Numerical solutions to some inverse problems

by

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AN ABSTRACT OF A DISSERTATION

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Department of Mathematics
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Abstract

In this dissertation, the author presents two independent researches on inverse problems: (1) creating materials in which heat propagates a long a line and (2) 3D inverse scattering problem with non-over-determined data. The theories of these methods were developed by Professor Alexander Ramm, see[7],[13] and are presented in Chapters 1 and 3. The algorithms and numerical results are taken from the papers of Professor Alexander Ramm and the author, see[19] and[20], and are presented in Chapters 2 and 4.

In Chapter 1, the theory of heat transfer in a complex medium consisting of many small impedance particles of an arbitrary shape is presented. Equation for the effective limiting temperature is derived when the characteristic size a of the particles tends to zero while the number of particles tends to infinity at a suitable rate, while the distance d between closest neighboring particles is much larger than a, $d \gg a$.

In Chapter 2, the theory is developed for creating a material in which the heat is transmitted along a given line. This gives a possibility to transfer information using heat signals. This seems to be a novel idea. The technical part of the theory is the construction of the potential $q(x)$. This potential describes the heat equation $u_t = \Delta u - q(x)u$ in the limiting medium which is obtained after the small impedance particles are distributed in a given domain. A numerical method is also established to construct numerically such a potential.

In Chapter 3, the Dynamical System Method (DSM) for solving ill-posed problems are presented. The corresponding theory is developed by Professor Ramm. The literature of the DSM is huge. In this chapter, we just present the results that will be used in chapter 4.

In Chapter 4, the 3D inverse scattering problem with non-over-determined scattering data is considered. The data are the scattering amplitude $A(\beta, \alpha_0, k)$ for all $\beta \in S^2_\beta$, where S^2_β is an open subset of the unit sphere S^2 in $\mathbb{R}^3$, $\alpha_0 \in S^2$ is fixed, and $k \in (a, b), 0 \leq a < b$.

In this chapter, a numerical method is given for solving the inverse scattering problem with non-over-determined scattering data and the numerical results are presented.

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Approved by:

Major Professor
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Chapter 1

Heat transfer in a complex medium

In this chapter the theory of heat transfer in a complex medium consisting of many small impedance particles of an arbitrary shape is presented. Equation for the effective limiting temperature is derived when the characteristic size a of the particles tends to zero while the number of particles tends to infinity at a suitable rate, while the distance d between closest neighboring particles is much larger than a, $d \gg a$.

These results are used for developing a method for creating materials in which heat is transmitted along a line in Chapter 2. Thus, the information can be transmitted by a heat signals.

The content of this chapter is based on Professor Ramm's papers cited in the bibliography, especially[6], [4] and[3].

1.1 Problem statement

Let many small bodies (particles) D_m, $1 \leq m \leq M$, of arbitrary shapes be distributed in a bounded domain $D \subset \mathbb{R}^3$, $a := \frac{1}{2} \max_m(\operatorname{diam} D_m)$, $\operatorname{diam} D_m$: diameter of D_m, and the boundary of D_m is denoted by $\mathcal{S}_m$ and is assumed twice continuously differentiable. The

small bodies are distributed according to the law

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(x)dx[1 + o(1)], \quad a \to 0. \tag{1.1}$$

Here $\Delta \subset D$ is an arbitrary open subdomain of D, $\kappa \in [0, 1)$ is a constant, $N(x) \geq 0$ is a continuous function, and $\mathcal{N}(\Delta)$ is the number of the small bodies D_m in Δ. The heat equation can be stated as follows:

$$u_t = \nabla^2 u + f(x) \quad \text{in} \quad \Omega := \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m, \quad u|_{t=0} = 0, \tag{1.2}$$

$$u_N = \zeta_m u \quad \text{on} \quad \mathcal{S}_m, \quad 1 \leq m \leq M, \qquad \text{Re}\zeta_m \geq 0. \tag{1.3}$$

Here N is the outer unit normal to $\mathcal{S}$,

$$\mathcal{S} := \bigcup_{m=1}^{M} \mathcal{S}_m, \quad \zeta_m = \frac{h(x_m)}{a^{\kappa}}, \quad x_m \in D_m, \quad 1 \leq m \leq M,$$

and $h(x)$ is a continuous function in D. Since one assumes $\text{Re}\zeta_m \geq 0$ in equation (1.3), one has the following assumption for $h(x)$:

$$\text{Re}h \geq 0. \tag{1.4}$$

Our goal is to study the limiting behavior as $a \to 0$ and $M = M(a) \to \infty$ of the heat distribution in Ω, to derive the equation of the limiting distribution of the temperature u, and to give numerical methods for calculating the solution to problem (1.2) - (1.3) when M is large. We do not assume periodicity in the location of the small bodies and our approach differs from the usual approach developed in homogenization theory.

In the next sections, we derive a linear algebraic system for finding the Laplace transform of the solution to problem (1.2) - (1.3), an integral equation for the limiting temperature as

$a \to 0$, and state the main theorem and result.

1.2 Derivation of the equation of the limiting temperature

Denote the Laplace transform of $u(x,t)$ by

$$\mathcal{U} := \mathcal{U}(x,\lambda) = \int_0^\infty e^{-\lambda t} u(x,t) dt.$$

Then, taking the Laplace transform of equations (1.2) - (1.3) one gets:

$$-\nabla^2 \mathcal{U} + \lambda \mathcal{U} = \lambda^{-1} f(x) \quad \text{in} \quad \Omega, \tag{1.5}$$

$$\mathcal{U}_N = \zeta_m \mathcal{U} \quad \text{on} \quad \mathcal{S}_m, \quad 1 \le m \le M. \tag{1.6}$$

Let

$$g(x,y) := g(x,y,\lambda) := \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|}, \tag{1.7}$$

$$F(x,\lambda) := \frac{1}{\lambda} \int_{\mathbb{R}^3} g(x,y) f(y) dy. \tag{1.8}$$

Look for the solution to (1.5) - (1.6) of the form

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} \int_{\mathcal{S}_m} g(x,s) \sigma_m(s) ds, \tag{1.9}$$

where

$$\mathcal{U}(x,\lambda) := \mathcal{U}(x) := \mathcal{U}, \tag{1.10}$$

and $\mathcal{U}(x)$ depends on λ.

The functions σ_m are unknown and should be found from the boundary conditions (1.6).

Equation (1.5) is satisfied by $\mathcal{U}$ of the form (1.9) with arbitrary continuous σ_m. Indeed, apply the operator $-\nabla^2 + \lambda$ to equation (1.9), one has

$$\begin{aligned}
&(-\nabla^2 + \lambda)\mathcal{U}(x, \lambda) \\
&\qquad = \frac{1}{\lambda} \int_{\mathbb{R}^3} (-\nabla^2 + \lambda) g(x, y) f(y) dy + \sum_{m=1}^{M} \int_{\mathcal{S}_m} (-\nabla^2 + \lambda) g(x, s) \sigma_m(s) ds \\
&\qquad = \frac{1}{\lambda} \int_{\mathbb{R}^3} \delta(x - y) f(y) dy + \sum_{m=1}^{M} \int_{\mathcal{S}_m} \delta(x - s) \sigma_m(s) ds \\
&\qquad = \lambda^{-1} f(x), \quad x \in \Omega.
\end{aligned}$$

Define the effective (self-consistent) field $\mathcal{U}_e(x)$ of $\mathcal{U}$, acting on the m-th particle, by the formula:

$$\mathcal{U}_e(x) := \mathcal{U}_{e,m}(x) := \mathcal{U}(x) - \int_{\mathcal{S}_m} g(x, s) \sigma_m(s) ds, \tag{1.11}$$

As $a \to 0$, the dependence on m disappears, i.e.

$$\lim_{a \to 0} \mathcal{U}_e(x) = \mathcal{U}(x), \tag{1.12}$$

since

$$\int_{\mathcal{S}_m} g(x, s) \sigma_m(s) ds \to 0 \text{ as } a \to 0.$$

Then the solution $\mathcal{U}(x)$ can be written as

$$\mathcal{U}(x) = \mathcal{U}_e(x) + \int_{\mathcal{S}_m} g(x, s) \sigma_m(s) ds. \tag{1.13}$$

To satisfy the boundary condition (1.6) one applies (1.13) to the boundary condition (1.6) and obtains

$$\frac{\partial \mathcal{U}_e(s)}{\partial N} + \frac{\partial}{\partial N} \int_{\mathcal{S}_m} g(s, s') \sigma_m(s') ds' = \zeta_m \mathcal{U}_e(s) + \zeta_m (T_m \sigma_m)(s), \tag{1.14}$$

where $s \in \mathcal{S}_m, 1 \leq m \leq M$ and the operator T_m is defined by the formula:

$$(T_m\sigma_m)(s) := \int_{\mathcal{S}_m} g(s,s')\sigma_m(s')ds'. \tag{1.15}$$

Let us apply the known formula for the outer limiting value on $\mathcal{S}_m$ of the normal derivative of a simple layer potential (see Chapter 11 of[6]):

$$\lim_{x\to s^-} \frac{\partial}{\partial N} \int_{\mathcal{S}_m} g(x,s')\sigma_m(s')ds' = \frac{(A_m\sigma_m)(s) - \sigma_m(s)}{2}, \tag{1.16}$$

where A_m is:

$$(A_m\sigma_m)(s) := 2\int_{\mathcal{S}_m} \frac{\partial g(s,s')}{\partial N_s}\sigma_m(s')ds', \tag{1.17}$$

and $x \to s^-$ means x goes to $s \in \mathcal{S}_m$ from the outside of D_m. Equation (1.14) then becomes

$$\frac{\partial \mathcal{U}_e(s)}{\partial N} + \frac{(A_m\sigma_m)(s) - \sigma_m(s)}{2} - \zeta_m\mathcal{U}_e(s) - \zeta_m(T_m\sigma_m)(s) = 0 \quad \text{on} \quad \mathcal{S}_m, \quad 1 \leq m \leq M. \tag{1.18}$$

We now apply the ideas and methods for solving many-body scattering problems developed in[6] -[5]. The idea has two parts: in the first part, one derives the asymptotic formula for $\mathcal{U}$:

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} g(x,x_m)Q_m, \quad a \to 0, \quad x_m \in D_m,$$

where $Q_m, 1 \leq m \leq M$ are scalars which are found asymptotically in the second part.

