Mini-Course on Limits and Sequences

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Abstract

The topic of limits and sequences run through the math syllabi from high school to graduate school. However rigorous proofs of this concept is not seen up until a students second year in college or even later. This text is aimed at presenting proofs of limits of sequences at a level accessible to high school and undergraduate students who are interested in learning such proofs. The "$\epsilon - N$" definition of the limit with proofs using this definition is presented in the text. We also look at properties of limits of sequences and their proofs as well as sequences without limits (that is sequences that diverge). We include some graphical representations of some sequences which can help one to determine whether a sequence will converge or diverge. Finally, the text contains a good number of exercises for the reader, some solved and others with hints to direct the reader in solving them.

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- 1. The sequence $\{x_n\}$ is given by the formula $x_n = \frac{2n+1}{n+3}$. Find
     (a) x_{10}
     (b) x_{25}
     (c) $x_{n+1} - x_n$

  2. The sequence $\{b_n\}$ is given by the formula $\{b_n\} = \frac{n}{3^n}$. Find
     (a) b_3,
     (b) b_5,
     (c) b_{n+1},
     (d) $\frac{b_{n+1}}{b_n}$.

  3. Find the first five terms of the sequence
     (a) $y_n = 1 + \frac{1}{2^n}$,
     (b) $x_n = 2 + \frac{1}{2^n}$,
     (c) $a_n = 2 + \frac{3}{n}$,
     (d) $b_n = \frac{2n+(-1)^n}{3n}$.

  4. For what values of n is $y_n > 200$ if $y_n = 2n - 5$?

  5. For what values of n is $y_n \leq 30$, if $y_n = 3n - 100$?

  6. Find the first five terms of the sequence $\{x_n\}$ and find the formula for the general term if
     (a) $x_1 = -10$, $x_{n+1} = x_n + 5$ for $n \geq 1$
     (b) $x_1 = 4$, $x_{n+1} = -x_n$ for $n \geq 1$

  7. Give an example of a whole number N such that for all $n > N$ the following inequality holds:
     (a) $(0.3)^n < 0.01$;
     (b) $(0.7)^n < 0.01$;
     (c) $(0.99)^n < 0.001$;
     (d) $(0.45)^n < 0.001$;
     (e) $(0.999)^n < 0.001$;

0.0.3 EXERCISES ON LIMITS OF SEQUENCES

1. Consider the sequence $\{x_n\} = \frac{1}{2n+1}$.
   (a) Compute the first several terms and guess the limit of the sequence.
   (b) Find the values of n such that the terms of the sequence lie in the range of zero within 0.1 (that is, for what values of n is $|x_n| \leq 0.1$?
   (c) Find the values of n such that the terms of the sequence lie in the range of zero within 0.01 (that is, for what values of n is $|x_n| \leq 0.01$) ?
2. (a) Compute the first few terms of the sequence $\{y_n\} = \frac{3n-1}{n}$ $(n = 1, 2, 3, ...)$.
   (b) Is it true that $\lim\limits_{n\to\infty}(x_n) = 3$? For what n is $|y_n - 3| < 0.01$?
3. Let $x_n = c$, where c is some constant number.
   Show that $\lim\limits_{n\to\infty}(x_n) = c$.
4. Prove
   (a) $\lim\limits_{n\to\infty} \frac{n+1}{n} = 1$,
   (b) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$,
   (c) $\lim\limits_{n\to\infty} \frac{5n-2}{n} = 5$,
   (d) $\lim\limits_{n\to\infty} \frac{3n-2}{2n} = 1.5$,
   (e) $\lim\limits_{n\to\infty} \frac{1}{n^2+1} = 0$,
   (f) $\lim\limits_{n\to\infty} 1 - \frac{1}{2^n} = 1$,

0.1 Introduction

This mini-course is addressed to undergraduate and high school students who would like to master rigorous proofs of limits of sequences based on the so-called "$\epsilon - N$" definition of a limit. While limits of most examples considered in standard calculus courses can be guessed intuitively, the accurate mathematical proof of such limits requires a careful reasoning and good understanding of the definition and properties of limits.
In this text we provide definitions and detailed proofs of some practice list problems.

Chapter 1

Sequences

1.1 Definition of Sequences

A sequence can informally be thought of as a list of ordered numbers, called elements of the sequence. Formally we define a sequence of (real) numbers as a function from the set of natural numbers ($\mathbb{N} = \{1, 2, 3, ...\}$) to the set of real numbers ($\mathbb{R}$). It assigns to each $n \in \mathbb{N}$ some real number x_n.

Sometimes people provide only the first few terms of a sequence as shown below with the assumption that the natural pattern to continue the sequence is clear:

(a) 1,2,3,4,...

(b) 1,$\frac{1}{2}$ $\frac{1}{4}$,$\frac{1}{8}$ $\frac{1}{16}$,... .

However in mathematics sequences are usually defined by formulas that would generate an element in the sequence when an integer is substituted into the formula. The two examples above could be the first terms of the sequences given by the respective general formulas

(a) $x_n = n$,

(b) $x_n = \frac{1}{2^n}$.

Observe that for any integer n substituted into the above formulas an element in the sequence is obtained. For $x_n = \frac{1}{2^n}$ by setting $n = 2$ we get the element $x_2 = \frac{1}{4}$.
Sequences however do not always have such nice formulas as we have seen above, some cannot even be represented by general formulas. Another common way to define a sequence is by recursion.

1.2 Recursive Process

This is a process that involves defining objects (in our case elements of sequences) in terms of preceding objects of the same kind.
It usually has an initial condition and a formula that reduces all other terms to the initial condition.
For example, given $x_0 = 0$ and $x_{n+1} = x_n + 1$, we compute the other terms of the sequence as follows.

$$x_1 = x_0 + 1 = 0 + 1 = 1,$$
$$x_2 = x_1 + 1 = 1 + 1 = 2,$$
$$x_3 = x_2 + 1 = 2 + 1 = 3.$$

And from this we can guess the general formula of the given recursive sequence to be $x_n = n$.

1.2.1 Exercise Set 1

1. The sequence $\{x_n\}$ is given by the formula $x_n = \frac{2n+1}{n+3}$. Find

   (a) x_{10},

   (b) x_{25},

   (c) $x_{n+1} - x_n$.

