The unfolding of third graders’ conceptualization of a function:

A collective case study

by

Jennifer Ellen Whitley

B.S., Black Hills State University, 2012
M.S., Kansas State University, 2015

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Curriculum and Instruction
College of Education

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

The goal of this study was to understand elementary students' meaning-making process of early algebra representations in the context of number patterns about even and odd numbers within their natural classroom setting. A particular emphasis was placed on understanding the role of the function as a way to generalize a relationship between inputs and outputs where every input has exactly one output, i.e. an even number can be defined as $f(n) = 2n$ for an integer n and an odd integer could be defined as $f(n) = 2n + 1$ for an integer n. Through a collective case study of individual students in a third-grade classroom, I qualitatively analyzed and characterized individual noticing patterns when given a function rule table and moreover, concluded that third-graders are able to conceptualize a function rule in the context of generating even integers and were able to act upon a function rule when generating odd integers. The study suggests that the benefit of integrating function tables into the early elementary instruction will support formal functional understanding in later grades.

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Chapter 1 - Introduction

Algebra, as a body of knowledge, is largely connected to all areas of mathematics and the understanding of algebraic representations, generalizations, and transformations develop mathematical proficiency, which is important for everyday problem solving within our current society (Kaput, 2008). Concurrently though, algebra has historically functioned inadvertently as a barrier for students wishing to complete their secondary education, enter college, and/or earn their college degree (Kilpatrick & Izsák, 2008; Moses, 2001; Schoenfield, 1995). The inequitable nature of the algebra curriculum including how and when algebra generally has been taught within schools has led to what math educators and researchers have deemed the "algebra problem" (Kaput, 2008, p. 5). Recent focus has been placed on "rethinking and reworking algebra" (Kaput, 2008, p. 6) within elementary, middle school, high school, and college to support all students in their current and future learning. In order to strive for equal access to economic opportunity and citizenship, ensuring algebra is accessible for all students is a major principle of the *Principles and Standards for School Mathematics* (NCTM, 2000) and more recently, the Common Core State Standards for Mathematics Initiative (CCSSI, 2010).

This rethinking and reworking algebra to ensure equitable opportunities further requires substantial research in the area of teaching and learning algebra in education (RAND Mathematics Study Panel, 2003). In response to limiting the role of algebra as a gatekeeper, current research has shown the positive influence of sustained algebra experiences beginning in elementary grades, i.e. *early algebra*. Early algebra focuses on building off of children's natural intuitions about patterns and relationships within arithmetic to begin formalizing algebraic representations and generalizations (Kaput, Carraher, & Blanton, 2008). Due to this shift in mathematics education, an emerging body of research has been devoted to children's algebraic thinking (Bastable & Schifter, 2008; Blanton, Stephens, Knuth, Gardiner, Isler, & Jee-Seon, 2015; Carpenter, Franke, & Levi, 2003; Dougherty, 2003, 2008; Schifter, 2009). In particular, Blanton and colleagues have shown that "children are capable of thinking algebraically across a broad and diverse set of big ideas" and argued the importance of a sustained and comprehensive approach to early algebra instruction for improvement in children's readiness for algebra in the middle grades (Blanton, Stephens, Knuth, Gardiner, Isler, & Jee-Seon, 2015, p. 76).

Since these shifts in mathematics education are still currently underway, although the importance of early algebra development is well established within the mathematics education community, it still remains to be seen how algebra is interpreted within the curriculum and classroom setting by both the students and teachers (Blanton, Stephens, Knuth, Gardiner, Isler, & Jee-Seon, 2015). The *Common Core State Standards for Mathematics* (CCSSI, 2010) support a comprehensive approach to early algebra and furthermore, support the use of various external representations to bridge the development of a conceptual understanding of arithmetic concepts with algebraic thinking. In order to improve the content and pedagogy of early algebra, it becomes imperative to understand how students interpret the algebra concepts presented within the classroom. This requires examining how elementary students are interpreting algebraic concepts, but also requires considering the minds of the students and viewing the algebra concepts from their own perspectives.

To begin addressing the open question as to how elementary students interpret and develop an understanding of algebra concepts presented within the classroom, I have situated the following research on student's mathematical visual representations within the area of early algebra. My hope is to better understand a selection of students' visual representations of algebra problems in order to advance the research guiding the instruction of algebra for all.

Moreover, I chose to situate the following research on student's mathematical visual representations within the area of function and the static external representation of function tables.

The choice of focusing on functions came to me after spending time in the classroom and many interviews during pilot studies where I interviewed selected third and fourth-grade students on various static external representations. Some of these included the area model to show distributive, associative, and commutative properties, or using the number line to show multiples of 3, 6, and 9. In my interviews, I found that the most interesting noticing patterns were appearing in function tables, which later developed into the lesson that I analyzed in the following research. Furthermore, since my prior Master's research (Johnson, 2015) was on College Algebra students' noticing patterns in linear function tables, I had prior knowledge that I could use in designing and analyzing the lesson study. It served as an interesting question to me as to whether a group of third-graders would have different noticing patterns when confronted with the details of the static external representation of the function table. As a college mathematics instructor, I knew the importance of advancing our understanding of student's internal representations within function tables so that we can better instruct our students.

Rationale

Research in cognitive psychology emphasizes that knowledge is formed through constructing mental representations of a concept, making connections between the representations of a concept, and developing a reasoning which links different pieces or schemas within the concept (Barmby, Harries, Higgins, & Suggate, 2009). This intricate process includes students modifying their "*internal* mental representations to construct mathematical relationships or structures that mirror those embodied in *external* instructional representations" (Cobb, Yackel,

& Wood, 1992, p. 2). Moreover, a deeper understanding of mathematical concepts and procedures is greatly related to the strength of connections among the internal and external representations of the students (Pape & Tchoshanov, 2001). Valuing the interaction between internal and external representations is essential to successful teaching and learning of mathematics (Goldin & Shetingold, 2001; Kilpatrick, Swafford, & Findell, 2001).

It has been shown through teacher anecdotes and education research that learning algebra at an exclusively abstract level without different visual representations and connections can cause many difficulties in learning mathematics, both cognitively and affectively (Tabach & Friedlander, 2008). As Tripathi (2008) metaphorically states, using different representations is like “examining the concept through a variety of lenses” (p. 439) with each lens providing a different perspective that creates a much deeper understanding of the concept. This focus on fostering “representational competence” has been shown to be crucial in not only the success of learning algebra but also the retaining of algebraic knowledge and algebraic reasoning skills (Boester & Lehrer, 2008, p. 212).

But aside from the advantages of the use of multiple visual representations in the teaching and learning of algebra, previous research has also shown that there continues to be a disconnect between what is being taught and what is being learned, specifically with visual representations in early algebraic understanding. Lobato and colleagues (2014) have shown that students’ internal representations in middle-school algebra were largely based on the cultural context of the classroom, but Boulton-Lewis (1998) found that children do not always make connections between their internal representations and the external representations that are offered in the culture of the classroom. Furthermore, Bolden and colleagues (2015) found that children generally did not view the external representations holistically and conceptually, and instead attended to only specific features, which often led to misunderstandings associated within their internal representations. These studies continue to remind educators and researchers that there lies a disconnect between student's internal and the external representations, and furthermore, this disconnect must be investigated by future research. The knowledge from such research has possibilities of transcending how teachers use visual representations in the classroom to support all students' learning of mathematics, and in particular, the learning of algebra.

Research Purpose

To begin addressing the question as to how elementary students interpret and develop an understanding of algebra concepts presented within the classroom both situationally and culturally, I have situated the following research on student's mathematical visual representations within the area of function relationships. The purpose of this study is to describe how individual third-graders make meaning, through mathematical noticing, of the external representations within the classroom and how this meaning-making process relates to a student's understanding of the algebra concepts. This meaning-making process translates student's mathematical noticing of external representations into their individual internal representations and is based on the situational context (external representations present within the classroom) and the cultural context of the classroom (see Figure 1-1 below). In the study, I qualitatively examined how third-grade students make meaning of early algebra representations of number patterns about even and odd numbers within their natural classroom setting. Furthermore, the study led me to understanding the role of the function as a way to generalize a relationship between inputs and outputs, i.e. an odd integer could be defined as $f(n) = 2n + 1$ for integer n. The collective case study draws upon classroom observations of one third-grade classroom and follow-up interviews with select students of that classroom who are of varying academic levels.

Rooted within interpretivism, symbolic interactionism provided the lens through which the research questions were viewed. Symbolic interactionism is an appropriate theoretical framework to guide my research because it aligns with my goal of understanding how students make meaning of mathematical representations. APOS Theory, which stands for Action, Process, Object, and Schema, is a theory of learning specific to mathematics education and was used as a micro-framework for analyzing students' understanding of visual representations. APOS Theory states that students build mathematical concepts by constructing mental actions, processes, and objects, and organizing them into schemas to make sense of the situations and to solve problems (Asiala et al., 1996). As a micro-framework, APOS serves as theoretical framework that guides my analysis of the case study student's understanding of the external representation of the function table.

Research Questions

One specific research question with two parts guided my research: Situated within a third-grade classroom, how do elementary student's mathematical noticing of external representations of algebraic problems relate to their (1) internal representations and (2) developed understanding of the concepts?

The following concept map helps visualize how I theoretically conjecture that student's mathematical noticing of external representations inform their "interpreted" student's internal representations which then inform student's understanding of the concepts. My research question addresses the goal of understanding both their "interpreted" internal representations and whether their understanding the concepts are correct or incorrect with what misconceptions. I conjecture that this meaning-making process of students interpreting the external representation of the static function “rule” table will relate to their understanding of the algebraic practice of generalizing about even and odd integers.

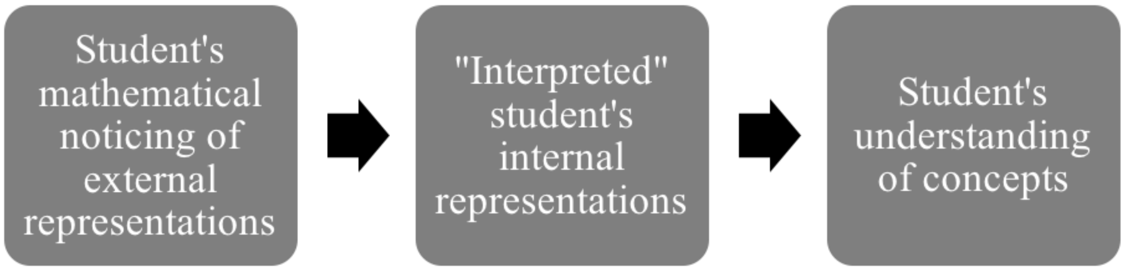


Figure 1-1 Concept Map of Research Question

Significance

The significance of this research would be supporting teachers in better understanding not only the external representations that they are teaching but how students are conceptualizing these representations. With this knowledge, teachers will be better able to support different students. Future research of my own may include developing a set instructional strategies that can be used to support all students in developing their mathematical noticing skills.

Aside from the many limitations that I will touch upon, this study has many possibilities in influencing mathematics education. As curriculum developers rework algebra with the inclusion of multiple representations, teachers are juggling how to facilitate students’ understandings of these representations. First and foremost, it will fill the gap in the research by gaining a deeper understanding of how students are making meaning of the visual representations being used in the classroom and furthermore, how that contributes to their understanding. This knowledge has possibilities of transcending how teachers use visual representations in the classroom to support all students’ learning of algebra concepts, which have been shown to be the gatekeeper for equitable outcomes in education.

Research Methodology

Grounded in interpretivist framework, collective case studies, as the methodology of this qualitative study, was done within one third-grade classroom from an elementary school in the Midwestern United States.

Data Collection Methods

Multiple sources of data were collected to provide for in-depth analysis and triangulation. The data collection methods for the purpose of this case study included in-depth and document elicitation participant interviews, classroom observations, researcher's field notes, and analysis of documents such as student work and teacher lesson plans. Observations and participant interviews were primarily carried out over a period of one semester (approximately four months).

Group interviews and individual participant interviews were a primary source of data in order to gain a deeper understanding of select students within the classroom. Classroom observations, natural conversations with the students, and researcher field notes were another primary source of data. As the researcher, my role was that of an active member in the classroom (Bhattacharya, 2012). It was important to me that I was both participating within the community of learners but also observing from the sideline as a peripheral member. Researcher field notes will reflect my reflexivity, that is, how my role and assumptions influence my interpretations of my observations (Preissle & Grant, 2014).

Operationalization of Constructs

For the purposes of this study, the following definitions apply:

1. *Algebra*: Refers to the body of knowledge and reasoning associated with "systematically symbolizing generalizations or regularities and constraints" and "syntactically guided…actions on generalizations" in mathematics (Kaput, 2008, p. 11).
2. *External Representations*: Visual external representations are those that are used in structured learning environments. This may include the use of symbolic (equations, tables, graphs), pictorial (including number lines, arrays, area, sets, etcetera), and concrete (manipulatives) representations of mathematics. Visual representations express features of both mathematics constructs and actions.
3. *Internal Representations*: Students' personal "symbolization constructions," and visual and spatial imagery formed by individual students (Goldin & Shteingold, 2001). Again, the visual representations express features of both mathematics constructs and actions.
4. *Problems:* Refer to a developmentally appropriate challenge, which in our case resides in algebra, for which students are given a goal but the means of achieving it may not be apparent.
5. *Function Tables:* A function is a relationship between an input (or independent variable) and an output (or dependent variable), with exactly one output for each input. Functions provide a means for describing and understanding relationships between variables. They can have multiple representations – algebraic symbols, graphs, tables, and verbal descriptions. In a function table, each column corresponds to the input and output variables. It is a static and discrete representation of the function relationship.

6. *Problem solving*: Refers the mental act of "making sense of the problem situation and the means necessary" in order to strategically make decisions, which will guide subsequent reasoning and understanding (Yee & Bostic, 2014, p. 2).
7. *Academic levels*: Refers to the academic achievement levels, i.e. low, middle, high, of students based on objective achievement scores and subjective teacher perceptions.
8. *Midwestern U.S.*: Refers to the region of the United States including the 12 states, Illinois, Indiana, Iowa, Kansas, Michigan, Minnesota, Missouri, Nebraska, North Dakota, Ohio, South Dakota, and Wisconsin.

Summary

This chapter introduces the rationale for this qualitative study along with the theoretical framework, symbolic interactionism, for which guides the research questions and methodology. The purpose of this collective case study is to describe how the situational context and cultural context inform upper elementary and middle school students' conceptual context of internally visualizing mathematical representations while the students engage in algebra problem solving. My hope was to better understand a selection of students' meanings of visual representations within the area of function tables to advance the research guiding the instruction of algebra for all.

Within Chapter 2, I elaborate on a theoretical framework, explore the literature related to early algebra development and visual representations related to early algebra development. In Chapter 3, I describe the research design including the, data collection, and methods of analysis general theory underlying the instructional design and analysis. Chapter 4 presents the results of my analyses and Chapter 5 concludes the dissertation, summarizing and discussing results and implications.

Chapter 2 - Literature Review

This following literature review is organized in four sections. The first section describes the general theoretical perspective that informs the present study. The second section outlines the recent shift in mathematics education to include early algebra instruction and current research on early algebra development. This includes importance of early algebra instruction as shown by empirical findings, as well as the possible gaps within current research. The third section defines the micro-framework, APOS Theory, that I used for analyzing students' meaning-making process.

Theoretical Perspective

The theoretical framework underlying my understanding of how students make meaning within algebra problem solving is that of symbolic interactionism. In the subsequent sections, I will discuss the foundation and basic assumptions of symbolic interactionism rooted within the theoretical perspective interpretivism, and how the theory aligns to this particular study.

As a theoretical perspective, interpretivism assumes the ontological belief that there are multiple realities (Crotty, 1998). As people interact with their world, these multiple realities are socially and experientially constructed through individual and collective meanings or understandings. Epistemologically, this aligns to the belief that the "truth, or meaning, comes into existence in and out of our engagement with the realities of the world" (Crotty, 1998, p. 8), i.e. a constructionist view of knowledge. Guided by these tenets, the interpretivist approach to inquiry is concerned with the unearthing, interpreting, and understanding of these meanings and the process of meaning making (Crotty, 1998; Schwandt, 2015). In particular, symbolic interactionism, a subsidiary of interpretivism, is largely focused on the social construction of participants' meanings or understandings (Reynolds & Herman-Kinney, 2003; Crotty, 1998).

The essence of the theory, which later came to be known as symbolic interactionism, was first articulated by George Herbert Mead, a pragmatic philosopher and social psychologist who worked closely with John Dewey (Reynolds & Herman-Kinney, 2003; Crotty, 1998). With the foundational lens of pragmatism, Mead perpetuates the influence of society or culture on the individual in his declaration *Mind, Self and Society*. Mead (1934) claims that the "mind can never find expression, and could never have come into existence at all, except in terms of a social environment" (p. 222) and he contributes the meaning-making process to the social relations and interactions, which largely consist of shared symbolic tools for that we communicate by.

Herbert Blumer (1969), a sociologist and student of Mead, coined the term symbolic interactionism and is often credited for the establishment of the theory within sociology (Crotty, 1998). Largely influenced by his mentor's ideas, Blumer's distinctive labeling and clear formation of symbolic interactionism allowed for the theory to become a methodologically guiding framework. Blumer (1969) outlines three key premises of the theoretical framework:

i. That human beings act toward things on the basis of the meaning that things have for them;

ii. The meaning of such things is derived from, or arises out of, the social interaction that one has with one's fellows;

iii. These meanings are handed in, and modified through, an interpretive process used by the person dealing with the things he encounters. (p. 2)

The central notion of symbolic interactionism is therefore communication, broadly defined to include written and spoken language, nonverbal and verbal cues, cultural norms and beliefs. Connecting this notion within the key concepts of interpretivism, symbolic interactionism positions that individuals are continuously making meaning of the world around them and the meaning-making process is an intersubjective and social interpretive process based on the nature of communication.

Alignment to the Study

Symbolic interactionism is an appropriate theoretical framework to guide the following study because it aligns with my research goal of understanding how students make meaning of mathematical representations within algebra problem solving. Voigt (1996) has premised that symbolic interactionist approach is a useful lens when studying students' learning in mathematics classrooms since it emphasizes both the cultural and individual components of learning. Furthermore, symbolic interactionism is compatible with the learning theory social constructivism, developed by Lev Vygotsky and assumes learning and knowledge construction takes place in a social arena, even though meaning is eventually internalized by those persons individually (Ernst, 2010). Students' meaning is dependent on the situational contexts (i.e., external representations or symbolic communication given in the problems) and cultural contexts (i.e., the verbal and nonverbal communication within the classroom and norms within the classroom environment). Guided by the premises of symbolic interactionism, the student's internal representations or meanings of algebra will be constructed by his or her interpretive understanding of the symbolic external representations and depend on communicative culture for which he or she engages in. Furthermore, the students will solve the algebra problem (i.e. act upon the representations) based on the meanings that the students have constructed.

Guided by the previous assumptions, the following research questions arise:

1. How do students make meaning of the external representations of algebraic problems while problem solving?

2. How do student's internal representations of the concept relate to an understanding of the concept?

The first question allows for me to investigate the culture of the classroom, which includes the verbal/nonverbal communication (i.e., social interaction), in order to understand student's meaning-making process of their interpretation of algebraic concepts given in representations. In particular, the research question is balanced upon the second premise of symbolic interactionism, "the meaning of such things is derived from, or arises out of, the social interaction that one has with one's fellows" (Blumer, 1969, p. 2).

The second question examines whether there is an understanding of the algebraic concepts but to a greater extent. The question aims to dig much deeper into how the student's interpretations and interactions formulate a student's understanding based on the meanings the student assigns to the representation. This stands upon the two remaining pillars of symbolic interactionism, i.e., that the students "act toward things on the basis of their meanings" and the "meanings are handed in, and modified through, an interpretive process" which is particular to the student (Blumer, 1969, p. 2). Overall, both research questions focus on the social construction of students' understandings and therefore are grounded within the symbolic interactionism framework.

Defining the Micro-Framework APOS Theory

Mathematical noticing further extends the notion of executive attention by basing it in the learning process of *reflective abstraction* (Campbell, 2001). Reflective abstraction is rooted in Piaget's work in cognitive development. As a learner attempts to understand new information, they go through the process of reflective abstraction as they construct meaning and form mathematical generalizations. Noticing therefore can be captured in these beginning stages as students begin to make sense of a mathematical problem by noticing features and patterns, which will lead the student to make meaning of the problem. Furthermore, by connecting noticing to reflective abstraction, Lobato and colleagues (2014) showed "that what one notices mathematically can serve as the rootstock upon which one constructs ways to reason in new situations" (p. 812). Therefore, we can connect mathematical noticing to learning.

In efforts to characterize students' cognitive learning processes in mathematics, we draw upon APOS Theory, a theory of learning specific to university level mathematics education but can be extended to all levels of mathematical learning. APOS Theory states that students build mathematical concepts by constructing mental actions, processes, and objects, and organizing them into schemas to make sense of the situations and to solve problems (Asiala et al., 1996). APOS Theory is based on Piaget's idea of relative abstraction, as extended to advanced mathematics primarily by Dubinsky (1991a, 1991b).

In relation to understanding functional relationships, the first level of understanding is that of action. A student in this level assumes a function is tied to specific rule or formula, which the answer depends on by manipulation of variables or replacing by numbers for calculations. Having a process conception of function, assumes a function is an input-output machine and is independent of the formula. An object is constructed from a process when the student becomes aware transformations, multiple representations, and properties, i.e. understanding the concept completely. Finally, a schema is developed from collection actions, processes, and objects across concepts, and thus building a framework, which will assist in problem solving in novel situations.

Early Algebra

Algebra continuously has a large role in mathematics education. Foundationally, the nature of algebraic reasoning has been seen as part of the organic development of human cognitive and communicative maturity (Kaput, 2008). Historically though, algebra has been seen as a gatekeeper (Kilpatrick & Izsák, 2008; Moses, 2001; Schoenfield, 1995) and in particular, the traditional arithmetic-then-algebra approach in mathematics education, where an arithmetic-only curriculum in the elementary grades is followed by a formal treatment of algebra in the secondary grades often beginning in 6th grade, has shown to be a major culprit of student's failure in retaining mathematics knowledge (Kaput, 2008; Schoenfield, 1995). This has led to the reexamining of traditional algebra education and the acceptance of inclusion of early algebra development (RAND Mathematics Study Panel, 2003).

One of the general arguments for reworking algebra is that the "arithmetic and then algebra" approach that has been traditionally taught does not give students the required time to conceptualize algebra over a long period. Constructivist learning theories suggest that students need multiple experiences in order to construct their learning and therefore theoretically algebra should be treated as a longitudinal content area with strands K-12 rather than one-subject content area starting in middle school. Students will then be able to develop the necessary experiences and in-depth conceptual understanding for supporting their algebraic reasoning. This has effort to rework algebra instruction with the include of early algebra development in elementary has been further pushed by The Common Core State Standards for Mathematics (CCSSO, 2010), which emphasizes the importance of early algebra in children's mathematics education beginning in kindergarten.

Defining Early Algebra

Kaput (2008) positions that the domain of algebra consists of both particular thinking practices and content strands. Specifically, he states that algebraic thinking involves (a) making and expressing generalizations in increasingly formal and conventional symbol systems and (b) reasoning with symbolic forms. He further argued that these practices take place across three content strands:

1. Algebra as the study of structures and systems abstracted from computations and relations, including those arising in arithmetic (algebra as generalized arithmetic) and quantitative reasoning.
2. Algebra as the study of functions, relations, and joint variation.
3. Algebra as the application of a cluster of modeling languages both inside and outside of mathematics. (Kaput, 2008, p. 11)

Research on Algebraic Understanding in Elementary Grades

Major research has delved into how elementary students conceptualize algebra concepts and how early algebra instruction can be integrated within the mathematics curriculum. A substantial amount of research in the last thirty years by multiple teams of mathematics education researchers have proven that not only will students be better prepared for formal algebra by learning and connecting arithmetic to algebra, but also that elementary arithmetic understanding is enhanced when connected the algebra reasoning.