Derivation of the asymptotic formula for $\mathcal{U}$:

One has, from (1.9),

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} g(x,x_m)Q_m + \sum_{m=1}^{M}\int_{\mathcal{S}_m} [g(x,s) - g(x,x_m)]\sigma_m(s)ds, \quad x_m \in D_m, \tag{1.19}$$

where

$$Q_m := \int_{\mathcal{S}_m} \sigma_m(s)ds,$$

Define

$$\mathcal{J}_1 := \sum_{m=1}^{M} g(x, x_m)Q_m, \tag{1.20}$$

$$\mathcal{J}_2 := \sum_{m=1}^{M} \int_{\mathcal{S}_m} [g(x, s) - g(x, x_m)]\sigma_m(s)ds. \tag{1.21}$$

One can prove that

$$|\mathcal{J}_2| \ll |\mathcal{J}_1| \text{ as } a \to 0 \tag{1.22}$$

provided that

$$\lim_{a\to 0} \frac{a}{d(a)} = 0, \tag{1.23}$$

where $d(a) = d$ is the minimal distance between neighboring particles.

Lemma 1.2.1. *One has*

$$J_1 := |g(x, x_m)Q_m| \gg \left| \int_{S_m} [g(x, s) - g(x, x_m)]\sigma_m(s)ds \right| := J_2, \quad a \to 0. \tag{1.24}$$

Proof. By (1.23) one has

$$|g(x, x_m)Q_m| \geq \frac{Q_m}{4\pi d}, \quad |x - x_m| \geq d. \tag{1.25}$$

Since $|x - x_m| \geq d \gg a$ and $|s - x_m| \leq a$, one has

$$|g(x, s) - g(x, x_m)| \leq \max\left\{ O\left(\frac{a}{d^2}\right), O\left(\frac{ka}{d}\right) \right\}, \tag{1.26}$$

and

$$\frac{J_2}{J_1} \leq O\left(\frac{a}{d} + ka\right) \ll 1. \tag{1.27}$$

Lemma 1.2.1 is proved. □

By lemma 1.2.1, each term in $\mathcal{J}_2$ is neglectible compared to the corresponding term in $\mathcal{J}_1$, so (1.22) holds. Thus, problem (1.5) - (1.6) is solved asymptotically by the formula

$$\mathcal{U}(x,\lambda) = F(x,\lambda) + \sum_{m=1}^{M} g(x,x_m)Q_m, \quad a \to 0, \tag{1.28}$$

provided that asymptotic formulas for Q_m, as $a \to 0$, are found.

Derivation of the formulas for Q_m:

To find the formulas for Q_m, let us integrate (1.18) over $\mathcal{S}_m$ and get

$$\begin{aligned} I_1 + I_2 + I_3 + I_4 := \int_{S_m} \frac{\partial \mathcal{U}_e(s)}{\partial N} ds + \int_{S_m} \frac{(A_m\sigma_m)(s) - \sigma_m(s)}{2} ds - \\ - \int_{S_m} \zeta_m \mathcal{U}_e(s) ds - \int_{S_m} \zeta_m (T_m\sigma_m)(s) ds = 0. \end{aligned} \tag{1.29}$$

The idea is to estimate the order of the terms in the equation (1.29) as $a \to 0$, and keep the main terms, that is, neglect the terms of higher order of smallness as $a \to 0$.

We get

$$I_1 := \int_{\mathcal{S}_m} \frac{\partial \mathcal{U}_e(s)}{\partial N} ds = \int_{D_m} \nabla^2 \mathcal{U}_e(x) dx = O(a^3). \tag{1.30}$$

Here we assumed that $|\nabla^2 \mathcal{U}_e| = O(1), a \to 0$. This assumption is valid since $\mathcal{U} = \lim_{a\to 0} \mathcal{U}_e$ is smooth as a solution to an elliptic equation.

One has

$$I_2 := \int_{\mathcal{S}_m} \frac{(A_m\sigma_m)(s) - \sigma_m(s)}{2} ds = -Q_m[1 + o(1)],\ a \to 0. \tag{1.31}$$

Indeed, one has

$$
\begin{aligned}
(A_m\sigma_m)(s) &:= 2\int_{S_m} \frac{\partial}{\partial N_S}\frac{e^{-\sqrt{\lambda}|s-t|}}{4\pi|s-t|}\sigma_m(t)dt \\
&= 2\int_{S_m} \frac{\partial}{\partial N_S}\left(\frac{1-\sqrt{\lambda}|s-t|+O(\lambda|s-t|^2)}{4\pi|s-t|}\right)\sigma_m(t)dt \\
&= (A_0\sigma_m)(s) + \int_{S_m} O(|s-t|)\sigma_m(t)dt \\
&\sim (A_0\sigma_m)(s) \quad \text{as } a\to 0, \qquad (1.32)
\end{aligned}
$$

where

$$
(A_0\sigma_m)(s) := 2\int_{S_m} \frac{\partial}{\partial N_S}\frac{1}{4\pi|s-t|}\sigma_m(t)dt.
$$

Let us show that

$$
\int_{S_m}(A_0\sigma_m)(s)ds = -\int_{S_m}\sigma_m(t)dt := -Q_m. \qquad (1.33)
$$

This is an exact relation. One has

$$
\begin{aligned}
\int_{S_m}(A_0\sigma_m)(s)ds &= 2\int_{S_m} ds\int_{S_m}\frac{\partial}{\partial N_S}\frac{1}{4\pi|s-t|}\sigma_m(t)dt \qquad (1.34)\\
&= \int_{S_m} dt\,\sigma_m(t)\,2\int_{S_m}\frac{\partial}{\partial N_S}\frac{1}{4\pi|s-t|}ds \\
&= -\int_{S_m}\sigma_m(t)dt, \qquad (1.35)
\end{aligned}
$$

here we have used the known formula:

$$
2\int_{S_m}\frac{\partial}{\partial N_S}\frac{1}{4\pi|s-t|}ds = -1, \quad t\in S_m. \qquad (1.36)
$$

Proof of this formula is given in Section 11.1 in[6].

Equation (1.33) implies (1.31).

Furthermore,

$$I_3 := -\zeta_m \int_{\mathcal{S}_m} \mathcal{U}_e(s)ds = -\zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m) = O(a^{2-\kappa}), \quad a \to 0, \tag{1.37}$$

where $|\mathcal{S}_m| = O(a^2)$ is the surface area of $\mathcal{S}_m$, $\zeta_m := \frac{h(x_m)}{a^\kappa}$.

Finally,

$$I_4 := -\zeta_m \int_{\mathcal{S}_m} ds \int_{\mathcal{S}_m} g(s,s')\sigma_m(s')ds' = -\zeta_m \int_{\mathcal{S}_m} ds' \sigma_m(s') \int_{\mathcal{S}_m} dsg(s,s')$$

$$= Q_m O(a^{1-\kappa}), \qquad a \to 0, \tag{1.38}$$

since

$$\int_{S_m} g(s,s')ds = O(a), \quad a \to 0.$$

Thus,

$$|I_1| \ll |I_3|, \quad |I_4| \ll |I_2|, \quad \text{as} \quad a \to 0, \quad \kappa \in [0,1).$$

Therefore, from equation (1.29) the main term of the asymptotic of Q_m, as $a \to 0$, is

$$Q_m = -\zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m). \tag{1.39}$$

Formulas (1.39) and (1.28) yield

$$\mathcal{U}(x,\lambda) = F(x,\lambda) - \sum_{m=1}^{M} g(x,x_m)\zeta_m |\mathcal{S}_m| \mathcal{U}_e(x_m,\lambda), \tag{1.40}$$

and (1.11) yields

$$\mathcal{U}_e(x_m,\lambda) = F(x_m,\lambda) - \sum_{m' \neq m, m'=1}^{M} g(x_m,x_{m'})\zeta_{m'} |\mathcal{S}_{m'}| \mathcal{U}_e(x_{m'},\lambda). \tag{1.41}$$

Denote

$$\mathcal{U}_e(x_m,\lambda) := \mathcal{U}_m, \quad F(x_m,\lambda) := F_m, \quad g(x_m,x_{m'}) := g_{mm'},$$

and write (1.41) as a linear algebraic system for $\mathcal{U}_m$:

$$\mathcal{U}_m = F_m - a^{2-\kappa}\sum_{m'\neq m} g_{mm'}h_{m'}c_{m'}\mathcal{U}_{m'}, \quad 1\leq m\leq M, \tag{1.42}$$

where $h_{m'} = h(x_{m'})$, $\zeta_{m'} = \frac{h_{m'}}{a^\kappa}$, $c_{m'} := |S_{m'}|a^{-2}$.

Solving the linear algebraic system (1.42) for $\mathcal{U}_m$ then substituting $\mathcal{U}_m$ into (1.40), one gets an analytic formula for $\mathcal{U}(x,\lambda)$. Taking Laplace inverse of $\mathcal{U}(x,\lambda)$, one obtains an analytic formula for $u(x,t)$.

1.3 Derivation of the integral equation of the limiting temperature

Consider a partition of the bounded domain D, in which the small bodies are distributed, into a union of $P << M$ small nonintersecting cubes Δ_p, $1\leq p\leq P$, of side b,

$$b >> d, \quad b = b(a)\to 0 \quad \text{as } a\to 0 \quad \lim_{a\to 0}\frac{d(a)}{b(a)} = 0.$$

Let $x_p\in\Delta_p$, $|\Delta_p|$ = volume of Δ_p. One obtains

$$a^{2-\kappa}\sum_{m'=1,m'\neq m}^{M} g_{mm'}h_{m'}c_{m'}\mathcal{U}_{m'} = a^{2-\kappa}\sum_{p'=1,p'\neq p}^{P} g_{pp'}h_{p'}c_{p'}\mathcal{U}_{p'}\sum_{x_{m'}\in\Delta_{p'}} 1 =$$

$$= \sum_{p'\neq p} g_{pp'}h_{p'}c_{p'}\mathcal{U}_{p'}N(x_{p'})|\Delta_{p'}|[1+o(1)], \quad a\to 0. \tag{1.43}$$

Thus, (1.42) yields a linear algebraic system (LAS) of order $P << M$ for the unknowns $\mathcal{U}_p$:

$$\mathcal{U}_p = F_p - \sum_{p' \neq p, p'=1}^{P} g_{pp'} h_{p'} c_{p'} N_{p'} \mathcal{U}_{p'} |\Delta_{p'}|, \quad 1 \leq p \leq P. \tag{1.44}$$

Since $P << M$, the order of the original LAS (1.42) is drastically reduced. This is crucial when the number of particles tends to infinity and their size a tends to zero. We have assumed that

$$h_{m'} = h_{p'}[1 + o(1)], \quad c_{m'} = c_{p'}[1 + o(1)], \quad \mathcal{U}_{m'} = \mathcal{U}_{p'}[1 + o(1)], \ a \to 0, \tag{1.45}$$

for $x_{m'} \in \Delta_{p'}$. This assumption is justified, for example, if the functions $h(x)$, $\mathcal{U}(x, \lambda)$,

$$c(x) = \lim_{x_{m'} \in \Delta_x, a \to 0} \frac{|S_{m'}|}{a^2},$$

and $N(x)$ are continuous, but these assumptions can be relaxed.

