Leibniz-type rules associated to bilinear pseudodifferential operators

by

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B.A., University of Nebraska at Kearney, 2013

M.A., Kansas State University, 2015

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Abstract

Leibniz-type rules associated to bilinear pseudodifferential operators have received considerable attention due to their applications in obtaining fractional Leibniz rules and the study of various partial differential equations. Generally speaking, fractional Leibniz rules provide a way of estimating the size and smoothness of a product of functions in terms of the size and smoothness of the individual functions themselves. Such rules are helpful in determining well-posedness results for solutions of PDEs modeling a variety of real world phenomena, ranging from Euler and Navier-Stokes equations (which model incompressible fluid flow, such as airflow over a wing) to Korteweg-de Vries equations (which model waves on shallow water surfaces).

Bilinear pseudodifferential operators act to combine two functions using their Fourier transforms and a symbol, which is a function that assigns different weights to the functions' frequency components as they are combined. Thus, Leibniz-type rules associated to bilinear pseudodifferential operators serve as a generalization of fractional Leibniz rules by providing estimates on the size and smoothness of some combination of two functions, for which pointwise multiplication is recoverable by choosing a symbol identically equal to one. A variety of function spaces may be used to measure the size and smoothness of functions involved, including Lebesgue spaces, Sobolev spaces, and Besov and Triebel-Lizorkin spaces. Further, bilinear pseudodifferential operators may be considered in association with different classes of symbols, which is to say that the symbol itself (and possibly its derivatives) will possess certain decay properties.

New Leibniz-type rules in two different settings will be presented in this manuscript. In the first setting, Leibniz-type rules associated to bilinear pseudodifferential operators with homogeneous symbols in a certain class are proved, where the sizes of the functions involved are measured using a combination of Lebesgue space norms and norms corresponding to function spaces admitting appropriate molecular decompositions, specifically focusing on the case of homogeneous Besov-type and Triebel-Lizorkin-type spaces. In the second setting, Leibniz-type rules and biparameter counterparts are proved in weighted Lebesgue and Sobolev spaces associated to Coifman-Meyer multiplier operators. All of the new Leibniz-type rules proved in the manuscript yield corresponding new fractional Leibniz rules, which are highlighted as appropriate. Various techniques from Fourier analysis serve as important tools in the proofs of these new results, such as obtaining paraproduct decompositions for bilinear pseudodifferential operators and utilizing Littlewood-Paley theory and square function-type estimates.

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Approved by:

Major Professor
Virginia Naibo

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New Leibniz-type rules in two different settings will be presented in this manuscript. In the first setting, Leibniz-type rules associated to bilinear pseudodifferential operators with homogeneous symbols in a certain class are proved, where the sizes of the functions involved are measured using a combination of Lebesgue space norms and norms corresponding to function spaces admitting appropriate molecular decompositions, specifically focusing on the case of homogeneous Besov-type and Triebel-Lizorkin-type spaces. In the second setting, Leibniz-type rules and biparameter counterparts are proved in weighted Lebesgue and Sobolev spaces associated to Coifman-Meyer multiplier operators. All of the new Leibniz-type rules proved in the manuscript yield corresponding new fractional Leibniz rules, which are highlighted as appropriate. Various techniques from Fourier analysis serve as important tools in the proofs of these new results, such as obtaining paraproduct decompositions for bilinear pseudodifferential operators and utilizing Littlewood-Paley theory and square function-type estimates.

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Dedication

To my family for all of their kind-hearted encouragement,

and especially to my mother and father for raising me to be curious and creative,

and to my wife for her constant support and occasional patience when math is brought up,

with love.

Chapter 1

An Overview of Leibniz-Type Rules

Our discussion about definitions, motivations, and the history of Leibniz-type rules must begin with some background on fractional Leibniz rules, the type of estimates which serve as a foundation for Leibniz-type rules. As the name suggests, fractional Leibniz rules are closely related to the general Leibniz rule, a formula which gives a representation for partial derivatives of products of functions. Considering the simplest case, the derivative of the product of two differentiable functions defined on $\mathbb{R}$, we obtain the product rule $(fg)' = f'g + fg'$. Notice that in this formula, the right-hand side has two terms, one of which has the derivative on f and no derivative on g, while the other has no derivative on f and one derivative on g. In its full generality, the Leibniz rule may be stated for two sufficiently differentiable functions f and g defined on $\mathbb{R}^n$ and any multi-index $\alpha \in \mathbb{N}_0^n$ as

$$\partial^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) = (\partial^\alpha f)g + f(\partial^\alpha g) + \dots .$$

Again, notice that as a part of this formula, there are two terms on the right-hand side, one of which has all α derivatives on f and no derivatives on g, and another which has no derivates on f and all α derivatives on g. We briefly note that all standard notation is collected in Appendix A, including definitions for the multi-indices mentioned above, along with definitions of function spaces and aspects of Fourier analysis that will be utilized below.

Fractional Leibniz rules, also known as Kato-Ponce inequalities (due to the pioneering work done by Kato-Ponce [39]), have the form

$$\|D^s(fg)\|_{L^r} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^s g\|_{L^{q_2}}, \tag{1.1}$$

$$\|J^s(fg)\|_{L^r} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|J^s g\|_{L^{q_2}}, \tag{1.2}$$

which hold for indices satisfying $1 < p_1, p_2, q_1, q_2 \leq \infty$, $\frac{1}{2} < r \leq \infty$ such that $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, and $s > \max\{0, n(\frac{1}{r} - 1)\}$ or $s \in 2\mathbb{N}_0$, and for functions f and g in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. The operators D^s and J^s are as defined below, but should be thought of, in general, as taking s derivatives of a function. Put simply, these fractional Leibniz rules involve measuring the size of s derivatives of a product of functions f and g, then bounding this quantity by the sum of two terms, one of which has all s derivatives on f and none on g, and the other having no derivatives on f and s derivatives on g. In this way, we see a parallel to the simpler notions of the product rule or general Leibniz rule, as discussed above.

For $s \in \mathbb{R}$, define the operator D^s via

$$\widehat{D^s f}(\xi) := |\xi|^s \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, f \in \mathcal{S}'(\mathbb{R}^n),$$

referred to as a homogeneous differentiation operator of order s if $s > 0$. There is a connection between the operator D^s and the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^n)$, which motivates thinking of D^s as taking s derivatives; see Section A.2 for more details. Similarly, define the operator J^s via

$$\widehat{J^s f}(\xi) := (1 + |\xi|^2)^{s/2} \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, f \in \mathcal{S}'(\mathbb{R}^n),$$

which is referred to as an inhomogeneous differentiation operator of order s if $s > 0$ and which shares a connection with the inhomogeneous Sobolev space $W^{s,p}(\mathbb{R}^n)$.

Inequalities of the forms (1.1) and (1.2) (and related commutator estimates) have received much attention due to their applicability to problems in partial differential equations. Techniques were initially developed for fractional Leibniz rules to handle the case $1 < r < \infty$.

Specifically, estimates similar to (1.2) were studied in Kato-Ponce [39] in relation to the Cauchy problem for the Euler and Navier-Stokes equations (which model incompressible fluid flow, such as airflow over a wing), with previous work in Strichartz [62] for the case $\frac{n}{p} < s < 1$. Further, in both Christ-Weinstein [14] and Kenig-Ponce-Vega [41], estimates along the lines of (1.1) were considered in connection to the Korteweg-de Vries equation (which models waves on shallow water surfaces), and in Gulisashvili-Kon [34], estimates of both forms were studied in relation to smoothing properties of Schrödinger semigroups. In recent years, the range for r has been extended to include $\frac{1}{2} < r \leq 1$, treated in Grafakos-Oh [30] and Muscalu-Schlag [52] (with related work in Koezuka-Tomita [42]), and the case $r = \infty$ was settled in Bourgain-Li [9] (see also Grafakos-Maldonado-Naibo [28] for related results). This is a small selection of previous work done relating to inequalities of the forms (1.1) and (1.2), and more history will be detailed throughout the manuscript once additional necessary concepts have been introduced.

On the left-hand side of inequalities (1.1) and (1.2), the functions f and g are combined via pointwise multiplication. It is natural to consider similar inequalities wherein the functions involved are combined using more versatile methods. In particular, we will combine functions using bilinear pseudodifferential operators.

Definition 1.1. Let $\sigma(x, \xi, \eta)$ be a complex-valued, smooth function defined for $x, \xi, \eta \in \mathbb{R}^n$. Define T_σ, the **bilinear pseudodifferential operator associated to** σ, by

$$T_\sigma(f, g)(x) := \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi \, d\eta, \quad \forall x \in \mathbb{R}^n.$$

In general, σ is referred to as the *symbol* associated with the operator T_σ. If σ does not depend on x, then σ is referred to as a *multiplier*, and T_σ is known as a **bilinear multiplier operator**.

When discussing bilinear pseudodifferential operators, certain decay estimates will be assumed for the associated symbol σ and its derivatives, which will result in σ lying in various symbol classes. These will be introduced in subsequent chapters as necessary. See Section A.5 for some simple examples of bilinear pseudodifferential operators, along with a remark on the connection between such operators and their linear counterparts.

Our main focus in this manuscript is proving various Leibniz-type rules, which are reminiscent of the Kato-Ponce inequalities introduced in (1.1) and (1.2), and which will often serve as complements and extensions of said equations. Leibniz-type rules are inequalitites of the form

$$\|T_\sigma(f,g)\|_X \lesssim \|f\|_{Y_1} \|g\|_{Z_1} + \|f\|_{Y_2} \|g\|_{Z_2}\,, \tag{1.3}$$

where T_σ is a bilinear pseudodifferential operator as in Definition 1.1 and X, Y_1, Y_2, Z_1, and Z_2 are function spaces measuring some sense of smoothness of the functions involved. In fact, since $T_\sigma(f,g) = fg$ for $\sigma \equiv 1$, we see that (1.1) and (1.2) may be regarded as Leibniz-type rules with $\sigma \equiv 1$ and X, Y_1, Y_2, Z_1, and Z_2 various Lebesgue spaces and homogeneous/inhomogeneous Sobolev spaces. The history given above for the development of (1.1) and (1.2) gives some background for Leibniz-type rules, but more can be said in general. Additional results of the form (1.3) in the case $\sigma \equiv 1$ with X, Y_1, and Z_2 Besov or Triebel-Lizorkin spaces and Y_2 and Z_1 Lebesgue spaces, along with applications to partial differential equations, may be found in Bahouri-Chemin-Danchin [2], Chae [12], and Runst-Sickel [59], while estimates of the form (1.3) with $\sigma \equiv 1$ involving weighted or variable exponent spaces were obtained in Cruz-Uribe-Naibo [17]. We give more history on the development of results of the form (1.3) for σ in certain bilinear homogeneous symbol classes in Chapter 2, and for σ in Coifman-Meyer or biparameter Coifman-Meyer multiplier classes in Chapter 3. Below, we state the main Leibniz-type rule results proved in Chapters 2 and 3, reserving technical definitions for the respective chapters. We note that the main results of Chapter 2 were originally published in Brummer-Naibo [11], while those of Chapter 3 are to appear in Brummer-Naibo [10].

In Chapter 2, we will present a unifying approach towards establishing Leibniz-type rules of the form (1.3) where T_σ is a bilinear pseudodifferential operator with bilinear symbol σ in the homogeneous symbol class $\dot{BS}^m_{1,1}$ for some $m \in \mathbb{R}$, and where Z_1 and Y_2 are standard Lebesgue spaces and X, Y_1, and Z_2 are function spaces admitting a molecular decomposition in the sense of Frazier-Jawerth [24; 25]. We demonstrate this unifying approach by proving explicit Leibniz-type rule results in the case where X, Y_1, and Z_2 are homogeneous Besov-type or Triebel-Lizorkin-type function spaces, denoted $\dot{B}^{s,\tau}_{p,q}$ and $\dot{F}^{s,\tau}_{p,q}$, respectively:

Theorem 1.2. *Let $m \in \mathbb{R}$ and $\sigma \in \dot{BS}^m_{1,1}$. If $0 < p, q \leq \infty$, $s_p < s < \infty$, and $0 \leq \tau < \frac{1}{p} + \frac{s-s_p}{n}$, it holds that*

$$\|T_\sigma(f,g)\|_{\dot{B}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{B}^{s+m,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{B}^{s+m,\tau}_{p,q}}, \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

If $0 < p < \infty$, $0 < q \leq \infty$, $s_{p,q} < s < \infty$, and $0 \leq \tau < \frac{1}{p} + \frac{s-s_{p,q}}{n}$, it holds that

$$\|T_\sigma(f,g)\|_{\dot{F}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{F}^{s+m,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s+m,\tau}_{p,q}}, \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

Note that s_p and $s_{p,q}$ are as in (2.19). We also note that there are homogeneous differentiation operators implicit in the results of Theorem 1.2, making them reminiscent of (1.1). Specifically, this is seen through the norm equivalences $\|D^s f\|_{\dot{B}^{0,\tau}_{p,q}} \sim \|f\|_{\dot{B}^{s,\tau}_{p,q}}$ and $\|D^s f\|_{\dot{F}^{0,\tau}_{p,q}} \sim \|f\|_{\dot{F}^{s,\tau}_{p,q}}$ (see Yang-Yuan [69, Proposition 3.5]). The proof of Theorem 1.2 is detailed in Section 2.4, utilizing as a primary tool Theorem 2.1. Theorem 2.1 establishes decay properties for certain families of functions relating to $T_\sigma(f,g)$, allowing us to utilize established theory for spaces admitting molecular decompositions based on the pioneering work of Frazier-Jawerth [24; 25] and therefore obtain the Leibniz-type rule results given in Theorem 1.2. Said results may be considered as bilinear counterparts to Grafakos-Torres [31, Theorems 1.1 and 1.2], wherein boundedness properties in homogeneous Besov and Triebel-Lizorkin spaces were addressed for linear pseudodifferential operators (and where such boundedness properties were extended to the setting of Besov-type and Triebel-Lizorkin-type spaces in Sawano-Yang-Yuan [61], again for linear pseudodifferential operators). In Subsection 2.1.2, we will discuss connections between Theorem 1.2 and Kato-Ponce inequalities in Besov-type and Triebel-Lizorkin-type spaces, and in Subsection 2.3.1, we detail a number of spaces which may be realized as particular cases of Besov-type and Triebel-Lizorkin-type spaces so that Theorem 1.2 will yield Leibniz-type rules and associated fractional Leibniz rules in such spaces.

In Chapter 3, one of our goals will be to prove Leibniz-type rules of the form (1.3) where T_σ is a bilinear multiplier operator with symbol σ a Coifman-Meyer multiplier, and where X, Y_1, Y_2, Z_1, and Z_2 are a combination of weighted Lebesgue spaces and weighted homogeneous/inhomogeneous Sobolev spaces:

Theorem 1.3. *Let $\sigma(\xi,\eta)$, $\xi,\eta\in\mathbb{R}^n$, satisfy (3.1) and consider $1<p,q\le\infty$, $\frac{1}{2}<r<\infty$ such that $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, and $s>\max\{0,n(\frac{1}{r}-1)\}$ or $s\in 2\mathbb{N}_0$. If $v\in A_p(\mathbb{R}^n)$ and $w\in A_q(\mathbb{R}^n)$, then for all $f,g\in\mathcal{S}(\mathbb{R}^n)$, it holds that*

$$\|D^s(T_\sigma(f,g))\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}\lesssim\|D^sf\|_{L^p(v)}\|g\|_{L^q(w)}+\|f\|_{L^p(v)}\|D^sg\|_{L^q(w)}, \tag{1.4}$$

$$\|J^s(T_\sigma(f,g))\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}\lesssim\|J^sf\|_{L^p(v)}\|g\|_{L^q(w)}+\|f\|_{L^p(v)}\|J^sg\|_{L^q(w)}, \tag{1.5}$$

where the implicit constant depends on p, q, s, $[v]_{A_p}$, $[w]_{A_q}$, and σ. If $v=w$, different choices of p and q are allowed in each term on the right-hand side of (1.4) and (1.5).

Estimates of the forms (1.4) and (1.5) for $\sigma\equiv 1$ and finite p and q were proved in Cruz-Uribe-Naibo [17], along with related weighted commutator estimates. Additionally, unweighted estimates in the spirit of (1.4) were proved in Hart-Torres-Wu [35] for certain multipliers with minimal smoothness assumptions, and estimates similar to (1.4) are proved in Naibo-Thomson [56] for A_∞ weights in the scales of weighted Besov/Triebel-Lizorkin and weighted Hardy spaces. Also in Chapter 3, we prove Leibniz-type rules relating to biparameter counterparts of the homogeneous differentiation operator D^s. For $s\in\mathbb{R}$ and $n_1,n_2\in\mathbb{N}$ such that $n=n_1+n_2$, define the operators D_1^s and D_2^s via

$$\widehat{D_\ell^s f}(\xi):=|\xi_\ell|^s\,\widehat{f}(\xi),\quad \xi=(\xi_1,\xi_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, f\in\mathcal{S}'(\mathbb{R}^n),\ell=1,2,$$

referred to as partial homogeneous s-th differentiation operators if $s>0$, and thought of as taking s partial derivatives in some subspace of $\mathbb{R}^n$. The Leibniz-type rules we prove relating to partial homogeneous differentiation operators are of the following form, where the symbol σ is a biparameter Coifman-Meyer multiplier:

Theorem 1.4. *Let $n = n_1 + n_2$ for $n_1, n_2 \in \mathbb{N}$, $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Assume $\sigma(\xi, \eta)$ satisfies (3.2) and consider $1 < p, q < \infty$, $\frac{1}{2} < r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $s_\ell > \max\{0, n_\ell(\frac{1}{r} - 1)\}$, $\ell = 1, 2$. If $v \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $w \in A_q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, it holds that*

$$\begin{aligned}\left\|D_1^{s_1} D_2^{s_2}(T_\sigma(f,g))\right\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})} \lesssim \left\|D_1^{s_1} D_2^{s_2} f\right\|_{L^p(v)} \|g\|_{L^q(w)} + \left\|D_1^{s_1} f\right\|_{L^p(v)} \left\|D_2^{s_2} g\right\|_{L^q(w)} \\ + \left\|D_2^{s_2} f\right\|_{L^p(v)} \left\|D_1^{s_1} g\right\|_{L^q(w)} + \|f\|_{L^p(v)} \left\|D_1^{s_1} D_2^{s_2} g\right\|_{L^q(w)},\end{aligned} \tag{1.6}$$

where the implicit constant depends on p, q, s_1, s_2, $[v]'_{A_p}$, $[w]'_{A_q}$, and σ. If $v = w$, different choices of p and q are allowed in each term on the right-hand side of (1.6).

For $\sigma \equiv 1$, biparameter results of the form (1.6) for $n_1 = n_2 = 2$ were studied in Muscalu-Pipher-Tao-Thiele [50] and for the general case $n_1, n_2 \in \mathbb{N}$ with $n_1 = n_2$ in Grafakos-Oh [30], while applications of (1.6) were studied in Kenig [40] in relation to local well-posedness results for the KP-I equation, which models capillary gravity waves. We detail the proofs of Theorem 1.3 and Theorem 1.4 in Section 3.4 and Section 3.3, respectively, following a similar procedure for each. Briefly, the shared procedure involves obtaining paraproduct decompositions for $T_\sigma(f, g)$, followed by analyzing multipliers associated with said decompositions and applying various square function-type estimates.

Chapter 2

Leibniz-Type Rules for Bilinear Operators with Homogeneous Symbols and Smooth Molecules

2.1 Introduction

Our main goal in this chapter will be to prove the Leibniz-type rules presented in Theorem 1.2. As a result, we obtain a process for proving general Leibniz-type rules of the form (1.3) involving function spaces which admit a molecular decomposition in the sense of Frazier-Jawerth [24; 25]. The main tool for proving such results is Theorem 2.1, which provides estimates necessary for verifying that certain families of functions associated to T_σ are families of smooth synthesis molecules:

Theorem 2.1. *Given $m \in \mathbb{R}$ and $\sigma \in \dot{BS}^m_{1,1}$, there exist $\sigma^1, \sigma^2 \in \dot{BS}^m_{1,1}$ with $T_\sigma = T_{\sigma^1} + T_{\sigma^2}$ and such that if $1 \leq r \leq \infty$, $0 < M < \infty$, $\Lambda \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{\Lambda}$ supported in $\{\xi \in \mathbb{R}^n : \frac{1}{2} < |\xi| < 2\}$, and $\gamma \in \mathbb{N}_0^n$, it holds that*

$$|\partial^\gamma T_{\sigma^1}(\Lambda_{\nu,k}, g)(x)| \lesssim \frac{2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} 2^{\frac{\nu n}{r}}}{(1+|2^\nu x - k|)^M} \|g\|_{L^r}, \qquad \forall x \in \mathbb{R}^n \tag{2.1}$$

and

$$|\partial^\gamma T_{\sigma^2}(f,\Lambda_{\nu,k})(x)| \lesssim \frac{2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} 2^{\frac{\nu n}{r}}}{(1+|2^\nu x-k|)^M} \|f\|_{L^r}, \qquad \forall x \in \mathbb{R}^n, \tag{2.2}$$

for every $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $f, g \in \mathcal{S}(\mathbb{R}^n)$, and where $\Lambda_{\nu,k}(x) = 2^{\frac{\nu n}{2}}\Lambda(2^\nu x - k)$.

With the decay properties given in Theorem 2.1, we are able to show that, up to multiplicative constants, we have families of smooth synthesis molecules in the settings of Besov-type and Triebel-Lizorkin-type spaces given by

$$\left\{\frac{T_{\sigma^1}(\Lambda_{\nu,k},g)}{2^{\nu m}\|g\|_{L^\infty}}\right\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \quad \text{and} \quad \left\{\frac{T_{\sigma^2}(f,\Lambda_{\nu,k})}{2^{\nu m}\|f\|_{L^\infty}}\right\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n},$$

where the implicit constants are uniform in $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $f, g \in \mathcal{S}(\mathbb{R}^n)$. The proof of Theorem 1.2 utilizes the molecular decomposition theory pioneered by Frazier-Jawerth [24; 25] in association with these families of smooth synthesis molecules to obtain the desired Leibniz-type rule results. Further, because smooth synthesis molecules also serve as building blocks for a variety of other function spaces, the procedures outlined in the proofs of Theorems 2.1 and 1.2 will apply to such spaces as well.

The outline for Chapter 2 is as follows: Subsection 2.1.1 details the bilinear homogeneous class of symbols $\dot{BS}^m_{1,1}$ appearing in the statement of Theorem 1.2, followed by Subsection 2.1.2, which examines Kato-Ponce inequalitites relating to Theorem 1.2. In Section 2.2, we give the proof for Theorem 2.1, the main tool used in the proof of Theorem 1.2. We provide the necessary background for Theorem 1.2 in Section 2.3, wherein we introduce full definitions for the function spaces $\dot{B}^{s,\tau}_{p,q}$ and $\dot{F}^{s,\tau}_{p,q}$ and the relevant material associated to families of smooth synthesis molecules and spaces admitting a molecular decomposition in the sense of Frazier-Jawerth [24; 25], along with examples of such spaces. Section 2.4 contains the proof of Theorem 1.2, and we conclude the chapter with some remarks in Section 2.5, including one about the situations where $s \le s_p$ and $s \le s_{p,q}$ (in which case analogous results may be obtained if some additional hypotheses are imposed upon the first adjoint of T_{σ^1} and the second adjoint of T_{σ^2}), and another which gives a version of the Leibniz-type rule results of Theorem 1.2 involving $L^r(\mathbb{R}^n)$-norms of f and g instead of their $L^\infty(\mathbb{R}^n)$-norms.

2.1.1 Class of symbols $\dot{BS}^m_{1,1}$

The class of bilinear symbols associated to results in this chapter are defined below, followed by two concrete examples of bilinear symbols relating to said class.

Definition 2.2. Let $\sigma(x,\xi,\eta) \in \mathcal{C}^\infty(\mathbb{R}^{3n} \setminus \{0\})$ and $m \in \mathbb{R}$. σ is in the bilinear class of symbols $\dot{BS}^m_{1,1}$ if, for all $\alpha,\beta,\gamma \in \mathbb{N}_0^n$, there exists $C_{\alpha,\beta,\gamma} > 0$ such that

$$\left|\partial_x^\gamma \partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\xi,\eta)\right| \leq \frac{C_{\alpha,\beta,\gamma}}{(|\xi|+|\eta|)^{|\alpha+\beta|-|\gamma|-m}}, \quad \forall (x,\xi,\eta) \neq (0,0,0). \tag{2.3}$$

The infimum over all such viable $C_{\alpha,\beta,\gamma}$ is denoted as $\|\sigma\|_{\gamma,\alpha,\beta}$.

As examples, we consider two classes of symbols closely related to those defined in Definition 2.2. First, setting $m = 0$, we have that the x-independent symbols in $\dot{BS}^0_{1,1}$ correspond exactly with the class of Coifman-Meyer multipliers, defined later in Definition 3.1. Coifman-Meyer multipliers are introduced in Subsection 3.1.1, and boundedness results for bilinear pseudodifferentail operators associated with this class of symbols are well-understood, due in large part to their categorization as bilinear Calderón-Zygmund operators (for a definition and treatment of such operators, see Grafakos-Torres [33]). As a second example, we consider a class which lacks some of the nice Lebesgue space boundedness properties exhibited by Coifman-Meyer multiplier operators and is a particular instance of a family of inhomogeneous classes of symbols closely related to those given in Definition 2.2, which we define next.

Definition 2.3. Let $\sigma(x,\xi,\eta) \in \mathcal{C}^\infty(\mathbb{R}^{3n} \setminus \{0\})$, $0 \leq \delta \leq \rho \leq 1$, and $m \in \mathbb{R}$. σ is in the bilinear Hörmander class of symbols $BS^m_{\rho,\delta}$ if, for all $\alpha,\beta,\gamma \in \mathbb{N}_0^n$, there exists $C_{\alpha,\beta,\gamma} > 0$ such that

$$\left|\partial_x^\gamma \partial_\xi^\alpha \partial_\eta^\beta \sigma(x,\xi,\eta)\right| \leq \frac{C_{\alpha,\beta,\gamma}}{(1+|\xi|+|\eta|)^{\rho|\alpha+\beta|-\delta|\gamma|-m}}, \quad \forall (x,\xi,\eta) \neq (0,0,0).$$

Specifically, we consider $BS^0_{1,1}$ for our second example, the so-called bilinear forbidden class of symbols. The relationship between symbols in $BS^0_{1,1}$ and $\dot{BS}^0_{1,1}$ comes from the fact that, given $\sigma \in BS^0_{1,1}$, we can decompose σ as a sum of one symbol which is supported within

$\{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\xi| + |\eta| \le 1\}$ (and is therefore well-behaved and smoothing) and another symbol which is in $\dot{BS}^0_{1,1}$. For pioneering work related to $BS^0_{1,1}$, see Coifman-Meyer [15] (and the references it contains). Symbols in the forbidden class $BS^0_{1,1}$ are known to produce bilinear pseudodifferential operators with a bilinear Calderón-Zygmund kernel, but they are not, in general, bilinear Calderón-Zygmund operators themselves, as they do not always possess mapping properties of the form $L^p \times L^q \to L^r$ for $1 < p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} > 0$ (for an explicit example, the construction of a symbol in $BS^0_{1,1}$ which fails to map $L^2 \times L^2$ into L^1 may be found in Bényi-Torres [7]). However, mapping properties (including those of type (1.3)) for bilinear pseudodifferential operators with symbols in $BS^0_{1,1}$ have been established in various other settings, including Sobolev spaces (see Bényi-Torres [7] and Bényi-Nahmod-Torres [6]) and Besov and Triebel-Lizorkin spaces (see Bényi [3], Naibo [53], and Koezuka-Tomita [42]). In general, much attention has been given to studying bilinear pseudodifferential operators with symbols in $BS^m_{\rho,\delta}$ for $0 \le \rho \le \delta \le 1$ and $m \in \mathbb{R}$ and related classes; see Bényi-Bernicot-Maldonado-Naibo-Torres [4], Bényi-Maldonado-Naibo-Torres [5], Bényi-Torres [7; 8], Herbert-Naibo [36; 37], Koezuka-Tomita [42], Michalowski-Rule-Staubach [46], Miyachi-Tomita [47–49], Naibo [53; 54], Naibo-Thomson [55], Rodríguez-López-Staubach [58], and references therein.

2.1.2 The case $\sigma \equiv 1$ and connections to Kato-Ponce inequalities

Considering the case $\sigma \equiv 1$ in Theorem 1.2, we obtain Kato-Ponce inequalities for Besov-type and Triebel-Lizorkin-type spaces, as highlighted in the following corollary:

Corollary 2.4. *If $0 < p, q \le \infty$, $s_p < s < \infty$, and $0 \le \tau < \frac{1}{p} + \frac{s - s_p}{n}$, it holds that*

$$\|fg\|_{\dot{B}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{B}^{s,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{B}^{s,\tau}_{p,q}}, \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

If $0 < p < \infty$, $0 < q \le \infty$, $s_{p,q} < s < \infty$, and $0 \le \tau < \frac{1}{p} + \frac{s - s_{p,q}}{n}$, it holds that

$$\|fg\|_{\dot{F}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{F}^{s,\tau}_{p,q}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{F}^{s,\tau}_{p,q}}, \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

As will be discussed in Subsection 2.3.1, homogeneous Sobolev spaces may be realized as particular cases of Besov-type or Triebel-Lizorkin-type spaces. Thus, we may recover (1.1) in the case $q_1 = p_2 = \infty$ directly from Corollary 2.4.

The proof of Theorem 1.2 may be regarded as a procedure for proving Leibniz-type rules relating to function spaces which admit a molecular decomposition as introduced in Frazier-Jawerth [24; 25]. Subsequently, the results of Corollary 2.4 yield the following type of estimates in a given function space X which measures smoothness in some sense and admits a molecular decomposition:

$$\|fg\|_X \lesssim \|f\|_X \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_X . \tag{2.4}$$

Chapter 1 details the historical development of results similar to (2.4) in the settings where X is the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^n)$ with $s > 0$ and $1 < p < \infty$ (in which case the $\dot{W}^{s,p}(\mathbb{R}^n)$-norm is defined via $\|D^s \cdot\|_{L^p}$), and where X is the inhomogeneous Sobolev space $W^{s,p}(\mathbb{R}^n)$ for the same range of parameters (in which case the $W^{s,p}(\mathbb{R}^n)$-norm is defined via $\|J^s \cdot\|_{L^p}$).

2.2 Proof of molecular estimates

In this section, we present a proof of Theorem 2.1, which we break into a few steps. We begin by obtaining a paraproduct decomposition for $T_\sigma(f,g)$, with T_σ a bilinear pseudodifferential operator as defined in Definition 1.1 having symbol $\sigma \in \dot{BS}^m_{1,1}$ as in Definition 2.2. Lemma 2.5 below will provide such a decomposition suited for our purposes and is proved in Subsection 2.2.1 (with ideas inspired by Coifman-Meyer [15]). We then procure formulas for the derivatives of the building blocks within the paraproduct decomposition for $T_\sigma(f,g)$, appropriately evaluated as in Theorem 2.1. Said formulas are stated in Lemma 2.6, which is proved in Subsection 2.2.2. Finally, we pull together the results of the lemmas in Subsection 2.2.3 to prove Theorem 2.1.

Throughout this section, we will make use of $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (A.5) and (A.6) and

$\Phi \in \mathcal{S}(\mathbb{R}^n)$ defined via (A.7), along with functions $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ as defined in (A.9) and (A.10) for which $\Psi = \psi\Psi$ and $\Phi = \phi\Phi$. We also note (A.3) and (A.4), which set notation for certain families of operators associated to the Ψ and Φ: Briefly, for $j \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$, and $f \in \mathcal{S}'(\mathbb{R}^n)$, we define $\widehat{\Delta_j^\Psi f}(\xi) \coloneqq \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi)$ and $\widehat{S_j^\Phi f}(\xi) \coloneqq \widehat{\Phi}(2^{-j}\xi)\widehat{f}(\xi)$

Finally, before beginning to state and prove lemmas, we give a decomposition for the symbol σ that will be used throughout the section. Let $\theta \in \mathcal{S}(\mathbb{R})$ be real-valued with

$$\operatorname{supp}(\theta) \subseteq (-2,2) \quad \text{ and } \quad \theta(t) + \theta\left(\frac{1}{t}\right) = 1, \quad \forall t > 0. \tag{2.5}$$

For $\sigma \in \dot{BS}^m_{1,1}$, $m \in \mathbb{R}$, we define

$$\sigma^1(x,\xi,\eta) \coloneqq \sigma(x,\xi,\eta)\theta\left(\frac{|\eta|}{|\xi|}\right) \text{ and } \sigma^2(x,\xi,\eta) \coloneqq \sigma(x,\xi,\eta)\theta\left(\frac{|\xi|}{|\eta|}\right), \quad \forall x,\xi,\eta \in \mathbb{R}^n,$$

so that $\sigma = \sigma^1 + \sigma^2$, and therefore,

$$T_\sigma(f,g) = T_{\sigma^1}(f,g) + T_{\sigma^2}(f,g), \quad \forall f,g \in \mathcal{S}(\mathbb{R}^n).$$

As shown in Lemma B.1, $\sigma^1, \sigma^2 \in \dot{BS}^m_{1,1}$, and we see by following the proof that

$$\left\|\sigma^1\right\|_{\gamma,\alpha,\beta}, \left\|\sigma^2\right\|_{\gamma,\alpha,\beta} \lesssim \sup_{\bar{\alpha}\le\alpha,\bar{\beta}\le\beta} \|\sigma\|_{\gamma,\bar{\alpha},\bar{\beta}}, \quad \forall \alpha,\beta,\gamma \in \mathbb{N}_0^n,$$

where the implicit constant depends only on α, β, γ, and θ. Also, by endowing $\mathcal{S}_0(\mathbb{R}^n)$ with the topology inherited from $\mathcal{S}(\mathbb{R}^n)$, a standard argument using integration by parts allows us to conclude that T_{σ^1} is continuous from $\mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and T_{σ^2} is continuous from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}_0(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$. With this decomposition for σ in mind, we may proceed with the proof of Theorem 2.1.

2.2.1 Construction of paraproduct decomposition

In this subsection, we state and prove Lemma 2.5, which gives a suitable paraproduct decomposition for the operators T_{σ^1} and T_{σ^2} with the symbols σ^1 and σ^2 as defined above, along with decay estimates for coefficients associated with the decompositions.

Lemma 2.5. *Let $m \in \mathbb{R}$ and $\sigma \in \dot{B}S^m_{1,1}$. With the notation introduced above and given $N > n$, there exist sequences of functions $\{\mathcal{C}^1[j](x,u,v)\}_{j\in\mathbb{Z}}$ and $\{\mathcal{C}^2[j](x,u,v)\}_{j\in\mathbb{Z}}$ defined for $x, u, v \in \mathbb{R}^n$ such that if $\gamma \in \mathbb{N}_0^n$, then*

$$\sup_{x,u,v\in\mathbb{R}^n} \left|\partial_x^\gamma \mathcal{C}^\ell[j](x,u,v)\right| \lesssim 2^{j(m+|\gamma|)}, \quad \forall j \in \mathbb{Z}, \ell = 1, 2, \tag{2.6}$$

and if $f \in \mathcal{S}_0(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, it holds that

$$T_{\sigma^1}(f,g)(x) = \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} \mathcal{C}^1[j](x,u,v)[\Delta_j^{\tau_u\Psi} f](x)[S_j^{\tau_v\Phi} g](x) \frac{du\,dv}{(1+|u|^2+|v|^2)^N} \tag{2.7}$$

and

$$T_{\sigma^2}(g,f)(x) = \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} \mathcal{C}^2[j](x,u,v)[S_j^{\tau_u\Phi} g](x)[\Delta_j^{\tau_v\Psi} f](x) \frac{du\,dv}{(1+|u|^2+|v|^2)^N}, \tag{2.8}$$

where $\widehat{\Delta_j^{\tau_u\Psi} f}(\xi) = \widehat{\tau_u\Psi}(2^{-j}\xi)\widehat{f}(\xi)$ and $\widehat{S_j^{\tau_v\Phi} g}(\xi) = \widehat{\tau_v\Phi}(2^{-j}\xi)\widehat{g}(\xi)$.

