Cohomological Hall algebras and 2 Calabi-Yau categories

by

Jie Ren

B.S., Beijing Normal University, 2008

M.S., Beijing Normal University, 2011

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
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Department of Mathematics
College of Arts and Sciences

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Abstract

The motivic Donaldson-Thomas theory of 2-dimensional Calabi-Yau categories can be induced from the theory of 3-dimensional Calabi-Yau categories via dimensional reduction. The cohomological Hall algebra is one approach to the motivic Donaldson-Thomas invariants. Given an arbitrary quiver one can construct a double quiver, which induces the preprojective algebra. This corresponds to a 2-dimensional Calabi-Yau category. One can further construct a triple quiver with potential, which gives rise to a 3-dimensional Calabi-Yau category. The critical cohomological Hall algebra (critical COHA for short) is defined for a quiver with potential. Via the dimensional reduction we obtain the cohomological Hall algebra (COHA for short) of the preprojective algebra. We prove that a subalgebra of this COHA consists of a semicanonical basis, thus is related to the generalized quantum groups. Another approach is motivic Hall algebra, from which an integration map to the quantum torus is constructed. Furthermore, a conjecture concerning some invariants of 2-dimensional Calabi-Yau categories is made.

We investigate the correspondence between the A_∞-equivalent classes of ind-constructible 2-dimensional Calabi-Yau categories with a collection of generators and a certain type of quivers. This implies that such an ind-constructible category can be canonically reconstructed from its full subcategory consisting of the collection of generators.

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Chapter 1

Introduction

The 2-dimensional Calabi-Yau categories and the 3-dimensional Calabi-Yau categories are related via dimensional reduction. The framework of 3-dimensional Calabi-Yau categories is appropriate for the theory of motivic Donaldson-Thomas invariants. The dimensional reduction from 3-dimensional Calabi-Yau categories (3CY categories for short) to 2-dimensional Calabi-Yau categories (2CY categories for short) gives rise to the corresponding theory of the latter. It is natural to ask about the meaning of the objects arising as a result of such dimensional reduction, and the relation between motivic Donaldson-Thomas theory and some invariants of 2CY categories, e.g. Kac polynomials. M. Kontsevich and Y. Soibelman established two theories to produce motivic Donaldson-Thomas invariants. One is using Cohomological Hall algebra (see [43]), and the other is via motivic Hall algebra (see [42]). Both theories give rise to the $\mathbb{Z}$-valued invariants as limits of motivic Donaldson-Thomas invariants. (See, e.g., [2] and [25–30] for $\mathbb{Z}$-valued Donaldson-Thomas invariants.)

Cohomological Hall algebra was first introduced in [43]. With a certain class of 4-dimensional quantum theories with $N = 2$ spacetime supersymmetry one should be able to associate the algebra of BPS states. The cohomological Hall algebra (COHA for short) is a rigorous mathematical definition related to this algebra. It can be defined in a wide class of situations including quivers with potential. A quiver with potential gives rise to an ind-constructible 3CY category. The heart of its t-structure consists of finite-dimensional representations of the quiver which are critical points of the potential. The 3CY category is related to 2CY category in the following way. Given a quiver Q with the set of vertices $I = \{i, \dots, n\}$ and arrows Ω, the associated preprojective algebra Π_Q gives rise to a 2CY category. This category can be upgraded to a 3CY category by constructing a "triple" quiver $\widehat{Q}$ with a cubic potential W. The critical cohomological Hall algebra (critical COHA for short) of $(\widehat{Q}, W)$ induces the COHA for Π_Q.

The 2CY categories, interesting on their own, are analogues of Kac-Moody algebras. Thus it is interesting to relate the COHA of 2CY categories to the generalized quantum groups. In particular, we give a construction of the semicanonical basis of a subalgebra of the COHA of the preprojective algebra Π_Q associated to a quiver Q.

The critical COHA of an arbitrary quiver with potential (Q, W) (not necessarily coming from the above upgrading), which is denoted by $\mathcal{H}_{Q,W}$, is an associative algebra structure on the dual space of the compactly supported critical cohomology (in other words, compactly supported equivariant cohomology with coefficients in the sheaf of vanishing cycles) of the stack $\mathcal{M}_Q$ of the representations of Q. The stack $\mathcal{M}_Q = \coprod_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} \mathcal{M}_{Q,\gamma} = \coprod_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} \mathbf{M}_{Q,\gamma}/\mathbf{G}_\gamma$ is a countable union of quotient stacks over dimension vectors γ, so

$$\mathcal{H}^{crit}_{Q,W} = \bigoplus_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} H_c^{\bullet, crit}(\mathcal{M}_{Q,\gamma}, W_\gamma)^\vee = \bigoplus_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} H^{\bullet, crit}_{c, \mathbf{G}_\gamma}(\mathbf{M}_{Q,\gamma}, W_\gamma)^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma,\gamma)}$$

is a direct sum of the dual of compactly supported critical equivariant cohomology. To define the multiplication, consider the diagram of stacks

$$\mathbf{M}_{Q,\gamma_1}/\mathbf{G}_{\gamma_1} \times \mathbf{M}_{Q,\gamma_2}/\mathbf{G}_{\gamma_2} \xleftarrow{p_1} \mathbf{M}_{Q,\gamma_1,\gamma_1}/\mathbf{G}_{\gamma_1,\gamma_2} \xrightarrow{p_2} \mathbf{M}_{Q,\gamma/}\mathbf{G}_\gamma$$

for $\gamma = \gamma_1 + \gamma_2$, where $\mathbf{M}_{Q,\gamma_1,\gamma_1}/\mathbf{G}_{\gamma_1,\gamma_2}$ is the stack parametrizing pairs (E, F) such that E is a representation of dimension γ and F is its subrepresentation of dimension γ_1. The multiplication is defined as $(p_2)_* p_1^*$ (see details in [43, Sec. 7.6]).

There is another version of the (non critical) COHA using rapid decay cohomology ([43, Sec. 4]). For the special case of a quiver without potential, an explicit formula for the multiplication was obtained using torus localization ([43, Sec. 2]).

We define the COHA of Π_Q in the following way. Since the critical loci of the trace of W in $\mathcal{M}_{\widehat{Q}}$ contains the stack of seminilpotent representations of Π_Q which is denoted by $\mathcal{M}^{sp}_{\Pi_Q}$, one can transport the multiplication of the critical COHA to the dual space of compactly supported cohomology of $\mathcal{M}^{sp}_{\Pi_Q} = \coprod_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} \mathcal{M}^{sp}_{\Pi_Q,\gamma} = \coprod_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} \mathbf{M}^{sp}_{\Pi_Q,\gamma}/\mathbf{G}_\gamma$ (which is a direct sum of the dual spaces of equivariant cohomology with compact support, or equivalently, the equivariant Borel-Moore homology). Indeed, by the detailed exposition of dimensional reduction in [12] (see [43] as well), for a fixed dimension vector γ there is an isomorphism

$$H^{\bullet, crit}_c(\mathcal{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) = H^{\bullet, crit}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) \simeq H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma},$$

where $\mathcal{M}^{sp}_{\widehat{Q},\gamma} \subset \mathcal{M}_{\widehat{Q},\gamma}$ is a substack, and $\mathbb{T}$ is the Tate motive. In this way we get a degree-preserving associative multiplication on

$$\mathcal{H}_{\Pi_Q} = \bigoplus_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q})^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma,\gamma)},$$

where χ_Q is the Euler form of Q. The above construction in particular proves the following statement:

- The zero degree part $\mathcal{H}^0_{\Pi_Q}$ is a subalgebra of $\mathcal{H}_{\Pi_Q}$, and admits a semicanonical basis consisting of classes of top dimensional irreducible components of $\mathcal{M}^{sp}_{\Pi_Q}$, analogous to Lusztig's semicanonical basis of generalized quantum groups (see [46], [47]). This semicanonical basis and its dual enjoy compatibility with a certain filtration.

In general, we expect intrinsic categorical meaning of the semicanonical basis for a certain class of 2CY categories.

We investigate the correspondence between the A_∞-equivalent classes of 2CY categories with a collection of generators and a certain type of quivers. More precisely, let $\mathbf{k}$ be a field of characteristic zero, and $\mathcal{C}$ a $\mathbf{k}$-linear triangulated 2CY A_∞-category. Assume that $\mathcal{C}$ is generated by a finite collection $\mathcal{E} = \{E_i\}_{i\in I}$ of generators satisfying

- $Ext^0(E_i, E_i) = \mathbf{k} \cdot id_{E_i}$,
- $Ext^0(E_i, E_j) = 0, \forall i \neq j$,
- $Ext^{<0}(E_i, E_j) = 0, \forall i, j$.

We prove that

- The equivalence classes of such categories with respect to A_∞-transformations preserving the Calabi-Yau structure and $\mathcal{E}$, are in one-to-one correspondence with finite symmetric quivers with even number of loops at each vertex.

The proof is based on the deformation theory of the canonical 2CY category. This deformation theory is controlled by a DG Lie algebra coming from all cyclic series in coordinates on $Ext^\bullet(\oplus E_i, \oplus E_i)[1]$. There is an analog in the 3CY case in [42, Sec. 8.1].

We also studied the motivic Donaldson-Thomas theory of a certain class of 2CY categories via motivic Hall algebras defined using motivic stack functions. Upgrading the above 2CY categories gives rise to a class of 3CY categories with trivial Euler classes. Motivic DT-invariants of such 3CY categories do not change inside of a connected component of the space of stability conditions. As a result, the DT-invariants are in fact invariants of the t-structure of the underlying 2CY category. We constructed motivic DT-series for 2CY categories and proved their factorization property. We also formulated a conjecture about an analog of the Kac polynomial of a 2CY category.

First, for an ind-constructible locally regular triangulated A_∞-category $\mathcal{C}$ over a field $\mathbf{k}$, an associative algebra $H(\mathcal{C})$ called motivic Hall algebra is defined on the space of motivic stack functions on its stack of objects, with negative powers of the Lefschetz motive $\mathbb{L}$

added (see [42]). Fix a constructible stability condition on $\mathcal{C}$ and a strict sector $V \subset \mathbb{C}$, we have the category $\mathcal{C}_V$ generated by semistable objects with the central charge in V, and the corresponding completed motivic Hall algebra $\widehat{H}(\mathcal{C}_V)$. There is an invertible element $A_V^{Hall} \in \widehat{H}(\mathcal{C}_V)$ roughly corresponding to the sum over all isomorphism classes of objects of $\mathcal{C}_V$, each counted with the weight given by the inverse to the motive of the group of automorphisms. For various strict sectors V the elements A_V^{Hall} satisfy the Factorization Property, namely, $A_V^{Hall} = A_{V_1}^{Hall} \cdot A_{V_2}^{Hall}$ where $V = V_1 \sqcup V_2$ is decomposed in the clockwise order. Next, let $\mathcal{C}$ be a 2CY category belonging to the class in Section 6.3, and $\mathcal{R}_\Gamma$ the quantum torus which is a commutative algebra. Then we proved in [58] the following theorems:

- The integration map $\Phi : H(\mathcal{C}) \to \mathcal{R}_\Gamma$ preserves the clockwise order multiplication, thus leads to the Factorization Property of the motivic DT series $A_V^{mot} = \Phi(A_V^{Hall})$: $A_V^{mot} = A_{V_1}^{mot} \cdot A_{V_2}^{mot}$.
- A_V^{mot} is constant on each connected component of the space of stability conditions on $\mathcal{C}$.