The continuity of the $\mathcal{U}(x, \lambda)$ is a consequence of the fact that this function satisfies elliptic equation, and the continuity of $c(x)$ is assumed. If all the small bodies are identical, then $c(x) = c_S = \text{const}$, so in this case the function $c(x)$ is certainly continuous.

The sum in the right-hand side of (1.44) is the Riemannian sum for the integral

$$\lim_{a \to 0} \sum_{p'=1, p' \neq p}^{P} g_{pp'} h_{p'} c_{p'} N(x_{p'}) \mathcal{U}_{p'} |\Delta'_p| =$$

$$\int_D g(x, y) h(y) c(y) N(y) \mathcal{U}(y, \lambda) dy,$$

and the left-hand side of (1.44), by (1.12), is

$$\lim_{a \to 0} \mathcal{U}_p = \mathcal{U}.$$

Therefore, the linear algebraic system (1.44) is a collocation method for solving integral equation

$$\mathcal{U}(x,\lambda) = F(x,\lambda) - \int_D g(x,y)c(y)h(y)N(y)\mathcal{U}(y,\lambda)dy. \tag{1.46}$$

Convergence of this method for solving equations with weakly singular kernels is proved in [14], see also [15] and [16]. If one solves equation (1.46) for $\mathcal{U}(x,\lambda)$, or the linear algebraic system (1.44) for $\mathcal{U}_p(\lambda)$, then one can Laplace-invert $\mathcal{U}(x,\lambda)$ for $u(x,t)$. Numerical methods for Laplace inversion from the real axis are discussed in [17] -[18].

1.4 Existence and uniqueness theorem of the limiting temperature

The following main theorem and its proof are taken from [6]

Theorem 1.4.1. *The limit $\mathcal{U}(x,\lambda)$ of the field $\mathcal{U}_a(x,\lambda)$ as $a \to 0$ does exist and solves equation (1.46). The solution to this equation is unique.*

Proof. Only the last statement of Theorem 1.4.1 is not yet proved. To prove it, it is sufficient to prove that the corresponding homogeneous equation (with $F = 0$) has only the trivial solution. Apply the operator $-\nabla^2 + \lambda$ to (1.46) with $F = 0$ and get

$$\left(-\nabla^2 + \lambda + c_S h(x)N(x)\right)\mathcal{U} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{1.47}$$

This is a Schrödinger equation with a potential $q(x) = c_S N(x)h(x)$ which has compact support Ω, and $\mathcal{U}$ decays at infinity exponentially as follows from the homogeneous equation (1.46). The assumption $\mathrm{Re}h \geq 0$ (see (1.4)) guarantees that the only solution to equation (1.47) is the trivial solution $\mathcal{U} = 0$. Let us prove this claim.

Multiply (1.47) by $\overline{\mathcal{U}}$ and integrate the result over $D_R = \{x : |x| \leq R\}$. The result is

$$\int_{D_R} |\nabla \mathcal{U}|^2 dx + \lambda \int_{D_R} |\mathcal{U}|^2 dx + \int_{D_R} q(x)|\mathcal{U}|^2 dx = 0. \tag{1.48}$$

Since $\lambda \geq 0$, the assumption $\mathrm{Re} q \geq 0$ and formula (1.48) imply $\mathcal{U} = 0$. The assumption $\mathrm{Re} q \geq 0$ is equivalent to the assumption $\mathrm{Re} h \geq 0$ which was done in formula (1.4). Theorem 1.4.1 is proved. □

1.5 Creating a medium in which heat transfers by a potential

Applying the operator $-\nabla^2 + \lambda$ to equation (1.46) one gets an elliptic differential equation:

$$(-\Delta + \lambda)\mathcal{U}(x, \lambda) = \frac{f(x)}{\lambda} - c(x)h(x)N(x)\mathcal{U}(x, \lambda). \tag{1.49}$$

Taking the inverse Laplace transform of this equation yields

$$u_t = \Delta u + f(x) - q(x)u, \quad q(x) := c(x)h(x)N(x). \tag{1.50}$$

Therefore, the limiting equation for the temperature contains the term $q(x)u$. Thus, the embedding of many small particles creates a distribution of source and sink terms in the medium, the distribution of which is described by the term $q(x)u$. We will use this result in Chapter 2 to create material in which heat propagates along a line.

Chapter 2

Creating materials in which heat propagates along a line

In this chapter a theory is developed for creating a material in which the heat is transmitted along a given line. This gives a possibility to transfer information using heat signals. This seems to be a novel idea. The technical part of the theory is the construction of the potential $q(x)$. This potential describes the heat equation $u_t = \Delta u - q(x)u$ in the limiting medium which is obtained after the small impedance particles are distributed in a given domain. A numerical method is also established to construct numerically such a potential. The content of this chapter is obtained from the paper[19] by the author and Professor A. G. Ramm.

2.1 Introduction

To create materials in which heat propagates along a line, one needs to create a medium in which the heat transfer is governed by the equation

$$u_t = \Delta u - q(x)u \quad \text{in } D, \quad u|_S = 0, \quad u|_{t=0} = f(x), \tag{2.1}$$

where D is a bounded domain with a piecewise-smooth boundary S, $D = D_0 \times [0, L]$, $D_0 \subset \mathbb{R}^2$ is a disc of radius R orthogonal to the axis x_1, $x = (x_1, x_2, x_3)$, $x_2, x_3 \in D_0$, $0 \leq x_1 \leq L$.

Such a medium is created by embedding many small impedance particles D_m, $1 \leq m \leq M$, into a given domain D filled with a homogeneous material. Let us assume that the distribution of small particles is:

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(x)dx(1 + o(1)), \quad a \to 0, \tag{2.2}$$

where $\mathcal{N}(\Delta)$ is the number of small particles in an arbitrary open subset $\Delta \in D$, $\kappa \in [0, 1)$ is a number that can be chosen by an experimenter as desired, $a = \frac{1}{2} \max\limits_{1 \leq m \leq M} \max\limits_{x,y \in D_m} |x - y|$ and $N(x) \geq 0$ is a continuous in D function that can be chosen by an experimenter as desired. As $a \to 0$, it is proved (see[4],[6]) that the solution $u(x, t, a)$ to the problem

$$u_t = \Delta u \quad \text{in } D \setminus \bigcup_{m=1}^{M} D_m, \; u_N = \zeta_m u \quad \text{on } S_m, \; 1 \leq m \leq M, \tag{2.3}$$

$$u|_{t=0} = f(x), \; \zeta_m := \frac{h(x_m)}{a^{\kappa}}, \; \text{Re}\zeta_m \geq 0, \tag{2.4}$$

where $x_m \in \mathbb{R}^3$ is an arbitrary point in D_m, has a limit, $u(x, t) = \lim_{a \to 0} u(x, t, a)$ which solves problem (2.1) with

$$q(x) = c_S N(x) h(x), \quad c_S := \frac{|S_m|}{a^2} = const, \tag{2.5}$$

so c_S does not depend on m, but the shape of S_m may depend on m. Therefore, given a potential $q(x)$ (which makes heat propagate along a line), one can choose an arbitrary continuous function $N(x) > 0$, can construct a continuous function $h(x) = \frac{q(x)}{c_S N(x)}$, and can distribute the small bodies according to the rule (2.2) to obtain the medium in which heat propagates along a line.

Suppose that

$$(-\Delta + q(x))\phi_n(x) = \lambda_n \phi_n, \quad \phi_n|_S = 0, \quad ||\phi_n||_{L^2(D)} = ||\phi_n|| = 1, \tag{2.6}$$

where $\{\phi_n\}$ is an orthonormal basis of $L^2(D) := H$, and $(f, \phi_n) := \int_D f(x)\overline{\phi_n(x)}dx$. Then the unique solution to (2.1) is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}(f, \phi_n)\phi_n(x). \tag{2.7}$$

If $q(x)$ is such that $\lambda_1 = 0$, $\lambda_2 \gg 1$, and $\lambda_2 \leq \lambda_3 \leq \dots$, then, as $t \to \infty$, the series (2.7) is well approximated by its first term (see[7]):

$$u(x,t) = (f, \phi_1)\phi_1 + O(e^{-10t}), \quad t \to \infty. \tag{2.8}$$

Thus, our problem is solved if $q(x)$ has the following property:

$$|\phi_1(x)| \text{ decays as } \rho \text{ grows}, \quad \rho = (x_2^2 + x_3^2)^{1/2}. \tag{2.9}$$

Since the eigenfunction is normalized, $||\phi_1|| = 1$, this function will not tend to zero in a neighborhood of the line $\rho = 0$, so information can be transformed by the heat signals along the line $\rho = 0$, that is, along $s-$axis. Here we use the cylindrical coordinates: $x = (x_1, x_2, x_3) = (s, \rho, \theta)$, $s = x_1$, $\rho = (x_2^2 + x_3^2)^{1/2}$. In Section 2 the potential $q(x)$ will not depend on θ.

The technical part of the solution is the construction of $q(x) = c_S N(x)h(x)$ such that

$$\lambda_1 = 0, \quad \lambda_2 \gg 1; \quad |\phi_1(x)| \text{ decays as } \rho \text{ grows}. \tag{2.10}$$

Since the function $N(x) > 0$ and $h(x), \mathrm{Re} h \geq 0$ are at our disposal, any desirable $q, \mathrm{Re}\, q \geq 0$, can be obtained by embedding many small impedance particles in a given domain D.
In section 2.2, the method for finding such a potential $q(x)$ is presented. In section 2.3, the numerical method is presented for finding this q, and in section 2.4, the numerical results are presented. In section 2.5, another numerical method is presented based on the method described in section 2.4.

2.2 Construction of $q(x)$

Let $q(x) = p(\rho) + Q(s)$, where $s := x_1$, $\rho := (x_2^2 + x_3^2)^{1/2}$. Then the solution to (2.1) is $u = v(\rho)w(s)$, where

$$-v_m'' - \rho^{-1}v_m' + p(\rho)v_m = \mu_m v_m,\ 0 \leq \rho \leq R,\ |v_m(0)| < \infty,\ v_m(R) = 0; \quad (2.11)$$

$$-w_j'' + Q(s)w_j = \nu_j w_j,\quad 0 \leq s \leq L,\ w_j(0) = w_j(L) = 0. \quad (2.12)$$

Our task is to find $Q(s)$ such that $\nu_1 = 0$, $\nu_2 \gg 1$ and $p(\rho)$ such that $\mu_1 = 0, \mu_2 \gg 1$, and $|v_m(\rho)|$ decays as ρ grows.