   Solution.
   As discussed earlier by substituting an integer n into the general formula we can determine the term in the sequence that corresponds to it, thus given

$$\begin{aligned} x_n &= \frac{2n+1}{n+3}, \\ \text{we find} \quad x_{10} &= \frac{2\cdot 10+1}{10+3}, \\ &= \frac{20+1}{13}, \\ &= \frac{21}{13}. \end{aligned}$$

Substituting $n = 25$ in the right side of the equation, just as we did for x_{10}, gives the solution for $x_{25} = \frac{51}{28}$. Now to solve for $x_{n+1} - x_n$ the first step would be to write an expression for x_{n+1}, this is done by replacing n in the formula for x_n with $n+1$. Doing this gives

$$\begin{aligned} x_{n+1} &= \frac{2(n+1)+1}{(n+1)+3}, \\ &= \frac{2n+2+1}{n+4}, \\ &= \frac{2n+3}{n+4}. \end{aligned}$$

Thus in solving for $x_{n+1} - x_n$ we end up with

$$\begin{aligned} x_{n+1} - x_n = \frac{2n+3}{n+4} - \frac{2n+1}{n+3} &= \frac{(n+3)(2n+3)-(n+4)(2n+1)}{(n+4)(n+3)}, \\ &= \frac{2n^2+3n+6n+9-2n^2-n-8n-4}{(n+4)(n+3)}, \\ &= \frac{5}{(n+4)(n+3)}. \end{aligned}$$

2. The sequence $\{b_n\}$ is given by the formula $\{b_n\} = \frac{n}{3^n}$. Find

   (a) b_3,

   (b) b_5,

   (c) b_{n+1},

(d) $\frac{b_{n+1}}{b_n}$.

Solution.

2(a) This follows the procedure in problem $1(c)$, which gives

$$b_3 = \frac{3}{3^3} = \frac{1}{3^2} = \frac{1}{9}.$$

2(b) $b_5 = \frac{5}{243}$.

2(c) This is similar to problem $1(c)$, we get

$$b_{n+1} = \frac{n+1}{3^{n+1}}.$$

2(d) Dividing our two terms and simplifying we get

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}}, \\ &= \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n}, \\ &= \frac{n+1}{3n}. \end{aligned}$$

3. Find the first five terms of the sequence

(a) $y_n = 1 + \frac{1}{2^n}$,

(b) $x_n = 2 + \frac{1}{2^n}$,

(c) $a_n = 2 + \frac{3}{n}$,

(d) $b_n = \frac{2n+(-1)^n}{3n}$.

Solution.
3(a)

$$\begin{aligned}
y_1 &= 1+\frac{1}{2^1}=1+\frac{1}{2}=\frac{3}{2},\\
y_2 &= 1+\frac{1}{2^2}=1+\frac{1}{4}=\frac{5}{4},\\
y_3 &= 1+\frac{1}{2^3}=1+\frac{1}{8}=\frac{9}{8},\\
y_4 &= 1+\frac{1}{2^4}=1+\frac{1}{16}=\frac{17}{16},\\
y_5 &= 1+\frac{1}{2^5}=1+\frac{1}{32}=\frac{33}{32}.
\end{aligned}$$

(b) $\frac{5}{2}$, $\frac{9}{4}$, $\frac{17}{8}$, $\frac{33}{16}$, $\frac{65}{32}$,
(c) 5, $\frac{7}{2}$, 3, $\frac{11}{4}$, $\frac{13}{5}$,
(d) $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{9}$, $\frac{2}{3}$, $\frac{9}{15}$.

4. For what values of n is $y_n > 200$ if $y_n = 2n-5$?

5. For what values of n is $y_n \le 30$, if $y_n = 3n-100$?

Solution.

4. We show how to solve problem 4, problem 5 is solved similarly.
We want to find n which satisfies

$$y_n = 2n-5 > 200.$$

Solve the inequality

$$2n-5>200$$

for the value of n. This gives

$$2n-5>200, \quad \text{which implies} \quad 2n > 200+5. \quad \text{Hence} \quad n > \frac{205}{2} = 102.5.$$

Thus for all integer values of $n > 102.5$, we have $y_n > 200$. Hence $n \geq 103$.

5. This holds for $n \leq 43$

6. Find the first five terms of the sequence $\{x_n\}$ and find the formula for the general term if

   (a) $x_1 = -10$, $x_{n+1} = x_n + 5$ for $n \geq 1$.

   (b) $x_1 = 4$, $x_{n+1} = -x_n$ for $n \geq 1$.

Solution.

As discussed in the introduction, some sequences can be expressed by a general mathematical formula, with which we can find a particular term in the sequence by inserting the value of a particular integer n.

$6(a)$: Just as in question 3, we find the first five terms of the sequence, only in this case we are already given the first term which is $x_1 = -10$. Using the first term and the formula of the $(n+1)st$ term we derive the second term through to the fifth term as described below.

$$x_{n+1} = x_n + 5.$$

If we substitute $n = 1$ into x_{n+1} we get

$$x_{1+1} = x_1 + 5 \quad \text{which implies} \quad x_2 = x_1 + 5.$$

We are given $x_1 = -10$, substitute this into the expression for x_2. Thus

$$x_2 = -10 + 5 = -5.$$

Next, if we substitute $n = 2$ into x_{n+1} we get

$$x_{2+1} = x_2 + 5 \quad \text{which implies} \quad x_3 = x_2 + 5.$$

We determined from our previous calculations that $x_2 = -5$. Substituting x_2 into the expression for x_3 gives

$$x_3 = -5 + 5 = 0.$$

Following this procedure for $n = 3$ and $n = 4$,

$$x_{3+1} = x_4 = x_3 + 5, \quad \text{which implies} \quad x_4 = 0 + 5 = 5;$$

$$x_{4+1} = x_5 = x_4 + 5, \quad \text{which implies} \quad x_5 = 5 + 5 = 10.$$

So the first five terms of the sequence for question $6(a)$ are -10, -5, 0, 5, 10.