We will restrict our proof to verifying (2.7) and (2.6) in the case $\ell = 1$, for the other results in the lemma follow analogously. We will consider

$$T_{\sigma^1}(f,g)(x) = \int_{\mathbb{R}^{2n}} \sigma^1(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta.$$

Taking into account (A.5), (A.8), and the fact that σ^1 is supported within $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : |\eta| \le 2|\xi|\}$ by definition, it is easily verified that $\widehat{\Phi}(2^{-j-6}\eta)$ is equal to 1 within the support of $\widehat{\Psi}(2^{-j}\xi)\sigma^1(x,\xi,\eta)$. To simplify notation, we will denote $\widehat{\Phi}(2^{-6}\cdot)$ as simply $\widehat{\Phi}$ going forward, as the only features that $\widehat{\Phi}$ possesses which are of importance are its membership in $\mathcal{S}(\mathbb{R}^n)$

and its compact support in a ball centered at the origin, both of which $\widehat{\Phi}(2^{-6}\cdot)$ possesses as well. Thus, we see that

$$\widehat{\Psi}(2^{-j}\xi)\sigma^1(x,\xi,\eta) = \widehat{\Psi}(2^{-j}\xi)\widehat{\Phi}(2^{-j}\eta)\sigma^1(x,\xi,\eta), \quad \forall x,\xi,\eta \in \mathbb{R}^n, j \in \mathbb{Z}.$$

By also considering (A.6) and the facts that $\Psi = \psi\Psi$ and $\Phi = \phi\Phi$, we have

$$\begin{aligned}
T_{\sigma^1}(f,g)(x) &= \int_{\mathbb{R}^{2n}} \left(\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi)\right) \sigma^1(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta \\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} \widehat{\Psi}(2^{-j}\xi)\widehat{\Phi}(2^{-j}\eta)\sigma^1(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta \\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} \sigma^1[j](x,2^{-j}\xi,2^{-j}\eta)\widehat{\Psi}(2^{-j}\xi)\widehat{\Phi}(2^{-j}\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta,
\end{aligned}$$

where $\sigma^1[j](x,\xi,\eta) \coloneqq \sigma^1(x,2^j\xi,2^j\eta)\psi(\xi)\phi(\eta)$.

We now analyze $\sigma^1[j]$ for $j \in \mathbb{Z}$. For any multi-indices $\alpha,\beta,\gamma \in \mathbb{N}_0^n$, an application of the Leibniz rule implies that $\partial_x^\gamma\partial_\xi^\alpha\partial_\eta^\beta\sigma^1[j](x,\xi,\eta)$ can be written as a linear combination of terms of the form

$$\partial^{\alpha_1}\widehat{\psi}(\xi)\partial^{\beta_1}\widehat{\phi}(\eta)[\partial_x^\gamma\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\sigma^1](x,2^j\xi,2^j\eta)2^{j|\alpha_2+\beta_2|}, \quad \alpha_1+\alpha_2=\alpha, \beta_1+\beta_2=\beta.$$

As mentioned in the introduction of Section 2.2, $\sigma^1 \in \dot{BS}^m_{1,1}$, so the absolute value of each term above is bounded by

$$\begin{aligned}
&2^{j|\alpha_2+\beta_2|}\left|\partial^{\alpha_1}\widehat{\psi}(\xi)\partial^{\beta_1}\widehat{\phi}(\eta)\right| \cdot \left|[\partial_x^\gamma\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\sigma^1](x,2^j\xi,2^j\eta)\right| \qquad (2.9)\\
&\quad\lesssim 2^{j|\alpha_2+\beta_2|}\frac{\left|\partial^{\alpha_1}\widehat{\psi}(\xi)\partial^{\beta_1}\widehat{\phi}(\eta)\right|}{(|2^j\xi|+|2^j\eta|)^{|\alpha_2+\beta_2|-|\gamma|-m}} = 2^{j(m+|\gamma|)}\frac{\left|\partial^{\alpha_1}\widehat{\psi}(\xi)\partial^{\beta_1}\widehat{\phi}(\eta)\right|}{(|\xi|+|\eta|)^{|\alpha_2+\beta_2|-|\gamma|-m}} \lesssim 2^{j(m+|\gamma|)},
\end{aligned}$$

where all implicit constants are independent of $j \in \mathbb{Z}$, and in the first inequality we have used (2.3) for σ^1, and in the last inequality we have used that $\psi(\xi)\phi(\eta)$ is supported in

$\{(\xi,\eta) \in \mathbb{R}^{2n} : \frac{1}{2} \leq |\xi| + |\eta| \leq \frac{9}{4}\}$, so that

$$\frac{\left|\partial^{\alpha_1}\widehat{\psi}(\xi)\partial^{\beta_1}\widehat{\phi}(\eta)\right|}{(|\xi|+|\eta|)^{|\alpha_2+\beta_2|-|\gamma|-m}} \leq \frac{\left\|\partial^{\alpha_1}\widehat{\psi}\right\|_{L^\infty}\left\|\partial^{\beta_1}\widehat{\phi}\right\|_{L^\infty}}{\min\left\{\left(\frac{1}{2}\right)^{|\alpha_2+\beta_2|-|\gamma|-m}, \left(\frac{9}{4}\right)^{|\alpha_2+\beta_2|-|\gamma|-m}\right\}}.$$

We now define coefficients in our paraproduct decomposition using $\sigma^1[j]$. For $u, v \in \mathbb{R}^n$, set

$$\mathcal{C}^1[j](x,u,v) \coloneqq (1+|u|^2+|v|^2)^N \mathcal{F}[\sigma^1[j](x,\cdot,\cdot)](u,v),$$

where by $\mathcal{F}[\sigma^1[j](x,\cdot,\cdot)](u,v)$, we mean to take the Fourier transform of $\sigma^1[j](x,\xi,\eta)$ with respect to $(\xi,\eta) \in \mathbb{R}^{2n}$ and evaluate at $(u,v) \in \mathbb{R}^{2n}$. Define the operator $1-\Delta_{\xi,\eta}$ as the identity minus the standard Laplacian operator with respect to both ξ and η, that is, $[1-\Delta_{\xi,\eta}](f) \coloneqq f - \sum_{j=1}^n \left(\frac{\partial^2 f}{\partial \xi_j^2} + \frac{\partial^2 f}{\partial \eta_j^2}\right)$. Since $[1-\Delta_{\xi,\eta}](e^{-2\pi i(u\cdot\xi+v\cdot\eta)}) = (1+4\pi^2|u|^2 + 4\pi^2|v|^2)e^{-2\pi i(u\cdot\xi+v\cdot\eta)}$, we have that

$$\begin{aligned}
\left|\partial_x^\gamma \mathcal{C}^1[j](x,u,v)\right| &= \left|\partial_x^\gamma\left[\int_{\mathbb{R}^{2n}} (1+|u|^2+|v|^2)^N \sigma^1[j](x,\xi,\eta)e^{-2\pi i(u\cdot\xi+v\cdot\eta)}\,d\xi\,d\eta\right]\right| \\
&\sim \left|\int_{\mathbb{R}^{2n}} \partial_x^\gamma \sigma^1[j](x,\xi,\eta)[1-\Delta_{\xi,\eta}]^N(e^{-2\pi i(u\cdot\xi+v\cdot\eta)})\,d\xi\,d\eta\right| \\
&= \left|\int_{\frac{1}{2}<|\xi|+|\eta|<\frac{9}{4}} [1-\Delta_{\xi,\eta}]^N(\partial_x^\gamma\sigma^1[j])(x,\xi,\eta)e^{-2\pi i(u\cdot\xi+v\cdot\eta)}\,d\xi\,d\eta\right| \\
&\leq \int_{\frac{1}{2}<|\xi|+|\eta|<\frac{9}{4}} \left|[1-\Delta_{\xi,\eta}]^N(\partial_x^\gamma\sigma^1[j])(x,\xi,\eta)\right|\,d\xi\,d\eta \\
&\lesssim 2^{j(m+|\gamma|)},
\end{aligned}$$

where in the third line we have taken into consideration the support of $\sigma^1[j]$ and done integration by parts, and in the last line we are using the fact that a finite sum of functions uniformly bounded, up to a constant, by $2^{j(m+|\gamma|)}$ (see (2.9)) integrated over a compact set is itself bounded by $2^{j(m+|\gamma|)}$.

Finally, considering the Fourier inversion formula with $\sigma^1[j]$, we obtain

$$\sigma^1[j](x,2^{-j}\xi,2^{-j}\eta) = \int_{\mathbb{R}^{2n}} \mathcal{C}^1[j](x,u,v)e^{2\pi i(u\cdot 2^{-j}\xi+v\cdot 2^{-j}\eta)}\frac{du\,dv}{(1+|u|^2+|v|^2)^N}.$$

By substituting back into the formula for T_{σ^1} and using property (A.3.3) of the Fourier transform, we obtain

$$\begin{aligned}
T_{\sigma^1}(f,g)(x) &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} \left[\int_{\mathbb{R}^{2n}} \mathcal{C}^1[j](x,u,v) e^{2\pi i (u\cdot 2^{-j}\xi + v\cdot 2^{-j}\eta)} \frac{du\,dv}{(1+|u|^2+|v|^2)^N} \right] \\
&\qquad \times \widehat{\Psi}(2^{-j}\xi)\widehat{\Phi}(2^{-j}\eta)\widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta, \\
&= \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} \mathcal{C}^1[j](x,u,v) \left[\int_{\mathbb{R}^n} e^{2\pi i u\cdot 2^{-j}\xi} \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi) e^{2\pi i x\cdot\xi}\,d\xi \right] \\
&\qquad \times \left[\int_{\mathbb{R}^n} e^{2\pi i v\cdot 2^{-j}\eta} \widehat{\Phi}(2^{-j}\eta)\widehat{g}(\xi) e^{2\pi i x\cdot\eta}\,d\eta \right] \frac{du\,dv}{(1+|u|^2+|v|^2)^N}, \\
&= \int_{\mathbb{R}^{2n}} \sum_{j\in\mathbb{Z}} \mathcal{C}^1[j](x,u,v) [\Delta_j^{\tau_u\Psi} f](x) [S_j^{\tau_v\Phi} g](x) \frac{du\,dv}{(1+|u|^2+|v|^2)^N},
\end{aligned}$$

thus completing the proof of the lemma.

2.2.2 Representation for derivatives of paraproduct building blocks

We now state and prove Lemma 2.6, which leads to pointwise estimates relating to the building blocks within the paraproduct decomposition established in Lemma 2.5. First, we set some notation by defining, for $u, v \in \mathbb{R}^n$,

$$\sigma^1[u,v](x,\xi,\eta) := \sum_{j\in\mathbb{Z}} \mathcal{C}^1[j](x,u,v) \widehat{\tau_u\Psi}(2^{-j}\xi)\widehat{\tau_v\Phi}(2^{-j}\eta),$$

so that $T_{\sigma^1[u,v]}(f,g)(x) = \sum_{j\in\mathbb{Z}} \mathcal{C}^1[j](x,u,v)\Delta_j^{\tau_u\Psi} f(x) S_j^{\tau_v\Phi} g(x)$ and

$$T_{\sigma^1}(f,g)(x) = \int_{\mathbb{R}^{2n}} T_{\sigma^1[u,v]}(f,g)(x) \frac{du\,dv}{(1+|u|^2+|v|^2)^N}. \tag{2.10}$$

Similarly define $\sigma^2[u,v]$. The following lemma deals with derivatives of $T_{\sigma^1[u,v]}$ and $T_{\sigma^2[u,v]}$ evaluated at certain functions in $\mathcal{S}(\mathbb{R}^n)$ as required for Theorem 2.1:

Lemma 2.6. *If $\gamma \in \mathbb{N}_0^n$, $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, $u, v \in \mathbb{R}^n$, and $g, \Lambda \in \mathcal{S}(\mathbb{R}^n)$ such that supp$(\widehat{\Lambda}) \subseteq$*

$\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$, *then*

$$\partial^\gamma T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x) = 2^{\frac{\nu n}{2}} \sum_{\substack{\delta=-1 \\ \gamma_1+\gamma_2+\gamma_3=\gamma}}^{1} 2^{\nu|\gamma_2+\gamma_3|} C_{\gamma_1,\gamma_2,\gamma_3} \partial_x^{\gamma_1} \mathcal{C}^1[\nu-\delta](x,u,v)$$
$$\times \Upsilon[\delta,\gamma_2](2^\nu x - k + 2^\delta u) \cdot [\Theta[\delta,\gamma_3] * g(2^{-\nu}\cdot)](2^\nu x + 2^\delta v),$$

where $\Upsilon[\delta,\gamma_2], \Theta[\delta,\gamma_3] \in \mathcal{S}(\mathbb{R}^n)$ *are independent of* ν, k, u, v, *and* g, *and* $\Lambda_{\nu,k}(x) = 2^{\frac{\nu n}{2}}\Lambda(2^\nu x - k)$. *An analogous formula holds for* $\partial^\gamma T_{\sigma^2[u,v]}(f, \Lambda_{\nu,k})$ *with* $f \in \mathcal{S}(\mathbb{R}^n)$.

We are interested in studying derivatives of

$$T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x) = \sum_{j\in\mathbb{Z}} \mathcal{C}^1[j](x,u,v)\Delta_j^{\tau_u\Psi}\Lambda_{\nu,k}(x)S_j^{\tau_v\Phi}g(x). \tag{2.11}$$

By properties (A.3.3) and (A.3.4) of the Fourier transform, we have

$$\widehat{\Lambda_{\nu,k}}(\xi) = 2^{\frac{\nu n}{2}}\mathcal{F}[\Lambda(2^\nu\cdot - k)](\xi) = 2^{\frac{\nu n}{2}} e^{2\pi i(-2^{-\nu}k)\cdot\xi}\widehat{\Lambda(2^\nu\cdot)}(\xi) = 2^{-\frac{\nu n}{2}} e^{2\pi i(-2^{-\nu}k)\cdot\xi}\widehat{\Lambda}(2^{-\nu}\xi), \tag{2.12}$$

so the support of $\Lambda_{\nu,k}$ coincides with the support of $\widehat{\Lambda}(2^{-\nu}\cdot)$, which is contained within $\{\xi \in \mathbb{R}^n : 2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\}$. Since $\mathrm{supp}(\widehat{\Psi}(2^{-j}\cdot)) \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, we may verify that $\widehat{\Psi}(2^{-j}\xi)\widehat{\Lambda_{\nu,k}}(\xi)$ is zero whenever $j < \nu - 1$ or $j > \nu + 1$. Thus, we simplify our sum over $j \in \mathbb{Z}$ in (2.11) to only run over $\nu - 1 \leq j \leq \nu + 1$, or equivalently, we consider

$$T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x) = \sum_{\delta=-1}^{1} \mathcal{C}^1[\nu-\delta](x,u,v)[(\Delta_{\nu-\delta}^{\tau_u\Psi}\Lambda_{\nu,k}) \cdot (S_{\nu-\delta}^{\tau_v\Phi}g)](x).$$

Using the Fourier inversion formula, we have

$$\Delta_{\nu-\delta}^{\tau_u\Psi}\Lambda_{\nu,k}(x) = \int_{\mathbb{R}^n} \widehat{\tau_u\Psi}(2^{-(\nu-\delta)}\xi)\widehat{\Lambda_{\nu,k}}(\xi)e^{2\pi i x\cdot\xi}\,d\xi,$$

$$S_{\nu-\delta}^{\tau_v\Phi}g(x) = \int_{\mathbb{R}^n} \widehat{\tau_v\Phi}(2^{-(\nu-\delta)}\eta)\widehat{g}(\eta)e^{2\pi i x\cdot\eta}\,d\eta.$$

We make these replacements in the representation for $T_{\sigma^1[u,v]}(\Lambda_{\nu,k},g)(x)$ obtained above, along with using (2.12) and changes of variables $\xi \mapsto 2^\nu \xi$ and $\eta \mapsto 2^\nu \eta$, to reach a form for $T_{\sigma^1[u,v]}(\Lambda_{\nu,k},g)(x)$ given by

$$\sum_{\delta=-1}^{1} \mathcal{C}^1[\nu-\delta](x,u,v)\left(2^{\frac{\nu n}{2}}\int_{\mathbb{R}^n}\widehat{\tau_u\Psi}(2^\delta\xi)\widehat{\Lambda}(\xi)e^{2\pi i(2^\nu x-k)\cdot\xi}\,d\xi\right)\left(2^{\nu n}\int_{\mathbb{R}^n}\widehat{\tau_v\Phi}(2^\delta\eta)\widehat{g}(2^\nu\eta)e^{2\pi i(2^\nu x)\cdot\eta}\,d\eta\right).$$

With this representation for $T_{\sigma^1[u,v]}(\Lambda_{\nu,k},g)$ in mind, we define

$$\begin{aligned} G_\delta[u,v,\nu,k](x) &:= \mathcal{C}^1[\nu-\delta](x,u,v)\left(\int_{\mathbb{R}^n}\widehat{\tau_u\Psi}(2^\delta\xi)\widehat{\Lambda}(\xi)e^{2\pi i(2^\nu x-k)\cdot\xi}\,d\xi\right) \\ &\quad\times\left(2^{\nu n}\int_{\mathbb{R}^n}\widehat{\tau_v\Phi}(2^\delta\eta)\widehat{g}(2^\nu\eta)e^{2\pi i(2^\nu x)\cdot\eta}\,d\eta\right), \end{aligned}$$

so that for any multi-index $\gamma \in \mathbb{N}_0^n$,

$$\partial^\gamma T_{\sigma^1[u,v]}(\Lambda_{\nu,k},g)(x) = 2^{\frac{\nu n}{2}}\sum_{\delta=-1}^{1}\partial^\gamma G_\delta[u,v,\nu,k](x).$$

We conclude the proof by analyzing $\partial^\gamma G_\delta[u,v,\nu,k](x)$, which by an application of the Leibniz rule, may be written as a linear combination of terms having the form

$$\begin{aligned} &C_{\gamma_1,\gamma_2,\gamma_3}\partial_x^{\gamma_1}\mathcal{C}^1[\nu-\delta](x,u,v)\left(\int_{\mathbb{R}^n}\widehat{\tau_u\Psi}(2^\delta\xi)\widehat{\Lambda}(\xi)[(2\pi i2^\nu\xi)^{\gamma_2}e^{2\pi i(2^\nu x-k)\cdot\xi}]\,d\xi\right) \\ &\times\left(2^{\nu n}\int_{\mathbb{R}^n}\widehat{\tau_v\Phi}(2^\delta\eta)\widehat{g}(2^\nu\eta)[(2\pi i2^\nu\eta)^{\gamma_3}e^{2\pi i(2^\nu x)\cdot\eta}]\,d\eta\right), \quad \gamma_1+\gamma_2+\gamma_3=\gamma. \end{aligned}$$

By associating the $(2\pi i)^{\gamma_2}$ and $(2\pi i)^{\gamma_3}$ terms with the constant $C_{\gamma_1,\gamma_2,\gamma_3}$ and using property (A.3.3) of the Fourier transform, we see that each term above may be expressed as

$$\begin{aligned} &2^{\nu|\gamma_2+\gamma_3|}C_{\gamma_1,\gamma_2,\gamma_3}\partial_x^{\gamma_1}\mathcal{C}^1[\nu-\delta](x,u,v)\left(\int_{\mathbb{R}^n}\xi^{\gamma_2}\widehat{\Psi}(2^\delta\xi)\widehat{\Lambda}(\xi)e^{2\pi i(2^\nu x-k+2^\delta u)\cdot\xi}\,d\xi\right) \\ &\times\left(2^{\nu n}\int_{\mathbb{R}^n}\eta^{\gamma_3}\widehat{\Phi}(2^\delta\eta)\widehat{g}(2^\nu\eta)e^{2\pi i(2^\nu x+2^\delta v)\cdot\eta}\,d\eta\right). \end{aligned}$$

Finally, utilizing properties (A.3.2) and (A.3.4) of the Fourier transform, we put everything together to obtain the desired result, that is

$$\partial^\gamma T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x) = 2^{\frac{\nu n}{2}} \sum_{\substack{\delta=-1 \\ \gamma_1+\gamma_2+\gamma_3=\gamma}}^{1} 2^{\nu|\gamma_2+\gamma_3|} C_{\gamma_1,\gamma_2,\gamma_3} \partial_x^{\gamma_1} \mathcal{C}^1[\nu-\delta](x,u,v)$$

$$\times \Upsilon[\delta,\gamma_2](2^\nu x - k + 2^\delta u) \cdot [\Theta[\delta,\gamma_3] * g(2^{-\nu}\cdot)](2^\nu x + 2^\delta v),$$

where $\Upsilon[\delta,\gamma_2], \Theta[\delta,\gamma_3] \in \mathcal{S}(\mathbb{R}^n)$ are defined via

$$\widehat{\Upsilon[\delta,\gamma_2]}(\xi) := \xi^{\gamma_2} \widehat{\Psi}(2^\delta \xi) \widehat{\Lambda}(\xi) \quad \text{and} \quad \widehat{\Theta[\delta,\gamma_3]}(\eta) := \eta^{\gamma_3} \widehat{\Phi}(2^\delta \eta).$$

By swapping the u and v variables, and replacing g with f, we obtain essentially the same result for $\partial^\gamma T_{\sigma^2[u,v]}(f, \Lambda_{\nu,k})(x)$.

2.2.3 Conclusion for molecular decay properties

We now pull together Lemmas 2.5 and 2.6 to complete the proof of Theorem 2.1. Let $\sigma \in \dot{BS}^m_{1,1}$, $1 \le r \le \infty$, $0 < M < \infty$, $\Lambda \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\Lambda}$ is supported within $\{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$, and $g \in \mathcal{S}(\mathbb{R}^n)$. By Lemma 2.6, we see that $\left|\partial^\gamma T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x)\right|$ is bounded by

$$2^{\frac{\nu n}{2}} \sum_{\substack{\delta=-1 \\ \gamma_1+\gamma_2+\gamma_3=\gamma}}^{1} 2^{\nu|\gamma_2+\gamma_3|} C_{\gamma_1,\gamma_2,\gamma_3} \left|\partial_x^{\gamma_1} \mathcal{C}^1[\nu-\delta](x,u,v)\right|$$

$$\times \left|\Upsilon[\delta,\gamma_2](2^\nu x - k + 2^\delta u)\right| \cdot \left\|\Theta[\delta,\gamma_3] * g(2^{-\nu}\cdot)\right\|_{L^\infty}.$$

By (2.6), we know $|\partial_x^{\gamma_1} \mathcal{C}^1[\nu-\delta](x,u,v)| \lesssim 2^{(\nu-\delta)(m+|\gamma_1|)}$. Using this identity and Young's inequality (see Section B.2), we bound $\left|\partial^\gamma T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x)\right|$ further by

$$2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} \sum_{\substack{\delta=-1 \\ \gamma_1+\gamma_2+\gamma_3=\gamma}}^{1} 2^{-\delta(m+|\gamma_1|)} C_{\gamma_1,\gamma_2,\gamma_3} \|\Theta[\delta,\gamma_3]\|_{L^{r'}} \cdot \left\|g(2^{-\nu}\cdot)\right\|_{L^r} \cdot \left|\Upsilon[\delta,\gamma_2](2^\nu x - k + 2^\delta u)\right|,$$

where $1 \le r' \le \infty$ satisfies $\frac{1}{r} + \frac{1}{r'} = 1$. We note that

$$2^{-\delta(m+|\gamma_1|)} \le 2^{|m|+|\gamma|}, \qquad C_{\gamma_1,\gamma_2,\gamma_3} \le \max_{\widetilde{\gamma_1}+\widetilde{\gamma_2}+\widetilde{\gamma_3}=\gamma} \{C_{\widetilde{\gamma_1},\widetilde{\gamma_2},\widetilde{\gamma_3}}\},$$

$$\|\Theta[\delta,\gamma_3]\|_{L^{r'}} \le \max_{\substack{\widetilde{\delta}=-1,0,1 \\ \widetilde{\gamma}\le\gamma}} \left\{ \left\|\Theta[\widetilde{\delta},\widetilde{\gamma}]\right\|_{L^{r'}} \right\}, \qquad \|g(2^{-\nu}\cdot)\|_{L^r} = 2^{\frac{\nu n}{r}} \|g\|_{L^r},$$

so that we have

$$\left|\partial^\gamma T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x)\right| \lesssim 2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} 2^{\frac{\nu n}{r}} \|g\|_{L^r} \sum_{\substack{\delta=-1 \\ \widetilde{\gamma}\le\gamma}}^{1} \left|\Upsilon[\delta,\widetilde{\gamma}](2^\nu x - k + 2^\delta u)\right|,$$

where the implicit constant depends on m and γ, but not on ν, k, u, v, or g. Finally, we use (A.2) to obtain the bound

$$\left|\Upsilon[\delta,\widetilde{\gamma}](2^\nu x - k + 2^\delta u)\right| \lesssim \frac{(1+|2^\delta u|)^M}{(1+|2^\nu x - k|)^M},$$

where the implicit constant depends on $\Upsilon[\delta,\widetilde{\gamma}]$ and M, but we may take the max of such constants over $\delta = -1, 0, 1$ and $\widetilde{\gamma} \le \gamma$ so that the implicit constant depends only on γ and M. Therefore, we see that

$$\left|\partial^\gamma T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x)\right| \lesssim \frac{2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} 2^{\frac{\nu n}{r}}}{(1+|2^\nu x - k|)^M} \|g\|_{L^r} \left(\sum_{\delta=-1}^{1} (1+|2^\delta u|)^M \right),$$

Plugging this result back into (2.10), we see that

$$\begin{aligned}
|\partial^\gamma T_{\sigma^1}(\Lambda_{\nu,k}, g)(x)| &\le \int_{\mathbb{R}^{2n}} \left|\partial^\gamma T_{\sigma^1[u,v]}(\Lambda_{\nu,k}, g)(x)\right| \frac{du\,dv}{(1+|u|^2+|v|^2)^N} \\
&\lesssim \frac{2^{\frac{\nu n}{2}} 2^{\nu(m+|\gamma|)} 2^{\frac{\nu n}{r}}}{(1+|2^\nu x - k|)^M} \|g\|_{L^r} \left(\int_{\mathbb{R}^{2n}} \frac{\sum_{\delta=-1}^{1} (1+|2^\delta u|)^M}{(1+|u|^2+|v|^2)^N} \, du\,dv \right).
\end{aligned}$$

Finally, letting $N > 0$ be large enough so that the integral converges yields the desired result for $T_{\sigma^1}(\Lambda_{\nu,k}, g)(x)$. The $T_{\sigma^2}(f, \Lambda_{\nu,k})(x)$ case follows by very similar calculations, thus completing the proof of Theorem 2.1.

2.3 Setting for Leibniz-type rules

In this section, we define a number of function spaces relevant to the Leibniz-type rule estimates stated in Theorem 1.2. We begin by introducing homogeneous Besov-type and Triebel-Lizorkin-type spaces in Subsection 2.3.1, which are the settings in which Theorem 1.2 is stated. Additionally, we give a number of special cases of such spaces, so that Theorem 1.2 will imply Leibniz-type rules in many familiar function spaces. In Subsection 2.3.2, we give details on the general class of function spaces for which the procedures outlined in the proofs of Theorems 2.1 and 1.2 may be utilized to obtain Leibniz-type rule estimates, with the class of interest being spaces which admit a molecular decomposition in the sense of Frazier-Jawerth [24; 25].

Before we start defining spaces, we set some notation, beginning with the following definition:

Definition 2.7. We denote by $\mathcal{D}$ the **collection of dyadic cubes in** $\mathbb{R}^n$. That is, $\mathcal{D} := \{Q_{\nu,k}\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n}$, where

$$Q_{\nu,k} := \{x \in \mathbb{R}^n : k_j \leq 2^\nu x_j < k_j + 1, j = 1, \dots, n\}.$$

Additionally, for any $Q \in \mathcal{D}$, we denote its *edge length* by $l(Q)$ and its *volume* by $|Q|$, so that $l(Q_{\nu,k}) = 2^{-\nu}$ and $|Q_{\nu,k}| = 2^{-\nu n}$. Also, for $Q = Q_{\nu,k}$, we denote $x_Q = x_{\nu,k} := 2^{-\nu}k$, the *"lower-left" corner* of the cube.

In the following subsections, we will require functions $\lambda, \Lambda \in \mathcal{S}(\mathbb{R}^n)$ satisfying some or all of the following properties:

$$\operatorname{supp}(\widehat{\lambda}), \operatorname{supp}(\widehat{\Lambda}) \subseteq \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\}, \tag{2.13}$$

$$\left|\widehat{\lambda}(\xi)\right|, \left|\widehat{\Lambda}(\xi)\right| > c \quad \text{for all } \xi \text{ such that } \tfrac{3}{5} < |\xi| < \tfrac{5}{3} \text{ and some } c > 0, \tag{2.14}$$

$$\sum_{j\in\mathbb{Z}} \overline{\widehat{\lambda}(2^{-j}\xi)}\widehat{\Lambda}(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0. \tag{2.15}$$

We may construct such a pair by considering real-valued $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (A.5), (A.6), and (2.14), then setting $\lambda = \Lambda = \Psi^{\frac{1}{2}}$.

2.3.1 Homogeneous Besov-type and Triebel-Lizorkin-type spaces

For the definitions below, we fix $\lambda \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.13) and (2.14). In Definition 2.10, the primary definition of this subsection, we define Besov-type and Triebel-Lizorkin-type spaces, the settings in which we obtain results in Theorem 1.2. But first, we define a few other important function spaces, which will help establish some context for the function spaces in Definition 2.10.

Homogeneous Besov and Triebel-Lizorkin spaces, as presented in Definition 2.8, serve to unify many well-known classical function spaces, including Lebesgue spaces, Sobolev spaces, Hardy spaces, and $BMO(\mathbb{R}^n)$. For a comprehensive overview of the aforementioned function spaces and some historical context on the development of homogeneous Besov and Triebel-Lizorkin spaces, see Triebel [63–65] and references therein.

Definition 2.8. Let $s \in \mathbb{R}$ and $0 < q \leq \infty$.

(a) For $0 < p \leq \infty$, the **homogeneous Besov space**, denoted $\dot{B}^s_{p,q}(\mathbb{R}^n)$, is the set of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \left\| \Delta_j^\lambda f \right\|_{L^p}^q \right)^{\frac{1}{q}} < \infty.$$

(b) For $0 < p < \infty$, the **homogeneous Triebel-Lizorkin space**, denoted $\dot{F}^s_{p,q}(\mathbb{R}^n)$, is the set of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{p,q}} := \left\| \left[\sum_{j \in \mathbb{Z}} (2^{js} \left| \Delta_j^\lambda f \right|)^q \right]^{\frac{1}{q}} \right\|_{L^p} < \infty.$$

In the case $p = \infty$, $\dot{F}^s_{\infty,q}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{\infty,q}} := \sup_{P \in \mathcal{D}} \left\{ \frac{1}{|P|} \int_P \sum_{j=-\log_2(l(P))}^{\infty} (2^{js} \left|\Delta_j^\lambda f\right|)^q \, dx \right\}^{\frac{1}{q}} < \infty.$$

We note that the spaces in Definition 2.8 are independent of the choice of the function λ (see, for example, Triebel [65]). Also, these spaces are in general quasi-Banach spaces, and in the case where $p, q \geq 1$, are Banach spaces, having $\mathcal{S}_0(\mathbb{R}^n)$ as a dense subspace if p and q are finite.

Somewhat more recently, there has been growing interest in a new family of function spaces called Q-spaces. Originally introduced in Aulaskari et al. [1] as Q_s, $0 < s < 1$, to be a Banach space of analytic functions in the unit disk satisfying

$$\sup_{w \in B(0,1)} \int_{B(0,1)} |f'(z)|^2 \, g(z,w)^s \, dz \leq \infty,$$

where $g(z,w) := \left|\frac{\log(1-\bar{w}z)}{w-z}\right|$ is the Green's function of $B(0,1)$, these spaces were further developed in Euclidean spaces in Essén et al. [21] and shown to constitute a nested family of nontrivial subspaces of $BMO(\mathbb{R}^n)$ (for $0 < s < 1$ if $n \geq 2$, or for $0 < s \leq \frac{1}{2}$ if $n = 1$). Stated in full generality, we make the following definition:

Definition 2.9. Let $0 < s < 1$, $0 < p \leq \infty$, and $1 \leq q < \infty$. The Q-**space**, denoted $Q_p^{s,q}$, is the set of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that $f(x) - f(y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\|f\|_{Q_p^{s,q}} := \sup_I |I|^{\frac{1}{p}-\frac{1}{q}} \left\{ \int_I \int_I \frac{|f(x) - f(y)|^q}{|x-y|^{n+qs}} \, dy \, dx \right\}^{\frac{1}{q}} < \infty,$$

where I ranges over all cubes in $\mathbb{R}^n$ with dyadic edge lengths.

As shown in Essén et al. [21], Q_s coincides with $Q^{s,2}_{\frac{n}{s}}$, and such spaces have applications in the study of Navier-Stokes equations (see Xiao [66; 67]).

We are now able to motivate the definition of our primary function spaces of interest. Homogeneous Besov-type and Triebel-Lizorkin-type spaces, defined below in Definition 2.10, were introduced and studied in Sawano-Yang-Yuan [61] and Yang-Yuan [68; 69] as natural spaces which extend and unify the scales of homogeneous Besov spaces, homogeneous Triebel-Lizorkin spaces, and Q-spaces, all as defined above, and therefore unify scales of many familiar function spaces obtained as particular cases, as detailed following Definition 2.10.

Definition 2.10. Let $s, \tau \in \mathbb{R}$ and $0 < q \leq \infty$.

(a) For $0 < p \leq \infty$, the **homogeneous Besov-type space**, denoted $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$, is the set of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{D}} \frac{1}{|P|^{\tau}} \left\{ \sum_{j=-\log_2(l(P))}^{\infty} \left[\int_P (2^{js} \left|\Delta_j^{\lambda} f(x)\right|)^p \, dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} < \infty.$$

(b) For $0 < p < \infty$, the **homogeneous Triebel-Lizorkin-type space**, denoted $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, is the set of all $f \in \mathcal{S}_0'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^{s,\tau}_{p,q}} := \sup_{P \in \mathcal{D}} \frac{1}{|P|^{\tau}} \left\{ \int_P \left[\sum_{j=-\log_2(l(P))}^{\infty} (2^{js} \left|\Delta_j^{\lambda} f(x)\right|)^q \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} < \infty.$$

From these definitions, it is easily seen that, for $s, \tau \in \mathbb{R}$ and $0 < p < \infty$, $\dot{B}^{s,\tau}_{p,p} = \dot{F}^{s,\tau}_{p,p}$. Also, as in the case with Definition 2.8, we note that the spaces in Definition 2.10 are independent of the choice of λ (see Yang-Yuan [69, Corollary 3.1]). As in [69], we will use $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ to denote either $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ or $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, excluding $p = \infty$ in the latter case. Additionally, for ease of notation we will often refer to $\|\cdot\|_{\dot{A}^{s,\tau}_{p,q}}$ as a norm throughout the remainder of the chapter, despite the fact that it is only a norm if $p, q \geq 1$ and is otherwise a quasi-norm.

Special cases of $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$. We refer the reader to Yang-Yuan [68, Section 3] and [69, Proposition 3.1] regarding the following statements about particular cases and unification properties of $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$. Whenever we say that two spaces coincide, we mean to say they are comprised of the same set of functions and their function space norms are equivalent.

(i) Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. If $\infty < \tau < 0$, then $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n) = \mathcal{P}(\mathbb{R}^n)$, where $\mathcal{P}(\mathbb{R}^n)$

denotes the set of all polynomials on $\mathbb{R}^n$. If $0 \le \tau < \infty$, then $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space with $\mathcal{S}_0(\mathbb{R}^n) \subseteq \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

(ii) If $0 < p, q \le \infty$ and $s \in \mathbb{R}$, then $\dot{B}^{s,0}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Besov space $\dot{B}^{s}_{p,q}(\mathbb{R}^n)$.

(iii) If $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$, then $\dot{F}^{s,0}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Triebel-Lizorkin space $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$, with $\dot{F}^{0}_{p,2}$ coinciding with the Hardy space $H^p(\mathbb{R}^n)$. Further, if $1 < p < \infty$, then $\dot{F}^{s}_{p,2}(\mathbb{R}^n)$ coincides with the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^n)$, and $\dot{F}^{0}_{p,2}(\mathbb{R}^n)$ coincides with $L^p(\mathbb{R}^n)$ (see Theorem A.3).

(iv) If $0 < p < \infty$, $0 < q \le \infty$, and $s \in \mathbb{R}$, then $\dot{F}^{s,\frac{1}{p}}_{p,q}(\mathbb{R}^n)$ coincides with the homogeneous Triebel-Lizorkin space $\dot{F}^{s}_{\infty,q}$. In particular, $\dot{F}^{0,\frac{1}{p}}_{p,2}(\mathbb{R}^n)$ coincides with $BMO(\mathbb{R}^n)$.

(v) If $0 < p \le \infty$, $1 \le q < \infty$, and $0 < s < 1$, then $\dot{F}^{s,\frac{1}{q}-\frac{1}{p}}_{q,q}(\mathbb{R}^n)$ coincides with the Q-space $Q^{s,q}_{p}(\mathbb{R}^n)$. In particular, $\dot{F}^{s,\frac{1}{2}-\frac{s}{n}}_{2,2}(\mathbb{R}^n)$ coincides with $Q_s(\mathbb{R}^n)$.

(vi) Further special cases of the spaces $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ involving homogeneous Besov-Morrey and Triebel-Lizorkin-Morrey spaces (along with definitions of said spaces) can be found in Sawano-Yang-Yuan [61, Theorem 1.1].