The DT-invariants of the 2CY category $\mathcal{C}$ are defined using (expected) quantum admissibility of A_V^{mot}. In the case when V is a ray, we conjectured that

- The DT-invariants are polynomials in the Lefschetz motive $\mathbb{L}$, and coincide with the motivic DT-invariants of some 3CY category.

The conjecture was motivated by [50], in which the motive of the stack of indecomposable representations of a quiver (Kac polynomial) was expressed in terms of the motives of stacks of representations of the corresponding preprojective algebra and the DT-invariants of the corresponding 3CY category. Some related results concerning DT-invariants can be found in the work of Hausel, Letellier and Rodriguez-Villegas [23], Joyce and Song [30], Reineke [56], Szendrői [67], etc..

Contents of the paper

The dissertation is organized as follows:

Chapter 2 gives basics about quivers, some algebras associated to quivers, and the stacks of their representations.

Chapter 3 is devoted to a reminder of critical COHA of I-bigraded smooth algebras with potential.

Chapter 4 is devoted to the explicit description of the multiplication of COHA of the preprojective algebra Π_Q, and the proof of the existence of the semicanonical basis, thus relate the COHA to the generalized quantum groups.

Chapter 5 introduces the ind-constructible 2 Calabi-Yau categories, and proves the correspondence between them and a certain type of quivers.

Chapter 6 introduces the motivic Donaldson-Thomas theory of a certain class of 2 Calabi-Yau categories via motivic Hall algebras.

Chapter 2

Quivers

Given a quiver or a quiver with relations, one considers the stacks of their representations. The cohomology of these stacks form the underlying vector space of the cohomological Hall algebras. In particular, we introduce the preprojective algebra associated to a quiver, and quivers with potential.

2.1 Quivers and the stack of representations

We introduce basic definitions and properties of quivers and their representations. We will basically follow [15] and [43].

Definition 2.1.1. *A quiver Q is a quadruple (I, Ω, s, t) consists of the set I of vertices, the set Ω of arrows, and the maps $s, t : \Omega \to I$ assigning source and target to each arrow.*

A quiver is called finite if both I and Ω are finite sets.

All quivers considered in the sequel are finite.

An arrow with source i and target j will be denoted by $a : i \to j$, where $i, j \in I$ are two vertices.

Definition 2.1.2. *A quiver Q is symmetric if it is endowed with an involution $*$ acting on both I and Ω such that $s(a^*) = t(a)^*, t(a^*) = s(a)^*$.*

Definition 2.1.3. *Let* $\boldsymbol{k}$ *be a field. A representation* $E = (E_i, E_a)$ *of a quiver* Q *over* $\boldsymbol{k}$ *consists of a family of* $\boldsymbol{k}$*-vector spaces* E_i *for* $i \in I$*, together with a family of* $\boldsymbol{k}$*-linear maps* $E_a : E_{s(a)} \to E_{t(a)}$ *for* $a \in \Omega$*. A subrepresentation* $E' = (E'_i, E'_a)$ *of* E *is a representation of* Q *such that* $E'_i \subset E_i, \forall i$*, and* E'_a *is the restriction of* E_a *to* $E'_{s(a)}$ *for each* $a \in \Omega$*.*

Definition 2.1.4. *A morphism* $f : E \to F$ *between two representations* E *and* F *is given by* $\boldsymbol{k}$*-linear maps* $f_i : E_i \to F_i$ *for all* $i \in I$*, satisfying* $F_a \circ f_{s(a)} = f_{t(a)} \circ E_a$ *for any* $a \in \Omega$*. Namely, the following diagram commutes:*

$$\begin{array}{ccc} E_{s(a)} & \xrightarrow{E_a} & E_{t(a)} \\ \downarrow{\scriptstyle f_{s(a)}} & & \downarrow{\scriptstyle f_{t(a)}} \\ F_{s(a)} & \xrightarrow{F_a} & F_{t(a)} \end{array}$$

A representation $E = (E_i, E_a)$ is regarded as *finite dimensional* if all $E_i, i \in I$ are finite dimensional over $\mathbf{k}$. In this case, the vector $\underline{\dim} E = (\dim E_i)_{i \in I}$ is called the *dimension vector* of E. Denote the category of finite dimensional representations of Q over $\mathbf{k}$ by $\mathrm{Rep}_{\mathbf{k}} Q$.

We call a sequence of arrows $a_l \cdots a_2 a_1$ such that $t(a_s) = s(a_{s+1})$ a *path of length* $l \geqslant 1$. If $t(a_s) = s(a_1)$, then the path is said to be an *oriented cycle*. In particular a loop is an oriented cycle. Besides paths of length $\geqslant 1$, we also consider the trivial path e_i, which is the path of length 0 with source and target $i \in \Omega$. Now we can define

Definition 2.1.5. *The path algebra* $\boldsymbol{k}Q$ *is the* $\boldsymbol{k}$*-algebra having a basis the set of all the paths in* Q*. The product is given by linearity and the following product rule for paths:*

$$(a_l^1 \cdots a_1^1)(a_r^2 \cdots a_1^2) = \begin{cases} a_l^1 \cdots a_1^1 a_r^2 \cdots a_1^2, & t(a_r^2) = s(a_1^1), \\ 0, & \textit{otherwise}. \end{cases}$$

Clearly, $\mathbf{k}Q$ is an associative algebra with the identity $1 = \Sigma_{i \in I} e_i$. We denote by $\mathbf{k}Q$-mod the category of finite dimensional left $\mathbf{k}Q$-modules. The following statement is well known.

Theorem 2.1.6. *The categories* $\mathrm{Rep}_{\boldsymbol{k}}Q$ *and* $\boldsymbol{k}Q$*-mod are equivalent. Furthermore,* $\mathrm{Rep}_{\boldsymbol{k}}Q$ *is an abelian category.*

Given a quiver Q one can define a bilinear form, which is called *Euler form*, as follows:

$$\begin{aligned}\chi_Q(\bullet,\bullet):\ & \mathbb{Z}I\times\mathbb{Z}I\to\mathbb{Z},\\ & (\alpha,\beta)\mapsto -\sum_{a\in\Omega}\alpha^{s(a)}\beta^{t(a)}+\sum_{i\in I}\alpha^i\beta^i,\end{aligned}$$

where $\alpha=(\alpha^i)_{i\in I}$, and $\beta=(\beta^i)_{i\in I}$ belong to $\mathbb{Z}I$.

Let's introduce the stack of representations of Q. Fix a dimension vector $\gamma=(\gamma^i)_{i\in I}$, and the complex coordinate vector spaces $\mathbb{V}_i:=\mathbb{C}^{\gamma^i}$ for all $i\in I$. We denote by $a_{ij}\in\mathbb{Z}_{\geqslant 0}$ the number of arrows from i to j for $i,j\in I$. Define an affine variety

$$\mathbf{M}_{Q,\gamma}:=\bigoplus_{a:i\to j}Hom_{\mathbb{C}}(\mathbb{C}^{\gamma^i},\mathbb{C}^{\gamma^j})\simeq\prod_{i,j}\mathbb{C}^{a_{ij}\gamma^i\gamma^j}.$$

The reductive linear algebraic group

$$\mathbf{G}_\gamma=\prod_{i\in I}GL(\gamma^i,\mathbb{C})$$

acts on $\mathbf{M}$ via base change

$$(g_i)_i\cdot(E_a)_a=(g_jE_ag_i^{-1})_{a:i\to j}.$$

Definition 2.1.7. *We call* $\boldsymbol{M}_{Q,\gamma}$ *the space of representations of* Q *of dimension* γ*, and* $\boldsymbol{G}_\gamma$ *the gauge group of* $\boldsymbol{M}_{Q,\gamma}$*. The quotient stack* $\boldsymbol{M}_{Q,\gamma}/\boldsymbol{G}_\gamma$ *is the stack of representations of* Q *with dimension* γ*.*

2.2 Quiver with relations

To any quiver Q, by giving relations we obtain some interesting algebras. In particular, we will define the Jacobi algebra and preprojective algebra.

Definition 2.2.1. *A relation of a quiver Q is a subspace of $\boldsymbol{k}Q$ spanned by linear combinations of paths having a common source and a common target, and of length at least 2.*

A quiver with relations is a pair (Q, R), where Q is a quiver, and R is a two-sided ideal of $\boldsymbol{k}Q$ generated by relations. The quotient algebra $\boldsymbol{k}Q/R$ is the path algebra of (Q, R).

A representation of (Q,R) is a $\boldsymbol{k}Q/R$-module.

Now let's define quivers with potential, which will give rise to a type of quivers with relations. Fix a quiver Q, and assume that we are given an element

$$W \in \mathbf{k}Q/[\mathbf{k}Q, \mathbf{k}Q]$$

represented by some element $\widetilde{W} \in \mathbf{k}Q$, i.e., $W = \widetilde{W} \pmod{[\mathbf{k}Q, \mathbf{k}Q]}$. The element W (or its lifting $\widetilde{W}$) is called a *potential*. Indeed, $\widetilde{W}$ is a linear combination of oriented cycles in $\mathbf{k}Q$. For an oriented cycle $p = a_l \cdots a_2 a_1$, let

$$\partial_a p = \begin{cases} a_{s-1} \cdots a_1 a_l \cdots a_{s+1}, & \exists s \in \{1, \dots, l\} \textit{ such that } a = a_s, \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 2.2.2. *The cyclic derivative of a potential W with respect to an arrow a is defined as*

$$\partial_a W = \sum_s \partial_a p_s,$$

if $\widetilde{W} = \sum_s p_s$ for oriented cycles p_s.