Guess: For $n \geq 1$

$$x_n = -10 + 5(n - 1).$$

We show our guess satisfies the base step and recursive formula.
Consider x_1. Then for $n = 1$ our guess gives

$$x_1 = -10 + 5(1 - 1) = -10.$$

So the initial condition for $x_1 = -10$ is satisfied.
We now check $x_{n+1} = x_n + 5$. From our guess we solve for x_{n+1} to get

$$x_{n+1} = -10 + 5(n + 1 - 1), \quad \text{which implies} \quad x_{n+1} = -10 + 5n.$$

Finally, we insert our guess into the right hand side of the recursive formula:

$$\begin{aligned} x_{n+1} &= x_n + 5, \\ x_{n+1} &= -10 + 5(n - 1) + 5, \\ x_{n+1} &= -10 + 5n, \end{aligned}$$

This proves that our guess satisfies both the initial condition and the recursive formula.

$6(b)$: The first five terms are: 4, -4, 4, -4, 4
and the formula for the general term is

$$x_n = (-1)^{n+1} \cdot 4 \quad (n \geq 1).$$

7. Give an example of a whole number N such that for all $n > N$ the following inequality holds:

   (a) $(0.3)^n < 0.01$;
   (b) $(0.7)^n < 0.01$;
   (c) $(0.99)^n < 0.001$;
   (d) $(0.45)^n < 0.001$;
   (e) $(0.999)^n < 0.001$.

   We state a proposition that will help solve the above problems.The proof of the proposition is given in Appendix 1

Proposition 1. *If* $|a| < 1$ *then for* $n > 0$, $|a|^n < \frac{1}{n} \cdot \frac{1}{1-|a|}$

Solution.

We solve problem $7(a)$, problems $7(b)$-$7(d)$ follow in the same manner.

(a) $|0.3| = 0.3 < 1$, so we can apply Proposition 1. We get

$$|0.3|^n < \frac{1}{n} \cdot \frac{1}{1-0.3},$$
$$\text{which implies} \quad \frac{1}{n} \cdot \frac{1}{0.7} < 0.01.$$

Taking reciprocals gives

$$0.7n > \frac{1}{0.01},$$
$$\text{which implies} \quad n > \frac{1}{0.7 \cdot 0.01},$$
$$\text{thus} \quad n > 142.8571.$$

So the whole number N that satisfies the inequality is $N = 143$.

7(b) $N = 334$;

7(c) $N = 100,001$;

7(d) $N = 1819$;

7(e) $N = 1,000,001$.

Chapter 2

Limits of Sequences

2.1 Definition of a Limit of a Sequence

We will now discuss the limit of a sequence $\{x_n\}$ and how to compute it, if it exists.
The limit of a given sequence $\{x_n\}$, if it exists, can be roughly described as the numerical value which the sequence tends to as we take n closer to infinity. If the limit of a sequence exists, we say that the sequence is *convergent*, otherwise we say that the sequence is *divergent*.

Formal Definition:
We say that L is the *limit* of the sequence $\{x_n\}$ when $n \to \infty$, if for any real number $\epsilon > 0$ there exists a whole number N such that for any $n > N$ we have $|x_n - L| < \epsilon$. In this case we write $\lim\limits_{n\to\infty} \{x_n\} = L$ or $\{x_n\} \to L$ when $n \to \infty$

For example, it is intuitively clear that the sequence $\{x_n\} = \frac{1}{2^n}$ converges to 0 as $n \to \infty$: write the first several terms of the sequence and observe that as n increases, the terms in the sequence keeps getting smaller (approaching 0). However, the rigorous proof that $\lim\limits_{n\to\infty} \frac{1}{2^n} = 0$ must be performed using the definition provided above. We will use the definition of the limit to show the number L is the limit of a given sequence $\{x_n\}$ in the examples that follow.

Example : Show using the definition that

$$\lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$

Solution:

To solve the problem, we first try to determine the value of N that depends on ϵ. Then the actual flow of the proof consists of the steps of this preliminary analysis performed in the "backward" direction. Fix $\epsilon > 0$, then to satisfy the condition $|a_n - 1| < \epsilon$ we must have

$$\left|\frac{n-1}{n+1} - 1\right| < \epsilon, \quad \text{for all} \quad n > N.$$

Thus we have:

$$\frac{n-1}{n+1} - 1 = \frac{n-1-n-1}{n+1} = \frac{-2}{n+1},$$

giving us

$$\left|\frac{n-1}{n+1} - 1\right| = \frac{2}{n+1}.$$

Our definition implies that $a_n \to 1$ for $n > N$ we must have

$$\frac{2}{n+1} < \epsilon, \quad \text{which implies} \quad \frac{2}{\epsilon} < n+1, \quad \text{or} \quad \frac{2}{\epsilon} - 1 < n$$

Therefore, we must take N to be the smallest positive integer such that $N > \frac{2}{\epsilon} - 1$. For example, taking $\epsilon = 0.1$ gives

$$\frac{2}{\epsilon} - 1 = \frac{2}{0.1} - 1 = 20 - 1 = 19.$$

So $N > \frac{2}{0.1} - 1 = 19$ and we can set $N = 20$. Similarly for $\epsilon = 0.001$, we can set $N = 2000$. So for any value of ϵ we can find N to satisfy our condition $|a_n - 1| < \epsilon$ for all $n > N$.

Now we are ready to write the argument of the actual proof that

$$\lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$

Indeed, suppose we choose some $\epsilon > 0$. Let us take $N-$ the smallest whole number that is greater than $\frac{2}{\epsilon} - 1$. That is

$$N > \frac{2}{\epsilon} - 1.$$

Then if $n > N$, we have

$$n > N > \frac{2}{\epsilon} - 1.$$

This implies $\frac{2}{n+1} < \epsilon$. Since

$$\left| \frac{n-1}{n+1} - 1 \right| = \frac{2}{n+1}$$

we get

$$\left| \frac{n-1}{n+1} - 1 \right| < \epsilon \quad \forall \quad n > N.$$

Thus, by the definition of the limit,

$$\lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$

□

2.1.1 Exercise Set 2

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1. Consider the sequence $\{x_n\} = \frac{1}{2n+1}$.

   (a) Compute the first several terms and guess the limit of the sequence.

   (b) Find the values of n such that the terms of the sequence lie in the range of zero within 0.1 (that is, for what values of n is $|x_n| \leq 0.1$?)