We make particular note of the new Leibniz-type rules obtained by realizing Q-spaces as particular cases of Triebel-Lizorkin-type spaces. Theorem 1.2 yields the following immediate corollary providing Leibniz-type rules for Q-spaces:

Corollary 2.11. *Let $s, s+m \in (0,1)$ and $\sigma \in \dot{BS}^{m}_{1,1}$. If $1 \le q \le p \le \infty$ and $q \ne \infty$, it holds that*

$$\|T_\sigma(f,g)\|_{Q^{s,q}_p} \lesssim \|f\|_{Q^{s+m,q}_p} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{Q^{s+m,q}_p}, \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n).$$

In addition, by considering the case $\sigma \equiv 1$, Corollary 2.11 yields fractional Leibniz rules for Q-spaces.

2.3.2 Families of smooth synthesis molecules and spaces admitting a molecular decomposition

The ideas in this subsection are based on the pioneering work done in Frazier-Jawerth [24; 25], with specific results as they relate to homogeneous Besov-type and Triebel-Lizorkin-type spaces studied in Yang-Yuan [69].

We begin this subsection with a type of wavelet decomposition for functions in a variety of spaces. Fix $\lambda \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.13) and (2.14), and define

$$\lambda_{\nu,k}(x) := 2^{\frac{\nu n}{2}}\lambda(2^\nu x - k) = 2^{\frac{\nu n}{2}}\lambda(2^\nu(x - x_{\nu,k})), \tag{2.16}$$

where we note that $\|\lambda\|_{L^2} = \|\lambda_{\nu,k}\|_{L^2}$. Given $\Lambda \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.13), (2.14), and (2.15), the following wavelet-type decomposition holds:

$$f = \sum_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \langle f, \lambda_{\nu,k}\rangle \Lambda_{\nu,k}, \tag{2.17}$$

where the series converges for $f \in L^2(\mathbb{R}^n)$ in the topology of $L^2(\mathbb{R}^n)$, for $f \in \mathcal{S}_0(\mathbb{R}^n)$ in the topology of $\mathcal{S}(\mathbb{R}^n)$, and for $f \in \mathcal{S}'(\mathbb{R}^n)$ in the the topology of $\mathcal{S}'(\mathbb{R}^n)$ modulo polynomials (see Frazier-Jawerth [24; 25] for details). Note that the notation $\langle\cdot,\cdot\rangle$ denotes the standard inner product for complex-valued functions.

For some needed results relating to the wavelet decomposition given in (2.17), we require sequence space analogs to Definition 2.10:

Definition 2.12. Let $s \in \mathbb{R}$, $0 \le \tau < \infty$, and $0 < q \le \infty$.

(a) For $0 < p \le \infty$, define the sequence space $\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)$ to be the collection of all sequences $t = \{t_Q\}_{Q\in\mathcal{D}} \subset \mathbb{C}$ indexed by the dyadic cubes such that

$$\|t\|_{\dot{b}^{s,\tau}_{p,q}} := \sup_{P\in\mathcal{D}} \frac{1}{|P|^\tau}\left\{\sum_{j=-\log_2(l(P))}^{\infty}\left[\int_P\left(\sum_{l(Q)=2^{-j}}|Q|^{-\frac{s}{n}-\frac{1}{2}}|t_Q|\,\chi_Q(x)\right)^p dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} < \infty.$$

(b) For $0 < p < \infty$, define the sequence space $\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)$ to be the collection of all sequences $t = \{t_Q\}_{Q\in\mathcal{D}} \subset \mathbb{C}$ indexed by the dyadic cubes such that

$$\|t\|_{\dot{f}^{s,\tau}_{p,q}} := \sup_{P\in\mathcal{D}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{Q\subset P} \left(|Q|^{-\frac{s}{n}-\frac{1}{2}} |t_Q| \chi_Q(x) \right)^q \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} < \infty.$$

As in the case of $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, we will use $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ to denote either $\dot{b}^{s,\tau}_{p,q}(\mathbb{R}^n)$ or $\dot{f}^{s,\tau}_{p,q}(\mathbb{R}^n)$, excluding $p = \infty$ in the latter case. A direct connection between the function spaces $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the sequence spaces $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ was established in Yang-Yuan [69, Theorem 3.1], wherein it was shown that the two spaces are related in the following way: If $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq \tau < \infty$, $f \in \mathcal{S}'_0(\mathbb{R}^n)$, and $\lambda \in \mathcal{S}(\mathbb{R}^n)$ satisfies (2.13) and (2.14), then

$$\|f\|_{\dot{A}^{s,\tau}_{p,q}} \sim \|\{\langle f, \lambda_{\nu,k}\rangle\}_{\nu,k}\|_{\dot{a}^{s,\tau}_{p,q}}. \tag{2.18}$$

For our proof of Theorem 1.2, we will make use of (2.18), along with one additional norm comparison property relating $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $\dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n)$ stated below as (2.20), for which we require a few additional definitions.

In Frazier-Jawerth [24; 25], the authors study sequence spaces characterizing homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}(\mathbb{R}^n)$, obtaining norm relationships akin to (2.18) and (2.20), with the latter requiring a notion of families of smooth synthesis molecules in $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$. Such families are related to almost-diagonal operators, another notion defined in [24; 25] which contributes to verifying boundedness properties similar to (2.20). In Definition 2.13 below, we give an analogous characterization for families of smooth synthesis molecules in $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, as presented in Yang-Yuan [69, Definition 4.2]. First, for $s \in \mathbb{R}$, define $s^* := s - \lfloor s \rfloor$, where $\lfloor s \rfloor$ denotes the largest integer smaller than or equal to s. Also define

$$\begin{aligned} s_p &:= n\left(\tfrac{1}{\min\{1,p\}} - 1\right), \\ s_{p,q} &:= n\left(\tfrac{1}{\min\{1,p,q\}} - 1\right), \end{aligned} \qquad J := \begin{cases} s_p + n & \text{if } \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n), \\ s_{p,q} + n & \text{if } \dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n). \end{cases} \tag{2.19}$$

Definition 2.13. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $0 \leq \tau < \infty$, and $Q \in \mathcal{D}$. A function $m_Q : \mathbb{R}^n \to \mathbb{C}$ is a **smooth synthesis molecule for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$** if there exist δ and M satisfying

$$\max\{s^*, (s+n\tau)^*\} < \delta \leq 1 \quad \text{and} \quad J < M < \infty,$$

such that the conditions given below hold for all $x, y \in \mathbb{R}^n$:

(i) For $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq \max\{\lfloor J - n - s \rfloor, -1\}$, m_Q satisfies a *vanishing moment condition* given by

$$\int_{\mathbb{R}^n} m_Q(z) z^\gamma \, dz = 0.$$

(ii) m_Q satisfies a size estimate given by

$$|m_Q(x)| \leq \frac{|Q|^{-\frac{1}{2}}}{(1 + l(Q)^{-1} |x - x_Q|)^{\max\{M, M-s\}}}.$$

(iii) For $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq \lfloor s + n\tau \rfloor$,

$$|\partial^\gamma m_Q(x)| \leq \frac{|Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}}}{(1 + l(Q)^{-1} |x - x_Q|)^M}.$$

(iv) For $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = \lfloor s + n\tau \rfloor$,

$$|\partial^\gamma m_Q(x) - \partial^\gamma m_Q(y)| \leq \sup_{|z| \leq |x-y|} \frac{|Q|^{-\frac{1}{2} - \frac{|\gamma|}{n} - \frac{\delta}{n}} |x - y|^\delta}{(1 + l(Q)^{-1} |x - z - x_Q|)^M}.$$

A collection $\{m_Q\}_{Q \in \mathcal{D}}$ indexed by the dyadic cubes is called a **family of smooth synthesis molecules for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$** if each m_Q is a smooth synthesis molecule for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

For a smooth synthesis molecule m_Q, property (i) of Definition 2.13 states that $\widehat{m_Q}$ and sufficiently many of its derivatives must be zero at the origin, since property (A.3.5) of the Fourier transform implies that, for any $\gamma \in \mathbb{N}_0^n$,

$$\partial^\gamma \widehat{m_Q}(0) \sim \int_{\mathbb{R}^n} x^\gamma m_Q(x) e^{-2\pi i 0 \cdot x} \, dx = \int_{\mathbb{R}^n} x^\gamma m_Q(x) \, dx.$$

The remainder of the properties imply that m_Q and sufficiently many of its derivatives decay away from the dyadic cube Q by which the function is indexed, where said decay is enough to verify a so-called almost-diagonal condition on certain operators associated with families of smooth synthesis molecules for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$. Below, we conclude the subsection with two remarks: Remark 2.3.1 shows that any m_Q which satisfies property (iii) of Definition 2.13 for all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq \lfloor s + n\tau \rfloor + 1$ necessarily satisfies property (iv) with $\delta = 1$, and Remark 2.3.2 verifies that $\{\lambda_{\nu,k}\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n}$ and $\{\Lambda_{\nu,k}\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n}$ (as defined in (2.16)) are families of smooth synthesis molecules for any $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ with parameters s, τ, p, and q as in Definition 2.13, $\delta = 1$, and any $M > J$.

In the proof of Theorem 1.2, we will make use of Theorem 2.14 below, which gives certain norm comparisons associated with families of smooth synthesis molecules on $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, and is proved in Yang-Yuan [69, Theorem 4.2] by analogous ideas on almost-diagonal operators used to prove Frazier-Jawerth [25, Theorem 3.5].

Theorem 2.14. *Let* $0 < p, q \leq \infty$*,* $s \in \mathbb{R}$*, and* $0 \leq \tau < \mathcal{T}$*, where if* $\max\{\lfloor J - n - s \rfloor, -1\} \geq 0$*,*

$$\mathcal{T} := \min\left\{\frac{1}{p} + \frac{M-J}{2n}, \frac{1}{p} + \frac{1-(J-s)^*}{n}\right\},$$

or if $\max\{\lfloor J - n - s \rfloor, -1\} < 0$*,*

$$\mathcal{T} := \min\left\{\frac{1}{p} + \frac{M-J}{2n}, \frac{1}{p} + \frac{s+n-J}{n}\right\}.$$

If $\{m_Q\}_{Q\in\mathcal{D}}$ *is a family of smooth synthesis molecules for* $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ *with parameters* δ *and* M *satisfying* $\max\{s^*, (s+n\tau)^*\} < \delta \leq 1$ *and* $J < M < \infty$*, then*

$$\left\| \sum_{Q\in\mathcal{D}} t_Q m_Q \right\|_{\dot{A}^{s,\tau}_{p,q}} \lesssim \|t\|_{\dot{a}^{s,\tau}_{p,q}}, \quad \forall t = \{t_Q\}_{Q\in\mathcal{D}} \in \dot{a}^{s,\tau}_{p,q}(\mathbb{R}^n), \tag{2.20}$$

where the implicit constant does not depend on the family of molecules.

Remark 2.3.1. Suppose $m_Q : \mathbb{R}^n \mapsto \mathbb{C}$ satisfies property (iii) of Definition 2.13 for some $M > J$ and all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq \lfloor s + n\tau \rfloor + 1$. We will see that m_Q also satisfies property (iv) with $\delta = 1$. Indeed, let $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = \lfloor s + n\tau \rfloor$. Then, for any $x, y \in \mathbb{R}^n$, the Mean Value Theorem implies the existence of some $t \in (0,1)$ with $z_{x,y} = tx + (1-t)y$ and

$$\nabla \partial^\gamma m_Q(z_{x,y}) = \frac{\partial^\gamma m_Q(x) - \partial^\gamma m_Q(y)}{x - y}.$$

Note that $|\nabla \partial^\gamma m_Q(z_{x,y})| \le \sum_{j=1}^n |\partial^{\gamma + e_j} m_Q(z_{x,y})|$ with $|\gamma + e_j| = \lfloor s + n\tau \rfloor + 1$, so that by property (iii), we see

$$|\partial^\gamma m_Q(x) - \partial^\gamma m_Q(y)| \le \sum_{j=1}^n \left|\partial^{\gamma+e_j} m_Q(z_{x,y})\right| \cdot |x - y| \le n \cdot \sup_{|z| \le |x-y|} \frac{|Q|^{-\frac{1}{2} - \frac{|\gamma|+1}{n}} |x - y|}{(1 + l(Q)^{-1} |x - z - x_Q|)^M},$$

where we have used that $z_{x,y} = x - z$ for some $z \in B(0, |x - y|)$. Therefore, property (iv) is satisfied, as desired.

Remark 2.3.2. Let p, q, s, and τ be as in Definition 2.13, and fix $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\lambda \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.13). We will show that $\lambda_{\nu,k}$, as defined in (2.16), is a smooth synthesis molecule for $\dot{A}_{p,q}^{s,\tau}$ with parameters $\delta = 1$ and any $M > J$. Considering the arguments made following Definition 2.13, we see that $\lambda_{\nu,k}$ satisfies property (i) since $\widehat{\lambda_{\nu,k}}$ has the same support as $\widehat{\lambda}(2^{-\nu}\cdot)$ (see (2.12)), which is compactly supported away from the origin. Next, let $\gamma \in \mathbb{N}_0^n$. Since

$$\partial^\gamma \lambda_{\nu,k}(x) = \partial^\gamma [2^{\frac{\nu n}{2}} \lambda(2^\nu(x - x_{\nu,k}))] = 2^{\frac{\nu n}{2}} [\partial^\gamma \lambda](2^\nu(x - x_{\nu,k})) 2^{\nu|\gamma|},$$

we use Definition 2.7 and (A.1) (since $\partial^\gamma \lambda \in \mathcal{S}(\mathbb{R}^n)$) to see that property (iii) holds for any $\gamma \in \mathbb{N}_0^n$ (and subsequently, property (iv) holds as reasoned in Remark 2.3.1, as does property (ii) by taking $\gamma = 0$ and $N = \max\{M, M - s\}$ in (A.1)).

2.4 Proof of Leibniz-type rules

We proceed with the proof of Theorem 1.2. Fix $\lambda, \Lambda \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.13), (2.14), and (2.15). We will consider only $T_{\sigma^1}(f, g)$; analogous steps apply to $T_{\sigma^2}(f, g)$. For $f, g \in \mathcal{S}_0(\mathbb{R}^n)$, we have

$$T_{\sigma^1}(f,g) = \sum_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \langle f, \lambda_{\nu,k}\rangle T_{\sigma^1}(\Lambda_{\nu,k}, g),$$

with convergence in $\mathcal{S}(\mathbb{R}^n)$, and where we have used that T_{σ^1} is continuous from $\mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$, that (2.17) converges in $\mathcal{S}_0(\mathbb{R}^n)$, and the linearity of T_{σ^1}.

Theorem 2.1 implies that, for a constant c_1 implicit in inequality (2.1), we have that

$$\left\{ \frac{c_1 2^{-\nu m} T_{\sigma^1}(\Lambda_{\nu,k}, g)}{\|g\|_{L^\infty}} \right\}_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n}$$

is a family of smooth synthesis molecules for any $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$ if $0 < p, q \le \infty$, $s > J - n$, and $0 \le \tau < \infty$ (with $\delta = 1$ and any $M > J$). Indeed, property (i) of Definition 2.13 is vacuously satisfied since $\lfloor J - n - s\rfloor < 0$. Further, by Theorem 2.1, we have for any $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\gamma \in \mathbb{N}_0^n$,

$$\left| \partial^\gamma \left[\frac{c_1 2^{-\nu m} T_{\sigma^1}(\Lambda_{\nu,k}, g)}{\|g\|_{L^\infty}} \right](x) \right| \lesssim \frac{2^{\frac{\nu n}{2}} 2^{\nu|\gamma|}}{(1 + |2^\nu x - k|)^M} = \frac{|Q|^{-\frac{1}{2}} |Q|^{-\frac{|\gamma|}{n}}}{(1 + l(Q)^{-1}|x - x_Q|)^M},$$

where we have used Definition 2.7. Since the above holds for all $\gamma \in \mathbb{N}_0^n$, we see that properties (ii), (iii), and (iv) hold (for (ii), consider $\gamma = 0$, keeping in mind that the result of Theorem 2.1 holds for any $0 < M < \infty$, that is, it holds for $\max\{M, M - s\}$; for (iv), see Remark 2.3.1).

We will apply (2.18) without any restrictions below, but to apply (2.20), we require that $0 \le \tau < \frac{1}{p} + \frac{s+n-J}{n}$, so as to satisfy the hypotheses of Theorem 2.14 (we may choose any $M > J$, so we let M be large enough so that the minimum in the expression for $\mathcal{T}$ in

Theorem 2.14 equals $\frac{1}{p} + \frac{s+n-J}{n}$). Considering $T_{\sigma^1}(f,g)$, we have

$$\begin{aligned}
\|T_{\sigma^1}(f,g)\|_{\dot{A}^{s,\tau}_{p,q}} &= \left\| \sum_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \langle f, \lambda_{\nu,k}\rangle T_{\sigma^1}(\Lambda_{\nu,k}, g) \right\|_{\dot{A}^{s,\tau}_{p,q}} \\
&\sim \left\| \sum_{\nu\in\mathbb{Z},k\in\mathbb{Z}^n} \left(2^{\nu m} \|g\|_{L^\infty} \langle f, \lambda_{\nu,k}\rangle\right) \left(\frac{c_1 2^{-\nu m} T_{\sigma^1}(\Lambda_{\nu,k}, g)}{\|g\|_{L^\infty}} \right) \right\|_{\dot{A}^{s,\tau}_{p,q}} \\
&\lesssim \|\{2^{\nu m} \|g\|_{L^\infty} \langle f, \lambda_{\nu,k}\rangle\}_{\nu,k}\|_{\dot{a}^{s,\tau}_{p,q}} \\
&= \|\{\langle f, \lambda_{\nu,k}\rangle\}_{\nu,k}\|_{\dot{a}^{s+m,\tau}_{p,q}} \|g\|_{L^\infty} \\
&\sim \|f\|_{\dot{A}^{s+m,\tau}_{p,q}} \|g\|_{L^\infty},
\end{aligned}$$

where in the first line we have used the decomposition for $T_{\sigma^1}(f,g)$ given above, in the third line we have used Theorem 2.14 and the fact that the piece in parentheses in the second line is a smooth synthesis molecule for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, in the fourth line we have used that, if $Q = Q_{\nu,k}$, then

$$|Q|^{-\frac{s}{n}} 2^{\nu m} = 2^{\nu s} 2^{\nu m} = 2^{\nu(m+s)} = |Q|^{-\frac{s+m}{n}}$$

(see Definition 2.12), and in the final line we have used (2.18). Following the above calculations, we obtain an analogous result for $T_{\sigma^2}(f,g)$, namely that $\|T_{\sigma^2}(f,g)\|_{\dot{A}^{s,\tau}_{p,q}} \lesssim \|f\|_{L^\infty} \|g\|_{\dot{A}^{s+m,\tau}_{p,q}}$. Since $T_\sigma(f,g) = T_{\sigma^1}(f,g) + T_{\sigma^2}(f,g)$, we use the subadditivity (or quasi-subadditivity, if p or q is less than 1) of the $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$-norm to obtain the desired result.

2.5 Remarks on Leibniz-type rules

We conclude this chapter with a few remarks regarding extensions of Theorem 1.2. Remark 2.5.1 below gives analogous results for the cases $s \le s_p$ and $s \le s_{p,q}$ by assuming some additional cancellation conditions on operators T_{σ^1} and T_{σ^2}, and Remark 2.5.2 discusses how the proof of Theorem 1.2 could be altered slightly to give results involving the $L^r(\mathbb{R}^n)$-norm of functions, $1 \le r < \infty$, instead of the $L^\infty(\mathbb{R}^n)$-norm of functions. Finally, Remark 2.5.3 examines how the implicit constants in Theorems 2.1 and 1.2 depend on the symbol σ.

Remark 2.5.1. Let $m \in \mathbb{R}$, $\sigma \in \dot{B}S^m_{1,1}$, and $0 < p, q \leq \infty$, as in the hypotheses for Theorem 1.2. Unlike the statement of Theorem 1.2, let $s \leq J - n$. The proof of Theorem 1.2 in Section 2.4 nearly works exactly in this case, with the only difference being that property (i) of Definition 2.13 is no longer vacuously true. By imposing some cancellation conditions on T_{σ^1} and T_{σ^2}, we will see that property (i) of Definition 2.13 is satisfied, which then implies the results of Theorem 1.2 in the case where $s \leq J - n$ and $0 \leq \tau < \frac{1}{p} + \frac{1-(J-s)^*}{n}$. The cancellation conditions we require are as follows:

$$T^{*1}_{\sigma^1}(x^\gamma, g) = T^{*2}_{\sigma^2}(f, x^\gamma) = 0, \quad \forall f, g \in \mathcal{S}_0(\mathbb{R}^n), \gamma \in \mathbb{N}_0^n \text{ with } |\gamma| \leq \lfloor J - n - s \rfloor,$$

where T^{*1} and T^{*2} denote the adjoint operators of a bilinear operator T. Specifically, if T is continuous from $\mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}_0(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$, then T^{*1} and T^{*2}, which map from $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}_0(\mathbb{R}^n)$ to $\mathcal{S}_0'(\mathbb{R}^n)$ and from $\mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}_0'(\mathbb{R}^n)$, respectively, are defined as

$$\langle h, T(f, g) \rangle = \langle T^{*1}(h, g), f \rangle = \langle T^{*2}(f, h), g \rangle.$$

The cancellation conditions above imply, for $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq \lfloor J - n - s \rfloor$,

$$\int_{\mathbb{R}^n} x^\gamma T_{\sigma^1}(\lambda_{\nu,k}, g)(x)\, dx = \langle x^\gamma, T_{\sigma^1}(\lambda_{\nu,k}, g) \rangle = \langle T^{*1}_{\sigma^1}(x^\gamma, g), \lambda_{\nu,k} \rangle = 0, \quad \forall g \in \mathcal{S}_0(\mathbb{R}^n),$$

and similarly for $T_{\sigma^2}(f, \lambda_{\nu,k})$. Thus, the families of functions indexed over dyadic cubes defined in Section 2.4 satisfy Definition 2.13 as families of smooth synthesis molecules for $\dot{A}^{s,\tau}_{p,q}$, so the proof given in Section 2.4 applies directly in this case.

Remark 2.5.2. Let m, σ, p, q, s, and τ be as in the hypotheses of Theorem 1.2 or Remark 2.5.1, and let $1 \leq r < \infty$. Since the families

$$\left\{ \frac{c_1 2^{-\nu m} 2^{-\frac{\nu n}{r}} T_{\sigma^1}(\Lambda_{\nu,k}, g)}{\|g\|_{L^r}} \right\}_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n} \quad \text{and} \quad \left\{ \frac{c_2 2^{-\nu m} 2^{-\frac{\nu n}{r}} T_{\sigma^2}(f, \Lambda_{\nu,k})}{\|f\|_{L^r}} \right\}_{\nu \in \mathbb{Z}, k \in \mathbb{Z}^n}$$

are also families of smooth synthesis molecules for $\dot{A}^{s,\tau}_{p,q}(\mathbb{R}^n)$, we may follow the reasoning in Section 2.4 to obtain

$$\|T_\sigma(f,g)\|_{\dot{A}^{s,\tau}_{p,q}} \lesssim \|f\|_{\dot{A}^{s+m+\frac{n}{r},\tau}_{p,q}} \|g\|_{L^r} + \|f\|_{L^r} \|g\|_{\dot{A}^{s+m+\frac{n}{r},\tau}_{p,q}} .$$

Remark 2.5.3. By carefully following the proofs of Theorem 2.1 and Theorem 1.2, we may verify that the implicit constants in the results of the theorems depend linearly on $\|\sigma\|_{K,L}$ for some $K, L \in \mathbb{N}$, where

$$\|\sigma\|_{K,L} \coloneqq \max_{|\gamma|\le K, |\alpha+\beta|\le L} \|\sigma\|_{\gamma,\alpha,\beta} ,$$

and $\|\sigma\|_{\gamma,\alpha,\beta}$ is as in Definition 2.2. We omit the careful tracking of constants here, but we may find that the implicit constants in the inequalities of Theorem 2.1 are multiples of $\|\sigma\|_{|\gamma|,2N}$ with $N \in \mathbb{N}$ such that $N > M + n$, and where γ and M are as in the statement of the theorem. In turn, this implies that the implicit constants in Theorem 1.2 can be taken to be multiples of $\|\sigma\|_{\lfloor s+n\tau\rfloor+1,2N}$ with $N > \max\{J+n, 2(s+n)-J+n\}$. The latter is also true for the inequalities from Remark 2.5.1 with $N > J + n + 2(1-(J-s)^*)$.

Chapter 3

Weighted Fractional Leibniz-Type Rules for Bilinear Multiplier Operators

3.1 Introduction

Our objective in this chapter is to prove the weighted Leibniz-type rules presented in Theorem 1.3 and Theorem 1.4. In Subsection 3.1.1, we introduce the classes of multipliers appearing in the statements of Theorems 1.3 and 1.4, along with a history of the development of boundedness properties for associated multiplier operators. Then, in Subsection 3.1.2, we highlight new Kato-Ponce inequalities relating to Theorems 1.3 and 1.4. We give the rest of the background for Theorems 1.3 and 1.4 in Section 3.2, including definitions and notation relating to weighted Lebesgue spaces and a number of lemmas containing square function-type estimates for weighted Lebesgue spaces which will be useful in proving the main Leibniz-type rule results. We conclude the chapter with Sections 3.3 and 3.4, wherein we prove Theorems 1.4 and 1.3, respectively.

3.1.1 Coifman-Meyer multipliers

In this section we introduce the classes of multipliers considered in Theorems 1.3 and 1.4.

DEFINITION 3.1. Let $\sigma(\xi,\eta) \in L^\infty(\mathbb{R}^{2n})$ be smooth away from the origin.

(a) σ is a **Coifman-Meyer multiplier** if, for all $\alpha, \beta \in \mathbb{N}_0^n$, there exists $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)\right| \leq \frac{C_{\alpha,\beta}}{(|\xi|+|\eta|)^{|\alpha|+|\beta|}}, \quad \forall (\xi,\eta) \neq (0,0). \tag{3.1}$$

(b) Let $n = n_1 + n_2$ for $n_1, n_2 \in \mathbb{N}$, $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. σ is a **biparameter Coifman-Meyer multiplier** if, for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}_0^{n_1} \times \mathbb{N}_0^{n_2}$, there exists $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)\right| \leq \frac{C_{\alpha,\beta}}{(|\xi_1|+|\eta_1|)^{|\alpha_1|+|\beta_1|}(|\xi_2|+|\eta_2|)^{|\alpha_2|+|\beta_2|}}, \quad \forall (\xi,\eta) \neq (0,0). \tag{3.2}$$

We note that symbols which satisfy (3.1) necessarily satisfy (3.2). As outlined in Subsection 2.1.1, the class of Coifman-Meyer multipliers coincides exactly with the x-independent symbols in $B\dot{S}^0_{1,1}$.

Boundedness properties associated to a general bilinear pseudodifferential operator T_σ have the following form in the setting of Lebesgue spaces:

$$\|T_\sigma(f,g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}. \tag{3.3}$$

When σ is a Coifman-Meyer multiplier, such estimates have been extensively studied; in particular, (3.3) holds for $1 < p, q < \infty$ and r such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. See Coifman-Meyer [15] for the introduction of Coifman-Meyer multipliers and the study of their boundedness properties on $L^2(\mathbb{R}^n)$, and see David-Journé [18] and Grafakos-Torres [33] for further work. Also relating to Coifman-Meyer multipliers, estimates similar to (3.3) in the context of weighted Lebesgue spaces are considered in Grafakos-Torres [32], with further development of results in Lerner et al. [45], wherein the authors prove weighted boundedness results for a more general class of operators, the class of bilinear Calderón-Zygmund operators. Analogous estimates relating to biparameter Coifman-Meyer multipliers have also been studied. In particular, boundedness properties for T_σ with σ satisfying (3.2) with $n_1 = n_2$ were proved in the unweighted setting in Muscalu et al. [50; 51] and Lacey-Metcalfe [44] (see also Journé [38] for some earlier specific case results) and in the weighted setting in Chen-Lu [13].

Remark 3.1.1. There is a natural connection between studying fractional Leibniz rules and Coifman-Meyer multipliers. For example, in studying (1.1), we may obtain a decomposition given by

$$D^s(fg)(x) = T_{\sigma^1}(D^s f, g)(x) + T_{\sigma^2}(f, D^s g)(x) + T_{\sigma^3}(D^s f, g)(x),$$

where σ^1 and σ^2 are Coifman-Meyer multipliers for $s > 0$, while σ^3 is a Coifman-Meyer multiplier if $s \geq 2n+1$ or $s \in 2\mathbb{N}_0$. Since boundedness properties relating to Coifman-Meyer multipliers are well-understood (based upon the works mentioned above), we may readily obtain bounds for some pieces of the decomposition above; the T_{σ^3} piece requires further analysis for $s < 2n + 1$, in which case σ^3 is not, in general, a Coifman-Meyer multiplier, nor does it belong to any class of symbols for which boundedness properties in the setting of Lebesgue spaces are known to hold. A similar connection may be drawn in relation to (1.2) by decomposing $J^s(fg)(x)$ as a sum of two Coifman-Meyer multipliers and a third multiplier operator requiring additional analysis, and in relation to fractional Leibniz rules associated to $D_1^{s_1} D_2^{s_2}(fg)$ by utilizing a decomposition involving biparameter Coifman-Meyer multipliers and other multiplier operators for which further analysis is needed.

3.1.2 The case $\sigma \equiv 1$ and connections to Kato-Ponce inequalitites

Restricting to the case $\sigma \equiv 1$ in Theorems 1.3 and 1.4, we obtain weighted counterparts to the fractional Leibniz rules, or Kato-Ponce inequalitites, introduced in Chapter 1. Even in this simplest case, where $T_\sigma(f, g)$ is replaced by the pointwise product fg, Theorem 1.3 and Theorem 1.4 yield new estimates, highlighted in Corollary 3.2 and Corollary 3.3 below. Specifically, the estimates in Corollary 3.2 are new for the cases $p = \infty$ or $q = \infty$ (for $1 < p, q < \infty$, see Cruz-Uribe-Naibo [17]), and the results in Corollary 3.3 are new.

Corollary 3.2. *Let $1 < p, q \leq \infty$, $\frac{1}{2} < r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $s > \max\{0, n(\frac{1}{r} - 1)\}$ or $s \in 2\mathbb{N}_0$. If $v \in A_p(\mathbb{R}^n)$ and $w \in A_q(\mathbb{R}^n)$, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, it holds that*

$$\|D^s(fg)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})} \lesssim \|D^s f\|_{L^p(v)} \|g\|_{L^q(w)} + \|f\|_{L^p(v)} \|D^s g\|_{L^q(w)},$$

$$\|J^s(fg)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})} \lesssim \|J^s f\|_{L^p(v)} \|g\|_{L^q(w)} + \|f\|_{L^p(v)} \|J^s g\|_{L^q(w)},$$

where the implicit constant depends on p, q, s, $[v]_{A_p}$, $[w]_{A_q}$, and σ. If $v = w$, different choices of p and q are allowed in each term on the right-hand side of the above inequalities.

Corollary 3.3. *Let $n = n_1 + n_2$ for $n_1, n_2 \in \mathbb{N}$, $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Further, let $1 < p, q < \infty$, $\frac{1}{2} < r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $s_\ell > \max\{0, n_\ell(\frac{1}{r} - 1)\}$, $\ell = 1, 2$. If $v \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $w \in A_q(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, it holds that*

$$\begin{aligned}\|D_1^{s_1} D_2^{s_2}(fg)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})} \lesssim \|D_1^{s_1} D_2^{s_2} f\|_{L^p(v)} \|g\|_{L^q(w)} + \|D_1^{s_1} f\|_{L^p(v)} \|D_2^{s_2} g\|_{L^q(w)} \\ + \|D_2^{s_2} f\|_{L^p(v)} \|D_1^{s_1} g\|_{L^q(w)} + \|f\|_{L^p(v)} \|D_1^{s_1} D_2^{s_2} g\|_{L^q(w)},\end{aligned}$$

where the implicit constant depends on p, q, s_1, s_2, $[v]'_{A_p}$, $[w]'_{A_q}$, and σ. If $v = w$, different choices of p and q are allowed in each term on the right-hand side of the above inequality.

Notice that, by considering weights identically equal to 1, the results of Corollary 3.2 recover (1.1) and (1.2), while the results of Corollary 3.3 would yield biparameter Kato-Ponce inequalities, as studied in [30; 50]. It is also worth noting that the assumptions on s, s_1, and s_2 in the corollaries above (and therefore in the main theorems) are sharp; the estimates do not necessarily hold if $s \leq \max\{0, n(\frac{1}{r} - 1)\}$ in Corollary 3.2 or if $s_\ell \leq \max\{0, n_\ell(\frac{1}{r} - 1)\}$ for $\ell = 1$ or $\ell = 2$ in Corollary 3.3. This was shown in the unweighted case in Grafakos-Oh [30].

3.2 Setting for Leibniz-type rules

In this section, we will introduce weighted Lebesgue spaces, the class of function spaces with which the main results of the chapter are concerned. Some background is needed first, beginning with the Hardy-Littlewood maximal operator:

Definition 3.4. For a locally integrable function f defined on $\mathbb{R}^n$, define the **Hardy-Littlewood maximal operator** through its action on f given by

$$\mathcal{M}(f)(x) \coloneqq \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

We will also require biparameter versions of the Hardy-Littlewood maximal operator:

Definition 3.5. Fix $n_1, n_2 \in \mathbb{N}$ such that $n = n_1 + n_2$. Define the **Hardy-Littlewood maximal operators $\mathcal{M}^1_{n_1}$ and $\mathcal{M}^2_{n_2}$** by their actions on certain f defined on $\mathbb{R}^n$ as follows: If $f(\cdot, x_2)$ is locally integrable as a function on $\mathbb{R}^{n_1}$ for a.e. $x_2 \in \mathbb{R}^{n_2}$, then

$$\mathcal{M}^1_{n_1}(f)(x_1, x_2) \coloneqq \sup_{Q_1 \ni x_1} \frac{1}{|Q_1|} \int_{Q_1} |f(y_1, x_2)| \, dy_1, \quad \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$

where the supremum is taken over all cubes $Q_1 \subset \mathbb{R}^{n_1}$ containing x_1. If $f(x_1, \cdot)$ is locally integrable as a function on $\mathbb{R}^{n_2}$ for a.e $x_1 \in \mathbb{R}^{n_1}$, then

$$\mathcal{M}^2_{n_2}(f)(x_1, x_2) \coloneqq \sup_{Q_2 \ni x_2} \frac{1}{|Q_2|} \int_{Q_2} |f(x_1, y_2)| \, dy_2, \quad \forall (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$

where the supremum is taken over all cubes $Q_2 \subset \mathbb{R}^{n_2}$ containing x_2.

Many of the results referenced throughout the paper pertaining to $\mathcal{M}$ will have biparameter analogs associated to $\mathcal{M}^1_{n_1}$ and $\mathcal{M}^2_{n_2}$, which will be detailed as necessary.

We next highlight a pointwise inequality in terms of $\mathcal{M}$ that will be useful throughout the chapter. Fix $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and begin with the pointwise inequality

$$\sup_{t>0} \left| \frac{1}{t^n} \varphi\left(\frac{\cdot}{t}\right) * f(x) \right| \leq \|\tilde{\varphi}\|_{L^1} \mathcal{M}f(x), \qquad \text{a.e. } x \in \mathbb{R}^n, \tag{3.4}$$

where $\tilde{\varphi}$ is an integrable radially-decreasing majorant of φ (for a proof, see Grafakos [26, Theorem 2.1.10]). Considering a translation of φ in the above inequality, we obtain the following estimate involving $\tau_u \varphi := \varphi(\cdot + u)$ and $\mathcal{M}$:

$$\sup_{t>0} \left| \frac{1}{t^n} \tau_u \varphi\left(\frac{\cdot}{t}\right) * f(x) \right| \lesssim (1 + |u|)^{n+1} \mathcal{M}f(x), \qquad \text{a.e. } x \in \mathbb{R}^n. \tag{3.5}$$

This comes as a result of (A.2), which implies the existence of an integrable radially-decreasing majorant given by $|\tau_u \varphi(x)| \lesssim \frac{(1+|u|)^{n+1}}{(1+|x|)^{n+1}}$, where the implicit constant depends only on n and φ. Thus,

$$\left\| \frac{(1 + |u|)^{n+1}}{(1 + |\cdot|)^{n+1}} \right\|_{L^1} = (1 + |u|)^{n+1} \int_{\mathbb{R}^n} \frac{dx}{(1 + |x|)^{n+1}} \lesssim (1 + |u|)^{n+1},$$

where the implicit constant depends only on the dimension n.