Given a dimension vector $\gamma \in \mathbb{Z}^I_{\geqslant 0}$ we obtain a function W_γ on $\mathbf{M}_{Q,\gamma}$, invariant under the action of $\mathbf{G}_\gamma$. The value of W_γ at any representation is given by the trace of the image of $\widetilde{W}$. For any short exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

of representations of Q with dimension vectors γ_1, $\gamma_1 + \gamma_2$ and γ_2 respectively, we have

$$W_{\gamma_1+\gamma_2}(E) = W_{\gamma_1}(E_1) + W_{\gamma_2}(W_2).$$

Given a quiver with potential (Q, W), the cyclic derivative $\partial_a W$ gives rise to a relation for any $a \in \Omega$. Let R be the ideal generated by $\{\partial_a W | a \in \Omega\}$, then (Q, R) is a quiver with relations.

Definition 2.2.3. *The quotient algebra*

$$\mathcal{J}(Q, W) := \boldsymbol{k}Q/R$$

is called the Jacobi algebra of (Q, W).

Thus the space of representations of $\mathcal{J}(Q, W)$ of dimension γ, which is denoted by $\mathbf{M}_{\mathcal{J}(Q,W),\gamma}$, is a closed subscheme of $\mathbf{M}_{Q,\gamma}$. Indeed, $\mathbf{M}_{\mathcal{J}(Q,W),\gamma} = Crit(W_\gamma)$.

Let Q be a quiver with the set of vertices I and the set of arrows Ω. One constructs a symmetric quiver called the double quiver $\overline{Q}$ as follows. $\overline{Q}$ has the set of vertices I, which is the same as the original quiver Q. The set of arrows is $\Omega \cup \overline{\Omega}$, where $\overline{\Omega}$ is the set of dual arrows, namely, for any arrow $a : i \to j \in \Omega$, we add an inverse arrow $a^* : j \to i \in \overline{\Omega}$ to Q. Thus $\sum_{a\in\Omega}[a, a^*]$ is a relation of $\overline{Q}$.

Definition 2.2.4. *The preprojective algebra associated to* Q *is the quotient algebra*

$$\Pi_Q := \boldsymbol{k}\overline{Q}/\sum_{a\in\Omega}[a, a^*].$$

Furthermore, we can construct a triple quiver with potential $(\widehat{Q}, W)$. The triple quiver $\widehat{Q}$ has the set of vertices I the same as Q. The set of arrows is $\Omega \cup \overline{\Omega} \cup L$. Namely, we add a loop $l_i : i \to i$ at each vertex $i \in I$ to $\overline{Q}$, and denote the set of added loops by $L = \{l_i : i \to i | i \in I\}$. The cubic potential W is defined to be $\sum_{a \in \Omega} [a, a^*]l$, where $l = \sum_{i \in I} l_i$. Then the preprojective algebra Π_Q is a subalgebra of $\mathcal{J}(\widehat{Q}, W)$.

Chapter 3

Critical COHA of smooth algebras with potential

The critical Cohomological Hall algebra of a smooth I-bigraded algebra with potential is defined in [43, Sec. 7]. We first remind the equivariant critical cohomology with compact support, which gives the underlying vector space of critical COHA. Then give the definition of the product. Thus the critical COHA of a smooth I-bigraded algebra with potential is a unital associative algebra. In particular, the critical COHA can be defined for quivers with potential.

For the convenience of the reader we will closely follow the very detailed exposition from [12], which contains proofs of several statements sketched in [43] as well as several useful improvements of the loc.cit.

3.1 Reminder on the critical cohomology

In this section we will first review the definition of vanishing cycles of sheaves and (equivariant) critical cohomology with compact support. Then the dimensional reduction relates the (equivariant) critical cohomology with compact support to ordinary (equivariant) cohomology with compact support. This will induce the product of COHA of preprojective algebras in the next chapter. The pullback and pushforward maps of (equivariant) critical cohomology with compact support associated to an affine or proper map, which are used in defining the product of critical COHA, are constructed.

3.1.1 Vanishing cycles of sheaves

Let Y be a complex manifold, and $Z \subset Y$ a closed subspace. Then for a sheaf $\mathcal{F}$ on Y, the functor Γ_Z is defined as

$$\Gamma_Z \mathcal{F}(U) = \mathrm{Ker}(\mathcal{F}(U) \to \mathcal{F}(U \setminus Z)).$$

Let $f : Y \to \mathbb{C}$ be a holomorphic function.

Definition 3.1.1. *The vanishing cycles functor φ_f is defined as follows:*

$$\varphi_f \mathcal{F}[-1] := (R\Gamma_{\{Re(f)\leq 0\}}\mathcal{F})_{f^{-1}(0)}.$$

Remark 3.1.2. *This is a nonstandard definition of this functor, which is equivalent to the usual one in the complex case.*

From now on we will abbreviate $R\mathscr{F}$ to $\mathscr{F}$ for any functor $\mathscr{F}$.

Recall that the Verdier dual $D\mathcal{F}$ of a sheaf $\mathcal{F}$ on Y is defined to be $Hom(\mathcal{F}, p_!\mathbb{Q})$, where $p : Y \to \mathrm{pt}$. For Y an equidimensional manifold there is a canonical isomorphism of functors

$$D(\bullet) \xrightarrow{\sim} (\bullet)^\vee \otimes \mathbb{T}^{\dim Y},$$

where $(\bullet)^\vee = Hom(\bullet, \mathbb{Q}_Y)$ is the duality functor, and $\mathbb{T} = \mathbb{Q}(-1)[-2]$ is the Tate motive, which is a mixed Hodge module of cohomological degree -2 and weight -2. The multiplication map $\mathbb{Q}_Y \otimes \mathbb{Q}_Y \to \mathbb{Q}_Y$ induces an isomorphism $\mathbb{Q}_Y^\vee \simeq \mathbb{Q}_Y$, so there is an isomorphism

$\mathbb{Q}_Y \otimes \mathbb{T}^{\dim Y} \to D\mathbb{Q}_Y$. It induces an isomorphism

$$\varphi_f \mathbb{Q}_Y \otimes \mathbb{T}^{\dim Y} \xrightarrow{\sim} \varphi_f D\mathbb{Q}_Y. \tag{3.1}$$

In general there is a natural isomorphism

$$\varphi_f D \simeq D\varphi_f.$$

If $g : Y' \to Y$ is a map between manifolds, then the natural transformation of functors

$$\Gamma_{\{Re(f)\leq 0\}} \longrightarrow g_* \Gamma_{\{Re(fg)\leq 0\}} g^*$$

induces a natural transformation

$$\varphi_f \longrightarrow g_* \varphi_{fg} g^*. \tag{3.2}$$

If g is an affine fibration then (3.2) is a natural equivalence. In general it is not an isomorphism.

On the other hand, if g is a closed embedding, then

$$\varphi_f g_* \xrightarrow{\sim} g_* \varphi_{fg} \tag{3.3}$$

is a natural isomorphism of functors.

Assume that g is an affine fibration, then by [12, Cor. 2.4] there is a natural equivalence of functors

$$\varphi_f g_! g_* \longrightarrow g_! \varphi_{fg} g_*. \tag{3.4}$$

3.1.2 Critical cohomology and dimensional reduction

Let's introduce the notion of (equivariant) critical cohomology with compact support. The dimensional reduction theorem relates the (equivariant) critical cohomology with compact support to the ordinary (equivariant) cohomology with compact support.

Definition 3.1.3. *For any submanifold $Y^{sp} \subset Y$, the critical cohomology with compact support $H_c^{\bullet,crit}(Y^{sp}, f)$ is defined as the cohomology of the following object in $\mathscr{D}^b(\boldsymbol{MMHS})$ ($\boldsymbol{MMHS}$ denotes the category of monodromic mixed Hodge structures):*

$$(\mathbb{C}^* \to \mathbb{A}^1)_!(Y^{sp} \times \mathbb{C}^* \to \mathbb{C}^*)_!(Y^{sp} \times \mathbb{C}^* \to Y \times \mathbb{C}^*)^* \varphi_{\frac{f}{u}} \mathbb{Q}_{Y \times \mathbb{C}^*},$$

where u is the coordinate on $\mathbb{C}^*$.

Let $Y = X \times \mathbb{A}^n$ be the total space of the trivial vector bundle, endowed with the $\mathbb{C}^*$-action that acts trivially on X and with weight one on $\mathbb{A}^n$. Let $f : Y \to \mathbb{A}^1$ be a $\mathbb{C}^*$-equivariant holomorphic function, where $\mathbb{C}^*$ acts with weight one on $\mathbb{A}^1$. Then $f = \sum_{k=1}^{k=n} f_k x_k$, where $\{x_k, k = 1, ..., n\}$ is a linear coordinate system on $\mathbb{A}^n$, and f_k are functions on X. Let $Z \subset X$ be the reduced scheme which is the vanishing locus of all functions f_k. Then Z is independent of the choice of x_k. Let $\pi : Y \to X$ be the natural projection, and $i : Z \to X$ be the closed inclusion. The following theorem is usually called dimensional reduction.

Theorem 3.1.4. *(see* [12, Cor. A.6]*)*

There is a natural isomorphism of functors in $\mathscr{D}^b(\boldsymbol{MHM}(X))$:

$$\pi_! \varphi_f \pi^* \xrightarrow{\sim} \pi_! \pi^* i_* i^*.$$

In particular,

$$H_c^{\bullet,crit}(Y, f) \simeq H_c^{\bullet}(Z \times \mathbb{A}^n, \mathbb{Q}) \simeq H_c^{\bullet}(Z, \mathbb{Q}) \otimes \mathbb{T}^n.$$

Here ***MHM***(X) *denotes the category of mixed Hodge modules on* X*.*

If $Y_i = X_i \times \mathbb{A}^{n_i}$ with $\mathbb{C}^*$-equivariant holomorphic functions f_i satisfy the above conditions for $i = 1, 2$, then we have

Theorem 3.1.5. *(see* [12, Prop. A.5]*)*

The following diagram of isomorphisms commutes:

$$\begin{array}{ccc} H_c^{\bullet,crit}(Y_1 \times Y_2, f_1 \boxplus f_2) & \xrightarrow{TS} & H_c^{\bullet,crit}(Y_1, f_1) \otimes H_c^{\bullet,crit}(Y_2, f_2) \\ \downarrow & & \downarrow \\ H_c^{\bullet}(Z_1 \times Z_2 \times \mathbb{A}^{n_1+n_2}, \mathbb{Q}) & \xrightarrow{Ku} & H_c^{\bullet}(Z_1 \times \mathbb{A}^{n_1}, \mathbb{Q}) \otimes H_c^{\bullet}(Z_2 \times \mathbb{A}^{n_2}, \mathbb{Q}) \end{array}$$

Here TS *denotes the Thom-Sebastiani isomorphism, and* Ku *the Künneth isomorphism (see loc.cit.).*

Corollary 3.1.6. *(see* [12, Cor. A.7]*)*

Let $X^{sp} \subset X$ *be a subvariety of* X *and* $Y^{sp} = X^{sp} \times \mathbb{A}^n$*,* $Z^{sp} = Z \cap X^{sp}$*. There is a natural isomorphism in* ***MMHS***

$$H_c^{\bullet,crit}(Y^{sp}, f) \simeq H_c^{\bullet}(Z^{sp} \times \mathbb{A}^n, \mathbb{Q}).$$

The above statements also hold in equivariant case. Let us recall that framework. Assume that Y is a G-equivariant vector bundle over X, where G is an algebraic group embedded in $GL(n, \mathbb{C})$, and $f : Y \to \mathbb{A}^1$ is G-invariant. Let $fr(n, N)$ be the space of n-tuples of linearly independent vectors in $\mathbb{C}^N$ for $N \geqslant n$, and $\overline{(Y, G)}_N := Y \times_G fr(n, N)$. We denote the induced function by $f_N : \overline{(Y, G)}_N \to \mathbb{A}^1$.