   (c) Find the values of n such that the terms of the sequence lie in the range of zero within 0.01 (that is, for what values of n is $|x_n| \leq 0.01$)?)

Solution: The first few terms of the sequence are

$$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \frac{1}{17}, \frac{1}{19}, \ldots \quad .$$

Observe that the terms in the sequence keep decreasing. Our guess is that $\lim_{n\to\infty} (x_n) = 0$.

(b) We will find values of n for which terms x_n are in the range $|x_n| \le \epsilon$ for general ϵ. Then for any values of n we use that result to find such a particular ϵ.
Consider $N > 0$ such that for $\epsilon > 0$

$$\left|\frac{1}{2N+1}\right| \le \epsilon.$$

Then

$$\left|\frac{1}{2N+1}\right| = \frac{1}{2N+1} \le \epsilon, \quad \text{which is equivallent to} \quad 2N+1 \ge \frac{1}{\epsilon},$$

hence we have

$$N \ge \frac{1-\epsilon}{2\epsilon}.$$

Now taking $\epsilon = 0.1$, we get

$$N \ge \frac{1-0.1}{2\cdot 0.1} = 4.5 \quad .$$

So for $\epsilon = 0.1$, and $N = 5$, taking $n \ge N$ gives

$$\left|\frac{1}{2n+1}\right| \le 0.1 \quad .$$

Thus the solution to problem (b) is all n such that $n > 5$. By a similar computation the solution to problem (c) is all n such that $n > 50$.

Remark:
Note that our work in (b) and (c) proves that $\lim_{n\to\infty} (x_n) = 0$.

2. (a) Compute the first few terms of the sequence $\{y_n\} = \frac{3n-1}{n}$ $(n = 1, 2, 3, \ldots)$.

(b) Is it true that $\lim_{n\to\infty}(x_n) = 3$? For what n is $|y_n - 3| < 0.01$?

Solution:

(a) Some terms of the sequence are:

$$2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5}, \frac{17}{6}, \frac{20}{7}, ... \quad .$$

(b) For $\lim_{n\to\infty}(y_n) = 3$ to hold, for any $\epsilon > 0$ there should exist $N > 0$ such that for all $n > N$ we have

$$\left|\frac{3n-1}{n} - 3\right| < \epsilon, \quad \text{which can be modified to}$$

$$\left|\frac{3n-1-3n}{n}\right| = \left|\frac{-1}{n}\right| = \frac{1}{n} < \epsilon.$$

Equivalently written as $\frac{1}{\epsilon} < n$.

$$\text{So for any} \quad \epsilon > 0 \quad \text{taking} \quad N > \frac{1}{\epsilon}$$

and for any $n > N$,

$$\text{since} \left|\frac{3n-1}{n} - 3\right| = \frac{1}{n}, \quad \text{we have} \quad \left|\frac{3n-1}{n} - 3\right| < \epsilon.$$

Hence it follows that $\lim_{n\to\infty}\left|\frac{3n-1}{n}\right| = 3$ by the definition. The graph of the sequence is shown below.
For the final part we want to find n such that

$$\left|\frac{3n-1}{n} - 3\right| < 0.01.$$

Replace ϵ in the above proof with 0.01. Thus we get

$$n > \frac{1}{0.01} = 100.$$

So for $n = 101, 102, 103, ...$ one has $|y_n - 3| < 0.01$.

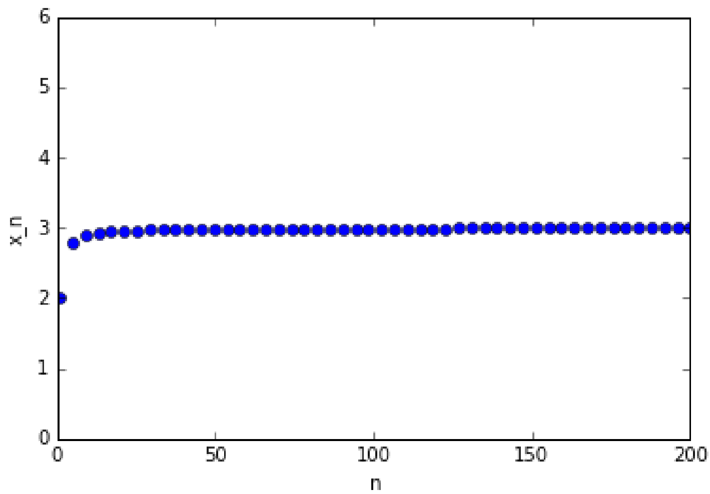

Figure 2.1: Graph showing that $\lim_{n\to\infty} \left|\frac{3n-1}{n}\right| = 3$

3. Let $x_n = c$, where c is some constant number. Show that $\lim_{n\to\infty}(x_n) = c$.

   Solution:
   By definition of the limit of a sequence, for any $\epsilon > 0$ there should exist $N > 0$, such that for every $n > N$ we have $|c - c| = 0 < \epsilon$. However, this holds for any value of N. Thus, by our definition it follows trivially that $\lim_{n\to\infty} c = c$.

4. Prove

   (a) $\lim_{n\to\infty} \frac{n+1}{n} = 1$,

   (b) $\lim_{n\to\infty} \frac{1}{n} = 0$,

   (c) $\lim_{n\to\infty} \frac{5n-2}{n} = 5$,

   (d) $\lim_{n\to\infty} \frac{3n-2}{2n} = 1.5$,

   (e) $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$,

(f) $\lim_{n\to\infty} 1 - \frac{1}{2^n} = 1.$

Solution:
We solve problem $4(a)$, problems $4(c)$ through $4(f)$ follow similarly.

4(a) To solve for the limit we first rewrite the sequence in an equivalent form by dividing the numerator by the highest power of n in the sequence. Thus

$$x_n = \frac{n+1}{n} = 1 \equiv \frac{1+\frac{1}{n}}{1} = 1 + \frac{1}{n}.$$

Now we prove using the definition of the limit that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$.
Take $\epsilon > 0$. Write

$$\left|1+\frac{1}{n}-1\right| < \epsilon, \quad \text{which is equivalent to} \quad \left|\frac{1}{n}\right| = \frac{1}{n} < \epsilon.$$

So for any $\epsilon > 0$ taking $N > \frac{1}{\epsilon}$ gives that for any $n > N > \frac{1}{\epsilon}$, one has
$\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| < \frac{1}{N} < \epsilon.$ Hence by the definition of the limit

$$\lim_{n\to\infty} \frac{n+1}{n} = 1.$$

The graph of the sequence is shown below.