We use the solution to inverse spectral problem for finding a potential which have the desired properties. The spectral function $\varrho(\lambda)$ of the Dirichlet differential operator $\ell w = -w'' + Q(s)w$ (see formula (2.12)) is defined by the formula

$$\varrho(\lambda) = \sum_{\nu_j < \lambda} \frac{1}{\alpha_j},$$

where α_j are normalizing constants. If $Q = 0$, then the eigenvalues of the corresponding

operator ℓ are $\nu_{j0} = \left(\frac{\pi j}{L}\right)^2$, $j = 1, 2, \dots$, the corresponding normalized eigenfunctions are $\sqrt{\frac{2}{L}} \sin \frac{j\pi x}{L}$, and the normalizing constants $\alpha_{j0} = \sqrt{\int_0^L \sin^2 \frac{j\pi x}{L} dx} = \sqrt{\frac{L}{2}}$. If Q is unknown, then the corresponding eigenfunctions are unknown and the role of the normalizing constants can play arbitrary positive numbers which have the right asymptotic. If $L = \pi$ then $\alpha_j = \sqrt{\frac{\pi}{2}} j\big(1 + O(\frac{1}{j})\big)$, $\sqrt{\nu_j} = j + \frac{c_1}{j} + O(j^{-2})$, and $w_j = \frac{\sin(jx)}{j} + O(j^{-2})$ as $j \to \infty$.

Let us recall the procedure, due to Gel'fand and Levitan (see[1],[2]) for finding the potential Q from the known spectral function. One defines the kernel

$$L(x, y) = \int_{-\infty}^{\infty} \frac{\sin(\sqrt{\lambda} x)}{\sqrt{\lambda}} \frac{\sin(\sqrt{\lambda} y)}{\sqrt{\lambda}} d(\varrho(\lambda) - \varrho_0(\lambda)),$$

where $\varrho(\lambda)$ is the spectral function of the operator ℓ with the potential $Q = Q(s)$, and $\varrho_0(\lambda)$ is the spectral function of the operator ℓ with the potential $Q = 0$ and the same boundary conditions.

Consider the Gel'fand-Levitan (GL) integral equation for the kernel $K(x, y)$:

$$K(x, y) + \int_0^x K(x, s) L(s, y) ds = -L(x, y), \qquad 0 \le y \le x.$$

The solution to this equation allows one to calculate the potential:

$$Q(x) = 2 \frac{dK(x, x)}{dx}.$$

From now on we set $L = \pi$. Then $\nu_{l0} = j^2$. Let $\nu_1 = 0, \nu_2 = 11, \nu_3 = 14, \nu_j = \nu_{j0}$ for $j > 3$.

Then the kernel $L(x,y)$ in the GL equation is defined as follows

$$L(x,y)=\frac{3xy}{\pi^3}+\frac{2}{\pi}\Big(\frac{\sin(\sqrt{\nu_2}x)}{\sqrt{\nu_2}}\frac{\sin(\sqrt{\nu_2}y)}{\sqrt{\nu_2}}+\frac{\sin(\sqrt{\nu_3}x)}{\sqrt{\nu_3}}\frac{\sin(\sqrt{\nu_3}y)}{\sqrt{\nu_3}}\Big)- \\ -\frac{2}{\pi}\Big(\sin x\sin y+\sin(2x)\sin(2y)+\sin(3x)\sin(3y)\Big), \quad (2.13)$$

where we set the normalizing constants $\alpha_j=\frac{\pi}{2}, j>3, \alpha_1=\frac{\pi^3}{3}$. The term xy is the value of the function $\frac{\sin\nu x}{\nu}\frac{\sin\nu y}{\nu}$ at $\nu=0$, and $\frac{\pi^3}{3}=||x||^2=\int_0^\pi x^2dx$.

Solve the GL equation:

$$K(s,\tau)+\int_0^s K(s,s')L(s',\tau)ds'=-L(s,\tau), \quad 0\le\tau\le s, \quad (2.14)$$

which is uniquely solvable (see [1], [2]). Equation (2.14) has finite-rank kernel and therefore can be solved analytically, being equivalent to a linear algebraic system. If $K(s,\tau)$ is found, then

$$Q(s)=2\frac{dK(s,s)}{ds}, \quad (2.15)$$

and this $Q(s)$ has the required properties: $\nu_1=0, \nu_2\gg 1, \nu_j\le\nu_{j+1}$.

Consider now the operator (2.11) for $v(\rho)$. We want to calculate $p(\rho)$ such that $\mu_1=0, \mu_2\gg 1, \mu_m\le\mu_{m+1}, |v_m(\rho)|$ decays as ρ grows.

We reduce this problem to the previous one that was solved. To do this, set $v=\frac{\psi}{\sqrt{\rho}}$. Then equation $-v''-\frac{1}{\rho}v'+p(\rho)v=\mu v$, is transformed to the equation

$$-\psi''-\frac{1}{4\rho^2}\psi+p(\rho)\psi=\mu\psi. \quad (2.16)$$

Let $p(\rho) = \frac{1}{4\rho^2} + Q(\rho)$, where $Q(\rho)$ is constructed above. Then equation (2.16) becomes

$$-\psi'' + Q(\rho)\psi = \mu\psi, \quad \psi(R) = 0, \quad \psi(0) = 0. \tag{2.17}$$

It has the desired eigenvalues $\mu_1 = 0, \mu_2 \gg 1, \mu_m \leq \mu_{m+1}$.

The eigenfunction $\phi_1(x) = v_1(\rho)w_1(s)$, where $v_1(\rho) = \frac{\psi_1(\rho)}{\sqrt{\rho}}$, decays as ρ grows, and the eigenvalues $\lambda_n = \mu_m + \nu_l$. Since $\mu_1 = \nu_1 = 0$ one has $\lambda_1 = 0$. Since $\nu_2 = 11, \mu_2 = 11, \lambda_2 = 11 \gg 1$. Thus, the desired potential is constructed: $q(x) = Q(s) + (\frac{1}{4\rho^2} + Q(\rho))$, where $Q(s)$ is given by formula (2.15).

This concludes the description of our procedure for the construction of q.

2.3 Numerical procedure

In section 2.3.1, a numerical method to construct $q(x)$ is presented. In section 2.3.2, a procedure is presented to check whether the constructed potential $q(x)$ is valid, by finding the eigenvalues of $q(x)$.

2.3.1 Numerical construction of q(x)

From the construction of $q(x)$ in Section 2.2, if one can construct $Q(s)$ then one gets $q(x) = Q(s) + \left(\frac{1}{4\rho^2} + Q(\rho)\right)$.

To construct $Q(s)$, one can use equation (2.15) and rewrite it as

$$Q(s) = 2\frac{dK(s,s)}{ds} = 2\left(\frac{\partial K(s,\tau)}{\partial s}\bigg|_{\tau=s} + \frac{\partial K(s,\tau)}{\partial \tau}\bigg|_{\tau=s}\right), \tag{2.18}$$

One can get $K_s := \frac{\partial K(s,\tau)}{\partial s}$ and $K_\tau := \frac{\partial K(s,\tau)}{\partial \tau}$ numerically by the following procedure.

The function $L(x,y)$ in (2.13) can be written as

$$L(x,y) = \sum_{j=1}^{6} a_j(x) b_j(y), \tag{2.19}$$

where $a_1(x) = \frac{3x}{\pi^3}$, $a_2(x) = \frac{2}{\pi}\frac{\sin(\sqrt{\nu_2}x)}{\nu_2}$, $a_3(x) = \frac{2}{\pi}\frac{\sin(\sqrt{\nu_3}x)}{\nu_3}$, $a_4(x) = -\frac{2}{\pi}\sin(x)$, $a_5(x) = -\frac{2}{\pi}\sin(2x)$, $a_6(x) = -\frac{2}{\pi}\sin(3x)$ and $b_1(x) = x$, $b_2(x) = \sin(\sqrt{\nu_2}x)$, $b_3(x) = \sin(\sqrt{\nu_3}x)$, $b_4(x) = \sin(x)$, $b_5(x) = \sin(2x)$, $b_6(x) = \sin(3x)$.

Then equation (2.14) becomes

$$K(s,\tau) + \sum_{j=1}^{6} b_j(\tau) \int_0^s K(s,s') a_j(s') ds' = -\sum_{j=1}^{6} a_j(s) b_j(\tau), \quad 0 \le \tau \le s. \tag{2.20}$$

Let $\psi_j(s) := \int_0^s K(s,s') a_j(s') ds'$, then equation (2.20) becomes

$$K(s,\tau) + \sum_{j=1}^{6} b_j(\tau) \psi_j(s) = -\sum_{j=1}^{6} a_j(s) b_j(\tau), \quad 0 \le \tau \le s. \tag{2.21}$$

Multiply (2.21) with $a_m(\tau), 1 \le m \le 6$ and integrate it with respect to τ to get

$$\psi_m(s) + \sum_{j=1}^{6} \left(\int_0^s b_j(\tau) a_m(\tau) d\tau \right) \psi_j(s) = -\sum_{j=1}^{6} a_j(s) \left(\int_0^s b_j(\tau) a_m(\tau) d\tau \right). \tag{2.22}$$

For a fix $s = s_0$, equation (2.22) is a 6×6 linear system which can be solved for $\psi_j(s_0), 1 \le j \le 6$. So, we can solve equation (2.22) to get $\psi_j(s)$ analytically and numerically. Differentiating equation (2.22), one can get a similar linear system to (2.22) and can get $\psi_j'(s)$

analytically and numerically from the linear system:

$$\psi'_m(s) + \sum_{j=1}^{6} \left(\int_0^s b_j(\tau) a_m(\tau) d\tau \right) \psi'_j(s) =$$
$$= -\sum_{j=1}^{6} \left(a_j(s) \left(\int_0^s b_j(\tau) a_m(\tau) d\tau \right) + (a_j(s) + \psi_j(s)) b_j(s) a_m(s) \right). \quad (2.23)$$

After finding $\psi_j(s)$ and $\psi'_j(s)$, one can find K_s and K_τ by differentiating equation (2.20) with respect to s and τ

$$K_s(s,\tau) = -\sum_{j=1}^{6} \left(a'_j(s) b_j(\tau) + b_j(\tau) \psi'_j(s) \right), \quad (2.24)$$

$$K_\tau(s,\tau) = -\sum_{j=1}^{6} \left(a_j(s) b'_j(\tau) + b'_j(\tau) \psi_j(s) \right). \quad (2.25)$$

From equation (2.24) - (2.25), one finds $K_s(s,\tau)|_{\tau=s}$ and $K_\tau(s,\tau)|_{\tau=s}$, and then finds $Q(s)$ numerically using equation (2.18).

2.3.2 Checking the eigenvalues of q(x)

To check whether the constructed potential $q(x)$ is the correct potential, one has to check whether the eigenvalues generated by the constructed potential $q(x)$ satisfy the conditions formulated in Section 2.2. It is sufficient to check the eigenvalues of $Q(s)$ are $\nu_1 = 0, \nu_2 = 11, \nu_3 = 14, \nu_j = j^2, j \geq 4$.