The study of number and operations is the core content of elementary grades, but decades of research have further emphasized that elementary students' arithmetic understanding is enhanced when connected to algebraic reasoning and understanding. The study of mathematics may be viewed as a way of thinking that involves studying patterns, looking for underlying structure and regularity within those patterns, making conjectures about the patterns, identifying and describing those relationships, and further developing mathematical arguments to justify why these relationships hold. These processes form what we call algebraic reasoning and are emphasized in the Common Core State Standards in Mathematics (CCSSI, 2010) as part of the eight *mathematical practices* taught K-12.

Generalizing Functional Relationships

An essential way to integrate early algebra thinking into the elementary classroom is through *functional thinking*. A function is defined as a relationship between an input (or independent variable) and an output (or dependent variable), with exactly one output for each input. In general, functions provide a means for describing and understanding relationships between two or more variables. The relationships can be very simple (such as a linear relationship) to very complex. Understanding functional relationships and how they behave is central to mathematical modeling. In mathematical modeling, a function serves as a model for describing how the variables change together in order to better understand and make decisions or predictions about the real-world phenomenon.

As with many different areas in mathematics, functions can have multiple representations – algebraic symbols, graphs, tables, and verbal or written descriptions are all used to describe functions in the classroom and later. One of the representations which will be a focus in this research, is the function table. In the function table, each column corresponds to the input (or independent) and output (or dependent) variables. In the figure below, we see that the function table is both a static and discrete representation of the function relationship. It is static because students are unable to manipulate the table. They may continue to fill out the table and are able to visualize different patterns, but it is not a manipulative representation. Furthermore, it is discrete as the table can only characterize a glimpse into the infinite relationship. Even though it serves as a discrete and static representation, a function table helps students become aware of how one quantity depends on the other. This dependency is made explicit by how the table is constructed and visually read.

Input ($\boldsymbol{n}$)	Output $\boldsymbol{f(n) = 2n}$
0	0
1	2
2	4
…	…

Figure 2-1 Structure of a Function Table

Blanton and colleagues (2008) have spent a considerable time on a multi-year project researching how function tables can be used in the elementary classroom. In particular, their substantial research has shown that what children and teachers do with the information in a function table varies significantly. Often, early elementary students first find a *recursive pattern,* which involves looking down the columns for a relationship within that sequence of numbers. For example, in Figure 2-1, the recursive relationship within the output column is “adding two.” Students would be able to use this information to predict what value comes after 4 to get 6, without knowing or using the multiplicative relationship with the input number.

To move students beyond recursive patterning and thinking, we want students to find or connect a *correspondence* between the input and output quantities. Blanton and Kaput (2005) gave the elementary students, as early as first grade, the problem of describing the total number of eyes that dogs have. For example, 3 dogs correspond to 6 eyes, or in symbols $E = 2n$ represents the number of eyes for n dogs. The researchers found that children as early as second grade are able to see a multiplicative relationship (similar to the table in Figure 2-1), and are able to describe this relationship as “multiply by two” (Blanton and Kaput, 2005). By third grade, students can begin to express this correspondence symbolically as $E = 2 \times n$.

Blanton and colleagues have defined a sequence of instruction, beginning as early as first grade, to introduce function tables in the mathematics classroom. Giving tasks such as the dog eyes task in first grade, will support students in counting and recording quantities that are related to each other (i.e. the number of dogs and number of eyes). This early step in functional understanding is critical for later more advanced function thinking. Later in second and third-grade, teachers should move students beyond using tables only to record number relationships and move towards analyzing and reasoning about the function relationships in the table. Finally, students will move from words to symbols in describing the functional relationship. Blanton and colleagues showed through their substantial project that this sequence and scaffolding of functional tables can guide to early elementary students (i.e. Kindergarten through 3rd grade) in using function tables representing linear, quadratic, and exponential functions to model contextual relationships. Blanton further emphasizes the importance of using function tables in the early grades – even though students may not be able to find the more complex functional relationships, students will still be organizing data, finding and interpreting recursive patterns all of which are current standards in those grades.

Summary and Limitations within Early Algebra Research

Since the NCTM *Principles and Standards for School Mathematics* (NCTM, 2000) first called for the inclusion of early algebra instruction as a K-12 strand of mathematics, a vast array of research has focused on answering what elementary classrooms and curriculum, which include algebraic instruction, look like and furthermore, how do students make meaning of these early algebra concepts. More focus has been on the first question, that is, how teachers can transform their arithmetic based curriculum to early algebra based curriculum and how such curriculum will support children's arithmetic skills. There is no doubt whether early algebra instruction is important and there are a multitude of resources available for elementary teachers in hoping to include instruction to support children in developing their algebraic thinking.

Blanton and colleagues' substantial research on early algebra instruction developed our understanding of what early algebra instruction looks like and further emphasizes the fact that the inclusion of algebra "makes mathematics richer, more connected, more general, and more explicit" for our students (p. 1). Their research was extensive and significant, but one limitation pertaining to functional relationships was that all the tasks and problems with function tables pertained to concrete and contextual situations and typically involved whole numbers (for example, the number of eyes on a dog or the number of handshakes in a group of people). Each situation began with students able to explore concretely the relationship. This was motivated by the fact that concrete situations will scaffold student's understandings to a more abstract situation. In my research, I chose a familiar (even and odd integers) but more abstract situation where students will explore the external representation of function tables in order to generalize patterns about even and odd integers. I was curious as to whether a non-contextual problem would still afford students with the possibilities that sustained early algebra instruction in contextual situations afforded students in prior research.

The Significance of Multiple Visual Representations

The use of multiple visual representations achieves a significant role in the teaching, learning, and doing of mathematics. In particular, algebra can be seen a representational system (Saul, 2001). For the purposes of this research, visual representations will be defined to include the use of symbolic (equations, tables, graphs), pictorial (including number lines, arrays, area, sets, etcetera), and concrete (manipulatives) representations of mathematics. Visual representations express features of both mathematics constructs and actions. Each one of these representations is important in the acquisition and communication of mathematical knowledge. With the view that a variety of visual representations supports students' ability to make connections and develop a deeper understanding, students' reasoning processes and mathematical discourse, and students' development and implementation of problem solving strategies, I will position the significance of using multiple visual representation in mathematics instruction based on research in both cognitive psychology and mathematics education.

Through my experiences in learning and teaching mathematics, I have found that a variety of visual representations to be an invaluable tool in supporting an equitable mathematics classroom. In efforts to support all students with different prior experiences, learning styles, cultures, etcetera, multiple visual representations serve as connectors for all students in their understanding of mathematics concepts. Being able to *visualize* mathematics opens many doors for students that may not have been given this opportunity before, and therefore, as a teacher, we are able to reach more students, including those who otherwise may struggle in mathematics. Furthermore, I have found even for high-achieving students, asking the students to represent their ideas in another away (i.e., different representation) pushes the students to a new level of understanding. I have learned that this teaching strategy relies on a strong pedagogical content knowledge (Shulman, 1986; Ball, Thames, Phelps, 2008) of the teacher, and when used effectively in the classroom how the strategy can transcend the learning of mathematics for all students.

Historically in mathematics, visual representations, which moved beyond symbols and words, were not abundant in textbooks or classrooms, especially when disregarding geometry topics. But in recent efforts to reform mathematics education, organizations such as the National Council of Teachers of Mathematics (NCTM) have highlighted the significance of representations in the teaching and learning of mathematics by including the process standard for representations in *Principles and Standards for School Mathematics* (NCTM, 2000). Similarly, the current Common Core State Standards for Mathematics (CCSSM) (CCSSI, 2010) emphasizes multiple representations, including a variety of visual representations, in both the practice and content standards. Students are expected to create and use multiple visual representations, while also forming connections among the different representations. Therefore, classrooms must provide students with many different opportunities with these representations.

The use of multiple visual representations in mathematics teaching has become critical in ensuring success for all students in learning mathematics. It is evident that the research on visual representations has pushed the mathematics education community to include this strategy in reformed instructional practices for all grade levels. After describing the relevant research, which supports three particular claims, I will discuss the implications of the research specifically in the elementary mathematics classroom. Following, I will suggest possible future research on the use of visual representations, which may further advance the teaching and learning of mathematics.

Situating my position within a constructivist view, the learner should be active in building their own knowledge, constructing relationships and developing conceptual understanding through the processes of assimilation and accommodation. Research in cognitive psychology, beginning with Piaget and Bruner, emphasizes that knowledge is formed through "connections between mental representations of a concept, with reasoning linking together the different parts of the concepts" (Barmby, Harries, Higgins, & Suggate, 2009, p. 220). Furthermore, this intricate process includes students modifying their "internal mental representations to construct mathematical relationships or structures that mirror those embodied in external instructional representations" (Cobb, Yackel, & Wood, 1992, p. 2). Through this theoretical perspective, teaching practices should be motivated by active learning where the learners are able to meaningfully construct mathematical relationships and concepts individually and as a community of learners. Teaching strategies such as the use of argumentation and problem solving are central to such a classroom (Yackel, et al, 1990).

In the following literature review, I will support my claim that the use of multiple visual representations not only plays a significant role in students' development of a deeper understanding of mathematical concepts and procedures, but also visual representations are important in stimulating students' reasoning, discourse, and problem solving.

Supporting the Development of a Deeper Understanding

Previously emphasized is the role that representations have on developing knowledge. The National Research Council (NRC) (2001) further emphasizes this fact by confirming that "because of the abstract nature of mathematics, people have access to mathematical ideas only through the representations of those ideas" (p. 94) and that a deeper understanding of mathematics concepts and procedures is greatly related to the strength of connections among the internal and external representations of the students (Pape & Tchoshanov, 2001). Multiple visual representations are key in developing that deeper understanding. As Tripathi (2008) metaphorically states, "using these different representations is like examining the concept through a variety of lenses, with each lens providing a different perspective that makes the picture (concept) richer and deeper" (p. 439).

Supporting Reasoning Processes and Mathematical Discourse

Strongly related to the development of a deeper understanding is the importance of reasoning in learning mathematics. Arcavi (2003) contends that visual representations are critical and legitimate elements of mathematics proof and argumentation, as visualization deeply engages "with the conceptual and not the merely perceptual" (p. 235). In other words, visual representations support discourse because it becomes a means to display students' understandings, which can then be discussed and critiqued. Visual representations give students a voice, both as the sharer and listener.

Supporting the Development and Implementation of Problem Solving Strategies

In examining the differences between expert and novice mathematical problem solvers, Stylianou and Silver (2004) found that problem solving success is largely related to students' ability to move flexibly among different representations. Rather than becoming stuck on one representation, strong problem solving strategies rely on switching viewpoints and approaching the problem with different representations until one (or more) leads to a solution. Therefore in the development of problem solving strategies, children need experience with multiple visual representations. This allows students to view visual representations as tools in solving problems, rather than representing only the answer or the problem.

Summary and Limitations within Research on Visual Representations to Support Children's Mathematics Learning

The literature highlighted strongly supports for the inclusion of visual representations in the mathematics classroom. In particular, the research implies that students must be given multiple opportunities to visualize mathematics through different representations. This visualization of concepts can be strengthened by the teacher when letting the students share their individual representations and supporting the students in drawing connections among the representations through discussion. The research also highlights the importance of viewing visual representations as tools for reasoning, discourse, and problem solving. And finally, we see the underlying message in the research, which positions the construction of representations as a social activity.

The following example better highlights how the use of multiple representations supports students in drawing connections, is a tool for reasoning, discourse, and problem solving, and is inherently a social activity. A teacher gives students a word problem representing a multiplication problem with an unknown product. The students use drawings, manipulatives, and symbols to solve the problem. After solving the problem, the students share their representations, which visualize the students' problem solving processes. Different representations, including grouping, array, area, and symbolic representations, are presented on the board. The teacher may further support his/her students in forming connections between the different representations that the students used to solve the multiplication problem. The teacher facilitates a discussion, which then reveals the mathematical structures of the concept multiplication. Students of all levels are able to engage in this discussion because each of their representations proved important in the learning process and the depth of understanding of multiplication is increased for all students.

Aside from the advantages of the use of multiple visual representations in mathematics instruction, previous research has also shown the possible limitations of visual representations, specifically in elementary instruction. For instance, Boulton-Lewis (1998) found that children do not always make connections between their internal representations, the representations they chose to solve problems with, and the representations the teachers would like the students to use. More recently, with the technological advances of eye-tracking methodology, Bolden and colleagues (2015) found that children, when given a number line, largely did not view the number line conceptually and instead attended to only one point on the number line.

Future research should address these limitations, asking (1) what students are attending to within a representation and (2) how instruction can better direct students to conceptually view representations and connect their internal representations with the external representations presented. Because of the complexities of many visual representations, *mathematical noticing*, which refers to "selecting, interpreting, and working with particular mathematical features" (Lobato, Hohensee, & Rhodenhamel, 2012, p. 438), offers a framework to attend to and reason within a visual representation. With this better understanding of student's internal representations, we can begin to focus our research attention to instruction, which develops reflective students who conceptually view representations and make connections. The knowledge has possibilities of transcending how teachers use visual representations in the elementary classroom to support students' learning of mathematics.

Summary of Literature Review

In this chapter I described the main themes of this research study: early algebra instruction and visual representations. The goal of this study was to describe how individual third-graders make-meaning, through mathematical noticing, of the external representations within the natural classroom and how this meaning-making process relates to a student's understanding of the algebra concepts. Within the current literature on early algebra instruction, it remains to be seen how algebra representations, in particular function tables, are interpreted within the curriculum and classroom setting by both the students and teachers. Blanton and colleagues call for a qualitative study with the purpose of describing elementary students' process of understanding function tables and functional relationships (Blanton, Stephens, Knuth, Gardiner, Isler, & Jee-Seon, 2015, p. 76). In the next chapter, I will discuss in detail the design of this collective case study and how the methodology was suitable for the purpose of understanding student's meaning-making process better.

Chapter 3 - Methodology

This collective case study involved a classroom lesson observation and in-depth and elicitation interviews of students within their perspective classroom. All aspects of the methodology of the study, including data collection, analytic methods, and validity and reliability of analysis, are discussed in the sections that follow.

Overview of Research Design

The purpose of the study was to understand elementary students' meaning-making of early algebra representations of number patterns about even and odd numbers within their natural classroom setting. A particular emphasis is placed on understanding the role of the function as a way to generalize a relationship between inputs and outputs where every input has exactly one output, i.e. an odd integer could be defined as $f(n) = 2n + 1$ for integer n.

Based on the purpose of my study, qualitative research methodology was deemed appropriate for this study. Qualitative research focuses on "emerging questions and procedures; data typically collected in the participant's setting, data analysis inductively building from particulars to general themes, and the researcher making interpretations of the meaning of data" (Creswell, 2014, p. 4). Interpretivisim, as a philosophy of inquiry that assumes the ontological belief that there are multiple realities, grounds the qualitative study (Crotty, 1998).

Specifically, the methodology of collective case study was employed in hopes to gain an in-depth understanding of students within their classroom context. Case studies are often used as a methodology guided by interpretivist framework, as they "seek to answer focused questions by producing in-depth descriptions and interpretations over a relatively short period of time" (Hays, p. 218). Generalization is not the goal of the study, but rather the goal the study is to gain multiple perspectives and multiple accounts of students' subjective interpretations of algebraic concepts within the culture of their classrooms to illuminate the understanding of the relationship between the students and classroom norms/instruction.

Yin (2014) defines case study as an empirical inquiry that "investigates a contemporary phenomenon (the "case") in depth and within its real-world context" (p. 16) – here the "case" refers elementary students and their early algebraic understanding within their classrooms and mathematics instruction. Yin further emphasizes that a case study is appropriate "especially when the boundaries between phenomenon and context may not be clearly evident" (p. 16) – which in my case refers to the relationship between elementary students' early algebra understanding and the representations presented within the cultural and situational context of the classroom. Within the current literature on early algebra instruction, it remains to be seen how algebra representations are interpreted within the curriculum and classroom setting by both the students and teachers (Blanton, Stephens, Knuth, Gardiner, Isler, & Jee-Seon, 2015, p. 76). In particular, Blanton and colleagues called for a qualitative study to expand on their findings of early functional understanding. A case study therefore becomes a suitable methodology to observe and interpret the relationship between the phenomenon of elementary students' early algebra understanding and the context of the classroom, specifically the visual representations within their mathematics instruction.

Pilot Study

Before landing on the decision to focus on the external representation of the function table, I conducted multiple individual interview with third-grade students in the same school that I later used for the remaining research study. Four students were individually interviewed for 15-minutes on six different occasions across a couple of months in the spring semester of 2017.

My purpose of the interviews was to better understand the students in the school and narrow down a task that would be appropriate for further study. The five tasks that were used in the pilot study can be found in Appendix G. In the formal elicitation interviews, students were given different representations of algebra concepts aligned to a problem-solving task that aimed to support the process of generalizing arithmetic properties based on arithmetic patterns. The external representations included the use of symbolic (equations, tables, graphs), pictorial (including number lines, arrays, area, sets, etcetera), and concrete (manipulatives) representations of mathematics. Open-ended probe questions were asked such as, “What do you notice about this picture?” “What do you see that makes you say that?” “What else do you notice?” “What connections can you make?” and additional questions used to clarify student’s thoughts and gain a more descriptive narrative. Students were instructed to verbalize their thought-process and explain their reasoning throughout the interview.

The pilot study helped narrow down my focus for the lesson that I used for my data collection. The students I interviewed during the pilot study had vast differences in noticing patterns when given a function table and I found it the most interesting representation to further explore. After deciding to focus on Task 3: *Generalizing about Even/Odd using Function Tables* from the pilot study, I expanded the task into a formal lesson (Appendix B). Before the official collection of data which included the lesson study, I taught the lesson in a fourth-grade classroom in the fall of 2017 in another school and then made additional changes based on feedback. The data collection process, which included the lesson taught in a third-grade classroom in the spring of 2018, is described in detail below following the description of the setting and participants.

Setting and Participants

The setting for this study was at one naturalistic classroom at one elementary school in a suburban setting in Midwestern United States. The participants of the study included a third-grade class and their perspective teacher within the suburban elementary school. After deciding on the school, I began volunteering in order learn about the classroom, students, and teacher. It should be noted that my relationship with the school principal and teacher was first through my current role as the mathematics instructional coach at the school.

Since a large part of my design is based on what is happening within the classroom, i.e. mathematics instruction, curriculum, classroom norms, it was essential to begin volunteering before I could design my interview tasks and protocol. Also, it was imperative to develop a rapport with the students and teachers to create a safe space. It should be noted that my volunteer time was also part of my required teacher observations through my mathematics instructional coach role. As the researcher, I viewed my role as an active member in the classroom. It was important that I was both participating within the community of learners but also observing from the sideline as a peripheral member. My previous experiences have proved to me that students are acutely aware of my presence in their classroom as I observe. Therefore, a crucial step before conducting research was to create a safe space for both the teacher and students within their natural setting.

The students, parents, teacher, and school principal signed informed consent forms (Appendix A) to be a part of the study. The students returned the forms to their classroom teacher, who collected them and passed them on to me. On the parental consent form the parents indicated whether they gave permission for the researcher to (a) use their child's written work, (b) interview and video-record the student individually, and (c) include the student in video of the classroom lessons-for data analysis purposes only.

School Demographics

The classroom lesson observation and the following student interviews took place during the Spring semester of 2018. Therefore, the population and demographics of the school setting reported are for the 2017-2018 school year. According to school documents, the elementary school serves 637 students in grades Kindergarten through Grade 5. Over a half of the students were white/non-Hispanic (61.4%) and about a quarter of the students were Hispanic (19.74%). Another 6.88% of students reported an ethnicity of African American and 9.2% of students reported "other" as their ethnicity. Almost half of the students (48.07%) reported low socio-economic status and therefore received free and reduced lunch in the school year.

Academically, 14.74% of school population received special education services during the 2017-2018 school year. Furthermore, the 2017 State Standardized Assessment Report, for Grade 3 showed 72% proficient in math and 56% proficiency in ELA which was above the state averages (52% proficient in math, 40% in ELA).

Classroom Setting and Norms

My observations of the teacher's third grade classroom took place within the spring semester of 2018. The participants of the study were the classroom teacher (CT) and the twelve students who had returned parental consent forms (there was a total of 18 students in the classroom but 5 students did not return their parental consent forms and therefore were excluded from the recorded interviews). Of the twelve students, 7 were girls and 5 were boys.

To help the reader gain an understanding of the norms of the classroom, what follows is a descriptive narrative of my classroom observations on the first day in January when I began observations in the CT's classroom. When I arrived on the first day, the classroom teacher (CT) was discussing a new seating arrangement. The first task of the day, as the students came in, was to move the students around. The seating arrangement had the students sitting in groups of four with plenty of room for the teachers to walk around to facilitate during individual and group activities. There was a rug in front of the SMART Board and after the students found their new seats, they were instructed to move quietly to the rug, sitting by their partners. During this time, I began looking around the classroom. The classroom was not only set up in a manner that was constructive to learning but also bright positive images about learning ("learning is NOT a spectator sport, so let's PLAY!") were posted around the room. This environment developed the image that the classroom was a place where true learning was *going* to occur, that is, students will be "learning from one another and continually attempting to improve", students have the "freedom to make mistakes in order to learn" and students will have to continually persevere (NRC, 2000, p. 144). The was math all around the room including a number line on all 4 walls, place value chart, the Standards for Mathematical Practice (Common Core State Standards - Mathematics, National Governors Association Center for Best Practices, 2010), and fraction number line work was written on the chalkboard.

In these first few moments, I also took notice of the CT's and student actions. The CT was very relaxed and positive with the students. As the observer, I could tell there was a relationship that had been developed over the year and the respect between the students and teacher was visible to me. As the students sat on the rug at the beginning of their math lesson on estimating measurements, the CT gave the students the opportunity to share while she questioned and probed their thinking. I was sitting in on the second lesson on estimation and measurement. The CT began by asking the students, "What are some things that we can estimate?" The students all shot up their hands, eager to talk and tell their stories. The students discussed how they have been estimating how much longer it was going to take for the butterflies, in the back of their classroom, to burst from their cocoons. Other students discussed estimating their mom's age, their own height, etc. The CT also told a story how he recently had to make estimations while driving to school today. These connections to the students' and teacher's daily lives opened the math lesson to the broader community.

During this conversation, the CT asked the students, "What do you think our brains were thinking when we estimated how much time it was going to take for the butterflies to come out of their cocoons?" The CT was emphasizing here the difference between *guessing* and *estimating* where "estimating is a guess that uses our brains!" To keep the learning active while ensuring all students are given the chance to talk, the CT used the Think-Pair-Share strategy to discuss the question. The CT was continually positive while his students were sharing their ideas and giving their opinions. Even when a student was incorrect, he would state, "That's a good idea" and continually used questioning to guide their thinking. There was an underlying persistence in the conversation, as the students were expected to think critically about everything. The students were also encouraged to state whether they agreed or disagreed using their "me-too" signals. The CT used higher-level questions (Hess, Jones, Carlock, & Walkup, 2009) to probe their thinking beyond their initial responses.

The remaining hour of the mathematics lesson was group work as the students worked on individual packets on estimating measurements. During this time, I became a participant of the classroom walking around and working with individuals or individual groups. I wanted to gain a feel for the individual students that I could only gain as a participant. The CT used questioning to facilitate the conversation and learning while the students were working. There were moments that I heard, in the background, praise such as "You used your brain!" when students were being critical thinkers about their estimations. Often, estimation can be taught as equivalent to guessing and I was relieved that the teacher was emphasizing the key difference. If students were struggling with their estimations, the teacher emphasized perseverance and asked, "How could we use our brains to figure it out?" or "Use your estimation brain." Furthermore, even when students could find the answer, questions were used to further their thinking. The CT may have asked, "How did you make that estimation?" or "Why did you choose to make that estimation?" Students were not expected to just stop learning as they finished the problems; they knew they were going to be asked deeper questions as we walked around. It is important to remark that the level of thinking required in the assessments given (the math packets) was low focusing on basic skills and concepts. But this required the teacher and teacher helpers to walk around and ask higher level questions, which focused on strategic thinking and analyzing.