3.2.1 Weights and weighted Lebesgue spaces

We begin by defining the weighted analogs to standard Lebesgue spaces, which essentially allow us to assign more or less weight to different regions of $\mathbb{R}^n$ when measuring size via integration:

Definition 3.6. A **weight on** $\mathbb{R}^n$ is a nonnegative locally integrable function defined on $\mathbb{R}^n$. Given a weight w on $\mathbb{R}^n$ and $0 < p < \infty$, define the **weighted Lebesgue space** $L^p(w)$ as the class of measurable functions defined on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, w(x) \, dx \right)^{\frac{1}{p}} < \infty.$$

We define $L^\infty(w) := L^\infty(\mathbb{R}^n)$.

Note that for $w \equiv 1$, $L^p(w)$ is simply the standard Lebesgue space $L^p(\mathbb{R}^n)$. The hypotheses of Theorems 1.3 and 1.4 reference the Muckenhoupt classes $A_p(\mathbb{R}^n)$ and the product Muckenhoupt classes $A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, collections of weights which are well-studied since they characterize boundedness properties of the Hardy-Littlewood maximal operator in weighted Lebesgue spaces. In particular, for $1 < p < \infty$,

$$w \in A_p(\mathbb{R}^n) \quad \Longleftrightarrow \quad \|\mathcal{M}f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \forall f \in L^p(w). \tag{3.6}$$

Classically, $A_p(\mathbb{R}^n)$ is defined as follows, which is equivalent to (3.6) (see, for example, Grafakos [26, Theorem 7.1.9]):

Definition 3.7. If $1 < p < \infty$, the **Muckenhoupt class** $A_p(\mathbb{R}^n)$ is comprised of all weights w on $\mathbb{R}^n$ satisfying

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. We define $A_\infty(\mathbb{R}^n) := \cup_{p>1} A_p(\mathbb{R}^n)$.

The biparameter analogs to these standard Muckenhoupt classes are then defined as follows:

Definition 3.8. Fix $n_1, n_2 \in \mathbb{N}$ such that $n = n_1 + n_2$. If $1 < p < \infty$, the **product Muckenhoupt class** $A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is comprised of all weights w on $\mathbb{R}^n$ satisfying

$$[w]'_{A_p} := \sup_R \left(\frac{1}{|R|}\int_R w(x)\,dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all sets $R = Q_1 \times Q_2$, with cubes $Q_1 \subset \mathbb{R}^{n_1}$ and $Q_2 \subset \mathbb{R}^{n_2}$. We define $A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) := \cup_{p>1} A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

It is worth noting that, if $w \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $w(\cdot, x_2) \in A_p(\mathbb{R}^{n_1})$ for a.e. $x_2 \in \mathbb{R}^{n_2}$, and $w(x_1, \cdot) \in A_p(\mathbb{R}^{n_2})$ for a.e. $x_1 \in \mathbb{R}^{n_1}$, with constants uniform in x_1 and x_2 (specifically, $[w(\cdot, x_2)]_{A_p}$ and $[w(x_1, \cdot)]_{A_p}$ are bounded uniformly in x_1 and x_2). Consequently, the operators $\mathcal{M}^1_{n_1}$ and $\mathcal{M}^2_{n_2}$ are bounded from $L^p(w)$ to $L^p(w)$, which can be seen by iterating norms.

For example,

$$\left\|\mathcal{M}^1_{n_1} f\right\|^p_{L^p(w)} = \int_{\mathbb{R}^{n_2}} \left\|\mathcal{M}^1_{n_1} f(\cdot, x_2)\right\|^p_{L^p(\mathbb{R}^{n_1}, w(\cdot,x_2))}\, dx_2 \lesssim \int_{\mathbb{R}^{n_2}} \|f(\cdot, x_2)\|^p_{L^p(\mathbb{R}^{n_1}, w(\cdot,x_2))}\, dx_2 = \|f\|^p_{L^p(w)}\,,$$

where we denote $\|f(\cdot, x_2)\|_{L^p(\mathbb{R}^{n_1}, w(\cdot,x_2))} \coloneqq \left(\int_{\mathbb{R}^{n_1}} |f(y_1, x_2)|^p\, w(y_1, x_2)\, dy_1\right)^{\frac{1}{p}}$ (and similarly for $\mathcal{M}^2_{n_2}$).

Remark 3.2.1. The process of iterating norms used above to verify biparameter results will be a common tool in proving many results in subsequent sections. The same reasoning may be applied, so long as the result corresponding to $\left\|\mathcal{M}^1_{n_1} f(\cdot, x_2)\right\|_{L^p(\mathbb{R}^{n_1}, w(\cdot,x_2))} \lesssim \|f(\cdot, x_2)\|_{L^p(\mathbb{R}^{n_1}, w(\cdot,x_2))}$ from above is uniform with respect to the necessary parameters (which will often result from the fact that $w \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ implies $[w(\cdot, x_2)]_{A_p}$ and $[w(x_1, \cdot)]_{A_p}$ are bounded uniformly in x_1 and x_2).

3.2.2 Littlewood-Paley operators and square function-type estimates

The following square function-type estimate, known as the weighted Fefferman-Stein maximal inequality (introduced in Fefferman-Stein [22]), is a vector-valued version of boundedness properties relating to the Hardy-Littlewood maximal operator: If $1 < p, q < \infty$ and $w \in A_p(\mathbb{R}^n)$, then for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions defined on $\mathbb{R}^n$, it holds that

$$\left\|\left(\sum_{j\in\mathbb{Z}} |\mathcal{M}(f_j)|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}} |f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p(w)}, \tag{3.7}$$

where the implicit constant depends on p, q, and $[w]_{A_p}$. By iterating norms just as we did in showing that $\mathcal{M}^1_{n_1}$ was a bounded operator (see Remark 3.2.1), we obtain that (3.7) is valid if $\mathcal{M}$ is replaced by either $\mathcal{M}^1_{n_1}$ or $\mathcal{M}^2_{n_2}$, a result which we will use while considering norm estimates in Subsection 3.3.3. Additionally, throughout the proofs of Theorems 1.3 and 1.4, many weighted estimates in the spirit of (3.7) which involve various Littlewood-Paley operators or other generalized operators will be needed. Here, we set notation for said

Littlewood-Paley operators and state a number of lemmas providing the necessary estimates.

We begin by defining families of operators analogous to those defined in (A.3) and (A.4), but which will be useful to us in the biparameter setting. Given functions $\Psi_1 \in \mathcal{S}(\mathbb{R}^{n_1})$ and $\Psi_2 \in \mathcal{S}(\mathbb{R}^{n_2})$ whose Fourier transforms are supported in annuli of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, define families of operators $\{\Delta_j^{\Psi_\ell}\}_{j\in\mathbb{Z}}$ by

$$\widehat{\Delta_j^{\Psi_\ell} f}(\xi) := \widehat{\Psi_\ell}(2^{-j}\xi_\ell)\widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \ell = 1, 2, \tag{3.8}$$

where $f \in \mathcal{S}'(\mathbb{R}^n)$. Similarly, given $\Phi_1 \in \mathcal{S}(\mathbb{R}^{n_1})$ and $\Phi_2 \in \mathcal{S}(\mathbb{R}^{n_2})$ whose Fourier transforms do not vanish at the origin and are supported in a ball centered at the origin in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, define families of operators $\{S_j^{\Phi_\ell}\}_{j\in\mathbb{Z}}$ by

$$\widehat{S_j^{\Phi_\ell} f}(\xi) := \widehat{\Phi_\ell}(2^{-j}\xi_\ell)\widehat{f}(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \ell = 1, 2. \tag{3.9}$$

Many of the lemmas that will be stated below involve $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying one or both properties (A.5) and (A.6). Some lemmas also involve functions which satisfy biparameter versions of said properties. Specifically, we will consider $\Psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$, $\ell = 1, 2$, which satisfy one or both of the following properties:

$$\operatorname{supp}(\widehat{\Psi_\ell}) \subseteq \{\xi_\ell \in \mathbb{R}^{n_\ell} : \frac{1}{2} < |\xi_\ell| < 2\}, \tag{3.10}$$

$$\sum_{j\in\mathbb{Z}} \widehat{\Psi_\ell}(2^{-j}\xi_\ell) = 1, \quad \xi_\ell \in \mathbb{R}^{n_\ell} \setminus \{0\}. \tag{3.11}$$

We defer to Section B.1 for the proofs or proof references of all of the following lemmas. We note that all lemmas stated below hold for slightly more general functions. In place of properties (A.5) and (3.10), it would suffice to have the Fourier transform of the functions supported in any annulus, and in place of properties (A.6) and (3.11), it would suffice for the sum to equal any constant (uniform in $\xi \neq 0$), not necessarily 1. The lemmas as stated here will be sufficient for our purposes.

The first lemma we will utilize gives a characterization of weighted Lebesgue spaces associated to weights in the Muckenhoupt classes in terms of square-function operators.

Lemma 3.9. *Let $1 < p < \infty$.*

(a) Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (A.5) and (A.6). If $w \in A_p(\mathbb{R}^n)$, it holds that

$$\|f\|_{L^p(w)} \sim \left\| \left(\sum_{j\in\mathbb{Z}} \left|\Delta_j^{\Psi} f\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}, \quad \forall f \in L^p(w), \tag{3.12}$$

where the implicit constant depends on p, Ψ, and $[w]_{A_p}$.

(b) For $\ell = 1,2$, let $\Psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ satisfy (3.10) and (3.11). If $w \in A_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, it holds that

$$\|f\|_{L^p(w)} \sim \left\| \left(\sum_{j_1\in\mathbb{Z}} \sum_{j_2\in\mathbb{Z}} \left|\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2} f\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}, \quad \forall f \in L^p(w), \tag{3.13}$$

where the implicit constant depends on p, Ψ_1, Ψ_2, and $[w]'_{A_p}$.

Notice in the statement of Lemma 3.9 that the result holds for weights in a specific Muckenhoupt class $A_p(\mathbb{R}^n)$ or $A_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ relating to the weighted Lebesgue space $L^p(w)$ in which the associated functions lie. By relaxing this condition and allowing the weights to be in any Muckenhoupt class (that is, in $A_\infty(\mathbb{R}^n)$ or $A_\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$), we may obtain results similar to (3.12) and (3.13) with the loss of one direction of norm comparability, as the following lemma states.

Lemma 3.10. *Let $0 < p < \infty$.*

(a) Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (A.5) and (A.6). If $w \in A_\infty(\mathbb{R}^n)$, it holds that

$$\|f\|_{L^p(w)} \lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \left|\Delta_j^{\Psi} f\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}, \quad \forall f \in L^2(\mathbb{R}^n).$$

(b) For $\ell = 1,2$, let $\Psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ satisfy (3.10) and (3.11). If $w \in A_\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, it holds that

$$\|f\|_{L^p(w)} \lesssim \left\| \left(\sum_{j_1\in\mathbb{Z}} \sum_{j_2\in\mathbb{Z}} \left|\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2} f\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}, \quad \forall f \in L^2(\mathbb{R}^n).$$

In proving Theorems 1.3 and 1.4, there will be situations where operators as defined in (A.3) and (3.8) will appear, but with a translation associated to the multiplier function, that is to say operators of the form $\Delta_j^{\tau_u \Psi}$ where $\tau_u \Psi(x) := \Psi(x+u)$ and u may depend on j. The following lemma says that one of the inequalities given in (3.12) and (3.13) holds for such operators with constants uniform with respect to the translations.

Lemma 3.11. *Let $1 < p < \infty$.*

(a) Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (A.5). Given a sequence $\bar{z} = \{z_{j,a}\}_{j\in\mathbb{Z}, a\in\mathbb{Z}^n} \subset \mathbb{R}^n$, define $\Psi^{\bar{z}}_{j,a}(x) := \Psi(x + z_{j,a})$ for $x \in \mathbb{R}^n$, $a \in \mathbb{Z}^n$, and $j \in \mathbb{Z}$. If $w \in A_p(\mathbb{R}^n)$, it holds that

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left| \Delta_j^{\Psi^{\bar{z}}_{j,a}} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \quad \forall f \in L^p(w),$$

where the implicit constant depends on Ψ, p, and $[w]_{A_p}$ but is independent of a and $\bar{z}$.

(b) For $\ell = 1, 2$, let $\Psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ satisfy (3.10). Given sequences $\bar{z}^\ell = \{z^\ell_{j,a}\}_{j\in\mathbb{Z}, a\in\mathbb{Z}^{n_\ell}} \subset \mathbb{R}^{n_\ell}$, define $\Psi^{\bar{z}^\ell}_{j,a}(x) := \Psi_\ell(x + z^\ell_{j,a})$ for $x \in \mathbb{R}^{n_\ell}$, $a \in \mathbb{Z}^{n_\ell}$, and $j \in \mathbb{Z}$. If $w \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, it holds that

$$\left\| \left(\sum_{j_1\in\mathbb{Z}} \sum_{j_2\in\mathbb{Z}} \left| \Delta_{j_1}^{\Psi^{\bar{z}^1}_{j_1,a_1}} \Delta_{j_2}^{\Psi^{\bar{z}^2}_{j_2,a_2}} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \quad \forall f \in L^p(w),$$

where the implicit constant may depend on Ψ_1, Ψ_2, p, and $[w]'_{A_p}$ but is independent of a_1, a_2, $\bar{z}^1$, and $\bar{z}^2$.

Next, we state two lemmas which give weighted square function-type estimates involving more general families of operators. Specifically, we detail Lemma 3.12 (which will be useful in the proofs of Lemmas 3.13 and 3.15 and Theorem 1.3) and Lemma 3.13 (which is utilized in the proofs of Lemmas 3.9 and 3.11).

Lemma 3.12. *Let $1 < r < \infty$ and assume $\{T_k\}_{k\in\mathbb{Z}}$ is a family of operators acting on* *functions defined on* $\mathbb{R}^n$ *such that if* $w \in A_r(\mathbb{R}^n)$, *it holds that*

$$\left\| \left(\sum_{k \in \mathbb{Z}} |T_k(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^r(w)} \lesssim \|f\|_{L^r(w)}, \quad \forall f \in L^r(w), \tag{3.14}$$

where the implicit constant may depend on r *and* $[w]_{A_r}$. *If* $1 < p < \infty$ *and* $w \in A_p(\mathbb{R}^n)$, *we have*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} |T_k(f_j)|^2 \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)},$$

with the implicit constant depending on p, r, *and* $[w]_{A_p}$.

Lemma 3.13. *Let* $1 < p < \infty$ *and* $n_1, n_2 \in \mathbb{N}$, *and assume* $\{T_j^1\}_{j \in \mathbb{Z}}$ *and* $\{T_j^2\}_{j \in \mathbb{Z}}$ *are families of operators defined for functions on* $\mathbb{R}^{n_1}$ *and* $\mathbb{R}^{n_2}$, *respectively, such that if* $w_1 \in A_2(\mathbb{R}^{n_1})$ *and* $w_2 \in A_p(\mathbb{R}^{n_2})$, *it holds that*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left|T_j^1(f)\right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w_1)} \lesssim \|f\|_{L^2(w_1)}, \quad \forall f \in L^2(w_1), \tag{3.15}$$

where the implicit constant may depend on $[w_1]_{A_2}$, *and*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left|T_j^2(f)\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w_2)} \lesssim \|f\|_{L^p(w_2)}, \quad \forall f \in L^p(w_2), \tag{3.16}$$

where the implicit constant may depend on p *and* $[w_2]_{A_p}$. *If* $w \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, *we have*

$$\left\| \left(\sum_{j_1 \in \mathbb{Z}} \sum_{j_2 \in \mathbb{Z}} \left|T_{j_1}^1 T_{j_2}^2(f)\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \quad \forall f \in L^p(w),$$

with the implicit constant depending on p *and* $[w]'_{A_p}$.

We conclude the section with two final lemmas giving weighted square function-type estimates relating to certain families of Littlewood-Paley-type operators whose associated multipliers have a translation and depend on the summation variable $j \in \mathbb{N}_0$. Lemma 3.15 requires Lemma 3.14 for its proof, and both lemmas will be utilized in the proof of Theorem 1.3.

Lemma 3.14. *Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (A.5). Given $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$, define $J_j^s\Psi$ via $\widehat{J_j^s\Psi}(\xi) \coloneqq (2^{-2j} + |\xi|^2)^{\frac{s}{2}}\widehat{\Psi}(\xi)$ for $\xi \in \mathbb{R}^n$. If $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$, it holds that*

$$\left\| \left(\sum_{j\in\mathbb{N}_0} \left| \Delta_j^{\tau_u J_j^s \Psi} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \quad \forall f \in L^p(w),\ u \in \mathbb{R}^n,$$

where the implicit constant depends on Ψ and $[w]_{A_p}$.

Lemma 3.15. *Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (A.5). Given $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$, define $J_j^s\Psi$ via $\widehat{J_j^s\Psi}(\xi) \coloneqq (2^{-2j} + |\xi|^2)^{\frac{s}{2}}\widehat{\Psi}(\xi)$ for $\xi \in \mathbb{R}^n$. If $1 < p, r < \infty$ and $w \in A_p(\mathbb{R}^n)$, it holds that*

$$\left\| \left(\sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{N}_0} \left| \Delta_j^{\tau_u J_j^s \Psi} f_k \right|^2 \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \right\|_{L^p(w)} \lesssim \left\| \left(\sum_{k\in\mathbb{Z}} |f_k|^r \right)^{\frac{1}{r}} \right\|_{L^p(w)},$$

where the implicit constant may depend on p, r, and $[w]_{A_p}$, but is independent of $u \in \mathbb{R}^n$ and $\{f_k\}_{k\in\mathbb{Z}}$.

3.3 Proof of weighted fractional Leibniz rules associated to biparameter Coifman-Meyer multipliers

In this section, we prove Theorem 1.4, which we will divide into a few pieces. In Subsection 3.3.1, we establish a biparaproduct decomposition of T_σ, similar to the ideas introduced in Subsection 2.2.1. In Subsection 3.3.2, we do some analysis on the multipliers associated with the decomposition, and we conclude by applying various norm estimates (primarily from Subsection 3.2.2) to obtain the desired results in Subsection 3.3.3.

Before beginning the proof of Theorem 1.4, we set some notation that will be used throughout the section. Let $n_1, n_2 \in \mathbb{N}$ with $n = n_1 + n_2$. For $\ell = 1, 2$, fix $\Psi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ satisfying (3.10) and (3.11). Define $\Phi_\ell \in \mathcal{S}(\mathbb{R}^{n_\ell})$ via

$$\widehat{\Phi_\ell}(\xi_\ell) \coloneqq \begin{cases} 1, & \xi_\ell = 0, \\ \sum_{j<-2} \widehat{\Psi_\ell}(2^{-j}\xi_\ell), & \xi_\ell \in \mathbb{R}^{n_\ell} \setminus \{0\}. \end{cases}$$

From this definition, we may conclude that for any $k \in \mathbb{Z}$,

$$\Phi_\ell(2^{-k}\xi_\ell) = \sum_{j<k-2} \widehat{\Psi_\ell}(2^{-j}\xi_\ell), \quad \forall \xi_\ell \in \mathbb{R}^{n_\ell} \setminus \{0\}. \tag{3.17}$$

Additional properties satisfied by Φ_ℓ include

$$\widehat{\Phi_\ell}\Big|_{B(0,\frac{1}{16})} \equiv 1, \qquad \operatorname{supp}(\widehat{\Phi_\ell}) \subset \{\xi_\ell \in \mathbb{R}^{n_\ell} : |\xi_\ell| < \tfrac{1}{4}\}. \tag{3.18}$$

Throughout the chapter, we will also require auxiliary functions $\psi_\ell, \phi_\ell \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\psi_\ell}$ and $\widehat{\phi_\ell}$ have slightly larger supports than $\widehat{\Psi_\ell}$ and $\widehat{\Phi_\ell}$, respectively, and

$$\widehat{\psi_\ell}\Big|_{\operatorname{supp}(\widehat{\Psi_\ell})} \equiv 1, \qquad \widehat{\phi_\ell}\Big|_{\operatorname{supp}(\widehat{\Phi_\ell})} \equiv 1. \tag{3.19}$$

We may without loss of generality choose $\psi_\ell, \phi_\ell \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}(\widehat{\psi_\ell}) \subseteq \{\xi_\ell \in \mathbb{R}^{n_\ell} : \tfrac{1}{2} < |\xi_\ell| < 2\}, \quad \operatorname{supp}(\widehat{\phi_\ell}) \subseteq \{\xi_\ell \in \mathbb{R}^{n_\ell} : |\xi_\ell| < \tfrac{1}{4}\}. \tag{3.20}$$

3.3.1 Paraproduct decomposition

Let σ be a biparameter Coifman-Meyer multiplier as in Definition 3.1. By (3.11), we have

$$\begin{aligned}
T_\sigma(f,g)(x) &= \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta \\
&= \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\left(\sum_{j_1,k_1\in\mathbb{Z}} \widehat{\Psi_1}(2^{-j_1}\xi_1)\widehat{\Psi_1}(2^{-k_1}\eta_1)\right)\left(\sum_{j_2,k_2\in\mathbb{Z}} \widehat{\Psi_2}(2^{-j_2}\xi_2)\widehat{\Psi_2}(2^{-k_2}\eta_2)\right) \\
&\qquad \times \widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi\, d\eta \\
&=: \sum_{t_1,t_2=1}^{3} \Pi_{t_1,t_2}(f,g)(x),
\end{aligned}$$

where for $t_1, t_2 \in \{1,2,3\}$, Π_{t_1,t_2} are bilinear multiplier operators with corresponding multipliers given by

$$\sigma_{t_1,t_2}(\xi,\eta) \coloneqq \sigma(\xi,\eta)M_1^{t_1}(\xi_1,\eta_1)M_2^{t_2}(\xi_2,\eta_2),$$

where

$$\begin{aligned}
M_\ell^1(\xi_\ell,\eta_\ell) &\coloneqq \sum_{\substack{j_\ell,k_\ell\in\mathbb{Z}\\ k_\ell<j_\ell-2}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-k_\ell}\eta_\ell) = \sum_{j_\ell\in\mathbb{Z}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Phi_\ell}(2^{-j_\ell}\eta_\ell), \\
M_\ell^2(\xi_\ell,\eta_\ell) &\coloneqq \sum_{\substack{j_\ell,k_\ell\in\mathbb{Z}\\ j_\ell<k_\ell-2}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-k_\ell}\eta_\ell) = \sum_{k_\ell\in\mathbb{Z}} \widehat{\Phi_\ell}(2^{-k_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-k_\ell}\eta_\ell), \\
M_\ell^3(\xi_\ell,\eta_\ell) &\coloneqq \sum_{j_\ell\in\mathbb{Z}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-j_\ell}\eta_\ell).
\end{aligned}$$

Technically, the actual form of M_ℓ^3 is given by

$$\sum_{\delta=-2}^{2}\left[\sum_{j_\ell\in\mathbb{Z}} \widehat{\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\Psi_\ell}(2^{-(j_\ell+\delta)}\eta_\ell)\right].$$

However, carrying the finite sum in δ throughout the proof does not affect the results, it merely changes some of the constants obtained in inequalities. We will not track the specific values of such constants, so for ease of notation, we will restrict to the case $\delta = 0$ (as done in the definition above for M_ℓ^3). Note that, due to the supports of $\widehat{\Psi_\ell}$ and $\widehat{\Phi_\ell}$, we have that $M_\ell^1(\xi_\ell,\eta_\ell)$ is supported where $|\eta_\ell| \le \frac{1}{2}|\xi_\ell|$, $M_\ell^2(\xi_\ell,\eta_\ell)$ is supported where $|\xi_\ell| \le \frac{1}{2}|\eta_\ell|$, and $M_\ell^3(\xi_\ell,\eta_\ell)$ is supported where $|\xi_\ell| \sim |\eta_\ell|$.

It will suffice to prove the desired boundedness result for each Π_{t_1,t_2}, $t_1,t_2 \in \{1,2,3\}$. In fact, due to similarities in arguments, we may restrict our analysis to $\Pi_{1,1}$ (also representing the process for $\Pi_{2,2}$), $\Pi_{1,2}$ (representing $\Pi_{2,1}$), $\Pi_{1,3}$ (representing $\Pi_{2,3}$, $\Pi_{3,1}$, and $\Pi_{3,2}$), and $\Pi_{3,3}$. In saying that we are representing the process for another operator, we mean that some functions' roles are interchanged initially, but by following the same steps outlined here, we obtain a different acceptable boundedness result for that piece. For example, $\Pi_{1,1}$ is a bilinear multiplier operator with associated multiplier

$$\sigma_{1,1}(\xi,\eta) = \sigma(\xi,\eta)\left(\sum_{j_1\in\mathbb{Z}} \widehat{\Psi_1}(2^{-j_1}\xi_1)\widehat{\Phi_1}(2^{-j_1}\eta_1)\right)\left(\sum_{j_2\in\mathbb{Z}} \widehat{\Psi_2}(2^{-j_2}\xi_2)\widehat{\Phi_2}(2^{-j_2}\eta_2)\right).$$

Ultimately, we will obtain a bound associated with $\Pi_{1,1}$ of the form

$$\|D_1^{s_1}D_2^{s_2}\Pi_{1,1}(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} \lesssim \|D_1^{s_1}D_2^{s_2}f\|_{L^p(v)}\|g\|_{L^q(w)}.$$

On the other hand, $\Pi_{2,2}$ is a bilinear multiplier operator with associated multiplier

$$\sigma_{2,2}(\xi,\eta) = \sigma(\xi,\eta)\left(\sum_{j_1\in\mathbb{Z}} \widehat{\Phi_1}(2^{-j_1}\xi_1)\widehat{\Psi_1}(2^{-j_1}\eta_1)\right)\left(\sum_{j_2\in\mathbb{Z}} \widehat{\Phi_2}(2^{-j_2}\xi_2)\widehat{\Psi_2}(2^{-j_2}\eta_2)\right).$$

Notice that $\widehat{\Psi_\ell}$ and $\widehat{\Phi_\ell}$, $\ell = 1,2$, are taking inputs ξ_ℓ and η_ℓ in the opposite way compared to the multiplier associated with $\Pi_{1,1}$. The result, by following the same procedure we use for $\Pi_{1,1}$, is a boundedness estimate of the form

$$\|D_1^{s_1}D_2^{s_2}\Pi_{2,2}(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} \lesssim \|f\|_{L^p(v)}\|D_1^{s_1}D_2^{s_2}g\|_{L^q(w)}.$$

Similar arguments may be made with other multiplier operators representing multiple cases.

First, we consider $\sigma_{1,1}$, the multiplier associated with $\Pi_{1,1}$. By (3.19), we have for $\ell = 1,2$

that $\widehat{\Psi_\ell} = \widehat{\psi_\ell}\widehat{\Psi_\ell}$ and $\widehat{\Phi_\ell} = \widehat{\phi_\ell}\widehat{\Phi_\ell}$, so we may rewrite $\sigma_{1,1}(\xi,\eta)$ as

$$\sum_{j_1,j_2\in\mathbb{Z}} \sigma[j_1,j_2](2^{-j_1}\xi_1, 2^{-j_2}\xi_2, 2^{-j_1}\eta_1, 2^{-j_2}\eta_2)\widehat{\Psi_1}(2^{-j_1}\xi_1)\widehat{\Phi_1}(2^{-j_1}\eta_1)\widehat{\Psi_2}(2^{-j_2}\xi_2)\widehat{\Phi_2}(2^{-j_2}\eta_2),$$

where

$$\sigma[j_1,j_2](\xi,\eta) \coloneqq \sigma(2^{j_1}\xi_1, 2^{j_2}\xi_2, 2^{j_1}\eta_1, 2^{j_2}\eta_2)\Gamma_{1,1}(\xi,\eta)$$

and $\Gamma_{1,1}(\xi,\eta) \coloneqq \widehat{\psi_1}(\xi_1)\widehat{\phi_1}(\eta_1)\widehat{\psi_2}(\xi_2)\widehat{\phi_2}(\eta_2)$. For multi-indices $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2) \in \mathbb{N}_0^{n_1} \times \mathbb{N}_0^{n_2}$, we have that $\left|\partial_\xi^\alpha\partial_\eta^\beta[\sigma(2^{j_1}\xi_1, 2^{j_2}\xi_2, 2^{j_1}\eta_1, 2^{j_2}\eta_2)]\right|$ is bounded by

$$\begin{aligned}
&2^{j_1(|\alpha_1+\beta_1|)+j_2(|\alpha_2+\beta_2|)}\left|[\partial_\xi^\alpha\partial_\eta^\beta\sigma](2^{j_1}\xi_1, 2^{j_2}\xi_2, 2^{j_1}\eta_1, 2^{j_2}\eta_2)\right| \\
&\quad\lesssim \left(\frac{2^{j_1}}{(|2^{j_1}\xi_1| + |2^{j_1}\eta_1|)}\right)^{|\alpha_1+\beta_1|}\left(\frac{2^{j_2}}{(|2^{j_2}\xi_2| + |2^{j_2}\eta_2|)}\right)^{|\alpha_2+\beta_2|} \\
&\quad= \frac{1}{(|\xi_1| + |\eta_1|)^{|\alpha_1+\beta_1|}}\frac{1}{(|\xi_2| + |\eta_2|)^{|\alpha_2+\beta_2|}},
\end{aligned}$$

where we have used that σ satisfies (3.2). Since $\Gamma_{1,1} \in \mathcal{S}(\mathbb{R}^{2n})$, we see that $\left|\partial_\xi^\gamma\partial_\eta^\delta\Gamma_{1,1}(\xi,\eta)\right| \le C_{\gamma,\delta}$ for any $\gamma,\delta \in \mathbb{N}_0^n$. Further, due to the supports of the functions which comprise $\Gamma_{1,1}$ (see (3.20)), it holds that

$$\operatorname{supp}(\Gamma_{1,1}) \subseteq \{(\xi,\eta) \in \mathbb{R}^{2n} : c_1 < |\xi_1| + |\eta_1| < C_1\} \cap \{(\xi,\eta) \in \mathbb{R}^{2n} : c_2 < |\xi_2| + |\eta_2| < C_2\},$$

where in the specific case of $\Gamma_{1,1}$, $c_1 = c_2 = \frac{1}{2}$ and $C_1 = C_2 = \frac{9}{4}$. Due to the lower bounds on $|\xi_1| + |\eta_1|$ and $|\xi_2| + |\eta_2|$ within the support of $\Gamma_{1,1}$, we may conclude that $\partial_\xi^\alpha\partial_\eta^\beta\sigma[j_1,j_2]$ is bounded uniformly in $j_1, j_2 \in \mathbb{Z}$.

Since $\sigma[j_1,j_2]$ is compactly supported (with support independent of j_1 and j_2), we may consider the Fourier series expansion of a periodic extension of $\sigma[j_1,j_2]$ (see Section A.3).

Let $H = [-\frac{h}{2}, \frac{h}{2}]^{2n}$, where h is large enough so that $\operatorname{supp}(\sigma[j_1, j_2]) \subseteq H$, then

$$\begin{aligned}
&\sigma[j_1, j_2](2^{-j_1}\xi_1, 2^{-j_2}\xi_2, 2^{-j_1}\eta_1, 2^{-j_2}\eta_2) = \\
&\quad \left(\sum_{a,b\in\mathbb{Z}^n} c[j_1, j_2, a, b] e^{\frac{2\pi i}{h}(a,b)\cdot(2^{-j_1}\xi_1, 2^{-j_2}\xi_2, 2^{-j_1}\eta_1, 2^{-j_2}\eta_2)}\right) \chi_H(2^{-j_1}\xi_1, 2^{-j_2}\xi_2, 2^{-j_1}\eta_1, 2^{-j_2}\eta_2),
\end{aligned} \tag{3.21}$$

where

$$c[j_1, j_2, a, b] = \frac{1}{h^{2n}} \int_{\mathbb{R}^{2n}} \sigma[j_1, j_2](\xi, \eta) e^{-\frac{2\pi i}{h}(a\cdot\xi + b\cdot\eta)}\, d\xi\, d\eta.$$

Fix $N \in \mathbb{N}$ sufficiently large (where we specify the necessary size for N in Subsection 3.3.3). Define

$$\mathcal{C}[j_1, j_2, a, b] \coloneqq (1 + |a|^2 + |b|^2)^N c[j_1, j_2, a, b].$$

Since $[1 - \Delta_{\xi,\eta}](e^{-\frac{2\pi i}{h}(a\cdot\xi+b\cdot\eta)}) = (1 + \frac{4\pi^2}{h^2}|a|^2 + \frac{4\pi^2}{h^2}|b|^2)e^{-\frac{2\pi i}{h}(a\cdot\xi+b\cdot\eta)}$, we have that

$$\begin{aligned}
|\mathcal{C}[j_1, j_2, a, b]| &= \frac{1}{h^{2n}} \left| \int_{\mathbb{R}^{2n}} (1 + |a|^2 + |b|^2)^N \sigma[j_1, j_2](\xi, \eta) e^{-\frac{2\pi i}{h}(a\cdot\xi+b\cdot\eta)}\, d\xi\, d\eta \right| \\
&\sim \left| \int_{\mathbb{R}^{2n}} \sigma[j_1, j_2](\xi, \eta) [1 - \Delta_{\xi,\eta}]^N (e^{-\frac{2\pi i}{h}(a\cdot\xi+b\cdot\eta)})\, d\xi\, d\eta \right| \\
&= \left| \int_{\substack{c_1<|\xi_1|+|\eta_1|<C_1 \\ c_2<|\xi_2|+|\eta_2|<C_2}} [1 - \Delta_{\xi,\eta}]^N (\sigma[j_1, j_2])(\xi, \eta) e^{-\frac{2\pi i}{h}(a\cdot\xi+b\cdot\eta)}\, d\xi\, d\eta \right| \\
&\leq \int_{\substack{c_1<|\xi_1|+|\eta_1|<C_1 \\ c_2<|\xi_2|+|\eta_2|<C_2}} \left| [1 - \Delta_{\xi,\eta}]^N (\sigma[j_1, j_2])(\xi, \eta) \right|\, d\xi\, d\eta,
\end{aligned}$$

where in the third line we have done integration by parts and taken into consideration the support of $\sigma[j_1, j_2]$. Considering the last line and using the fact that a finite sum of uniformly bounded functions integrated over a compact set is itself bounded, we have that $|\mathcal{C}[j_1, j_2, a, b]| \lesssim 1$, where the implicit constant is independent of $j_1, j_2 \in \mathbb{Z}$ and $a, b \in \mathbb{Z}^n$. We obtain a paraproduct representation for $\Pi_{1,1}(f, g)$ by substituting (3.21) back into the formula for $\sigma_{1,1}$ (and dropping the χ_H piece, which is redundant due to the supports of Ψ_ℓ and Φ_ℓ, $\ell = 1, 2$, in the formula for $\sigma_{1,1}$). By also using property (A.3.3) of the Fourier transform, the result is a paraproduct representation for $\Pi_{1,1}(f,g)(x)$ given by

$$\sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_1,j_2\in\mathbb{Z}} \mathcal{C}[j_1,j_2,a,b][\Delta_{j_1}^{\tau_{a_1}\Psi_1}\Delta_{j_2}^{\tau_{a_2}\Psi_2} f](x)[S_{j_1}^{\tau_{b_1}\Phi_1} S_{j_2}^{\tau_{b_2}\Phi_2} g](x), \tag{3.22}$$

where $a=(a_1,a_2), b=(b_1,b_2)\in\mathbb{Z}^{n_1}\times\mathbb{Z}^{n_2}$ and $\tau_u F(\cdot) = F(\cdot + \frac{u}{h})$. In fact, throughout the remainder of our arguments, our translation operators will always implicity divide the shift by h, but we relax this notation since the division by h in the shift is negligible.

Obtaining paraproduct representations for $\Pi_{1,2}$, $\Pi_{1,3}$, and $\Pi_{3,3}$ involves exactly the same steps used in considering the multiplier associated with $\Pi_{1,1}$, with a few slight adjustments. First, $\Gamma_{1,1}$ is replaced by a product of appropriate auxiliary functions, where

$$\Gamma_{1,2}(\xi,\eta) := \widehat{\psi_1}(\xi_1)\widehat{\phi_1}(\eta_1)\widehat{\phi_2}(\xi_2)\widehat{\psi_2}(\eta_2),$$

$$\Gamma_{1,3}(\xi,\eta) := \widehat{\psi_1}(\xi_1)\widehat{\phi_1}(\eta_1)\widehat{\psi_2}(\xi_2)\widehat{\psi_2}(\eta_2),$$

$$\Gamma_{3,3}(\xi,\eta) := \widehat{\psi_1}(\xi_1)\widehat{\psi_1}(\eta_1)\widehat{\psi_2}(\xi_2)\widehat{\psi_2}(\eta_2),$$

corresponding to $\Pi_{1,2}$, $\Pi_{1,3}$, and $\Pi_{3,3}$, respectively. The other differences in these cases from that of $\Pi_{1,1}$ are the specific values of c_ℓ and C_ℓ, $\ell=1,2$. However, $c_1,c_2>0$ and $C_1,C_2<\infty$, which is all that is necessary for the analysis. Finally, we obtain the following paraproduct decompositions:

$$\Pi_{1,2}(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_1,j_2\in\mathbb{Z}} \mathcal{C}[j_1,j_2,a,b][\Delta_{j_1}^{\tau_{a_1}\Psi_1} S_{j_2}^{\tau_{a_2}\Phi_2} f](x)[S_{j_1}^{\tau_{b_1}\Phi_1}\Delta_{j_2}^{\tau_{b_2}\Psi_2} g](x),$$

$$\Pi_{1,3}(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_1,j_2\in\mathbb{Z}} \mathcal{C}[j_1,j_2,a,b][\Delta_{j_1}^{\tau_{a_1}\Psi_1} \Delta_{j_2}^{\tau_{a_2}\Psi_2} f](x)[S_{j_1}^{\tau_{b_1}\Phi_1}\Delta_{j_2}^{\tau_{b_2}\Psi_2} g](x),$$

$$\Pi_{3,3}(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_1,j_2\in\mathbb{Z}} \mathcal{C}[j_1,j_2,a,b][\Delta_{j_1}^{\tau_{a_1}\Psi_1} \Delta_{j_2}^{\tau_{a_2}\Psi_2} f](x)[\Delta_{j_1}^{\tau_{b_1}\Psi_1}\Delta_{j_2}^{\tau_{b_2}\Psi_2} g](x),$$

where the coefficients $\mathcal{C}[j_1,j_2,a,b]$ are defined slightly differently in each line but are nonetheless uniformly bounded in $j_1,j_2\in\mathbb{Z}$ and $a,b\in\mathbb{Z}^n$.

