, \\
& 2) \sum_{v} f_i(s_i,v) \geq zd_i, \quad \forall i = 1,\dots,k, \\
& 3) \sum_{i=1}^{k} |f_i(e)| \leq \sigma(e), \quad \forall e \in E, \\
& 4) f_i(u,v) = -f_i(v,u), \quad \forall i = 1,\dots,k,\ \forall \{u,v\} \in E.
\end{aligned} \tag{4.1}$$

The concurrent problem can also be written in path form. To do so, we let

$$\Gamma_i = \Gamma(s_i,t_i) := \{ \text{ the family of paths from } s_i \text{ to } t_i \text{ , for all } i\},$$

and let $\Gamma := \overset{k}{\underset{i=1}{\cup}} \Gamma_i$ be their disjoint union. Again, since cycle flows add no value and use resources, we can restrict attention to flows f_i that can be written as a sum of path flows along paths in Γ_i. For each $\gamma \in \Gamma$, we introduce a variable $x_\gamma \geq 0$ to represent the path flow strength.

$$f(u,v) = \sum_{\substack{\gamma \in \Gamma \\ (u,v) \in \gamma}} x(\gamma) - \sum_{\substack{\gamma \in \Gamma \\ (v,u) \in \gamma}} x(\gamma). \tag{4.2}$$

With this insight, the path formulation of concurrent max flow problem can be written as

$$\begin{aligned}
&\underset{z, x(\gamma)}{\text{maximize}} \quad z \\
&\text{subject to} \qquad 1') \sum_{\gamma \in \Gamma_i} x(\gamma) \geq z d_i, \quad i = 1, \ldots, k, \\
&\qquad\qquad\qquad 2') \sum_{\substack{\gamma \in \Gamma \\ e \in \gamma}} x(\gamma) \leq \sigma(e), \quad \forall e \in E, \\
&\qquad\qquad\qquad 3') \quad x(\gamma) \geq 0, \quad \forall \gamma \in \Gamma.
\end{aligned} \tag{4.3}$$

The problems in (4.1) and in (4.3) are equivalent problems. To see this, first let $\{x(\gamma)\}_{\gamma \in \Gamma}$ be a feasible choice for the problem in (4.3), we show that there exists a feasible choice for the problem in (4.1) with the same or higher value.

Constraint 1 in problem (4.1) follows from problem (4.2) and the divergence-free condition in Definition 3.2.1:

$$\begin{aligned}
\sum_{u \sim v} f_i(u, v) &= \sum_{u \sim v} \Big(\sum_{\substack{\gamma \in \Gamma_i \\ (u,v) \in \gamma}} x(\gamma) - \sum_{\substack{\gamma \in \Gamma_i \\ (v,u) \in \gamma}} x(\gamma) \Big) \\
&= \sum_{u \sim v} \sum_{\substack{\gamma \in \Gamma_i \\ (u,v) \in \gamma}} x(\gamma) - \sum_{u \sim v} \sum_{\substack{\gamma \in \Gamma_i \\ (v,u) \in \gamma}} x(\gamma) \\
&= 0.
\end{aligned}$$

Constraint 2 follows from the constraint $1'$ in (4.3) and also the fact that there are no flows entering the sources s_i:

$$\sum_{v} f_i(s_i, v) = \sum_{v} \sum_{\substack{\gamma \in \Gamma_i \\ (s_i,v) \in \gamma}} x(\gamma) - \sum_{v} \sum_{\substack{\gamma \in \Gamma_i \\ (v,s_i) \in \gamma}} x(\gamma) = \sum_{v} \sum_{\substack{\gamma \in \Gamma_i \\ \{s_i,v\} \in \gamma}} x(\gamma) \geq z d_i.$$

Constraint 3 in problem (4.1) follows from the triangle inequality, and constraint $2'$ in (4.3).

The absolute values can be dropped since the flows $x(\gamma)$ are non-negative.

$$\begin{aligned}\sum_{i=1}^{k}|f_i(u,v)| &= \sum_{i=1}^{k}\Big|\sum_{\substack{\gamma\in\Gamma_i\\(u,v)\in\gamma}}x(\gamma)-\sum_{\substack{\gamma\in\Gamma_i\\(v,u)\in\gamma}}x(\gamma)\Big| \leq \\ &\leq \sum_{i=1}^{k}\left(\left|\sum_{\substack{\gamma\in\Gamma_i\\(u,v)\in\gamma}}x(\gamma)\right|+\left|\sum_{\substack{\gamma\in\Gamma_i\\(v,u)\in\gamma}}x(\gamma)\right|\right) \\ &= \sum_{\substack{\gamma\in\Gamma\\(u,v)\in\gamma}}x(\gamma)+\sum_{\substack{\gamma\in\Gamma\\(v,u)\in\gamma}}x(\gamma) \\ &\leq \sigma(u,v).\end{aligned}$$

Finally constraint 4 holds because

$$f_i(u,v)=\sum_{\substack{\gamma\in\Gamma_i\\(u,v)\in\gamma}}x(\gamma)-\sum_{\substack{\gamma\in\Gamma_i\\(v,u)\in\gamma}}x(\gamma)=-\Big(\sum_{\substack{\gamma\in\Gamma_i\\(v,u)\in\gamma}}x(\gamma)-\sum_{\substack{\gamma\in\Gamma_i\\(u,v)\in\gamma}}x(\gamma)\Big)=-f_i(v,u).$$

So, the problem in (4.1) is an upper bound for problem in (4.3). Next, we show that the problem in (4.3) is a upper bound for the problem in (4.1); if $f_i, i=1,\dots,k$ are s_it_i-flows, and are feasible for the problem in (4.1), then there is a feasible choice for the problem in (4.3) with the same or higher value. This follows from the following lemma.

Lemma 4.1.1. *Let f be an st-flow. Then there exists a subfamily $\{\gamma_1,\dots,\gamma_k\}\subset\Gamma$ and positive $x_1,\dots,x_k$ such that, if $\tilde f$ is the flow $\sum_{i=1}^{k}x_k\gamma_k$, then $\mathrm{Val}(\tilde f)\geq\mathrm{Val}(f)$ and $|\tilde f|\leq|f|$.*

Proof. We construct the flow $\tilde f$ by first replacing the graph $G=(V,E)$ by a directed graph $\vec G=(V,\vec E)$ as follows. For each $e=\{u,v\}\in E$ we set $\vec e$ to be exactly one of (u,v) or (v,u) in such a way that $f(\vec e)\geq 0$. Then $\vec E:=\{\vec e : e\in E\}$. Thus, f becomes a flow on a directed graph. We then construct a new flow $\tilde f$ by Algorithm 1. This flow can then be transferred back to the undirected graph by the antisymmetry property. We claim that $\tilde f$ has the properties described in the lemma.

First, observe that, on each iteration, the value of r on at least one edge becomes 0.

Algorithm 1 Constructs the flow $\tilde{f}$ for Lemma 4.1.1

$r \leftarrow f$
$S \leftarrow \varnothing$
while $\exists \gamma \in \Gamma(s,t)$, s.t. $r(u,v) > 0,\ \forall (u,v) \in \gamma$ **do**
  $x \leftarrow \min_{(u,v)\in\gamma} r(u,v)$
  $S \leftarrow S \cup \{(\gamma, x)\}$
  $r(u,v) \leftarrow r(u,v) - x, \quad \forall (u,v) \in \gamma$
end while
$\tilde{f} \leftarrow \sum_{(\gamma,x)\in S} x\mathbb{1}_\gamma$

Thus, the algorithm must terminate in no more than $|E|$ iterations. Moreover, throughout the algorithm, the relation

$$f = r + \sum_{(\gamma,x)\in S} x\mathbb{1}_\gamma,$$

is maintained, where $\mathbb{1}_\gamma$ is the unit path flow along the path γ. In particular, this implies that, when the algorithm terminates,

$$\mathrm{Val}(f) = \mathrm{Val}(r) + \sum_{(\gamma,x)\in S} x = \mathrm{Val}(r) + \mathrm{Val}(\tilde{f}).$$

When the algorithm terminates, $\mathrm{Val}(r) \leq 0$ since otherwise we would be able to find a path from s to t with positive r-flow, contradicting the stopping condition of the algorithm. Thus, $\mathrm{Val}(\tilde{f}) \geq \mathrm{Val}(f)$. Finally, the property $|\tilde{f}| \leq |f|$ holds by construction. □

Using the usage matrix in (3.5) the concurrent max flow problem in (4.3) can be written as

$$\begin{aligned} \underset{z,x(\gamma)}{\text{maximize}} \quad & z \\ \text{subject to} \quad & \sum_{\gamma\in\Gamma_i} x(\gamma) \geq z d_i, \qquad i = 1,\dots,k, \\ & \sum_{\gamma\in\Gamma} \mathcal{N}(\gamma,e)x(\gamma) \leq \sigma(e), \qquad \forall e \in E, \\ & x(\gamma) \geq 0, \qquad \forall \gamma \in \Gamma. \end{aligned} \tag{4.4}$$

For each $(s_i, t_i) \in D$, define the indicator function $\mathbb{1}_{\Gamma_i} : \Gamma \to \{0, 1\}$ as

$$\mathbb{1}_{\Gamma_i}(\gamma) = \begin{cases} 1 & \text{if} \quad \gamma \in \Gamma_i, \\ 0 & \text{otherwise.} \end{cases}$$

Define usage matrix $\mathcal{N}$ of the graph G with dimension $|\Gamma| \times |E|$ as

$$\mathcal{N}(\gamma, e) = \begin{cases} 1 & \text{if} \quad e \in \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

and demand matrix M with dimension $k \times |\Gamma|$ as

$$M = (\mathbb{1}_{\Gamma_1}, \dots, \mathbb{1}_{\Gamma_k})^T. \tag{4.5}$$

Also, define vectors

$$x = [x(\gamma)], \quad \forall \gamma \in \Gamma,$$

$$d = [d_i], \quad \forall i = 1, \dots, k,$$

$$\sigma = [\sigma(e)], \quad \forall e \in E.$$

Then, the constraints in (4.4) can be written as

$$\sum_{\gamma \in \Gamma_i} x_\gamma = \mathbb{1}_{\Gamma_i}^T x = (Mx)_i, \quad i = 1, \dots, k,$$

$$\sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e) x_\gamma = (\mathcal{N}^T x)_e, \quad \forall e \in E.$$

So, the problem in (4.4) can be expressed in the form

$$\begin{aligned}
\underset{z,x}{\text{maximize}} \quad & z \\
\text{subject to} \quad & Mx \geq zd, \\
& \mathcal{N}^T x \leq \sigma, \\
& x \geq 0.
\end{aligned} \tag{4.6}$$

4.2 Connecting maximum concurrent flow problem to the modulus

Inspired by the fact that 1-modulus is the dual of the single-source single-sink max flow problem (see Remark 5.1.3), we now consider the dual to the concurrent max flow problem in (4.6).