Another key component that I witnessed was the conversations on the tables as the students worked together (even when no teacher was present). The students not only stayed on track in the math lesson but also were able to discuss mathematically while pushing each other. They used "me-too" hand signals when the teachers were not asking them to and they asked each other meaningful questions like "How did you get that?" or "I don't understand, can you explain that to me?" This norm, which focuses on active listening and meaningful discussion, can only be built from the teacher emphasizing it throughout the year. This keyed me in on what the students believed about learning.

My observations in the same math classroom took place once more in February and twice in March for the final observation in late March that was video recorded for research purposes. The mathematics being learned in the following observations was fractions. I noticed the same classroom norms were set up, with students discussing in groups before exploring their thoughts in whole-class discussions. What follows is a descriptive narrative of that late March lesson on even and odd numbers.

Classroom Lesson Narrative

The classroom lesson I chose to video record and analyze focused on students analyzing patterns about even and odd numbers with the goal of generalizing the pattern. I was curious in how third graders may approach the idea of a function as defined as a rule that matches two sets of numbers, inputs and outputs, such that for each input there is only one possible output according to the rule. Therefore, the tables used in Figure 1 and Figure 2 were used to not only elicit students thinking about generalizing even and odds but also to analyze their noticing patterns within the table.

The written lesson is included in Appendix B. The lesson objectives were that students will be able to generate and analyze number patterns about even and odds and students will begin to define what it means to generalize about a pattern. The lesson was aligned to the fourth-grade Common Core State Standard in Mathematics CCSS.4.0A.5: Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule “Add 3” and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

My choice on choosing a fourth-grade standard in a third-grade lesson was two-fold. After observing the teacher and students in the classroom, I knew that the students were already beginning to generalize from patterns, although not formally introduced. Moreover, it was important to me that I gain an understanding of student’s internal representations before any formal instruction on generalization patterns. Also, it should be noted that even and odd numbers, patterns, and pattern rules were all concepts that the students have seen throughout third grade and previous grades.

Task 1 Generalizing about evens. In the lesson, after reviewing what a pattern is and looking at a few pattern strings, I had the students look at a table of values given below. Students are given some background that tables help us organize patterns, but they may be unfamiliar with the input and output formula. This gives us a chance, as a researcher, to understand how early elementary students understand input and output tables before any introduction to function rules. This function table may be characterized as an "abstract" representation. Furthermore, using the abstract table to characterize even and odd patterns is an abstract problem. This was a key difference between my task and Blanton and colleagues' work which used function tables to characterize concrete situations (such as number of eyes on a dog) (Blanton et al., 2015).

Input: Number	Output: Number × 2
1	2
2	4
3	6
4	8
5	10

Figure 3-1 Lesson, Task 1 Table

The middle of the lesson had students exploring the above table of numbers at each of their grouped tables. Each table included three students and either the classroom teacher or a teacher helper to help facilitate the discussion. The group discussions were recorded individually and transcribed. Students individually wrote then shared in the group, their “notice and wonders” about the table of numbers.

a) What do you notice about the numbers in the table? Share with a partner.

b) What do you wonder about the number in the table? Share with a partner.

After approximately five minutes, the classroom teacher gathered the students back together to discuss Task 1. One of the students shared that they noticed the pattern of looking down the input column (i.e. recursive pattern) and shared that the input numbers went “odd, even, odd, even, odd”. Another student shared that they noticed all of the output numbers are even, again students were noticing the recursive patterns in the table. The CT was careful to use the language of input and output and pushed the students to question why all of the outputs were odd. This connects the recursive pattern to a generalized pattern. An important moment of the lesson was when one student shared that they thought all of the outputs were odd “because if you add the same input number together, it will be an even” and another student connected her statement to multiplication of two. This led students to the noticing pattern of reading the table “across” from inputs to outputs (i.e. correspondence pattern). It also led students to generalizing that if we continued the pattern (or table), when multiplying by two, we will always have an even number. This progression of noticing patterns both across and down the columns and connecting this noticing patterns to being able to generalize about the function rule mirrors the work of Blanton and colleagues (2015).

Task 2 Generalizing about odds.

Task 2 asked students to first individually fill in the following table and record their observations below their table. Students worked in their groups for approximately 10 minutes on Task 2. These group discussions were recorded separately.

INPUT: Number	**OUTPUT:** Double the number, plus 1
1	
2	
3	
4	
5	
10	
25	
100	

Figure 3-2 Lesson, Task 2 Table

After working in their groups, the whole class came back together and the CT had students share the patterns they saw. Again, students began by sharing that all of the outputs were odd "because when you plus 1, it's odd" (i.e. generalized pattern). Another student comparing the two tables from Task 1 and 2 noticing that in the first table we were just doubling and in the second table, we are additionally adding one. The CT repeated this, "so you are saying that an odd plus an odd is even but when you add the one it gives you an odd." A student could further generalize this statement to "if you add an even number and an odd number it is odd".

Without probing from the classroom teacher, students continued to explain why this is true. One student explained why "even + odd = odd" by stating that "one is left out" when dividing by two and therefore is odd. Although it was not drawn on the board, that particular student was referring to their internal representation from their small group discussions shown below.

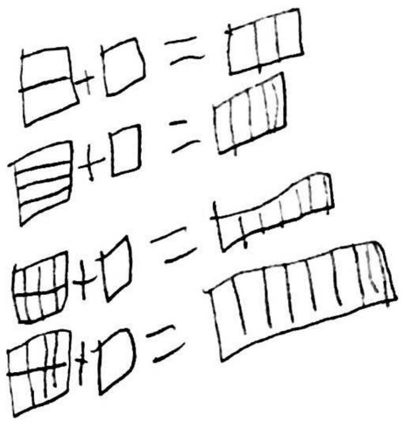
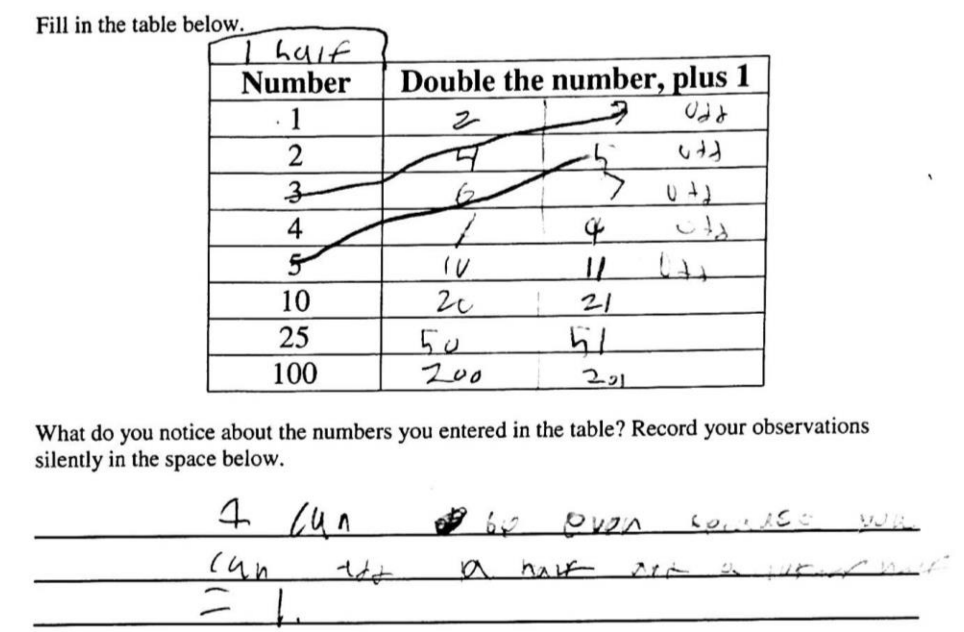


Figure 3-3 Student Work showing Even + 1

A student from another group shared that he/she noticed that "half plus half equals one so one can be either even or odd. The written work of that student is shown below.



Fill in the table below.

1 half

Number	Double the number, plus 1		
1	2	3	odd
2	4	5	odd
3	6	7	odd
4	/	9	odd
5	10	11	odd
10	20	21	
25	50	51	
100	200	201	

What do you notice about the numbers you entered in the table? Record your observations silently in the space below.

Figure 3-4 Student Work showing "One can be Even"

In efforts to wrap the lesson up, the classroom teacher dismissed the student's idea that one can be both even and odd by explaining that the two equal parts (½ and ½) are not whole numbers and therefore does not follow our definition of even numbers. In closure, the CT had students connect the odd, even, odd… pattern to the number line above the board in the classroom.

Focus Students for Interview

For the follow-up interviews I used purposeful sampling, based on my observations of student's understandings, to select students for interviews following the above described lesson. Four students (only three were interviewed as one student was absent the day I came back for interviews) from the classroom were selected based on my observations of their work during the lesson with the goal of selecting students with varying understandings of the early algebra concepts. My focus students all had returned parental consent forms. My observations during the classroom lesson and the input from the classroom teacher were the determining factors of my selection as my goal was to choose students that were willing to talk during the small groups and had an interesting insight to the even and odd problem. Students were individually interviewed within the next week following the classroom lesson and those transcripts are included in Appendix F.

Research Questions

Before discussing the data collection methods and analysis, it is beneficial to revisit the research questions along with a discussion. Situated within a third grade classroom, how do elementary student's mathematical noticing of external representations of algebraic problems relate to their (1) internal representations and (2) developed understanding of the concepts?

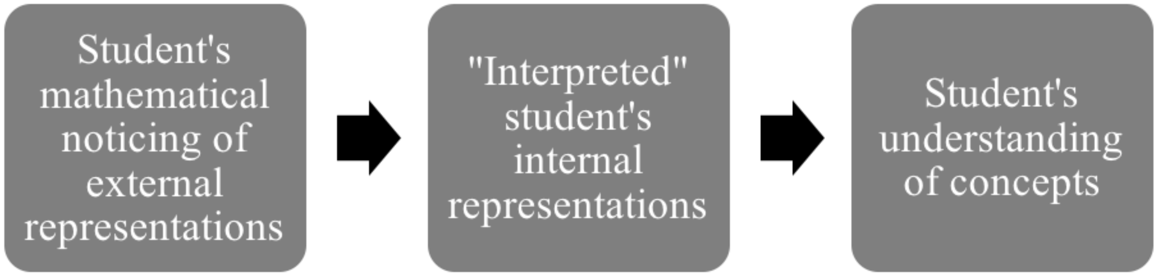


Figure 3-5 Concept Map of Research Question

First, I hoped to address the gap I have previously identified within the current literature on early algebra instruction, that is, how visual representations, within the curriculum and classroom setting which support early algebraic reasoning, are interpreted by the students. I focused on the external representation of tables as these are static external representations used to make meaning of generalization of patterns and early understanding of a function. The generalization of numeric patterns has been shown as a key practice in supporting algebraic understanding at the elementary level (Bastable & Schifter, 2008; Schifter, 1999; Schifter, Monk, Russell, & Bastable, 2008). In particular, function tables are used as a static external representation to describe a rule or functional relationship of two quantities. I will be describing student's meaning-making process of the external representations and individually what features of the external representations do the students attend to and or do not attend to along with their meaning. This hopes to extend previous research on early functional understanding (Blanton et al., 2015).

Secondly, I conjecture that this meaning-making process of individual students will affect their understanding of the algebraic practice of generalization. For this question, I will be looking at whether student's mathematical noticing of the external representation function table differ between those that correctly and incorrectly understand the algebra concept of even and odd numbers and the generalization of the pattern made by even and odds. Defined by Lobato, Hohensee, and Rhodenhamel (2012), *mathematical noticing* refers to "selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for students' attention" (p. 438). Although it is nearly impossible to gain an in-depth understanding of student's internal representations as observers, we can infer student's internal representations by asking students to describe their process of interpreting the external representations and drawing their own representations of concepts. Through these visual reports, gestures, and written student representations, we begin to connect student's internal representations of those concepts represented in the classroom and with their subsequent algebraic reasoning and understanding. To interpret student's internal representation, I aimed to describe student's noticing-patterns and further examined how these noticing-patterns are related to student's understanding of generalizing about even and odd integers.

For purposes of addressing the above questions, I employed established qualitative data collection methods and analysis. In the following sections, I define these data collection methods and analysis.

Data Collection

Data was collected over the course of a semester (approximately three months). Multiple sources of data were collected to provide for an in-depth analysis and triangulation. The data collection methods for this case study included classroom observations along with researcher's field notes, in-depth and document elicitation participant interviews, and analysis of student work. In efforts to fully answer my research questions from different angles and perspectives, these methods were used in conjunction and the data collection was an iterative process. Because of the complexity of analyzing the meaning-making process, it was important for me to gain a broad but also precise understanding of the classroom and participants. For example, the observations gave me a broader understanding, while the focus interviews were used to specifically seek out information to better understand individual student's mathematical understanding within the classroom.

Classroom Observations

Classroom observations, natural conversations with the students, and researcher field notes were primary source of data. As the researcher, my role was that of an active member in the classroom (Bhattacharya, 2012) but I chose to be a peripheral participant, i.e. researcher-observer versus a teacher-researcher. Classroom data included partial audio/video recordings of a select mathematics lesson that I wrote specifically to address my research question.

A four-component focusing framework adapted from Lobato and colleagues (2013) was used to focus my observations and subsequently my analysis on that within the classroom which the literature has shown influences student's mathematical noticing and meaning-making process of mathematical representations (Lobato, Hohensee & Rhodehamel, 2013; Hatano & Greeno, 1999). This piece of data therefore informed the first portion of my research question, that is, how students make meaning of external representations within the classroom. Lobato and colleagues (2013) claimed that students' meaning-making process of representations arises through the "interplay between a set of discourse practices, i.e. focusing interactions, and the features of mathematical tasks during engagement in particular types of mathematical activity" (p. 814). These three features of each classroom interaction, i.e. (1) focusing interactions, (2) mathematical tasks, (3) mathematical activity, inform the fourth component of the framework - "centers of focus", i.e. "the properties, features, regularities, or conceptual objects that students notice" within an external representation.

It should be remarked that teachers often play the important role in directing students' attention toward these features, but we must also pay attention to the influence of peers within these social interactions. Therefore, focusing interactions include both teacher to student interactions and student to student interactions and moreover, include talk, gestures, and diagrams written on the board or student work. The mathematical task will also be relevant as it is generally assumed that features of the tasks presented within the classroom will influence student's internal representations of a concept. For example, if area models are often presented as a representation of the distributive property, when visualizing the distributive property, students may be more likely to use an area model that they are familiar with. Finally, the nature of the mathematical activity and classroom norms can influence how student's notice features of the representations.

In the Figure 3-3, the framework is used to illustrate the complexity of the classroom in relation to student's mathematical noticing. Within a given math task, the task gives rise to multiple centers of foci (here the arrows represent three noticing patterns that arose from the task, i.e. three centers of foci in the task). Different students may notice different features of the representations when working on their own but through classroom interactions, these students shift in time to the noticing patterns that the teacher alludes to.

Number	Number x 2
1	2
2	4
3	6
4	8
5	10

Mathematical Activity: Task 1 Table

Figure 3-6 Noticing Framework

Descriptive field notes were a secondary source of data and therefore much of the time spent in the classroom, I will take field notes on my laptop computer. The descriptive field notes will include details about the class setting, the student's attitudes on that particular day, the type of work and instruction that was given leading toward the nature of the mathematical activity, the content focus of the lesson including all mathematical tasks/activities, and student-student and teacher-student interactions relating to the focusing interactions of centers of focus. In addition, reflective notes in the form of journaling were important to reflect on the lesson and the students' reactions to content. Researcher field notes reflect my reflexivity, that is, how my role and assumptions influence my interpretations of my observations (Preissle & Grant, 2014).

Interview Data and Written Work

Small Group Interviews During the lesson, four groups of 3 students and a teacher-helper will be interviewed and transcribed as they work on Task 1 and 2. The interview protocol that the teacher-helpers used is included in Appendix D. The transcriptions of the small group interviews are included in Appendix E.

Further individual participant interviews with purposefully chosen focus students were another primary source of data to gain a deeper understanding of select students within the classrooms. Four students were purposefully chosen based on observations of student understanding and teacher input. Only three were interviewed as one students was absent the day of the interviews. Each individual interview took place one week from the lesson and took about 15 - 20 minutes. The students revisited the tasks from the lesson and were asked open-ended questions pertaining their noticing patterns.

Open-ended probes such as, “What do you notice about this picture?” “What do you see that makes you say that?” “What else do you notice?” “What connections can you make?” and additional questions were asked to clarify student’s thoughts and gain a more descriptive narrative. Students were instructed to verbalize their thought-process and explain their reasoning throughout the interview. I was looking for what features of the representations (table) the individual students are noticing, which has not previously been researched, but also their reasoning patterns which may be similar to those identified in previous studies of early functional understanding. For purposes of transcribing the interview later for analysis, the interviews were audio recorded and student work was included. The transcripts are included in Appendix D.

The purpose of both the small group and individual interviews was to gain an inferred understanding of students’ internal representations that have been informed by the external representations and classroom interactions identified previously. As students talked through their noticing patterns, I recorded gestures and speech that gave light to their mathematical noticing which form their internal representations. Furthermore, by asking questions pertaining to their reasoning and understanding, I aimed to connect these noticing patterns to their correct or incorrect interpretations of the external representations.

Student Work

Another piece of data will be the inclusion of documents such as the lesson plan (Appendix B) and student work (Appendix D). Since students wrote down specifically what they noticed and wondered during the lesson, I have a qualitative record of noticing patterns within the external representation. This was important when conducting the analysis of the noticing patterns.

Data Analysis

The whole class lesson was video recorded and the small group and individual interviews were also audio recorded. Both the lesson and individual interviews were transcribed by me. Since I knew the students well by the time of the lesson, when transcribing I was able to hear and decipher the different student's voices.

For my analysis, inductive open-coding (Corbin & Strauss, 2007) was used for analyzing the transcripts of student interviews and classroom videos to determine salient points in the discussions connected to the visual representations. The emergent APOS theory was used as a micro-framework for analyzing students' understanding of the visual representations. The purpose of this study is to describe how the situational context (i.e., external representations or symbolic communication given in the problems) and cultural context (i.e., the verbal and nonverbal communication within the classroom and norms within the classroom environment) inform upper elementary students' conceptual context of internally visualizing mathematical representations (function table) while students engage in algebra problem solving tasks. The situational contexts, cultural contexts, and conceptual contexts will each be determined through separate but also, interrelated inductive analyses.

Data reduction was done to reduce vast amount of interview and classroom data to a manageable amount with a narrowed focus for analysis (Miles & Huberman, 2013). I made and ad hoc decision to focus on the use of the function table in the lesson. This decision was based on themes that emerge through my observations and reflexive notes within the data collection and analysis period.

Analyzing Noticing Patterns

In answering the first part of my research question, that is, how do the students make meaning of the external representations of algebraic problems while problem solving, I coded student's responses as "noticing patterns" during the small group and whole group discussions. Open-coding from grounded theory involved chunking the data (talk, gestures, written work during classroom observations) and then identifying and naming categories into which the observed phenomenon could be grouped. In determining the students' conceptual context of internally visualizing mathematical representations of algebra concepts, I analyzed the transcribed small group interviews by inferring categories of reasoning, using a mixed approach (Miles, Huberman, & Saladana, 2013). A mixed approach of coding reflects the idea of using some codes that were previously identified from the literature (e.g. "across" or "down" the function table from Johnson, 2015) while other codes (e.g., the output number is transferring to the input number") were induced using open-coding from grounded theory (Corbin & Strauss, 2007). When inducing codes, I used student language to retain their understanding and internal representations. Whether students spoke, wrote down, or gestured towards a noticing pattern, these were all coded as "noticing patterns."

I focused on one task at a time and one student at a time to be careful not to count a response twice. For each student, I coded and counted each noticing pattern on Task 1. Then I started on Task 2 for each student. I used both the transcriptions and student work simultaneously to gain a better picture of what had transpired through the small group discussion. If a student both wrote, spoke, and gestured the same noticing pattern, I only counted the noticing pattern once for the student. To ensure internal consistency of the codes, I repeated this process again while listening and reading through the transcripts.

Task 1 (Evens)

The following noticing-patterns were coded through a mixed approach process on the first task, which had student notice and wonder about a function table on evens.

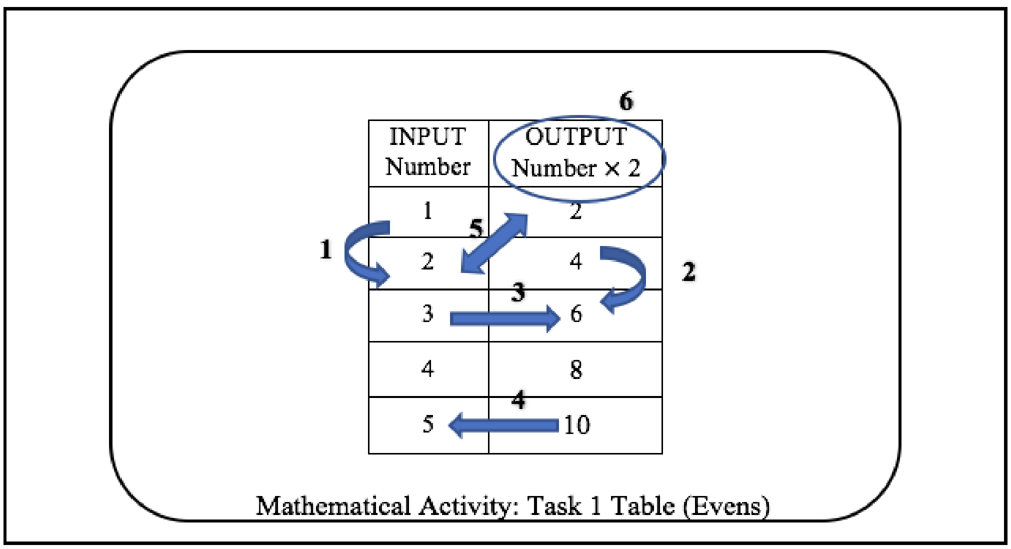


Figure 3-7 Coded Noticing Patterns in Task 1

Through a mixed approach, I coded the twelve third-graders noticing patterns in one of the six codes:

Noticing Pattern 1 (*Recursive*): Down Input Column ("odd, even, odd, even, odd")
Noticing Pattern 2 (*Recursive*): Down Output Column ("all even")
Noticing Pattern 3 (*Correspondence*): Across Input to Output ("Output = Input number times 2")
Noticing Pattern 4 (*Inverse Correspondence*): Across Output to Input ("Input = Output number times ½ =")
Noticing Pattern 5: (Even) Output numbers are "*transferred*" to the same number in the Input column.
Noticing Pattern 6: The "Rule"

Noticing patterns 1 through 3 were the most obvious to code and were noticing patterns I had previously assumed. Because of this, I went through and listened to each clip again while assigning these single codes to each response or gesture by a student. If a student recorded the "notice" both verbally and on their worksheet, I only coded that noticing pattern once. Furthermore, students may have interpreted the recursive noticing pattern of down the input/output columns differently. For example, one student may have stated that he/she saw that the "output numbers were all even" while the other student may have stated that he/she way that they were "counting by twos." Since these were the same noticing pattern, they were coded the same. I was not concerned about student's mathematical understandings, I was just looking for how students were interpreting the table through their noticing patterns.

Noticing Pattern 4 “Inverse Correspondence” at first glance may appear to be the same as Noticing Pattern 3 “Correspondence” but I decided to code these as different noticing patterns. It seemed to be a more sophisticated understanding of the function table that a student can use the output to generate an input number. Only two of the twelve students stated that they noticed this pattern and they noticed it much differently. For example, one student was able to understand the inverse relationship between the input and output. He/she saw that the output number was two times the input number and inversely, the input number was ½ of the output number. Another student also gestured in the interview that he/she was looking across the output to the input. He/she recognize that they could subtract the input from the output to obtain the input number, i.e. 2 – 1 = 1, 4 – 2 = 2, 6 – 3 = 3. Both of these student reasoning’s were a sophisticated way to read the table columns inversely.