3.3.2 Analysis of multipliers

The following functions will be integral components of the multipliers we will come across in applying the norm estimates in Subsection 3.3.3:

$$N_\ell^1[j_\ell, a_\ell, b_\ell](\xi_\ell, \eta_\ell) := |\xi_\ell + \eta_\ell|^{s_\ell}\, \widehat{\Psi_\ell}(2^{-j_\ell}(\xi_\ell + \eta_\ell))\widehat{\tau_{a_\ell}\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\tau_{b_\ell}\Phi_\ell}(2^{-j_\ell}\eta_\ell),$$
$$N_\ell^2[j_\ell, a_\ell, b_\ell](\xi_\ell, \eta_\ell) := |\xi_\ell + \eta_\ell|^{s_\ell}\, \widehat{\Psi_\ell}(2^{-j_\ell}(\xi_\ell + \eta_\ell))\widehat{\tau_{a_\ell}\Phi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\tau_{b_\ell}\Psi_\ell}(2^{-j_\ell}\eta_\ell),$$
$$N_\ell^3[j_\ell, a_\ell, b_\ell](\xi_\ell, \eta_\ell) := |\xi_\ell + \eta_\ell|^{s_\ell}\, \widehat{\tau_{a_\ell}\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\tau_{b_\ell}\Psi_\ell}(2^{-j_\ell}\eta_\ell),$$

for $\ell = 1, 2$, $j_\ell \in \mathbb{Z}$, and $a, b \in \mathbb{Z}^{n_\ell}$. In this section, we do some preparatory analysis on the above functions.

We first consider $N_\ell^1[j_\ell, a_\ell, b_\ell](\xi_\ell, \eta_\ell)$. Notice that

$$\begin{aligned}
&N_\ell^1[j_\ell, a_\ell, b_\ell](\xi_\ell, \eta_\ell)\\
&\quad = |\xi_\ell + \eta_\ell|^{s_\ell}\, \widehat{\Psi_\ell}(2^{-j_\ell}(\xi_\ell + \eta_\ell))\widehat{\tau_{a_\ell}\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\tau_{b_\ell}\Phi_\ell}(2^{-j_\ell}\eta_\ell)\\
&\quad = |\xi_\ell|^{s_\ell}\left[\left|2^{-j_\ell}(\xi_\ell + \eta_\ell)\right|^{s_\ell}\widehat{\Psi_\ell}(2^{-j_\ell}(\xi_\ell + \eta_\ell))\right]\left[\left|2^{-j_\ell}\xi_\ell\right|^{-s_\ell}\widehat{\tau_{a_\ell}\Psi_\ell}(2^{-j_\ell}\xi_\ell)\right]\widehat{\tau_{b_\ell}\Phi_\ell}(2^{-j_\ell}\eta_\ell)\\
&\quad = |\xi_\ell|^{s_\ell}\,\mathcal{F}[D^{s_\ell}\Psi_\ell](2^{-j_\ell}(\xi_\ell + \eta_\ell))\mathcal{F}[D^{-s_\ell}\tau_{a_\ell}\Psi_\ell](2^{-j_\ell}\xi_\ell)\widehat{\tau_{b_\ell}\Phi_\ell}(2^{-j_\ell}\eta_\ell).
\end{aligned}$$

We consider the Fourier series expansion of a periodic extension of the compactly supported function $\mathcal{F}[D^{s_\ell}\Psi_\ell]$ (whose support is identical to that of $\widehat{\Psi_\ell}$). Let $H_\ell = [-\frac{h}{2}, \frac{h}{2}]^{n_\ell}$ with $\operatorname{supp}(\mathcal{F}[D^{s_\ell}\Psi_\ell]) \subseteq H_\ell$, then

$$\mathcal{F}[D^{s_\ell}\Psi_\ell](2^{-j_\ell}\omega) = \left[\sum_{m_\ell \in \mathbb{Z}^{n_\ell}} c^{s_\ell}[m_\ell]e^{\frac{2\pi i}{h}2^{-j_\ell}\omega\cdot m_\ell}\right]\chi_{H_\ell}(2^{-j_\ell}\omega),$$

where

$$c^{s_\ell}[m_\ell] = \frac{1}{h^{n_\ell}}\int_{H_\ell}|\zeta|^{s_\ell}\,\widehat{\Psi_\ell}(\zeta)e^{-\frac{2\pi i}{h}\zeta\cdot m_\ell}\,d\zeta,$$

with $c^{s_\ell}[m_\ell] = \mathcal{O}((1 + |m_\ell|)^{-K})$ for any $K > 0$ by (A.1) (replace $m_\ell \in \mathbb{Z}^{n_\ell}$ by a continuous variable so we regard c^{s_ℓ} as a continuous function defined on $\mathbb{R}^{n_\ell}$; then $c^{s_\ell} \in \mathcal{S}(\mathbb{R}^{n_\ell})$ is the inverse Fourier transform of $|\cdot|^{s_\ell}\,\widehat{\Psi_\ell}$, which is in $\mathcal{S}(\mathbb{R}^{n_\ell})$ since $\widehat{\Psi_\ell}\in\mathcal{S}(\mathbb{R}^{n_\ell})$ is supported away from the origin). Substituting in this Fourier series expansion formula for $\mathcal{F}[D^{s_\ell}\Psi_\ell](2^{-j_\ell}\cdot)$ and using property (A.3.3) of the Fourier transform, we obtain a final representation given by

$$N_\ell^1[j_\ell,a_\ell,b_\ell](\xi_\ell,\eta_\ell)=|\xi_\ell|^{s_\ell}\sum_{m_\ell\in\mathbb{Z}^{n_\ell}}c^{s_\ell}[m_\ell]\mathcal{F}[D^{-s_\ell}\tau_{a_\ell+m_\ell}\Psi_\ell](2^{-j_\ell}\xi_\ell)\mathcal{F}[\tau_{b_\ell+m_\ell}\Phi_\ell](2^{-j_\ell}\eta_\ell). \quad (3.23)$$

By the same process, we may obtain a representaiton for $N_\ell^2[j_\ell,a_\ell,b_\ell](\xi_\ell,\eta_\ell)$ given by

$$N_\ell^2[j_\ell,a_\ell,b_\ell](\xi_\ell,\eta_\ell)=|\eta_\ell|^{s_\ell}\sum_{m_\ell\in\mathbb{Z}^{n_\ell}}c^{s_\ell}[m_\ell]\mathcal{F}[\tau_{a_\ell+m_\ell}\Phi_\ell](2^{-j_\ell}\xi_\ell)\mathcal{F}[D^{-s_\ell}\tau_{b_\ell+m_\ell}\Psi_\ell](2^{-j_\ell}\eta_\ell), \quad (3.24)$$

where the coefficients $c^{s_\ell}[m_\ell]$ are defined slightly differently, but satisfy the same decay estimate.

Next, we consider $N_\ell^3[j_\ell,a_\ell,b_\ell](\xi_\ell,\eta_\ell)$. Define $\varphi_\ell\in\mathcal{S}(\mathbb{R}^{n_\ell})$ via $\widehat{\varphi}_\ell:=\widehat{\Phi_\ell}(2^{-6}\cdot)$, so that $\widehat{\varphi}_\ell$ is identically 1 on $\{\xi_\ell\in\mathbb{R}^{n_\ell}:|\xi_\ell|\le 4\}$ and is supported within $\{\xi_\ell\in\mathbb{R}^{n_\ell}:|\xi_\ell|\le 16\}$ by (3.18). It is easily verified that $\widehat{\varphi}_\ell(2^{-j_\ell}(\xi_\ell+\eta_\ell))$ equals 1 within the support of $N_\ell^3[j_\ell,a_\ell,b_\ell](\xi_\ell,\eta_\ell)$, so we have

$$\begin{aligned}
&N_\ell^3[j_\ell,a_\ell,b_\ell](\xi_\ell,\eta_\ell)\\
&\qquad=|\xi_\ell+\eta_\ell|^{s_\ell}\,\widehat{\varphi}_\ell(2^{-j_\ell}(\xi_\ell+\eta_\ell))\widehat{\tau_{a_\ell}\Psi_\ell}(2^{-j_\ell}\xi_\ell)\widehat{\tau_{b_\ell}\Psi_\ell}(2^{-j_\ell}\eta_\ell)\\
&\qquad=|\xi_\ell|^{s_\ell}\left[\left|2^{-j_\ell}(\xi_\ell+\eta_\ell)\right|^{s_\ell}\widehat{\varphi}_\ell(2^{-j_\ell}(\xi_\ell+\eta_\ell))\right]\left[\left|2^{-j_\ell}\xi_\ell\right|^{-s_\ell}\widehat{\tau_{a_\ell}\Psi_\ell}(2^{-j_\ell}\xi_\ell)\right]\widehat{\tau_{b_\ell}\Psi_\ell}(2^{-j_\ell}\eta_\ell)\\
&\qquad=|\xi_\ell|^{s_\ell}\,\mathcal{F}[D^{s_\ell}\varphi_\ell](2^{-j_\ell}(\xi_\ell+\eta_\ell))\mathcal{F}[D^{-s_\ell}\tau_{a_\ell}\Psi_\ell](2^{-j_\ell}\xi_\ell)\widehat{\tau_{b_\ell}\Psi_\ell}(2^{-j_\ell}\eta_\ell).
\end{aligned}$$

We consider the Fourier series expansion of a periodic extension of the compactly supported function $\mathcal{F}[D^{s_\ell}\varphi_\ell]$. This results in

$$\mathcal{F}[D^{s_\ell}\varphi_\ell](2^{-j_\ell}\omega)=\left[\sum_{m_\ell\in\mathbb{Z}^{n_\ell}}\widetilde{c^{s_\ell}}[m_\ell]e^{\frac{2\pi i}{h}2^{-j_\ell}\omega\cdot m_\ell}\right]\chi_{H_\ell}(2^{-j_\ell}\omega),$$

where

$$\widetilde{c^{s_\ell}}[m_\ell] = \frac{1}{h^{n_\ell}} \int_{H_\ell} |\zeta|^{s_\ell} \varphi_\ell(\zeta) e^{-\frac{2\pi i}{h}\zeta\cdot m_\ell}\, d\zeta$$

with $\widetilde{c^{s_\ell}}[m_\ell] = \mathcal{O}((1+|m_\ell|)^{-n_\ell - s_\ell})$ (see Grafakos-Oh [30, Lemma 1]). We substitute this Fourier series expansion back into the formula found above for $N_\ell^3[j_\ell, a_\ell, b_\ell](\xi_\ell, \eta_\ell)$ to obtain a final representation given by

$$N_\ell^3[j_\ell, a_\ell, b_\ell](\xi_\ell, \eta_\ell) = |\xi_\ell|^{s_\ell} \sum_{m_\ell \in \mathbb{Z}^{n_\ell}} \widetilde{c^{s_\ell}}[m_\ell] \mathcal{F}[D^{-s_\ell}\tau_{a_\ell + m_\ell}\Psi_\ell](2^{-j_\ell}\xi_\ell)\mathcal{F}[\tau_{b_\ell+m_\ell}\Psi_\ell](2^{-j_\ell}\eta_\ell). \tag{3.25}$$

3.3.3 Norm estimates

In this subsection, we consider $\|D_1^{s_1} D_2^{s_2}(T_\sigma(f,g))\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}$ and obtain the desired boundedness results, recalling that the indices p, q, r, s_1, s_2, and the weights v and w are as in the statement of Theorem 1.4. Before beginning, define $r^* := \min\{1, r\}$, and note that $v^{\frac{r}{p}} w^{\frac{r}{q}} \in A_{\max\{p,q\}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subset A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ (see Lemma B.2 with $\theta_1 = \frac{r}{p}$ and $\theta_2 = \frac{r}{q}$; the same reasoning holds for product Muckenhoupt weight classes). Following the first line of inequalities in Subsection 3.3.1, we see that

$$\|D_1^{s_1} D_2^{s_2}(T_\sigma(f,g))\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})} \lesssim \sum_{t_1, t_2 = 1}^{3} \|D_1^{s_1} D_2^{s_2} \Pi_{t_1,t_2}(f,g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}.$$

As reasoned in Subsection 3.3.1, we need only obtain the desired results for the pieces associated with $\Pi_{1,1}$, $\Pi_{1,2}$, $\Pi_{1,3}$, and $\Pi_{3,3}$.

We begin with the $\Pi_{1,1}$ term. By Lemma 3.10, we have

$$\|D_1^{s_1} D_2^{s_2} \Pi_{1,1}(f,g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*} \lesssim \left\| \left(\sum_{j_1, j_2 \in \mathbb{Z}} \left| \Delta_{j_1}^{\Psi_1} \Delta_{j_2}^{\Psi_2} D_1^{s_1} D_2^{s_2} \Pi_{1,1}(f,g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*}. \tag{3.26}$$

By recalling the paraproduct representation obtained for $\Pi_{1,1}$ in (3.22), we see that $\Delta_{j_1}^{\Psi_1} \Delta_{j_2}^{\Psi_2} D_1^{s_1} D_2^{s_2} \Pi_{1,1}$

is a bilinear multiplier operator with associated multiplier given by

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{\widetilde{j_1},\widetilde{j_2}\in\mathbb{Z}}\mathcal{C}[\widetilde{j_1},\widetilde{j_2},a,b]$$
$$\times|\xi_1+\eta_1|^{s_1}\,\widehat{\Psi_1}(2^{-j_1}(\xi_1+\eta_1))\widehat{\tau_{a_1}\Psi_1}(2^{-\widetilde{j_1}}\xi_1)\widehat{\tau_{b_1}\Phi_1}(2^{-\widetilde{j_1}}\eta_1)$$
$$\times|\xi_2+\eta_2|^{s_2}\,\widehat{\Psi_2}(2^{-j_2}(\xi_2+\eta_2))\widehat{\tau_{a_2}\Psi_2}(2^{-\widetilde{j_2}}\xi_2)\widehat{\tau_{b_2}\Phi_2}(2^{-\widetilde{j_2}}\eta_2).$$

By considering the supports of Ψ_ℓ and Φ_ℓ for $\ell=1,2$, it may easily be verified that for each fixed $j_\ell\in\mathbb{Z}$, the sum in $\widetilde{j_\ell}\in\mathbb{Z}$ actually need only run over $j_\ell-3\le\widetilde{j_\ell}\le j_\ell+3$. This simplies the form of the multiplier associated to $\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2}D_1^{s_1}D_2^{s_2}\Pi_{1,1}$ to be as follows:

$$\sum_{\delta_1,\delta_2=-3}^{3}\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\mathcal{C}[j_1+\delta_1,j_2+\delta_2,a,b]$$
$$\times|\xi_1+\eta_1|^{s_1}\,\widehat{\Psi_1}(2^{-j_1}(\xi_1+\eta_1))\widehat{\tau_{a_1}\Psi_1}(2^{-j_1-\delta_1}\xi_1)\widehat{\tau_{b_1}\Phi_1}(2^{-j_1-\delta_1}\eta_1)$$
$$\times|\xi_2+\eta_2|^{s_2}\,\widehat{\Psi_2}(2^{-j_2}(\xi_2+\eta_2))\widehat{\tau_{a_2}\Psi_2}(2^{-j_2-\delta_2}\xi_2)\widehat{\tau_{b_2}\Phi_2}(2^{-j_2-\delta_2}\eta_2).$$

The next step would be to split the ℓ^2-norm and weighted L^r-norm in (3.26) across the sum in $\delta_1,\delta_2=-3,\dots,3$. Since the analysis works in the same way for all δ_ℓ cases, we simplify notation by only considering the case $\delta_1=\delta_2=0$, a case which yields the following form for the multiplier associated to $\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2}D_1^{s_1}D_2^{s_2}\Pi_{1,1}$:

$$\sum_{a,b\in\mathbb{Z}^n}\frac{\mathcal{C}[j_1,j_2,a,b]}{(1+|a|^2+|b|^2)^N}N_1^1[j_1,a_1,b_1](\xi_1,\eta_1)N_2^1[j_2,a_2,b_2](\xi_2,\eta_2),$$

where $N_\ell^1[j_\ell,a_\ell,b_\ell]$ for $\ell=1,2$ is as defined in Subsection 3.3.2. With this multiplier in mind, (3.23) implies that $\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2}D_1^{s_1}D_2^{s_2}\Pi_{1,1}(f,g)(x)$ has the form

$$\sum_{a,b\in\mathbb{Z}^n}\frac{\mathcal{C}[j_1,j_2,a,b]}{(1+|a|^2+|b|^2)^N}\sum_{m_1\in\mathbb{Z}^{n_1}}\sum_{m_2\in\mathbb{Z}^{n_2}}c^{s_1}[m_1]c^{s_2}[m_2]$$
$$\times\left[(\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}\Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2}D_1^{s_1}D_2^{s_2}f)\cdot(S_{j_1}^{\tau_{b_1+m_1}\Phi_1}S_{j_2}^{\tau_{b_2+m_2}\Phi_2}g)\right](x).$$

Substituting this into (3.26), and taking into account that $\mathcal{C}[j_1, j_2, a, b]$ are uniformly bounded, we see that $\|D_1^{s_1} D_2^{s_2} \Pi_{1,1}(f,g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*}$ is bounded by

$$\sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^{Nr^*}} \sum_{m_1\in\mathbb{Z}^{n_1}} \sum_{m_2\in\mathbb{Z}^{n_2}} |c^{s_1}[m_1]|^{r^*} |c^{s_2}[m_2]|^{r^*}$$
$$\times \left\| \left(\sum_{j_1,j_2\in\mathbb{Z}} \left| [\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1} \Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2} D_1^{s_1} D_2^{s_2} f] \cdot [S_{j_1}^{\tau_{b_1+m_1}\Phi_1} S_{j_2}^{\tau_{b_2+m_2}\Phi_2} g] \right|^2 \right)^{\frac{1}{2}} \right\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*},$$

where we have used Minkowski's integral inequality (see Section B.2). Recall that we may choose $N > 0$ as large as we wish, which only would have affected the definition of the $\mathcal{C}[j_1, j_2, a, b]$, but not their uniform boundedness property. Also recall the decay properties of the coefficients $c^{s_\ell}[m_\ell]$, for which $c^{s_\ell}[m_\ell] = \mathcal{O}((1+|m_\ell|)^{-K})$ for any $K > 0$. We will see that, if we can obtain some $\tilde{K} > 0$ for which the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm piece above is bounded by something of the form

$$(1+|a|+|b|+|m|)^{\tilde{K}} \|D_1^{s_1} D_2^{s_2} f\|_{L^p(v)} \|g\|_{L^q(w)},$$

where $a = (a_1, a_2), b = (b_1, b_2), m = (m_1, m_2) \in \mathbb{Z}^{n_1} \times \mathbb{Z}^{n_2}$, our work will be complete, for if we were able to do so, that would imply that $\|D_1^{s_1} D_2^{s_2} \Pi_{1,1}(f,g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*}$ is bounded by

$$\sum_{a,b,m\in\mathbb{Z}^n} \left(\frac{(1+|a|+|b|+|m|)^{\tilde{K}}}{(1+|a|^2+|b|^2)^N (1+|m_1|)^K (1+|m_2|)^K} \right)^{r^*} \|D_1^{s_1} D_2^{s_2} f\|_{L^p(v)}^{r^*} \|g\|_{L^q(w)}^{r^*}.$$

Choosing $K, N > 0$ large enough would allow the above summation in $a, b, m \in \mathbb{Z}^n$ to converge, so that we would conclude the desired result, namely

$$\|D_1^{s_1} D_2^{s_2} \Pi_{1,1}(f,g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*} \lesssim \|D_1^{s_1} D_2^{s_2} f\|_{L^p(v)}^{r^*} \|g\|_{L^q(w)}^{r^*}.$$

With this goal in mind, we use (3.5) to obtain a pointwise bound (uniform in $j_1, j_2 \in \mathbb{Z}$)

given by

$$\left|[S_{j_1}^{\tau_{b_1+m_1}\Phi_1} S_{j_2}^{\tau_{b_2+m_2}\Phi_2} g](x)\right| \lesssim (1+|b_1|+|m_1|)^{n_1+1}(1+|b_2|+|m_2|)^{n_2+1}\mathcal{M}_{n_1}^1\mathcal{M}_{n_2}^2 g(x).$$

Using this pointwise bound and an application of Hölder's inequality on the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm piece of interest, we obtain the following bound for the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm term:

$$\begin{aligned}&(1+|b_1|+|m_1|)^{n_1+1}(1+|b_2|+|m_2|)^{n_2+1}\left\|\mathcal{M}_{n_1}^1\mathcal{M}_{n_2}^2 g\right\|_{L^q(w)}\\ &\quad\times\left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|[\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}\Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2}D_1^{s_1}D_2^{s_2}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)}.\end{aligned}$$

Finally, an application of Lemma 3.11 on the $L^p(v)$-norm term, along with the facts that $\mathcal{M}_{n_1}^1$ and $\mathcal{M}_{n_2}^2$ are bounded operators on $L^q(w)$ (as discussed in Subsection 3.2.1) and $(1+|b_1|+|m_1|)^{n_1+1}(1+|b_2|+|m_2|)^{n_2+1}\leq(1+|a|+|b|+|m|)^{n+2}$, concludes the case for $\Pi_{1,1}$.

We now move on to the $\Pi_{1,2}$ term, which will be handled very similarly to the $\Pi_{1,1}$ term. We again begin with an application of Lemma 3.10 to obtain

$$\|D_1^{s_1}D_2^{s_2}\Pi_{1,2}(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}\lesssim\left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2}D_1^{s_1}D_2^{s_2}\Pi_{1,2}(f,g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}. \quad (3.27)$$

Following the arguments in the $\Pi_{1,1}$ case, we see that we must examine $\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2}D_1^{s_1}D_2^{s_2}\Pi_{1,2}$, a bilinear multiplier operator with associated multiplier given by

$$\begin{aligned}&\sum_{\delta_1,\delta_2=-3}^{3}\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\mathcal{C}[j_1+\delta_1,j_2+\delta_2,a,b]\\ &\quad\times|\xi_1+\eta_1|^{s_1}\,\widehat{\Psi_1}(2^{-j_1}(\xi_1+\eta_1))\widehat{\tau_{a_1}\Psi_1}(2^{-j_1-\delta_1}\xi_1)\widehat{\tau_{b_1}\Phi_1}(2^{-j_1-\delta_1}\eta_1)\\ &\quad\times|\xi_2+\eta_2|^{s_2}\,\widehat{\Psi_2}(2^{-j_2}(\xi_2+\eta_2))\widehat{\tau_{a_2}\Phi_2}(2^{-j_2-\delta_2}\xi_2)\widehat{\tau_{b_2}\Psi_2}(2^{-j_2-\delta_2}\eta_2).\end{aligned}$$

As reasoned in the $\Pi_{1,1}$ case, we may disregard the sums in δ_1 and δ_2. By utilizing the functions studied in Subsection 3.3.2, we see that the multiplier associated with $\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2}D_1^{s_1}D_2^{s_2}\Pi_{1,2}$

has the form

$$\sum_{a,b\in\mathbb{Z}^n}\frac{\mathcal{C}[j_1,j_2,a,b]}{(1+|a|^2+|b|^2)^N}N_1^1[j_1,a_1,b_1](\xi_1,\eta_1)N_2^2[j_2,a_2,b_2](\xi_2,\eta_2).$$

Thus, we may expand the multiplier using (3.23) and (3.24) to see that $\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2}D_1^{s_1}D_2^{s_2}\Pi_{1,2}(f,g)(x)$ has the form

$$\sum_{a,b\in\mathbb{Z}^n}\frac{\mathcal{C}[j_1,j_2,a,b]}{(1+|a|^2+|b|^2)^N}\sum_{m_1\in\mathbb{Z}^{n_1}}\sum_{m_2\in\mathbb{Z}^{n_2}}c^{s_1}[m_1]c^{s_2}[m_2]$$
$$\times\left[(\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}S_{j_2}^{\tau_{a_2+m_2}\Phi_2}D_1^{s_1}f)\cdot(S_{j_1}^{\tau_{b_1+m_1}\Phi_1}\Delta_{j_2}^{D^{-s_2}\tau_{b_2+m_2}\Psi_2}D_2^{s_2}g)\right](x).$$

As in the $\Pi_{1,1}$ case, we now substitute back into (3.27) to see that $\|D_1^{s_1}D_2^{s_2}\Pi_{1,2}(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}$ is bounded by

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^{Nr^*}}\sum_{m_1\in\mathbb{Z}^{n_1}}\sum_{m_2\in\mathbb{Z}^{n_2}}|c^{s_1}[m_1]|^{r^*}|c^{s_2}[m_2]|^{r^*}$$
$$\times\left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|[\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}S_{j_2}^{\tau_{a_2+m_2}\Phi_2}D_1^{s_1}f]\cdot[S_{j_1}^{\tau_{b_1+m_1}\Phi_1}\Delta_{j_2}^{D^{-s_2}\tau_{b_2+m_2}\Psi_2}D_2^{s_2}g]\right|^2\right)^{\frac{1}{2}}\right\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}.$$

Thus, if there exists some $\tilde{K}>0$ for which the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm piece above is bounded by something of the form $(1+|a|+|b|+|m|)^{\tilde{K}}\|D_1^{s_1}f\|_{L^p(v)}\|D_2^{s_2}g\|_{L^q(w)}$, the $\Pi_{1,2}$ case will be complete, as it would imply $\|D_1^{s_1}D_2^{s_2}\Pi_{1,2}(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}\lesssim\|D_1^{s_1}f\|_{L^p(v)}^{r^*}\|D_2^{s_2}g\|_{L^q(w)}^{r^*}$, again as reasoned in the $\Pi_{1,1}$ case. First, note that by (3.5), we have the following pointwise bounds (uniform in j_2 and j_1, respectively):

$$\left|[\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}S_{j_2}^{\tau_{a_2+m_2}\Phi_2}D_1^{s_1}f](x)\right|\lesssim(1+|a_2|+|m_2|)^{n_2+1}\left|[\mathcal{M}_{n_2}^2\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}D_1^{s_1}f](x)\right|,$$

$$\left|[S_{j_1}^{\tau_{b_1+m_1}\Phi_1}\Delta_{j_2}^{D^{-s_2}\tau_{b_2+m_2}\Psi_2}D_2^{s_2}g](x)\right|\lesssim(1+|b_1|+|m_1|)^{n_1+1}\left|[\mathcal{M}_{n_1}^1\Delta_{j_2}^{D^{-s_2}\tau_{b_2+m_2}\Psi_2}D_2^{s_2}g](x)\right|.$$

With these pointwise bounds in mind, we may use them with the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm piece above, followed by Hölder's inequality on the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm, to obtain a bound for the

$L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm term given by

$$(1+|a_2|+|m_2|)^{n_2+1}\left\|\left(\sum_{j_1\in\mathbb{Z}}\left|\mathcal{M}^2_{n_2}\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}D_1^{s_1}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)}$$
$$\times(1+|b_1|+|m_1|)^{n_1+1}\left\|\left(\sum_{j_2\in\mathbb{Z}}\left|\mathcal{M}^1_{n_1}\Delta_{j_2}^{D^{-s_2}\tau_{b_2+m_2}\Psi_2}D_2^{s_2}g\right|^2\right)^{\frac{1}{2}}\right\|_{L^q(w)}.$$

Finally, applications of the weighted Fefferman-Stein inequality (3.7) followed by Lemma 3.11 on each weighted term (applying part (a) while iterating norms; see Remark 3.2.1) yield the desired result, thus concluding the $\Pi_{1,2}$ case.

We now consider the $\Pi_{1,3}$ term. In this case, we apply Lemma 3.10 while iterating norms (see Remark 3.2.1) to obtain

$$\|D_1^{s_1}D_2^{s_2}\Pi_{1,3}(f,g)\|^{r^*}_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}\lesssim\left\|\left(\sum_{j_1\in\mathbb{Z}}\left|\Delta_{j_1}^{\Psi_1}D_1^{s_1}D_2^{s_2}\Pi_{1,3}(f,g)\right|^2\right)^{\frac{1}{2}}\right\|^{r^*}_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}. \tag{3.28}$$

Using the paraproduct representation obtained for $\Pi_{1,3}$ in Subsection 3.3.1, we see that the bilinear multiplier operator $\Delta_{j_1}^{\Psi_1}D_1^{s_1}D_2^{s_2}\Pi_{1,3}$ has an associated multiplier given by

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{\widetilde{j_1},\widetilde{j_2}\in\mathbb{Z}}\mathcal{C}[\widetilde{j_1},\widetilde{j_2},a,b]$$
$$\times|\xi_1+\eta_1|^{s_1}\,\widehat{\Psi_1}(2^{-j_1}(\xi_1+\eta_1))\widehat{\tau_{a_1}\Psi_1}(2^{-\widetilde{j_1}}\xi_1)\widehat{\tau_{b_1}\Phi_1}(2^{-\widetilde{j_1}}\eta_1)$$
$$\times|\xi_2+\eta_2|^{s_2}\,\widehat{\tau_{a_2}\Psi_2}(2^{-\widetilde{j_2}}\xi_2)\widehat{\tau_{b_2}\Psi_2}(2^{-\widetilde{j_2}}\eta_2).$$

By the same reasoning as in the $\Pi_{1,1}$ case, we may, without loss of generality, equate $\widetilde{j_1}$ with j_1 and disregard the sum in $\widetilde{j_1}\in\mathbb{Z}$. Also, for ease of notation, we rewrite $\widetilde{j_2}$ as simply j_2. Thus, using the notation of functions introduced in Subsection 3.3.2, we obtain the following form for the multiplier associated to $\Delta_{j_1}^{\Psi_1} D_1^{s_1} D_2^{s_2} \Pi_{1,3}$:

$$\sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_2\in\mathbb{Z}} \mathcal{C}[j_1,j_2,a,b] N_1^1[j_1,a_1,b_1](\xi_1,\eta_1) N_2^3[j_2,a_2,b_2](\xi_2,\eta_2).$$

Expanding the multiplier using (3.23) and (3.25), we see that $\Delta_{j_1}^{\Psi_1} D_1^{s_1} D_2^{s_2} \Pi_{1,3}(f,g)(x)$ has the form

$$\sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j_2\in\mathbb{Z}} \mathcal{C}[j_1,j_2,a,b] \sum_{m_1\in\mathbb{Z}^{n_1}} \sum_{m_2\in\mathbb{Z}^{n_2}} c^{s_1}[m_1]\widetilde{c^{s_2}}[m_2]$$
$$\times \left[(\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1} \Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2} D_1^{s_1} D_2^{s_2} f) \cdot (S_{j_1}^{\tau_{b_1+m_1}\Phi_1} \Delta_{j_2}^{\tau_{b_2+m_2}\Psi_2} g) \right](x).$$

We substitute this formulation back into (3.28), from which we obtain that $\|D_1^{s_1} D_2^{s_2} \Pi_{1,3}(f,g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*}$ is bounded by

$$\sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^{Nr^*}} \sum_{m_1\in\mathbb{Z}^{n_1}} \sum_{m_2\in\mathbb{Z}^{n_2}} |c^{s_1}[m_1]|^{r^*} \left|\widetilde{c^{s_2}}[m_2]\right|^{r^*}$$
$$\times \left\| \left(\sum_{j_1\in\mathbb{Z}} \left(\sum_{j_2\in\mathbb{Z}} \left| [\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1} \Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2} D_1^{s_1} D_2^{s_2} f] \cdot [S_{j_1}^{\tau_{b_1+m_1}\Phi_1} \Delta_{j_2}^{\tau_{b_2+m_2}\Psi_2} g] \right| \right)^2 \right)^{\frac{1}{2}} \right\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*},$$

where we have used the uniform bound on $\mathcal{C}[j_1,j_2,a,b]$ and Minkowski's integral inequality (see Section B.2). Due to the decay of the coefficients $\widetilde{c^{s_2}}[m_2]$, namely $\widetilde{c^{s_\ell}}[m_\ell] = \mathcal{O}((1+|m_\ell|)^{-n_\ell-s_\ell})$, the existence of some $\tilde{K}>0$ for which the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm piece above was bounded by something of the form $(1+|a|+|b|+|m_1|)^{\tilde{K}} \|D_1^{s_1} D_2^{s_2} f\|_{L^p(v)} \|g\|_{L^q(w)}$ would imply that $\|D_1^{s_1} D_2^{s_2} \Pi_{1,3}(f,g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*}$ is bounded by

$$\sum_{a,b,m\in\mathbb{Z}^n} \left(\frac{(1+|a|+|b|+|m_1|)^{\tilde{K}}}{(1+|a|^2+|b|^2)^N (1+|m_1|)^K (1+|m_2|)^{n_2+s_2}} \right)^{r^*} \|D_1^{s_1} D_2^{s_2} f\|_{L^p(v)}^{r^*} \|g\|_{L^q(w)}^{r^*}.$$

For $N,K>0$ sufficiently large, the sum in $a,b\in\mathbb{Z}^n$ and $m_1\in\mathbb{Z}^{n_1}$ converges, while the sum in $m_2\in\mathbb{Z}^{n_2}$ converges due to the hypothesis in Theorem 1.4 that $s_2>\max\{0,n_2(\frac{1}{r}-1)\}$

(that is, this hypothesis guarantees that $(n_2+s_2)r^*>n_2$). With the sum in $a,b,m\in\mathbb{Z}^n$ above converging, we would obtain $\|D_1^{s_1}D_2^{s_2}\Pi_{1,3}(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}\lesssim\|D_1^{s_1}D_2^{s_2}f\|_{L^p(v)}^{r^*}\|g\|_{L^q(w)}^{r^*}$, thus completing the $\Pi_{1,3}$ case. To analyze the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm term above, we begin by using (3.5) for the operator $S_{j_1}^{\tau_{b_1+m_1}\Phi_1}$ (obtaining a bound independent of $j_1\in\mathbb{Z}$), followed by an application of Hölder's inequality on the ℓ^1-norm in $j_2\in\mathbb{Z}$, which gives the following bound for the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm piece:

$$(1+|b_1|+|m_1|)^{n_1+1}$$
$$\times\left\|\left[\sum_{j_1\in\mathbb{Z}}\left(\sum_{j_2\in\mathbb{Z}}\left|\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}\Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2}D_1^{s_1}D_2^{s_2}f\right|^2\right)\left(\sum_{j_2\in\mathbb{Z}}\left|\mathcal{M}_{n_1}^1\Delta_{j_2}^{\tau_{b_2+m_2}\Psi_2}g\right|^2\right)\right]^{\frac{1}{2}}\right\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}.$$

After pulling the second ℓ^2-norm in $j_2\in\mathbb{Z}$ outside the sum in $j_1\in\mathbb{Z}$, an application of Hölder's inequality on the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm yields an upper bound for the original $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm term of the form

$$(1+|b_1|+|m_1|)^{n_1+1}\left\|\left(\sum_{j_2\in\mathbb{Z}}\left|\mathcal{M}_{n_1}^1\Delta_{j_2}^{\tau_{b_2+m_2}\Psi_2}g\right|^2\right)^{\frac{1}{2}}\right\|_{L^q(w)}^{r^*}$$
$$\times\left\|\left(\sum_{j_1,j_2\in\mathbb{Z}}\left|\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}\Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2}D_1^{s_1}D_2^{s_2}f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)}^{r^*}.$$

We complete the $\Pi_{1,3}$ case with an application of the weighted Fefferman-Stein inequality (3.7) on the $L^q(w)$-norm piece, followed by an application of Lemma 3.11 on both pieces, noting that we iterate norms on the $L^q(w)$-norm piece (see Remark 3.2.1).