4(b) Take $\epsilon > 0$ we want to show that

$$\left|\frac{1}{n} - 0\right| < \epsilon,$$

this gives

$$\left|\frac{1}{n}\right| < \epsilon,$$

which we from problem $4(a)$ holds for any $n > N > \frac{1}{\epsilon}$. Hence by the definition of the limit $\lim_{n\to\infty} \frac{1}{n} = 0$.

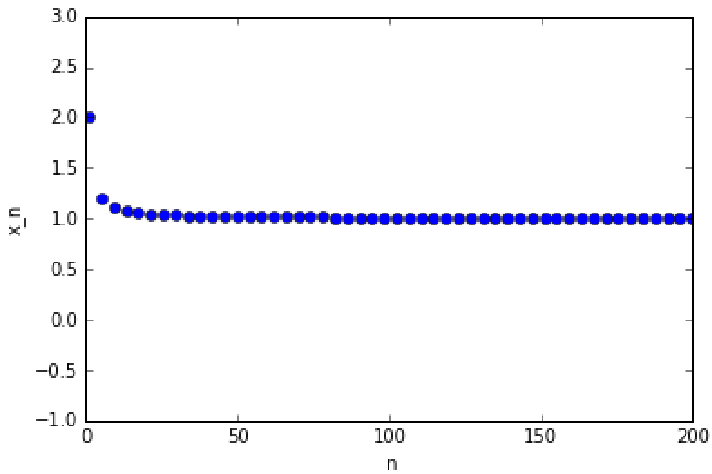

Figure 2.2: Graph showing that $\lim\limits_{n\to\infty} \left|1 + \frac{1}{n}\right| = 1$

4(c) Hint: Rewrite $\frac{5n-2}{n}$ as $5 - \frac{2}{n}$.

4(d) Hint: Rewrite $\frac{3n-2}{2n}$ as $\frac{3}{2} - \frac{1}{n}$.

4(e) Hint: $n > \sqrt{\frac{1}{\epsilon} - 1}$.

4(f) Hint: Since $\left|\frac{1}{2}\right| < 1$, we can use Proposition 1 on page 9 to find N that will satisfy $\left|\frac{1}{2}\right|^n < \epsilon$. Let $\epsilon > 0$, take $N = \frac{2}{\epsilon}$ then for any $n > N$,

$$\left|1 - \frac{1}{2^n} - 1\right| = \left|-\frac{1}{2}\right|^n =< \frac{2}{n} < \epsilon.$$

Hence it follows from the definition of the limit that

$$\lim_{n\to\infty} 1 - \frac{1}{2^n} = 1.$$

2.2 Properties of Limits

In practice one can find limits of many sequences by reducing them to simple ones. This is based on the properties of limits which we prove in this section. Let c be any constant and suppose $\{x_n\}$ and $\{y_n\}$ are two sequences such that their limits exist and $\lim_{n\to\infty}(x_n) = L$ and $\lim_{n\to\infty}(y_n) = K$. Then the following properties hold.

1. Multiplication by Constant:
   $\lim_{n\to\infty}(c \cdot x_n) = c \cdot \lim_{n\to\infty}(x_n) = c \cdot L.$

2. Sum and Difference rule:
   $\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty}(x_n) \pm \lim_{n\to\infty}(y_n) = L \pm K.$

3. Product rule:
   $\lim_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty}(x_n) \cdot \lim_{n\to\infty}(y_n) = LK.$

4. Quotient rule:
   Suppose $K \neq 0$ and $y_n \neq 0$ for all n. Then $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{\lim_{n\to\infty}(x_n)}{\lim_{n\to\infty}(y_n)} = \frac{L}{K}.$

5. Power rule:
   Suppose m is a whole number and $m \geq 2$.
   Then $\lim_{n\to\infty}(x_n)^m = (\lim_{n\to\infty} x_n^m) = L^m.$

6. Square root property :
   Suppose $x_n \geq 0$ for all $n \geq 0$ and $L \geq 0$.
   Then $\lim_{n\to\infty}\sqrt{(x_n)} = \sqrt{\lim_{n\to\infty}(x_n)} = \sqrt{L}.$

Remark: In general, it can be proved that if the condition $x_n \geq 0$ for all n holds, then it implies that the limit $L = \lim_{n\to\infty} x_n \geq 0$ if it exists.

Proof of the Properties

We will now prove the above properties using the definition of the limit. We first state important facts which we will use in some of the proofs.

1. Multiplication by Constant :
   Suppose $\lim_{n\to\infty}(x_n) = L$. If $c = 0$ then it holds trivially since
   $$\lim_{n\to\infty} 0 \cdot x_n = \lim_{n\to\infty} 0.$$
   But since 0 is a constant it follows from Exercise 2 problem 3 that $\lim_{n\to\infty} 0 \cdot x_n = 0 = 0 \cdot x_n$. Assume $c \neq 0$. Now fix $\epsilon > 0$. Then by the definition of the limit there exist an $N > 0$ such that for $n > N$
   $$|x_n - L| < \frac{\epsilon}{|c|}.$$
   Then for all $n > N$
   $$|cx_n - cL| = |c|\,|x_n - L| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon.$$
   So $\lim_{n\to\infty} cx_n = cL$ by definition.

2. Sum and Difference rule :
   We prove the sum rule, the proof of the difference rule follows similarly. We are given that $\lim_{n\to\infty}(x_n) = L$ and $\lim_{n\to\infty}(y_n) = K$.
   Let $\epsilon > 0$, by the definition of the limit there is $N_1 > 0$ such that for all $n > N_1$
   $$|x_n - L| < \frac{\epsilon}{2},$$
   and there is an $N_2 > 0$ such that for all $n > N_2$
   $$|y_n - K| < \frac{\epsilon}{2}.$$
   We want to show that given $\epsilon > 0$ there is $N > 0$ such that for all $n > N$
   $$|(x_n + y_n) - (L + K)| < \epsilon.$$
   To prove the sum and difference rules, we need the following proposition.