One can find numerically the eigenvalues of the Dirichlet operator $-\frac{d^2}{dx^2} + Q(s)$ on the interval $[0, \pi]$ by minimizing the following functional where u is taken from an N-dimensional subspace of functions u_N, N is a large integer:

$$\frac{\int_0^\pi (|u'|^2 + Q|u|^2) dx}{\int_0^\pi |u|^2 dx}. \quad (2.26)$$

We take

$$u_N = \sum_{n=1}^{N} c_n \varphi_n(x),$$

and $\varphi_n(x) := \sqrt{\frac{2}{\pi}} \sin(nx), 1 \le n \le N$. The minimization is taken over parameters c_n. The eigenvalues of the resulting matrix approximate the eigenvalues of the Dirichlet operator $-\frac{d^2}{dx^2} + Q(s)$ on the interval $[0, \pi]$. Finding minima of the functional (2.26) is equivalent to finding the minima of the quadratic form

$$\sum_{n=1}^{N} n^2 |c_n|^2 + \sum_{n,m=1}^{N} c_n c_m q_{nm}, \text{ under the restriction } \sum_{n=1}^{N} |c_n|^2 = 1, \tag{2.27}$$

where $q_{nm} := \frac{2}{\pi} \int_0^\pi Q(x) \sin(nx) \sin(mx) dx$, $q_{nm} = q_{mn}$. Minimizing (2.27) is equivalent to minimizing

$$f(c_1, \ldots, c_N) = \sum_{n=1}^{N} n^2 |c_n|^2 + \sum_{n,m=1}^{N} c_n c_m q_{nm} - \nu \left(\sum_{n=1}^{N} c_n^2 - 1 \right). \tag{2.28}$$

A necessary condition for a smooth function $f(c_1, \ldots, c_N)$ to have minima is $\frac{\partial f}{\partial c_n} = 0$. This leads to a linear system with respect to $c_1, \ldots, c_N$:

$$n^2 c_n + \sum_{m=1}^{N} q_{nm} c_m - \nu c_n = 0. \tag{2.29}$$

Linear system (2.29) can be written as

$$PC = \nu C, \tag{2.30}$$

where P is a symmetric matrix with entries $P_{nm} := n^2 \delta_{nm} + q_{nm}$, and C is a column vector $C = (c_1, \ldots, c_N)$. Then the eigenvalues of the Dirichlet operator $-\frac{d^2}{dx^2} + Q(s)$ on the interval $[0, \pi]$ are approximated by the eigenvalues of the matrix P. The approximation is the better

the larger is N.

2.3.3 Calculating q_{nm}

In section 2.3.2, one needs to calculate

$$q_{nm} = \frac{2}{\pi}\int_0^\pi Q(x)\sin(nx)\sin(mx)dx$$

to construct the matrix P. One can calculate the matrix P by using the formula $2\sin A\sin B = \cos(A-B)-\cos(A+B)$. So, one first calculates

$$\tilde{q}(k) := \frac{1}{\pi}\int_0^\pi Q(x)\cos(kx)dx, \quad 0 \le k \le 2N,$$

and then calculates q_{nm} by the formula

$$q_{nm} = \tilde{q}(|n-m|) - \tilde{q}(n+m).$$

2.4 Numerical results

Based on the numerical procedure in section 2.3, a computer algorithm/program is developed with the following main steps:

1. Partition the interval $[0,\pi]$ into $\mathcal{M}$ equal-distanced intervals with the endpoints x_i, $1 \le i \le \mathcal{M}+1$.

2. For each $x_i, 1 \le i \le \mathcal{M}+1$, one solves the linear systems (2.22) and (2.23) for $\psi_j(x_i)$ and $\psi_j'(x_i)$, $1 \le j \le 6$.

3. Find $K_s(x_i,x_i)$ and $K_\tau(x_i,x_i)$ by using equations (2.24) and (2.25).

4. Find $Q(x_i), 1 \le i \le \mathcal{M}+1$.

5. Construct the matrix P in equation (2.30) by calculating q_{nm} using the procedure in section 3.3, where $\tilde{q}(k)$ is calculated using the Riemann sum

$$\tilde{q}(k) = \frac{1}{\pi} \sum_{i=1}^{\mathcal{M}} \frac{Q(x_i)\cos(kx_i) + Q(x_{i+1})\cos(kx_{i+1})}{2}(x_{i+1} - x_i).$$

6. Find the eigenvalues of P using the Jacobi eigenvalue algorithm see, for example[8].

The above algorithm is run five times with $\mathcal{M} = 100, 150, 200, 250$, and 300. The constructed potentials $Q(s)$ are as in the following graph.

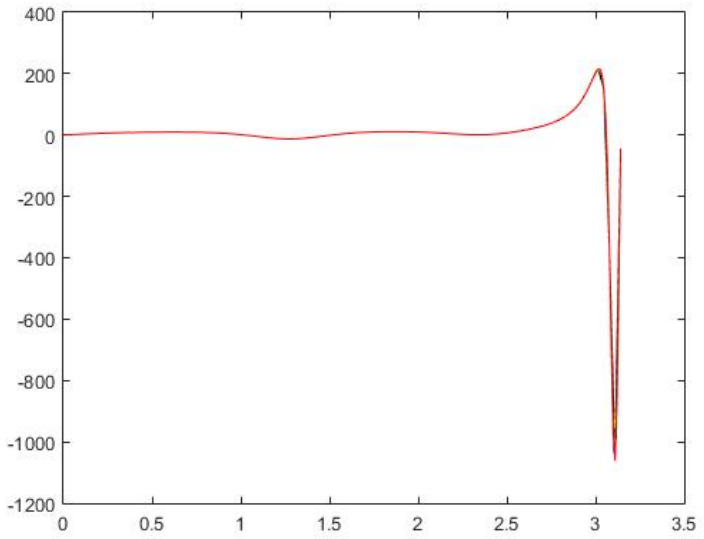

Figure 2.1: *Numerically constructed potentials* $Q(s)$.

The relative error of the eigenvalues is calculated by the following formula:

$$\delta_{\mathcal{M}} = \max_{1 \le j \le \mathcal{M}} \frac{|\nu_j^{(\mathcal{M})} - \nu_j|}{\nu_j}.$$

The following table gives the relative errors of the eigenvalues of the constructed potentials $Q(s)$ for $\mathcal{M} = 100, 150, 200, 250$, and 300.

$\mathcal{M}$	Relative errors $\delta_{\mathcal{M}}$
100	57.12%
150	5.01%
200	1.65%
250	0.67%
300	0.32%

Table 2.1: Relative errors for equal-distance small intervals.

From the above table, one can construct the potential $Q(s)$ with $\mathcal{M} = 250$ equal-distance small intervals and gets the relative error of the eigenvalues less than 1%. The above result also shows that the constructed $Q(s)$ is valid.

2.5 Another method to calculate eigenvalues

In the numerical results in section 2.4, one needs $\mathcal{M} = 250$ equal-distance small intervals to get the relative error less than 1%. From the graph of the constructed $Q(s)$, since $Q(s)$ is pretty steep close to π, one can improve the method in section 2.4 by distributing more equal-distanced intervals $\mathcal{M}_2$ in the interval $[\frac{9\pi}{10}, \pi]$ and less equal-distanced intervals $\mathcal{M}_1$ in the interval $[0, \frac{9\pi}{10}]$.

The following result is obtained

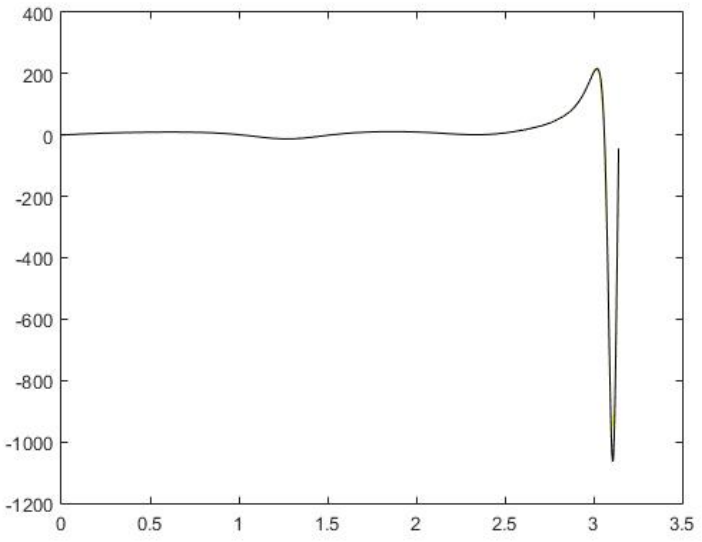

Figure 2.2: *Numerically constructed potential $Q(s)$ for non-uniform small intervals.*

$\mathcal{M}_1$	$\mathcal{M}_2$	Relative errors
50	50	4.68%
50	75	0.94%
50	100	0.23%

Table 2.2: Relative errors for non-uniform small intervals.

Remark 2.5.1. About calculating $\tilde{q}(k)$

In the constructed potential $Q(s)$ in Figure 2, let $Q_{max} := \max_{1\leq i\leq \mathcal{M}+1} Q(x_i)$, $x_{max} := \{x_i : Q(x_i) = Q_{max}\}$, $Q_{min} := \min_{1\leq i\leq \mathcal{M}+1} Q(x_i)$, and $x_{min} := \{x_i : Q(x_i) = Q_{min}\}$. The parts of the potential from x_{max} to x_{min} and from x_{min} to π look like straight lines and one may try to

calculate $\tilde{q}(k)$ by

$$\tilde{q}(k) = \frac{1}{\pi}\left(\int_0^{x_{max}} Q(x)\cos(kx)dx + \right.$$
$$\left. + \int_{x_{max}}^{x_{min}} Q(x)\cos(kx)dx + \int_{x_{min}}^{\pi} Q(x)\cos(kx)dx\right)$$
$$:= I_1 + I_2 + I_3. \quad (2.31)$$

If $Q(x)$ is a straight line from x_{max} to x_{min} and from x_{min} to π, one can calculate I_2 and I_3 analytically. However, this does not provide the desired numerical accuracy as the following numerical experiment shows.

In this experiment, P_1 is the numerical matrix in equation (2.30) obtained in the experiment described in Section 5 for $\mathcal{M}_1 = 50$ and $\mathcal{M}_2 = 75$, and P_2 is the numerical matrix obtained by considering $Q(x)$ as a straight line from x_{max} to x_{min} and from x_{min} to π. The relative error matrix E is calculated by $E_{nm} = \dfrac{|P_{1,nm} - P_{2,nm}|}{|P_{1,nm}|}$. Then $\min(E) = 0.37$ but $\max(E) = 510.17$. So, although the parts of the potential look like straight lines from x_{max} to x_{min} and from x_{min} to π, one cannot consider them as straight lines in numerical calculations.