The Lagrangian of this problem is

$$\begin{aligned}
L(z,x,w,\rho,\lambda) &= -z - w^T(Mx - zd) + \rho^T(\mathcal{N}^T x - \sigma) - \lambda^T x \\
&= x^T(-M^T w + \mathcal{N}\rho - \lambda) + z(-1 + w^T d) - \rho^T \sigma,
\end{aligned}$$

where

$$w = [w_i], \quad \rho = [\rho(e)], \quad \lambda = [\lambda_\gamma], \quad d = [d_i], \quad \forall e \in E,\ \gamma \in \Gamma_i,\ \forall i = 1,\ \ldots, k.$$

The dual function $g(w,\rho,\lambda)$ is

$$\begin{aligned}
g(w,\rho,\lambda) &= \inf_{z,x} L(z,x,w,\rho,\lambda) \\
&= \begin{cases} -\rho^T\sigma & -M^T w + \mathcal{N}\rho - \lambda = 0 \quad \text{and} \quad -1 + w^T d = 0, \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore the dual problem for the problem in (4.4) takes the following form

$$
\begin{aligned}
\underset{\rho,w}{\text{minimize}} \quad & \sigma^T \rho \\
\text{subject to} \quad & \mathcal{N}\rho \geq M^T w, \\
& w^T d = 1, \\
& w \in \mathbb{R}^k_{\geq 0}, \\
& \rho \in \mathbb{R}^E_{\geq 0},
\end{aligned}
\tag{4.7}
$$

where

$$(\mathcal{N}\rho)(\gamma) = \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) = \ell_\rho(\gamma), \quad \gamma \in \Gamma_i,\ i = 1, \ldots, k,$$

$$(M^T w)(\gamma) = \sum_{i=1}^{k} \mathbb{1}^T_{\Gamma_i} w(s_i, t_i) = w_\gamma, \quad \gamma \in \Gamma_i,\ i = 1, \ldots, k.$$

So, the problem in (4.7) can be written as

$$
\begin{aligned}
\underset{\rho,w}{\text{minimize}} \quad & \sigma^T \rho \\
\text{subject to} \quad & \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) \geq w_i, \qquad \forall \gamma \in \Gamma_i,\ i = 1, \ldots, k, \\
& \sum_{i=1}^{k} w_i d_i \geq 1, \\
& w \in \mathbb{R}^k_{\geq 0}, \qquad \rho \in \mathbb{R}^E_{\geq 0}.
\end{aligned}
\tag{4.8}
$$

The usage vectors of the paths γ in the usage matrix $\mathcal{N}$ can be scaled by their corresponding path weight w_i. Let

$$\mathcal{N}_w(\gamma, e) = \frac{\mathcal{N}(\gamma, e)}{w_i} \tag{4.9}$$

when for $\gamma \in \Gamma_i$. (See Remark 4.2.1 for the interpretation when $w_i = 0$.)

So the equivalent form of the problem in (4.8) will be

$$
\begin{aligned}
\underset{\rho,w}{\text{minimize}} \qquad & \sigma^T\rho \\
\text{subject to} \qquad & \sum_{e\in E}\mathcal{N}_w(\gamma,e)\rho(e) \geq 1, \qquad \forall\gamma\in\Gamma_i,\ i=1,\ldots,k, \\
& \sum_{i=1}^{k} w_i d_i \geq 1, \\
& w\in\mathbb{R}^k_{\geq 0}, \qquad \rho\in\mathbb{R}^E_{\geq 0}.
\end{aligned}
\tag{4.10}
$$

which is a parameterized 1-modulus problem.

Remark 4.2.1. The case $w_i = 0$, requires a special interpretation. In this case, we may think of $\mathcal{N}_w(\gamma,e) = +\infty$ if $e\in\gamma\in\Gamma_i$ and $\mathcal{N}_w(\gamma,e) = 0$ if $e\notin\gamma\in\Gamma_i$. Moreover, when $\gamma\in\Gamma_i$ in this case, we define

$$\sum_{e\in E}\mathcal{N}_w(\gamma,e)\rho(e) \geq 1,$$

to be true for any $\rho\in\mathbb{R}^E_{\geq 0}$. This is consistent with the original formulation (4.8) because, when $w_i = 0$, the objects in Γ_i do not provide any additional constraints on ρ not already implied by $\rho \geq 0$.

4.3 Karush-Kuhn-Tucker conditions and optimality

At this point we study the strong duality between the primal problem in (4.4) and its dual in (4.8). First we check the Slater's condition for (4.8), and then look at the Karush-Kuhn-Tucker (KKT) optimality conditions.

The objective in (4.8) is a convex function and the constraints are inequality constraints and are convex functions, therefore the problem is a convex optimization problem. The objective and the constraints are also differentiable. We can find an strictly feasible point in $x\in\mathbb{R}^n$ such that all constraints hold strictly, so the problem satisfies in Slater's condition, therefore the strong duality holds by Theorem 2.3.1. Hence by Corollary 2.4.7 the KKT conditions provide necessary and sufficient conditions for optimality.

The Karush-Kuhn-Tucker (KKT) optimality conditions for (4.4) and its dual in (4.8) include stationary and complementary slackness conditions, in addition to the feasibility conditions for the primal and dual problems.

$$\begin{aligned} L(\rho, w, x, z, \lambda, \mu) &= \rho^T\sigma + x^T(M^T w - \mathcal{N}\rho) + z(1 - w^T d) - \lambda^T w - \mu^T \rho \\ &= \rho(\sigma^T - \mathcal{N}^T x - \mu) + w^T(Mx - zd - \lambda^T \mathbb{1}) + z \end{aligned} \tag{4.11}$$

The gradients with respect to variables ρ and w vanish, so:

$$\nabla_\rho L = \sigma + \mathcal{N}^T x - \mu = 0,$$
$$\nabla_w L = Mx - zd - \lambda = 0,$$

therefore the stationary conditions are:

$$\begin{aligned} &\sigma(e) - \sum_{i=1}^{k} \sum_{\gamma \in \Gamma_i} \mathcal{N}(\gamma, e) x(\gamma) = \mu(e) \geq 0, \quad \forall\, e \in E, \\ &\sum_{\gamma \in \Gamma_i} x(\gamma) - d_i z = \lambda_i \geq 0, \quad \forall i = 1, \ldots, k. \end{aligned} \tag{4.12}$$

The complementary slackness conditions are:

$$\begin{aligned} &\sum_{\gamma \in \Gamma_i} x(\gamma)(w_i - \ell_\rho(\gamma)) = 0, \quad \forall\, i = 1, \ldots k, \\ &z(1 - \sum_{i=1}^{k} d_i w_i) = \sum_{e \in E} \mu(e)\rho(e) = \sum_{i=1}^{k} \lambda_i w_i = 0, \end{aligned} \tag{4.13}$$

Where the quantities $\mu(e)$ and λ_i can be interpreted as the residual capacity of the edge e, and the residual demand for the pair $s_i t_i$ respectively.

Chapter 5

Concurrent modulus and generalizations of maximum concurrent flow

In this chapter, we show that the dual form of the maximum concurrent flow problem in (4.8) can be represented in the form of a general parametrized modulus problem. The flexibility of the modulus framework can be used to introduce a new class of generalized forms of this problem. This is done by considering a more general form of energy in the objective function. We call this generalized problem *concurrent modulus problem*. This chapter studies this generalization along with maximum concurrent p-modulus flow problems for certain values of p and analyzes their properties. At the end of the chapter some examples are presented to illustrate this theory.

5.1 Concurrent p-modulus

Using the notation of Section 4.1, suppose Γ is the disjoint union of the families $\Gamma_i = \Gamma(s_i, t_i)$ for $i = 1, \ldots, k$. We shall call Γ a *concurrent family* of paths. If we now fix a positive demand vector $d \in \mathbb{R}^k_{>0}$ and exponent $p \in [1, \infty]$, we can define the *concurrent p-modulus*, $\mathrm{Mod}_{p,\sigma,d}(\Gamma)$

as the value of the following optimization problem:

$$
\begin{aligned}
\underset{\rho, w}{\text{minimize}} \quad & \mathcal{E}_{p,\sigma}(\rho) \\
\text{subject to} \quad & \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) \geq w_i, \qquad \forall \gamma \in \Gamma_i,\ \forall i = 1, \ldots, k, \\
& \sum_{i=1}^{k} w_i d_i \geq 1, \\
& w \in \mathbb{R}^k_{\geq 0}, \quad \rho \in \mathbb{R}^E_{\geq 0}.
\end{aligned}
\tag{5.1}
$$

Remark 5.1.1. Note that if $r > 0$, then (ρ, w) is admissible for $\operatorname{Mod}_{p,\sigma,d}(\Gamma)$ if and only if $(\rho/r, w/r)$ is admissible for $\operatorname{Mod}_{p,\sigma,rd}(\Gamma)$. Moreover, $\mathcal{E}_{p,\sigma}(\rho/r) = r^{-p}\mathcal{E}_{p,\sigma}(\rho)$. This shows the scaling

$$
\operatorname{Mod}_{p,\sigma,rd}(\Gamma) = r^{-p} \operatorname{Mod}_{p,\sigma,d}(\Gamma).
$$

This justifies the normalization adopted in this dissertation that the d_i sum to 1.

Remark 5.1.2. In fact, the definition in (5.1) can be easily extended to more general families Γ_i, beyond just path families. The results of this section apply with only minor modifications.

5.1.1 Inner and outer problems

This problem can be better understood by considering the optimization in two stages. In the first stage (the *inner problem*), we consider minimizing only over the density ρ, where w is frozen. This gives a *w-parameterized concurrent modulus* problem:

$$
\begin{aligned}
\underset{\rho}{\text{minimize}} \quad & \mathcal{E}_{p,\sigma}(\rho) \\
\text{subject to} \quad & \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) \geq w_i, \qquad \forall \gamma \in \Gamma_i,\ \forall i = 1, \ldots, k, \\
& \rho \in \mathbb{R}^E_{\geq 0}.
\end{aligned}
\tag{5.2}
$$

We shall use the notation $\mathrm{Mod}_{p,\sigma}(\Gamma_w)$ to represent the value of this problem. The meaning behind this notation can be understood as follows. If $\gamma \in \Gamma_i$, then

$$\sum_{e\in E}\mathcal{N}(\gamma,e)\rho(e) \geq w_i \quad \iff \quad \sum_{e\in E} w_i^{-1}\mathcal{N}(\gamma,e)\rho(e) \geq 1.$$

(See Remark 4.2.1 for the interpretation when $w_i = 0$.) Thus, the inner problem can be viewed as a standard modulus problem wherein the objects in each Γ_i have been re-weighted so that their edge usages are now w_i^{-1} times the previous values.