Proposition 2. *Let x and y be two real numbers. Then the triangle holds:*

$$|x + y| \leq |x| + |y|\,.$$

Choose $N = \max(N_1, N_2)$ (i.e the largest of N_1 and N_2). Then for all $n > N$

$$\begin{aligned} |x_n + y_n - L - K| &= |(x_n - L) + (y_n - K)| \\ &\leq |x_n - L| + |y_n - K| \quad \text{by triangle inequality} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

3. Product rule :
We are given that $\lim_{n\to\infty}(x_n) = L$ and $\lim_{n\to\infty}(y_n) = K$.
Let $\epsilon > 0$, by the definition of the limit this implies that there is $N_1 > 0$ such that for all $n > N_1$,

$$|x_n - L| < \sqrt{\epsilon}.$$

And there is $N_2 > 0$ such that for all $n > N_2$,

$$|y_n - K| < \sqrt{\epsilon}.$$

Choosing $N = \max(N_1, N_2)$ for $n > N$ gives

$$\begin{aligned} |(x_n - L) \cdot (y_n - K) - 0| &= |(x_n - L)| \cdot |(y_n - K)| \\ &< \sqrt{\epsilon}\sqrt{\epsilon} \\ &= \epsilon. \end{aligned}$$

Hence we have that

$$\lim_{n\to\infty}[(x_n - L)(y_n - K)] = 0 \quad .$$

Next consider

$$(x_n - L) \cdot (y_n - K) = x_n y_n - K x_n - L y_n + KL,$$

From which we get

$$(x_n - L) \cdot (y_n - K) + Kx_n + Ly_n - KL = x_n y_n.$$

Finally, taking

$$\begin{aligned}
\lim_{n\to\infty}(x_n y_n) &= \lim_{n\to\infty}[(x_n - L)\cdot(y_n - K) + x_n K + y_n L - KL] \\
&= \lim_{n\to\infty}[(x_n - L)\cdot(y_n - K)] + \lim_{n\to\infty}(x_n K) + \lim_{n\to\infty}(y_n L) + \lim_{n\to\infty}(-KL) \\
&= 0 + LK + KL - KL \\
&= LK
\end{aligned}$$

$\square$

4. Quotient rule :
For this we first prove that $\lim_{n\to\infty}\frac{1}{y_n} = \frac{1}{K}$ and apply rule 3 to show that $\lim_{n\to\infty}(x_n)\cdot\lim_{n\to\infty}\left(\frac{1}{y_n}\right) = \frac{L}{K}$. Let $\epsilon > 0$. We want to show that there is $N > 0$ such that for all $n > N$

$$\left|\frac{1}{y_n} - \frac{1}{K}\right| < \epsilon.$$

Since $\lim_{n\to\infty}(y_n) = K$, by the definition of the limit we know

$$|y_n - K| < \tilde{\epsilon}. \quad \text{for big enough} \quad n$$

Now consider

$$\begin{aligned}
|K| &= |K - y_n + y_n| \\
&\leq |K - y_n| + |y_n| \quad \text{by the triangle inequality} \\
&< \tilde{\epsilon} + |y_n|.
\end{aligned}$$

Thus we have

$$|K| - \tilde{\epsilon} < |y_n| \quad \text{which implies} \quad \frac{1}{|y_n|} < \frac{1}{|K| - \tilde{\epsilon}}, \quad \text{which gives a bound for} \quad \frac{1}{|y_n|}.$$

Hence for big enough n,

$$\begin{aligned}\left|\frac{1}{y_n}-\frac{1}{K}\right| &= \left|\frac{K-y_n}{Ky_n}\right| \\ &= \left|\frac{1}{Ky_n}\right|\cdot|K-y_n| \\ &< \frac{|K-y_n|}{|K|\cdot(|K|-\tilde{\epsilon})} \quad \text{using the bound above} \\ &< \frac{\tilde{\epsilon}}{|K|\cdot(|K|-\tilde{\epsilon})}\end{aligned} \tag{2.1}$$

Thus we proved that for any $\tilde{\epsilon}>0$ there exists N such that for $n>N$

$$\left|\frac{1}{y_n}-\frac{1}{k}\right| < \frac{\tilde{\epsilon}}{|K|(|K|-\tilde{\epsilon})}.$$

But we need to show that for any $\epsilon>0$

$$\left|\frac{1}{y_n}-\frac{1}{K}\right| < \epsilon \quad \text{for big enough} \quad n.$$

Let us see how $\tilde{\epsilon}$ and ϵ could be related; assume that

$$\frac{\tilde{\epsilon}}{|K|(|K|-\tilde{\epsilon})} = \epsilon$$

Then

$$\begin{aligned}\tilde{\epsilon} &= \epsilon|K|^2-|K|\epsilon\tilde{\epsilon} \\ \tilde{\epsilon} &= \frac{\epsilon|K|^2}{(1+|K|\epsilon)}\end{aligned}$$

This analysis allows us to proceed to fix $\epsilon>0$. Consider $\tilde{\epsilon}=\frac{\epsilon|K|^2}{1+|K|\epsilon}$. Note that $\tilde{\epsilon}>0$. Then there exists N such that for all $n>N$.

$$\left|\frac{1}{y_n}-\frac{1}{K}\right| < \frac{\tilde{\epsilon}}{|K|(|K|-\tilde{\epsilon})}.$$

But

$$\begin{aligned}
\frac{\tilde{\epsilon}}{|K|(|K|-\tilde{\epsilon})} &= \frac{\epsilon|K|^2}{|K|(1+|K|\epsilon)(|K|-\frac{\epsilon|K|^2}{1+|K|\epsilon})} \\
&= \frac{\epsilon|K|^2(1+|K|\epsilon)}{|K|(1+|K|\epsilon)|K|} \\
&= \epsilon,
\end{aligned}$$

So

$$\left|\frac{1}{y_n}-\frac{1}{K}\right| < \epsilon \quad \text{for} \quad n > N$$

Hence by our definition $\lim\limits_{n\to\infty} \frac{1}{y_n} = \frac{1}{K}$. Finally using rule 3 we get

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) &= \lim_{n\to\infty}(x_n)\cdot\lim_{n\to\infty}\left(\frac{1}{y_n}\right) \\
&= L\cdot\frac{1}{K} \\
&= \frac{L}{K}
\end{aligned}$$