Chapter 3

Dynamical System Method (DSM)

The Dynamical System Method (DSM) is developed by Professor A. G. Ramm. The literature of the DSM is huge, see, for example,[9],[12]. In this chapter, we just present the results that will be used in chapter 4.

3.1 Problem Statement

Consider an equation

$$Au = f, \tag{3.1}$$

where A is a linear operator in a Hilbert space H. Let f_δ be the "noise data", i.e., an element which satisfies the inequality $||f_\delta - f|| \leq \delta$ and which is given together with the "noise level" $\delta > 0$, while the exact data f is not known. The equation with the noisy data is:

$$Au_\delta = f_\delta. \tag{3.2}$$

Our goal is to find a stable solution to (3.2), which is an element u_δ such that

$$\lim_{\delta \to 0} ||u_\delta - y|| = 0,$$

where $Ay = f$ and y is the unique minimal-norm solution of the linear equation (3.1). The minimal-norm solution is the solution which is orthogonal to the null space N of A, $N = N(A) = \{u : Au = 0\}$.

3.2 DSM for solving equation (3.2)

Theorem 3.2.1. *The solution $u_\delta(t)$ to the following dynamical system*

$$\dot{u}_\delta = -u_\delta + T^{-1}_{a(t)} A^* f_\delta, \quad u(0) = 0, \tag{3.3}$$

satisfies

$$\lim_{\delta \to 0} ||u_\delta(t_\delta) - y|| = 0, \tag{3.4}$$

where $T_a := A^*A + aI$, $f = Ay, y \perp N$, and the following assumptions hold:

$$a(t) > 0 \text{ is a monotonically decaying continuous function}, \tag{3.5}$$

$$\lim_{t \to \infty} a(t) = 0, \tag{3.6}$$

$$\int_0^\infty a(t)dt = \infty, \tag{3.7}$$

$$\lim_{\delta \to 0} t_\delta = \infty, \quad \lim_{\delta \to 0} \frac{\delta}{\sqrt{a(t_\delta)}} = 0. \tag{3.8}$$

Proof. One has, by the triangle inequality,

$$\begin{aligned} ||u_\delta(t) - y|| &\leq ||u_\delta(t) - u(t)|| + ||u(t) - y|| \\ &:= I_1 + I_2, \end{aligned} \tag{3.9}$$

where $u(t)$ solves problem (3.3) with exact data

$$\dot{u} = -u + T^{-1}_{a(t)} A^* f, \quad u(0) = 0. \tag{3.10}$$

The solutions to problems (3.3) and (3.10) are of the form

$$u(t) = \int_0^t e^{-(t-s)} T_{a(s)}^{-1} A^* f \, ds, \tag{3.11}$$

$$u_\delta(t) = \int_0^t e^{-(t-s)} T_{a(s)}^{-1} A^* f_\delta \, ds. \tag{3.12}$$

So, one has

$$I_1 := ||u_\delta(t) - u(t)|| = \left|\left| \int_0^t e^{-(t-s)} T_{a(t)}^{-1} A^* (f_\delta - f) ds \right|\right|. \tag{3.13}$$

By Lemma 3.2.2 below, $||T_{a(t)}^{-1} A^*|| \leq \frac{1}{2\sqrt{a(t)}}$, so

$$\begin{aligned} I_1 &\leq \int_0^t e^{-(t-s)} \frac{||f_\delta - f||}{2\sqrt{a(s)}} ds \\ &\leq \int_0^t e^{-(t-s)} \frac{\delta}{2\sqrt{a(s)}} ds. \end{aligned} \tag{3.14}$$

By assumption (3.5), which implies $a(t) \leq a(s)$ if $t \geq s$,

$$\begin{aligned} I_1 &\leq \frac{\delta}{2\sqrt{a(t)}} \int_0^t e^{-(t-s)} ds \\ &= \frac{\delta}{2\sqrt{a(t)}} (1 - e^{-t}). \end{aligned} \tag{3.15}$$

So, if one chooses t_δ such that assumption (3.8) holds, then $I_1 \to 0$ as $\delta \to 0$.

Let us now estimate I_2 as $\delta \to 0$. Since $A^* A = T_a - aI$, $f = Ay$, (3.11) becomes

$$u(t) = \int_0^t e^{-(t-s)} y \, ds - \int_0^t e^{-(t-s)} a(s) T_{a(s)}^{-1} y \, ds. \tag{3.16}$$

Thus,

$$I_2 := ||u(t) - y|| \leq e^{-t} ||y|| + \int_0^t e^{-(t-s)} a(s) ||T_{a(s)}^{-1} y|| ds. \tag{3.17}$$

By Lemma 3.2.3 below, one has

$$\lim_{t\to\infty}\int_0^t e^{-(t-s)}a(s)T_{a(s)}^{-1}y\,ds=\lim_{t\to\infty}a(t)||T_{a(t)}^{-1}y||. \tag{3.18}$$

By the spectral theorem, one has

$$a^2||T_a^{-1}y||^2=\int_0^\infty\frac{a^2}{(s+a)^2}d(E_sy,y)\to 0,\quad a\to 0. \tag{3.19}$$

Assumption (3.6) and (3.17)-(3.19) imply $I_2\to 0$ as $t\to\infty$.

Therefore, theorem 3.2.1 is proved. □

Lemma 3.2.2. *For T_a, A, a as in Theorem 3.2.1, one has*

$$||T_a^{-1}A^*||\le\frac{1}{2\sqrt{a}}. \tag{3.20}$$

Proof. One has

$$\begin{aligned}A^*(AA^*+aI)&=(A^*A+aI)A^*\\ A^*Q_a&=T_aA^*,\quad Q_a:=Q+aI,\ Q:=AA^*\\ T_a^{-1}A^*&=A^*Q_a^{-1},\end{aligned} \tag{3.21}$$

since Q_a and T_a are boundedly invertible.So,

$$||T_a^{-1}A^*||=||A^*Q_a^{-1}||. \tag{3.22}$$

Also, by the polar decomposition,

$$A^*=U\sqrt{Q},$$

where U is a partial isometry,

$$\begin{aligned} ||T_a^{-1}A^*|| &= ||U\sqrt{Q}Q_a^{-1}|| \\ &\leq ||\sqrt{Q}Q_a^{-1}||, \quad \text{since } U \text{ is a partial isometry} \\ &= \sup_{s\geq 0}\frac{\sqrt{s}}{s+a} = \frac{1}{2\sqrt{a}}. \end{aligned} \tag{3.23}$$

Here we use the spectral theorem for the selfadjoint operator Q in the last equality, namely $||g(Q)|| = \sup_{s\geq 0}|g(s)|$.

Therefore, Lemma 3.2.2 is proved. □

Lemma 3.2.3. *Let $f(x)$ be a continuous function on $[0,\infty)$ such that $f(\infty) := \lim_{x\to\infty} f(x)$ exists, then*

$$\lim_{y\to\infty}\int_0^y e^{-(y-x)}f(x)dx = f(\infty). \tag{3.24}$$

Proof. One has

$$\begin{aligned} \int_0^y e^{-(y-x)}f(x)dx &= \int_0^c e^{-(y-x)}f(x)dx + \int_c^y e^{-(y-x)}f(x)dx \\ &:= I_1 + I_2, \end{aligned} \tag{3.25}$$

where $c < y$ is fixed. One has $\lim_{y\to\infty} I_1 = 0$ since $[0,c]$ is compact and $f(x)$ is continuous on $[0,\infty)$. Moreover, for any $\epsilon > 0$, one can choose sufficiently large $c < y$, such that $|f(c) - f(\infty)| < \epsilon$, and for such c, one has

$$\begin{aligned} \lim_{y\to\infty} I_2 &= f(\infty)\lim_{y\to\infty}\int_c^y e^{-(y-x)}dx \\ &= f(\infty)\lim_{y\to\infty}(1 - e^{c-y}) = f(\infty). \end{aligned} \tag{3.26}$$

So, Lemma 3.2.3 is proved. □

3.3 Iterative method to solve (3.2) and suggested choice of $a(t)$ and t_δ

To use the theory in section 3.2, one needs to choose the function $a(t)$ satisfying the assumptions (3.5)-(3.7). We suggest the following $a(t)$:

$$a(t) = \frac{1}{1+t}. \tag{3.27}$$

One can derive an iterative method to solve (3.2) from equation (3.11) by choosing a step function $\tilde{a}(t)$ approximating $a(t)$:

$$\tilde{a}(t) = a_n, \quad t_n \leq t < t_{n+1}, \tag{3.28}$$

where $a_n = a(t_n)$ and we choose t_n such that $t_{n+1} - t_n = h_n, h_n = q^n, q = 2$. From equation (3.11), one computes $u_n = u_\delta(t_n)$ by:

$$u_n = e^{-t_n} \sum_{i=1}^{n} (e^{t_i} - e^{t_{i-1}})(T + a_{i-1})^{-1} A^* f_\delta, \tag{3.29}$$

and one gets an iterative method for solving (3.2):

$$u_{n+1} = e^{-h_n} u_n + (1 - e^{-h_n})(T + a_n)^{-1} A^* f_\delta, \quad h_n = t_{n+1} - t_n. \tag{3.30}$$

To choose t_δ, we use a relaxed discrepancy principle (see[9]):

$$0.9\delta \leq ||Au_n - f_\delta|| \leq 1.001\delta. \tag{3.31}$$

One stops the iterative method (3.30) when condition (3.31) is satisfied.

Chapter 4

3D inverse scattering problem with non-over-determined data

The 3D inverse scattering problem with non-over-determined scattering data is considered. The data are the scattering amplitude $A(\beta, \alpha_0, k)$ for all $\beta \in S^2_\beta$, where S^2_β is an open subset of the unit sphere S^2 in $\mathbb{R}^3$, $\alpha_0 \in S^2$ is fixed, and $k \in (a, b), 0 \leq a < b$. In this paper, a numerical method is given for solving the inverse scattering problem with non-over-determined scattering data and the numerical results are presented. The content of this chapter is obtained from the paper[20] by the author and Professor A. G. Ramm.

4.1 Introduction

Let us first consider the direct scattering problem:

$$(\nabla^2 + k^2 - q(x))u = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{4.1}$$

$$u = e^{ik\alpha\cdot x} + v, \tag{4.2}$$

$$v = A(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \frac{x}{r} = \beta., \tag{4.3}$$

where $\alpha, \beta \in S^2$ are the directions of the incident wave and scattered wave correspondingly, S^2 is the unit sphere, $k^2 > 0$ is energy, $k > 0$ is a constant, $A(\beta, \alpha, k)$ is the scattering amplitude (scattering data), which can be measured, and $q(x) \in Q$, where Q is a set of C^1-smooth real-valued compactly supported functions, $q = 0$ for $\max_j |x| \geq R$, $R > 0$ is a constant. By D the support of q is denoted.
The direct scattering problem (4.1) - (4.3) has a unique solution (see, e.g., [2]).