Noticing Pattern 5 “Even Outputs “transferred” to same number in Input” was another surprising and interesting noticing pattern from a student. When I first heard this pattern in the video, I thought the student was just jumping around the table. But then I realized that they were seeing a relationship between the even integers in the output column to the even integers in the input column. I used the student’s word “transferred” when naming the noticing pattern.

And finally, some students were very aware of the “rule” given at the top of the table. In various ways students alluded to the rule (either circling the rule in the table) or sharing that they knew the rule or used the rule to determine the pattern. Of course, this noticing pattern is directly related to Noticing Pattern 3 “Across Input to Output” but I coded it separately as they were gesturing towards the labels in the table. It seemed to me a different understanding of the table is being used whether students are noticing the rule from the label at the top of the table versus students are understanding the rule from the generalized correspondence pattern.

The following table breaks down for each coded noticing pattern both the number of recorded student responses and an example student response. Further discussion on these results will follow in the next chapter.

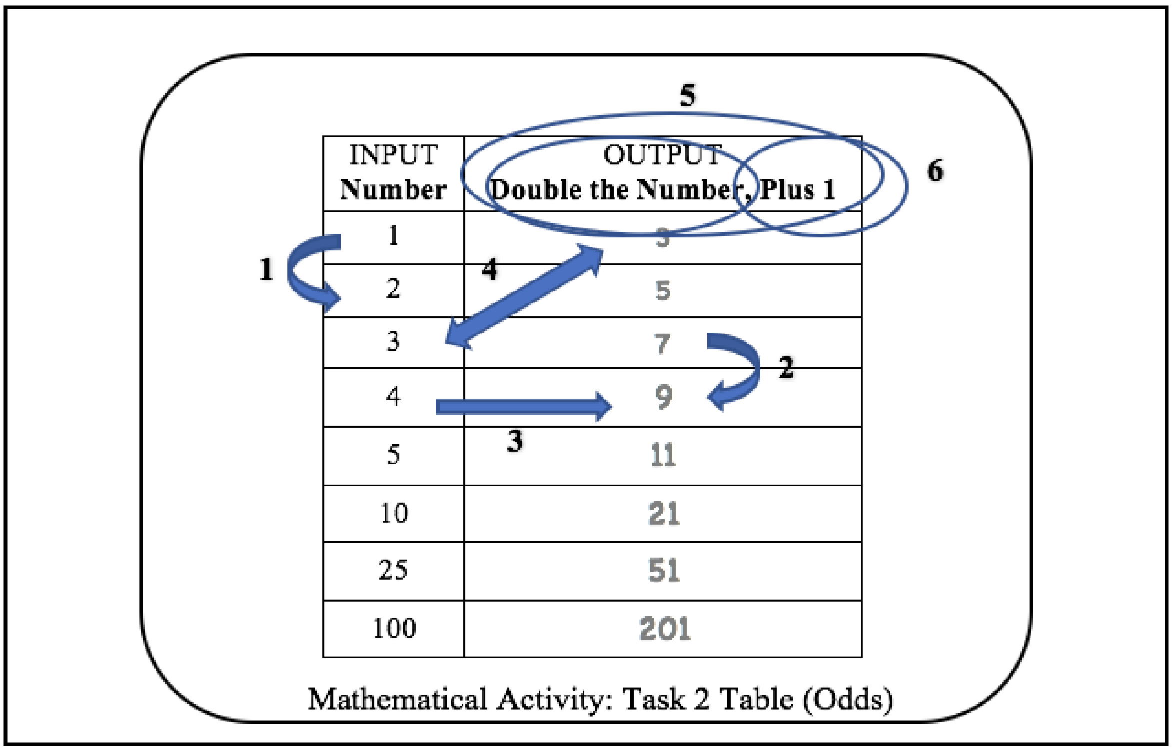
Table 3.1 Noticing Patterns within Task 1 “Generalizing Even Integers”

Noticing Pattern	**Example Student Responses**	**Number of Recorded Student Responses**
1: Down Input Column: *Recursive*	“Counting by ones” “Input numbers are odd, even, odd, even, odd”	6
2: Down Output Column: *Recursive*	“Counting by twos” “Output numbers are all even” “They are getting bigger”	12
3: Across Input to Output: *Correspondence*	“When you double the input number you get the output number” “The numbers are doubling themselves” “If you go sideways, it goes 1 × 2 = 2, 2 × 2 = 4, 3 × 2 = 6, …” “Input numbers = Outputs numbers × 2” “Adding the same number 2 and 2 is 4…”	8
4: Across Output to Input: *Inverse Correspondence*	“Input numbers are half of the output numbers” “2 – 1 = 1, 4 – 2 = 2, 6 – 3 = 3, etc.”	3
5: Even Outputs “transferred”	“The output number is	1

to same number in Input	transferring to the input number"	
6: The "Rule"	"The rule is times two"	3

Task 2 (Odds)

The following noticing-patterns were coded through a mixed approach on the second task, which first had student fill out the table by doubling the input number and adding one and then students recorded and discussed what they noticed individually and in the small group.



Mathematical Activity: Task 2 Table (Odds)

Figure 3-8 Coded Noticing Patterns in Task 2

Again, through a mixed approach, I coded the twelve third-graders noticing patterns within Task 2 in one of the six codes:

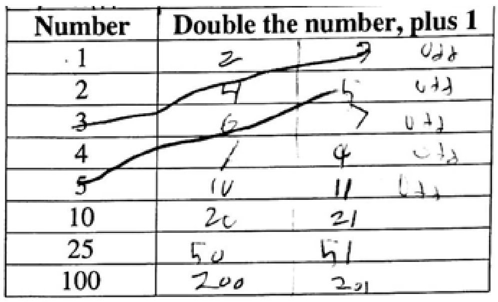
Noticing Pattern 1 (*Recursive*): Down Input Column ("odd, even, odd, even, odd" &

"Skips")
Noticing Pattern 2 (*Recursive*): Down Output Column ("all odd")
Noticing Pattern 3 (*Correspondence*): Across Input to Output ("double then number, plus one")
Noticing Pattern 4: Odd Outputs "*transferred*" to same number in Output
Noticing Pattern 5: The "Rule"
Noticing Pattern 6: Two Parts of the Rule

Noticing patterns 1 through 2 (recursive patterns) were again the most obvious to code and were noticing patterns I had previously assumed. Similarly, as in Task 1, students may have interpreted the noticing pattern of down the input/output columns differently. For example, one student may have stated that he/she saw that the "output numbers were all odd" while the other student may have stated that he/she way that they were "counting by twos." Since these were the same noticing pattern, they were coded the same. When students saw that the numbers in the input column "skipped" started at 5, I coded this as noticing pattern 1 as they were looking down the input column. Again, I was not concerned about student's mathematical understandings, I was just looking for how students were interpreting the table through their noticing patterns. Later in my findings, I aimed to connect these recursive patterns to the generalized pattern of all input numbers are odd.

Noticing Pattern 3 (correspondence pattern) was more difficult for students to see than equivalent correspondence pattern in Task 1. But a couple of students did allude to "doubling the number and adding one to get the input". I coded these responses as Noticing Pattern 3. While if students focused specifically on the rule at the top of the table, I coded these responses as Noticing Pattern 5.

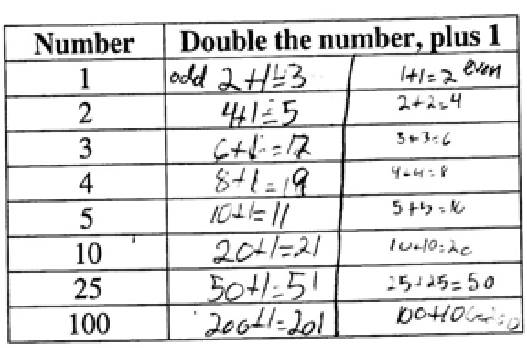
Noticing Pattern 4 “Odd Outputs “transferred” to same number in Output” was another sophisticated pattern similar to Noticing Pattern 5 in Task 1. Below is an image from a student’s work where he/she draws arrows connecting each odd output to the same odd number in the input column.



Number	Double the number, plus 1		
1	2	3	odd
2	4	5	odd
3	6	7	odd
4	/	9	odd
5	10	11	odd
10	20	21	
25	50	51	
100	200	201	

Figure 3-9 Student Work of Noticing Pattern 4 “Transferred” in Task 2

Finally, Noticing Pattern 6 was one of the more interesting patterns to arise in the interviews and student work. Some students noticed two parts of the rule. The first part was “doubling the number” and the students remembered that this was similar to Task 1. The second part was adding 1 to the doubled number to obtain the output number. Four students including the student work in Figure 3-6 above, used this method of understanding the function rule.



Number	Double the number, plus 1	
1	odd 2+1=3	1+1=2 even
2	4+1=5	2+2=4
3	6+1=7	3+3=6
4	8+1=9	4+4=8
5	10+1=11	5+5=10
10	20+1=21	10+10=20
25	50+1=51	25+25=50
100	200+1=201	100+100=200

Figure 3-10 Student Work of Noticing Pattern 4 in Task 2

The following table breaks down for each coded noticing-pattern both the number of recorded student responses and an example student response. Further discussion on these results will follow in the next Chapter.

Table 3.2 Noticing Patterns within Task 2 "Generalizing Odd Integers"

Noticing Pattern	Example Student Response	Number of Recorded Student Responses
1: Down Input Column	"Odd, even, odd…" "Counting wrong" or "Skips at 6"	5
2: Down Output Column	"Output numbers are all odd"	8
3: Across Input to Output	"(3 + 3) + 1 = 7" "You double the number and then add one to get the output"	5
4: Odd Outputs "transferred" to same number in Output	"The 1 (input) equals 3 (output) and there is a 3 here (input). The 2 equals 5 and there is a 5 here too!"	2
5: The "Rule"		0
6. Two Parts of the Rule, "doubling" and then "add 1"	Insert image (Jada, Lilana) "one side is even" and "one side is odd"	4

Analyzing Student's Generalization of Evens and Odds as related to Functions

Recalling my research question, "How do elementary student's mathematical noticing of external representations of algebraic problems relate to their (1) internal representations and (2) developed understanding of the concepts?" I hypothesized that student's noticing patterns influenced their understanding of the algebraic concept of generalizing even and odd integers as related to functional understanding. Being able to show student *understanding* of concept can be vague so I used a micro-framework in order to characterize students' understanding of even and odd integers as related to functional understanding.

APOS Theory, which stands for Action, Process, Object, and Schema, is a theory of learning specific to mathematics education. APOS Theory states that students build mathematical concepts by constructing mental actions, processes, and objects, and organizing them into schemas to make sense of the situations and to solve problems (Asiala et al., 1996). APOS Theory is based on Piaget's idea of relative abstraction, as extended to advanced mathematics primarily by Dubinsky (1991a, 1991b). In relation to understanding functions, the first level of understanding is that of action. A student in this level assumes a function is tied to specific rule or formula, which the answer depends on by manipulation of variables or replacing variables by numbers for calculations. Having a process conception of function, assumes a function is an input-output machine and is independent of the formula. An object is constructed from a process when the student becomes aware transformations, multiple representations, and properties, i.e. understanding the concept completely. Finally, a schema is developed from collection actions, processes, and objects across concepts, and thus building a framework, which will assist in problem solving in novel situations.

My first round of analysis looked at only student's noticing patterns pertaining to the external representations given to the students. Therefore to answer my research question, I had to embark on additional data collection and analysis to connect these noticing patterns to the student's internal representations and developed understandings of the concepts. Open-coding from grounded theory was used again to analyze the interviews and student work. My goal this time was to look for themes in how students were conceptualizing even and odd integers but moreover, I wanted to connect individual student's noticing patterns to their internal representations and conceptualizations of even and odd integers. Therefore, I tried to conduct my analysis by focusing on one student at a time and matched their noticing patterns to their interpreted understanding of even and odd integers. With the twelve different students interviewed in the small group, there were too many confounding variables to accurately analyze how students noticing patterns connected to their internal representations. Although, a lot of great definitions and conceptions of even and odds came out of the lesson, there was not a rigorous way to connect the definitions to individual noticing-patterns. After much frustration that I would be unable to show any connection between individual student's noticing patterns and their conceptualizations, I turned my focus on focusing on just a selection of students. This narrative inquiry on just three students was used to finish out my collective case study.

To limit the confounding variables that arise in a social interaction of a classroom setting, I used a follow-up interview to focus on one student from each group (I began with four students, one from each group, but one student was eliminated as he was sick the day of the interviews). My inductive analysis for each individual students' interview was guided by APOS theory. I inductively worked up from the data to discover themes within student's voices and understanding. I realized that it was not too surprising that the students were able to understand the definition and generalize about even and odd integers (as third-graders they already knew these definitions) but it was more interesting and ground-breaking that the third-graders were starting to conceptualize on their own what a function means. Therefore, I made the ad hoc decision to use APOS theory in guiding my analysis in how the individual students conceptualized a function.

In the following chapter, I will present each theme along with a thematic narrative giving voice to the three students along the way. Throughout, I will connect the findings with respect to my research question.

Chapter 3 Summary

I began this chapter by describing the case study methodology that I used in the collective case study. The participants included a third-grade classroom at a small suburban elementary school in Midwestern United States. The teacher was guided by a lesson that I wrote specifically with the goal of eliciting student's noticing patterns of a function "rule" table with the purpose of generalizing patterns of even and odd integers. During the lesson, small groups interviews were done following an interview protocol. Following the lesson, three students were chosen for individual follow-up interviews. Student work from the lesson was also included in the data collection. An important aspect of this study was the open coding model I used to identify student's noticing patterns which I hypothesized connected to how students were conceptualizing the function table in generalizing even and odd integers. Finally, I used APOS theory as a micro-framework in analyzed three student's understandings of even and odd integers as defined through functional relationships.

Chapter 4 - Results

In order to describe how students developed an understanding of a functional relationship based on the abstract external representation of a function table, I collected several pieces of data focusing on observations and interviews. In this chapter I present the results of my analysis of the student noticing patterns during the lesson.

Noticing Patterns within Function Tables

The purpose of the study was to understand elementary students' meaning-making of early algebra representations (function rule table) of number patterns about even and odd numbers within their natural classroom setting. A particular emphasis was placed on understanding the role of the function as a way to generalize a relationship between inputs and outputs where every input has exactly one output, i.e. an odd integer could be defined as $f(n) = 2n + 1$ for integer n. Most of the data used in the study and the analysis was based on the small group interviews that were recorded and transcribed during one third grade lesson taught by the classroom teacher. It was important for the purpose of the research to obtain the naturalistic setting of students engaged in the social activity of noticing and wondering about external representations. First, I identified the different noticing patterns within the function table which served as an eye-opener into student's internal representations. Then I deconstructed each noticing pattern with the purpose of constructing student's conceptions of a function in terms of generalizing even and odd integers.

The following table summarizes the noticing patterns within Task 1 on Generalizing Evens:

Table 4.1 Noticing Patterns within Task 1 "Generalizing Even Integers"

Noticing Pattern	Example Student Responses	Number of Recorded Student Responses
1: Recursive Pattern (Down Input Column)	"Counting by ones" "Input numbers are odd, even, odd, even, odd"	6
2: Recursive Pattern (Down Output Column)	"Counting by twos" "Output numbers are all even" "They are getting bigger"	12
3: Correspondence Pattern (Across Input to Output)	"When you double the input number you get the output number" "The numbers are doubling themselves" "If you go sideways, it goes $1 \times 2 = 2$, $2 \times 2 = 4$, $3 \times 2 = 6$, …" "Input numbers = Outputs numbers × 2" "Adding the same number 2 and 2 is 4…"	8
4: Inverse Correspondence Pattern (Across Output to Input)	"Input numbers are half of the output numbers" "$2 – 1 = 1$, $4 – 2 = 2$, $6 – 3 = 3$, etc."	3
5: Even Outputs "transferred" to same number in Input	"The output number is transferring to the input number"	1
6: The "Rule"	"The rule is times two"	3

The following table summarizes the noticing patterns within Task 2 on Generalizing Odds:

Table 4.2 Noticing Patterns within Task 2 "Generalizing Odd Integers"

Noticing Pattern	Example Student Response	Number of Recorded Student Responses
1: Recursive Pattern (Down Input Column)	"Odd, even, odd…" "Counting wrong" or "Skips at 6"	5
2: Recursive Pattern (Down Output Column)	"Output numbers are all odd"	8
3: Correspondence Pattern (Across Input to Output)	"(3 + 3) + 1 = 7" "You double the number and then add one to get the output"	5
4: Odd Outputs "transferred" to same number in Output	"The 1 (input) equals 3 (output) and there is a 3 here (input). The 2 equals 5 and there is a 5 here too!"	2
5: The "Rule"		0
6. Two Parts of the Rule, "doubling" and then "add 1"	"one side is even" and "one side is odd"	4

The results above show that third-graders had many different noticing patterns when given a function table. Of the twelve students included in the study, most the students (9 students in Task 1 and 8 students in Task 2) responded that they noticed and generalized on their own that all of the output numbers were even in Task 1 and all of the output numbers were odd in Task 2. In the whole-class discussion, the class concluded that these patterns (of going down the columns) would continue indefinitely in both tables. This gives insight to student's understanding of how a function table gives a pattern that can be generalized and continued indefinitely.

Less common but of significance was the pattern of noticing across the columns within in each row. In the first task, eight students mentioned noticing the rule of doubling across the row and but less than half (4) of the students mentioned noticing across the row in Task 2. In comparison to the previous result, this shows that it may be easier for students to look down the columns to see patterns than to look across. Furthermore, when comparing Task 1 to Task 2, we may assume that it is easier to notice Pattern 3 “Across Input to Output” directly from the table because it is a proportional relationship versus in Table 2 the addition of 1 adds a complexity to the rule. This could also explain why Pattern 4 “Across Output to Input” arose in Task 1, again since it was a proportional relationship.

Pattern 5 in Task 1 and Pattern 4 in Task 2 can be described as “finding evens and odd” across the columns. In Task 1, a few students noticed that there was a pattern of evens in the output column that show up later in the input column. Similarly, for odd integers in Task 2. Although this is a very interesting noticing pattern, I was not sure what we could conclude about student’s understanding of the table based on this pattern.

Finally, there were two noticing patterns based on the “rule” at the top of the table. It was interesting to me that some students were more focused on the rule than others. In the first task, the rule and outputs were given the students. Therefore, more students ended up first discovering the pattern (Pattern 3: Across Input to Output) and later in the interviews they remarked that the table was informing them of this fact all along based on the rule. This proved to me that it is possible with a proportional relationship, elementary students should be able to deduce a function rule based on Pattern 3. Based on APOS theory, I would characterize these students in the Process level of understanding a function when given a multiplicative or proportional relationship. Students in *Object* level of understanding were able to understand the equivalencies of “doubling” to “adding the same number to itself” to “multiplying by two” when generating an even number.

Because of the complexities of the Task 2 table, students were less aware or less focused on the function rule. In Task 2, students were first having to use the rule “double the number, plus one” to generate the outputs. Every student did perform task correctly. Based on APOS theory, we therefore could conclude that all the students are at least within the *Action* level of understanding when given a non-proportional function table such as in Task 2. To deal with the complexity of the non-proportional relationship of $f(n) = 2n + 1$, four students broke the function rule into two parts, first doubling and then adding one. This helped these students in generalizing why all of the outputs were even (since every number doubled is even and an even + odd is odd). In the next section, I explore this more by looking at specific student’s narrative as they grapple with the functional understanding.

Student’s Generalization of Evens and Odds as related to Functions

Recalling my research question, “How do elementary student’s mathematical noticing of external representations of algebraic problems relate to their (1) internal representations and (2) developed understanding of the concepts?” I hypothesized that student’s noticing patterns influenced their understanding of the algebraic concept of generalizing even and odd integers as related to functional understanding. Being able to show student *understanding* of concept can be vague so I used a micro-framework, APOS theory, in order to characterize students’ understanding of even and odd integers as related to functional understanding. What follows is a descriptive narrative inquiry of three students interviewed individually one week following the lesson study. Students were given and reminded about the table in Task 2 on generating odd integers (see Figure 4-1).

INPUT **Number**	OUTPUT **Double the Number, Plus 1**
1	3
2	5
3	7
4	9
5	11
6	13

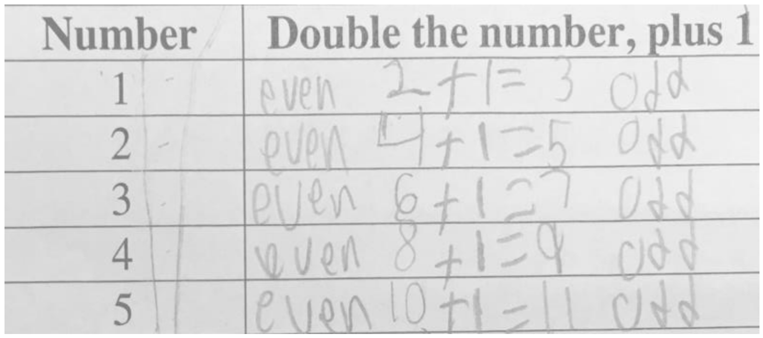
Figure 4-1 Task 2 Table: Generating Odd Integers

For the purposes of the analysis and discussion, the three students are given pseudonyms, A, P, and O (it will soon come to light as to why I chose these pseudonyms). What follows is a thematic narrative for each student interviewed.

Thematic Narrative of *A* – Struggling to Make Meaning of Function

The first student interviewed was *A*. She was quiet student in both her group work during the lesson and during her individual interview. *A* began the interview by remembering the table and first noticing the correspondence pattern “across” the columns. She started at the top row and noticed that you could add one and two to get three. She continued down the row and paused when she got to the six. Here she noticed that you could add 6 + 6 and then add 1 more to get 13. It was interesting that she noticed the correspondence pattern before any of the recursive patterns. *A* was struggling to find any more patterns and was having a difficult time recalling how we generated the outputs previously in class, so I had her fill out the table again to help her.

I was hoping that she would noticed more patterns with the evens and odds. It took some guidance to help her fill out the table. We discussed what it means to double and we went through each row together.



Number	Double the number, plus 1
1	even 2+1=3 odd
2	even 4+1=5 odd
3	even 6+1=7 odd
4	even 8+1=9 odd
5	even 10+1=11 odd

Figure 4-2 *A*'s Work on Table

After completing the table and with some more nudging, *A* started to notice the evens (when doubling) and the odds (when adding the one). She wrote those patterns in the table. She agreed that the pattern should continue but at first was unable or reluctant to describe how or why it would continue. After asking her to draw a picture, she drew the following picture to show how doubling and adding one will always be odd. Her image shows why $2(3) + 1 = 6 + 1 = 7$ is odd. As she explained, "when you double the number it's even. Then the plus one will always make it odd because it is leftover".

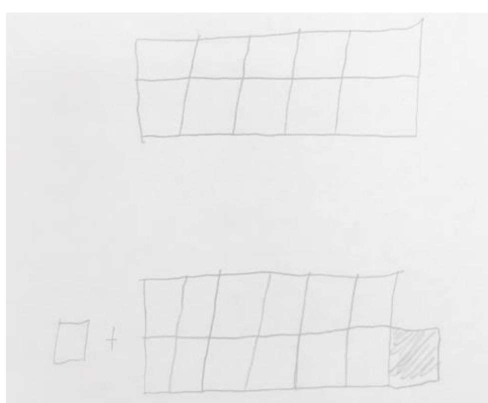


Figure 4-3 *A*'s Work Showing Why 6 + 1 = 7 is Odd

It was interesting to me that *A* first noticed the correspondence pattern "across" the columns before noticing the recursive patterns down the columns (which many the students noticed first). As we keep working on the table problem, I began to characterize *A* at the *Action* level of understanding, which is the first level of understanding of a function based on APOS theory. In the first level of understanding a function, a student may assume that a function is tied to a specific rule or formula. *A* was very aware that there was a rule that she must follow to fill out the table. But she struggled to make further sense of the table in understanding the relationship between the two numbers in the row. In conclusion, *A* could act on the function rule, which could be connected to her correspondence pattern across each row, but needed further guidance in determining any recursive or generalized pattern in the table.