With $\mathrm{Mod}_{p,\sigma}(\Gamma_w)$ defined as in (5.2), we can now recognize the concurrent modulus $\mathrm{Mod}_{p,\sigma,d}(\Gamma)$ as the value of the problem

$$\begin{aligned} \underset{w}{\text{minimize}} \quad & \mathrm{Mod}_{p,\sigma}(\Gamma_w) \\ \text{subject to} \quad & \sum_{i=1}^{k} w_i d_i \geq 1, \\ & w \in \mathbb{R}^k_{\geq 0}. \end{aligned} \tag{5.3}$$

We shall refer to (5.3) as the *outer problem* for concurrent modulus.

For any parameter $1 \leq p < \infty$, the concurrent p-modulus problem takes the form:

$$\begin{aligned} \underset{\rho,w}{\text{minimize}} \quad & \sum_{e\in E}\sigma(e)\rho(e)^p \\ \text{subject to} \quad & \sum_{e\in E}\mathcal{N}(\gamma,e)\rho(e) \geq w_i, \qquad \forall\gamma\in\Gamma_i,\ \forall i = 1,\dots,k, \\ & \sum_{i=1}^{k} w_i d_i \geq 1, \\ & w \in \mathbb{R}^k_{\geq 0}, \quad \rho \in \mathbb{R}^E_{\geq 0}, \end{aligned} \tag{5.4}$$

and *concurrent* ∞*-modulus* has the form:

$$\begin{aligned}
\underset{\rho,w}{\text{minimize}} \quad & \max_e \quad \sigma(e)\rho(e) \\
\text{subject to} \quad & \sum_{e\in E} \mathcal{N}(\gamma,e)\rho(e) \geq w_i, \qquad \forall\gamma\in\Gamma_i,\ \forall i=1,\dots,k, \\
& \sum_{i=1}^{k} w_i d_i \geq 1, \\
& w\in\mathbb{R}^k_{\geq 0}, \quad \rho\in\mathbb{R}^E_{\geq 0},
\end{aligned} \tag{5.5}$$

Remark 5.1.3. If $p = 1$, then (5.4) is just (4.8). Thus, we may think of the concurrent modulus problem as a generalization of the maximum concurrent flow problem. Moreover, if $k = 1$ and $d_1 = 1$, then (5.4) reduces to (3.11) with $\mathcal{N}$ the usage matrix for the family of paths connecting s_1 to t_1. Thus, concurrent modulus is also a generalization of p-modulus.

5.1.2 Dual problems

Next, we drive the dual form of the problem to study the dual theory on concurrent modulus.

The case $1 < p < \infty$. For $1 < p < \infty$, it is convenient to replace the energy $\mathcal{E}_{p,\sigma}$ by $\tilde{\mathcal{E}}_{p,\sigma}(\rho) := \sum_{e\in E} \sigma(e)|\rho(e)|^p$. Since $\mathcal{E}_{p,\sigma} = \tilde{\mathcal{E}}_{p,\sigma}$ on $\mathbb{R}^E_{\geq 0}$, this does not change the value of modulus. With this replacement, the Lagrangian for (5.4) is

$$
\begin{aligned}
L(\rho, w, x, z, \lambda, \mu) = &\sum_{e \in E} \sigma(e)|\rho(e)|^p + \sum_{i=1}^{k} \sum_{\gamma \in \Gamma_i} x(\gamma) \left(w_i - \sum_{e \in E} \mathcal{N}(\gamma, e)\rho(e) \right) \\
&+ z(1 - \sum_{i=1}^{k} d_i w_i) - \sum_{i=1}^{k} \lambda_i w_i - \sum_{e \in E} \mu(e)\rho(e) \\
&= \sum_{e \in E} \sigma(e)|\rho(e)|^p \\
&- \sum_{e \in E} \rho(e)\big(\sum_{i=1}^{k} \sum_{\gamma \in \Gamma_i} \mathcal{N}(\gamma, e)x(\gamma) + \mu(e) \big) \\
&+ \sum_{i=1}^{k} w_i \Big(\sum_{\gamma \in \Gamma_i} x(\gamma) - z d_i - \lambda_i \Big) \\
&+ z,
\end{aligned}
$$

which is defined for all $\rho \in \mathbb{R}^E$.

To get the dual function $g(x, z, \lambda, \mu)$, we find the infimum value of the Lagrangian over the variables ρ and w. Stationary condition can be used to get the extermal values ρ^* and w^*.

The gradient of the Lagrangian with respect to variable ρ should vanish at extemal point, i.e. $\nabla_\rho L(\rho, w, x, z, \lambda, \mu) = 0$, which yields

$$
p\sigma(e)|\rho(e)|^{p-1} \operatorname{sgn}(\rho(e)) - \sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e)x(\gamma) - \mu(e) = 0, \quad \forall e \in E,
$$

or, equivalently,

$$
p\sigma(e)|\rho(e)|^{p-1} \operatorname{sgn}(\rho(e)) = \sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e)x(\gamma) + \mu(e), \quad \forall e \in E.
$$

By dual feasibility and the non-negativity of $\mathcal{N}$, we see that an optimal ρ must be non-negative, yielding the relation

$$p\sigma(e)\rho(e)^{p-1} = \sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)x(\gamma) + \mu(e), \quad \forall e \in E. \tag{5.6}$$

If $\mu(e) > 0$, for some $e \in E$, then (5.6) would imply that $\rho(e) > 0$. Since this contradicts the complementary slackness condition

$$\mu(e)\rho(e) = 0, \quad \forall e \in E,$$

it follows that $\mu = 0$. Therefore the extremal density ρ is,

$$\rho^*(e) = \left(\frac{1}{p\sigma(e)} \sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)x(\gamma)\right)^{\frac{1}{p-1}}, \quad \forall e \in E. \tag{5.7}$$

Next, we find the gradient of the Lagrangian with respect to variable w which should also vanish at the extemal point

$$\nabla_w L(\rho, w, x, z, \lambda, \mu) = 0,$$

so,

$$\sum_{\gamma\in\Gamma_i} x(\gamma) - zd_i - \lambda_i = 0, \quad \forall i = 1, \ldots, k.$$

Since $\lambda \geq 0$, therefore

$$\sum_{\gamma\in\Gamma_i} x(\gamma) - zd_i \geq 0, \quad \forall i = 1, \ldots, k. \tag{5.8}$$

Then the dual function will be

$$\begin{aligned} g(x, z, \lambda, \mu) &= L(\rho^*, w^*, x, z, \lambda, \mu) \\ &= \sum_{e\in E} \sigma(e)\rho^{*^p}(e) - \sum_{e\in E} \rho^*(e) \left(\sum_{\gamma\in\Gamma} \mathcal{N}(\gamma, e)x(\gamma)\right) + z. \end{aligned}$$

Using (5.7), we get

$$\begin{aligned} g(x,z,\lambda,\mu) &= \sum_{e\in E}\sigma(e)\rho^{*^p}(e) - \sum_{e\in E}\rho^*(e)\left(\rho^{*^{p-1}}p\sigma(e)\right) + z \\ &= \sum_{e\in E}\sigma(e)\rho^{*^p}(e) - \sum_{e\in E}p\sigma(e)\rho^{*^p}(e) + z \\ &= (1-p)\sum_{e\in E}\sigma(e)\rho^{*^p}(e) + z. \end{aligned}$$

Therefore the dual function takes the form

$$g(x,z,\lambda,\mu) = (1-p)\sum_{e\in E}\sigma(e)\left(\frac{1}{p\sigma(e)}\sum_{\gamma\in\Gamma}\mathcal{N}(\gamma,e)x(\gamma).\right)^{\frac{p}{p-1}} + z,$$

and the dual form of the problem (5.4) will be

$$\begin{aligned} \underset{x,z,\mu}{\text{maximize}} \quad & g(x,z,\mu) \\ \text{subject to} \quad & \sum_{\gamma\in\Gamma_i} x(\gamma) \geq z d_i, \quad i = 1,\ldots,k, \\ & x \geq 0. \end{aligned} \tag{5.9}$$

The case $p = 1$. When $p = 1$, the dual problem is

$$\begin{aligned} \underset{z,x_\gamma}{\text{maximize}} \quad & z \\ \text{subject to} \quad & \sum_{\gamma\in\Gamma_i} x_\gamma \geq z d_i, \qquad i = 1,\ldots,k, \\ & \sum_{\gamma\in\Gamma}\mathcal{N}(\gamma,e)x_\gamma \leq \sigma(e), \qquad \forall e\in E, \\ & x_\gamma \geq 0, \qquad \forall\gamma\in\Gamma. \end{aligned}$$

The case $p = \infty$. And as we will see in (5.21), the dual of $p = \infty$ case takes the form

$$\begin{aligned} \underset{\rho,w}{\text{maximize}} \quad & z \\ \text{subject to} \quad & \sum_{e\in E} \beta(e) = 1 \\ & \sum_{\gamma\in\Gamma_i} x(\gamma) \geq z d_i, \qquad \forall i = 1, \dots, k, \\ & \sum_{\gamma\in\Gamma} N(\gamma, e)\rho(e) \leq \beta(e)\sigma(e), \quad \forall e \in E. \end{aligned}$$

5.2 Concurrent p-modulus and some special cases

By definition, the concurrent 1-modulus is the dual of the maximum concurrent flow problem. Thus, we already have a first analog of the single source-single sink theory.

Theorem 5.2.1. *Let Γ be a concurrent family with k source-sink pairs and let $d \in \mathbb{R}^k_{>0}$. Then the corresponding maximum concurrent flow problem with demand vector d has the value* $\mathrm{Mod}_{1,\sigma,d}(\Gamma)$.

5.2.1 Concurrent resistance

Next we turn our attention to the $p = 2$ case. For $p = 2$ the problem in (5.1) takes the form:

$$\begin{aligned} \underset{\rho,w}{\text{minimize}} \quad & \sum_{e\in E} \sigma(e)\rho(e)^2 \\ \text{subject to} \quad & \sum_{e\in E} \mathcal{N}(\gamma, e)\rho(e) \geq w_i, \qquad \forall\gamma \in \Gamma_i,\ \forall i = 1, \dots, k, \\ & \sum_{i=1}^{k} w_i d_i \geq 1, \\ & w \in \mathbb{R}^k_{\geq 0}, \quad \rho \in \mathbb{R}^E_{\geq 0}. \end{aligned} \tag{5.10}$$

The equivalent matrix form for (5.10) can be written as:

$$
\begin{aligned}
\underset{\rho, w}{\text{minimize}} \quad & \rho^T \Sigma \rho \\
\text{subject to} \quad & \mathcal{N}\rho \geq M^T w, \\
& d^T w \geq 1, \\
& w \geq 0, \quad \rho \geq 0,
\end{aligned}
\tag{5.11}
$$

where Σ is a diagonal matrix with dimension $|E| \times |E|$:

$$
\Sigma = \text{Diag}(\sigma(e_1), \ldots, \sigma(e_{|E|})),
$$

and matrix M defined in (4.5).