Consider now the *inverse scattering problem*:
Find the potential $q(x) \in Q$ from the scattering data $A(\beta, \alpha, k)$.

The uniqueness of the inverse scattering problem with fixed-energy data ($k = k_0 > 0$ is fixed) is proved by A.G.Ramm[2]: $q(x) \in Q$ is uniquely determined by the scattering data $A(\beta, \alpha, k_0)$ for a fixed $k = k_0 > 0$ and all $\alpha, \beta \in S^2$. A.G.Ramm also gave a method for solving the inverse scattering problem with fixed-energy data and obtained an error estimate for the solution for exact data and also for noisy data, [2], Chapter 5.

In this paper, we give a numerical implementation of the method proposed by A. G. Ramm in [13] for solving the inverse scattering problem with non-over-determined data, that is, finding $q(x) \in Q$ from the scattering data $A(\beta, k) := A(\beta, \alpha_0, k)$ for a fixed $\alpha_0 \in S^2$, all $\beta \in S^2$, and all $k \in (a, b), 0 \leq a < b$. The basic uniqueness theorem for this problem is proved in [11]. In Section 4.2, the idea of this numerical method is described. This idea and the method described in Section 4.2 belong to A. G. Ramm. In Section 4.3, the DSM algorithm used to solve ill-posed linear system is described. In Section 4.4, the numerical procedure is presented and in Section 4.5, some examples of the numerical inversion are given.

The essential novel features of this inversion method are:

1) The data are non-over-determined. So these are minimal scattering data that allow one to uniquely recover the potential.

2) The inverse scattering problem is highly non-linear. Nevertheless, our method reduces the inversion to stable solution of linear algebraic system (4.8).

3) The numerical difficulty comes from the fact that system (4.8) is highly ill-conditioned.

4.2 Inversion method

Let $q \in Q$. The unique solution to (4.1) - (4.3) solves the integral equation:

$$u(x,k) = e^{ik\alpha_0 \cdot x} - \int_D g(x,y,k)q(y)u(y,k)dy, \quad g(x,y,k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{4.4}$$

$k > 0$ is a constant, $\alpha_0 \in S^2$ is fixed. This equation is uniquely solvable for u under our assumptions on q. Let $h(x,k) = q(x)u(x,k)$. Then (4.4) implies

$$h(x,k) = q(x)e^{ik\alpha_0 \cdot x} - q(x)\int_D g(x,y,k)h(y,k)dy. \tag{4.5}$$

Equations (4.3) and (4.4) yield the exact formula for the scattering amplitude:

$$-4\pi A(\beta,k) = \int_D e^{-ik\beta \cdot y}h(y,k)dy. \tag{4.6}$$

From equation (4.5) one derives the formula for $q(x)$ if $h(k,x)$ is found:

$$q(x) = h(x,k)[e^{ik\alpha_0 \cdot x} - \int_D g(x,y,k)h(y,k)dy]^{-1}. \tag{4.7}$$

The idea of our inversion method is: by the uniqueness theorem from[11], equation (4.6) has a unique solution $h(x,k)$. Solve numerically equation (4.6) for $h(x,k)$. To do this, discretize (4.6) and get a linear algebraic system for $h_{pm} := h(y_p, k_m)$. If h_{pm} are found, then $q(x_p)$

are found by formula (4.7), see also formula (4.9) below.

Let us partition D into P small cubes with volume $\Delta_p, 1 \leq p \leq P$. Let y_p be any point inside the small cube Δ_p. Choose P different numbers $k_m \in (a,b), 1 \leq m \leq P$, and P different vectors $\beta_j \in S^2, 1 \leq j \leq P$. Then discretize (4.6) and get a linear algebraic system for finding h_{pm}:

$$-4\pi A(\beta_j, k_m) = \sum_{p=1}^{P} e^{-ik_m \beta_j \cdot y_p} h_{pm} \Delta_p, \quad 1 \leq j, m \leq P, \tag{4.8}$$

where $h_{pm} := h(y_p, k_m)$. Solve the linear algebraic system (4.8) numerically, then use equation (4.7) to find the values of the unknown potential $q(x_p)$

$$q(x_p) = h_{pm} \left[e^{ik_m \alpha_0 \cdot x_p} - \sum_{p'=1, p' \neq p}^{P} g(x_p, y_{p'}, k_m) h_{p'm} \Delta_{p'} \right]^{-1}, \quad 1 \leq p \leq P. \tag{4.9}$$

Remark 1. *Note that the right hand side of* (4.9) *should not depend on* m *or* j*. This independence is an important requirement in the numerical solution of our inverse scattering problem, a compatibility condition for the data. This requirement is automatically satisfied for the limiting integral equation* (4.6)*, see formula* (4.7)*.*

The values of $q(y_p)$ essentially determine the unknown potential $q(x)$ if P is large and q is continuous. The potential q is unique by the uniqueness theorem in[11].

Note that one can choose β_j and k_m so that the determinant of the system (4.8) is not equal to zero (see[13]), so that the system (4.8) is uniquely solvable, but the numerical difficulty is unavoidable: the system (4.8) is very ill-conditioned because it comes from an integral equation (4.6) of the first kind with an analytic kernel. We use the dynamical system method (DSM) from[9] to solve stably the ill-conditioned system (4.8).

4.3 Dynamical System Method (DSM)

Equation (4.8) with noisy data is a linear algebraic system of the form

$$Mu_\delta = f_\delta \tag{4.10}$$

where M is the matrix of the size $P \times P$ and f_δ is the noisy data, that is, the noisy values of $-4\pi A(\beta_j, k_m)$ in equation (4.8), $||f_\delta - f|| = \delta$ for a fixed m and $1 \leq j \leq P$. To solve this ill-posed system we use the Dynamical Systems Method (DSM) in chapter 3:

$$u'_\delta(t) = -u_\delta(t) + (T + a(t))^{-1} M^* f_\delta, \quad u(0) = u_0, \tag{4.11}$$

where $T := M^*M$, M^* is the adjoint matrix and $a(t) > 0$ is a non-increasing function such that $a(t) \to 0$ as $t \to \infty$. Equation (4.11) has a unique solution

$$u_\delta(t) = u_0 e^{-t} + e^{-t} \int_0^t e^s (T + a(s))^{-1} M^* f_\delta ds. \tag{4.12}$$

To use the DSM method, we need to choose $a(t)$ and find a stopping time t_δ so that $u_\delta(t_\delta)$ approximates the solution of $Au = f$, so that $\lim_{\delta \to 0} ||u_\delta - u|| = 0$, where the norm is in $\mathbb{R}^P$.

Choice of $a(t)$ and t_δ:

In[9], necessary conditions for $a(t)$ are: $a(t)$ is a nonincreasing function and $\lim_{t\to\infty} a(t) = 0$. In our experiments, we choose $a(t) = \frac{1}{1+t}$. Consider a step function $\tilde{a}(t)$ approximating $a(t)$:

$$\tilde{a}(t) = a_n, \quad t_n \leq t < t_{n+1}, \tag{4.13}$$

where $a_n = a(t_n)$ and we choose t_n such that $t_{n+1} - t_n = h_n, h_n = q^n, q = 2$. From equation

(4.12), one computes $u_n = u(t_n)$ by:

$$u_n = u_0 e^{-t_n} + e^{-t_n} \sum_{i=1}^{n} (e^{t_i} - e^{t_{i-1}})(T + a_{i-1})^{-1} M^* f_\delta. \tag{4.14}$$

and one gets an iterative method to solve (4.12):

$$u_{n+1} = e^{-h_n} u_n + (1 - e^{-h_n})(T + a_n)^{-1} M^* f_\delta, \quad h_n = t_{n+1} - t_n. \tag{4.15}$$

To choose t_δ, we use a relaxed discrepancy principle (see[9]):

$$0.9\delta \leq ||Mu_n - f_\delta|| \leq 1.001\delta, \tag{4.16}$$

where the norm is the vector norm: $||\overrightarrow{x}|| := \sqrt{x_1^2 + \cdots + x_P^2}$, $\overrightarrow{x} := (x_1, \ldots, x_P) \in \mathbb{R}^P$.
One stops the iterative method (4.15) when condition (4.16) is satisfied.

4.4 Numerical procedure and results

In practice, one can measure the noisy scattering data experimentally. For our numerical experiments, we need to construct the noisy scattering data $A(\beta_j, k_m)$.

4.4.1 Constructing noisy scattering data

Given a potential $q(x)$, let us first construct the exact scattering data $\tilde{A}(\beta_j, k_m)$. We partition D into P small cubes and discretize equation (4.4) to get

$$u(x_p, k_m) = e^{ik_m \alpha_0 \cdot x_p} - \sum_{j=1}^{P} g(x_p, y_j, k_m) q(y_j) u(y_j, k_m) \Delta_j. \tag{4.17}$$

One solves this linear system for $u(x_p, k_m)$ assuming that $q(x)$ is known. Then the exact scattering data are calculated by the following formula:

$$\tilde{A}(\beta_j, k_m) = -\frac{1}{4\pi} \sum_{p=1}^{P} e^{-ik\beta_j \cdot y_p} q(y_p) u(y_p, k_m) \Delta_p. \tag{4.18}$$

Then one can randomly perturb each $\tilde{A}(\beta_j, k_m), 1 \leq j \leq P$ by $\tilde{\delta} = \text{const} > 0$ to get the noisy scattering data $A(\beta_j, k_m) = \tilde{A}(\beta_j, k_m) \pm \tilde{\delta}$, where the plus or minus sign is randomly choosen. Then $\delta := ||A(\beta_j, k) - \tilde{A}(\beta_j, k)|| := \sqrt{\sum_{j=1}^{P} |A(\beta_j, k) - \tilde{A}(\beta_j, k)|^2}$, for a fixed k, is the noise level of the data in equation (4.10).

4.4.2 Numerical procedure

The following steps are implemented in each experiment:

1. Choose D, α_0, P, $q(x)$, k_m, and $\tilde{\delta}$.

2. Use (4.18) to find the exact scattering data $\tilde{A}(\beta_j, k_m)$, $1 \leq j \leq P$. Try different values of k_m so that the determinant of the system in (4.8) is not zero. Let k be the found value of k_m.

3. Use the procedure in Section 4.4.1 to get the noisy scattering data $A(\beta_j, k), 1 \leq j \leq P$.

4. Solve the linear algebraic system (4.8) to get $h_{pm} = h(y_p, k_m)$. Here we use the DSM method in Section 3 with the noise

$$\delta := ||A(\beta_j, k_m) - \tilde{A}(\beta_j, k_m)|| := \sqrt{\sum_{j=1}^{P} |A(\beta_j, k_m) - \tilde{A}(\beta_j, k_m)|^2}, \quad 1 \leq j \leq P.$$

5. Calculate the potential $\tilde{q}(x_p)$ by formula (4.9):

$$\tilde{q}(x_p) = h_p \left[e^{ik\alpha_0 \cdot x_p} - \sum_{p'=1, p' \neq p}^{P} g(x_p, y_{p'}, k_m) h_{p'm} \Delta_{p'} \right]^{-1}, \quad 1 \leq p \leq P.$$

See Remark 1 below formula (4.9).