Finally, we consider the last case, which relates to $\Pi_{3,3}$. From the paraproduct representation for $\Pi_{3,3}$ given in Subsection 3.3.1, we see that $D_1^{s_1}D_2^{s_2}\Pi_{3,3}$ is a bilinear multiplier operator with the following associated multiplier:

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{j_1,j_2\in\mathbb{Z}}\mathcal{C}[j_1,j_2,a,b]$$
$$\times\left[|\xi_1+\eta_1|^{s_1}\,\widehat{\tau_{a_1}\Psi_1}(2^{-j_1}\xi_1)\widehat{\tau_{b_1}\Psi_1}(2^{-j_1}\eta_1)\right]\cdot\left[|\xi_2+\eta_2|^{s_2}\,\widehat{\tau_{a_2}\Psi_2}(2^{-j_2}\xi_2)\widehat{\tau_{b_2}\Psi_2}(2^{-j_2}\eta_2)\right].$$

Again switching to notation using functions from Subsection 3.3.2, we may express the multiplier as follows:

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{j_1,j_2\in\mathbb{Z}}\mathcal{C}[j_1,j_2,a,b]N_1^3[j_1,a_1,b_1](\xi_1,\eta_1)N_2^3[j_2,a_2,b_2](\xi_2,\eta_2).$$

By (3.25), we see that $D_1^{s_1}D_2^{s_2}\Pi_{3,3}(f,g)(x)$ has the form

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{j_1j_2\in\mathbb{Z}}\mathcal{C}[j_1,j_2,a,b]\sum_{m_1\in\mathbb{Z}^{n_1}}\sum_{m_2\in\mathbb{Z}^{n_2}}\widetilde{c^{s_1}}[m_1]\widetilde{c^{s_2}}[m_2]$$
$$\times\left[(\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}\Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2}D_1^{s_1}D_2^{s_2}f)\cdot(\Delta_{j_1}^{\tau_{b_1+m_1}\Psi_1}\Delta_{j_2}^{\tau_{b_2+m_2}\Psi_2}g)\right](x).$$

Therefore, by the uniform bound on $\mathcal{C}[j_1,j_2,a,b]$ and Minkowski's integral inequality (see Section B.2), we see that we may bound $\|D_1^{s_1}D_2^{s_2}\Pi_{3,3}(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}$ by

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^{Nr^*}}\sum_{m_1\in\mathbb{Z}^{n_1}}\sum_{m_2\in\mathbb{Z}^{n_2}}\left|\widetilde{c^{s_1}}[m_1]\right|^{r^*}\left|\widetilde{c^{s_2}}[m_2]\right|^{r^*}$$
$$\times\left\|\sum_{j_1,j_2\in\mathbb{Z}}\left|[\Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1}\Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2}D_1^{s_1}D_2^{s_2}f]\cdot[\Delta_{j_1}^{\tau_{b_1+m_1}\Psi_1}\Delta_{j_2}^{\tau_{b_2+m_2}\Psi_2}g]\right|\right\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}.$$

If we could bound the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm piece above by $\|D_1^{s_1}D_2^{s_2}f\|_{L^p(v)}\|g\|_{L^q(w)}$, this would imply that $\|D_1^{s_1}D_2^{s_2}\Pi_{3,3}(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*}$ is bounded by

$$\sum_{a,b,m\in\mathbb{Z}^n}\left(\frac{1}{(1+|a|^2+|b|^2)^N(1+|m_1|)^{n_1+s_1}(1+|m_2|)^{n_2+s_2}}\right)^{r^*}\|D_1^{s_1}D_2^{s_2}f\|_{L^p(v)}^{r^*}\|g\|_{L^q(w)}^{r^*}.$$

For N sufficiently large, the sum in $a, b \in \mathbb{Z}^n$ converges, and as reasoned in the $\Pi_{1,3}$ case, the sum in $m \in \mathbb{Z}^n$ converges due to the hypotheses on s_ℓ, $\ell = 1, 2$, in the statement of Theorem 1.4, which guarantee $(n_\ell + s_\ell)r^* > n_\ell$. With all sums converging, we would obtain

$$\|D_1^{s_1} D_2^{s_2} \Pi_{3,3}(f, g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*} \lesssim \|D_1^{s_1} D_2^{s_2} f\|_{L^p(v)}^{r^*} \|g\|_{L^q(w)}^{r^*},$$

which completes not only the $\Pi_{3,3}$ case, but also the proof of the theorem. In considering the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm term above, we begin with two applications of Hölder's inequality, first on the ℓ^1-norm in $j_1, j_2 \in \mathbb{Z}$, then on the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm, which gives the following bound for the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm piece:

$$\left\| \left(\sum_{j_1, j_2 \in \mathbb{Z}} \left| \Delta_{j_1}^{D^{-s_1}\tau_{a_1+m_1}\Psi_1} \Delta_{j_2}^{D^{-s_2}\tau_{a_2+m_2}\Psi_2} D_1^{s_1} D_2^{s_2} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(v)} \left\| \left(\sum_{j_1, j_2 \in \mathbb{Z}} \left| \Delta_{j_1}^{\tau_{b_1+m_1}\Psi_1} \Delta_{j_2}^{\tau_{b_2+m_2}\Psi_2} g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w)}.$$

An application of Lemma 3.11 on each piece then concludes the estimate for $\Pi_{3,3}$.

3.4 Proof of weighted fractional Leibniz rules associated to Coifman-Meyer multipliers

In this section, we present the proof of Theorem 1.3, broken up as follows. In Subsection 3.4.1, we establish an appropriate paraproduct decomposition for the bilinear multiplier operator T_σ, similar to the work done in Subsection 3.3.1 to determine a biparaproduct decomposition. Subsection 3.4.2 deals with the proof of the homogeneous estimates (1.4), while Subsection 3.4.3 considers the inhomogeneous estimates (1.5).

Some boundedness results relating to Coifman-Meyer multiplier operators were mentioned in Subsection 3.1.1. Before beginning the proof of Theorem 1.3, we state specifically one such result for weighted Lebesgue spaces which will be useful in proving both the homogeneous and inhomogeneous estimates:

Theorem 3.16. *Let $\sigma(\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, satisfy (3.1) for $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha + \beta| \leq 2n+1$, and* *consider* $1 < p, q \leq \infty$ *and* $\frac{1}{2} < r < \infty$ *such that* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *If* $v \in A_p(\mathbb{R}^n)$ *and* $w \in A_q(\mathbb{R}^n)$, *then for all* $f \in L^p(v)$ *and* $g \in L^q(w)$,

$$\|T_\sigma(f,g)\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})} \lesssim \|f\|_{L^p(v)} \|g\|_{L^q(w)},$$

where the implicit constant depends on p, q, $[v]_{A_p}$, $[w]_{A_q}$, *and* σ.

For a proof of Theorem 3.16, see Grafakos-Martell [29, Corollary 8.2] or Lerner et al. [45, Corollary 3.9], both of which give estimates in weighted Lebesgue spaces for the more general class of bilinear Calderón-Zygmund operators. Note that the statements of the corollaries mentioned do not include the cases $p = \infty$ or $q = \infty$. However, keeping in mind that $L^\infty(w) = L^\infty$ for any Muckenhoupt weight w, it is easily seen in the succinct proof of [29, Corollary 8.2] that the operators involved are also bounded from $L^p(v) \times L^\infty$ to $L^p(v)$ and from $L^\infty \times L^p(v)$ to $L^p(v)$ for $v \in A_p(\mathbb{R}^n)$ with $1 < p < \infty$, as stated in Theorem 3.16.

Throughout the following sections, we will require auxiliary functions $\Psi, \Phi \in \mathcal{S}(\mathbb{R}^n)$. Specifically, fix $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (A.5) and (A.6), and define $\Phi \in \mathcal{S}(\mathbb{R}^n)$ via (A.7). Additionally, we will require auxiliary functions $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ as in (A.9) and (A.10), so that $\Psi = \psi\Psi$ and $\Phi = \phi\Phi$.

3.4.1 Paraproduct decomposition

Let σ be a Coifman-Meyer multiplier, as defined in Definition 3.1. By (A.6), we have

$$\begin{aligned} T_\sigma(f,g)(x) &= \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta) \left(\sum_{j,k \in \mathbb{Z}} \widehat{\Psi}(2^{-j}\xi)\widehat{\Psi}(2^{-k}\eta) \right) \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x \cdot (\xi+\eta)} \, d\xi \, d\eta \\ &=: \Pi_1(f,g)(x) + \Pi_2(f,g)(x) + \Pi_3(f,g)(x), \end{aligned}$$

where the Π pieces are bilinear multiplier operators given by

$$\begin{aligned}
\Pi_1(f,g)(x) &= \sum_{\substack{j,k\in\mathbb{Z}\\ k<j-2}} \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{\Psi}(2^{-j}\xi)\widehat{\Psi}(2^{-k}\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta \\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{\Psi}(2^{-j}\xi)\widehat{\Phi}(2^{-j}\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta, \\
\Pi_2(f,g)(x) &= \sum_{\substack{j,k\in\mathbb{Z}\\ j<k-2}} \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{\Psi}(2^{-j}\xi)\widehat{\Psi}(2^{-k}\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta \\
&= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{\Phi}(2^{-k}\xi)\widehat{\Psi}(2^{-k}\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta, \\
\Pi_3(f,g)(x) &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} \sigma(\xi,\eta)\widehat{\Psi}(2^{-j}\xi)\widehat{\Psi}(2^{-j}\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta.
\end{aligned}$$

Technically, Π_3 is a multiplier operator whose associated multiplier has the form

$$\sum_{\delta=-2}^{2}\left[\sum_{j\in\mathbb{Z}} \sigma(\xi,\eta)\widehat{\Psi}(2^{-j}\xi)\widehat{\Psi}(2^{-j-\delta}\eta)\right].$$

However, keeping track of this extra finite sum in δ throughout the proof does not affect the results, but merely changes some of the constants obtained in inequalities. Since we are not tracking the specific values of such constants, we will, without loss of generality, consider only the case $\delta = 0$. Note that, due to the supports of $\widehat{\Psi}$ and $\widehat{\Phi}$, the integrand in the formula for $\Pi_1(f,g)$ is supported where $|\eta| \leq \frac{1}{2}|\xi|$, the integrand for $\Pi_2(f,g)$ is supported where $|\xi| \leq \frac{1}{2}|\eta|$, and the integrand for $\Pi_3(f,g)$ is supported where $|\xi| \sim |\eta|$.

It will suffice to prove the desired boundedness result for each Π piece above, and due to the fact that the Π_1 and Π_2 cases are handled with very similar arguments, we will restrict our analysis to the Π_1 and Π_3 cases (see Subsection 3.3.1 for an example relating to the proof of biparameter estimates which justifies considering only a few cases). We will first establish a paraproduct representation for $\Pi_1(f,g)$. Using that $\Psi = \psi\Psi$ and $\Phi = \phi\Phi$, we see that the multiplier associated with Π_1 may be expressed as

$$\sum_{j\in\mathbb{Z}} \sigma[j](2^{-j}\xi, 2^{-j}\eta)\widehat{\Psi}(2^{-j}\xi_1)\widehat{\Phi}(2^{-j}\eta),$$

where $\sigma[j](\xi,\eta) \coloneqq \sigma(2^j\xi, 2^j\eta)\Gamma_1(\xi,\eta)$ with $\Gamma_1(\xi,\eta) = \widehat{\psi}(\xi)\widehat{\phi}(\eta)$. At this point, we follow through with much of the same type of arguments as used in Subsection 3.3.1, so we will omit the exact details here. To give an idea of how things proceed, we would next verify that $\sigma[j]$ and its derivatives are bounded uniformly in $j \in \mathbb{Z}$, due to the support of Γ_1 and the fact that σ is a Coifman-Meyer multiplier. Then, we would consider the Fourier series expansion of a periodic extension of the compactly supported $\sigma[j]$, using the Fourier series coefficients $\{c[j,a,b]\}_{a,b\in\mathbb{Z}^n}$ to define another family of coefficients for some fixed $N \in \mathbb{N}$ sufficiently large (the exact size of which will be specified in Subsections 3.4.2 and 3.4.3), given by $\mathcal{C}[j,a,b] \coloneqq (1+|a|+|b|)^N c[j,a,b]$ and satisfying $|\mathcal{C}[j,a,b]| \lesssim 1$ uniformly in $j \in \mathbb{Z}$ and $a, b \in \mathbb{Z}^n$. Finally, by substituting the Fourier series expansion of $\sigma[j]$ back into the multiplier for Π_1, we arrive at the following paraproduct representation:

$$\Pi_1(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b][\Delta_j^{\tau_a\Psi} f](x)[S_j^{\tau_b\Phi} g](x).$$

Obtaining a paraproduct representation for $\Pi_3(f,g)$ follows the same steps, with the notable difference being the use of $\Gamma_3(\xi,\eta) = \widehat{\psi}(\xi)\widehat{\psi}(\eta)$ in place of Γ_1. In this case, we obtain the following paraproduct representation:

$$\Pi_3(f,g)(x) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b][\Delta_j^{\tau_a\Psi} f](x)[\Delta_j^{\tau_b\Psi} g](x),$$

where the coefficients $\mathcal{C}[j,a,b]$ are defined slightly differently, but are still bounded uniformly in $j \in \mathbb{Z}$ and $a, b \in \mathbb{Z}^n$.

For both Subsections 3.4.2 and 3.4.3, let p, q, r, s, and the weights v and w be as in the statement of Theorem 1.3. Note that $v^{\frac{r}{p}} w^{\frac{r}{q}} \in A_{\max\{p,q\}}(\mathbb{R}^n) \subset A_\infty(\mathbb{R}^n)$ (see Lemma B.2 with $\theta_1 = \frac{r}{p}$ and $\theta_2 = \frac{r}{q}$), and as before, set $r^* = \min\{1, r\}$.

3.4.2 Homogeneous estimates

We begin in the same way as in Subsection 3.3.3, noting that

$$\begin{aligned}
&\|D^s(T_\sigma(f,g))\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} \\
&\qquad \lesssim \|D^s\Pi_1(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} + \|D^s\Pi_2(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} + \|D^s\Pi_3(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}.
\end{aligned}$$

Our goal is to verify the desired result for each of the Π pieces above (or more precisely, for the Π_1 and Π_3 pieces, as justified in Subsection 3.4.1). We will first study the Π_1 piece. Since $D^s\Pi_1$ is itself a bilinear multiplier operator, we may consider its associated multiplier o_1, given by

$$\begin{aligned}
o_1(\xi,\eta) &:= \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b]\,|\xi+\eta|^s\,\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Phi}(2^{-j}\eta) \\
&= \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b]\frac{|\xi+\eta|^s}{|\xi|^s}\left(\widehat{\tau_a\Psi}(2^{-j}\xi)\,|\xi|^s\right)\widehat{\tau_b\Phi}(2^{-j}\eta).
\end{aligned}$$

Thus, we see that $D^s\Pi_1(f,g)(x) = T_{o_1}(f,g)(x) = T_{\widetilde{o_1}}(D^s f,g)(x)$, where

$$\widetilde{o_1}(\xi,\eta) := \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b]\frac{|\xi+\eta|^s}{|\xi|^s}\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Phi}(2^{-j}\eta).$$

In fact, it can be verified that $\widetilde{o_1}$ satisfies (3.1) for all multi-indices $\alpha,\beta\in\mathbb{N}_0^n$ (using techniques similar to those used to prove Lemma B.1), so an application of Theorem 3.16 yields that

$$\|D^s\Pi_1(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} = \|T_{\widetilde{o_1}}(D^s f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} \lesssim \|D^s f\|_{l^p(v)}\|g\|_{L^q(w)},$$

thus completing the Π_1 case. Note that the restriction on s in the statement of Theorem 1.3 was not needed for the study of Π_1.

We now move on to the Π_3 case. Following the reasoning in the Π_1 case, it can be seen that $D^s\Pi_3(f,g) = T_{\widetilde{o_3}}(D^s f, g)$, where

$$\widetilde{o_3}(\xi,\eta) := \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b] \frac{|\xi+\eta|^s}{|\xi|^s} \widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta).$$

In the instances where $s > 2n+1$ or $s \in 2\mathbb{N}_0$, it may be verified that $\widetilde{o_3}$, like $\widetilde{o_1}$ above, satisfies the hypotheses on the multiplier in Theorem 3.16, so that an application of said theorem yields the desired result just as in the Π_1 case. More specifically, the hypotheses on the multiplier require that (3.1) be satisfied for $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha+\beta| \leq 2n+1$. Recall that in the Π_3 case, $|\xi| \sim |\eta|$ within the support of the multiplier $\widetilde{o_3}$. Thus, taking derivatives of the piece $|\xi+\eta|^s$ in $\widetilde{o_3}$ may prove problematic while checking whether the multiplier satisfies the hypotheses of Theorem 3.16. However, for $s > 2n+1$, not enough derivatives need to be taken for a singularity to develop (since we only need to check $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha+\beta| \leq 2n+1$), and for $s \in 2\mathbb{N}_0$, no singularity develops no matter how many derivatives are taken. For the remainder of the subsection, we deal with Π_3 in the general case $s > \max\{0, n(\frac{1}{r}-1)\}$. We split our analysis into three cases: p and q finite, $q = \infty$, and $p = \infty$.

Case 1: $\frac{1}{2} < r < \infty$, $1 < p, q < \infty$. The reasoning for this case follows very closely to that of Subsection 3.3.2, so we will omit some of the details which have already been carefully laid out in that subsection. Note that we may rewrite $\widetilde{o_3}$ as

$$\widetilde{o_3}(\xi,\eta) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b] N^3[j,a,b](\xi,\eta),$$

where

$$N^3[j,a,b](\xi,\eta) := \frac{|\xi+\eta|^s}{|\xi|^s} \widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta). \tag{3.29}$$

For $\widehat{\varphi} = \widehat{\Phi}(2^{-6}\cdot)$, it is easily verified that $\widehat{\varphi}(2^{-j}(\xi+\eta)) = 1$ for all $(\xi,\eta) \in \operatorname{supp}(N^3[j,a,b])$.

With this in mind, we may express $N^3[j,a,b]$ as

$$\begin{aligned}
N^3[j,a,b](\xi,\eta) &= \frac{|\xi+\eta|^s}{|\xi|^s}\widehat{\varphi}(2^{-j}(\xi+\eta))\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta) \\
&= \left[\left|2^{-j}(\xi+\eta)\right|^s \widehat{\varphi}(2^{-j}(\xi+\eta))\right]\left[\left|2^{-j}\xi\right|^{-s}\widehat{\tau_a\Psi}(2^{-j}\xi)\right]\widehat{\tau_b\Psi}(2^{-j}\eta) \\
&= \mathcal{F}[D^s\varphi](2^{-j}(\xi+\eta))\mathcal{F}[D^{-s}\tau_a\Psi](2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta).
\end{aligned}$$

Due to the compact support of $\mathcal{F}[D^s\varphi]$, say within $H=[-\frac{h}{2},\frac{h}{2}]^n$ with h sufficiently large, we consider the Fourier series expansion of a periodic extension of $\mathcal{F}[D^s\varphi]$, obtaining

$$\mathcal{F}[D^s\varphi](2^{-j}\omega) = \left[\sum_{m\in\mathbb{Z}^n}\widetilde{c^s}[m]e^{\frac{2\pi i}{h}2^{-j}\omega\cdot m}\right]\chi_H(2^{-j}\omega),$$

where the Fourier series coefficients are given by

$$\widetilde{c^s}[m] = \frac{1}{h^n}\int_H |\zeta|^s\,\varphi(\zeta)e^{-\frac{2\pi i}{h}\zeta\cdot m}\,d\zeta,$$

with $\widetilde{c^s}[m]=\mathcal{O}((1+|m|)^{-n-s})$ (see Grafakos-Oh [30, Lemma 1]). We substitute this Fourier series expansion for $\mathcal{F}[D^s\varphi]$ back into the formula above for $N^3[j,a,b]$, keeping in mind property (A.3.3) of the Fourier transform, to obtain a final representation given by

$$N^3[j,a,b](\xi,\eta) = \sum_{m\in\mathbb{Z}^n}\widetilde{c^s}[m]\mathcal{F}[D^{-s}\tau_{a+m}\Psi](2^{-j}\xi)\mathcal{F}[\tau_{b+m}\Psi](2^{-j}\eta).$$

With this identity for the multiplier $\widetilde{\sigma_3}$, we see that $D^s\Pi_3(f,g)=T_{\widetilde{\sigma_3}}(D^sf,g)$ has the form

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{j\in\mathbb{Z}}\mathcal{C}[j,a,b]\sum_{m\in\mathbb{Z}^n}\widetilde{c^s}[m]\left[(\Delta_j^{D^{-s}\tau_{a+m}\Psi}D^sf)\cdot(\Delta_j^{\tau_{b+m}\Psi}g)\right](x).$$

We may now consider $\|D^s\Pi_3(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}$. Using the fact that the coefficients $\mathcal{C}[j,a,b]$ are bounded uniformly in $j \in \mathbb{Z}$ and $a, b \in \mathbb{Z}^n$, we see that $\|D^s \Pi_3(f,g)\|^{r^*}_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}$ is controlled by

$$\sum_{a,b,m \in \mathbb{Z}^n} \frac{|\widetilde{c^s}[m]|^{r^*}}{(1+|a|^2+|b|^2)^{Nr^*}} \left\| \sum_{j \in \mathbb{Z}} \left| [\Delta_j^{D^{-s}\tau_{a+m}\Psi} D^s f] \cdot [\Delta_j^{\tau_{b+m}\Psi} g] \right| \right\|^{r^*}_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}.$$

Consider just the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm piece above. Two applications of Hölder's inequality, first on the ℓ^1-norm in $j \in \mathbb{Z}$, then on the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm, imply that the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm term above is bounded by

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left| \Delta_j^{D^{-s}\tau_{a+m}\Psi} D^s f \right|^2 \right)^{\frac{1}{2}} \right\|^{r^*}_{L^p(v)} \left\| \left(\sum_{j \in \mathbb{Z}} \left| \Delta_j^{\tau_{b+m}\Psi} g \right|^2 \right)^{\frac{1}{2}} \right\|^{r^*}_{L^q(w)}.$$

Applications of Lemma 3.11 on each weighted norm piece then imply that $\|D^s \Pi_3(f,g)\|^{r^*}_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}$ itself is bounded by

$$\sum_{a,b,m \in \mathbb{Z}^n} \left(\frac{1}{(1+|a|^2+|b|^2)^N (1+|m|)^{n+s}} \right)^{r^*} \|D^s f\|^{r^*}_{L^p(v)} \|g\|^{r^*}_{L^q(w)}.$$

For N sufficiently large, the sum in $a, b \in \mathbb{Z}^n$ converges, and the sum in $m \in \mathbb{Z}^n$ converges due to the hypothesis that $s > \max\{0, n(\frac{1}{r} - 1)\}$ in the statement of Theorem 1.3. Thus, $\|D^s \Pi_3(f,g)\|^{r^*}_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})} \lesssim \|D^s f\|^{r^*}_{L^p(v)} \|g\|^{r^*}_{L^q(w)}$, which concludes case 1 for Π_3.

Case 2: $q = \infty$, $p = r$, $1 < p < \infty$. In this case, we desire to show that, for $v \in A_p(\mathbb{R}^n)$,

$$\|D^s \Pi_3(f,g)\|_{L^p(v)} \lesssim \|D^s f\|_{L^p(v)} \|g\|_{L^\infty}.$$

We begin by applying Lemma 3.10 to obtain

$$\|D^s \Pi_3(f,g)\|_{L^p(v)} \lesssim \left\| \left(\sum_{k \in \mathbb{Z}} \left| \Delta_k^\Psi D^s \Pi_3(f,g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(v)}. \tag{3.30}$$

Fix $k \in \mathbb{Z}$, then $\Delta_k^\Psi D^s \Pi_3(f,g)$ corresponds with $T_{\widetilde{o_3}[k]}(D^s f, g)$, where

$$\widetilde{o_3}[k](\xi,\eta) = \widehat{\Psi}(2^{-k}(\xi+\eta)) \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b] N^3[j,a,b](\xi,\eta)$$

with $N^3[j,a,b]$ as defined in (3.29). It is easily verified that, due to the support of $\widehat{\Psi}$, the summation over $j \in \mathbb{Z}$ needs only run over $j \geq k-3$. With this in mind, and expanding $N^3[j,a,b]$ using its definition, we see that

$$\begin{aligned}
&\widetilde{o_3}[k](\xi,\eta) \\
&\quad = \widehat{\Psi}(2^{-k}(\xi+\eta)) \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\geq k-3} \mathcal{C}[j,a,b] \frac{|\xi+\eta|^s}{|\xi|^s} \widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta) \\
&\quad = \left[2^{ks}\left|2^{-k}(\xi+\eta)\right|^s \widehat{\Psi}(2^{-k}(\xi+\eta))\right] \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \\
&\qquad \times \sum_{j\geq k-3} \mathcal{C}[j,a,b] \left[2^{-js}\left|2^{-j}\xi\right|^{-s} \widehat{\tau_a\Psi}(2^{-j}\xi)\right] \widehat{\tau_b\Psi}(2^{-j}\eta) \\
&\quad = 2^{ks}\widehat{D^s\Psi}(2^{-k}(\xi+\eta)) \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \\
&\qquad \times \sum_{j\geq k-3} 2^{-js}\mathcal{C}[j,a,b]\mathcal{F}[D^{-s}\tau_a\Psi](2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta).
\end{aligned}$$

With this representation in mind for the multiplier $\widetilde{o_3}[k]$, we see that $\left|\Delta_k^\Psi D^s\Pi_3(f,g)(x)\right| = \left|T_{\widetilde{o_3}[k]}(D^s f, g)(x)\right|$ is bounded by

$$2^{ks} \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\geq k-3} 2^{-js}\left|\mathcal{C}[j,a,b]\right| \cdot \left|\Delta_k^{D^s\Psi}\left[(\Delta_j^{D^{-s}\tau_a\Psi} D^s f)\cdot(\Delta_j^{\tau_b\Psi} g)\right](x)\right|.$$

By an application of Hölder's inequality on the ℓ^1-norm in $j \geq k-3$, we see that $\left|\Delta_k^\Psi D^s\Pi_3(f,g)(x)\right|$

is further bounded by

$$2^{ks}\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\left(\sum_{j\geq k-3}2^{-2js}\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{j\geq k-3}|\mathcal{C}[j,a,b]|^2\cdot\left|\Delta_k^{D^s\Psi}\left[(\Delta_j^{D^{-s}\tau_a\Psi}D^sf)\cdot(\Delta_j^{\tau_b\Psi}g)\right]\right|^2\right)^{\frac{1}{2}}.$$

Using the facts that the coefficients $\mathcal{C}[j,a,b]$ are uniformly bounded and $\sum_{j\geq k-3}(2^{-2s})^j\sim 2^{-2sk}$, we obtain

$$\left|\Delta_k^{\Psi}D^s\Pi_3(f,g)(x)\right|\lesssim\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\left(\sum_{j\geq k-3}\left|\Delta_k^{D^s\Psi}\left[(\Delta_j^{D^{-s}\tau_a\Psi}D^sf)\cdot(\Delta_j^{\tau_b\Psi}g)\right]\right|^2\right)^{\frac{1}{2}}.$$

Substituting back into (3.30) and applying the triangle inequality for Lebesgue space norms and Minkowski's integral inequality (see Section B.2), we see that $\|D^s\Pi_3(f,g)\|_{L^p(v)}$ is bounded by

$$\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\left\|\left(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\left|\Delta_k^{D^s\Psi}\left[(\Delta_j^{D^{-s}\tau_a\Psi}D^sf)\cdot(\Delta_j^{\tau_b\Psi}g)\right]\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)},$$

If we were able to bound the $L^p(v)$-norm piece above by something of the form $\|D^sf\|_{L^p(v)}\|g\|_{L^\infty}$, the case would be complete, since the sum in $a,b\in\mathbb{Z}^n$ converges for sufficiently large N. First, we apply Lemma 3.12 with $\{T_k\}_{k\in\mathbb{Z}}=\{\Delta_k^{D^s\Psi}\}_{k\in\mathbb{Z}}$ and $r=2$, which is possible since (3.14) holds with $r=2$ by Lemma 3.9, implying that the $L^p(v)$-norm piece is bounded by

$$\left\|\left(\sum_{j\in\mathbb{Z}}\left|(\Delta_j^{D^{-s}\tau_a\Psi}D^sf)\cdot(\Delta_j^{\tau_b\Psi}g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)}\leq\sup_{j\in\mathbb{Z}}\left\{\left\|\Delta_j^{\tau_b\Psi}g\right\|_{L^\infty}\right\}\left\|\left(\sum_{j\in\mathbb{Z}}\left|\Delta_j^{D^{-s}\tau_a\Psi}D^sf\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)}.$$

Finally, we may bound the supremum above by $\|g\|_{L^\infty}$ since

$$\left\|\Delta_j^{\tau_b\Psi}g\right\|_{L^\infty}=\left\|[2^{jn}\tau_b\Psi(2^j\cdot)]*g\right\|_{L^\infty}\leq\left\|2^{jn}\tau_b\Psi(2^j\cdot)\right\|_{L^1}\|g\|_{L^\infty}=\|\Psi\|_{L^1}\|g\|_{L^\infty}\sim\|g\|_{L^\infty},\tag{3.31}$$

where we have used Hölder's inequality and we note that the above string of inequalities is independent of $j \in \mathbb{Z}$, and we bound the $L^p(v)$-norm piece by $\|D^s f\|_{L^p(v)}$ using Lemma 3.11, thus concluding this case.

Case 3: $p = \infty$**,** $q = r$**,** $1 < q < \infty$**.** In this last case, we desire to show that, for $w \in A_q(\mathbb{R}^n)$,

$$\|D^s \Pi_3(f,g)\|_{L^q(w)} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^q(w)} .$$

We follow the exact procedure from case 2 (replacing $L^p(v)$ with $L^q(w)$) until we see that we may complete the case by verifying

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left| (\Delta_j^{D^{-s}\tau_a\Psi} D^s f) \cdot (\Delta_j^{\tau_b\Psi} g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w)} \lesssim \|D^s f\|_{L^\infty} \|g\|_{L^q(w)} .$$

Similar to the arguments made in case 2, we have

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left| (\Delta_j^{D^{-s}\tau_a\Psi} D^s f) \cdot (\Delta_j^{\tau_b\Psi} g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w)} \le \sup_{j\in\mathbb{Z}} \left\{ \left\| \Delta_j^{D^{-s}\tau_a\Psi} D^s f \right\|_{L^\infty} \right\} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \Delta_j^{\tau_b\Psi} g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w)} .$$

We bound the $L^q(w)$-norm piece by $\|g\|_{L^q(w)}$ using Lemma 3.11, and we bound the supremum by $\|D^s f\|_{L^\infty}$, using the fact that, independent of $j \in \mathbb{Z}$,

$$\begin{aligned} \left| \Delta_j^{D^{-s}\tau_a\Psi} D^s f(x) \right| &= \left| [2^{jn} D^{-s}\tau_a \Psi(2^j \cdot)] * D^s f(x) \right| \\ &\le \left\| 2^{jn} D^{-s}\tau_a \Psi(2^j \cdot) \right\|_{L^1} \|D^s f\|_{L^\infty} = \left\| D^{-s}\Psi \right\|_{L^1} \|D^s f\|_{L^\infty} . \end{aligned}$$

Finally, the case and the proof of the homogeneous estimates are concluded by noting that $\|D^{-s}\Psi\|_{L^1} < \infty$ since, in view of the support of $\widehat{\Psi}$, $\widehat{D^{-s}\Psi} \in \mathcal{S}(\mathbb{R}^n)$.

3.4.3 Inhomogeneous estimates

In this final subsection, we seek to prove the inhomogeneous estimates given by (1.5). As in Subsection 3.4.2, we begin by noting that

$$\begin{aligned}&\|J^s(T_\sigma(f,g))\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}\\&\qquad\lesssim \|J^s\Pi_1(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}+\|J^s\Pi_2(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}+\|J^s\Pi_3(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})},\end{aligned}$$

and our goal reduces to verifying the desired result for the Π_1 (which also represents the Π_2 piece) and Π_3 pieces. We begin with the Π_1 piece, whose treatment will be very similar to the Π_1 piece in Subsection 3.4.2. $J^s\Pi_1$ is a bilinear multiplier operator, whose associated multiplier o_1' is given by

$$o_1'(\xi,\eta)\coloneqq\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{j\in\mathbb{Z}}\mathcal{C}[j,a,b](1+|\xi+\eta|^2)^{\frac{s}{2}}\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Phi}(2^{-j}\eta).$$

We see that $J^s\Pi_1(f,g)(x)=T_{\widetilde{o_1}'}(J^sf,g)(x)$, where

$$\widetilde{o_1}'(\xi,\eta)\coloneqq\sum_{a,b\in\mathbb{Z}^n}\frac{1}{(1+|a|^2+|b|^2)^N}\sum_{j\in\mathbb{Z}}\mathcal{C}[j,a,b]\frac{(1+|\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}}\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Phi}(2^{-j}\eta).$$

Again, we may apply Theorem 3.16 to obtain the desired estimates in the Π_1 case, for it can be verified that $\widetilde{o_1}'$ satisfies (3.1) for any $\alpha,\beta\in\mathbb{N}_0^n$ (using techniques similar to those used to prove Lemma B.1). Theorem 3.16 then yields

$$\|J^s\Pi_1(f,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}=\|T_{\widetilde{o_1}'}(J^sf,g)\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}\lesssim\|J^sf\|_{l^p(v)}\|g\|_{L^q(w)}.$$

As in the homogeneous case, the restriction on s in the statement of Theorem 1.3 was not needed with the Π_1 piece.

We now consider the Π_3 case. Similar to the calculations in the Π_1 case, it can be reasoned that $J^s\Pi_3(f,g) = T_{\widetilde{o_3}'}(J^s f, g)$, where

$$\widetilde{o_3}'(\xi,\eta) \coloneqq \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \sum_{j\in\mathbb{Z}} \mathcal{C}[j,a,b] \frac{(1+|\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}} \widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta).$$

If $s > 2n+1$ or $s \in 2\mathbb{N}_0$, then similar to the $\widetilde{o_1}'$ case, we can show that $\widetilde{o_3}'$ satisfies the hypotheses on the multiplier in Theorem 3.16, so an application of this theorem takes care of these values of s. Thus, it remains to analyze Π_3 in the general case $s > \max\{0, n(\frac{1}{r}-1)\}$, which we split further into three separate cases, just as in the proof of the homogeneous results. In each case, it will be necessary to decompose $\widetilde{o_3}'$ as follows:

$$\widetilde{o_3}'(\xi,\eta) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \left[\widetilde{o_3}^+[a,b](\xi,\eta) + \widetilde{o_3}^-[a,b](\xi,\eta)\right],$$

where

$$\widetilde{o_3}^+[a,b](\xi,\eta) \coloneqq \frac{(1+|\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}} \sum_{j\geq 0} \mathcal{C}[j,a,b]\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta),$$

$$\widetilde{o_3}^-[a,b](\xi,\eta) \coloneqq \frac{(1+|\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}} \sum_{j< 0} \mathcal{C}[j,a,b]\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta).$$

Since

$$J^s\Pi_3(f,g) = T_{\widetilde{o_3}'}(J^s f,g) = \sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^N} \left[T_{\widetilde{o_3}^+[a,b]}(J^s f,g) + T_{\widetilde{o_3}^-[a,b]}(J^s f,g)\right],$$

we see that $\|J^s\Pi_3(f,g)\|^{r^*}_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}$ is bounded by

$$\sum_{a,b\in\mathbb{Z}^n} \frac{1}{(1+|a|^2+|b|^2)^{Nr^*}} \left[\left\|T_{\widetilde{o_3}^+[a,b]}(J^s f,g)\right\|^{r^*}_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} + \left\|T_{\widetilde{o_3}^-[a,b]}(J^s f,g)\right\|^{r^*}_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}\right].$$

If we are able to obtain bounds of the form

$$\left\|T_{\widetilde{o_3}^+[a,b]}(J^s f,g)\right\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})} \lesssim P^+(|a|,|b|)\,\|J^s f\|_{L^p(v)}\,\|g\|_{L^q(w)},$$

$$\left\| T_{\widetilde{o_3}^-[a,b]}(J^s f, g) \right\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})} \lesssim P^-(|a|, |b|) \left\| J^s f \right\|_{L^p(v)} \|g\|_{L^q(w)},$$

where P^+ and P^- are polynomial functions, the result would be complete by taking N sufficiently large to counteract the polynomial growth and ensure convergence in $a, b \in \mathbb{Z}^n$.