Definition 3.1.7. *For a* G*-invariant closed subset* $Y^{sp} \subset Y$*, we define the equivariant* *critical cohomology with compact support by*

$$H^{\bullet,crit}_{c,G}(Y^{sp},f) := \lim_{N\to\infty} H^{\bullet,crit}_{c}(Y^{sp}_N, f_N) \otimes \mathbb{T}^{-dim(fr(n,N))},$$

where $Y^{sp}_N \subset \overline{(Y,G)}_N$ *is the subspace of points projected to* Y^{sp}*.*

Theorem 3.1.8. *(see* [12, Cor. A.8]*)*

Let $Y^{sp} = X^{sp} \times \mathbb{A}^n$ *be the total space of a sub G-bundle. Then there is an isomorphism in* ***MMHS***

$$H^{\bullet,crit}_{c,G}(Y^{sp},f) \simeq H^{\bullet}_{c,G}(Z^{sp}\times \mathbb{A}^n,\mathbb{Q}).$$

Moreover, the following diagram of isomorphisms commutes:

$$\begin{array}{ccc} H^{\bullet,crit}_{c,G}(Y_1^{sp}\times Y_2^{sp}, f_1 \boxplus f_2) & \xrightarrow{TS} & H^{\bullet,crit}_{c,G}(Y_1^{sp},f_1)\otimes H^{\bullet,crit}_{c,G}(Y_2^{sp},f_2) \\ \downarrow & & \downarrow \\ H^{\bullet}_{c,G}(Z_1^{sp}\times Z_2^{sp}\times \mathbb{A}^{n_1+n_2},\mathbb{Q}) & \xrightarrow{Ku} & H^{\bullet}_{c,G}(Z_1^{sp}\times \mathbb{A}^{n_1},\mathbb{Q})\otimes H^{\bullet}_{c,G}(Z_2^{sp}\times \mathbb{A}^{n_2},\mathbb{Q}) \end{array}$$

Remark 3.1.9. *For a general* Y *endowed with a* $\boldsymbol{G}$*-action, and a* $\boldsymbol{G}$*-invariant function* f*, the dual of the equivariant critical compactly supported cohomology* $H^{\bullet,crit}_{c,\boldsymbol{G}}(Y,f)^{\vee}$ *admits a* $H^{\bullet}_{\boldsymbol{G}}(\mathrm{pt},\mathbb{Q})$*-module structure. This module structure is constructed via*

$$\begin{aligned} \Delta_N: \quad & (Y\times_{\boldsymbol{G}} fr(n,N)) \to (Y\times_{\boldsymbol{G}} fr(n,N))\times(\mathrm{pt}\times_{\boldsymbol{G}} fr(n,N)), \\ & (y,z)\mapsto ((y,z),(\mathrm{pt},z)). \end{aligned}$$

More generally, by the diagonal embedding

$$\overline{\Delta}_N : (Y \times_{\boldsymbol{G}} fr(n,N)) \to (Y \times_{\boldsymbol{G}} fr(n,N)) \times (Y \times_{\boldsymbol{G}} fr(n,N))$$

an extended action

$$H^{\bullet}_{\boldsymbol{G}}(Y,\mathbb{Q}) \otimes H^{\bullet,crit}_{c,\boldsymbol{G}}(Y,f)^{\vee} \longrightarrow H^{\bullet,crit}_{c,\boldsymbol{G}}(Y,f)^{\vee}$$

of $H^{\bullet}_{\boldsymbol{G}}(Y,\mathbb{Q})$ can be built in the same way.

See the details in [12, Sec. 2.6].

3.1.3 Pullback and pushforward maps

Let $g : X \to Y$ be a $\mathbf{G}$-equivariant morphism between complex algebraic manifolds, and $f : Y \to \mathbb{A}^1$. Let $Y^{sp} \subset Y$ be $\mathbf{G}$-invariant, and $X^{sp} = g^{-1}(Y^{sp})$. We wish to have maps going both ways between $H^{\bullet,crit}_{c,\mathbf{G}}(Y^{sp},f)^{\vee}$ and $H^{\bullet,crit}_{c,\mathbf{G}}(X^{sp},f)^{\vee}$. We will assume that g is of two types: affine fibration and proper.

First, let g be an affine fibration. Then the pullback

$$g^* : H^{\bullet,crit}_{c,\mathbf{G}}(Y^{sp},f)^{\vee} \otimes \mathbb{T}^{\mathrm{dimg}} \xrightarrow{\sim} H^{\bullet,crit}_{c,\mathbf{G}}(X^{sp},fg)^{\vee} \tag{3.5}$$

is an isomorphism. Indeed, let $g_N : \overline{(X,\mathbf{G})}_N \to \overline{(Y,\mathbf{G})}_N$, there is a natural isomorphism

$$\mathbb{Q}_{\overline{(Y,\mathbf{G})}_N} \xrightarrow{\sim} (g_N)_* \mathbb{Q}_{\overline{(X,\mathbf{G})}_N}.$$

Applying φ_{F_N} to the Verdier dual of the above isomorphism we obtain a map

$$\varphi_{f_N}(g_{N,!} D\mathbb{Q}_{\overline{(X,\mathbf{G})}_N} \xrightarrow{\sim} D\mathbb{Q}_{\overline{(Y,\mathbf{G})}_N}),$$

which by (3.1) gives us an isomorphism

$$\varphi_{f_N}(g_{N,!}\mathbb{Q}_{\overline{(X,\mathbf{G})}_N} \xrightarrow{\sim} \mathbb{Q}_{\overline{(Y,\mathbf{G})}_N} \otimes \mathbb{T}^{-\dim(g)}).$$

By (3.4) we obtain an isomorphism

$$g_{N,!}\varphi_{f_N g_N}\mathbb{Q}_{\overline{(X,\mathbf{G})}_N} \xrightarrow{\sim} \varphi_{f_N}\mathbb{Q}_{\overline{(Y,\mathbf{G})}_N} \otimes \mathbb{T}^{-\dim(g)}.$$

Thus by restricting to $\mathbb{Q}_{\overline{(Y^{sp},\mathbf{G})}_N}$, taking compactly supported cohomology, passing to the limit, and taking duals, we have the pullback isomorphism.

To define the pushforward we first define the Euler characteristic of g as follows. Let $V = T_{X/Y}$ be the relative tangent bundle of g, and $z : X \to V$ be the inclusion of the zero section. Consider the composition

$$z_*\mathbb{Q}_{\overline{(X,\mathbf{G})}_N} \to \mathbb{Q}_{\overline{(V,\mathbf{G})}_N} \otimes \mathbb{T}^{\dim}(g) \to z_*\mathbb{Q}_{\overline{(X,\mathbf{G})}_N} \otimes \mathbb{T}^{\dim(g)}$$

where the first morphism is obtained by taking the Verdier dual of the second. Taking cohomology and using the isomorphism $H^\bullet_{\mathbf{G}}(Y,\mathbb{Q}) \simeq H^\bullet_{\mathbf{G}}(X,\mathbb{Q})$ gives us the map

$$\mathfrak{eu}_g : H^\bullet_{\mathbf{G}}(Y,\mathbb{Q}) \longrightarrow H^\bullet_{\mathbf{G}}(Y,\mathbb{Q}).$$

We further assume that $\mathfrak{eu}_g(1)$ is not a zero divisor in $H_{c,\mathbf{G}}(Y,f)^\vee$ for the extended action in Remark 3.1.9. Then the pushforward map associated to g is defined as

$$g_* := (g^*)^{-1} \cdot \mathfrak{eu}_g(1)^{-1} : H^{\bullet,crit}_{c,\mathbf{G}}(X^{sp}, fg)^\vee \longrightarrow H^{\bullet,crit}_{c,\mathbf{G}}(Y^{sp}, f)^\vee[\mathfrak{eu}_g(1)^{-1}].$$

Note that the pushforward preserves degree.

Next, assume that g is proper, which induces proper maps $g_N : \overline{(X,\mathbf{G})}_N \to \overline{(Y,\mathbf{G})}_N$.

Using

$$\varphi_{f_N}(\mathbb{Q}_{\overline{(Y,\mathbf{G})}_N} \to g_{N,*}\mathbb{Q}_{\overline{(X,\mathbf{G})}_N})$$

and (3.4), we obtain the pushforward

$$g_* : H^{\bullet,crit}_{c,\mathbf{G}}(X^{sp}, fg)^\vee \longrightarrow H^{\bullet,crit}_{c,\mathbf{G}}(Y^{sp}, f)^\vee. \tag{3.6}$$

3.2 I-bigraded smooth algebras

Let's recall the notion of an I-bigraded smooth algebra, where I is a finite set.

Definition 3.2.1. *An associative unital algebra R over a field $\mathbf{k}$ is called smooth if it is finitely generated and formally smooth in the sense of D. Quillen and J. Cuntz, i.e., if the bimodule $\Omega^1_R := \mathrm{Ker}(R \otimes_{\boldsymbol{k}} R \xrightarrow{\mathrm{mult}} R)$ is projective. Here* mult $: R \otimes_{\boldsymbol{k}} R \to R$ *is the multiplication.*

The property of formal smoothness is equivalent to the following lifting property for non-commutative nilpotent extentions: for any associative unital algebra A over $\mathbf{k}$, a nilpotent two-sided ideal $J \subset A$ (i.e., $J^n = 0$ for some $n > 0$), and a homomorphism $\phi : R \to A/J$, there exists a lifting of ϕ to a homomorphism $R \to A$.

Definition 3.2.2. *Given a finite set I, an unital associative algebra over $\mathbf{k}$ is I-bigraded if $R = \oplus_{i,j\in I} R_{ij}$ such that $R_{ij} \cdot R_{jk} \subset R_{ik}$.*

Equivalently, R is I-bigraded if there is a morphism of unital algebras $\mathbf{k}^I \to R$.