The Lagrangian for this problem is

$$
\begin{aligned}
L(\rho, w, x, z, \lambda, \mu) &= \rho^T \Sigma \rho + x^T (M^T w - \mathcal{N}\rho) + z(1 - d^T w) - \lambda^T w - \mu^T \rho \\
&= \rho^T \Sigma \rho - (\mathcal{N}^T x + \mu)^T \rho + (Mx - zd - \lambda)^T w + z.
\end{aligned}
$$

To compute the dual function $g(z, \lambda, \mu)$ for this problem, the infimum of Lagrangian over variables ρ and w is needed. We can use Stationary from KKT conditions to find the extermal ρ^* and w^*.

$$
\nabla_\rho L = 2\Sigma\rho - N^T + \mu = 0,
$$

$$
\nabla_w L = Mx - zd - \lambda = 0.
$$

Therefore we get optimal density as

$$
\rho^* = \frac{1}{2}\Sigma^{-1}(N^T x + \mu),
$$

and, from complementary slackness $\mu = 0$, so

$$\rho^* = \frac{1}{2}\Sigma^{-1}(N^T x).$$

Also, since $\lambda \geq 0$, therefore $Mx \geq zd$.

The dual function $g(x, z, \mu)$ will be

$$\begin{aligned} g(x, z, \mu) &= \inf_{\rho, w} L(\rho, w, x, z, \mu) \\ &= \frac{1}{4}(\mathcal{N}^T x)^T \Sigma^{-1}(\mathcal{N}^T x) - \frac{1}{2}(\mathcal{N}^T x)^T \Sigma^{-1}(\mathcal{N}^T x) + z \\ &= z - \frac{1}{4}(\mathcal{N}^T x)^T \Sigma^{-1}(\mathcal{N}^T x). \end{aligned}$$

So the dual problem for (5.11) takes the following form:

$$\begin{aligned} \underset{x,z}{\text{maximize}} \qquad & z - \frac{1}{4}(\mathcal{N}^T x)^T \Sigma^{-1}(\mathcal{N}^T x) \\ \text{subject to} \qquad & Mx \geq zd, \\ & x \geq 0, \end{aligned} \tag{5.12}$$

or equivalently

$$\begin{aligned} \underset{x,z}{\text{maximize}} \qquad & z - \frac{1}{4}\sum_{e \in E} \sigma(e)^{-1} \left(\sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e) x(\gamma) \right)^2 \\ \text{subject to} \qquad & \sum_{\gamma \in \Gamma_i} x(\gamma) \geq z d_i, \quad \forall i = 1, \ldots, k, \\ & x \in \mathbb{R}^{\Gamma}_{\geq 0}. \end{aligned} \tag{5.13}$$

In order to make sense of this formula, let us introduce another optimization problem:

$$\begin{aligned} \underset{\mu}{\text{minimize}} \quad & \sum_{e \in E} \sigma(e)^{-1} \left(\sum_{i=1}^{k} d_i \sum_{\gamma \in \Gamma_i} \mathcal{N}(\gamma, e) \mu(\gamma) \right)^2 \\ \text{subject to} \quad & \sum_{\gamma \in \Gamma_i} \mu(\gamma) = 1, \quad \forall i = 1, \dots, k, \\ & \mu \in \mathbb{R}^{\Gamma}_{\geq 0}. \end{aligned} \tag{5.14}$$

Noting the similarity between (5.14) and (3.18), we may interpret the restriction $\mu_i := \mu|_{\Gamma_i}$ as a pmf on the family Γ_i, giving rise to a random object $\underline{\gamma_i} \in \Gamma_i$. This provides an alternative interpretation of the objective function as

$$\sum_{e \in E} \sigma(e)^{-1} \left(\sum_{i=1}^{k} d_i \mathbb{P}_{\mu_i}(e \in \underline{\gamma}_i) \right)^2, \tag{5.15}$$

where the inner sum can be interpreted as a demand-weighted average of the edge usage probabilities corresponding to each subfamily $\Gamma_i \subset \Gamma$.

Remark 5.2.2. To simplify notation in what follows, we shall denote the value of (5.14) by $R_{\sigma,d}(\Gamma)$, since, in light of Remark 5.1.3, when $k = 1$ and $d_1 = 1$,

$$R_{\sigma,d}(\Gamma) = R_{\text{eff}}(s_1, t_1),$$

which is the effective resistance between s_1 and t_1, with the edge weights σ treated as edge conductances.

One way to interpret the concurrent effective resistance is as follows. For each source/sink pair (s_i, t_i), we wish to choose a flow with value d_i, defined through the corresponding path flows as

$$f_i = \sum_{\gamma \in \Gamma_i} d_i \mu_i(\gamma) f(\gamma),$$

where, as before, $f(\gamma)$ refers to the unit path flow along the path γ. The energy defined in (5.15) can then be interpreted as a multi-commodity analog of dissipated power, with

$\sigma(e)^{-1}$ acting as the resistance of the associated resistor and the sum inside the parentheses interpreted as the total current flowing through the corresponding resistor. In this way, (5.14) can be seen as a multi-commodity version of Thompson's energy minimization principle for resistor networks.

The dual problem (5.13) provides the following duality theorem on concurrent modulus which resembles the connection between 2-modulus and effective resistance described in Section 3.6.

Theorem 5.2.3. *Let Γ be a concurrent family with k source-sink pairs and let $d \in \mathbb{R}^k_{\geq 0}$. Then*

$$\mathrm{Mod}_{2,\sigma,d}(\Gamma)^{-1} = R_{\sigma,d}(\Gamma). \tag{5.16}$$

Moreover, if ρ^* is optimal for (5.1) and μ^* is optimal for (5.14), then

$$\rho^*(e) = \frac{\mathrm{Mod}_{2,\sigma,d}(\Gamma)}{\sigma(e)} \sum_{i=1}^{k} d_i \sum_{\gamma\in\Gamma_i} \mathcal{N}(\gamma,e)\mu^*(\gamma). \tag{5.17}$$

Proof. Let x, z be admissible for the problem in (5.13). Define

$$\nu := \min_i \frac{\sum\limits_{\gamma\in\Gamma_i} x(\gamma)}{d_i} > 0,$$

and

$$\mu(\gamma) := \frac{x(\gamma)}{\nu d_i}, \qquad \text{for } \gamma \in \Gamma_i.$$

Then μ satisfies

$$\sum_{\gamma\in\Gamma_i} \mu(\gamma) \geq 1, \quad \forall i = 1, \ldots, k.$$

With this decomposition, (5.13) can be rewritten as

$$
\begin{aligned}
\underset{\nu,\mu,z}{\text{maximize}} \quad & z - \frac{\nu^2}{4}\sum_{e\in E}\sigma(e)^{-1}\left(\sum_{i=1}^{k} d_i \sum_{\gamma\in\Gamma_i}\mathcal{N}(\gamma,e)\mu(\gamma)\right)^2 \\
\text{subject to} \quad & \sum_{\gamma\in\Gamma_i}\mu(\gamma)\geq 1, \quad \forall i=1,\ldots,k, \\
& \nu\geq z, \quad \mu\in\mathbb{R}^{\Gamma}_{\geq 0}.
\end{aligned}
\tag{5.18}
$$

The value of z is positive, since if we set z to zero that makes all $\mu(\gamma)$ zero. By construction, (ν,μ,z) are admissible for this problem, and they give the same energy as (x,z) in (5.13), so (5.13) will be a lower bound for (5.18).

Now, let (ν,μ,z) be feasible for the problem (5.18), and define $x(\gamma) := \nu d_i\mu(\gamma)$, for $\gamma\in\Gamma_i$, then,

$$\sum_{\gamma\in\Gamma_i} x(\gamma) = \sum_{\gamma\in\Gamma_i}\nu d_i\mu(\gamma)\geq \nu d_i \geq z d_i,$$

which shows (x,z) are feasible for (5.13). Also,

$$\sum_{\gamma\in\Gamma}\mathcal{N}(\gamma,e)x(\gamma) = \sum_{i}\nu d_i\sum_{\gamma\in\Gamma_i}\mathcal{N}(\gamma,e)\mu(\gamma),$$

which shows that (5.13) and (5.18) have same energies, so (5.18) will be a lower bound for (5.13), therefore both problems are equivalent.

We also need to show that the value of the objective function in (5.13) is non-negative. If $z=0$, then the maximum value of x will be zero, which makes the energy zero. Let $z>0$, and set $z:=\varepsilon$ so,

$$x(\gamma) = \frac{\varepsilon d_i}{|\Gamma_i|},$$

is feasible, and the energy of the problem is

$$
\begin{aligned}
&\varepsilon - \frac{1}{4}\sum_e \sigma^{-1}(e)\left(\sum_i \frac{\varepsilon d_i}{|\Gamma_i|}\sum_{\gamma\in\Gamma_i}\mathcal{N}(\gamma,e)\right)^2 \\
=&\varepsilon - \frac{\varepsilon^2}{4}\sum_e \sigma^{-1}(e)\left(\sum_i \frac{d_i}{|\Gamma_i|}\sum_{\gamma\in\Gamma_i}\mathcal{N}(\gamma,e)\right)^2,
\end{aligned}
$$

where the last equation has non-negative value for $\varepsilon << 1$. This restricts z to be non-negative.

Maximizing first over μ in problem (5.18) shows that the optimal μ^* solves (5.14). Maximizing over z next, shows that we should select $z = \nu$, and we are left with the problem of maximizing

$$\nu - \frac{\nu^2}{4} R_{\sigma,d}(\Gamma),$$

with respect to ν.

The maximum is attained at $\nu^* = 2/R_{\sigma,d}(\Gamma)$, giving the value

$$\nu^* - \frac{(\nu^*)^2}{4} R_{\sigma,d}(\Gamma) = \frac{1}{R_{\sigma,d}(\Gamma)}$$

for (5.13), which proves (5.16).

Moreover, Lagrangian duality shows that

$$\rho^*(e) = \frac{1}{2\sigma(e)}\sum_{\gamma\in\Gamma}\mathcal{N}(\gamma,e)x^*(\gamma) = \frac{\nu^*}{2\sigma(e)}\sum_{i=1}^k d_i \sum_{\gamma\in\Gamma_i}\mathcal{N}(\gamma,e)\mu^*(\gamma),$$

which proves (5.17). $\square$

We say that $R_{\sigma,d}$ is a *concurrent resistance*. Hence, in view of Theorem 5.2.3, the concurrent modulus $\mathrm{Mod}_{2,\sigma,d}(\Gamma)$ can be thought as a concurrent conductance.