Finally, before breaking into separate cases, define $\varphi \in \mathcal{S}(\mathbb{R}^n)$ via $\widehat{\varphi} := \widehat{\Phi}(2^{-7}\cdot)$ (so that $\widehat{\varphi}$ is identically 1 on $\{\xi \in \mathbb{R}^n : |\xi| \leq 8\}$ and is supported within $\{\xi \in \mathbb{R}^n : |\xi| \leq 32\}$). Also, define the multiplier operator J_j^t for $j \in \mathbb{N}_0$, $t \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^n)$ via

$$\widehat{J_j^t f}(\xi) := (2^{-2j} + |\xi|^2)^{\frac{t}{2}} \widehat{f}(\xi).$$

Case 1: $\frac{1}{2} < r < \infty$, $1 < p, q < \infty$. For this case, we will find that $P^+ = P^- \equiv 1$. We first treat $\widetilde{o_3}^+[a,b](\xi, \eta)$. For any $j \in \mathbb{Z}$, it is easily verified that $\widehat{\varphi}(2^{-j}(\xi + \eta)) = 1$ within the support of $\widehat{\Psi}(2^{-j}\xi)\widehat{\Psi}(2^{-j}\eta)$, so

$$\widetilde{o_3}^+[a,b](\xi,\eta) = \frac{(1 + |\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}} \sum_{j \geq 0} \mathcal{C}[j,a,b] \widehat{\varphi}(2^{-j}(\xi+\eta)) \widehat{\tau_a \Psi}(2^{-j}\xi) \widehat{\tau_b \Psi}(2^{-j}\eta).$$

Consider the identity

$$\frac{(1 + |\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}} = \frac{(2^{-2j} + |2^{-j}(\xi+\eta)|^2)^{\frac{s}{2}}}{(2^{-2j} + |2^{-j}\xi|^2)^{\frac{s}{2}}},$$

along with the fact $\widehat{\Psi} \equiv \widehat{\psi}\widehat{\Psi}$, which yield

$$\begin{aligned}
\widetilde{o_3}^+[a,b](\xi,\eta) &= \sum_{j \geq 0} \mathcal{C}[j,a,b] \left[(2^{-2j} + \left|2^{-j}(\xi+\eta)\right|^2)^{\frac{s}{2}} \widehat{\varphi}(2^{-j}(\xi+\eta)) \right] \\
&\qquad \times \left[(2^{-2j} + \left|2^{-j}\xi\right|^2)^{-\frac{s}{2}} \widehat{\psi}(2^{-j}\xi) \right] \widehat{\tau_a \Psi}(2^{-j}\xi) \widehat{\tau_b \Psi}(2^{-j}\eta) \\
&= \sum_{j \geq 0} \mathcal{C}[j,a,b] \widehat{J_j^s \varphi}(2^{-j}(\xi+\eta)) \widehat{J_j^{-s} \psi}(2^{-j}\xi) \widehat{\tau_a \Psi}(2^{-j}\xi) \widehat{\tau_b \Psi}(2^{-j}\eta).
\end{aligned}$$

Since both $\widehat{J_j^s \varphi}$ and $\widehat{J_j^{-s} \psi}$ are compactly supported, we may consider the Fourier series expansions of periodic extensions of each. Suppose h is large enough so that each function is supported within $H = [-\frac{h}{2}, \frac{h}{2}]^n$, then

$$\widehat{J_j^s\varphi}(2^{-j}\omega) = \left[\sum_{m\in\mathbb{Z}^n} \widetilde{c^s}[m,j]e^{\frac{2\pi i}{h}2^{-j}\omega\cdot m}\right]\chi_H(2^{-j}\omega),$$

$$\widehat{J_j^{-s}\psi}(2^{-j}\omega) = \left[\sum_{\mu\in\mathbb{Z}^n} c^{-s}[\mu,j]e^{\frac{2\pi i}{h}2^{-j}\omega\cdot\mu}\right]\chi_H(2^{-j}\omega),$$

where

$$\widetilde{c^s}[m,j] = \frac{1}{h^n}\int_{\mathbb{R}^n}(2^{-2j}+|\zeta|^2)^{\frac{s}{2}}\widehat{\varphi}(\zeta)e^{-\frac{2\pi i}{h}\zeta\cdot m}\,d\zeta,$$

$$c^{-s}[\mu,j] = \frac{1}{h^n}\int_{\mathbb{R}^n}(2^{-2j}+|\zeta|^2)^{-\frac{s}{2}}\widehat{\psi}(\zeta)e^{-\frac{2\pi i}{h}\zeta\cdot\mu}\,d\zeta.$$

With these Fourier series coefficients in mind, define

$$\widetilde{b^s}[m] := \sup_{j\geq 0}\left|\widetilde{c^s}[m,j]\right| \quad\text{ and }\quad b^{-s}[\mu] := \sup_{j\geq 0}\left|c^{-s}[\mu,j]\right|.$$

We note decay properties associated with these coefficients given by $\widetilde{b^s}[m] = \mathcal{O}((1+|m|)^{-n-s})$ (Grafakos-Oh [30, Lemma 2]) and $b^{-s}[\mu] = \mathcal{O}((1+|\mu|)^{-K})$ for any $K > 0$ (Grafakos-Oh [30, Lemma 3]). Replacing $\widehat{J_j^s\varphi}$ and $\widehat{J_j^{-s}\psi}$ with their Fourier series representaitons and using property (A.3.3) of the Fourier transform, we see that

$$\begin{aligned}
&\widetilde{o_3}^+[a,b](\xi,\eta)\\
&\quad = \sum_{j\geq 0}\mathcal{C}[j,a,b]\left[\sum_{m\in\mathbb{Z}^n}\widetilde{c^s}[m,j]e^{\frac{2\pi i}{h}2^{-j}(\xi+\eta)\cdot m}\right]\left[\sum_{\mu\in\mathbb{Z}^n}c^{-s}[\mu,j]e^{\frac{2\pi i}{h}2^{-j}\xi\cdot\mu}\right]\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta)\\
&\quad = \sum_{j\geq 0}\mathcal{C}[j,a,b]\sum_{m,\mu\in\mathbb{Z}^n}\widetilde{c^s}[m,j]c^{-s}[\mu,j]\mathcal{F}[\tau_{a+m+\mu}\Psi](2^{-j}\xi)\mathcal{F}[\tau_{b+m}\Psi](2^{-j}\eta).
\end{aligned}$$

Using this representation for the multiplier $\widetilde{o_3}^+[a,b]$, we have

$$T_{\widetilde{o_3}^+[a,b]}(J^sf,g)(x) = \sum_{j\geq 0}\mathcal{C}[j,a,b]\sum_{m,\mu\in\mathbb{Z}^n}\widetilde{c^s}[m,j]c^{-s}[\mu,j]\left[(\Delta_j^{\tau_{a+m+\mu}\Psi}J^sf)\cdot(\Delta_j^{\tau_{b+m}\Psi}g)\right](x).$$

Taking absolute values, and using the fact that $\mathcal{C}[j,a,b]$ are bounded uniformly in $j \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$, along with the formulas given above for $\widetilde{b^s}[m]$ and $b^{-s}[\mu]$, we see that

$$\left| T_{\widetilde{o_3}^+[a,b]}(J^s f, g)(x) \right| \lesssim \sum_{m,\mu \in \mathbb{Z}^n} \widetilde{b^s}[m] b^{-s}[\mu] \sum_{j \geq 0} \left| \left[(\Delta_j^{\tau_{a+m+\mu}\Psi} J^s f) \cdot (\Delta_j^{\tau_{b+m}\Psi} g) \right](x) \right|.$$

To wrap things up with $\widetilde{o_3}^+[a,b]$, we have

$$\begin{aligned}
&\left\| T_{\widetilde{o_3}^+[a,b]}(f,g) \right\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*} \\
&\quad \leq \sum_{m,\mu \in \mathbb{Z}^n} \widetilde{b^s}[m]^{r^*} b^{-s}[\mu]^{r^*} \left\| \sum_{j \geq 0} \left| [\Delta_j^{\tau_{a+m+\mu}\Psi} J^s f] \cdot [\Delta_j^{\tau_{b+m}\Psi} g] \right| \right\|_{L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})}^{r^*} \\
&\quad \leq \sum_{m,\mu \in \mathbb{Z}^n} \widetilde{b^s}[m]^{r^*} b^{-s}[\mu]^{r^*} \left\| \left(\sum_{j \geq 0} \left| \Delta_j^{\tau_{a+m+\mu}\Psi} J^s f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(v)}^{r^*} \left\| \left(\sum_{j \geq 0} \left| \Delta_j^{\tau_{b+m}\Psi} g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q(w)}^{r^*} \\
&\quad \lesssim \sum_{m,\mu \in \mathbb{Z}^n} \widetilde{b^s}[m]^{r^*} b^{-s}[\mu]^{r^*} \left\| J^s f \right\|_{L^p(v)}^{r^*} \left\| g \right\|_{L^q(w)}^{r^*} \\
&\quad \lesssim \left\| J^s f \right\|_{L^p(v)}^{r^*} \left\| g \right\|_{L^q(w)}^{r^*},
\end{aligned}$$

where in the second inequality we have applied Hölder's inequality twice (first on the ℓ^1-norm in $j \geq 0$, then on the $L^r(v^{\frac{r}{p}} w^{\frac{r}{q}})$-norm), in the third inequality we have used Lemma 3.11, and in the last inequality we have used the decay properties of $\widetilde{b^s}[m]$ and $b^{-s}[\mu]$ (where for $\widetilde{b^s}[m]$ we require the hypothesis that $s > \max\{0, n(\frac{1}{r} - 1)\}$ for convergence, and for $b^{-s}[\mu]$ we use $b^{-s}[\mu] = \mathcal{O}((1+|\mu|)^{-K})$ with K sufficiently large).

We now consider $\widetilde{o_3}^-[a,b]$. For any $j \in \mathbb{Z}$ with $j < 0$, it is easily verified that $\widehat{\varphi}(\xi + \eta) = \widehat{\varphi}(\xi) = 1$ within the support of $\widehat{\Psi}(2^{-j}\xi)\widehat{\Psi}(2^{-j}\eta)$, so

$$\begin{aligned}
\widetilde{o_3}^-[a,b](\xi,\eta) &= \frac{(1+|\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}} \widehat{\varphi}(\xi+\eta)\widehat{\varphi}(\xi) \sum_{j<0} \mathcal{C}[j,a,b] \widehat{\tau_a \Psi}(2^{-j}\xi) \widehat{\tau_b \Psi}(2^{-j}\eta) \\
&= \widehat{J^s \varphi}(\xi+\eta) \widehat{J^{-s}\varphi}(\xi) \sum_{j<0} \mathcal{C}[j,a,b] \widehat{\tau_a \Psi}(2^{-j}\xi) \widehat{\tau_b \Psi}(2^{-j}\eta). \qquad (3.32)
\end{aligned}$$

Since $\widehat{J^s \varphi}$ and $\widehat{J^{-s}\varphi}$ are compactly supported, say within $H = [-\frac{h}{2}, \frac{h}{2}]$ where h is large enough to contain the support of both functions, we may consider the Fourier series expansions of periodic extensions of both functions:

$$\widehat{J^s\varphi}(\omega) = \left[\sum_{m\in\mathbb{Z}^n} c^s[m] e^{\frac{2\pi i}{h}\omega\cdot m}\right]\chi_H(\omega), \qquad \widehat{J^{-s}\varphi}(\omega) = \left[\sum_{\mu\in\mathbb{Z}^n} c^{-s}[\mu] e^{\frac{2\pi i}{h}\omega\cdot \mu}\right]\chi_H(\omega),$$

where

$$c^s[m] = \frac{1}{h^n}\int_{\mathbb{R}^n}(1+|\zeta|^2)^{\frac{s}{2}}\widehat{\varphi}(\zeta)e^{-\frac{2\pi i}{h}\zeta\cdot m}\,d\zeta, \qquad c^{-s}[\mu] = \frac{1}{h^n}\int_{\mathbb{R}^n}(1+|\zeta|^2)^{-\frac{s}{2}}\widehat{\varphi}(\zeta)e^{-\frac{2\pi i}{h}\zeta\cdot \mu}\,d\zeta,$$

and $c^s[m] = \mathcal{O}((1+|m|)^{-K})$, $c^{-s}[\mu] = \mathcal{O}((1+|\mu|)^{-K})$ for any $K>0$ by (A.1) (replace $m,\mu\in\mathbb{Z}^n$ by continuous variables so we regard c^s, c^{-s} as continuous functions defined on $\mathbb{R}^n$; then $c^s, c^{-s}\in\mathcal{S}(\mathbb{R}^n)$ are the inverse Fourier transforms of the smooth and compactly supported functions $(1+|\cdot|^2)^{\frac{s}{2}}\widehat{\varphi}, (1+|\cdot|^2)^{-\frac{s}{2}}\widehat{\varphi}\in\mathcal{S}(\mathbb{R}^n)$). We replace $\widehat{J^s\varphi}$ and $\widehat{J^{-s}\varphi}$ with their Fourier series expansions and use property (A.3.3) of the Fourier transform to obtain

$$\begin{aligned}\widetilde{o_3}^-[a,b](\xi,\eta) &= \left[\sum_{m\in\mathbb{Z}^n} c^s[m]e^{\frac{2\pi i}{h}(\xi+\eta)\cdot m}\right]\left[\sum_{\mu\in\mathbb{Z}^n} c^{-s}[\mu]e^{\frac{2\pi i}{h}\xi\cdot\mu}\right]\sum_{j<0}\mathcal{C}[j,a,b]\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta)\\ &= \sum_{m,\mu\in\mathbb{Z}^n} c^s[m]c^{-s}[\mu]\sum_{j<0}\mathcal{C}[j,a,b]\mathcal{F}[\tau_{a+2^j(m+\mu)}\Psi](2^{-j}\xi)\mathcal{F}[\tau_{b+2^jm}\Psi](2^{-j}\eta).\end{aligned}$$

With this representation for the multiplier $\widetilde{o_3}^-[a,b]$, we have

$$T_{\widetilde{o_3}^-[a,b]}(J^sf,g)(x) = \sum_{m,\mu\in\mathbb{Z}^n} c^s[m]c^{-s}[\mu]\sum_{j<0}\mathcal{C}[j,a,b]\left[(\Delta_j^{\tau_{a+2^j(m+\mu)}\Psi}J^sf)\cdot(\Delta_j^{\tau_{b+2^jm}\Psi}g)\right](x).$$

Therefore, we obtain

$$
\begin{aligned}
&\left\|T_{\widetilde{o_3}^-[a,b]}(f,g)\right\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*} \\
&\quad\leq \sum_{m,\mu\in\mathbb{Z}^n} |c^s[m]|^{r^*}\left|c^{-s}[\mu]\right|^{r^*}\left\|\sum_{j<0}\left|[\Delta_j^{\tau_{a+2^j(m+\mu)}\Psi}J^sf]\cdot[\Delta_j^{\tau_{b+2^jm}\Psi}g]\right|\right\|_{L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})}^{r^*} \\
&\quad\lesssim \sum_{m,\mu\in\mathbb{Z}^n} |c^s[m]|^{r^*}\left|c^{-s}[\mu]\right|^{r^*}\left\|\left(\sum_{j<0}\left|\Delta_j^{\tau_{a+2^j(m+\mu)}\Psi}J^sf\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)}^{r^*}\left\|\left(\sum_{j<0}\left|\Delta_j^{\tau_{b+2^jm}\Psi}g\right|^2\right)^{\frac{1}{2}}\right\|_{L^q(w)}^{r^*} \\
&\quad\lesssim \sum_{m,\mu\in\mathbb{Z}^n} |c^s[m]|^{r^*}\left|c^{-s}[\mu]\right|^{r^*}\|J^sf\|_{L^p(v)}^{r^*}\|g\|_{L^q(w)}^{r^*} \\
&\quad\lesssim \|J^sf\|_{L^p(v)}^{r^*}\|g\|_{L^q(w)}^{r^*}
\end{aligned}
$$

where in the first inequality we have used the fact that the coefficients $\mathcal{C}[j,a,b]$ are bounded uniformly in $j\in\mathbb{Z}$ and $a,b\in\mathbb{Z}^n$, in the second inequality we have applied Hölder's inequality twice (first on the ℓ^1-norm in $j>0$, then on the $L^r(v^{\frac{r}{p}}w^{\frac{r}{q}})$-norm), in the third inequality we have used Lemma 3.11 on each piece, and in the final inequality we have used the decay properties of $c^s[m]$ and $c^{-s}[\mu]$, thus concluding our analysis of $\widetilde{o_3}^-[a,b]$ and completing case 1.

Case 2: $q=\infty$, $p=r$, $1<p<\infty$. We begin by considering $\widetilde{o_3}^+[a,b]$, in which case our goal is to show that, for $v\in A_p(\mathbb{R}^n)$,

$$
\left\|T_{\widetilde{o_3}^+[a,b]}(J^sf,g)\right\|_{L^p(v)}\lesssim P^+(|a|,|b|)\,\|J^sf\|_{L^p(v)}\,\|g\|_{L^\infty},
$$

where P^+ is a polynomial function. An application of Lemma 3.10 gives

$$
\left\|T_{\widetilde{o_3}^+[a,b]}(J^sf,g)\right\|_{L^p(v)}\lesssim\left\|\left(\sum_{k\in\mathbb{Z}}\left|\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]}(J^sf,g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)}. \tag{3.33}
$$

Fix $k \in \mathbb{Z}$, then we see that $\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]}(J^s f, g)$ corresponds with $T_{\widetilde{o_3}^+[k,a,b]}(J^s f, g)$, where

$$\widetilde{o_3}^+[k,a,b](\xi,\eta) := \frac{(1+|\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}}\widehat{\Psi}(2^{-k}(\xi+\eta))\sum_{j\geq 0}\mathcal{C}[j,a,b]\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta).$$

Due to the support of $\widehat{\Psi}$, it is easily verified that we may adjust the summation in $\widetilde{o_3}^+[k,a,b]$ to run over $j \geq \max\{0, k-3\}$.

We now analyze $\widetilde{o_3}^+[k,a,b]$, first for the case $k \geq 3$, so that $\max\{0,k-3\} = k-3$, and

$$\begin{aligned}
&\widetilde{o_3}^+[k,a,b](\xi,\eta) \\
&\quad = \frac{(1+|\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}}\widehat{\Psi}(2^{-k}(\xi+\eta))\sum_{j\geq k-3}\mathcal{C}[j,a,b]\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta) \\
&\quad = \left[2^{ks}\left|(2^{-2k}+\left|2^{-k}(\xi+\eta)\right|^2\right|^{\frac{s}{2}}\widehat{\Psi}(2^{-k}(\xi+\eta))\right] \\
&\qquad \times \sum_{j\geq k-3}\mathcal{C}[j,a,b]\left[2^{-js}(2^{-2j}+\left|2^{-j}\xi\right|^2)^{-\frac{s}{2}}\widehat{\tau_a\Psi}(2^{-j}\xi)\right]\widehat{\tau_b\Psi}(2^{-j}\eta) \\
&\quad = 2^{ks}\widehat{J_k^s\Psi}(2^{-k}(\xi+\eta))\sum_{j\geq k-3}2^{-js}\mathcal{C}[j,a,b]\mathcal{F}[J_j^{-s}\tau_a\Psi](2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta).
\end{aligned}$$

With this representation for the multiplier $\widetilde{o_3}^+[k,a,b]$, and since $\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]} = T_{\widetilde{o_3}^+[k,a,b]}$, we then obtain

$$\left|\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]}(J^s f, g)(x)\right| \leq 2^{ks}\sum_{j\geq k-3}2^{-js}\left|\mathcal{C}[j,a,b]\right|\left|\Delta_k^{J_k^s\Psi}\left[(\Delta_j^{J_j^{-s}\tau_a\Psi}J^s f)\cdot(\Delta_j^{\tau_b\Psi}g)\right](x)\right|.$$

Through an application of Hölder's inequality on ℓ^1-norm in $j \geq k-3$ and by recalling that $\mathcal{C}[j,a,b]$ is bounded uniformly in $j \in \mathbb{Z}$ and $a,b \in \mathbb{Z}$, we see that

$$\left|\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]}(J^s f, g)(x)\right| \lesssim 2^{ks}\left(\sum_{j\geq k-3}2^{-2js}\right)^{\frac{1}{2}}\left(\sum_{j\geq k-3}\left|\Delta_k^{J_k^s\Psi}\left[(\Delta_j^{J_j^{-s}\tau_a\Psi}J^s f)\cdot(\Delta_j^{\tau_b\Psi}g)\right](x)\right|^2\right)^{\frac{1}{2}}.$$

Further, we have $\sum_{j\geq k-3} 2^{-2js} \sim 2^{-2ks}$, so that

$$\left|\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]}(J^s f, g)(x)\right| \lesssim \left(\sum_{j\geq k-3} \left|\Delta_k^{J_k^s\Psi}\left[(\Delta_j^{J_j^{-s}\tau_a\Psi} J^s f)\cdot(\Delta_j^{\tau_b\Psi} g)\right](x)\right|^2\right)^{\frac{1}{2}}, \quad \forall k\geq 3.$$

Next, we analyze $\widetilde{o_3}^+[k,a,b]$ for $k<3$, in which case $\max\{0,k-3\}=0$. For $k<3$, it is easily verified that $\widehat{\varphi}$ is identically 1 within the support of $\widehat{\Psi}(2^{-k}\cdot)$, so

$$\begin{aligned}
&\widetilde{o_3}^+[k,a,b](\xi,\eta)\\
&\quad = \frac{(1+|\xi+\eta|^2)^{\frac{s}{2}}}{(1+|\xi|^2)^{\frac{s}{2}}}\widehat{\varphi}(\xi+\eta)\widehat{\Psi}(2^{-k}(\xi+\eta))\sum_{j\geq 0}\mathcal{C}[j,a,b]\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta)\\
&\quad = \left[(1+|\xi+\eta|^2)^{\frac{s}{2}}\widehat{\varphi}(\xi+\eta)\right]\widehat{\Psi}(2^{-k}(\xi+\eta))\\
&\qquad\quad \times\sum_{j\geq 0}\mathcal{C}[j,a,b]\left[2^{-js}(2^{-2j}+\left|2^{-j}\xi\right|^2)^{-\frac{s}{2}}\widehat{\tau_a\Psi}(2^{-j}\xi)\right]\widehat{\tau_b\Psi}(2^{-j}\eta)\\
&\quad = \widehat{J^s\varphi}(\xi+\eta)\widehat{\Psi}(2^{-k}(\xi+\eta))\sum_{j\geq 0}2^{-js}\mathcal{C}[j,a,b]\mathcal{F}[J_j^{-s}\tau_a\Psi](2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta).
\end{aligned}$$

Since $\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]} = T_{\widetilde{o_3}^+[k,a,b]}$, we have

$$\left|\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]}(J^s f, g)(x)\right| \leq \sum_{j\geq 0} 2^{-js}\left|\mathcal{C}[j,a,b]\right|\left|S_0^{J^s\varphi}\Delta_k^\Psi\left[(\Delta_j^{J_j^{-s}\tau_a\Psi} J^s f)\cdot(\Delta_j^{\tau_b\Psi} g)\right](x)\right|.$$

By applying Hölder's inequality on the ℓ^1-norm in $j\geq 0$, using the uniform bound on $\mathcal{C}[j,a,b]$ in $j\in\mathbb{Z}$ and $a,b\in\mathbb{Z}^n$, and using that $\sum_{j\geq 0} 2^{-2js}\sim 1$, we obtain

$$\left|\Delta_k^\Psi T_{\widetilde{o_3}^+[a,b]}(J^s f, g)(x)\right| \lesssim \left(\sum_{j\geq 0}\left|S_0^{J^s\varphi}\Delta_k^\Psi\left[(\Delta_j^{J_j^{-s}\tau_a\Psi} J^s f)\cdot(\Delta_j^{\tau_b\Psi} g)\right](x)\right|^2\right)^{\frac{1}{2}}, \quad \forall k<3.$$

We now combine results from the $k\geq 3$ case and the $k<3$ case to finish our analysis of

$\widetilde{o_3}^+[a,b]$. Returning our attention to (3.33), we have

$$\begin{aligned}
&\left\| T_{\widetilde{o_3}^+[a,b]}(J^s f, g) \right\|_{L^p(v)} \\
&\quad \lesssim \left\| \left(\sum_{k\geq 3} \left| \Delta_k^{\Psi} T_{\widetilde{o_3}^+[a,b]}(J^s f, g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(v)} + \left\| \left(\sum_{k\leq 2} \left| \Delta_k^{\Psi} T_{\widetilde{o_3}^+[a,b]}(J^s f, g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(v)} \\
&\quad \lesssim \left\| \left(\sum_{k\geq 3} \sum_{j\geq k-3} \left| \Delta_k^{J_k^s \Psi} \left[(\Delta_j^{J_j^{-s}\tau_a \Psi} J^s f) \cdot (\Delta_j^{\tau_b \Psi} g) \right] \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(v)} \\
&\qquad + \left\| \left(\sum_{k\leq 2} \sum_{j\geq 0} \left| S_0^{J^s \varphi} \Delta_k^{\Psi} \left[(\Delta_j^{J_j^{-s}\tau_a \Psi} J^s f) \cdot (\Delta_j^{\tau_b \Psi} g) \right] \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(v)},
\end{aligned}$$

where in the first inequality, we have used the triangle inequality twice, first on the ℓ^2-norm, then on the $L^p(v)$-norm. To handle the two summands following the final inequality above, we will use a few lemmas introduced in Subsection 3.2.2. We begin with an applicaiton of Lemma 3.15 (with $u = 0$ and $r = 2$) to see that the first summand is bounded by some constant (independent of $a, b \in \mathbb{Z}^n$) multiplied by

$$\left\| \left(\sum_{j\geq 0} \left| (\Delta_j^{J_j^{-s}\tau_a \Psi} J^s f) \cdot (\Delta_j^{\tau_b \Psi} g) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(v)}. \tag{3.34}$$

For the second summand, we apply Lemma 3.12 with $\{T_k\}_{k\in\mathbb{Z}} = \{S_0^{J^s\varphi}\Delta_k^{\Psi}\}_{k\in\mathbb{Z}}$, $\{f_j\}_{j\in\mathbb{Z}} = \{(\Delta_j^{J_j^{-s}\tau_a \Psi} J^s f) \cdot (\Delta_j^{\tau_b \Psi} g)\}_{j\in\mathbb{N}_0}$, and $r = 2$ to obtain that the summand is also bounded by some constant (independent of $a, b \in \mathbb{Z}^n$) multiplied by (3.34). To apply Lemma 3.12, we must verify that $\{S_0^{J^s\varphi}\Delta_k^{\Psi}\}_{k\in\mathbb{Z}}$ satisfies (3.14) with $r = 2$. For $\Theta \in L^2(\omega)$ and $\omega \in A_2(\mathbb{R}^n)$, we see that

$$\left\| \left(\sum_{k\in\mathbb{Z}} \left| S_0^{J^s\varphi} \Delta_k^{\Psi} \Theta \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\omega)} = \left\| \left(\sum_{k\in\mathbb{Z}} \left| \Delta_k^{\Psi} (S_0^{J^s\varphi} \Theta) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\omega)} \lesssim \left\| S_0^{J^s\varphi} \Theta \right\|_{L^2(\omega)}$$

by Lemma 3.9. Finally, $S_0^{J^s\varphi}$ is a bounded operator on $L^2(\omega)$ (verified as follows), so that

$\{S_0^{J^s\varphi}\Delta_k^\Psi\}_{k\in\mathbb{Z}}$ satisfies (3.14) as desired. Indeed, we actually have that $S_0^{J^s\varphi}$ is a bounded operator on $L^\rho(\nu)$ for any $s>0$, $1<\rho\leq\infty$, and $\nu\in A_\rho(\mathbb{R}^n)$, since $|(J^s\varphi)*\Theta(x)|\lesssim|\mathcal{M}(\Theta)(x)|$ by (3.4) and $\mathcal{M}$ is a bounded operator on $L^\rho(\nu)$.

Having bounded both summands by a term of the form (3.34), we continue from the string on inequalities above to conclude the $\widetilde{o_3}^+[a,b]$ case:

$$\left\|T_{\widetilde{o_3}^+[a,b]}(f,g)\right\|_{L^p(v)}\lesssim\left\|\left(\sum_{j\geq 0}\left|(\Delta_j^{J_j^{-s}\tau_a\Psi}J^sf)\cdot(\Delta_j^{\tau_b\Psi}g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)}$$
$$\leq\sup_{j\geq 0}\left\{\left\|\Delta_j^{\tau_b\Psi}g\right\|_{L^\infty}\right\}\left\|\left(\sum_{j\geq 0}\left|\Delta_j^{J_j^{-s}\tau_a\Psi}J^sf\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(v)},$$

where the supremum is bounded by $\|g\|_{L^\infty}$ due to (3.31) and the $L^p(v)$-norm piece is bounded by $\|J^sf\|_{L^p(v)}$ due to Lemma 3.14 (and all implicit constants are independent of $a,b\in\mathbb{Z}^n$).

We now move on to the consider the $\widetilde{o_3}^-[a,b]$ case, for which we must verify that, for $v\in A_p(\mathbb{R}^n)$,

$$\left\|T_{\widetilde{o_3}^-[a,b]}(J^sf,g)\right\|_{L^p(v)}\lesssim P^-(|a|,|b|)\,\|J^sf\|_{L^p(v)}\,\|g\|_{L^\infty},$$

where P^- is a polynomial function. Using (3.32), we obtain

$$T_{\widetilde{o_3}^-[a,b]}(J^sf,g)(x)=S_0^{J^s\varphi}\left[\sum_{j<0}\mathcal{C}[j,a,b][(\Delta_j^{\tau_a\Psi}S_0^{J^{-s}\varphi}J^sf)\cdot(\Delta_j^{\tau_b\Psi}g)]\right](x).$$

We note that, by the reasoning following (3.34), both $S_0^{J^s\varphi}$ and $S_0^{J^{-s}\varphi}$ are bounded operators on $L^p(v)$. Further, the symbol $\sum_{j<0}\mathcal{C}[j,a,b]\widehat{\tau_a\Psi}(2^{-j}\xi)\widehat{\tau_b\Psi}(2^{-j}\eta)$ satisfies (3.1) for $\alpha,\beta\in\mathbb{N}_0^n$ such that $|\alpha+\beta|\leq 2n+1$ with constants $C_{a,b}\leq(1+|a|)^{2n+1}(1+|b|)^{2n+1}C'$, which may be seen by considering property (A.3.3) of the Fourier transform and techniques similar to those used in the proof of Lemma B.1. Therefore,

$$\begin{aligned}\left\|T_{\widetilde{o_3}^-[a,b]}(J^s f, g)\right\|_{L^p(v)} &= \left\| S_0^{J^s\varphi}\left[\sum_{j<0}\mathcal{C}[j,a,b][(\Delta_j^{\tau_a\Psi} S_0^{J^{-s}\varphi} J^s f)\cdot(\Delta_j^{\tau_b\Psi} g)]\right]\right\|_{L^p(v)} \\ &\lesssim \left\|\sum_{j<0}\mathcal{C}[j,a,b][(\Delta_j^{\tau_a\Psi} S_0^{J^{-s}\varphi} J^s f)\cdot(\Delta_j^{\tau_b\Psi} g)]\right\|_{L^p(v)} \\ &\lesssim (1+|a|)^{2n+1}(1+|b|)^{2n+1}\left\|S_0^{J^{-s}\varphi} J^s f\right\|_{L^p(v)}\|g\|_{L^\infty} \\ &\lesssim (1+|a|)^{2n+1}(1+|b|)^{2n+1}\|J^s f\|_{L^p(v)}\|g\|_{L^\infty},\end{aligned}$$

where in the second line we have used the boundedness of $S_0^{J^s\varphi}$ on $L^p(v)$, in the third line we have used the uniform boundedness of $\mathcal{C}[j,a,b]$ and applied Theorem 3.16, and in the last line we have used the boundedness of $S_0^{J^{-s}\varphi}$ on $L^p(v)$. This then concludes our analysis of $\widetilde{o_3}^-[a,b]$ and subsequently case 2.

Case 3: $p=\infty$**,** $q=r$**,** $1<q<\infty$**.** In this final case, we must show that, for $w\in A_q(\mathbb{R}^n)$,

$$\|J^s\Pi_3(f,g)\|_{L^q(w)} \lesssim \|J^s f\|_{L^\infty}\|g\|_{L^q(w)},$$

In the $\widetilde{o_3}^+[a,b]$ case, we follow the exact calculations from case 2 until the final set of inequalities, replacing $L^p(v)$-norms with $L^q(w)$-norms. A slight alteration to the last chain of inequalities yields

$$\begin{aligned}\left\|T_{\widetilde{o_3}^+[a,b]}(f,g)\right\|_{L^q(w)} &\lesssim \left\|\left(\sum_{j\geq 0}\left|(\Delta_j^{J_j^{-s}\tau_a\Psi} J^s f)\cdot(\Delta_j^{\tau_b\Psi} g)\right|^2\right)^{\frac{1}{2}}\right\|_{L^q(w)} \\ &\leq \sup_{j\geq 0}\left\{\left\|\Delta_j^{J_j^{-s}\tau_a\Psi} J^s f\right\|_{L^\infty}\right\}\left\|\left(\sum_{j\geq 0}\left|\Delta_j^{\tau_b\Psi} g\right|^2\right)^{\frac{1}{2}}\right\|_{L^q(w)},\end{aligned}$$

where the $L^q(w)$-norm piece is bounded by $\|g\|_{L^q(w)}$ due to Lemma 3.14 and the supremum is bounded by $\|J^s f\|_{L^\infty}$ (justified as follows), and all implicit constants are independent of

$a, b \in \mathbb{Z}^n$. By Young's inequality, we have

$$\left\| \Delta_j^{J_j^{-s}\tau_a\Psi} J^s f \right\|_{L^\infty} \leq \left\| 2^{jn} J_j^{-s}\tau_a\Psi(2^j\cdot) \right\|_{L^1} \|J^s f\|_{L^\infty} = \left\| J_j^{-s}\Psi \right\|_{L^1} \|J^s f\|_{L^\infty},$$

so that

$$\begin{aligned}
\sup_{j\geq 0} \left\{ \left\| \Delta_j^{J_j^{-s}\tau_a\Psi} J^s f \right\|_{L^\infty} \right\} &\leq \sup_{j\geq 0} \left\{ \left\| J_j^{-s}\Psi \right\|_{L^1} \right\} \|J^s f\|_{L^\infty} \\
&\leq \left(\int_{\mathbb{R}^n} \sup_{j\geq 0} \left| J_j^{-s}\Psi(x) \right| \, dx \right) \|J^s f\|_{L^\infty} \\
&\leq \left(\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^K} \right) \|J^s f\|_{L^\infty}
\end{aligned}$$

for any $K > 0$ (see Grafakos-Oh [30, Lemma 3]). Taking K large enough ensures that the integral in the final line is finite.

All that remains is to consider the $\widetilde{o_3}^-[a,b]$ case. Again, we follow the calculations from case 2 until the final set of inequalities, then conclude with

$$\begin{aligned}
\left\| T_{\widetilde{o_3}^-[a,b]}(J^s f, g) \right\|_{L^q(w)} &= \left\| S_0^{J^s\varphi} \left[\sum_{j<0} \mathcal{C}[j,a,b][(\Delta_j^{\tau_a\Psi} S_0^{J^{-s}\varphi} J^s f) \cdot (\Delta_j^{\tau_b\Psi} g)] \right] \right\|_{L^q(w)} \\
&\lesssim \left\| \sum_{j<0} \mathcal{C}[j,a,b][(\Delta_j^{\tau_a\Psi} S_0^{J^{-s}\varphi} J^s f) \cdot (\Delta_j^{\tau_b\Psi} g)] \right\|_{L^q(w)} \\
&\lesssim (1+|a|)^{2n+1}(1+|b|)^{2n+1} \left\| S_0^{J^{-s}\varphi} J^s f \right\|_{L^\infty} \|g\|_{L^q(w)} \\
&\lesssim (1+|a|)^{2n+1}(1+|b|)^{2n+1} \|J^s f\|_{L^\infty} \|g\|_{L^q(w)},
\end{aligned}$$

where in the second and last lines we have used the boundedness of $S_0^{J^s\varphi}$ and $S_0^{J^{-s}\varphi}$ on $L^q(w)$ and $L^\infty(\mathbb{R}^n)$, respectively, and in the third line we have applied Theorem 3.16 (with the same justification as given in case 2 at that step). This wraps up our analysis on $\widetilde{o_3}^-[a,b]$, thus concluding case 3 and completing all inhomogeneous estimates.