$\square$

5. Power rule :
   To prove this we use a method of proof known as induction. We first consider a base case where our hypothesis holds. In this case $m = 2$ is our base case since it breaks it down to product property. Thus

$$\lim_{n\to\infty}(x_n^2) = \lim_{n\to\infty}(x_n\cdot x_n) = \lim_{n\to\infty}(x_n)\cdot\lim_{n\to\infty}(x_n) = L\cdot L = L^2.$$

   Now assume the power rule holds for $m-1$. Then we have

$$\lim_{n\to\infty}(x_n^{m-1}) = L^{m-1}.$$

   Thus applying rule 3 we have

$$\begin{aligned}
\lim_{n\to\infty}(x_n^m) &= \lim_{n\to\infty}(x_n^{m-1}\cdot x_n) \\
&= \lim_{n\to\infty}(x_n^{m-1})\lim_{n\to\infty}(x_n) \\
&= L^{m-1}\cdot L \\
&= L^m.
\end{aligned}$$

$\square$

6. Root rule:
   Since $\lim_{n\to\infty} x_n = L$, then by the definition of the limit, for every $\epsilon > 0$, there is $N > 0$ such that for all $n > N$ we have
   $$|x_n - L| < \epsilon\sqrt{L}.$$
   Notice that
   $$|\sqrt{x_n} + \sqrt{L}| \geq \sqrt{L} \quad \text{so} \quad \frac{1}{|\sqrt{x_n} + \sqrt{L}|} \leq \frac{1}{\sqrt{L}}.$$
   Thus we have
   $$\begin{aligned} |\sqrt{x_n} - \sqrt{L}| &= \left|\frac{(\sqrt{x_n} - \sqrt{L}) \cdot (\sqrt{x_n} + \sqrt{L})}{(\sqrt{x_n} + \sqrt{L})}\right| \\ &= \frac{|x_n - L|}{|\sqrt{x_n} + \sqrt{L}|} \\ &\leq \frac{|x_n - L|}{\sqrt{L}} \\ &< \frac{\epsilon\sqrt{L}}{\sqrt{L}} = \epsilon. \end{aligned}$$
   Thus by the definition of the limit $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}$.

Examples

We now look at some examples where we make use of the properties discussed above.

1. Prove that $\lim_{n\to\infty} \frac{n+1}{n} = 1$.

   Solution: First rewrite $\frac{n+1}{n}$ as $1 + \frac{1}{n}$ and take the limit of the new expression. This gives
   $$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right).$$

Applying rules 1 and 2 we get

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{n}\right) &= \lim_{n\to\infty} 1 + \lim_{n\to\infty} (\frac{1}{n}) \\
&= 1+0 \\
&= 1.
\end{aligned}$$

2. Prove that $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$.

   Solution: We rewrite the sequence $\frac{1}{n^2}$ as $\left(\frac{1}{n}\cdot\frac{1}{n}\right)$. Applying rule 3 on page 18 then gives

$$\lim_{n\to\infty}\frac{1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n} = 0\cdot 0 = 0.$$

3. Prove that $\lim\limits_{n\to\infty} \frac{4n^2+n-1}{n^2+n} = 4$.
   We prove the limit in two ways. Solution 1 :
   Note that the above sequence is the sum of $\frac{n-1}{n+1}$ and $\frac{3n-1}{n}$. That is

$$\frac{n-1}{n+1}+\frac{3n-1}{n} = \frac{n^2-n+3n^2+3n-n-1}{n^2+n} = \frac{4n^2+n-1}{n^2+n}.$$

   Applying rule 2 on page 18 to $\left(\frac{n-1}{n+1}+\frac{3n-1}{n+1}\right)$ gives

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{n-1}{n+1}+\frac{3n-1}{n}\right) &= \lim_{n\to\infty}\frac{n-1}{n+1} + \lim_{n\to\infty}\frac{3n-1}{n} \\
&= \lim_{n\to\inf}\frac{1-\frac{1}{n}}{1+\frac{1}{n}} + \lim_{n\to\infty}\frac{3n-1}{n} \\
&= 1+3 \\
&= 4
\end{aligned}$$

   Solution 2 : Assuming we had no idea that $\frac{4n^2+n-1}{n^2+n}$ is the sum of $\frac{n-1}{n+1}$ and $\frac{3n-1}{n}$, then using the limit definition to prove the above might be quiet cumbersome. However, rewriting the sequence and applying the rules, we get

$$
\begin{aligned}
\lim_{n\to\infty} \frac{4n^2+n-1}{n^2+n} &= \lim_{n\to\infty} \frac{4+\frac{1}{n}-\frac{1}{n^2}}{1+\frac{1}{n}} \\
&= \frac{\lim_{n\to\infty}\left(4+\frac{1}{n}-\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)} \quad \text{by rule 4 page 18} \\
&= \frac{\lim_{n\to\infty} 4+\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{1}{n^2}}{\lim_{n\to\infty} 1+\lim_{n\to\infty}\frac{1}{n}} \quad \text{by rule 2 page 18} \\
&= \frac{4+0+0}{1+0} \quad \text{using rule 1 and the known limits} \\
&= 4.
\end{aligned}
$$

2.3 Non-Existing Limits

We defined the limit of the sequence to be the numerical value L that the sequence tends to as n approaches infinity. This value may exist or not. Here we look at two of such examples and observe how they behave graphically.

1. Consider the sequence $\{x_n\} = (-1)^n$. Investigate the existence of the limit of this sequence as $n \to \infty$.

   Solution:
   We start by writing the first few terms of the sequence.
   $\{x_n\} = 1, -1, 1, -1, 1, -1, 1, -1, ... \quad (n = 1, 2, 3, ...) \quad .$

   Thus the terms of the sequence takes values between these two numbers infinitely many times as $n \to \infty$ (thus it fails to tend to a particular value L). In this case we say the sequence oscillates and hence it diverges.