For a quiver $Q = (I, \Omega)$, the path algebra $\mathbf{k}Q$ is an I-bigraded smooth algebra. Indeed, $\mathbf{k}Q = \oplus_{i,j\in I}(\mathbf{k}Q)_{ij}$ where $(\mathbf{k}Q)_{ij}$ is the set of paths with source i and target j.

The notion of potential (see 2.2) can be generalized to an I-bigraded smooth algebra R: $W \in R/[R,R]$ and $W = \widetilde{W}$ (mod$[R,R]$).

3.3 Critical COHA

Let's recall the definition of the critical COHA of an I-bigraded smooth algebra with potential following [43].

For any I-bigraded smooth algebra with potential (R, W) and any dimension vector $\gamma = (\gamma^i)_{i\in I} \in \mathbb{Z}^I_{\geqslant 0}$, the scheme $\mathbf{M}_\gamma = \mathbf{M}_{R,\gamma}$ of representations of R in coordinate spaces $\mathbb{V}_i = \mathbf{k}^{\gamma^i}, i \in I$ is a smooth affine scheme. Any choice of a finite set of I-bigraded generators of R gives a closed embedding of $\mathbf{M}_\gamma$ into the affine space $\mathbf{M}_{Q,\gamma}$ for some quiver Q with the set of vertices equal to I.

Assume that we are given a bilinear form $\chi_R : \mathbb{Z}^I \otimes \mathbb{Z}^I \to \mathbb{Z}$ such that for any two dimension vectors $\gamma_1, \gamma_2 \in \mathbb{Z}^I_{\geqslant 0}$ and any two representations $E_i \in \mathbf{M}_{\gamma_i}(\overline{\mathbf{k}})$, we have

$$\dim Hom(E_1, E_2) - \dim Ext^1(E_1, E_2) = \chi_R(\gamma_1, \gamma_2).$$

This implies that the smooth scheme $\mathbf{M}_\gamma$ is equidimensional for any γ and

$$\dim \mathbf{M}_\gamma = -\chi_R(\gamma, \gamma) + \sum_{i\in I} (\gamma^i)^2.$$

In the case when R is the path algebra of a quiver Q, recall $a_{ij} \in \mathbb{Z}_{\geqslant 0}$ the number of arrows from i to j for $i, j \in I$. Then

$$\chi_Q(\gamma_1, \gamma_2) = \chi_{\mathbf{k}Q}(\gamma_1, \gamma_2) = -\sum_{i,j\in I} a_{ij}\gamma_1^i \gamma_2^j + \sum_{i\in I} \gamma_1^i \gamma_2^i$$

is the Euler form.

Fix a dimension vector $\gamma \in \mathbb{Z}^I_{\geqslant 0}$, and assume that a complex algebraic group $\mathbf{G}_\gamma$ acts on $\mathbf{M}_\gamma$. The potential W gives rise to a $\mathbf{G}_\gamma$-invariant function $W_\gamma : \mathbf{M}_\gamma \to \mathbf{k}$ as in 2.2. Consider a $\mathbf{G}_\gamma$-invariant subvariety $\mathbf{M}^{sp}_\gamma \subset \mathbf{M}_\gamma$ satisfying the following conditions $(**)$:

- $\mathbf{M}^{sp}_\gamma \subset Crit(W_\gamma)$, i.e., the 1-form vanishes at $\mathbf{M}^{sp}_\gamma$,

- for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of representations of $\overline{\mathbf{k}} \otimes_{\mathbf{k}} R$ with dimension vectors $\gamma_1, \gamma := \gamma_1 + \gamma_2, \gamma_2$ respectively, $E \in \mathbf{M}^{sp}_{\gamma}$ if and only if $E_1 \in \mathbf{M}^{sp}_{\gamma_1}$, and $E_2 \in \mathbf{M}^{sp}_{\gamma_2}$.

The second condition implies that the representations in $\mathbf{M}^{sp}_{\gamma}(\overline{\mathbf{k}})$ for all $\gamma \in \mathbb{Z}^I_{\geqslant 0}$ form an abelian category, which is a Serre subcategory of the abelian category $Crit(W)(\overline{\mathbf{k}}) := \sqcup_{\gamma} Crit(W_{\gamma})(\overline{\mathbf{k}})$, which is itself a full subcategory of $\overline{\mathbf{k}} \otimes_{\mathbf{k}} R$-mod. One may always choose $\mathbf{M}^{sp}_{\gamma} = Crit(W_{\gamma}), \forall \gamma \in \mathbb{Z}^I_{\geqslant 0}$.

Example 3.3.1. *For a quiver with potential (Q, W), let $\mathbb{H}_+ := \{re^{i\theta} | r \in \mathbb{R}_{>0}, \theta \in (0, \pi]\}$, and $\zeta \in \mathbb{H}^I_+$. Such a ζ gives rise a Bridgeland stability condition for* $\mathrm{Rep}_{\boldsymbol{k}} Q$*. The slope of a representation E of Q is defined to be $\mu(E) :=$* $\mathrm{Arg}(\dim(E) \cdot \zeta)$*. A representation E of Q is called ζ-semistable if for all nonzero subrepresentations $E' \subset E$, there is an inequality $\zeta(E') \leqslant \zeta(E)$. It is called ζ-stable if this inequality is strict for all proper $E' \subset E$. Fix a* $\theta \in (0, \pi]$*, one can check that the condition on a Q-representation E of being ζ-semistable and with $\mu(E) = \theta$ satisfies the second condition of $(**)$. The ζ-stable representations with a fixed slope θ do not satisfy this condition. For instance it's not closed under taking direct sum.*

In this case, we can take $\boldsymbol{M}^{sp}_{Q,\gamma}$ to be $\boldsymbol{M}^{\zeta-ss}_{Q,\gamma}$, the space of ζ-semistable representations.

Fix any $\gamma_1, \gamma_2 \in \mathbb{Z}^I_{\geqslant 0}$ and let $\gamma = \gamma_1 + \gamma_2$. Denote by $\mathbf{M}_{\gamma_1,\gamma_2}$ the space of representations of R in coordinate spaces of dimensions $(\gamma_1^i + \gamma_2^i)_{i \in I}$ such that the subspaces of dimensions $(\gamma_1^i)_{i \in I}$ form a subrepresentation. The space $\mathbf{M}_{\gamma_1,\gamma_2}$ is a closed subspace of $\mathbf{M}_{\gamma}$. The group $\mathbf{G}_{\gamma_1,\gamma_2} \subset \mathbf{G}_{\gamma}$ consisting of elements preserving subspaces $(\mathbf{k}^{\gamma_1^i} \subset \mathbf{k}^{\gamma^i})_{i \in I}$ acts on $\mathbf{M}_{\gamma_1,\gamma_2}$.

The coproduct on $\bigoplus_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} H^{\bullet, crit}_{c, \mathbf{G}_{\gamma}}(\mathbf{M}^{sp}_{\gamma}, W_{\gamma})$ is defined in the following way:

- $H^{\bullet, crit}_{c, \mathbf{G}_{\gamma}}(\mathbf{M}^{sp}_{\gamma}, W_{\gamma}) \to H^{\bullet, crit}_{c, \mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\gamma}, W_{\gamma})$, which is the pullback associated with the embedding of groups $\mathbf{G}_{\gamma_1,\gamma_2} \to \mathbf{G}_{\gamma}$ with proper quotient.

- $H^{\bullet, crit}_{c, \mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\gamma}, W_{\gamma}) \to H^{\bullet, crit}_{c, \mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\gamma_1,\gamma_2}, W_{\gamma})$, where $\mathbf{M}^{sp}_{\gamma_1,\gamma_2} := \mathbf{M}^{sp}_{\gamma} \cap \mathbf{M}_{Q,\gamma_1,\gamma_2}$, is given by the pullback of the closed embedding $\mathbf{M}_{\gamma_1,\gamma_2} \hookrightarrow \mathbf{M}^{sp}_{Q,\gamma}$.

- $H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\gamma_1,\gamma_2}, W_\gamma) \simeq H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\gamma_1,\gamma_2}, W_\gamma)$, where $\widetilde{\mathbf{M}}^{sp}_{\gamma_1,\gamma_2} \subset \mathbf{M}_{\gamma_1,\gamma_2}$ is the pullback of $\mathbf{M}^{sp}_{\gamma_1} \times \mathbf{M}^{sp}_{\gamma_2}$ under the projection $\mathbf{M}_{\gamma_1,\gamma_2} \to \mathbf{M}_{\gamma_1} \times \mathbf{M}_{\gamma_2}$. The isomorphism follows from the fact that $\mathbf{M}^{sp}_{\gamma_1,\gamma_2} = Crit(W_\gamma) \cap \widetilde{\mathbf{M}}^{sp}_{\gamma_1,\gamma_2}$, by the conditions $(**)$. Hence the sheaf of vanishing cycles of W_γ vanishes on $\widetilde{\mathbf{M}}^{sp}_{\gamma_1,\gamma_2} - \mathbf{M}^{sp}_{\gamma_1,\gamma_2}$.

- $H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\gamma_1,\gamma_2}, W_\gamma) \to H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\gamma_1,\gamma_2}, W_{\gamma_1,\gamma_2})$, where W_{γ_1,γ_2} is the restriction of W_γ to $\mathbf{M}_{\gamma_1,\gamma_2}$.

- $H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\gamma_1,\gamma_2}, W_{\gamma_1,\gamma_2}) \simeq H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\gamma_1} \times \mathbf{M}^{sp}_{\gamma_2}, W_{\gamma_1} \boxplus W_{\gamma_2}) \otimes \mathbb{T}^c$. This isomorphism comes from the following facts: there is a homotopy equivalence $\mathbf{G}_{\gamma_1} \times \mathbf{G}_{\gamma_2} \sim \mathbf{G}_{\gamma_1,\gamma_2}$, and $\widetilde{\mathbf{M}}^{sp}_{\gamma_1,\gamma_2}$ is a bundle over $\mathbf{M}_{\gamma_1} \times \mathbf{M}_{Q,\gamma_2}$ with affine fibers, and moreover, W_{γ_1,γ_2} is the pullback of $W_{\gamma_1} \boxplus W_{\gamma_2}$. The shift is given by

$$c = \dim\mathbf{M}_{\gamma_1,\gamma_2}/\mathbf{G}_{\gamma_1,\gamma_2} - \dim\mathbf{M}_{\gamma_1}/\mathbf{G}_{\gamma_1} - \dim\mathbf{M}_{\gamma_2}/\mathbf{G}_{\gamma_2} = -\chi_R(\gamma_2, \gamma_1).$$

- $H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\gamma_1} \times \mathbf{M}^{sp}_{\gamma_2}, W_{\gamma_1} \boxplus W_{\gamma_2}) \simeq H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\gamma_1}, W_{\gamma_1}) \otimes H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\gamma_2}, W_{\gamma_1})$. This is the Thom-Sebastiani isomorphism.