Thematic Narrative of *P* – Processing the Function

Our second student, *P*, was a talkative girl with quick number sense. She began similarly to *A*, by looking across the rows to find what you could add to the input number to produce the output number. She noticed that "1 + 2 equals 3, 2 + 3 equals 5, 3 + 4 equals 7, 4 + 5 equals 9, 5 + 6 equals 11" and said she was looking across the rows to figure that out. Then I questioned her how she would figure out the next number in the table, and input of 6. Here her thinking was more sophisticated as she used the recursive pattern of the outputs to know that the output corresponding the input 6 will be two more than the previous output, so 13. Then she went back and thought what number added to 6 would give 13. In her own words, "Two more than 11 is 13. So, I just think 6 + 7 equals 13." Her flexible thinking between the recursive pattern and the correspondence pattern was key in helping her develop a more conceptual understanding of the functional relationships in the pattern. *P*'s sophisticated thinking assumes that a function has a relationship between the input and outputs. She was able to observe that she may move between the input and output numbers and that there is a correspondence and inverse correspondence between the numbers.

The remainder of *P*'s interview focused on the pattern of evens and odds. When asked to justify why every output is odd (as she previously stated), she showed the following work (Figure 4-4). Again, she was clear in her wording that she was using the input to generate an odd output number. As she wrote the string of her equations, she emphasizes that no matter if she started with an odd number (e.g. 1) or even number (e.g. 4), when doubling the number, we get an even and when adding one (which is odd) we will always get an odd number. This is true because of the fact that an odd plus an even is always odd.

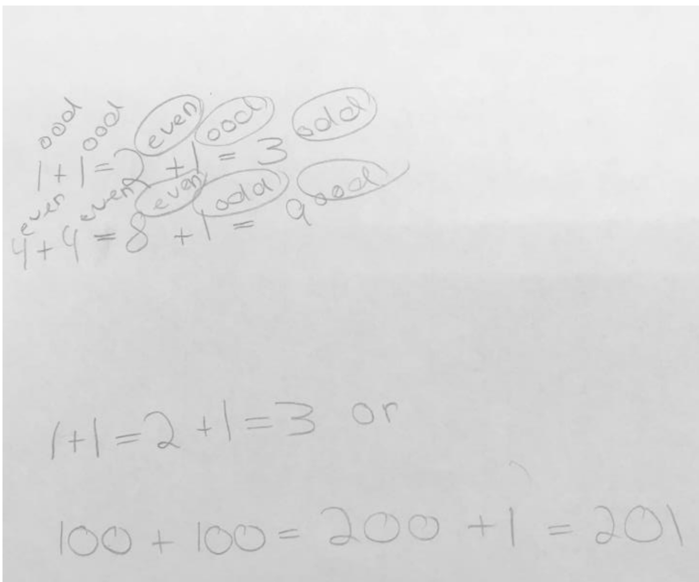


Figure 4-4 *P*'s Work Showing Why Every Output is Odd

We could characterize that *P*'s understanding of a function relationship is at the *Process* level, which is the second level of understanding of a function based on APOS theory. At this level, a student assumes a function is an input-output machine and is independent of the formula. Throughout the interview, *P* was less focused on the "rule" (doubling the number, plus one) and she used the table in a more sophisticated way by using both recursive and correspondence patterns to understand the relationship between the input and output. In the end, this sophisticated noticing of the table helped her generalizing that every output is odd because and even + odd is odd.

Thematic Narrative of *O* – Awareness of Multiple Representations

During the small group interviews and the lesson, the final student interviewed, named *O*, showed some abstract thinking and therefore, I was curious if she could expand on her thinking in a separate individual interview. *O* quickly remembered all the patterns that she saw in the original task during the lesson. She noticed the recursively patterns, i.e. the inputs are "counting by ones" and that the outputs are "counting by twos". She also quickly noticed or recalled that the outputs were all odd. When asked how to find the corresponding output for the input 6, she quickly used the rule. In her own words, "I would just double 6, so 6 + 6 = 12 and then add one more so 13." I asked about another number, 1000, and again she doubled and added one to get 2001. In only five minutes, *O* gave reasoning behind each noticing pattern from my first analysis.

At this point, I shifted the interview and asked why the rule always gives us an odd number. I recalled that *O* had drawn the following pictorial during the small group discussion when explaining why doubling the number and adding one was odd (Figure 4-5). *O* was able to explain that when doubling, we always get an even number, because we can divide it evenly in half. Then the adding of one more square would make the number odd. You can see O's reasoning in her internal representation of $2(1) + 1$ and $2(2) + 1$ that she drew.

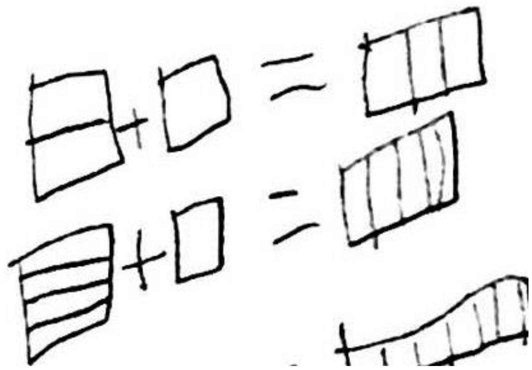


Figure 4-5 *O*'s Work Showing Why Every Output is Odd

One of the most interesting internal representations from the class, was also from *O*. When asked if she could show me the next number (input of 3) she wrote the following equation (Figure 4-6). *O* could translate the pictorial above into an abstract representation of the same pattern. When asked why she used parenthesis, she stated that, "it is easier to see the even part." And similarly to *P*, she concluded that the output will always be odd because an even plus an odd is always odd.

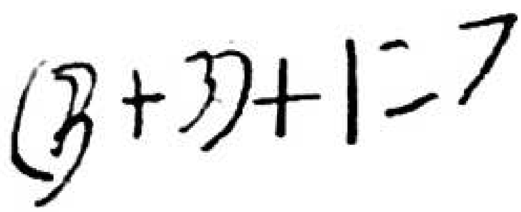


Figure 4-6 *O*'s Work Showing the Equation $(3 + 3) + 1 = 7$

In characterizing *O*'s understanding of functions, her noticing patterns and reasoning could be characterized similarly to *P*, that is, the process level of understanding. Her ability to write the abstract equation in Figure 4-6 shows me that she may be on her way to an Object understanding of a function. With an object understanding, a student becomes aware of transformations, multiple representations, and properties of the function, i.e. understands the concept completely.

Chapter 4 Summary

In this chapter, I presented the data I collected in order to investigate how third-graders make meaning of a function table when used to help generalize even and odd integers. First, I analyzed the small group discussions to characterize each noticing pattern when given a function table. The frequency of these noticing patterns start to paint a picture as to what students are attending to within a function table. With hopes of connecting individual student noticing patterns to their understanding of the function table, I conducted a separate analysis of three student's individual interviews following the lesson. A thematic narrative helps give a voice to students at different levels of understanding of a function, i.e. from *A*'s Action level to *P*'s Process level to *O*'s emerging Object level of understanding.

Chapter 5 - Conclusion

This chapter begins with an overview of the study and a discussion of findings based on the emerging themes. I then will present implications, discuss limitations and constraints, and make recommendations for future research.

Overview of Research

The design of my study was largely influenced by my own experiences teaching mathematics and my philosophical stance of mathematics education. As a mathematics student and teacher, I understand that "seeing is understanding" when it comes to learning mathematics. Recent research in neuroimaging has shown that mathematical thinking is grounded in the visual processing networks of our brain (Boaler, 2017). These new results further accentuate my personal mathematics teaching philosophy that centers on visuals in the classroom. Furthermore, since my research during my Master's program, I have been acutely aware that my students may not be internally visualizing the concepts like I am able to do as an expert. The NRC (National Research Council) elaborates that "experts notice features and meaningful patterns of information that are not noticed by novices" (NRC, 2000, p. 31). Therefore, I hoped to address how visual representations, within the curriculum and classroom setting which support early algebraic reasoning, are interpreted by the students. Specifically, I was guided by the following question: Situated within a third-grade classroom, how do elementary student's mathematical noticing of external representations of algebraic problems relate to their (1) internal representations and (2) developed understanding of the concepts?

First, I hoped to address the gap I have previously identified within the current literature on early algebra instruction, that is, how visual representations, within the curriculum and classroom setting which support early algebraic reasoning, are interpreted by the students. I focused on the external representation of tables as these are static external representations used to make meaning of generalization of patterns and early understanding of a function. The generalization of numeric patterns has been shown as a key practice in supporting algebraic understanding at the elementary level (Bastable & Schifter, 2008; Schifter, 1999; Schifter, Monk, Russell, & Bastable, 2008). In particular, function tables are used as a static external representation to describe a rule or functional relationship of two quantities. Although tables are a static and abstract representation of functions, being able to make meaning of a table and understand the relationships within the table is crucial in later mathematics. Furthermore, tables are generally only used as a form of recording (like a t-chart) in elementary grades. Generally, it is not until middle school that tables begin to be used as a means to represent functional relationships between to variables. Therefore, I hoped to describe student's meaning-making process of the external representations of the function table and individually what features (noticing-patterns) of the function table do the students attend to and or do not attend to along with their meaning.

Secondly, I conjecture that this meaning-making process of individual students will relate to their understanding of the algebraic practice of generalization (refer to Figure 5-1). For this question, I looked at whether student's mathematical noticing of the function table differ between those that correctly and incorrectly understand the algebra concept of even and odd numbers and the generalization of the even and odd number patterns. Defined by Lobato, Hohensee, and Rhodenhamel (2012), *mathematical noticing* refers to "selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for students' attention" (p. 438). Although it is nearly impossible to gain an in-depth understanding of student's internal representations as observers, we can infer student's internal representations by asking students to describe their process of interpreting the external representations and drawing their own representations of concepts. Through these visual reports, gestures, and written student representations, I tried to connect student's internal representations of those concepts represented in the classroom and with their subsequent algebraic reasoning and understanding.

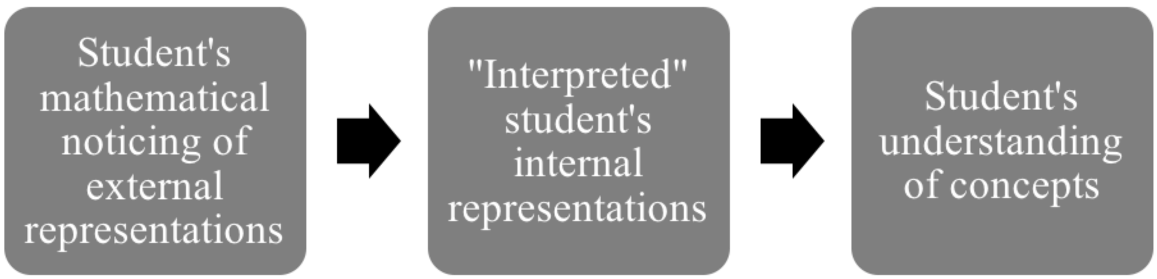


Figure 5-1 Concept Map of Research Question

Symbolic interactionism, a subsidiary of interpretivism, which is focused on the social construction of participants' meanings or understandings (Reynolds & Herman-Kinney, 2003; Crotty, 1998), provided a sound overarching theoretical perspective in which to base this study. Guided by the premises of symbolic interactionism, the student's internal representations or meanings of algebra will be constructed by his or her interpretive understanding of the symbolic external representations and depend on communicative culture for which he or she engages in. Furthermore, the students will solve the algebra problem (i.e. act upon the representations) based on the meanings that the students have constructed. This process can be seen in Figure 5-1.

Since my goal was to describe students meaning-making process, the methodology of collective case study was appropriate in order to gain in-depth understanding of individual students within their classroom context. After a pilot study which included multiple interviews testing out different algebra tasks and external representations, I decided to focus on the external representation of the function table and the algebraic concept of generalizing about even and odd integers. At this point in the semester, I designed a whole-class lesson that included individual, small group, and whole group discussions focused on noticing patterns within a function table with a lesson goal of generalizing about the even and odd number patterns. With the help of the classroom teacher and three other teacher helpers, the whole class and small group discussions were all audio-recorded and later transcribed by me.

I analyzed the transcripts of the small group discussions and concurrently analyzed the individual student work in order to narrow down student's noticing patterns to 6 coded noticing-patterns for each task. Another portion of my analysis was comparing the individual noticing-patterns to student's interpreted internal representations and their generalization of even and odd number patterns.

Discussion of Findings

The major finding for this study was being able to characterize elementary students' mathematical noticing patterns regarding a function table. In the following section, I summarize the findings from the study and also discuss why some components of the original research question could not be answered with the data I collected.

Noticing Patterns within Function Tables

In my research question, I sought to describe how elementary student's mathematical noticing of external representations (function table) of algebraic problems relate to their (1) internal representations and (2) developed understanding of the concept (generalizing even and odd integers). The first part of the research question was focused on describing student's mathematical noticing of the static external representation of a function table.

The results of the study show that third-graders had many different noticing patterns when given a function table. Of the twelve students included in the study, most the students (9 students in Task 1 and 8 students in Task 2) responded that they noticed on their own that all of the output numbers were even in Task 1 and all of the output numbers were odd in Task 2. We may characterize this noticing-pattern as a *generalized* pattern and this finding aligns with Blanton and colleagues work which similarly showed how function tables can be used to teach early generalized patterns. This further gives evidence supporting the importance of using function tables as a representation for generalizing arithmetic patterns in the early elementary classroom.

The other common noticing pattern was also by looking down the columns, i.e. *recursive pattern*. Six students in Task 1 and five students in Task 2 noticed that the input numbers count by ones and the output numbers skip count by twos. In the whole-class discussion, the class concluded that these recursive patterns would continue indefinitely in both tables. This important finding gives insight how student's may use recursive patterns to develop an understanding of how a function table gives a recursive pattern that can be generalized and continued indefinitely.

Less common but of significance was the correspondence pattern of looking across the input to output within in each row. In the first task, eight students mentioned noticing the rule of doubling the input to obtain the output and three students were also able to notice that inversely they could halve the output to obtain the input. Due to the proportional relationship of the function $f(x) = 2n$, it was easier for students to notice these patterns directly from the table. Moreover, this supported the students understanding of the function rule as defined by inputs and outputs and based on APOS theory. Based on this finding, we may conclude that the majority of the third-graders in the study were in the Process level of understanding a function when given a multiplicative or proportional relationship. Some students were even able to understand the equivalencies of "doubling" to "adding the same number to itself" to "multiplying by two" when generating an even number, which suspect that they are beginning to develop an Object level of understanding of functions.

In Task 2, students were given a non-proportional function, $f(n) = 2n + 1$ to generate odd integers. Based on complexity of the non-proportional relationship, fewer students than in Task 1 mentioned using the correspondence-pattern to understand the function rule. But the small group interviews did show that in all of the groups, students were able to understand how to use the function rule to generate an odd number. This shows that all third-graders in the study were able to *act* on the function but some may have been still grappling with the ability to *process* the functional relationship when given a non-proportional relationship. To deal with the complexity of the non-proportional relationship of $f(n) = 2n + 1$, a quarter of the third-graders broke the function rule into two parts, first doubling and then adding one. This helped these students in generalizing why all of the outputs were even (since every number doubled is even and an even + odd is odd).

The Conceptualization of a Function

The second part of my research question pursued specifically *how* elementary student's noticing patterns are related to their internal representations and developed understanding of the concept. Although, I began to answer this question in the discussion above, in order to specifically relate student's noticing patterns to their understanding of the function as a concept, I used the individual interviews to bring focus on this connection. Based on the individual interviews following the lesson, I conjecture that the more noticing patterns that a student internalizes in the function table is positively correlated to the student's understanding of a function relationship, that is, the more noticing-patterns within the function table the higher the student's level of understanding of a functional relationship.

In particular, *O* and *P* both noticed all six noticing patterns previously identified during the lesson and was able to accurately describe their internal representations of how they generalized an odd number. When interviewing the students, you are able to follow along as the flexibly move between noticing patterns to describe the functional relationships within the table. I characterized their functional understanding as the more advanced Process level of understanding of a function and were on their way to develop a Object level of understanding of a function. To further support a Object level of understanding, I would work with the students in generalizing an abstract equation for the function related to their examples and further connect the table, pictorial, and equation representations of the pattern.

On the other hand, *A* was limited to one noticing-pattern which utilized the given rule in the function table. With probing, she was able to eventually notice recursive patterns but unable to relate between the two patterns. This led to my characterization of her functional understanding to be at an Action level. Moreover, this may infer that the fewer frequency of noticing-patterns may be related to a lower level of understanding of a functional relationships.

Limitations

Time Constraints of the Study

A very significant limitation of this study was that the data collection occurred in a relatively short period of time. Due to both personal time constraints and with the constraint of not interfering with the classroom schedule, I was limited to one specific lesson that I could control the design and purpose of. Thus I was limited to studying only one specific instance of student's noticing patterns rather than a more comprehensive design that would encapsulate student's longitude development of their mathematical understandings. Even within this short-time period, I feel I was able to show that third-graders are able to conceptualize the foundations of a function when related to a concept pattern they are familiar with. With more time and resources, it would be interesting to see how this understanding can be developed further within the elementary classroom in order to inform early algebra instruction.

Teacher-Researcher versus Researcher-Observer

The choice of being a "researcher-observer" versus the "teacher-researcher" in my study was two-fold. First, since I am not an elementary teacher, I did not feel that it was appropriate for me to teach the lesson. Furthermore, as a researcher I viewed my role as an observer in a naturalistic setting. My choices in the design of the lesson study were specifically done to not interfere with the natural classroom. Although, I stand by my decision in this design choice, I also understand that there may be limitations when the researcher does not teach the lesson in a lesson study. For instance, the classroom teacher may have made different choices than me which ended up influencing my findings. We know that the situational context (i.e., external representations or symbolic communication given in the problems) and cultural context (i.e., the verbal and nonverbal communication within the classroom and norms within the classroom environment) inform upper students' conceptual context of internally visualizing mathematical representations, therefore, there were confounding variables when trying to distinguish student's individual internal representations and their conceptual understandings of even and odd integers as defined by a function relationship.

Implications

Even with the limitations described above, I feel the results of this study have implications for teachers and curriculum developers both at the elementary level and later curriculum when a function is traditionally introduced (i.e. middle school, high school, or developmental math in college).

Implications for Teachers and Curriculum Developers

This research has the potential to influence how teachers and curriculum developers first introduce functional relationships in both elementary and later curriculum when a function is traditionally introduced. Without using the notation generally used to introduce a function, the study proves that third-graders were able to act upon and conceptualize a function at a complex level of understanding. As a university instructor who develops curriculum and teaches developmental math courses, I know that the traditional $y = f(x)$ notation of a function is the first obstacle for many of my students to overcome. By instead introducing functions by exploring patterns similar to those in this study, students are encouraged to first explore the patterns and discover the relationships within the table which then would lead to a more generalized notation for functions. Furthermore, exploring function tables within a setting that is familiar to students such as even and odd integers again can help students discover the input-output pattern.

Another implication of this study reaffirms previous research on external representations, including my previous Master's research findings, that is, students may need support or just more frequent experiences in noticing patterns when given complex representations such as tables. The literature stemming from Dubinsky's APOS theory, suggests that students' experiences with multiple forms of representations (i.e. graphs, tables, and symbolic representations) help build the schema needed to develop the whole picture of mathematical relations (Eisenberg, 1992). Particularly concerning instruction in the domain of functions, starting in middle school we represent and communicate the concept using words, symbols, graphs, and tables, although the representations may not be treated equally or the connections between them may not be made explicit. Following Adu-Gyamfi and Bossé's (2013) results, my Master's research found that undergraduate students notably struggled more on items that asked to identify linear functions given a tabular representation (Johnson, 2015). We can assume that when we introduce functions in a tabular setting earlier and more often in the elementary grades, students will be more familiar with this external representation.

Recommendations for Future Research

Although you could characterize the results of this collective case study as preliminary, I feel they draw our attention to more unanswered questions in many different directions. Future work should continue to look at the development of conceptualizing a function given the external representation of a function table in the context of early algebra. This would be require an extensive study that may begin in early elementary with the introduction of the function table similar to Tasks 1 and 2 and continue into middle school as we move students through different scenarios of the function table. For instance, a repeated-cross sectional interview study could look at the relationship between noticing patterns within a function table (and other representations) and student's understanding of functional relationships. The goal would be to characterize the progression of student's understanding of a function table based on APOS levels of cognitive development and quantitative analysis to explore my conjecture that the frequency of noticing patterns are positively correlated to the student's understanding of a functional relationship.

The anticipated broader impact of my work is to move beyond understanding student's visual patterns in multi-representational problems to developing an effective instructional environment based on these findings. This research would be based on qualitative research characterizing the progression of a student's understanding of function in multiple representations but would use a mixed methods lesson study design to explore the benefit of certain tasks and instruction strategies such as open-ended notice and wonder questioning and visual cueing may support all students in developing an advanced schema understanding of a function.

There is still much work to be done to understand how students attend to features of a table or other external representations of a function before we can begin constructing effective cues, but this study offers hope that we can begin to understand students' internal representations of what they are seeing when grappling with the math.

Closing Thoughts

In many ways, this dissertation taught me more about myself and my research than it did about the students in that particular third-grade classroom. Since my Master's research, I have continued to be interested in visual attention and noticing patterns. As an undergraduate mathematics instructor, I have learned the importance of specifically asking about student's noticing patterns and I am acutely aware of explicitly directing students to expert noticing patterns in order to support their own internal representations and conceptions. This largely influenced my choice of my graduate research topic and furthermore, guided my research design. I began this study attempting to understand how elementary students would notice and interpret a function table before any type of direct instruction on functions has been given. I was motivated by the assumption that if I could write a lesson that connects to a concept students are familiar with, i.e. even and odd integers, the students would offer great insight into their noticing patterns within a function table. I soon learned though my aspirations were larger than I could attempt to accomplish with the limited resources and time I had available to me. I learned that a quality qualitative research design requires more time and resources than are usually available to a graduate student. After the lesson, it became clear to me that although the lesson was successful, it was going to be difficult to characterize and separate individual student noticing patterns to student's conceptualizations. With more time and resources, I would like to take this problem further with a large collection of interview studies that involved students of different ages.

As I look back at the choices I made two years ago when I first began designing my study, I am also acutely aware that my attempt to become fluent in qualitative design and analysis led to a more difficult and often frustrating doctoral research experience. All prior research experiences I had had included quantitative or mixed methods design and analysis. My thoughts were that I wanted to become a more well-rounded researcher, but I learned through this experience that individuals may be more drawn to one epistemological design than the other. As a teacher and scholar of education research, I have found that I am drawn to qualitative studies as they give me more insight into the students' mindsets. This is important to me as I apply the research to my instructional decisions. But as an education researcher, I feel more qualified and actually enjoy more the process of quantitative design and analysis. This was a surprising and undoubtedly important result of this study. Gratefully, I am still in love with my topic of understanding student's noticing patterns related to functions and I now have a better understanding how I may accomplish my goals with a mixed method interview study where I can both qualitatively and quantitatively measure student's noticing patterns.

In closing, I would like to leave the reader with this quote from the author Malcom Gladwell, "It takes 10,000 hours of deliberate practice to become an expert" (Gladwell, 2008, p. 42). I began this doctoral research experience with hopes of becoming an expert in mathematics education research and qualitative research design. The defeating feeling of a dissertation is when you realize you are not an expert and you may not be an expert by the time of your defense. But with more deliberate practice and with a better understanding of who I am as a researcher, I am now one step closer.