6. Find the relative error of the estimate of the potential:

$$\text{err} := \frac{||q(x) - \tilde{q}(x)||}{||q(x)||} = \sqrt{\frac{\sum_{p=1}^{P} |q(x_p) - \tilde{q}(x_p)|^2 \Delta_p}{\sum_{p=1}^{P} |q(x_p)|^2 \Delta_p}}. \tag{4.19}$$

4.5 Numerical results

In these experiments, we choose D to be the unit cube centered at the origin, with sides parallel to coordinate planes, $P = 1000$, $\alpha_0 = (1, 0, 0)$, and $50 \leq k_m \leq 100$ is chosen so that the determinant of the system in (4.8) is not zero. The condition number of A is $\sim 10^{13}$.
In the following graphs, the x-axis is the index of the collocation points (this index varies from 1 to 1000) and the y-axis is the value of the potential.

4.5.1 Constant potential with compact support

The inverse scattering problem with constant in D potential is used to test our inversion method.
In this experiment, we take $q(x) = 10$. The following results are obtained:

Constant potential $q(x) = 10$		
$\tilde{\delta}$	$\delta = \|\|A(\beta_j, k) - \tilde{A}(\beta_j, k)\|\|$	relative error
0.04	0.4348	0.0566
0.02	0.2174	0.0037
0.01	0.1087	0.00065

Table 4.1: Numerical results for constant potential $q(x) = 10$.

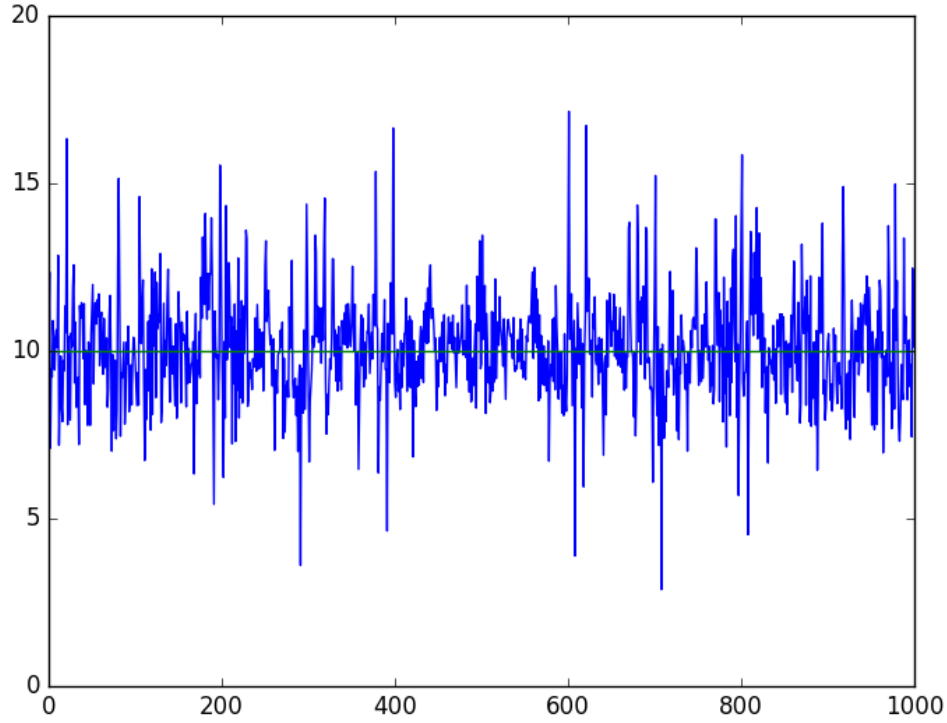

Figure 4.1: *Constructed potential vs original constant potential* $q(x) = 10$ *when* $\tilde{\delta} = 0.01$

In the figure above, the x-axis is the index p of the small cube, $1 \leq p \leq P$ and the y-axis is the value of the potential at the middle of the small cube p. The potential is a function $\mathbb{R}^3 \to \mathbb{R}$, but we do 1D plot here so one can see the point-wise comparison of the potentials.

4.5.2 Potential $q(x) = \frac{\exp(-|x|)}{|x|}$

In this experiment, we take $q(x) = \frac{\exp(-|x|)}{|x|}$. The following results are obtained:

Potential $q(x) = \frac{\exp(-\|x\|)}{\|x\|}$		
$\tilde{\delta}$	$\delta = \|\|A(\beta_j, k) - \tilde{A}(\beta_j, k)\|\|$	relative error
0.04	0.0806	0.1284
0.02	0.0403	0.0547
0.01	0.0201	0.0367

Table 4.2: Numerical results for the potential $q(x) = \frac{\exp(-|x|)}{|x|}$.

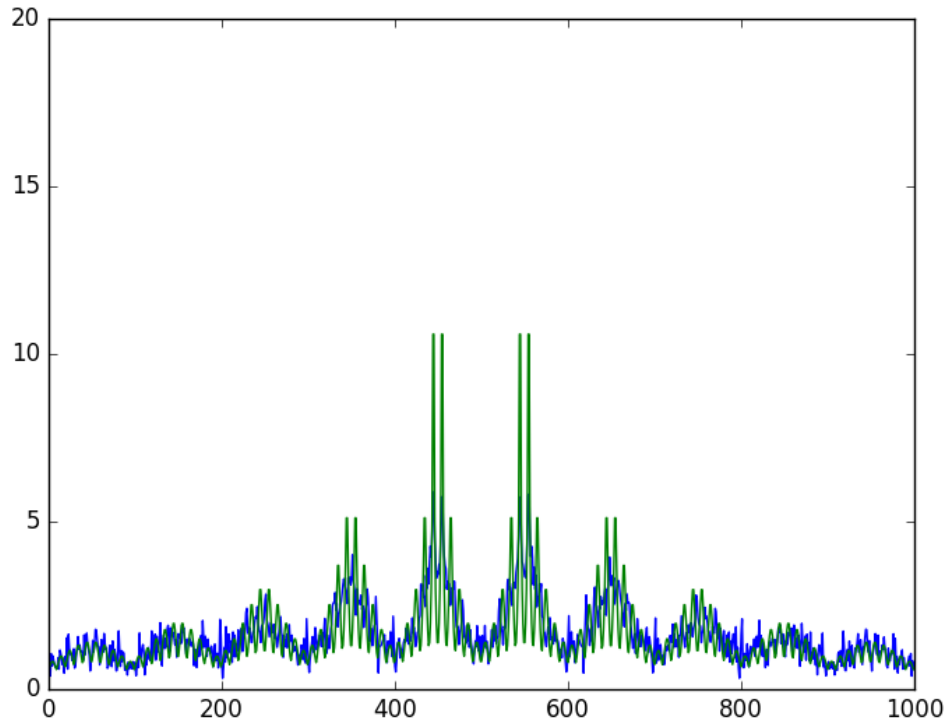

Figure 4.2: *Constructed potential vs original potential* $q(x) = \frac{\exp(-|x|)}{|x|}$ *when* $\tilde{\delta} = 0.01$

In the figure above, the x-axis is the index p of the small cube, $1 \leq p \leq P$ and the y-axis is the value of the potential at the middle of the small cube p. The potential is a function $\mathbb{R}^3 \to \mathbb{R}$, but we do 1D plot here so one can see the point-wise comparison of the potentials.

Bibliography

[1] B.M. Levitan, Inverse Sturm-Liouville problems, VNU Press, Utrecht, 1987.

[2] A.G. Ramm, Inverse problems, Springer, New York, 2005.

[3] A.G. Ramm, Creating materials in which heat propagates a long a line, Boll Union. Math. Ital. (BUMI), 8, N3, (2015), 165-168

[4] A.G. Ramm, Heat transfer in a medium in which many small particles are embedded, Math. Model. Nat. Phenom., 8, N1, (2013), 193-199.

[5] A. G. Ramm, Wave scattering by many small bodies: transmission boundary conditions, Reports on Math. Physics, 71, N3, (2013), 279-290.

[6] A.G. Ramm, Scattering of Acoustic and Electromagnetic Waves by Small Bodies of Arbitrary Shapes. Applications to Creating New Engineered Materials, Momentum Press, New York, 2013.

[7] A.G. Ramm, Heat transfer in a complex medium, Chaotic Modeling and Simulation (CMSIM), 1, (2016), 3-17.

[8] B.N. Parlett, The symmetric eigenvalue problem, Prentice-Hall, Englewood Cliffs, NJ, 1980

[9] A.G.Ramm, Dynamical systems method for solving operator equations, Elsevier, Amsterdam, 2007.

[10] A.G.Ramm, Uniqueness theorem for inverse scattering problem with non-overdetermined data, *J.Phys. A, FTC,* 43, (2010), 112001.

[11] A.G.Ramm, Uniqueness of the solution to inverse scattering problem with scattering data at a fixed direction of the incident wave, *J. Math. Phys.,* 52, 123506, (2011).

[12] A.G. Ramm, N. S. Hoang, Dynamical Systems Method and Applications. Theoretical Developments and Numerical Examples. Wiley, Hoboken, 2012, ISBN-13: 978-1-118-02428-7

[13] A.G.Ramm, A numerical method for solving 3D inverse scattering problem with non-over-determined data. J. Pure Appl. Math., 1, N1, (2017), 1-2.

[14] A. G. Ramm, A collocation method for solving integral equations, Internat. Journ. Comp. Sci and Math., 3, N2, (2009), 222-228.

[15] A. G. Ramm, Collocation method for solving some integral equations of estimation theory, Internat. Journ. of Pure and Appl. Math., 62, N1, (2010), 57-65.

[16] A. G. Ramm, S. Indratno, A collocation method for solving some integral equations in distributions, Journ. Comp. Appl. Math., 236, (2011), 1296-1313.

[17] A. G. Ramm, Inversion of the Laplace transform from the real axis, Inverse problems, 2, (1986), L55-59.

[18] A. G. Ramm, S. Indratno, Inversion of the Laplace transform from the real axis using an adaptive iterative method, Internat. Jour. Math. Math. Sci (IJMMS), Vol. 2009, Article 898195, 38 pages.

[19] A. G. Ramm, C. Van, Creating materials in which heat propagates along a line: theory and numerical results, Pure and Applied Functional Analysis, Vol. 2, N. 4, (2017), 639-648.

[20] A. G. Ramm, C. Van, A numerical algorithm for solving 3D inverse scattering problem with non-over-determined data, submitted.