The composition of the above maps gives us a coproduct

$$m^\vee_{\gamma_1,\gamma_2} : H^{\bullet,crit}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_\gamma, W_\gamma) \to H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\gamma_1}, W_{\gamma_1}) \otimes H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\gamma_2}, W_{\gamma_2}) \otimes \mathbb{T}^{-\chi_R(\gamma_2,\gamma_1)}.$$

By letting

$$\mathcal{H}^{crit}_\gamma := H^{\bullet,crit}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_\gamma, W_\gamma)^\vee \otimes \mathbb{T}^{\dim\mathbf{M}_\gamma/\mathbf{G}_\gamma},$$

we obtain a product

$$m_{\gamma_1,\gamma_2} : \mathcal{H}^{crit}_{\gamma_1} \otimes \mathcal{H}^{crit}_{\gamma_2} \longrightarrow \mathcal{H}^{crit}_\gamma \otimes \mathbb{T}^d$$

on the space $\mathcal{H}^{crit} = \bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} \mathcal{H}^{crit}_\gamma$.

Theorem 3.3.2. *The product m_{γ_1,γ_2} on the space $\mathcal{H}^{crit}$ is associative.*

Proof. See [43]. □

In the case of quivers with potential, since $\dim \mathbf{G}_\gamma = \sum_{i\in I}(\gamma^i)^2$, we have that the dimension of the stack

$$\dim \mathbf{M}_{\mathrm{Q},\gamma}/\mathbf{G}_\gamma = \dim \mathbf{M}_{\mathrm{Q},\gamma} - \dim \mathbf{G}_\gamma = -\chi_Q(\gamma,\gamma).$$

Thus

$$\mathcal{H}_\gamma^{crit} = H_{c,\mathbf{G}_\gamma}^{\bullet,crit}(\mathbf{M}_\gamma^{sp}, W_\gamma)^\vee \otimes \mathbb{T}^{-\chi_Q(\gamma,\gamma)},$$

and the critical COHA

$$\mathcal{H}^{crit} = \bigoplus_{\gamma\in\mathbb{Z}_{\geqslant 0}^I} \mathcal{H}_\gamma^{crit}$$

of the triple quiver with potential induces the COHA of the preprojective algebra in the next chapter.

Chapter 4

Cohomological Hall algebras and semicanonical basis

In this chapter we will give a detailed description of the product of the COHA of the preprojective algebra, which is induced by the critical COHA of a quiver with potential. Then show that the degree zero part is a subalgebra of COHA. Moreover, this subalgebra admits a semicanonical basis, which enjoys the same properties as those of the semicanonical basis of the generalized quantum groups.

4.1 COHA of preprojective algebras

Let Q be a quiver with the set of vertices I and the set of arrows Ω. Recall the double quiver $\overline{Q}$, the preprojective algebra Π_Q, and the triple quiver with potential $(\widehat{Q}, W)$ (see Section 2.2).

- $\overline{Q}$ has the set of vertices I, which is the same as the original quiver Q. The set of arrows is $\Omega \cup \overline{\Omega}$, where $\overline{\Omega}$ is the set of dual arrows, namely, for any arrow $a : i \to j \in \Omega$, we add an inverse arrow $a^* : j \to i \in \overline{\Omega}$ to Q.

- $\Pi_Q = \mathbb{C}\overline{Q}/\sum_{a\in\Omega}[a,a^*]$.

- $\widehat{Q}$ has the set of vertices I the same as Q as well. The set of arrows is $\Omega \cup \overline{\Omega} \cup L$. Namely, we add a loop $l_i : i \to i$ at each vertex $i \in I$ to $\overline{Q}$, and denote the set of added loops by $L = \{l_i : i \to i | i \in I\}$.

  It is endowed with the cubic potential $W = \sum_{a\in\Omega}[a,a^*]l$, where $l = \sum_{i\in I} l_i$.

For any dimension vector $\gamma = (\gamma^i)_{i\in I} \in \mathbb{Z}^I_{\geqslant 0}$ we have the following algebraic varieties:

a) the space $\mathbf{M}_{\overline{Q},\gamma}$ of representations of the double quiver $\overline{Q}$ in the coordinate spaces $(\mathbb{C}^{\gamma^i})_{i\in I}$;

b) the similar space of representations $\mathbf{M}_{\Pi_Q,\gamma}$ of Π_Q;

c) the similar space of representations $\mathbf{M}_{\widehat{Q},\gamma}$ of $\widehat{Q}$.

All these spaces of representations are endowed with the action by conjugation of the complex algebraic group $\mathbf{G}_\gamma = \prod_{i\in I} GL(\gamma^i, \mathbb{C})$.

In the context of Section 3.1.2, let $X = \mathbf{M}_{\overline{Q},\gamma}$, $Y = \mathbf{M}_{\widehat{Q},\gamma} = \mathbf{M}_{\overline{Q},\gamma} \times \mathbb{A}^{\gamma\cdot\gamma}$ (dot denotes the inner product), and $f = Tr(W)_\gamma = \sum_{i\in I, k=1,\dots,(\gamma^i)^2} f_{ik}x_{ik}$, where f_{ik} are functions on $\mathbf{M}_{\overline{Q},\gamma}$, and $\{x_{ik}\}$ is a linear coordinate system on $\mathbb{A}^{\gamma\cdot\gamma}$. Then $Z = \mathbf{M}_{\Pi_Q,\gamma}$. Denote by $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$ the space of representations of $\overline{Q}$ in coordinate spaces of dimension $\gamma_1 + \gamma_2$ such that the standard coordinate subspaces of dimension γ_1 form a subrepresentation, and the restriction of $\rho \in \mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$ on the block-diagonal part is an element in $\mathbf{M}_{\Pi_Q,\gamma_1} \times \mathbf{M}_{\Pi_Q,\gamma_2}$. The group $\mathbf{G}_{\gamma_1,\gamma_2} \subset \mathbf{G}_\gamma$ consisting of transformations preserving subspaces $(\mathbb{C}^{\gamma_1^i} \subset \mathbb{C}^{\gamma^i})_{i\in I}$ acts on $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$. Suppose that we are given a collection of $\mathbf{G}_\gamma$-invariant closed subsets $\mathbf{M}^{sp}_{\overline{Q},\gamma} \subset \mathbf{M}_{\overline{Q},\gamma}$ satisfying the following condition:

(∗) For any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of representations of $\overline{Q}$ with dimension vectors $\gamma_1, \gamma := \gamma_1 + \gamma_2, \gamma_2$ respectively, $E \in \mathbf{M}^{sp}_{\overline{Q},\gamma}$ if and only if $E_1 \in \mathbf{M}^{sp}_{\overline{Q},\gamma_1}$, and $E_2 \in \mathbf{M}^{sp}_{\overline{Q},\gamma_2}$.

Then $\mathbf{M}^{sp}_{\widehat{Q},\gamma} = \mathbf{M}^{sp}_{\overline{Q},\gamma} \times \mathbb{A}^{\gamma\cdot\gamma}$ satisfy the conditions (∗∗).

We describe the product of the COHA of Π_Q explicitly. First, The critical COHA of $(\widehat{Q}, W)$ (see [43]) induces the coproduct on the vector space

$$\bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}} H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})$$

as follows:

- $H^\bullet_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q}) \to H^\bullet_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})$, which is the pullback associated to the closed embedding of groups $\mathbf{G}_{\gamma_1,\gamma_2} \to \mathbf{G}_\gamma$ with proper quotient.

  The projections

  $$pr_{\gamma_1,\gamma_2,N} : \overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \to \overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_\gamma)}_N$$

  induce natural transformations of functors

  $$\varphi_{\gamma,N} \to (pr_{\gamma_1,\gamma_2,N})_! \varphi_{\gamma,\gamma_1,\gamma_2,N} (pr_{\gamma_1,\gamma_2,N})^*$$

  by (3.2) and properness of $pr_{\gamma_1,\gamma_2,N}$, thus give us

  $$(\pi_{\gamma,N})_!\varphi_{\gamma/u,N}(\pi_{\gamma,N})^*[-1] \to (\pi_{\gamma,N})_!(pr_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma,\gamma_1,\gamma_2)/u,N}(pr_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma,N})^*[-1].$$

  Here $\varphi_{\gamma,N} = \varphi_{Tr(W)_{\gamma,N}}$ is the vanishing cycles functor of the function $tr(W)_{\gamma,N}$ on $\overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_\gamma)}_N$, and $\varphi_{\gamma,\gamma_1,\gamma_2,N}$ corresponds to $Tr(W)_{\gamma,\gamma_1,\gamma_2,N}$ on $\overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N$. (Note that in subscript of $\varphi_{\gamma,\gamma_1,\gamma_2,N}$, γ indicates the dimension vector of $\mathbf{M}_{\widehat{Q},\gamma}$, and γ_1, γ_2 indicate those of $\mathbf{G}_{\gamma_1,\gamma_2}$. We will use similar notations in the subsequent steps.)

Since the following diagram commutes:

$$\begin{array}{ccc}
\overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{pr_{\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\widehat{Q},\gamma}, \mathbf{G}_{\gamma})}_N \times \mathbb{C}^* \\
\Big\downarrow \pi_{\gamma,\gamma_1,\gamma_2,N} & & \Big\downarrow \pi_{\gamma,N} \\
\overline{(\mathbf{M}_{\overline{Q},\gamma}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{pr_{\overline{Q},\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\overline{Q},\gamma}, \mathbf{G}_{\gamma})}_N \times \mathbb{C}^*
\end{array}$$

we have

$$\begin{aligned}
& (\pi_{\gamma,N})_!(pr_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma,\gamma_1,\gamma_2)/u,N}(pr_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma,N})^*[-1] \\
\simeq \quad & (pr_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma,\gamma_1,\gamma_2,N})_!\varphi_{(\gamma,\gamma_1,\gamma_2)/u,N}(\pi_{\gamma,\gamma_1,\gamma_2,N})^*(pr_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1].
\end{aligned}$$

By Theorem 3.1.4, we have two isomorphisms:

$$(\pi_{\gamma,N})_!\varphi_{\gamma/u,N}(\pi_{\gamma,N})^*[-1] \simeq (\pi_{\gamma,N})_!(\pi_{\gamma,N})^*(i_{\gamma,N})_*(i_{\gamma,N})^*$$

and

$$\begin{aligned}
& (pr_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma,\gamma_1,\gamma_2,N})_!\varphi_{(\gamma,\gamma_1,\gamma_2)/u,N}(\pi_{\gamma,\gamma_1,\gamma_2,N})^*(pr_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1] \\
\simeq \quad & (pr_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma,\gamma_1,\gamma_2,N})_!(\pi_{\gamma,\gamma_1,\gamma_2,N})^*(i_{\gamma,\gamma_1,\gamma_2,N})_*(i_{\gamma,\gamma_1,\gamma_2,N})^*(pr_{\overline{Q},\gamma_1,\gamma_2,N})^*.
\end{aligned}$$

Here $i_{\gamma,N}$ and $i_{\gamma,\gamma_1,\gamma_2,N}$ are inclusions, and the subscripts have the same meaning as the vanishing cycles functors above.