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Appendix A

A Glossary for Notation

A.1 Frequently used notation

For the following quick definitions, let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$:

$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of positive integers
$\mathbb{N}_0$	set of non-negative integers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\lvert x\rvert$	$\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
$\lvert\alpha\rvert$	$\alpha_1 + \alpha_2 + \cdots + \alpha_n$
x^α	$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$
$\partial_{x_j}^k f(x)$	the k-th partial derivative of $f(x_1, x_2, \ldots, x_n)$ with respect to x_j
$\partial^\alpha f(x)$	$\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n} f(x)$
$k!$	$k(k-1)\ldots 2 \cdot 1$ for $k \in \mathbb{N}$ (and $0! = 1$)
$\alpha!$	$\alpha_1! \cdot \alpha_2! \ldots \alpha_n!$
$\binom{\alpha}{\beta}$	$\frac{\alpha!}{\beta!(\alpha-\beta)!}$ for multi-indices $\alpha, \beta \in \mathbb{N}_0^n$
$B(x,r)$	the Euclidean ball of radius $r > 0$ centered at x

S^n	the n-dimensional unit sphere; the boundary of $B(0,1) \subset \mathbb{R}^{n+1}$
$\langle f, g\rangle$	$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$
$\tau_u f(x)$	$f(x+u)$ for any $u \in \mathbb{R}^n$
$a \lesssim b$	$a \leq Cb$, where $C > 0$ may depend on some parameters, but not on functions or parameters being tracked
$a \sim b$	$a \lesssim b$ and $b \lesssim a$

A.2 Function spaces

For more detailed background and results relating to the following function spaces, see for example Grafakos [27, Chapters 1-3]:

Space of smooth functions. The space $\mathcal{C}^\infty(\mathbb{R}^n)$ denotes the set of complex-valued infinitely differentiable functions defined on $\mathbb{R}^n$.

Lebesgue spaces. For $0 < p < \infty$, we define the $L^p(\mathbb{R}^n)$-norm of a function f (which is actually a quasi-norm if $p < 1$) via

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{\frac{1}{p}},$$

where dx denotes the standard n-dimensional Lebesgue measure, and we define the $L^\infty(\mathbb{R}^n)$-norm via

$$\|f\|_{L^\infty} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| = \inf\{K > 0 : d_K(f) = 0\}$$

where $d_K(f)$ denotes the standard Lebesgue measure of $\{x \in \mathbb{R}^n : |f(x)| > K\}$. We then denote $L^p(\mathbb{R}^n)$ to be the space of all functions with finite $L^p(\mathbb{R}^n)$-norm, where we consider two functions equal if they differ on a set of Lebesgue measure zero. By virtue of being a norm, it is well known that the triangle inequality holds for $1 \leq p \leq \infty$, implying for $f, g \in L^p(\mathbb{R}^n)$ that $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.

By replacing $\mathbb{R}^n$ with $\mathbb{Z}$ and the standard Lebesgue measure with the counting measure in the definition for $L^p(\mathbb{R}^n)$ above, we obtain the Lebesgue sequence space ℓ^p. Specifically, this space consists of all sequences $\{x_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ with finite ℓ^p-norm, defined for $0 < p < \infty$ via

$$\|\{x_n\}_{n\in\mathbb{Z}}\|_{\ell^p} \coloneqq \left(\sum_{n\in\mathbb{Z}} |x_n|^p\right)^{\frac{1}{p}}$$

and for $p = \infty$ via

$$\|\{x_n\}_{n\in\mathbb{Z}}\|_{\ell^\infty} \coloneqq \sup_{n\in\mathbb{Z}} |x_n|\,.$$

Schwartz class. The space $\mathcal{S}(\mathbb{R}^n)$ denotes the set of functions in $\mathcal{C}^\infty(\mathbb{R}^n)$ which decrease rapidly at infinity. Specifically, $f \in \mathcal{S}(\mathbb{R}^n)$ if for every $\alpha, \beta \in \mathbb{N}_0^n$,

$$\rho_{\alpha,\beta}(f) \coloneqq \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta f(x)\right| < \infty.$$

The set $\{\rho_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}_0^n}$ form a collection of seminorms on $\mathcal{S}(\mathbb{R}^n)$ through which we may define a notion of convergence within $\mathcal{S}(\mathbb{R}^n)$. For $\{f_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$, we say that the sequence f_j converges to $f \in \mathcal{S}(\mathbb{R}^n)$ if $\rho_{\alpha,\beta}(f_j - f) \xrightarrow[j\to\infty]{} 0$ for every $\alpha, \beta \in \mathbb{N}_0^n$. An equivalent characterization of $\mathcal{S}(\mathbb{R}^n)$ may be given as follows: $f \in \mathcal{S}(\mathbb{R}^n)$ if for every $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, there exists some constant $\widetilde{C}_{\alpha,N} > 0$ such that for all $x \in \mathbb{R}^n$,

$$|\partial^\alpha f(x)| \leq \frac{\widetilde{C}_{\alpha,N}}{(1+|x|)^N}. \tag{A.1}$$

Thus, the Schwartz class is the collection of all smooth functions which decay faster than the reciprocal of any polynomial at infinty. We give a few examples of Schwartz class functions below.

- Any smooth function with compact support lies in $\mathcal{S}(\mathbb{R}^n)$, since such functions vanish at infinity.
- The function $e^{-|x|^2}$ is in $\mathcal{S}(\mathbb{R}^n)$, despite not being compactly supported.
- Suppose $f \in \mathcal{S}(\mathbb{R}^n)$. It is easily verified that $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$ for any multi-index $\alpha \in \mathbb{N}_0^n$ and that $\mathcal{P}f \in \mathcal{S}(\mathbb{R}^n)$ for any polynomial function $\mathcal{P}$ defined on $\mathbb{R}^n$. Further, by the Leibniz rule for differentiation, $fg \in \mathcal{S}(\mathbb{R}^{n+m})$ for any $g \in \mathcal{S}(\mathbb{R}^m)$.

Finally, we verify a useful property about Schwartz class functions. Given $f \in \mathcal{S}(\mathbb{R}^n)$, $u \in \mathbb{R}^n$, and $N \in \mathbb{N}_0$, we show that

$$|\tau_u f(x)| \leq \widetilde{C}_{0,N} \left(\frac{1+|u|}{1+|x|} \right)^N . \tag{A.2}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$, we see that

$$|\tau_u f(x)| = |f(x+u)| \leq \frac{\widetilde{C}_{0,N}}{(1+|x+u|)^N}.$$

We need only show that $\frac{1+|u|}{1+|x|}$ is a majorant of $\frac{1}{1+|x+u|}$. Indeed, by the triangle inequality, we see that $1+|x| \leq 1+|x+u|+|u| \leq (1+|u|)(1+|x+u|)$ so that

$$\frac{1}{1+|x+u|} = \frac{1+|u|}{(1+|u|)(1+|x+u|)} \leq \frac{1+|u|}{1+|x|}.$$

Space of tempered distributions. The space $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual of $\mathcal{S}(\mathbb{R}^n)$, that is, the space of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. For $T \in \mathcal{S}'(\mathbb{R}^n)$, we denote its action on a given test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ via $T(\varphi)$. Convergence in $\mathcal{S}'(\mathbb{R}^n)$ is defined as expected for the dual of a function space: for $\{T_j\}_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$, $T_j \xrightarrow[j\to\infty]{} T$ if $T \in \mathcal{S}'(\mathbb{R}^n)$ and $T_j(\varphi) \xrightarrow[j\to\infty]{} T(\varphi)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We give a few examples of distributions below, defined for $\varphi \in \mathcal{S}(\mathbb{R}^n)$:

- The Dirac mass at a point $x_0 \in \mathbb{R}^n$ is denoted $\delta_{x_0} \in \mathcal{S}'(\mathbb{R}^n)$ and defined as $\delta_{x_0}(\varphi) := \varphi(x_0)$.
- Any $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, can be identified with a tempered distribution $T_f \in \mathcal{S}'(\mathbb{R}^n)$ by defining $T_f(\varphi) := \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$.
- Any finite Borel measure μ can be identified with a tempered distribution $T_\mu \in \mathcal{S}'(\mathbb{R}^n)$ by defining $T_\mu(\varphi) := \int_{\mathbb{R}^n} \varphi(x)\,d\mu(x)$.

We make the following definitions associated to a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ acting on a test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$:

- Define a notion of derivatives for tempered distributions via $[\partial^\alpha T](\varphi) := (-1)^{|\alpha|} T(\partial^\alpha \varphi)$ for any multi-index $\alpha \in \mathbb{N}_0^n$.
- Define a notion of the Fourier transform for tempered distributions via $\widehat{T}(\varphi) := T(\widehat{\varphi})$ and $[\mathcal{F}^{-1}T](\varphi) := T(\mathcal{F}^{-1}\varphi)$.

Sobolev spaces. For $0 < p < \infty$ and $s \in \mathbb{R}$, the spaces $\dot{W}^{s,p}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$ denote the homogeneous and inhomogeneous Sobolev spaces, respectively, where the corresponding function space norms (or quasi-norms, if $p < 1$) are defined via

$$\|f\|_{\dot{W}^{s,p}} := \|D^s f\|_{L^p} \quad \text{and} \quad \|f\|_{W^{s,p}} := \|J^s f\|_{L^p} .$$

Through certain parallels involving the Sobolev function space norms, we may think of the operators D^s and J^s as taking s derivatives of a function. Specifically, for $k \in \mathbb{N}$, it is well-known that $f \in \dot{W}^{k,p}(\mathbb{R}^n)$ if and only if $\sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} < \infty$ (so that all k-th order derivatives are in $L^p(\mathbb{R}^n)$), while $f \in W^{k,p}(\mathbb{R}^n)$ if and only if $\sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p} < \infty$ (so that all derivatives up to order k are in $L^p(\mathbb{R}^n)$).

Hardy spaces. Let $0 < p \leq \infty$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)\, dx \neq 0$. The space $H^p(\mathbb{R}^n)$ denotes the Hardy space, having function space norm given by

$$\|f\|_{H^p} := \left\| \sup_{t>0} |\varphi_t * f| \right\|_{L^p} ,$$

where $\varphi_t(x) := t^{-n}\varphi(x/t)$. We note that different choices of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ yield equivalent norms and that $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$.

Space of bounded mean oscillation. The space $BMO(\mathbb{R}^n)$ denotes the space of all locally integrable functions for which the supremum of their mean oscillations over cubes in $\mathbb{R}^n$ is finite, measured via

$$\|f\|_{BMO} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx < \infty,$$

where $|Q|$ denotes the volume of Q and f_Q is the average of f on the cube Q, that is

$f_Q := \frac{1}{|Q|}\int_Q |f(x)|\, dx$. We also note that $BMO(\mathbb{R}^n)$ is the dual vector space of the Hardy space $H^1(\mathbb{R}^n)$.

A.3 Fourier analysis background

Fourier coefficients and series. Suppose that $f \in \mathrm{L}^1(\mathbb{R})$ is periodic of period h, that is to say $f(x) = f(x+h)$ for any $x \in \mathbb{R}$. Define the Fourier series expansion of f by $\sum_{a\in\mathbb{Z}} c[a]e^{\frac{2\pi i}{h}ax}$, where $\{c[a]\}_{a\in\mathbb{Z}}$ are the corresponding Fourier series coefficients, defined by

$$c[a] = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} f(y)e^{-\frac{2\pi i}{h}ay}\, dy.$$

Now suppose that f is a continuous compactly-supported function defined on $\mathbb{R}^n$, say with compact support inside $H = [-\frac{h}{2}, \frac{h}{2}]^n$ for some $h > 0$ large enough. By extending f periodically and iterating the formula for the Fourier series expansion of a function defined on $\mathbb{R}$ given above, we obtain the following pointwise equality:

$$f(x) = \left(\sum_{a\in\mathbb{Z}^n} c[a]e^{\frac{2\pi i}{h}a\cdot x}\right)\chi_H(x),$$

where

$$c[a] = \frac{1}{h^n}\int_H f(y)e^{-\frac{2\pi i}{h}a\cdot y}\, dy.$$

Fourier transform on $\mathcal{S}(\mathbb{R}^n)$. For a function $f \in \mathcal{S}(\mathbb{R}^n)$, define its Fourier transform by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x}\, dx.$$

The definition of the Fourier transform is extended to $\mathcal{S}'(\mathbb{R}^n)$ by duality (see Section A.2). Note that $\mathcal{F}$ is a bijection on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ and an isometry on $L^2(\mathbb{R}^n)$. Moreover,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi, \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Many other convenient properties hold for the Fourier transform as well, listed below for $f, g \in \mathcal{S}(\mathbb{R}^n)$:

(A.3.1) $\mathcal{F}[af + bg] = a\widehat{f} + b\widehat{g}$ for constants $a, b \in \mathbb{C}$

(A.3.2) $\mathcal{F}[f * g] = \widehat{f} \cdot \widehat{g}$

(A.3.3) $\widehat{\tau_u f}(\xi) = e^{2\pi i u \cdot \xi}\widehat{f}(\xi)$ for any $u \in \mathbb{R}^n$

(A.3.4) $\mathcal{F}[f(\frac{1}{t}\cdot)](\xi) = t^n \widehat{f}(t\xi)$ for any $t > 0$

(A.3.5) $\frac{\partial \widehat{f}}{\partial \xi_j}(\xi) = -2\pi i \mathcal{F}[x_j f(x)](\xi)$ for $1 \leq j \leq n$, which implies $\partial^\alpha \widehat{f}(\xi) = (-2\pi i)^{|\alpha|}\mathcal{F}[(\cdot)^\alpha f](\xi)$ for any $\alpha \in \mathbb{N}_0^n$

(A.3.6) $\mathcal{F}[\frac{\partial f}{\partial x_j}](\xi) = 2\pi i \xi_j \widehat{f}(\xi)$ for $1 \leq j \leq n$, which implies $\widehat{\partial^\alpha f}(\xi) = (2\pi i)^{|\alpha|}\xi^\alpha \widehat{f}(\xi)$

For further Fourier analysis background information, including proofs of many of the statements above, see for example Duoandikoetxea [20, Chapter 1] or Grafakos [26, Chapters 2 and 3].

A.4 Auxiliary functions and associated operators

While determining paraproduct decompositions in various situations throughout the manuscript, certain operators will frequently appear, which we define as follows. Given $\Psi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform is supported in an annulus, we define the family of operators $\{\Delta_j^\Psi\}_{j\in\mathbb{Z}}$ by

$$\widehat{\Delta_j^\Psi f}(\xi) := \widehat{\Psi}(2^{-j}\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, \tag{A.3}$$

where $f \in \mathcal{S}'(\mathbb{R}^n)$. Similarly, for $\Phi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform does not vanish at the origin and is supported in a ball centered at the origin, we define the family of operators $\{S_j^\Phi\}_{j\in\mathbb{Z}}$ by

$$\widehat{S_j^\Phi f}(\xi) := \widehat{\Phi}(2^{-j}\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n. \tag{A.4}$$

The operator Δ_j^Ψ acts to effectively zero out all but a specific band of frequencies of order 2^j of the function to which it is applied, while S_j^Φ isolates all frequencies of order below 2^j.

Specific instances of auxiliary functions $\Psi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying the support conditions mentioned above will also be useful throughout proofs in the manuscript. Fix a real-valued $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}(\widehat{\Psi}) \subseteq \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\}, \tag{A.5}$$

$$\sum_{j\in\mathbb{Z}} \widehat{\Psi}(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{A.6}$$

Having fixed such a Ψ, we define $\Phi \in \mathcal{S}(\mathbb{R}^n)$ via

$$\widehat{\Phi}(\xi) := \begin{cases} 1, & \xi = 0, \\ \sum_{j<-2} \widehat{\Psi}(2^{-j}\xi), & \xi \in \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{A.7}$$

Obtaining Ψ satisfying (A.5) and (A.6) can be done in a straightforward manner. For example, take any real-valued $\varphi \in \mathcal{S}(\mathbb{R}^n)$ which is non-negative, radial, decreasing, and for which $\varphi|_{B(0,\frac{1}{2})} \equiv 1$ and $\operatorname{supp}(\varphi) \subseteq B(0,1)$. By defining

$$\widehat{\Psi}(\xi) := \varphi(\xi/2) - \varphi(\xi),$$

it is easily verified that Ψ satisfies (A.5) and (A.6). Further, it may be verified that Φ, as defined in (A.7), has a Fourier transform which does not vanish at the origin and is supported in a ball centered at the origin:

$$\widehat{\Phi}\Big|_{B(0,\frac{1}{16})} \equiv 1, \qquad \operatorname{supp}(\widehat{\Phi}) \subset \{\xi \in \mathbb{R}^n : |\xi| < \tfrac{1}{4}\}. \tag{A.8}$$

We will also require functions $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$ relating to Ψ and Φ as defined above satisfying

$$\widehat{\psi}\Big|_{\operatorname{supp}(\widehat{\Psi})} \equiv 1, \qquad \widehat{\phi}\Big|_{\operatorname{supp}(\widehat{\Phi})} \equiv 1, \tag{A.9}$$

and satisfying

$$\operatorname{supp}(\widehat{\psi}) \subseteq \{\xi \in \mathbb{R}^n : \tfrac{1}{2} < |\xi| < 2\}, \quad \operatorname{supp}(\widehat{\phi}) \subseteq \{\xi \in \mathbb{R}^n : |\xi| < \tfrac{1}{4}\}. \tag{A.10}$$

A.5 Pseudodifferential operator background

We give a few specific examples of symbols and the bilinear pseudodifferential operators with which they are associated, all of which may be verified using properties in Section A.3:

- If σ is independent of both ξ and η, that is, $\sigma(x,\xi,\eta) = \sigma(x)$, then $T_\sigma(f,g)(x) = \sigma(x)f(x)g(x)$. Of particular note is the symbol $\sigma \equiv 1$, for which $T_\sigma(f,g)(x) = f(x)g(x)$. In discussing Leibniz-type rule results relating to bilinear pseudodifferential operators, we can often obtain results more reminiscent to (1.1) and (1.2) by considering the symbol $\sigma \equiv 1$, so that $T_\sigma(f,g)$ appears as pointwise multiplication.

- If σ is a multiplier of the form $\sigma(\xi,\eta) = \xi^\alpha \eta^\beta$ for multi-indices $\alpha, \beta \in \mathbb{N}_0$, then

$$T_\sigma(f,g)(x) = \frac{1}{(2\pi i)^{|\alpha+\beta|}} (\partial^\alpha f)(x)(\partial^\beta g)(x).$$

  That is, products of derivatives of functions may be expressed via bilinear pseudodifferential operators with the right choice of symbol.

- Combining the previous two examples, we note that if σ is of the form

$$\sigma(x,\xi,\eta) = \sum_{\alpha,\beta} K_{\alpha,\beta}(x) \xi^\alpha \eta^\beta,$$

  where the sum is taken over some finite set of multi-indices, then

$$T_\sigma(f,g)(x) = \sum_{\alpha,\beta} \frac{K_{\alpha,\beta}(x)}{(2\pi i)^{|\alpha+\beta|}} (\partial^\alpha f)(x)(\partial^\beta g)(x).$$

Remark A.5.1. Definition 1.1 is naturally motivated by the analogous definition for linear pseudodifferential operators which act on a single function, as we examine in this remark. For $m(x,\xi)$ a complex-valued, smooth function defined for $x,\xi \in \mathbb{R}^n$, the linear pseudodifferential operator associated to m is defined via

$$T_m(f)(x) \coloneqq \int_{\mathbb{R}^n} m(x,\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi, \quad \forall x \in \mathbb{R}^n.$$

The following identity holds for $f, g \in \mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned}T_m(fg)(x) &= \int_{\mathbb{R}^n} m(x,\xi)\widehat{fg}(\xi)e^{2\pi i x\cdot\xi}\,d\xi = \int_{\mathbb{R}^n} m(x,\xi)[\widehat{f} * \widehat{g}](\xi)e^{2\pi i x\cdot\xi}\,d\xi \\ &= \int_{\mathbb{R}^n} m(x,\xi)\left[\int_{\mathbb{R}^n} \widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta\right]e^{2\pi i x\cdot\xi}\,d\xi = \int_{\mathbb{R}^{2n}} m(x,\xi+\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta,\end{aligned}$$

where in the second equality we have used property (A.3.2) of the Fourier transform, and in the last equality we did a change of variables $\xi \mapsto \xi+\eta$. Thus, we may recover the action of any linear pseudodifferential operator acting on a product of functions using an appropriate bilinear pseudodifferential operator: $T_m(fg) = T_\sigma(f,g)$ where $\sigma(x,\xi,\eta) \coloneqq m(x,\xi+\eta)$.

A.6 Littlewood-Paley theory background

We briefly introduce Littlewood-Paley theory, from which many techniques and results will be useful throughout discussions of our main results. Littlewood-Paley theory is built upon the idea of decomposing a function into a sum of functions with localized frequencies using the operators Δ_j^Ψ defined in (A.3). One way to obtain such a localization is by considering square functions:

Definition A.1. The **square function** associated with a family of Littlewood-Paley operators $\{\Delta_j^\Psi\}_{j\in\mathbb{Z}}$ is defined by

$$f \mapsto \left(\sum_{j\in\mathbb{Z}} \left|\Delta_j^\Psi f\right|^2\right)^{1/2}.$$

Another aspect of Littlewood-Paley theory concerns itself with relating the size of a function to the strength of the frequencies from which it is built. For functions in $L^2(\mathbb{R}^n)$, there is a straightforward relationship:

Theorem A.2 (Plancherel theorem)**.** *The Fourier transform is an isometry on $L^2(\mathbb{R}^n)$, that is, for any $f \in L^2(\mathbb{R}^n)$,*

$$\|f\|_{L^2} = \left\|\widehat{f}\right\|_{L^2}.$$

However, for functions in general $L^p(\mathbb{R}^n)$, no such simple isometry holds. The following theorem serves as somewhat of an analog for $L^p(\mathbb{R}^n)$ functions, relating the sizes of a function and its corresponding square function as in Definition A.1:

Theorem A.3 (Littlewood-Paley theorem)**.** *Let $1 < p < \infty$ and $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (A.5). Then for any $f \in L^p$,*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left| \Delta_j^{\Psi} f \right|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.$$

If in addition Ψ satisfies (A.6), then the reverse inequality is true, that is,

$$\|f\|_{L^p} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \left| \Delta_j^{\Psi} f \right|^2 \right)^{1/2} \right\|_{L^p}.$$

Theorem A.3 serves as a basis for many square function-type estimates we will find useful in proving results throughout the manuscript. For more extensive background on Littlewood-Paley theory, including a proof of this result, see for example Duoandikoetxea [20, Chapter 8] or Grafakos [26, Chapter 6].

Appendix B

Useful Results

B.1 Proofs of various lemmas

Lemma B.1. *Let $\sigma \in \dot{BS}^m_{1,1}$, and let $\theta \in \mathcal{S}(\mathbb{R})$ be real-valued satisfying (2.5). Then $\sigma^1, \sigma^2 \in \dot{BS}^m_{1,1}$, where*

$$\sigma^1(x,\xi,\eta) := \sigma(x,\xi,\eta)\theta\left(\tfrac{|\eta|}{|\xi|}\right) \quad \text{and} \quad \sigma^2(x,\xi,\eta) := \sigma(x,\xi,\eta)\theta\left(\tfrac{|\xi|}{|\eta|}\right).$$

Proof. We consider only the σ^1 case, as the σ^2 case follows analogously. Fix $\alpha, \beta, \gamma \in \mathbb{N}_0^n$. We wish to show that

$$\left|\partial_x^\gamma \partial_\xi^\alpha \partial_\eta^\beta \sigma^1(x,\xi,\eta)\right| \lesssim \frac{1}{(|\xi|+|\eta|)^{|\alpha+\beta|-|\gamma|-m}}, \quad \forall (x,\xi,\eta) \neq (0,0,0),$$

where the implicit constant may depend on the multi-indices involved. Due to (2.5), θ is constant on $B(0,\frac{1}{2}) \cup (\mathbb{R} \setminus B(0,2))$, so the result clearly holds within $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : \frac{|\eta|}{|\xi|} < \frac{1}{2} \text{ or } \frac{|\eta|}{|\xi|} > 2\}$. Thus, we restrict our attention to $\{(x,\xi,\eta) \in \mathbb{R}^{3n} : \frac{1}{2}|\xi| \leq |\eta| \leq 2|\xi|\}$, a region wherein $|\xi| \sim |\eta|$. By an application of the Leibniz rule for differentiation, we see that $\partial_x^\gamma \partial_\xi^\alpha \partial_\eta^\beta \sigma^1(x,\xi,\eta)$ may be expressed as a linear combination of terms of the form

$$\partial_x^\gamma \partial_\xi^{\alpha_1} \partial_\eta^{\beta_1} \sigma(x,\xi,\eta) \partial_\xi^{\alpha_2} \partial_\eta^{\beta_2} \tau(\xi,\eta), \qquad \alpha_1+\alpha_2=\alpha, \beta_1+\beta_2=\beta,$$

where $\tau(\xi,\eta) := \theta(\frac{|\eta|}{|\xi|})$. Note that $\tau(\lambda\xi,\lambda\eta) = \tau(\xi,\eta)$ for any $\lambda > 0$, which implies that for any $\alpha_2, \beta_2 \in \mathbb{N}_0^n$,

$$\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\tau(\xi,\eta) = \lambda^{|\alpha_2+\beta_2|}[\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\tau](\lambda\xi,\lambda\eta), \qquad \forall\lambda > 0.$$

Taking $\lambda = |(\xi,\eta)|^{-1}$, we then have

$$\begin{aligned}\left|\partial_x^\gamma\partial_\xi^\alpha\partial_\eta^\beta\sigma^1(x,\xi,\eta)\right| &\lesssim \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} \left|\partial_x^\gamma\partial_\xi^{\alpha_1}\partial_\eta^{\beta_1}\sigma(x,\xi,\eta)\right| \frac{\left|[\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\tau]\left(\frac{(\xi,\eta)}{|(\xi,\eta)|}\right)\right|}{|(\xi,\eta)|^{|\alpha_2+\beta_2|}}\\ &\lesssim \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} \frac{1}{(|\xi|+|\eta|)^{|\alpha_1+\beta_2|-|\gamma|-m}} \frac{\left|[\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\tau]\left(\frac{(\xi,\eta)}{|(\xi,\eta)|}\right)\right|}{(|\xi|+|\eta|)^{|\alpha_2+\beta_2|}}\\ &\sim \frac{1}{(|\xi|+|\eta|)^{|\alpha+\beta|-|\gamma|-m}},\end{aligned}$$

where in the second line we have used (2.3) and the fact that $|(\xi,\eta)| \sim |\xi| + |\eta|$, and in the last line we have used the fact that $\tau(\xi,\eta)$ is smooth away from the ξ- and η-axes by definition, and therefore, $\partial_\xi^{\alpha_2}\partial_\eta^{\beta_2}\tau$ is bounded on the compact subset of the unit sphere $\{(\xi,\eta) \in S^{2n-1} : \frac{1}{2}|\xi| \le |\eta| \le 2|\xi|\}$, the closure of which is disjoint from the ξ- and η-axes.

□

Lemma B.2. *Let $v \in A_p(\mathbb{R}^n)$ and $w \in A_q(\mathbb{R}^n)$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then $v^{\theta_1}w^{\theta_2} \in A_{\max\{p,q\}}(\mathbb{R}^n)$ for any $0 < \theta_1, \theta_2 < 1$ with $\theta_1 + \theta_2 = 1$.*

Proof. Without loss of generality, we suppose $p \ge q$. By standard nesting properties of Muckenhoupt weights, $w \in A_p(\mathbb{R}^n)$ (see, for example, Grafakos [26]). To satisfy Definition 3.7, we desire to show that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q v(x)^{\theta_1}w(x)^{\theta_2}\,dx\right)\left(\frac{1}{|Q|}\int_Q (v(x)^{\theta_1}w(x)^{\theta_2})^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Consider the first piece in parentheses, which by an application of Hölder's inequality (see Section B.2) and the fact that $\theta_1 + \theta_2 = 1$

yields

$$\frac{1}{|Q|}\int_Q v(x)^{\theta_1}w(x)^{\theta_2}\,dx \le \frac{1}{|Q|^{\theta_1+\theta_2}}\left(\int_Q (v(x)^{\theta_1})^{\frac{1}{\theta_1}}\,dx\right)^{\theta_1}\left(\int_Q (w(x)^{\theta_2})^{\frac{1}{\theta_2}}\,dx\right)^{\theta_2}$$
$$= \left(\frac{1}{|Q|}\int_Q v(x)\,dx\right)^{\theta_1}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)^{\theta_2}.$$

Similar calculations on the second piece in parentheses give

$$\left(\frac{1}{|Q|}\int_Q (v(x)^{\theta_1}w(x)^{\theta_2})^{-\frac{1}{p-1}}\,dx\right)^{p-1} \le \left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\,dx\right)^{(p-1)\theta_1}\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{(p-1)\theta_2}.$$

Finally, we obtain the desired result by noting that v and w are both in $A_p(\mathbb{R}^n)$.

□

Proof of Lemma 3.9. For part (a), see Rychkov [60, Proposition 1.9] (see also Kurtz [43]); part (b) is a consequence of part (a), Lemma 3.13, and a duality argument (see also Fefferman-Stein [23, pp. 128–129]).

□

Proof of Lemma 3.10. Part (a) follows from the facts that if $\Phi \in \mathcal{S}(\mathbb{R}^n)$ has integral 1 and $f \in L^2(\mathbb{R}^n)$, then $|f| \le \sup_{t>0}\left|\frac{1}{t^n}\Phi(\frac{\cdot}{t}) * f\right|$ pointwise, and

$$\left\|\sup_{t>0}\left|\frac{1}{t^n}\Phi\left(\frac{\cdot}{t}\right) * f\right|\right\|_{L^p(w)} \sim \left\|\left(\sum_{k\in\mathbb{Z}}\left|\Delta_k^{\Psi} f\right|^2\right)^{\frac{1}{2}}\right\|_{L^p(w)}$$

for $0 < p < \infty$ and $w \in A_\infty(\mathbb{R}^n)$ (see Qui [57, Theorem 1.4] for such norm equivalence). Part (b) was proved in Ding et al. [19, Theorem 3.5] for $0 < p \le 1$; the case $p > 1$ follows from the latter along with an extrapolation result on A_∞ weights (see Cruz-Uribe-Martell-Pérez [16, Theorem 2.1] with $p_0 = p$ for any $0 < p \le 1$ and the family of functions $\mathscr{F} = \left\{f, \left(\sum_{j_1,j_2\in\mathbb{Z}}\left|\Delta_{j_1}^{\Psi_1}\Delta_{j_2}^{\Psi_2} f\right|^2\right)^{\frac{1}{2}}\right\}_{f\in L^2(\mathbb{R}^n)}$).

□

Proof of Lemma 3.11. For part (a), see Cruz-Uribe-Naibo[17, Theorem 2.1]; part (b) is a consequence of part (a) and Lemma 3.13.

□

Proof of Lemma 3.12. The case $p = r$ is easily verified:

$$\begin{aligned}
\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |T_k(f_j)|^2 \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \right\|_{L^r(w)}^r &= \int_{\mathbb{R}^n} \left[\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |T_k(f_j)(x)|^2 \right)^{\frac{r}{2}} \right] w(x)\, dx \\
&= \sum_{j\in\mathbb{Z}} \left\| \left(\sum_{k\in\mathbb{Z}} |T_k(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^r(w)}^r \\
&\lesssim \sum_{j\in\mathbb{Z}} \|f_j\|_{L^r(w)}^r \\
&= \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^r(w)}^r ,
\end{aligned}$$

where in the third inequality we have used (3.14), which is independent of f_j. The $p \neq r$ case follows from the latter case and extrapolation (see, for example, Duoandikoetxea [20, Theorem 7.8]).

□

Proof of Lemma 3.13. This is an immediate consequence of Lemma 3.12 with an iteration of norms. We have

$$\begin{aligned}
\left\| \left(\sum_{j_1\in\mathbb{Z}} \sum_{j_2\in\mathbb{Z}} \left| T^1_{j_1} T^2_{j_2}(f) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}^p &\lesssim \int_{\mathbb{R}^{n_2}} \left\| \left(\sum_{j_2\in\mathbb{Z}} \left| T^2_{j_2} f(\cdot, x_2) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{n_1}, w(\cdot, x_2))}^p dx_2 \\
&= \int_{\mathbb{R}^{n_1}} \left\| \left(\sum_{j_2\in\mathbb{Z}} \left| T^2_{j_2} f(x_1, \cdot) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{n_2}, w(x_1, \cdot))}^p dx_1 \\
&\lesssim \int_{\mathbb{R}^{n_1}} \|f(x_1, \cdot)\|_{L^p(\mathbb{R}^{n_2}, w(x_1, \cdot))}^p \, dx_1 \\
&= \|f\|_{L^p(w)} ,
\end{aligned}$$

where in the first line we have applied Lemma 3.12 with $r = 2$ (which is applicable due to

(3.15) and the fact that $w(\cdot, x_2) \in A_p(\mathbb{R}^{n_1})$ uniformly in $x_2 \in \mathbb{R}^{n_2}$), and in the third line we have used (3.16), noting again that $w(x_1, \cdot) \in A_p(\mathbb{R}^{n_2})$ uniformly in $x_1 \in \mathbb{R}^{n_1}$.

$\square$

Proof of Lemma 3.14. As reasoned in (A.10), consider $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (A.5) and such that $\widehat{\psi}\widehat{\Psi} = \widehat{\Psi}$. For $s \in \mathbb{R}$, $j \in \mathbb{N}$, and $u \in \mathbb{R}^n$, we have

$$\begin{aligned}
\Delta_j^{\tau_u J_j^s \Psi} f(x) &= \int_{\mathbb{R}^n} e^{2\pi i (2^{-j}\xi)\cdot u} (2^{-2j} + \left|2^{-j}\xi\right|^2)^{\frac{s}{2}} \widehat{\Psi}(2^{-j}\xi) \widehat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi \\
&= \int_{\mathbb{R}^n} \widehat{J_j^s \psi}(2^{-j}\xi) \widehat{\tau_u \Psi}(2^{-j}\xi) \widehat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi,
\end{aligned}$$

where $\widehat{J_j^s \psi}(\xi) = (2^{-2j} + |\xi|^2)^{\frac{s}{2}} \widehat{\psi}(\xi)$. Let $\{c^s[m,j]\}_{m\in\mathbb{Z}^n}$ be the Fourier series coefficients of the periodic extension of $\widehat{J_j^s \psi}\chi_H$, where $H = [-\frac{h}{2}, \frac{h}{2}]^n$ with h large enough so that $\operatorname{supp}(\widehat{\psi}) \subset H$. By Grafakos-Oh [30, Lemma 3], it follows that $\sup_{j\geq 0} |c^s[m,j]| = \mathcal{O}((1+|m|)^{-K})$ for any $K \in \mathbb{N}$. Then, we get

$$\begin{aligned}
\left|\Delta_j^{\tau_u J_j^s \Psi} f(x)\right| &= \left| \int_{\mathbb{R}^n} \left(\sum_{m\in\mathbb{Z}^n} c^s[m,j] e^{\frac{2\pi i}{h}(2^{-j}\xi)\cdot m} \right) \widehat{\tau_u \Psi}(2^{-j}\xi) \widehat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi \right| \\
&= \left| \sum_{m\in\mathbb{Z}^n} c^s[m,j] \int_{\mathbb{R}^n} \mathcal{F}[\tau_{u+\frac{m}{h}} \Psi](2^{-j}\xi) \widehat{f}(\xi) e^{2\pi i x\cdot\xi} \, d\xi \right| \\
&= \left| \sum_{m\in\mathbb{Z}^n} c^s[m,j] \Delta_j^{\tau_{u+\frac{m}{h}} \Psi} f(x) \right| \\
&\lesssim \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^K} \left| \Delta_j^{\tau_{u+\frac{m}{h}} \Psi} f(x) \right|,
\end{aligned}$$

where we have used property (A.3.3) from Section A.3 in the second equality. Therefore,

$$\left\| \left(\sum_{j\in\mathbb{N}_0} \left| \Delta_j^{\tau_u J_j^s \Psi} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \sum_{m\in\mathbb{Z}^n} \frac{1}{(1+|m|)^K} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \Delta_j^{\tau_{u+\frac{m}{h}} \Psi} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim \|f\|_{L^p(w)},$$

where we have applied the triangle inequality for weighted Lebesgue spaces, Lemma 3.11, and assumed $K > n$ to ensure convergence in the sum in $m \in \mathbb{Z}^n$.

$\square$

Proof of Lemma 3.15. We merely apply Lemma 3.12, noting that (3.14) holds by Lemma 3.14.

□

B.2 Known results

Theorem B.3 (Hölder's inequality)**.** *Let $1 \leq p, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. The for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$,*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Theorem B.4 (Young's inequality)**.** *Let $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then for all $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, we have*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Theorem B.5 (Minkowski's integral inequality)**.** *Let $0 < p_1, p_2 < \infty$, and suppose (S_1, μ_1) and (S_2, μ_2) are two σ-finite measure spaces with $F : S_1 \times S_2 \to \mathbb{R}$ measurable. Then*

$$\left(\int_{S_2} \left|\int_{S_1} F(x,y)^{p_1} \, d\mu_1(x)\right|^{\frac{p_2}{p_1}} d\mu_2(y)\right)^{\frac{1}{p_2}} \leq \left[\int_{S_1} \left(\int_{S_2} |F(x,y)|^{p_2} \, d\mu_2(y)\right)^{\frac{p_1}{p_2}} d\mu_1(x)\right]^{\frac{1}{p_1}}.$$