   Next we would show that this sequence indeed fails to satisfy the definition of the limit. Recall that by definition, a sequence x_n converges to the limit L, if for every $\epsilon > 0$ there exists $N > 0$, such that for all $n \geq N$ we have $|x_n - L| < \epsilon$. Since the inequality holds for every ϵ in the definition of the limit, it will be enough to show that the definition

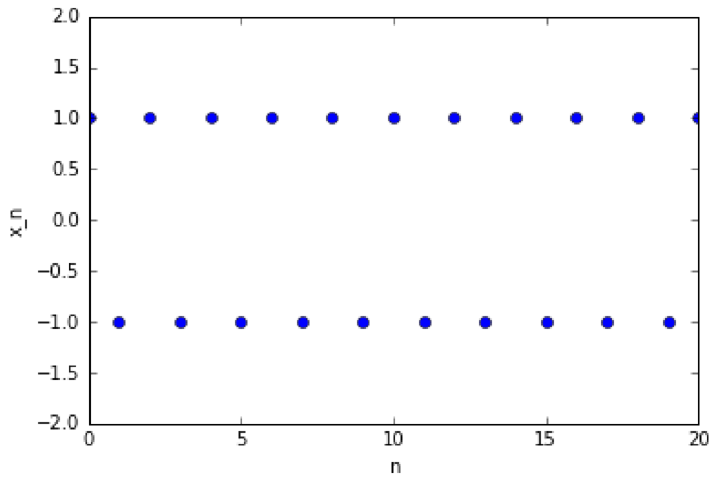

Figure 2.3: Graph of the sequence $\{x_n\} = (-1)^n$

is violated if we can find $\epsilon > 0$ such that for any $N > 0$ there exist $n \geq N$ such that $|x_n - L| \geq \epsilon$.
Now fix $\epsilon = \frac{1}{2}$. It is clear that if the limit of $x_n = (-1)^n$ exists, it would either be $L = 1$ or $L = -1$.

$$|-1-1| = |-2| = 2 > \frac{1}{2} \quad \text{for infinitely many values of } n \quad (n = 1, 3, 5, ...).$$

Hence, it fails to satisfy the conditions of the definition of the limit.

Suppose the limit for the sequence is -1. Then we have

$$|1-(-1)| = |2| = 2 > \frac{1}{2} \quad \text{for infinitely many values of } n \quad (n = 0, 2, 4, 6, ...).$$

2. Consider the sequence $\{x_n\} = n^2$. investigate the existence of the limit of this sequence as $n \to \infty$.

Solution: The first few terms of the sequence are

$$\{x_n\} = 0, 1, 4, 9, 16, ... \quad .$$

We observe that the terms keep increasing without upper bound as we increase n. This indicates that this sequence does not posses a finite limit.

Now we show that no number indeed satisfies the definition of the limit of this sequence.
Suppose L is the limit of the sequence. Then for all $\epsilon > 0$ there exists $N > 0$ such that for every $n > N$ we have $|n^2 - L| < \epsilon$. Now observe that for $n > L + 1$

$$|n^2 - L| > |L + 1 - L| = 1.$$

Taking $\epsilon = \frac{1}{2}$ we get $|n^2 - L| > 1 > \frac{1}{2}$, which violates the definition of the limit.

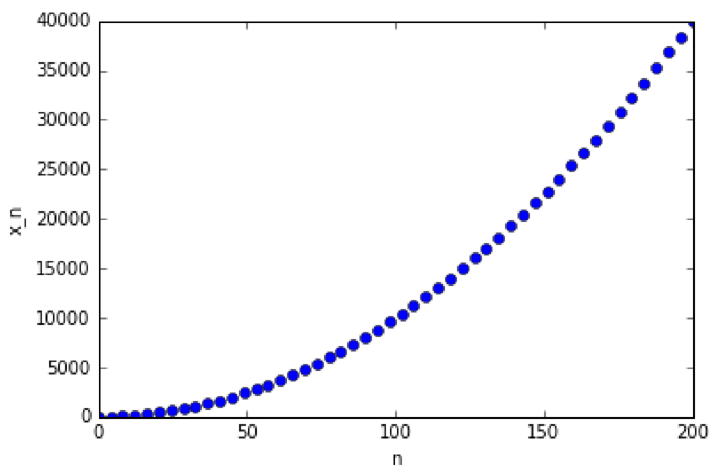

Figure 2.4: Graph of the sequence $\{x_n\} = n^2$

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APPENDIX

Appendix A

Proof of Proposition 1.

Proposition 1 : If $|a| < 1$ then for $n > 0$, $|a|^n < \frac{1}{n} \cdot \frac{1}{1-|a|}$.

Proof :
Suppose $a = 0$. Then we have

$$|0|^n = 0 < \frac{1}{n} \cdot \frac{1}{1-0} = \frac{1}{n},$$

thus the inequality holds for $a = 0$.
Now suppose $a > 0$ and consider the following expression:

$$(1 + a + a^2 + ... + a^{n-1}) \cdot (1 - a).$$

Expanding the above product we get

$$\begin{aligned} (1 + a + a^2 + ... + a^{n-1}) \cdot (1 - a) &= (1 + a + a^2 + ... + a^{n-1}) - (a + a^2 + a^3 + ... + a^{n-1} + a^n) \\ &= 1 - a^n. \end{aligned}$$

So for $|a| < 1$ we have

$$\begin{aligned} n|a|^n &= |a|^n + |a|^n + |a|^n + ... + |a|^n \quad (\text{ sum}) \quad n \quad \text{times} \\ &\leq 1 + |a| + |a|^2 + ... + |a|^{n-1} \\ &= \frac{1 - |a|^n}{1 - |a|} \\ &< \frac{1}{1 - |a|}. \end{aligned}$$

Combining the inequalities above we get

$$\begin{aligned} n|a|^n &< \frac{1}{1-|a|} \quad \text{which implies} \\ |a|^n &< \frac{1}{n} \cdot \frac{1}{1-|a|}. \end{aligned}$$

$\square$

Appendix B

Proof of Proposition 2

Proposition 2: Let x and y be two real numbers then the triangle inequality holds

$$|x+y| \leq |x| + |y| \,.$$

Proof :
By definition

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Thus $|x| \geq \pm x$. This implies that

$$x + y \leq |x| + y \leq |x| + |y|$$

and

$$-x - y \leq |x| - y \leq |x| + |y|$$

Combining the two results above we get

$$|x+y| \leq |x| + |y|.$$

$\square$