Pulling back to $\mathbf{M}^{sp}_{\overline{Q},\gamma,N} \times \mathbb{C}^*$ gives us the commutative diagram

$$\begin{array}{ccc}
H^{\bullet,crit}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) & \longrightarrow & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) \\
\Big\downarrow \wr & & \Big\downarrow \wr \\
H^{\bullet}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma} & \longrightarrow & H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma}
\end{array}$$

- $H^\bullet_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q}) \to H^\bullet_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q}) \otimes \mathbb{T}^{-\gamma_1\cdot\gamma_2}$, where $\mathbf{M}^{sp}_{\Pi_Q,\gamma_1,\gamma_2} = \mathbf{M}^{sp}_{\Pi_Q,\gamma} \cap \mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$, and $\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2} \subset \mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2}$ is the pullback of $\mathbf{M}^{sp}_{\Pi_Q,\gamma_1} \times \mathbf{M}^{sp}_{\Pi_Q,\gamma_2}$ under the projection $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2} \to \mathbf{M}_{\Pi_Q,\gamma_1} \times \mathbf{M}_{\Pi_Q,\gamma_2}$. This is the pullback associated to the closed embedding $\mathbf{M}_{\Pi_Q,\gamma_1,\gamma_2} \to \mathbf{M}_{\Pi_Q,\gamma}$.

The inclusions

$$j_{\gamma_1,\gamma_2,N} : \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \to \overline{(\mathbf{M}_{\widehat{Q},\gamma},\mathbf{G}_{\gamma_1,\gamma_2})}_N$$

induce natural transformations of functors

$$\varphi_{\gamma,\gamma_1,\gamma_2,N} \to (j_{\gamma_1,\gamma_2,N})_*\varphi_{\gamma_1,\gamma_2,N}(j_{\gamma_1,\gamma_2,N})^*$$

by (3.2). So we have

$$\begin{aligned} & (\pi_{\gamma,\gamma_1,\gamma_2,N})_!\varphi_{(\gamma,\gamma_1,\gamma_2)/u,N}(\pi_{\gamma,\gamma_1,\gamma_2,N})^*[-1] \\ \to\ & (\pi_{\gamma,\gamma_1,\gamma_2,N})_!(j_{\gamma_1,\gamma_2,N})_*\varphi_{(\gamma_1,\gamma_2)/u,N}(j_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma,\gamma_1,\gamma_2,N})^*[-1]. \end{aligned}$$

By the commutative diagram

$$\begin{array}{ccc} \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{j_{\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\widehat{Q},\gamma},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* \\ \downarrow{\scriptstyle \pi_{\gamma_1,\gamma_2,N}} & & \downarrow{\scriptstyle \pi_{\gamma,\gamma_1,\gamma_2,N}} \\ \overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* & \xrightarrow{j_{\overline{Q},\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\overline{Q},\gamma},\mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^* \end{array}$$

we have

$$\begin{aligned} & (\pi_{\gamma,\gamma_1,\gamma_2,N})_!(j_{\gamma_1,\gamma_2,N})_*\varphi_{(\gamma_1,\gamma_2)/u,N}(j_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma,\gamma_1,\gamma_2,N})^*[-1] \\ \simeq\ & (j_{\overline{Q},\gamma_1,\gamma_2,N})_*(\pi_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,N})^*(j_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1]. \end{aligned}$$

Then the isomorphisms

$$\begin{aligned} & (\pi_{\gamma,\gamma_1,\gamma_2,N})_! \varphi_{(\gamma,\gamma_1,\gamma_2)/u,N} (\pi_{\gamma,\gamma_1,\gamma_2,N})^*[-1] \\ \simeq \ & (\pi_{\gamma,\gamma_1,\gamma_2,N})_! (\pi_{\gamma,\gamma_1,\gamma_2,N})^* (i_{\gamma,\gamma_1,\gamma_2,N})_* (i_{\gamma,\gamma_1,\gamma_2,N})^* \end{aligned}$$

and

$$\begin{aligned} & (j_{\overline{Q},\gamma_1,\gamma_2,N})_* (\pi_{\gamma_1,\gamma_2,N})_! \varphi_{(\gamma_1,\gamma_2)/u,N} (\pi_{\gamma_1,\gamma_2,N})^* (j_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1] \\ \simeq \ & (j_{\overline{Q},\gamma_1,\gamma_2,N})_* (\pi_{\gamma_1,\gamma_2,N})_! (\pi_{\gamma_1,\gamma_2,N})^* (i_{\gamma_1,\gamma_2,N})_* (i_{\gamma_1,\gamma_2,N})^* (j_{\overline{Q},\gamma_1,\gamma_2,N})^* \end{aligned}$$

obtained from the theorem give us the commutative diagram by pulling back to $\mathbf{M}^{sp}_{\overline{Q},\gamma,\gamma_1,\gamma_2,N} \times \mathbb{C}^*$:

$$\begin{array}{ccc} & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\widehat{Q},\gamma_1,\gamma_2}, W_\gamma) \xrightarrow{\sim} H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2}, W_\gamma) & \\ \nearrow & & \searrow \\ H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\widehat{Q},\gamma}, W_\gamma) & & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2}, W_{\gamma_1,\gamma_2}) \\ \downarrow \wr & & \downarrow \wr \\ H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q}) \otimes \mathbb{T}^{\gamma\cdot\gamma} & \longrightarrow & H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \otimes \mathbb{T}^{l_1} \end{array}$$

where $l_1 = \gamma \cdot \gamma - \gamma_1 \cdot \gamma_2$.

- $H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \xrightarrow{\sim} H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2}, \mathbb{Q}) \otimes \mathbb{T}^{-\gamma_1\cdot\gamma_2}$.

The affine fibrations

$$q_{\gamma_1,\gamma_2,N} : \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1} \times \mathbf{G}_{\gamma_2})}_N \to \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1,\gamma_2})}_N$$

induce isomorphisms

$$\varphi_{(\gamma_1,\gamma_2)/u,N}(\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1,\gamma_2})}_N \times \mathbb{C}^*}) \xrightarrow{\sim} (q_{\gamma_1,\gamma_2,N})_* \mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2}, \mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N \times \mathbb{C}^*}).$$

By applying Verdier duality we get

$$\varphi_{(\gamma_1,\gamma_2)/u,N}((q_{\gamma_1,\gamma_2,N})_! D\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*} \xrightarrow{\sim} D\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*}).$$

Then

$$\varphi_{(\gamma_1,\gamma_2)/u,N}((q_{\gamma_1,\gamma_2,N})_! \mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*} \xrightarrow{\sim} \mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*}\otimes\mathbb{T}^{\gamma_1\cdot\gamma_2})$$

by (3.1), and

$$\begin{aligned}
& (q_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}\\
\simeq\quad & (q_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(q_{\gamma_1,\gamma_2,N})^*\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*}\\
\xrightarrow{\sim}\quad & \varphi_{(\gamma_1,\gamma_2)/u,N}(\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*}\otimes\mathbb{T}^{\gamma_1\cdot\gamma_2})
\end{aligned}$$

by (3.4). Then we have isomorphisms

$$\begin{aligned}
& (\pi_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,N})^*(\mathbb{Q}_{\overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*}\otimes\mathbb{T}^{\gamma_1\cdot\gamma_2})\\
\to\quad & (\pi_{\gamma_1,\gamma_2,N})_!(q_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(q_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma_1,\gamma_2,N})^*\mathbb{Q}_{\overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*}.
\end{aligned}$$

The commutative diagram

$$\begin{array}{ccc}
\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^* & \xrightarrow{q_{\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*\\
\downarrow{\scriptstyle \pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N}} & & \downarrow{\scriptstyle \pi_{\gamma_1,\gamma_2,N}}\\
\overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^* & \xrightarrow{q_{\overline{Q},\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1,\gamma_2})}_N\times\mathbb{C}^*
\end{array}$$

gives us isomorphisms

$$\begin{aligned}
& (\pi_{\gamma_1,\gamma_2,N})_!(q_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(q_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma_1,\gamma_2,N})^*[-1]\\
\simeq\quad & (q_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(q_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1].
\end{aligned}$$

Theorem 3.1.4 implies isomorphisms

$$(\pi_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,N})^*[-1] \simeq (\pi_{\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,N})^*(i_{\gamma_1,\gamma_2,N})_*(i_{\gamma_1,\gamma_2,N})^*$$

and

$$\begin{aligned} & (q_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(q_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1] \\ \simeq\ & (q_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(i_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_*(i_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(q_{\overline{Q},\gamma_1,\gamma_2,N})^*. \end{aligned}$$

Pulling back to $\mathbf{M}^{sp}_{\overline{Q},\gamma_1,\gamma_2,N}\times\mathbb{C}^*$ gives us the commutative diagram

$$\begin{array}{ccc} H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2},W_{\gamma_1,\gamma_2}) & \xrightarrow{\sim} & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2},W_{\gamma_1,\gamma_2})\otimes\mathbb{T}^{-\gamma_1\cdot\gamma_2} \\ \downarrow\wr & & \downarrow\wr \\ H^{\bullet}_{c,\mathbf{G}_{\gamma_1,\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{\gamma\cdot\gamma-\gamma_1\cdot\gamma_2} & \xrightarrow{\sim} & H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{\gamma\cdot\gamma-2\gamma_1\cdot\gamma_2} \end{array}$$

- $H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\xrightarrow{\sim}H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1}\times\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{\Sigma a_{ij}\gamma_1^i\gamma_2^j+\Sigma a_{ij}\gamma_2^i\gamma_1^j}$.

  Similar as the previous step, the affine fibrations

$$p_{\gamma_1,\gamma_2,N}:\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\to\overline{(\mathbf{M}_{\widehat{Q},\gamma_1}\times\mathbf{M}_{\widehat{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N$$

induce isomorphisms

$$\begin{aligned} & (p_{\gamma_1,\gamma_2,N})_!\varphi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N}(p_{\gamma_1,\gamma_2,N})^*\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1}\times\mathbf{M}_{\widehat{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*} \\ \xrightarrow{\sim}\ & \varphi_{\gamma_1\boxplus\gamma_2,N}(\mathbb{Q}_{\overline{(\mathbf{M}_{\widehat{Q},\gamma_1}\times\mathbf{M}_{\widehat{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}\otimes\mathbb{T}^l), \end{aligned}$$

where $l=\sum_{a:i\to j\in\widetilde{Q}_1}\gamma_1^j\gamma_2^i$.