5.2.2 Concurrent shortest path

Finally, we turn our attention to the $p = \infty$ case. Problem (5.1) takes the following form when $p = \infty$:

$$
\begin{aligned}
\underset{\rho,w}{\text{minimize}} \quad & \max_{e\in E} \sigma(e)\rho(e) \\
\text{subject to} \quad & \sum_{e\in E} \mathcal{N}(\gamma, e)\rho(e) \geq w_i, \qquad \forall\gamma \in \Gamma_i,\ \forall i = 1, \ldots, k, \\
& \sum_{i=1}^{k} w_i d_i \geq 1, \\
& w \in \mathbb{R}^k_{\geq 0}, \quad \rho \in \mathbb{R}^E_{\geq 0}.
\end{aligned}
\tag{5.19}
$$

Let $t = \max_{e\in E} \sigma(e)\rho(e)$, so

$$\sigma(e)\rho(e) \leq t, \quad \forall e \in E.$$

Therefore the problem can be written as:

$$
\begin{aligned}
\underset{t,\rho,w}{\text{minimize}} \quad & t \\
\text{subject to} \quad & \sum_{e\in E} \mathcal{N}(\gamma, e)\rho(e) \geq w_i, \qquad \forall\gamma \in \Gamma_i,\ \forall i = 1, \ldots, k, \\
& \sum_{i=1}^{k} w_i d_i \geq 1, \\
& \sigma(e)\rho(e) \leq t, \qquad \forall e \in E, \\
& w \in \mathbb{R}^k_{\geq 0}, \quad \rho \in \mathbb{R}^E_{\geq 0}.
\end{aligned}
\tag{5.20}
$$

To find the dual form of this problem, first we find the Lagrangian:

$$
\begin{aligned}
L(t, \rho, w, x, z, \lambda, \mu, \beta) =& t + \sum_{i=1}^{k} \sum_{\gamma \in \Gamma_i} x(\gamma) \left(w_i - \sum_{e \in E} \mathcal{N}(\gamma, e) \rho(e) \right) + z(1 - \sum_{i=1}^{k} d_i w_i) \\
& + \sum_{e \in E} \beta(e) \left(\sigma(e) \rho(e) - t \right) - \sum_{i=1}^{k} \lambda_i w_i - \sum_{e \in E} \mu(e) \rho(e) \\
=& t \big(1 - \sum_{e \in E} \beta(e) \big) \\
& + \sum_{e \in E} \rho(e) \Big(\beta(e) \sigma(e) - \sum_{i=1}^{k} \sum_{\gamma \in \Gamma_i} \mathcal{N}(\gamma, e) x(\gamma) - \mu(e) \Big) \\
& + \sum_{i=1}^{k} w_i \Big(\sum_{\gamma \in \Gamma_i} x(\gamma) - z d_i - \lambda_i \Big) \\
& + z.
\end{aligned}
$$

The dual function is:

$$
\begin{aligned}
g(x, z, \lambda, \mu, \beta) &= \inf_{t, \rho, w} L(t, \rho, x, z, \lambda, \mu, \beta) \\
&= \begin{cases} z & \text{if} \quad \begin{cases} \sum_{e \in E} \beta(e) = 1, \quad \text{and,} \\ \sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e) x(\gamma) - \mu(e) = \beta(e) \sigma(e), \quad \text{and,} \\ \sum_{\gamma \in \Gamma_i} x(\gamma) - z d_i - \lambda_i = 0, \end{cases} \\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

From the complementary slackness $\mu = 0$, so

$$
\sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e) x(\gamma) = \beta(e) \sigma(e), \quad \forall e \in E,
$$

and since $\lambda \geq 0$, therefore

$$
\sum_{\gamma \in \Gamma_i} x(\gamma) \geq z d_i.
$$

So the dual problem for problem in (5.19) will be:

$$\begin{aligned}
\underset{\rho,w}{\text{maximize}} \quad & z \\
\text{subject to} \quad & \beta \in \mathbb{P}(E), \\
& \sum_{\gamma\in\Gamma} N(\gamma,e)\rho(e) \leq \beta(e)\sigma(e), \quad \forall e \in E, \\
& \sum_{\gamma\in\Gamma_i} x(\gamma) \geq z d_i, \qquad \forall i = 1,\dots,k.
\end{aligned} \tag{5.21}$$

Where $\mathbb{P}$ is the probability mass distribution on Γ.

Theorem 5.2.4. *When $p = \infty$ the inner problem (5.2) has value*

$$\mathrm{Mod}_{\infty,\sigma,w} = \max_i w_i \ell_{\sigma^{-1}}(\Gamma_i)^{-1}.$$

Proof. To establish an upper bound, define

$$\alpha =: \max_i w_i \ell_{\sigma^{-1}}(\Gamma_i)^{-1}, \quad \text{and let} \quad \rho = \alpha\sigma(e)^{-1}.$$

Note that $\mathcal{E}_{\infty,\sigma}(\rho) = \alpha$. Thus, α is an upper bound for the inner problem provided we show that ρ is admissible. To see this, let γ be an object in some Γ_j. Then, since

$$\alpha \geq w_j \ell_{\sigma^{-1}}(\Gamma_j)^{-1}, \quad \text{and} \quad \ell_{\sigma^{-1}}(\gamma) \geq \ell_{\sigma^{-1}}(\Gamma_j)$$

it follows:

$$\ell_\rho(\gamma) = \alpha\ell_{\sigma^{-1}}(\gamma) \geq w_j \ell_{\sigma^{-1}}(\Gamma_j)^{-1}\ell_{\sigma^{-1}}(\Gamma_j) = w_j$$

establishing the upper bound.

To establish the lower bound, choose j and $\gamma \in \Gamma_j$ such that

$$w_j \ell_{\sigma^{-1}}(\gamma)^{-1} = \alpha$$

Suppose $\rho \geq 0$ satisfies $\ell_\rho(\gamma) \geq w_j$. Then we have

$$w_j \leq \sum_{e\in\gamma} \rho(e) = \sum_{e\in\gamma} \sigma^{-1}(e)\sigma(e)\rho(e) \leq \mathcal{E}_{\infty,\sigma}(\rho) \sum_{e\in\gamma} \sigma^{-1}(e) = \mathcal{E}_{\infty,\sigma}(\rho)\ell_{\sigma^{-1}}(\gamma)$$

Thus,

$$\text{Mod}_{\infty,\sigma,w} \geq w_j \ell_{\sigma^{-1}}(\gamma^{-1}) = \alpha.$$

$\square$

Theorem 5.2.5. *When $p = \infty$, the outer problem (5.3) has value*

$$\Big(\sum_{j=1}^{k} d_j \ell_{\sigma^{-1}}(\Gamma_j)\Big)^{-1}. \tag{5.22}$$

Proof. Applying Theorem 5.2.4 shows that the problem with $p = \infty$ case can be written as

$$\begin{array}{lll} \underset{w}{\text{minimim}} & \underset{i}{\text{maximum}} & w_i \ell_{\sigma^{-1}}(\Gamma_{\text{sp}}(s_i,t_i))^{-1} \\ \text{subject to} & \displaystyle\sum_{i=1}^{k} w_i d_i \geq 1, & \\ & w \geq 0. & \end{array}$$

To establish an upper bound, define

$$\beta =: \left(\sum_{j=1}^{k} d_j \ell_{\sigma^{-1}}(\Gamma_j\right)^{-1}.$$

To simplify notation, define

$$\xi_i = \ell_{\sigma^{-1}}(\Gamma_{\text{sp}}(s_i,t_i))$$

and let

$$w_i = \Big(\sum_{j=1}^{k} \xi_j d_j\Big)^{-1} \xi_j.$$

Then

$$\max_i w_i \xi_i^{-1} = \left(\sum_{j=1}^{k} \xi_j d_j\right)^{-1} = \beta$$

so

$$\min_w \max_i w_i \xi_i^{-1} \leq \beta$$

Moreover, this w is admissible since

$$\sum_i w_i d_i = \sum_i \left(\sum_j \xi_j d_j\right)^{-1} \xi_i d_i = 1$$

showing that β is an upper bound for the outer problem.

To see that β is also a lower bound, suppose $w \geq 0$ is admissible. Then

$$1 \leq \sum_i w_i d_i = \sum_i w_i \xi_i^{-1} \xi_i d_i \leq \left(\max_i w_i \xi^{-1}\right) \sum_i x i_i d_i,$$

showing that

$$\max_i w_i \xi^{-1} \geq \left(\sum_i \xi_i d_i\right)^{-1}$$

so

$$\min_i \max_i w_i \xi^{-1} \geq \min_i \left(\sum_i \xi_i d_i\right)^{-1} = \beta$$

□

So, $\text{Mod}_{\infty,\sigma,d}^{-1}(\Gamma)$ is the demand-weighted sum of shortest path lengths between the pairs.

5.3 Convexity of the maximum concurrent flow problem

Lemma 5.3.1. *Each of the problems in* (5.1), (5.2), *and* (5.3) *is a convex optimization problem.*

Proof. Since all constraints in all three cases are affine, it remains only to verify that the objective functions are all convex. Convexity in (5.1) and (5.2) comes from the fact that the p-energy is convex.

For (5.3), we need to show that the function

$$w \mapsto \operatorname{Mod}_{p,\sigma}(\Gamma_w)$$

is convex on $\mathbb{R}^k_{\geq 0}$. To do so, let $w^1, w^2 \in \mathbb{R}^k_{\geq 0}$ and $0 \leq \theta \leq 1$. Let ρ^1 and ρ^2 be optimal densities for $\operatorname{Mod}_{p,\sigma}(\Gamma_{w^1})$ and $\operatorname{Mod}_{p,\sigma}(\Gamma_{w^2})$ respectively. Define $w = \theta w^1 + (1-\theta)w^2$ and $\rho = \theta\rho^1 + (1-\theta)\rho^2$.

Note that ρ is admissible for $\operatorname{Mod}_{p,\sigma}(\Gamma_w)$ since, for all $\gamma \in \Gamma_i$,

$$\ell_\rho(\gamma) = \theta\ell_{\rho^1}(\gamma) + (1-\theta)\ell_{\rho^2}(\gamma) \geq \theta w_i^1 + (1-\theta)w_i^2 = w_i.$$

Thus, by the convexity of the p-energy,

$$\begin{aligned}
\operatorname{Mod}_{p,\sigma}(\Gamma_{\theta w^1+(1-\theta)w^2}) &\leq \mathcal{E}_{p,\sigma}(\theta\rho^1 + (1-\theta)\rho^2) \\
&\leq \theta\mathcal{E}_{p,\sigma}(\rho^1) + (1-\theta)\mathcal{E}_{p,\sigma}(\rho^2) \\
&= \theta \operatorname{Mod}_{p,\sigma}(\Gamma_{w^1}) + (1-\theta)\operatorname{Mod}_{p,\sigma}(\Gamma_{w^2}).
\end{aligned}$$

□

5.4 The probabilistic interpretation

Modulus has a probabilistic interpretation that is very useful to gain additional intuition. We explored this formulation in Section 3.7 which is particularly simple case $p = 2$, see [4, Theorem 5.1].