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Appendix A - Parental Letter and Consent Form

Dear Parent/ Guardian: March 1, 2018

My name is Jennifer Johnson-Whitley and I am a doctoral candidate at Kansas State University in the department of Curriculum and Instruction. I am a former math teacher, currently graduate teaching assistant at KSU, interested in mathematics education, specifically how students perceive mathematics. I am interested in learning more about how elementary students visualize mathematics concepts in their own way. I would like permission for your child to participate in my research which I will be conducting during the month of March. **This letter contains a description of the project, "The Unfolding of Student's Internal Representations of Early Algebra: A Case Study of Elementary Students", followed by permission forms for you to sign.**

In the new mathematics standards that Kansas has adopted, there is an emphasis on early algebra development in order to support students' arithmetic understanding and procedural fluency and also, prepare students for middle school algebra. I want to better understand how students in elementary school make meaning through internal visualization of the algebra concepts we introduce in elementary math. **During a class lesson taught Thursday, March 8th by their teacher Mr. Shawn Ryan, I will be audio recording students conversations concerning their meaning-making process.** The audio recording will not interfere with their learning. **The lesson will be video recorded**, but the focus will be on the teacher during the video recording. Following the lesson, I will be **conducting one individual interview for about 6 students throughout the month of March and would like your permission to work with your child on math tasks for about 15 minutes. I would like to collect their student work and video record the interviews.** This will allow me to analyze and characterize student's meaning-making process during a math lesson and during one-on-one sessions. The math tasks will be based on and an extension of their math lessons during my time in the classroom. Students will therefore benefit from extra time being spent on the math already being focused in the classroom and no risks are expected. Before beginning my interviews, I will be volunteering in the classroom to develop a rapport with the students and teachers.

I also would like your permission to use selected video and audio clips in professional conference presentations and/or publications such as my dissertation. These video and audio clips would be available for your preview should you wish to see them prior to giving your approval. Please consider if you would be comfortable with this use of video exemplars that include your child. **In any publications associated with this research, pseudonyms will be used in place of your child's name, the teacher's name, and the name of the school.**

If you choose not to have your child participate in any aspects of this study or if you choose to

withdraw your permission at any time, there will be no penalty. Participation in the project will not affect your child's grades, participation in class, or placement for the following school year. **If you have any questions concerning the research study or your child's participation in this study, please call me at 785-532-1436.** If you have any questions about you or your child's rights as a participant in this research, or if you feel you or your child have been placed at risk, you can contact the IRB Office:

- Rick Scheidt, Chair, Committee on Research Involving Human Subjects, 203 Fairchild Hall, Kansas State University, Manhattan, KS 66506, (785) 532-3224.
- Cheryl Doerr, Associate Vice President for Research Compliance and University Veterinarian, 203 Fairchild Hall, Kansas State University, Manhattan, KS 66506, (785) 532-3224.

To give consent for your child to participate in this study, please complete that attached form and have your child return it to Mr. Ryan by Thursday, March 8.

Sincerely,

Jennifer Johnson-Whitley, MS
Doctoral Candidate, Graduate Teaching Assistant
Kansas State University | College of Education
(785) 532-1436 | jejohnson@ksu.edu

*Please complete this form and return it to your Mr. Shawn Ryan by **Thursday, March 8**.*

Parent/Guardian Consent Form

I have read the information presented above and have had an opportunity to ask questions and receive answers pertaining to this research project. I am aware that my permission is voluntary and that I am free to withdraw my permission at any time without any penalties to my child or me.

I **give permission** for Jennifer Johnson-Whitley to: (please check all that apply)

______Collect student work and include my child in video recordings of the individual interview. She may use the video clips and/or audio recordings that include my child during conference presentations or academic publications.

______I **do not give permission** for my child to participate in any aspect of the data collection for this study.

____

Child's Name Grade Classroom Teacher

____

Signature of Parent/Guardian

Appendix B - Lesson Plan

Generalizing about Even/Odds using Tables

Objective: **Students will be able to generate and analyze number patterns about even and odds.** Students will begin to define what it means to generalize about a pattern.

CCSS.4.0A.5: Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

MP.7: Look for and make use of structure.

Beginning: Whole Class (5 minutes)

1. Today we are going to explore numbers and the patterns that we see in numbers. Do any of you know what a pattern is? What is an example of a pattern? "Patterns are cool because they help us see interesting things about how our numbers work."
2. Start by giving students a few easy patterns, and asking them to find the missing terms. For example: 3, 6, 9, 12, 15, ___, ___, ___. Tell students that patterns are when you do anything repeatedly (you repeat an operation). Ask students to identify the "rule" in the pattern about (A: plus 3).

Middle: Partner/Small Group

3. Tables can be used to help us organize our patterns. Have you used tables before in math? Give students another pattern, this time in a table. For example:

__**Task 1**__: (20 minutes)

Input: Number	Output: Number x 2
1	2
2	4

3	6
4	8
5	10

4. ***Notice and Wonder*: (Think-Pair-Share) –** *Partner Interview Audio Recorded*
    a. *What do you notice about the numbers in the table?* Share with a partner.
    b. *What do you wonder about the number in the table?* Share with a partner.

5. Have some students share their notice and wonder to the whole class.

Whole Class Discussion – Guiding Questions:

a. What patterns did you find? How did you find the patterns? (All numbers on the right are even. When the number on the left increases by 1, the number on the right increases by 2.) (Use arrows to point out student's patterns - Point out both the horizontal and vertical patterns.)

When we are looking at the rows we are looking at the relationship between input and output (rule). When we are looking at the columns we are looking at just the input or just the outputs.

b. Can we add any number on the first column and generate/produce a number in the second column?
c. What if we are given a number in the second column, can we find the number in the right column?
d. When students point out that all the numbers on the left are even, have students state a claim about numbers being multiplied by 2.

6. ***IF TIME PERMITS:* Guiding questions in small group**. Students may use pictorial or concrete justification. Give students two-colored chips and/or square tiles for a concrete justification.

    a. Does an even number multiplied by 2 result in an even or odd number? Why do you think this is? Use the manipulatives or draw a picture.
    b. Does an odd number multiplied by 2 result in an even or odd number? Why do you think this is? Use the manipulatives or draw a picture.

7. **Whole Group**: Patterns can be more simple like the one above (where we only had one operation), but they can also become complex. With your partner, ***fill in the table at your desk.***

Task 2: 20 minutes

Input: Number	Output: Double the number, plus 1
1	
2	
3	
4	
5	
10	
25	
100	

8. Give students time to complete the table, and review for accuracy, if needed.
9. **Think-Pair-Share in small groups:**
    a. Ask students what they notice about the numbers they entered into the table. *Have students silently write all their observations on the handout and then have them share with a partner.*

10. Guiding questions in small group: *Group Discussion Audio Recorded*

- What do you notice about the numbers in the table? (All numbers on right are odd. When the number on the left increases by 1, the number on the right increases by 2.)
- Point out to me where you see your pattern? What made you think to look for that?
- How do you know that pattern will continue?
- Does an even number plus 1 result in an even or odd number? Why do you think this is? Use the manipulatives or draw a picture.
- Does an odd number plus 1 result in an even or odd number? Why do you think this is? Use the manipulatives or draw a picture.
- Explain why the numbers you entered in the table are all odd. How does your picture show this property?

Class Debrief: 10 minutes

1. Have students share what they learned today to the rest of the class. *Create an anchor chart about the properties of even and odd numbers that the students found.*

Guiding Questions:

- What patterns did you notice? What helped you see that pattern?
- How can we generalize what we saw in words and using a number sentence?

Handouts

Task 1:

Input: Number	Output: Number x 2
1	2
2	4
3	6
4	8
5	10

What do you **notice** about the numbers in the table?

What do you **wonder** about the numbers in the table?

Task 2: Fill in the table below.

INPUT: Number	**OUTPUT:** Double the number, plus 1
1	
2	
3	
4	
5	
10	
25	
100	

What do you **notice** about the numbers you entered in the table? Record your observations silently in the space below.

Appendix C - Lesson Transcript

Generalizing about Even and Odd Numbers Using Tables (3rd Grade Math Lesson)

Objective: **Students will be able to generate and analyze number patterns about even and odds.** Students will begin to define what it means to generalize about a pattern.

CCSS.4.0A.5: Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.

MP.7: Look for and make use of structure.

BEGINNING OF THE LESSON:

CT: So, ladies and gentlemen. Today for math group, we are taking it a little bit of a different direction then what we have been doing. We have been talking a lot about fractions. With fractions and math all year round, we have been talking about patterns. You have seen patterns all year round in math, right?

Students: Yeah!

CT: We have seen different patterns in mathematics. Well, our objective today is we are going to generate and analyze number patterns about even and odd numbers. Do you guys know what even and odd numbers are?

Various Students: Yes, I do.

CT: Okay, so we are going to do that. And with that, you will be able to find what it means to generalize about patterns. So, that is what our objective is today. We are going to talk about even and odd numbers and find the pattern about even and odd numbers. OKay, Ready?

01:02 CT: Okay, so right here on the board you have the pattern. On your (individual) whiteboards, can you repeat the pattern that I have?

------------------Written on Board: 3, 6, 9, 12, ___, ___, ___.

-----------------*Students working on their own and in groups. CT walks around to tables. After about 1 minute quietly working, individual groups start discussing the pattern. A teacher helper is sitting with each group to facilitate the groups.*

02:32 CT: Okay, so what is the pattern? How can we complete this pattern?

Student - Phillip: You can complete it ....complete it by counting by three.

CT: Counting by three. So what would be our next number? Madison?

Student - Madison: 15.

CT: And after that? Alexa?

Student - Alexa: 18.

CT: And after that? Sterling?

Student - Sterling: 21.

CT: And after that? Haley?

Student - Haley: 24.

CT: So, I have another question. What is the rule for this pattern? When we have a pattern, we have a rule that we must follow. What is that rule? Colton?

Student - Colton: Plus 3.

CT: Plus 3. So we are adding three every time. Can you can me anything else about this pattern? Meliana?

Student - Meliana: It goes odd, even, odd.

CT: Odd, even, odd - okay. Madison, you see that too? So every other number is an even number. It goes odd, even, odd, even, odd. Very good.

03:35 CT: What about this pattern?

------------------Written on Board: 2, 6, 8, 12, ___, ___, ___.

CT: What numbers are missing there?

------------------*Students begin discussing in their groups.*

04:30 ------------------*CT made a mistake in his next pattern. The pattern switches from counting by 4's (starting at 2) to counting by 2's. Student exclaims, "It's impossible!" A discussion follows about generating the pattern. "After you make a mistake, the following numbers will be also incorrect." The students move to generating patterns in order to fix it.*

CT: How about this? The rule I was going for was "add 4". If I were to add 4, what would be the pattern? What number is in there that shouldn't be there?

Students: 8.

CT: 8 shouldn't be there. What should it be?

Student: 10.

CT: So Julian says that my first mistake was the 8. That it should be a 10. So....okay. Colton, what did you say?

Student - Colton: That is is supposed to be a 10. And if you add 4, it is supposed to be 14.

CT: There is truth to that statement. So because I made a mistake here, it messed it up the entire way through. Isn't that what a pattern does? If you make a mistake early, it is going to throw it off the entire way? Does this remind you about something? How did I learn sevens?

CT: How do I learn sevens? By always counting an extra point in football. 7, 14, 21,....But if I miss an extra point, then I would always be off. Much like I did here.

CT: I wish I had done that on purpose, but I did not.

MIDDLE OF THE LESSON:
07:51 CT: With patterns, we can see patterns in lots of different ways. The way we are going to see patterns today is with tables. What I want you to do at your tables is to look at the table on the paper. What does this table say? What is the rule? So I am going to be quiet for a minute. Just take a look at it and see what you see. You are welcome to write things down but work quietly on your own for a minute.

Input: Number	Output: Number x 2
1	2
2	4
3	6
4	8
5	10

09:49 CT: So ladies and gentlemen, now share with your partners at your table.

------------- Individual Partner/Small Group Discussions recorded separately at the tables. (about 5 minutes)

16:09CT: Da, da, da, da,

Students: Da, Da.

CT: Okay, ladies and gentlemen. I heard some great conversations at your tables. Would any of you be willing to share some ideas to the whole class? What are some of the things you noticed about this table? Stephanie?

Student - Stephanie: I noticed that the input numbers go odd, even, odd, even, odd.

CT: Okay, so the input numbers ----- *CT writes on the board what they noticed about the input.*

CT: Kaley, what do you notice?

Student - Kaley: The output numbers are all even.

CT: The output numbers are all even! ------ *CT writes on the board what they noticed about the outputs.*

CT: Do you wonder why? Why is it all even over here?

Student: Because it......*(Pause)*

17:57 CT: Do you need help? Can someone help Haley? She is wondering why the inputs go odd, even, odd but every number in the outputs are even. Jala, what do you think?

Student - Jala: I think that it will be the same number.

CT: Okay, one more time because I don't think I am following that correctly?

Student - Jala: Because if you add the same input number together, it will be an even.

CT: Okay, so if you add input number 1 + 1, the output will be even. Okay, good. Alright. Does anyone want to add to that? Colton?

Student - Colton: That um, it just times 2. Because of the input is times 2, you are going to end up with an even number because two odd numbers is even.

CT: Okay, because you are multiplying times 2. What do you mean by "times 2"?

Student - Colton: You don't have to! You can add the number to the same number.

CT: Okay, you don't have to multiply. Alright. Jaslyn?

Student - Jaslyn: The input numbers are 1/2 the output numbers.

CT: You are seeing some fractions there? I like that. Jada?

Student - Jada: I saw the that 1 times 2 is 2, 2 times 2 is 4, 3 times 2 is 6.

CT: So you are noticing that you can multiply across by 2. -----CT gestures across the table rows.

CT: Good. So what if we continued this pattern? Would we ever have an odd number in the output column?

Student(s): No.

CT: No, you don't think so?

CT: Yes, mam?

Student - Because an odd number plus an odd number is an even number and an even number plus an even number is even.

CT: Okay so odd number plus an odd number is an even number and an even number plus an even number is even. Does everyone agree with that?

CT: Okay, so let's move onto the next one. ------------*When we are looking at the rows we are looking at the relationship between input and output or the rule. When we are looking at the columns we are looking at just the input or just the outputs.*

23:13 CT: Alright, we have another task. Another pattern at your desks. Ladies and gentlemen, you have another pattern sheet here. What you need to do is to take a look at it and fill in the pattern. This pattern will use the input and what you need to figure out is what is your output going to be.

INPUT: Number	**OUTPUT:** Double the number, plus 1
1	
2	
3	
4	
5	
10	
25	
100	

-------------- Individual Partner/Small Group Discussions recorded separately at the tables. (about 5 minutes)

35:31 CT: Da, da, da, da

Students: Da, da.

CT: Okay, ladies and gentlemen, eyes up here, please. Okay, we heard a lot of great conversation about this table. Can someone share with me the patterns they see? Can you help me fill this out real quick?

CT: If we double this number, plus 1, what do we have? Alexa?

Student - Alexa: 3, 5, 7, 9, 11, 51, 101.

CT: Is that right? Can someone show me what you noticed? Jala?

Student - Jala: I noticed that the output numbers are odd. It says double the numbers and then add 1, and when you plus 1 and I got 3.

CT: Good. So these numbers are odd (gesturing to the outputs) Why are these numbers odd? Can someone expand on that? Kathleen?

Student: In the table before, we were just doubling. But now with the plus one, that equals odd.

CT: So you are saying that an odd plus an odd number is even but then you add the one which gives you odd.

38:02 **Student - Jada: I noticed that if you add a even number plus an even number is even. And if you add an odd number plus an odd number is even. But if you add an even number plus an odd number is odd number.**

CT: So Jada is saying...this goes for both tables. Both tables taught us that if you add two even numbers you get an even number. When you add two odd numbers you get an odd number. And when you add an odd and even number you get an odd number. I like that. Good thinking Jada.

CT: Yes, ma'am?

38:49 **Student: You get an odd number...an odd number plus an even number because one of those odd numbers and the even number, the one is left out. So it is an odd number.**

------------ *CT makes "mind blown" gesture.*

CT: I love it. Great thinking guys! Sterling?

40:54 **Student - Sterling: I noticed, um, when I first looked at it. I saw the one and it reminded me of the 1/2 plus 1/2 is one. So one can be even.**

CT: What? Repeat that one more time Sterling.

Student - Sterling: When I first saw it, I noticed that 1/2 plus 1/2 equals one. And I know that it can be even.

CT: So you noticed that one can be an even number too because you get equal parts. Like 1/2 and 1/2. That is a very interesting thought! That is definitely thinking about fractions.

------------- CT discusses whole numbers versus fractions when looking at the definition of even and odd numbers.

CT: I love your thinking Sterling. Can you break one into two equal groups? Yes, if we use fractions. So what Sterling is saying because 1 can be broken into 1/2 and 1/2 which are equal parts it is even. But are those equal parts whole numbers? No.

Student - Alexa: I was thinking that fractions are just division.

CT: You are absolutely right. Okay.

CT: Well, ladies and gentlemen. What did we notice about odd and even numbers in a pattern like this? Can someone tell me about a pattern like this? All together?

Student - Meliana: If you are going to like 5 plus 5 will equal an even number because 10 is right after an odd number. Because it goes odd, even, odd.

CT: Okay, what about instead of adding 5 plus 5, what if I did 5 times 2?

Student - Meliana: 5 times 2 will be an even number too. It will equal 10.

CT: What if I did 25 times 2?

Student - Meliana: 25 times 2 is 50 which is still an even number.

CT: So it is still an even number.

Sterling: Whenever you count, no matter what number you start with you always go 1 - odd, 2 - even, 3 - odd, 4 - even, 5 - odd ------- *looking at number line above board in the classroom.*

CT: And why is 3 odd? Can three be divided evenly into to parts?

CT: Alright, ladies and gentlemen you did a fantastic day today. Later on today we will get back to fractions. Thank you so much!

Appendix D - Interview Protocol

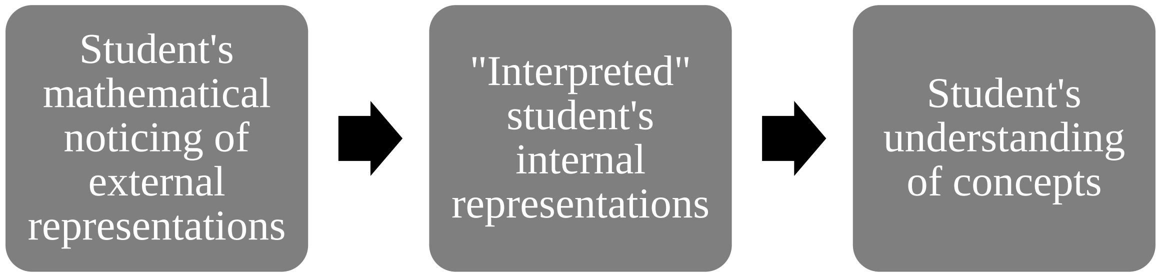
Generalizing about Even/Odds using Tables

Research Purpose:

To describe how individual students make meaning, through mathematical noticing, of the external representations within the classroom and how this meaning-making process relates to a student's understanding of the algebra concepts. This meaning-making process translates student's mathematical noticing of external representations into their individual internal representations and is based on the situational context (external representations present within the classroom) and the cultural context of the classroom. The focus is on describing individual student's visual process within one classroom. We are not necessarily studying the classroom context, just describing it and inferring how it relates to individual student's mathematical noticing.

The central goal of algebraic thinking is to get children to think about, describe, and justify what is going on in general (in other words, *generalize*) with regards to some mathematical situation. The mathematical situation or concept for which we are attending to in the study is the use of input-output tables which are a representation of a given rule. This aligns to the CCSS mathematics standard for 4gh grade (CCSS.4.0A.5). The CCSS mathematics standard for 4^{th} grade states: "Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. For example, given the rule 'Add 3' and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way."

Research Question: How do elementary student's mathematical noticing of external representations of algebraic problems relate to their (1) internal representations and (2) developed understanding of the concepts?



Student's mathematical noticing of external representations inform student's internal representations through the meaning-making process. These internal representations then inform student's correct or incorrect understanding of the concepts.

Definitions:

Mathematical noticing refers to "selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for students' attention" (Lobato, Hohensee, & Rhodenhamel, 2012, p. 438). Visual *external representations* are those that are used in structured learning environments. This may include the use of symbolic (equations, tables, graphs), pictorial (including number lines, arrays, area, sets, etcetera), and concrete (manipulatives) representations of mathematics. Visual representations express features of both mathematics constructs and actions. *Internal Representations* are students' personal "symbolization constructions," and visual and spatial imagery formed by individual students (Goldin & Shteingold, 2001). Again, the internal representations express features of both mathematics constructs and actions.

Interview Guidelines:

Student will be interviewed during a 3rd grade mathematical lesson on the standard CCSS.4.0A.5. The lesson will be taught by their normal classroom teacher. The interviews will be semi-structured (i.e. the interviews use an interview protocol to help guide the interview through the interview process but it also provides the interviewer with the ability to probe the student for additional details) and will take place during normal "Think-Pair-Shares" throughout the lesson. Students in the classroom are often asked to discuss their mathematical thinking with partners and in this case, their partners will be the interviewers, rather than their classmates.

<u>Task 1</u>: Students are shown the following table with the rule "multiplied by 2" at the front of the classroom. Students are instructed to think quietly to themselves what they notice and wonder about the numbers in the table and then share with their partner.

a. *What do you notice about the numbers in the table?* Share with a partner.
b. *What do you wonder about the numbers in the table?* Share with a partner.

Number	Number x 2
1	2
2	4
3	6
4	8
5	10

Guiding Questions during Partner Talk: (AUDIO RECORDED)

1. What did you notice about the numbers in the table?
2. How did you notice that? Could you point out to me where you noticed that pattern? What made you think to look for that? (Encourage students to draw on the table if needed.)
3. What does that make you wonder? What does that tell you about the numbers?
4. What else did you notice about the numbers in the table? How did you notice that? What does that make you wonder?
5. Continue the questions in this manner – probing students for their additional thinking.

**It may help to model to students "thinking aloud".*

Guiding Questions during Whole Group Discussion: (SWIVL VIDEO RECORDED)

a. What patterns did you find? How did you find the patterns? (All numbers on the right are even. When the number on the left increases by 1, the number on the right increases by 2.) (Use arrows to point out student's patterns - Point out both the horizontal and vertical patterns.)
b. Can we add any number on the first column and generate/produce a number in the second column?
c. What if we are given a number in the second column, can we find the number in the right column?
d. When students point out that all the numbers on the left being even. Have students state a claim about numbers being multiplied by 2.
e. *IF WE HAVE TIME: Students may use pictorial or concrete justification. Give students two-colored chips and/or square tiles for a concrete justification.*

i. *Does an even number multiplied by 2 result in an even or odd number? Why do you think this is? Use the manipulatives or draw a picture.*
ii. *Does an odd number multiplied by 2 result in an even or odd number? Why do you think this is? Use the manipulatives or draw a picture.*

<u>Task 2</u>: Students are given the following empty table and are instructed to fill out the table on their own. Give students time to complete the table, and review for accuracy, if needed. Then students are instructed to silently write all their observations on the handout and then have them share with their groups.

Number	Double the number, plus 1
1	
2	
3	
4	
5	
10	
25	
100	

Guiding Questions during Small Group Discussion: (AUDIO RECORDED)

1. What did you notice about the numbers in the table?
2. Point out to me where you see your pattern? What made you think to look for that?
3. Will that pattern continue? How do you know?
4. What else did you notice about the numbers in the table? How did you notice that? Will that pattern continue? Explain.
5. Continue the questions in this manner – probing students for their additional thinking.
6. If students bring up that all the numbers on the right are odd (similar to the previous table when all numbers on the left were even):
    a. Does an even number plus 1 result in an even or odd number? Why do you think this is? Use the manipulatives or draw a picture.
    b. Does an odd number plus 1 result in an even or odd number? Why do you think this is? Use the manipulatives or draw a picture.
    c. Explain why the numbers you entered in the table are all odd. How does your picture show this property?
7. How can we generalize what we saw in words and using a number sentence?