In analogy with the probabilistic interpretation of 2-modulus, we develop probabilistic interpretation of the concurrent resistance defined in (5.14). Given a concurrent family Γ with k source-sink pairs, we will only consider pmfs μ which are trivial couplings of sequences of

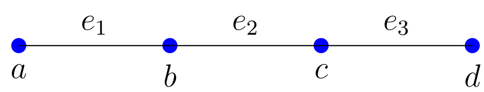

Figure 5.1: *The graph for Example 5.5.1.*

pmf μ_i for each family Γ_i, $i = 1, \ldots, k$. In other words, μ is the law for sequences $(\underline{\gamma}_1, \ldots, \underline{\gamma}_k)$ of random paths, sampled independently and such that $\underline{\gamma}_i$ has law μ_i. Let

$$\mathcal{P}_0(\Gamma) = \{\mu = (\mu_1, \ldots, \mu_k) : \text{ independent, } \mu_i \in \mathcal{P}(\Gamma_i)\}$$

Then, in the unweighted case, we can write

$$\begin{aligned} \mathrm{Mod}_{2,\sigma,d}(\Gamma)^{-1} &= \min_{\mu \in \mathcal{P}_0(\Gamma)} \sum_{e \in E} \left[\mathbb{E}_\mu \left(\sum_{i=1}^{k} d_i \mathcal{N}\left(\underline{\gamma}_i, e\right) \right) \right]^2 \\ &= \min_{(\mu_1, \ldots, \mu_k) \in \mathcal{P}_0(\Gamma)} \sum_{e \in E} \left[\sum_{i=1}^{k} d_i \mathbb{P}_{\mu_i} \left(e \in \underline{\gamma}_i \right) \right]^2 . \end{aligned} \tag{5.23}$$

Comparing (5.23) with (3.18), we see that the expected number of random objects in Γ that use the edge e, i.e., the expected usage of e, has been replaced by the expected number of "demand-weighted" independent source-sink paths that contain the edge e, i.e., the expected usage of e by sequences of independent source-sink paths.

5.5 Examples

Example 5.5.1. Consider the concurrent max flow problem for graph G in Figure 5.1. We want to send a flow from source a to target c and from source b to target d. Let the respective demands for these two pairs be $d_1 = d$ and $d_2 = 1 - d$, where $0 < d < 1$ is a parameter of the problem. We assume that the capacities on e_1, e_2 and e_3 are σ_1, σ_2, and σ_3 respectively, with each $\sigma_i > 0$.

The case $\mathbf{p = 1}$. Since there is only one path between each source-sink pair, the associated dual problem (4.8) is relatively simple.

$$
\begin{aligned}
\underset{\rho,w}{\text{minimize}} \quad & \sigma_1\rho_1 + \sigma_2\rho_2 + \sigma_3\rho_3, \\
\text{subject to} \quad & \rho_1 + \rho_2 \geq w_1, \\
& \rho_2 + \rho_3 \geq w_2, \\
& dw_1 + (1-d)w_2 \geq 1, \\
& \rho \in \mathbb{R}^3_{\geq 0}, \\
& w \in \mathbb{R}^2_{\geq 0}.
\end{aligned} \tag{5.24}
$$

One can verify directly from the KKT conditions that the maximum concurrent flow value is

$$
z^* = \min\left\{\frac{\sigma_1}{d}, \sigma_2, \frac{\sigma_3}{1-d}\right\}. \tag{5.25}
$$

Each of the three possible values for z^* corresponds to the saturation of one of the three edges by the maximum concurrent flow. For example, consider the case that the minimum in (5.25) is attained at $\frac{\sigma_1}{d}$, corresponding to the saturation of edge e_1. This is equivalent to the condition that $d \geq \max\left\{\frac{\sigma_1}{\sigma_2}, \frac{\sigma_1}{\sigma_1+\sigma_3}\right\}$. Suppose we let $x_1 = \sigma_1$ and $x_2 = \frac{1-d}{d}\sigma_1$, where x_1 is the value of the path flow connecting a to c and x_2 is the value of the path flow connecting b to d. This flow is feasible since

$$
x_1 = \sigma_1 \leq \sigma_1, \qquad x_1 + x_2 = \frac{\sigma_1}{d} \leq \sigma_2, \qquad \text{and} \qquad x_2 = \frac{1-d}{d}\sigma_1 \leq \sigma_3.
$$

To verify that this flow is maximal, it is sufficient to construct dual variables ρ and w that, together with this x satisfy the KKT conditions. It is straightforward to check that the choices

$$
\rho = \left(\frac{1}{d}, 0, 0\right), \quad w = \left(\frac{1}{d}, 0\right)
$$

have this property.

Similarly, the minimum in (5.25) equals σ_2 when

$$\frac{\sigma_2 - \sigma_3}{\sigma_2} \le d \le \frac{\sigma_1}{\sigma_2},$$

which is only possible if $\sigma_2 \le \sigma_1 + \sigma_3$. This case corresponds to the saturation of e_2. One can verify that a solution is

$$\rho^* = (0, 1, 0)\,, \quad w^* = (1, 1)\,, \quad x^* = \sigma_2(d, 1-d).$$

Finally, the minimum in (5.25) equals $\frac{\sigma_3}{1-d}$ when

$$d \le \min\left\{\frac{\sigma_1}{\sigma_1 + \sigma_3}, \frac{\sigma_2 - \sigma_3}{\sigma_2}\right\}.$$

In this case, the critical edge is e_3 and a solution is

$$\rho^* = \left(0, 0, \frac{1}{1-d}\right), \quad w^* = \left(0, \frac{1}{1-d}\right), \quad x^* = \sigma_3\left(\frac{d}{1-d}, 1\right)$$

The case $\mathbf{p = 2}$. In this case we can find μ^* from the optimization problem (5.14), so

$$(\mu^*(\gamma_1), \mu^*(\gamma_2)) = (1, 1).$$

The value of this optimization problem will be

$$R_{\sigma,d}(\Gamma) = \frac{d^2}{\sigma_1} + \frac{1}{\sigma_2} + \frac{(1-d)^2}{\sigma_3}.$$

And the values of ρ^* using the equation (5.17) are:

$$\rho^* = \frac{1}{\frac{d^2}{\sigma_1} + \frac{1}{\sigma_2} + \frac{(1-d)^2}{\sigma_3}}\left(\frac{d}{\sigma_1}, \frac{1}{\sigma_2}, \frac{1-d}{\sigma_3}\right).$$

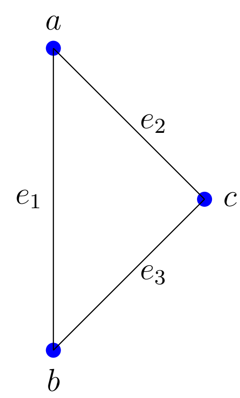

Figure 5.2: *The graph for Example 5.5.2.*

and

$$w^* = \frac{1}{\frac{d^2}{\sigma_1} + \frac{1}{\sigma_2} + \frac{(1-d)^2}{\sigma_3}} \left(\frac{d}{\sigma_1} + \frac{1}{\sigma_2}, \frac{1}{\sigma_2} + \frac{1-d}{\sigma_3} \right)$$

We got ν^* as $\frac{1}{R_{\sigma,d}(\Gamma)}$, so

$$\nu^* = \frac{1}{\frac{d^2}{\sigma_1} + \frac{1}{\sigma_2} + \frac{(1-d)^2}{\sigma_3}}.$$

then

$$x^* = \frac{1}{\frac{d^2}{\sigma_1} + \frac{1}{\sigma_2} + \frac{(1-d)^2}{\sigma_3}} (d, 1-d).$$

The case $\mathbf{p} = \infty$. In this case we compute the $\mathrm{Mod}_{\infty,\sigma,d}(\Gamma)$ using (5.22).

$$\mathrm{Mod}_{\infty,\sigma,d}(\Gamma) = \left(\frac{d}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1-d}{\sigma_3} \right)^{-1}.$$

The next example demonstrates the use of the probabilistic interpretation of modulus in the case $p = 2$.

Example 5.5.2. Consider the graph in Figure 5.2. Suppose the demand pairs are $D = \{(a,b),(a,c)\}$ with demands $d_1 = 1$, $d_2 = 2$. Here, the capacities of all the edges are 1.

To compute the concurrent conductance in this case, note that there are two families each containing two paths:

$$\Gamma_1 = \{\gamma_1 = ab,\ \gamma_2 = acb\} \qquad \text{and} \qquad \Gamma_2 = \{\gamma_1' = ac,\ \gamma_2' = abc\}$$

So to define $\mu = (\mu_1, \mu_2) \in \mathcal{P}_0(\Gamma)$ and the associated random object $(\underline{\gamma}, \underline{\gamma}') \in \Gamma$ in this case it is enough to pick parameters $p_1, p_2 \in [0,1]$, so that $\mathbb{P}_{\mu_1}(\underline{\gamma} = \gamma_1) = p_1$ and $\mathbb{P}_{\mu_2}(\underline{\gamma}' = \gamma_1') = p_2$. With this parameterization, the demand-weighted usage for edge $e_1 = (a,b)$ is

$$\begin{aligned} d_1\mathbb{P}_{\mu_1}(e_1 \in \underline{\gamma}) + d_2\mathbb{P}_{\mu_2}(e_1 \in \underline{\gamma}') &= d_1\mathbb{P}_{\mu_1}(\underline{\gamma} = \gamma_1) + d_2\mathbb{P}_{\mu_2}(\underline{\gamma}' = \gamma_2') \\ &= p_1 + 2(1-p_2). \end{aligned}$$

Similarly,

$$\begin{aligned} d_1\mathbb{P}_{\mu_1}(e_2 \in \underline{\gamma}) + d_2\mathbb{P}_{\mu_2}(e_2 \in \underline{\gamma}') &= (1-p_1) + 2p_2 \qquad \text{and} \\ d_1\mathbb{P}_{\mu_1}(e_3 \in \underline{\gamma}) + d_2\mathbb{P}_{\mu_2}(e_3 \in \underline{\gamma}') &= (1-p_1) + 2(1-p_2). \end{aligned}$$

By squaring these usages and adding them up, we find that we want to

$$\text{minimize } 14 - 4p_1 - 16p_2 + 3p_1^2 + 12p_2^2 - 4p_1p_2, \quad \text{subject to } 0 \le p_1,\ p_2 \le 1.$$

Solving this gives

$$p_1 = 1 \qquad \text{and} \qquad p_2 = \frac{5}{6}$$

Also the minimum is computed as

$$\mathrm{Mod}_{2,d}(\Gamma)^{-1} = \frac{14}{3}.$$

Using (5.17) we get that

$$\rho^*(e_1) = \frac{4}{14}, \qquad \rho^*(e_2) = \frac{5}{14}, \qquad \rho^*(e_3) = \frac{1}{14},$$

and one can check that $\rho^*(e_1)^2 + \rho^*(e_2)^2 + \rho^*(e_3)^2 = 3/14$. Also, again $w_1^* = 4/14$ and $w_2^* = 5/14$, so ρ^* is a feasible solution.

Example 5.5.3. We consider a complete bipartite graph $K_{m,n}$, and compute the concurrent

p-modulus for this graph for the values $p = 1, 2, \infty$. Assume that all of the capacities of all of the edges of the graph are one, and all the demands between pairs are the same and they sum to one.

In $K_{m,n}$, the vertex set V is naturally partitioned into two disjoint subsets $V = V_\ell \cup V_r$, which we shall refer to as the "left side" and "right side" of the graph. In $K_{m,n}$ every vertex in V_ℓ is connected to every vertex in V_r and there are no other edges. This also leads to a natural partition on the pairs of distinct vertices (s_i, t_i) into three classes. If both s_i and t_i are in V_ℓ, we refer to the pair as a *left-left pair*. Similarly, if both are in V_r, we say the pair is a *right-right pair*. All other pairs have one vertex in V_ℓ and one in V_r, which we shall call a *left-right pair*. Note that $K_{m,n}$ has

$$\begin{aligned} \binom{m}{2} &\quad \text{left-left pairs,} \\ \binom{n}{2} &\quad \text{right-right pairs,} \\ mn &\quad \text{left-right pairs, and} \\ \binom{m+n}{2} &\quad \text{total pairs.} \end{aligned} \tag{5.26}$$

These counts imply the following property of binomial coefficients that will be useful.

$$\binom{m}{2} + \binom{n}{2} = \binom{m+n}{2} - mn. \tag{5.27}$$

To establish the values of p-modulus for this problem, we construct an upper bound on modulus using the modulus problem in (5.1) and lower bound on modulus using the problem in (4.3) and show that they coincide.

Since the demands between the pairs in the bipartite graph sum to one, i.e. $\sum_{i=1}^{k} d = 1$, we must have:

$$d = \binom{m+n}{2}^{-1}. \tag{5.28}$$

Upper bound. We can obtain an upper bound for the concurrent modulus by making an admissible choice of ρ and w for (5.1). First we guess values of the density ρ on the edges of the graph. By symmetry, we guess that the density is a constant number ρ for all the edges of the graph. Next we guess the values of an admissible choice for w. Let (s_i, t_i) be a pair of nodes on the bipartite graph $K_{m,n}$. We expect w to take three different values $w_{\ell\ell}, w_{rr}$, and $w_{\ell r}$, depending on whether the pair is a left-left pair, a right-right pair, or a left-right pair.

Using the admissibility constraint of the problem in (5.1) we can make guesses for the values of w. Since the shortest path connecting a left-left or a right-right pair requires two hops, a reasonable guess is that $w_{\ell\ell} = w_{rr} = 2\rho$ (recalling that we have guessed that ρ is constant on the edges). On the other hand, each left-right pair is connected by a single edge, suggesting the guess $w_{\ell r} = \rho$.

Moreover, from the constraint $\sum\limits_{i=1}^{k} w_i d_i = 1$ in problem (5.1) and the fact that $d_i = d$ for all i, we get

$$\sum_{i=1}^{k} w_i d_i = d \sum_{i=1}^{k} w_i = 1$$

Using the counts in (5.26), we see that this requires

$$\begin{aligned} 1 = \sum_{i=1}^{k} w_i d_i &= d\left(\binom{m}{2} w_{\ell\ell} + \binom{n}{2} w_{rr} + mn w_{\ell r}\right) \\ &= d\left(\binom{m}{2} 2\rho + \binom{n}{2} 2\rho + mn\rho\right). \end{aligned}$$

So, using (5.27) and (5.28), our guess for the constant vector ρ is

$$\rho = \frac{d^{-1}}{2\left(\binom{m}{2} + \binom{n}{2}\right) + mn} = \frac{d^{-1}}{2d^{-1} - mn} = \frac{1}{2 - mnd}. \tag{5.29}$$

Note that this guess is independent of the value of the exponent p.

Now we are ready to produce upper and lower bounds for concurrent modulus on $K_{m,n}$. By construction, the choice of ρ and w above is admissible for (5.1) and, therefore,

$$\mathrm{Mod}_{p,\sigma,d}(\Gamma) \le \mathcal{E}_{p,\sigma}(\rho) = \frac{|E|}{(2 - mnd)^p} = \frac{mn}{(2 - mnd)^p} \tag{5.30}$$

when $1 \le p < \infty$ and

$$\mathrm{Mod}_{\infty,\sigma,d}(\Gamma) \le \mathcal{E}_{\infty,\sigma}(\rho) = \frac{1}{2 - mnd}. \tag{5.31}$$

We can establish the value of $\mathrm{Mod}_{p,\sigma,d}$ for the cases $p \in \{1, 2, \infty\}$ by finding a lower bound that coincides with this upper bound.

Lower bound for $p = \infty$. The value of concurrent ∞-modulus computed directly using (5.22):

$$\begin{aligned}
\mathrm{Mod}_{\infty,\sigma,d}(\Gamma)^{-1} &= \sum_{i=1}^{k} d\,\mathrm{dist}_{\sigma^{-1}}(s_i, t_i) \\
&= d\left(mn + 2\binom{m}{2} + 2\binom{n}{2}\right) \\
&= d(2d^{-1} - mn).
\end{aligned}$$

Therefore, for our choice of σ and d,

$$\mathrm{Mod}_{\infty,\sigma,d}(\Gamma) = (2 - dmn)^{-1} = \frac{\binom{m+n}{2}}{\binom{m+n}{2} + \binom{m}{2} + \binom{n}{2}}. \tag{5.32}$$

By (5.31), our guess for ρ and w attains this value and, therefore, are optimal for ∞-modulus.

Lower bound for $p = 1$. Next, we obtain a lower bound on the 1-modulus by making a guess for the values x_γ in the dual problem (4.3). We make the following guess: $x_\gamma > 0$ only if γ is either a one-hop or a two-hop path. In other words, we guess that x is supported on the disjoint union $\Gamma_{\ell\ell} \cup \Gamma_{rr} \cup \Gamma_{\ell r}$, where $\Gamma_{\ell\ell}$ is the family of two-hop paths connecting a left-left pair, Γ_{rr} is the family of two hop paths connecting a right-right pair, and $\Gamma_{\ell r}$ is the family of one-hop paths. By symmetry, we guess an x of the form

$$x_\gamma = \begin{cases} \alpha & \text{if } \gamma \in \Gamma_{\ell\ell}, \\ \beta & \text{if } \gamma \in \Gamma_{rr}, \\ \delta & \text{if } \gamma \in \Gamma_{\ell r}, \\ 0 & \text{otherwise.} \end{cases}$$

The following counts can be verified for an arbitrary $e \in E$.

$$\begin{aligned} |\{\gamma \in \Gamma_{\ell\ell} : e \in \gamma\}| &= m - 1, \\ |\{\gamma \in \Gamma_{rr} : e \in \gamma\}| &= n - 1, \\ |\{\gamma \in \Gamma_{\ell r} : e \in \gamma\}| &= 1. \end{aligned} \tag{5.33}$$

If we assume that each edge is used to full capacity in the dual problem, then we should satisfy

$$\alpha(m-1) + \beta(n-1) + \delta = 1. \tag{5.34}$$

Now, consider the constraint $\sum_{\gamma \in \Gamma_i} x_\gamma \geq z d_i$ in problem (4.3) for each source-sink pair (s_i, t_i). Again by the symmetry of the problem, we expect that the optimal choice of x will realize this inequality as equality for each pair. For a left-left pair, then, we guess that

$$zd = \sum_{\gamma \in \Gamma_i} x_\gamma = \sum_{\gamma \in \Gamma_i \cap \Gamma_{\ell\ell}} x_\gamma = n\alpha,$$

since there are exactly n two-hop paths connecting s_i to t_i. By similar arguments for right-right and left-right pairs, we assume that

$$zd = m\beta = \delta.$$

Substituting these guesses into (5.34) yields the relation

$$zd\left(\frac{n-1}{m}+\frac{m-1}{n}+1\right)=1,$$

which we can use to produce a guess for z.

Rewriting this shows that we should guess z so that

$$\begin{aligned}1&=\frac{zd}{mn}\left(n(n-1)+m(m-1)+mn\right)\\&=\frac{zd}{mn}\left(2\binom{m}{2}+2\binom{n}{2}+mn\right)\\&=\frac{zd}{mn}(2d^{-1}-mn),\end{aligned}$$

where the last equality follows from (5.27). This provides the guess

$$z=\frac{mn}{2-mnd}.$$

This choice of x and z is admissible for (4.3), showing that

$$\mathrm{Mod}_{1,\sigma,d}(\Gamma)\geq\frac{mn}{2-mnd}.$$

This lower bound coincides with the upper bound in (5.30) when $p=1$. Therefore, for our choice of σ and d,

$$\mathrm{Mod}_{1,\sigma,d}(\Gamma)=\frac{mn}{2-dmn}=\frac{mn\binom{m+n}{2}}{\binom{m+n}{2}+\binom{m}{2}+\binom{n}{2}}.\tag{5.35}$$

Lower bound for $p=2$. In the $p=2$ case, we use Theorem 5.2.3. In order to establish the bound, we need to choose a PMF μ_i for each $\Gamma(s_i,t_i)$. We do this as follows. For a left-left pair, we distribute the probability uniformly on the n two-hop paths. For a right-right pair, we distribute the probability uniformly on the m two hop paths. For a left-right pair, we concentrate the probability on the single one-hop connecting path. For a given pair (s_i,t_i) and a given edge e, this results in the following probabilities.