Appendix E - Small Group Interview Transcripts and Student Work

Group 1 (Scott): Jada, Sterling, Liliana

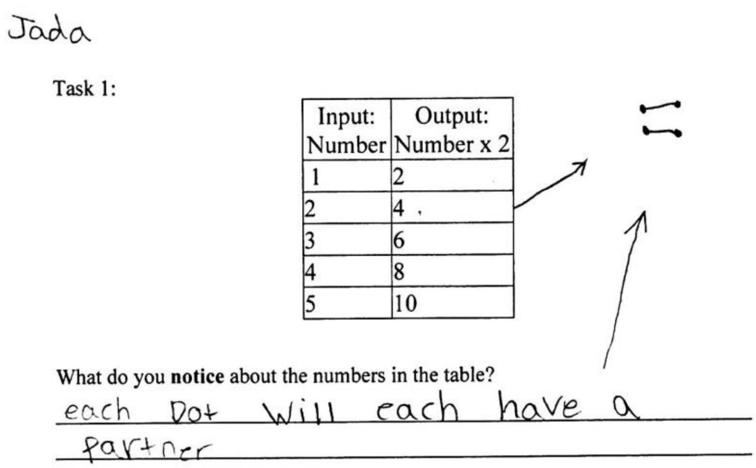
Group 2 (Jenny): Jazzy, Hailey, DJ

Group 3 (Julie): Phillip, Stephanie, Madison

Group 4 (Sherri): Brianna, Julian, Colton

Group 1: Jada, Sterling, Liliana

Task 1: Notice and Wonder (Evens)


Jada

Task 1:

Input: Number	Output: Number x 2
1	2
2	4
3	6
4	8
5	10

What do you **notice** about the numbers in the table?

each Dot Will each have a partner

(Sterling and Lilianna did not write anything down for Task 1)

Teacher: Jada, what did you notice?

Sterling: I noticed that the numbers go by one (inputs) and that these numbers are doubled.

Jada: I noticed...This numbers are doubled.

Teacher: Okay, how did you come to that? How did you get that idea? What did you notice?

Teacher: Anything else did you notice? Anything that you wonder?

Jada: They are getting bigger. The next one will be 12. (after 10).

Teacher: So what do you think the rule is?

Sterling: It's times 2! 1 times 2 is 2, 2 times 2 is 4.

Jada: This one is counting by twos.

Teacher: How did you notice that? Say that again.

Jada: I saw that one was counting by twos and that one is counting up by ones.

Teacher: Keep talking, I know that you are doing good. I like that you noticed that they are counting by twos going down (output). Anything else that you noticed?

Task 2: Fill in the Table (Odds)

Jada

Fill in the table below.

Number	Double the number, plus 1
1	1+1=2 2+1=3
2	2+2=4 4+1=5
3	3+3=6 6+1=7
4	4+4=8 8+1=9
5	5+5=10 10+1=11
10	10+10=20 20+1=21
25	25+25=50 50+1=51
100	100+100=200 200+1=201

What do you notice about the numbers you entered in the table? Record your observations silently in the space below.

I Noticed that the output Numbers are all odd I Noticed because it Says Double the number. So I did 1+1=2 then it Says Plus 1 So then I did 2+1=3 so 3 was My output number.

Liliana

Fill in the table below.

Number	Double the number, plus 1	
1	odd 2+1=3	1+1=2 even
2	4+1=5	2+2=4
3	6+1=7	3+3=6
4	8+1=9	4+4=8
5	10+1=11	5+5=10
10	20+1=21	10+10=20
25	50+1=51	25+25=50
100	200+1=201	100+100=200

What do you notice about the numbers you entered in the table? Record your observations silently in the space below.

I notice because 1+1=2 then you add 2+1=3 because odd + even = odd number so one side is even and one side is odd.

Steling Sterling

Fill in the table below.

1 half

Number	Double the number, plus 1		
1	2	3	odd
2	4	5	odd
3	6	7	odd
4	/	9	odd
5	10	11	odd
10	20	21	
25	50	51	
100	200	201	

What do you notice about the numbers you entered in the table? Record your observations silently in the space below.

4 can be even because you can [illegible] a half [illegible]
= 1.

Task 2: Fill in the Table (Odds)

Jada: It's counting wrong (pointing at input column).

Teacher: What do you mean it's counting wrong?

Jada: It's going 1, 2, 3, 4, 5, and then 10, 25, 100.

Teacher: Oh. Did anyone else notice that?

Lilliana: They forget the 6, 7, 8, 9. Also, they are doing even numbers here.

Teacher: Jada, what did you just notice?

Jada: It says double the number and then add one. So the first one would be 2 and then add one so it will be 3.

Teacher: Do you agree with that Sterling? Did you notice that pattern? Because you have 2 on yours?

Jada: It says double the number so I got 2 and then it says plus one and I got 3.

Sterling: Ohh!

Teacher: Don't erase. Just add another column.

------ students working --------

Teacher: What do you notice sterling?

Sterling: That all of these numbers are odd (outputs).

Teacher: How did you notice that?

Sterling: Because I was doing what Jada did from 3. And then adding one, I figured out that it was odd.

Teacher: It looks like you were able to go down that column pretty easy. Why is that?

Sterling:....

Teacher: What do you think? Did you notice anything else Lilliana?

Lilliana: That when you do 2 plus 2 it is even but then when you do 4 plus one it is odd.

Teacher: So when you just double the number, it is even. So it's like the other one (task 1).

Lilliana: Yes. And then the rest will be odd.

Teacher: Why?

Lilliana: Because 1 is an odd number. And all of these numbers are even (doubling) And odd plus even is odd.

Teacher: That's interesting. When did you notice that they were all going to be odd?

Jada: When I....

Lilliana: Just like what I said, when it was 4 plus odd is odd.

Teacher: What is different from this chart and the last one? What makes this numbers here (outputs on Task 1) even?

Sterling: Because it is the same number. You can add 16 plus 16 is 32 but you can't add 16 to 17, it's not going to be the same number. You could do that. But it is has to be the same number.

Teacher: So in order to double it, you have to add the same number.

Sterling: Yah. Two of the same number equals this (outputs of Task 1).

Teacher: So can you generalize that in another way? Can you pick me another number.

Jada: 200.

Teacher. 200. So if I follow this rule, should it give me

Sterling: 401.

Teacher: So is it even or odd?

Sterling: Odd.

Teacher: What if I do 201? 201 is odd so is it going to come out even?

Sterling: No, it's going to be odd because one is an odd number.

Lilliana: 201 plus 201 is 402 plus 1 is 403.

Teacher: So because odd plus even gives me an odd.

Sterling: Do you want us to write it down?

Teacher: You can.

------students writing----

Sterling: Any number can be even. You have to think about it!....Like 25 is an even number.

Teacher: Why is 25 an even number?

Sterling: 25 could be an even number because you can do 12 and half plus 12 and half.

Teacher: Oh, so if I can add the same thing it must be even. So one is even because of 1/2 plus 1/2.

Sterling: Yeah.

Teacher: Cool. That's interesting.

Teacher: What do you guys think about Sterling's rule. Could one be even?

Lilliana: No, it's odd. It starts at one and then two is even and 3 is odd.

Teacher: Okay, so you are saying that an even number always comes after an odd number.

Lilliana: Yah.

Teacher: Interesting. Is 3 even or odd?

Sterling: It could be both! Because 1 1/2 plus 1 1/2 is 3. But it can be an odd number also because if you are not going to use the 1/2 it would be odd.

Sterling: Did I just learn something?

Teacher: Is that surprising?

Sterling: No, but I would be surprised if I taught something.

Sterling: So any number can be even and odd.

Teacher: Liliana doesn't agree with you. Every time you say that she shakes her head.

Sterling: **Just think about it!** half plus half equals 1.

Teacher: So what part of the rule are you doing?

Lilliana: If you do double number it is even. But 1 plus 2 equals an odd number.

Teacher: So any number can be doubled? So then it is even?

-----CT gathers students back together ----

Group 2: Jaslene (Jazzy), Hailey, DJ

Task 1: Notice and Wonder (Evens)

Haily

Task 1:

Input: Number	Output: Number x 2
1	2
2	4
3	6
4	8
5	10

What do you **notice** about the numbers in the table?

What I notice about the numbers in thes table is it is conting By 2 and the other one is nots they conting By 1. of the output Numbes aare all even and the Input Number are odde and even. add todd= even. the output Numbers are counting By 2 that is whcy thay are not the Same from 2X2 So that is whay thay Did that.

Jaslene

Task 1:

Input: Number	Output: Number x 2
1	2
2	4
3	6
4	8
5	10

What do you **notice** about the numbers in the table?

I see that the input numbers are half of the output numbers. all the outputs are even. the inputs go odd even. the output count by 2twos

Input: Number	Output: Number x 2
1	2
2	4
3	6
4	8
5	10

What do you **notice** about the numbers in the table?

1+1=2
2+2=4
3+3=6
4+4=8
5+5=10
the input numbers = output number
x 2. Input numbers

Teacher: You can start writing down what you notice.

------- *Students writing*

01:04 Teacher: So we are going to now share your notice and wonders in a group. We are going to talk together.

DJ: I noticed that the input numbers are half of the output numbers times 2.

Teacher: How did you notice that?

DJ: Because 1+1 is 2. 2+2 is 4. 3+3 is 6. 4+4 is 8. 5+5 is 10.

Teacher: So that means the same things as half this number (outputs) is this (input).

DJ: Yes, 1x2 is 2. 2x2 is 4. 3x3 is 6. 4x (4...)x2 is 8. 5x2 is 10.

Teacher. Did you guys notice anything different?

Hailey: I did! This one (output) the pattern is we are counting by 2's .

Jaslene: This one (output) goes 2, 4, 6, 8, 10. This one (input) goes odd, even, odd, even, odd.

Teacher: So the outputs you are counting by 2's. What are you counting by for the inputs?

Jaslene & Haley together: Ones. 1, 2, 3, 4, 5.

DJ: 1, 2, 3, 4, 5. 2, 4, 6, 8, 10. 10, 8, 6, 4, 2. 5, 4, 3, 2, 1.

Teacher: And what else did you notice about the even and odds?

DJ: Only the output numbers have odds.

Teacher: Oh, did anyone else notice that?

Jaslene: Me and Hailey already talked about that.

Teacher: Point that out to me. ------*Students pointed out the different patterns. Chanting the odd/even pattern in input and output.*

Teacher: That's cool. Why do you think that is. How do you explain that?

Jaslene: Because when you add an odd number.....

Hailey: Because when you count by 2....they are all even.

Jaslene: Because an odd number plus an odd number is an even number.

Teacher: Write that down. That is a good observation.

DJ: Because if you do 6 + 6 you get 12. That's even.

Teacher: Yeah.

------------Students begin writing more observations.

Task 2: Complete the Table (Odds)

Jaslene

Fill in the table below.

Number	Double the number, plus 1
1	3
2	5
3	7
4	9
5	11
10	21
25	51
100	201

25

What do you notice about the numbers you entered in the table? Record your observations silently in the space below. Me Jaslene

the Doubled numbers are all odd. the numbers go odd even until

DJ

Fill in the table below.

Number	Double the number, plus 1
1	3
2	5
3	7
4	9
5	11
10	21
25	51
100	201

Hailey

Fill in the table below.

Number	Double the number, plus 1
1	3
2	5
3	7
4	9
5	11
10	21
25	51
100	201

What do you notice about the numbers you entered in the table? Record your observations silently in the space below.

What I think the Double number that are couting by odd. Me and Jazzy also notes the Numbers aso are oddeven odd even. what we saw was 1+1=3 then for the Numbers is 3. odd + even = odd 1+1=2

Jaslene: It's counting by 3's too!

Hailey: No because after 3 would be 6.

.....

Jaslene: I don't know what 25 doubled.

Teacher: What if you think of 25 as a quarter. If you have 2 quarters how many cents do you have.

Jaslene: 50!

Teacher: Good so you just doubled 25.

Teacher: Good job guys. Thank you for working hard.

Teacher: While DJ is finishing filling out the table, I want you to be really quiet and record what you notice about the table. DJ is working really hard so be quiet.

.....

Teacher: Okay, let's come together now. What are some things that you noticed in your table?

Hailey: I don't like focusing on the table, but I like the writing. Writing is fun.

Teacher: Well, that is why we need to always have some writing in math. Okay, so let's talk about it.

Jaslene: Me and Hailey noticed that the "Double the number, plus 1" are all odd (outputs).

Teacher: All of these guys are odd. Did you notice that DJ?

Jaslene: And we notice that these (input) go odd, even, odd, even, odd.

Teacher: What is something else that you noticed?

Haley: We also noticed that this is the same (referring to task 1 table) because we could have just gone 1x2 is 2 and then we add 1.

Jaslene: Oh...I just noticed something! That the 1 (input) equals the 3 (output) and there is a 3 here (input)! And that the 2 equals the 5 and there is a 5 here! Cool. ------- Hailey is using her fingers to bounce between the rows and columns.

Teacher: Cool. How did you notice that?

Jaslene: Me and Hailey were just talking about the odd and evens and that they were all odd here (outputs) and then I just looked at and I noticed that these matched and I wanted to see if they all do the same. And then it goes like that.

Hailey: I don't think I can write all this down. I'm forgetting it.

Teacher: That's okay. We can just talk about it. I am recording everything.

Hailey and Jaslene: You are!? How? Cool. Hi!!!!

Teacher: So DJ, do you notice anything?

DJ: Every time they are adding up by 2 (outputs) until it gets to 21.

Teacher: Why does that happen? I noticed that too - there's a jump. You really are on to something.

DJ: Maybe so that we wouldn't have all easy numbers to figure out.

Teacher: So you think I wanted you to find bigger numbers.

Hailey: It skips from 5 to 10 and then 10 to 25.

CT: What's different from this pattern from the last one? Because the last one the outputs were all even.

Hailey: Yeah, and this one, they are all odd.

Teacher: Why is that?

Hailey: The ones (inputs) are all the same. But these are all even (task 1) and these are all odds (task 2)

CT: What makes them different?

Teacher: So you noticed in this table, all the outputs are even (task 1) but in this table, all of the outputs are odd (task 2). What changed?

DJ: Maybe....when we added the plus 1 all of the numbers are odd.

Teacher: Okay, so you are saying the difference between this table and this one is we are adding the one. So do you think the plus 1 is what makes it odd?

Hailey: Maybe.

Jaslene: I agree with DJ.

Teacher: I still don't understand what makes adding one makes it odd.

DJ: Because odd + odd makes it even and when you add the one, it makes it even + odd which makes it odd. Like 7 + 6 is odd.

Group 3: Phillip, Stephanie, Madison

Task 1: Notice and Wonder (Evens)

Input: Number	Output: Number x 2
1 o	2
2 i	4
3 o	6
4 i	8
5 o	10

What do you **notice** about the numbers in the table?

1+1=2

2+2=4

3+3=6

4+4=8

5+5=10

the input numbers = output number x 2. Input numbers

Stephanie

Task 1: Philip

Input: Number	Output: Number x 2
1 0	2 e
2 e	4 e
3 0	6 e
4 e	8 e
5 0	10 e
6 e	12 e
7	14 e

What do you **notice** about the numbers in the table?

I see that the input number is doubling to equal the output number and then the output number is transfering to the input number

Madison group 1

Task 1:

Input: Number	Output: Number x 2
1 odd	2 even
2 even	4 even
3 odd	6 even
4 even	8 even
5 odd	10 even

What do you **notice** about the numbers in the table?

I see they are counting by 2 and by ones
and I see 1+1=2, 1+1=2 4
2+2=4
3+3=6
4+4=8
5+5=10 and when you double the input
number you get the output number
the input goes even odd and the
output number goes even even

Teacher: Write down what you see. For one minute.

-----students writing ------

Teacher: Okay, so now we are going to talk about what we see.

Stephanie: So what I noticed about these numbers, the 2 in the output is transferred to the input number?

Teacher: Does that happen for every number?

Philip: No, it skips 3....so every other one.

Teacher: How did you notice that?

Philip: I noticed that. Because 1 and 1 is 2. 2 and 2 is 4. 3 and 3 is 6. 5 and 5 is 10.

Teacher: What does that say?

Phillip: Doubling. You are doubling the numbers.

Teacher: Okay. Did you notice anything else about the numbers?

Philip: This number is counting by one. The input numbers are counting by ones and then output numbers are counting by two.

Teacher: Why do you think the output numbers are counting up by two?

Philip: Because all of these numbers are just counting by two. 1 times 2, 2 times 2, 3 times 2, 4 times 2, 5 times 2.

Teacher: Did you notice any other patterns?

Philip: Nope.

Teacher: Okay. What did others notice? What about any wonders. So you were talking about where the six would be?

Philip (extending his table): So this would be a twelve and here would be the six.

Phillip: I don't wonder anything else. Wait, can I extend the table?

Teacher: So if I give you any input number can you figure out the output.

Philip: Yeah, I can do that. I know math.

Teacher: So if I give you 20, what is the output?

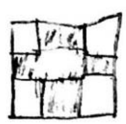
Philip: 40.

Teacher: How did you get that?

Phillip: I just times 2.

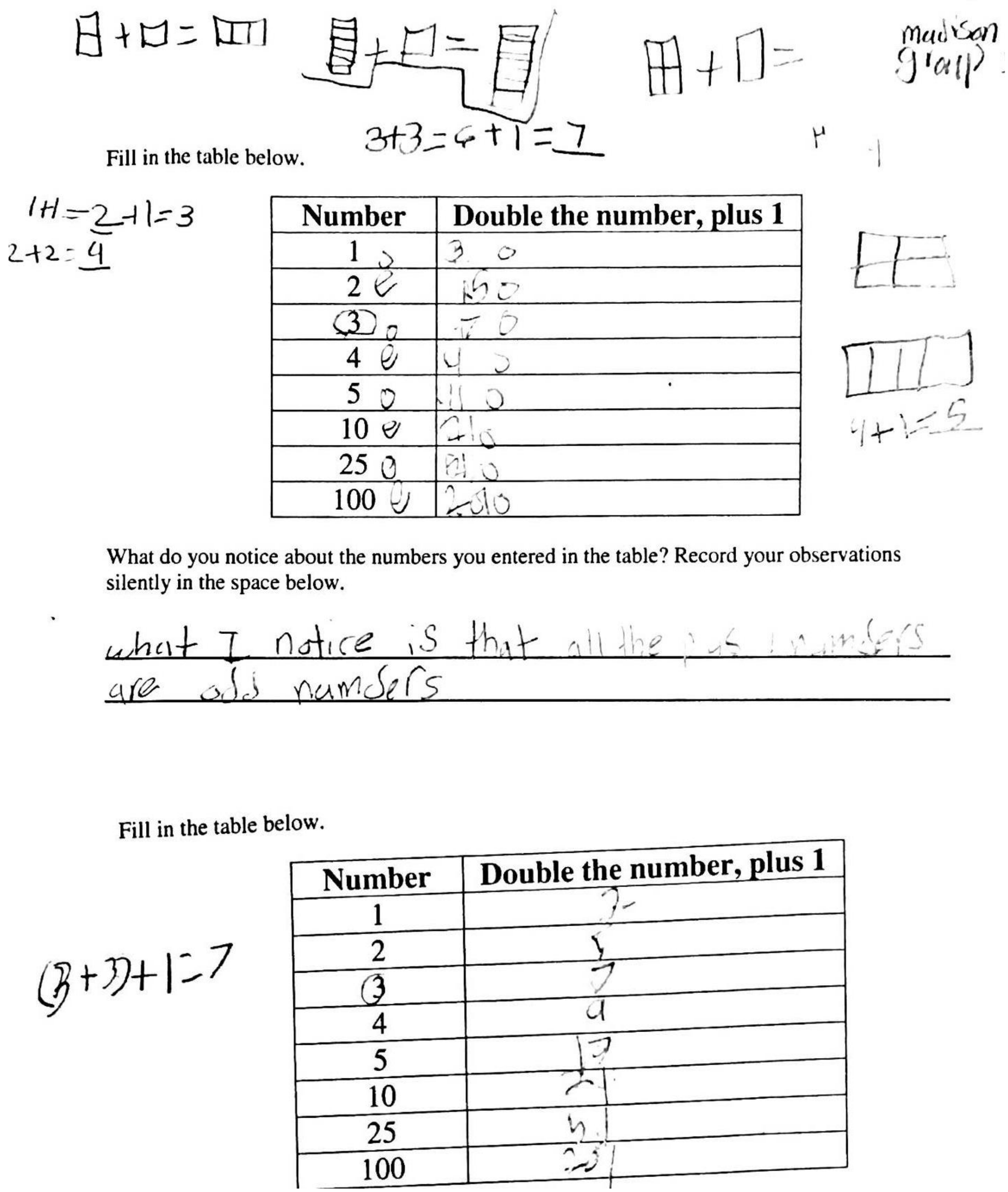
Task 2: Fill in the Table (Odds)

1 Phillip



Fill in the table below.

Number	Double the number, plus 1
1	3
2	5
3	7
4	9
5	11
10	21
25	51
100	201



madison graph

3+3=6+1=7

Fill in the table below.

1+1=2+1=3
2+2=4

Number	Double the number, plus 1
1	3
2	5
3	7
4	9
5	11
10	21
25	51
100	201

4+1=5

What do you notice about the numbers you entered in the table? Record your observations silently in the space below.

what I notice is that all the plus 1 numbers are odd numbers

Fill in the table below.

(3+3)+1=7

Number	Double the number, plus 1
1	3
2	5
3	7
4	9
5	11
10	2[illegible]
25	5[illegible]
100	20[illegible]

Stephanie

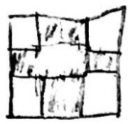
Teacher: This one is a little bit more tricky isn't it. It says double the number and then plus one.

Madison: They are all odd. (looking at outputs)

Teacher: How did you notice that?

Madison: 1 + 1 is 2 and then add another one it's three.

Phillip: Wait, I notice something. Odd, even, odd, even, odd (looking at inputs). So you know how you can make squares and color them in pattern. Yeah, it's like that.



Teacher: While he's drawing that. Stephanie, what made you notice that pattern?

Stephanie: I thought of to make the pattern. Because I thought I didn't understand it because I didn't understand the plus one.

Teacher: So how did you figure out the plus one?

Stephanie: I figured it out that 1 + 1 is two and then plus one meant to add one to the answer you got.

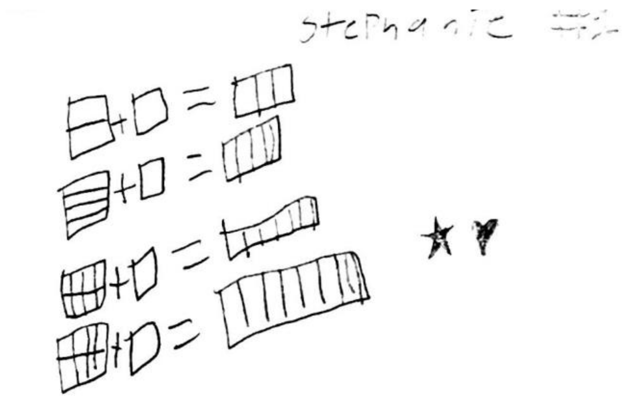
Teacher: Oh, okay. Because yeah, then it would make the second number different.

Teacher: You guys said that this (the rule) always gives us an odd number. Why does it do that?

Stephanie: Because you have an even number and then when you add an odd number you get an odd number.

Teacher: How did you figure that out?

Stephanie: When you add 1 and 1 it gives you 2 because it says to double and then when you add one it goes to 3.



Teacher: So what would that look like for the next number?

-----Steph drawing-----

Stephanie: We have the 4 here.

Teacher: And how did you get from the two to the four?

Stephanie: 4 can be half. I got the two from the four because 2 plus 2 is four and then I add the 1 and I get 5.

Teacher: So if I get any number, could we figure it out?

Philip: But Steph, I have a question. I don't understand how half equals 3.

Stephanie: The half....I mean the denominator is 2. Two plus one equals 3 and the two can be divided into two.

Teacher: So you got the half from dividing the two square here.

Steph: Yeah.

Teacher: So then what would the next number be?

Madison: I did 3 plus 3 is 6 but then it says to plus one so I added that one to get 7.

Teacher: So does an even number plus one always result in an odd number?

Phillip: Odd.

Teacher: You all think odd? Why do you think that? Can you justify it for me?

Stephanie: I did 3 + 3 in parenthesis and then 1.

$$(3+3)+1=7$$

Teacher: Why did you use the parenthesis?

Stephanie: It is easier to see the even part.

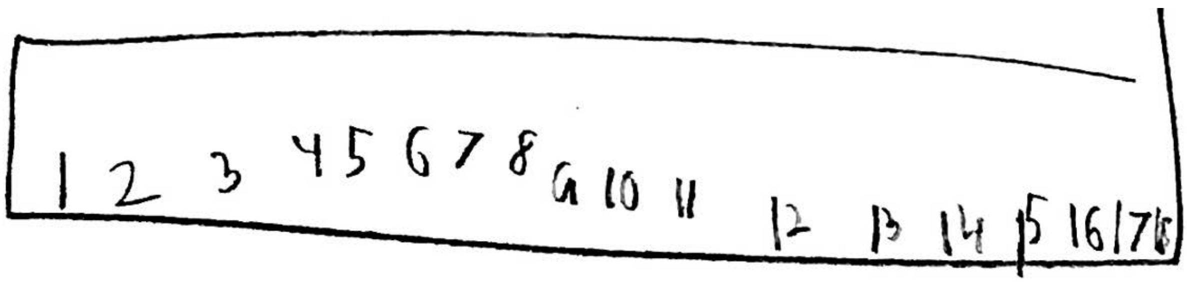
Phillip: If I start with 10 squares and then add one it equals 11.

Teacher: So does that work for every even number. If you go one more, then it will be odd?

Students: Yes!

Teacher: Okay, I have another question. If we do an odd number plus one then it will be even.

Pointing to the number line in the classroom.



Madison: So the next number is even.

Group 4: Brianna, Julian, Colton

Task 1: Notice and Wonder (Evens)

Brianna

Gr. 2

1 2

Task 1:

Input: Number			Output: Number x 2
1	6	11	2
2	7	12	4
3	8	13	6
4	9	14	8
5	10	15	10

What do you **notice** about the numbers in the table?

1+1=2 2+2=4 3+3=6

4+4=8 5+5=10 and

2-1=1 2-4=2 3-6=3 4-8=4

5-10=5

Task 1:

Input: Number	Output: Number x 2
1	2
2	4
3	6
4	8
5	10

Kolton
Group 2

What do you **notice** about the numbers in the table?

they are doing the rule X2
and on one side its rule is +1
and on the other its rule is +2

Task 1:

Input: Number	Output: Number x 2
1	2
2	4
3	6
4	8
5	10

What do you **notice** about the numbers in the table?

The numbers are dubling them selfs

-----Students Writing-------

Teacher: So tell me about what you notice.

Brianna: Right here (input column) you are counting by ones. Right here you go backwards and go...right here you can add 5 add 5 is 10 and 4 and 4 is 8 and 3 and 3 equals 6. and 2 and 2 equals 4 and 1 and 1 equals 2.

Teacher: So you are saying that this first column in the input is going by ones. And then when you look for the output to the second column, you are just taking this number (input) and adding it to itself. Is that what I am saying?

Brianna: Hm, hm.

Teacher: Anything else that you notice?

Brianna: Um...

Teacher: So it looks like you noticed a pattern down the table. And you noticed the pattern across. Anything else you noticed?

Brianna: Um...

Teacher: What about the numbers in this column (outputs) do you notice anything about these numbers.

Brianna: two, four, six, ...

Teacher: You noticed that it is going by twos.

Brianna: yah.

Teacher: Anything else about those numbers? Anything you wondered about? You have so extra numbers you filled in the middle here?

Teacher: What about you guys? Did you notice anything else?

Julian: The numbers are doubling.

Kolton: The one side it is counting by ones. And the other side is counting by two. Across the rules is times 2.

Teacher: Ohh, what did you say the rule was? (to Brianna)

Brianna: That 2 and 2 is 4. 3 and 3 is 6. ...And here 2 subtract 1 equals 1. 2 subtract 4 is 2. 3 subtract 6 equals 3...4 subtract 8 equals 4.

Julian: That's impossible.8 subtract 4.

Teacher: So listen to what he noticed.

Kolton: I noticed that it is just timesing by 2.

Brianna: He is doing the same thing as what I was doing. But it is different.

Teacher: Is it different? Or is it the same? She is doing 2 and 2 is 4. 3 and 3 is 6. He is saying 2 times 3 is 6.

Brianna: I'm adding two. Oh wait.

Teacher: You are not adding two when you are going this direction...across.

Julian: Adding the same thing to the same thing is times 2.

Teacher: Okay.

Task 2: Complete the Table (Odds)

Brianna
Gr. 2

Fill in the table below.

Number	Double the number, plus 1
1	3
2	5
3	7
4	9
5	11
10	21
25	31
100	201

1000

... entered in the table? Record your observations

Jollan
Gr. 2

Fill in the table below.

Number	Double the number, plus 1
1 o	1+1 = 2+1 = 3
2 e	2+2 = 4+1 = 5
3 o	3+3 = 6+1 = 7
4 e	4+4 = 8+1 = 9
5 o	5+5 = 10+1 = 11
10 e	10+10 = 20+1 = 21
25 o	25+25 = 50+1 = 51
100 e	100+100 = 200+1 = 201

What do you notice about the numbers you entered in the table? Record your observations silently in the space below.

I notisd that the ones in the 1st colom is even od and the ones in the 2nd colom are All od

Kolton
Gr. 2

Fill in the table below.

Number	Double the number, plus 1
1	3
2	5
3	7
4	9
5	11
10	21
25	51
100	201

What do you notice about the numbers you entered in the table? Record your observations silently in the space below.

I notice that you do x2 +1 evry number

Teacher: Fill in the table.

Kolton: 2, 3,

Teacher: What does it say here?

Kolton: Plus 1.

Teacher: What does it say?

Kolton Double the number, plus 1. So 2, 4, 6, 8, 10,...

Teacher: So if you take 1 and you double it, what do you get?

Kolton: 2

Teacher: And if you plus one, what do you get?

Kolton: 3.

Teacher: Okay. What if you have 2.

Kolton: So 4. Now, 5.

---------Students working on the table-------

Brianna: I'm done.

Teacher (to Brianna): Let's take a look. If you have 10 and then double it, what do you get?

Brianna: 11.

Teacher: What does double mean?

Brianna: ....

Teacher: What do you notice about the number in the first column?

Julian: Ahhh....They are all even.

Teacher: In the first column?

Julian: No.

Students: They're all odd!

Teacher: Yes, in the second column. What about the numbers in the first column?

Julian: They go odd/even/odd/even/odd.

Teacher: Okay.

Kolton: There's no six though. (referring to the first column)

Teacher: So what is it doing?

Teacher (to Brianna): If you have 10 and you doubled it.

Brianna: um....

Teacher: What did you do to 5 when you doubled it?

Brianna: I don't know.

Teacher: How do you double a number.

Brianna: 3 and 2 equals 5.

Teacher: Correct. I think you are not remembering what it means to double a number.

Brianna: Did I get 5 wrong?

Teacher: No, you don't have them wrong. But when I listen to you explain it, your explanation doesn't match your numbers.

Brianna: Did I get 10 wrong?

Teacher: Yes, you were doing great until you got to 5 and then the pattern was changing. So it's starting to trick you.

------Students working-----

Teacher: Julian, what are you noticing? How are you getting the numbers over here?

Julian: I'm adding the number plus the number and then I added one. So 2 + 2 = 4 plus 1 equals 5. 3 + 3 equals 6 plus 1 is 7.

Teacher: You see how he is doing that? (to Brianna) So how would you double 10.

Brianna: 10 plus 10.

Teacher: So how would you double 25?

------ Brianna working ------

Teacher to Kolton: How about you? I haven't heard from you yet. How did you get your numbers?

Kolton: Um, I just did what they said up here.

Teacher: Okay, so how did you do that. For example 5. What did you do with the five.

Kolton: Added 5 to 5 and made 10. And then it said plus 1 so added 10 plus 1.

Teacher: Okay. I have a question. What do you notice about the numbers in this column? (right)

Kolton: They're all odd.

Teacher: Why do you think that happens? That they are all odd?

Brianna: 6 is even. and 5 is odd. And odd added even makes more odd.

Teacher: Why did you pick 6 and 5?

Brianna: An even and an odd.

Teacher: Where in your table did you do 6 plus 5.

Brianna: Nowhere.

Teacher: Did you do 6 + 5? (to Julian)

Julian: Yes. Here (pointing at the input 5).

Teacher (to Brianna): Do you agree with Julian? He says that it is here that you are doing 6+5? Do you agree?

Brianna: Because you are doing 5+5 and that equals 10 but you have to add one more and double it. But that's not equal.

Teacher: Do you notice it now? Who got your idea started? You started with the 6 plus 5. But how did you notice that?

Brianna: So me!

Teacher: I do think that it started with you. I think that you saw a pattern here. What's the pattern here.

Brianna: Every number here adding is going to be even but then you add one more so it is going to be odd.

Teacher: So let's pick a number 3.

Brianna: So 6 but then you add one more which is 7. **And 7 is not equal so it is odd.**

Teacher: So when you double it you have an even number but when you add one it is odd.

Brianna: Because even plus odd becomes more odd.

Teacher: Isn't that interesting. I wonder if that always happens.

Julian: It does.

Teacher: So let's say you have 1000. What would be "double the number plus 1"

Brianna: That would be 2001.

Teacher: Wow. That's interesting.

CT: What about million?

Julian: 2 million and one!

Brianna: Uh ah.

CT: Okay, I think you figured out the pattern. So what's different between this patter (Task 1) and this pattern (Task 2).

Julian: This one goes all even and this one goes all odd.

CT: Why are these ones all even?

Julian: Because they don't add one. Because an even number plus an even number is even but even plus odd is odd.

Appendix F - Follow-Up Individual Interview Transcripts

INPUT **Number**	OUTPUT **Double the Number, Plus 1**
1	3
2	5
3	7
4	9
5	11
6	13

Stephanie

Me: Hi Stephanie, do you remember what we did last week when I was in your classroom?

Stephanie: Yes, we talked about even and odd numbers.

Me: Good. So today we are going to look at that problem again and I want to ask you a few more questions.

Stephanie: Okay.

Me: So here is your table that you filled out in class. What do you notice in the table. Can you talk me through everything that you see.

Stephanie: I noticed that these numbers (input) count by ones but skip here (at 5).

Me: Okay, what else do you notice?

Stephanie: I noticed that these numbers (outputs) are all odd.

Me: You said that the inputs had a pattern where you would add 1, what is the pattern in the outputs?

Stephanie: The pattern is counting by twos.

Me: So you could add two.

Stephanie: Yeah.

Me: Good. What else do you noticed?

Stephanie: To get 3 here, I add 1 + 1 and then add one more.

Me: What made you notice that pattern?

Stephanie: I figured it out that 1 + 1 is two and then plus one meant to add one to the answer you got.

Me: Oh, okay. So earlier you said that we skipped 6. Could we use six to find what corresponding number would be here?

Stephanie: Yeah.

Me: Can you try that.

Stephanie: It’s 13.

Me: How did you get that?

Stephanie: I would just double 6, so 6 + 6 = 12 and then add one more so 13.

Me: That's good. Could we do that for any number…any input number?

Stephanie: Yeah.

Me: How about 1000?

Stephanie: It would be….2001.

Me: Good!

Me: So earlier, you said that this (the rule) always gives us an odd number. Why does it do that?

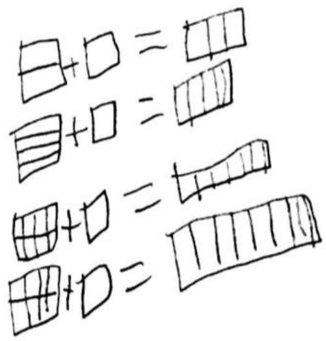
Stephanie: Because you have an even number and then when you add an odd number you get an odd number.

Me: How did you figure that out?

Stephanie: When you add 1 and 1 it gives you 2 because it says to double and then when you add one it goes to 3.

Me: So what would that look like for the next number?

-----Steph drawing-----



Stephanie: We have the 4 here.

Me: And how did you get from the two to the four?

Stephanie: 4 can be half. I got the two from the four because 2 plus 2 is four and then I add the 1 and I get 5.

Me: So you got the half from dividing the two square here.

Steph: Yeah.

Me: Could you show me for the next number.

Stephanie: I did 3 + 3 in parenthesis and then 1.

$$(3+3)+1=7$$

Me: Why did you use the parenthesis?

Stephanie: It is easier to see the even part.

Me: That makes sense. So…if I give you any input number, could we figure out the output?

Stephanie: Yeah.

Me: And will it always be and odd number?

Stephanie: Yeah because an even number plus one always result in an odd number?

Me: So you are saying when you double it will be even and then when you add one it will always be odd?

----END-----ran out of time.

Jazzy

Me: Hi Jasmine, do you remember what we did last week when I was in your classroom?

Me: Good. So today we are going to look at that problem again and I want to ask you a few more questions.

Jaz: Ok.

Me: So you guys filled out this table if you remember, but here I have the same table already filled out for you. Um, so I want to ask you again to look at the table and see those relationships that you remember from class. And as you are looking just keep talking about everything that you see.

Jaz: I see that 1 + 2 equals 3. 2 + 3 equals 5, 3 + 4 equals 7, 4 + 5 equals 9, 5 + 6 equals 11.

Me: Okay, did you see a pattern? Are you looking across to figure that out?

Jaz: Yeah.

Me: So to figure out for six, how would you do that?

Jaz: Two more than 11 is 13. So I just think what two numbers give me 11. So 6 + 5 equals 11.

Me: Okay, so first you knew that the number after 11 in the output column was 13 and you just needed to find how to make 13 from six.

Jaz: yeah.

Me: Okay, so these are increasing by one, and these are increasing by two.

Jaz: I also noticed that these were all odd numbers. 1 is odd, 2 is even, 3 is odd….

Me: Okay good. So these (inputs) alternate odd/even but these (outputs) are all odd.

Me: So in class, we talked a little bit about how we know that. How do you we know that these are all odd.

Jaz: Because we are doubling the number to get an even number and then adding an odd number, which is one and you get an odd number.

Me: Very cool. So no matter what number we start with here (input) we double the number and we will always get an even.

Jaz: mmhmm.

Me: Okay. Then adding one gets you an odd?

Jaz. Yeah.

Me: So could you explain that more to me. Why is that true?

Jaz: Because 1 is not an even number but two is. So you have to skip count by two to get another even number.

Me: Could you draw a picture to help me see what you are trying to do?

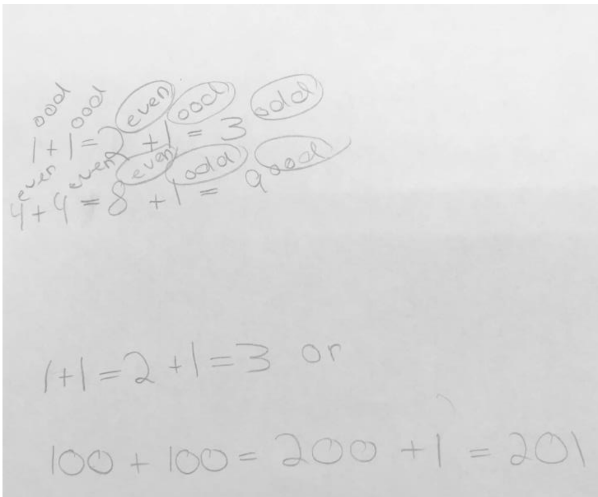
-----drawing----

Jaz: 1 plus 1 is 2 which is even and add one and it's odd.

Me: Does that work for all numbers. Could I give you 4?

Jaz: So it’s even + even is 8 which is even. And add 1 which is odd and then 9 is odd.

Jaz: So these stay the same (circles even/odd/odd)



Me: So these depend on whether I give you an odd or even (the first two numbers). Will this (doubled number) always be even.

Jaz: Yeah.

Me: So do you see a pattern.

Jaz: Yeah.

Me: Could you do it with a bigger number? Like 100? Will the output be odd or even.

Jaz: It will be odd.

----draws on the paper 100 + 100 = 200 + 1 = 201

Me: Looks good.

Me: So we have been looking at general rules that work for all numbers. Like the commutative property and distributive property are general rules in math. Is there a general rule that you can state from this table.

Jaz: Probably.

Me: So these tables help us organize the rule for us.

Jaz: …..

Me: So what are you thinking about?

Jaz: I'm thinking if you double the number you get 2 and then you add 1 and you got 3. We could do that for any of these numbers.

Me: So is there an equation you could write that would work for any number?

Jaz: Um…I'm a little too tired.

Me: Okay. That is okay. We have had a long interview. You did such a great job!

Jada (Struggling Learner)

Me: If you remember we filled in a table like this. Now, let's look at this table again. And as you are looking at the table, just tell me what you notice about the numbers and the table.

Jada: You added by one (inputs). And you added by 2 (outputs)

Me: Good. So what were you looking at?

Jada: (pointing to across).

Me: Oh so you saw that you can add 1 and 2 across to give you 3.

Jada: Yeah.

…..

Me: Can I ask what you are looking at now?

Jada: *Points to the 6 input.*

Me: Okay, so you are trying to figure out how these two are connected?

Jada: You could add 6 + 6. That gives you 12 and then add 1 more to get 13.

Me: Good. So you double that and then added one more to give you 13.

Me: Does that work for other numbers?

Jada: Yeah.

Me: How about you fill out the table one more time to help us remember how we did this.

----I could tell Jada was struggling with this in her head, so I gave her the blank table to fill out.

Jada: *writing in the table*

Number	Double the number, plus 1
1	even 2+1=3 odd
2	even 4+1=5 odd
3	even 6+1=7 odd
4	even 8+1=9 odd
5	even 10+1=11 odd

Me: Good job, Jada! Do you notice any patterns?

----Now she fills in the even and odd pattern.

Jada: Yeah, all of these numbers are even. All of these numbers are odd.

Me: That's good. Do you think that pattern will continue.

Jada: Yeah.

Me: Can you explain that to me? Or could you draw a picture?

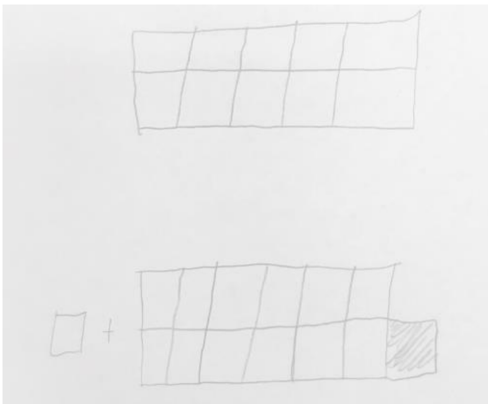
Jada: *draws the picture below*

Jada: When you double the number here it's even. Then the plus one will always make it odd because it is leftover.

Me: That's good. Do you think that pattern will continue? We are almost done.

Jada: Yeah.

Me: Okay good. So I want to thank you for the awesome job you did today!



Appendix G - Pilot Study Tasks

Generalizing about Arithmetic Properties

Task 1: Commutative Property

What do you notice about these equations? What makes you say that? What else do you notice? Is the equation true or false? Why?

39 + 121 = 121 + 39

78 + 93 = 93 + 78

178 + 232 = 232 + 178

Do you think the addends can always be switched for any number? Why? Describe in your own words what we are doing here.

23 x 15 = 15 x 23

18 x 36 = 36 x 18

…

Do you think the factors can always be switched? Why?

Could you draw a picture as to why this is true?

Describe your picture. How does your picture help explain why we can switch the numbers and still get the same result?

Task 2: Distributive Property

What do you notice about this equation? What makes you say that? What else do you notice? Is the equation true or false? Why?

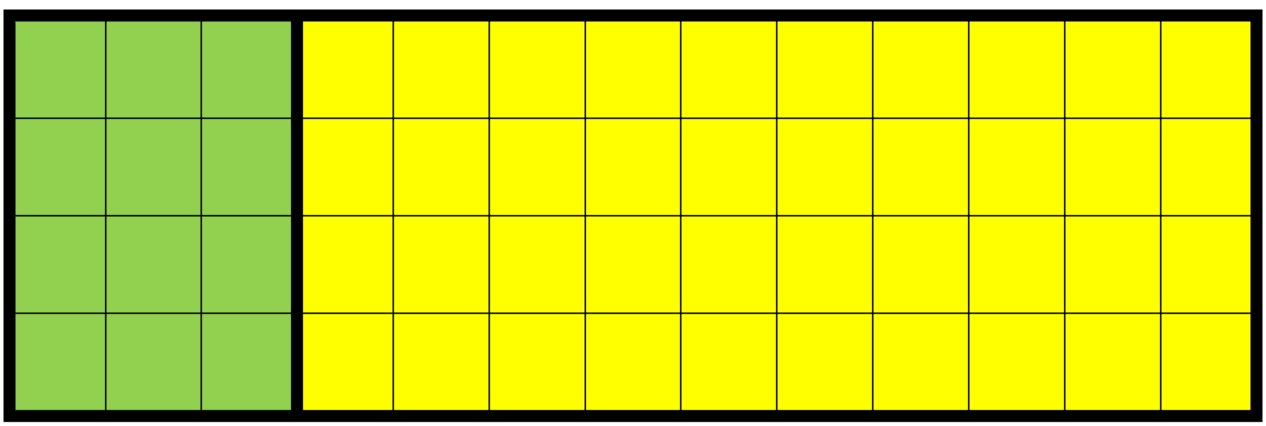
4 x 13 = 4 x (10 + 3)

Describe in your own words what we are doing in the equation and why it works.

Could you draw a picture as to why this is true?

Describe your picture. How does your picture help explain this property?

Given the following picture, what do you notice? How does this picture help you understand the equation above?



Generalizing Properties

Can you describe in general what is going on here in the equation ***4 x 13 = 4 x (10 + 3)?***

Will this "trick" we discovered above work for more numbers? How can we find out?

Generate a set of equations using the distributive property and have the student check whether each equation is true or false. *How does this convince you whether our "trick" works for all numbers? How does your picture (or the picture above) help convince you?*

Task 3: Generalizing about Even/Odds using Tables

The table below shows a list of numbers. For every number listed in the table, multiply it by 2 and add 1. Record the result on the right.

Number	Double the number, plus 1
1	
2	
3	
4	
5	
6	

What do you notice about the numbers you entered into the table?

1. *Does an even number multiplied by 2 result in an even or odd number? Why do you think this is? Draw a picture.*
2. *Does an odd number multiplied by 2 result in an even or odd number? Why do you think this is? Draw a picture.*
3. *Does an even number plus 1 result in an even or odd number? Why do you think this is?*
4. *Does an odd number plus 1 result in an even or odd number? Why do you think this is?*
5. *Explain why the numbers you entered in the table are all odd. How does your picture show this property?*
6. *Can we write an equation to generalize the pattern?*

https://www.illustrativemathematics.org/content-standards/4/OA/C/5/tasks/487

<u>Task 4: Generalizing about 3, 6, 9</u>

Part I

1. Make a list of the first ten multiples of 3.
2. Which of the numbers in your list are multiples of 6? What pattern do you see in where the multiples of 6 appear in the list?
3. Which numbers in the list are multiples of 9? Can you predict when multiples of 9 will appear in the list of multiples of 3? Explain your reasoning.

Part I

1. Starting with 9, list the first 10 multiples of 9.

2. In the list in part (a) what patterns do you see with the digits in the 10's place? What patterns do you see with the digits in the 1's place?

3. Using pictures, words, or equations, explain the patterns you observed in part (b).

<u>Task 5: Generalizing a Pattern Task</u>

Purpose: To move beyond arithmetic to generalizing algebraic thinking.

Even/Odd Numbers

What happens when you add two even numbers? What about three odd numbers? An infinite number of odd numbers?

What happens when you add two odd numbers? What about three odd numbers? An infinite number of odd numbers?

What happens when you add an odd number and an even number? Is the result even or odd?

Make a conjecture that shows what you found. How do you know your conjecture is true? Will it always work?

1,895 + 1,987 + 2,073 + 5,999

Without calculating these numbers, is this set of numbers added together going to be odd or even and why?