If the pair is a left-left pair, then

$$\mathbb{P}_{\mu_i}(e \in \underline{\gamma}_i) = \begin{cases} \frac{1}{n} & \text{if } e \text{ is incident on } s_i \text{ or } t_i, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for a right-right pair,

$$\mathbb{P}_{\mu_i}(e \in \underline{\gamma}_i) = \begin{cases} \frac{1}{m} & \text{if } e \text{ is incident on } s_i \text{ or } t_i, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for a left-right pair,

$$\mathbb{P}_{\mu_i}(e \in \underline{\gamma}_i) = \begin{cases} 1 & \text{if } e = \{s_i, t_i\}, \\ 0 & \text{otherwise.} \end{cases}$$

Using the counts in (5.33), this choice yields the inequality

$$\begin{aligned} \mathrm{Mod}_{2,\sigma,d}(\Gamma)^{-1} &\leq \sum_{e\in E}\left(d\sum_{i=1}^{k}\mathbb{P}_{\mu_i}(e \in \underline{\gamma}_i)\right)^2 \\ &= d^2 mn\left(\frac{m-1}{n} + \frac{n-1}{m} + 1\right)^2 \\ &= \frac{d^2}{mn}\left(2\binom{m}{2} + 2\binom{n}{2} + mn\right)^2 \\ &= \frac{d^2}{mn}\left(2d^{-1} - mn\right)^2 \\ &= \frac{(2-mnd)^2}{mn} \end{aligned}$$

where we have used (5.27). This proves the lower bound

$$\mathrm{Mod}_{2,\sigma,d}(\Gamma) \geq \frac{mn}{(2-mnd)^2},$$

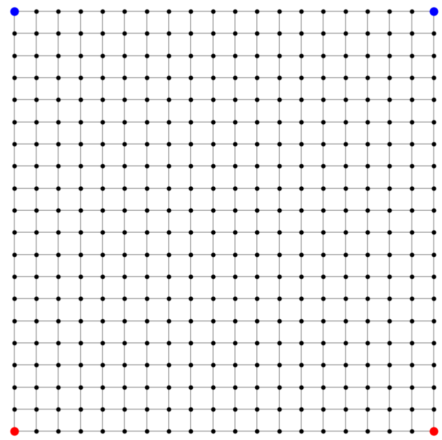

Figure 5.3: *Grid graph with source-sink pairs in adjacent corners*

which coincides with (5.30) when $p = 2$. Therefore, for our choice of σ and d,

$$\mathrm{Mod}_{2,\sigma,d}(\Gamma) = \frac{mn}{(2 - dmn)^2} = \frac{mn\binom{m+n}{2}^2}{\left(\binom{m+n}{2} + \binom{m}{2} + \binom{n}{2}\right)^2}. \tag{5.36}$$

Example 5.5.4. Consider a square grid graph with 400 nodes, and let (s_1, t_1) and (s_2, t_2) be source-sink pairs with demands d_1 and d_2 respectively. Consider a set of different demands between nodes s_1 and t_1 as:

$$d_1 \in \mathcal{D} := \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right\},$$

where $d_2 = 1 - d_1$. The graph is unit capacity and all the edges have capacity one. Consider the following two cases for the positions of the source-sink pairs:

Pairs in adjacent corners: Each source node and its corresponding sink node are in adjacent corners of the graph as shown in Figure 5.3. The nodes in red color represents the pair (s_1, t_1) and the nodes in blue represent the pair (s_2, t_2). Here we compute the concurrent 2-modulus values numerically for the source-sink pairs $\{(s_1, t_1), (s_2, t_2)\}$. Figure 5.4 shows the amount of flows for different demands in set $\mathcal{D}$. When there is no demand in pair (s_1, t_1), i.e. $d_1 = 0$, the value the flow f_1 from s_1 to t_1 is zero, and f_2 uses all of the resources to send the maximum flow from s_2 to t_2. As the demand d_1 for the pair (s_1, t_1) increases, the amount of flow f_1 also increases. Finally when the demands d_1 and d_2 are same and both are equal

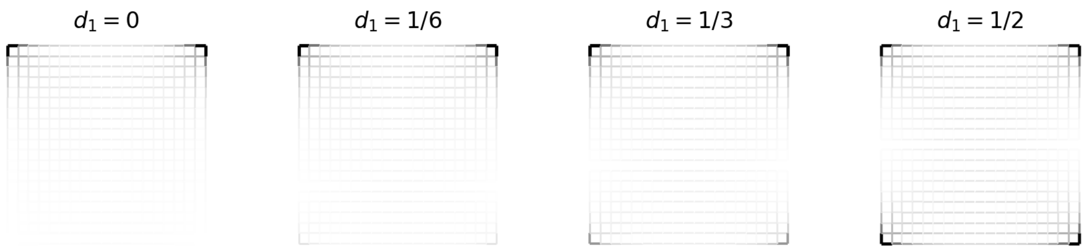

Figure 5.4: *Flows f_1 and f_2 for some values of demands with pairs on adjacent corners*

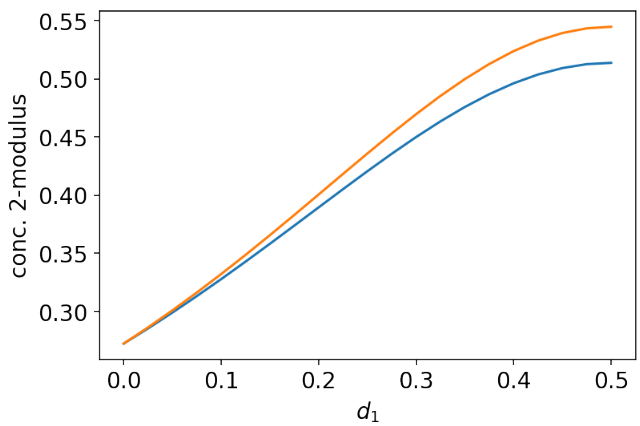

Figure 5.5: *Concurrent 2-modulus v.s demand d_1 demands on adjacent corners*

to half, the amount of flows f_1 and f_2 are also same. The reason is the symmetry in the grid graph and also positions of the source-sink pairs in adjacent corners, where the flows f_1 and f_2 don't compete much over the resources. Figure 5.5 shows the values of concurrent 2-modulus as a function of demand d_1 over interval $[0, 0.5]$ in two case; first when the families of paths between pairs (s_1, t_1) and (s_2, t_2) have some interactions. The value of concurrent 2-modulus in this case is shown in blue. Second when the families of paths between pairs (s_1, t_1) and (s_2, t_2) have no interactions. The value of concurrent 2-modulus in this case have shown in orange which is just slightly higher than the case where families have some interactions. When the pairs are in adjacent corners, they don't compete over the resources very much therefore the concurrent 2-modulus for these two cases are not very different.

Pairs in opposite corners: Next consider each source node and its corresponding sink node are in opposite corners of the graph as shown in Figure 5.6. The nodes in red color represents the pair (s_1, t_1) and the nodes in blue shows (s_2, t_2). As figure 5.7 shows when we

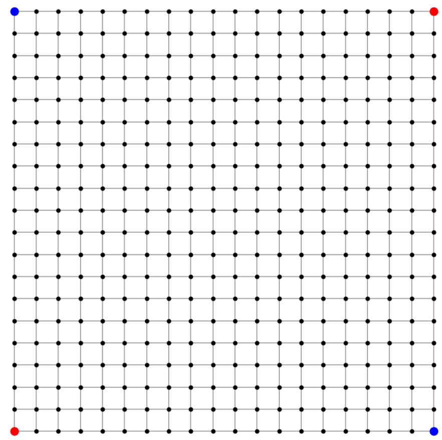

Figure 5.6: *Grid graph with source-sink pairs in apposite corners*

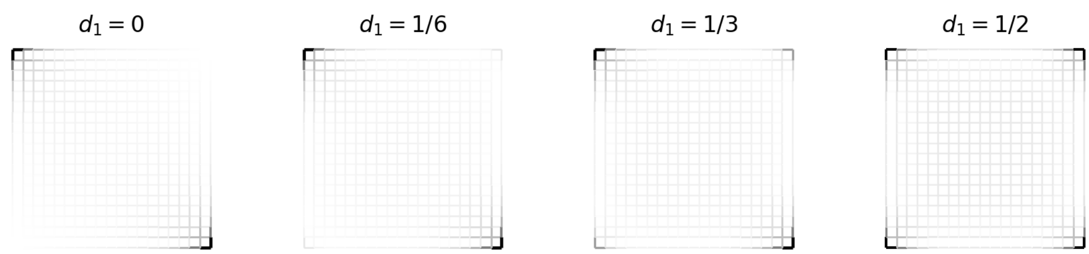

Figure 5.7: *Flows f_1 and f_2 for some values of demands with pairs on apposite corners*

increase the demand d_1 from zero to $\frac{1}{2}$, the flow f_1 from s_1 to t_1 also increases.

Also, Figure 5.8 compares the concurrent 2-modulus as a function of demand d_1 on interval $[0, 0.5]$ for two cases. First the families between pairs have some interactions and the concurrent 2-modulus is graphed in blue v.s when the families between pairs have no interaction and the concurrent 2-modulus is graphed in orange. As the graph shows the flows f_1 and f_2 in this graph compete over the given resources therefore concurrent 2-modulus increases when there is no interaction between families.

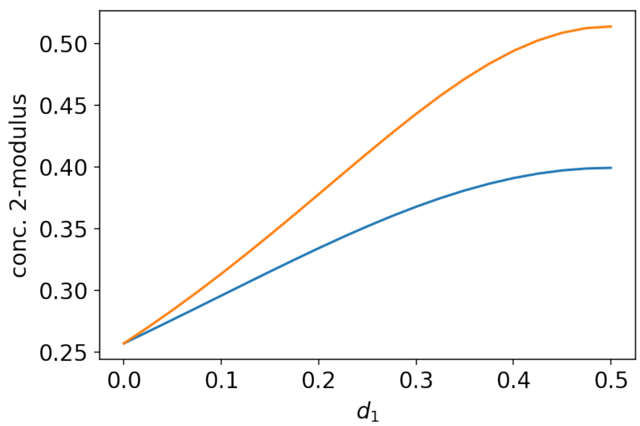

Figure 5.8: *Concurrent 2-modulus v.s demand d_1 demands on apposite corners*

Chapter 6

Conclusion and future work

In this dissertation we showed that maximum concurrent flow problem can be embedded into a family of p-modulus problems. In particular we studied $p = 1, 2$ and ∞ cases and also developed the probabilistic interpretation of p-modulus problems. The key results are summarized in Table 6.

Items of particular interest for future research include

- a better understanding of concurrent modulus problems for general p,
- and exploration of the dependence of the concurrent modulus problem on the parameters σ and ρ,
- an exploration of the properties of concurrent modulus problem using different families of objects,
- centrality measures for networks with multiple sources and targets, and
- applications of concurrent modulus.

	1 **pair**	k **pairs**
$p = 1$	maximum flow (minimum cut) problem $\sim$ 1-modulus problem	maximum concurrent flow problem $\sim$ **concurrent 1-modulus problem**
$p = 2$	effective resistance problem $\sim$ 2-modulus problem	**concurrent resistance** problem $\sim$ **concurrent 2-modulus problem**
$p = \infty$	shortest path problem $\sim$ ∞-modulus problem	**weighted shortest path problem** $\sim$ **concurrent ∞-modulus problem**

Table 6.1: *Summary of connections between modulus and single- and multi-commodity flow problems. The items in bold indicate the contributions of this dissertation.*

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