Then we have isomorphisms

$$\begin{aligned}&(\pi_{\gamma_1\times\gamma_2,N})_!(p_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(p_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma_1\times\gamma_2,N})^*\mathbb{Q}_{\overline{(\mathbf{M}_{\overline{Q},\gamma_1}\times\mathbf{M}_{\overline{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}\\ &\xrightarrow{\sim}(\pi_{\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1\boxplus\gamma_2)/u,N}(\pi_{\gamma_1\times\gamma_2,N})^*(\mathbb{Q}_{\overline{(\mathbf{M}_{\overline{Q},\gamma_1}\times\mathbf{M}_{\overline{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*}\otimes\mathbb{T}^l).\end{aligned}$$

The commutative diagram

$$\begin{array}{ccc}\overline{(\mathbf{M}_{\widehat{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^* & \xrightarrow{p_{\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\widehat{Q},\gamma_1}\times\mathbf{M}_{\widehat{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*\\ \downarrow{\scriptstyle \pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N}} & & \downarrow{\scriptstyle \pi_{\gamma_1\times\gamma_2,N}}\\ \overline{(\mathbf{M}_{\overline{Q},\gamma_1,\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^* & \xrightarrow{p_{\overline{Q},\gamma_1,\gamma_2,N}} & \overline{(\mathbf{M}_{\overline{Q},\gamma_1}\times\mathbf{M}_{\overline{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*\end{array}$$

implies isomorphisms

$$\begin{aligned}&(\pi_{\gamma_1\times\gamma_2,N})_!(p_{\gamma_1,\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(p_{\gamma_1,\gamma_2,N})^*(\pi_{\gamma_1\times\gamma_2,N})^*[-1]\\ \simeq\ &(p_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(p_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1].\end{aligned}$$

By Theorem 3.1.4, we have

$$(\pi_{\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1\boxplus\gamma_2)/u,N}(\pi_{\gamma_1\times\gamma_2,N})^*[-1]\simeq(\pi_{\gamma_1\times\gamma_2,N})_!(\pi_{\gamma_1\times\gamma_2,N})^*(i_{\gamma_1\times\gamma_2,N})_*(i_{\gamma_1\times\gamma_2,N})^*$$

and

$$\begin{aligned}&(p_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!\varphi_{(\gamma_1,\gamma_2,\gamma_1\times\gamma_2)/u,N}(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(p_{\overline{Q},\gamma_1,\gamma_2,N})^*[-1]\\ \simeq\ &(p_{\overline{Q},\gamma_1,\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_!(\pi_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(i_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})_*(i_{\gamma_1,\gamma_2,\gamma_1\times\gamma_2,N})^*(p_{\overline{Q},\gamma_1,\gamma_2,N})^*.\end{aligned}$$

By pulling back to $\overline{(\mathbf{M}^{sp}_{\overline{Q},\gamma_1}\times\mathbf{M}^{sp}_{\overline{Q},\gamma_2},\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2})}_N\times\mathbb{C}^*$, we have

$$\begin{array}{ccc} H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\widehat{Q},\gamma_1,\gamma_2},W_{\gamma_1,\gamma_2}) & \xrightarrow{\sim} & H^{\bullet,crit}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\widehat{Q},\gamma_1}\times\mathbf{M}^{sp}_{\widehat{Q},\gamma_2},W_{\gamma_1}\boxplus W_{\gamma_2})\otimes\mathbb{T}^{l} \\ \downarrow\wr & & \downarrow\wr \\ H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\widetilde{\mathbf{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{l_1} & \xrightarrow{\sim} & H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1}\times\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{l_2} \end{array}$$

where $l_1=\gamma\cdot\gamma-\gamma_1\cdot\gamma_2$, and $l_2=\gamma_1\cdot\gamma_1+\gamma_2\cdot\gamma_2+l$.

- $H^{\bullet}_{c,\mathbf{G}_{\gamma_1}\times\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1}\times\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\xrightarrow{\sim}H^{\bullet}_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})\otimes H^{\bullet}_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})$.

  This is the Künneth isomorphism compatible with the Thom-Sebastiani isomorphism by Theorem 3.1.5.

The above computations can be summarized for convenience of the reader in the form of the following statement.

Proposition 4.1.1. *The coproduct making the vector space* $\bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}}H^{\bullet}_{c,\boldsymbol{G}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})$ *into a coalgebra is given by the composition of the maps*

$$\begin{aligned} & H^{\bullet}_{c,\boldsymbol{G}_\gamma}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})\to H^{\bullet}_{c,\boldsymbol{G}_{\gamma_1,\gamma_2}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q}) \\ \longrightarrow\ & H^{\bullet}_{c,\boldsymbol{G}_{\gamma_1,\gamma_2}}(\widetilde{\boldsymbol{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-\gamma_1\cdot\gamma_2} \\ \xrightarrow{\sim}\ & H^{\bullet}_{c,\boldsymbol{G}_{\gamma_1}\times\boldsymbol{G}_{\gamma_2}}(\widetilde{\boldsymbol{M}}^{sp}_{\Pi_Q,\gamma_1,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-2\gamma_1\cdot\gamma_2} \\ \xrightarrow{\sim}\ & H^{\bullet}_{c,\boldsymbol{G}_{\gamma_1}\times\boldsymbol{G}_{\gamma_2}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_1}\times\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-\chi_Q(\gamma_1,\gamma_2)-\chi_Q(\gamma_2,\gamma_1)} \\ \xrightarrow{\sim}\ & H^{\bullet}_{c,\boldsymbol{G}_{\gamma_1}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})\otimes H^{\bullet}_{c,\boldsymbol{G}_{\gamma_2}}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})\otimes\mathbb{T}^{-\chi_Q(\gamma_1,\gamma_2)-\chi_Q(\gamma_2,\gamma_1)}. \end{aligned}$$

Now let

$$\mathcal{H}_\gamma:=H^{\bullet}_{c,\mathbf{G}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})^\vee\otimes\mathbb{T}^{-\chi_Q(\gamma,\gamma)},$$

and

$$\mathcal{H}=\bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}}\mathcal{H}_\gamma.$$

Then the above coproduct makes $\mathcal{H}$ an associative algebra with product

$$
\begin{aligned}
& \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} = H^{\bullet}_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})^{\vee} \otimes \mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)} \otimes H^{\bullet}_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})^{\vee} \otimes \mathbb{T}^{-\chi_Q(\gamma_2,\gamma_2)} \\
= \ & H^{\bullet}_{c,\mathbf{G}_{\gamma_1}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1},\mathbb{Q})^{\vee} \otimes H^{\bullet}_{c,\mathbf{G}_{\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_2},\mathbb{Q})^{\vee} \otimes \mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)-\chi_Q(\gamma_2,\gamma_2)} \\
\to \ & H^{\bullet}_{c,\mathbf{G}_{\gamma_1+\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1+\gamma_2},\mathbb{Q})^{\vee} \otimes \mathbb{T}^{-\chi_Q(\gamma_1,\gamma_2)-\chi_Q(\gamma_2,\gamma_1)} \otimes \mathbb{T}^{-\chi_Q(\gamma_1,\gamma_1)-\chi_Q(\gamma_2,\gamma_2)} \\
= \ & H^{\bullet}_{c,\mathbf{G}_{\gamma_1+\gamma_2}}(\mathbf{M}^{sp}_{\Pi_Q,\gamma_1+\gamma_2},\mathbb{Q})^{\vee} \otimes \mathbb{T}^{-\chi_Q(\gamma_1+\gamma_2,\gamma_1+\gamma_2)} = \mathcal{H}_{\gamma_1+\gamma_2}.
\end{aligned}
$$

Definition 4.1.2. *The associative algebra $\mathcal{H}$ is called the Cohomological Hall algebra of the preprojective algebra Π_Q associated with the quiver Q.*

Remark 4.1.3. *In the framework of equivariant K-theory a similar notion was introduced in [72].*

Corollary 4.1.4. *This product preserves the modified cohomological degree, thus the zero degree part*

$$\mathcal{H}^0 = \bigoplus_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} \mathcal{H}^0_\gamma = \bigoplus_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} H^{-2\chi_Q(\gamma,\gamma)}_{c,\boldsymbol{G}_\gamma}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma},\mathbb{Q})^{\vee} \otimes \mathbb{T}^{-\chi_Q(\gamma,\gamma)}$$

is a subalgebra of $\mathcal{H}$.

Remark 4.1.5. *We can reformulate the definition of COHA of Π_Q using language of stacks. The natural morphism of stacks*

$$\boldsymbol{M}_{\Pi_Q,\gamma_1,\gamma_2}/\boldsymbol{G}_{\gamma_1,\gamma_2} \to \boldsymbol{M}_{\Pi_Q,\gamma}/\boldsymbol{G}_\gamma$$

is proper, hence it induces the pushforward map on $\mathcal{H}$. Composting it with the pullback by the morphism

$$\boldsymbol{M}_{\Pi_Q,\gamma_1,\gamma_2}/\boldsymbol{G}_{\gamma_1,\gamma_2} \to \boldsymbol{M}_{\Pi_Q,\gamma_1}/\boldsymbol{G}_{\gamma_1} \times \boldsymbol{M}_{\Pi_Q,\gamma_2}/\boldsymbol{G}_{\gamma_2},$$

we obtain the product.

4.2 Lusztig's seminilpotent Lagrangian subvariety

In this section we work in the framework close to the one from [6].

Let Q be a quiver (possibly with loops) with vertices I and arrows Ω, and denote by Ω_i the set of loops at $i \in I$. We call i imaginary if the number of loops $\omega_i = |\Omega_i| \geqslant 1$, and real if $\omega_i = 0$. Let I^{im} be the set of imaginary vertices and I^{re} real vertices.

Definition 4.2.1. *A representation $x \in \boldsymbol{M}_{\overline{Q},\gamma}$ is seminilpotent if there is an I-graded filtration $W = (W_0 = V_\gamma \supset \ldots \supset W_r = \{0\})$ of the representation space $V_\gamma = (V_i)_{i\in I}$, such that $x_{a^*}(W_\bullet) \subseteq W_{\bullet+1}$, and $x_a(W_\bullet) \subseteq W_\bullet$ for $a \in \Omega$.*

Remark 4.2.2. *Our definition of seminilpotency is slightly different from that in [6]. We put nilpotent condition on the dual arrows a^* rather than a. But main results of [6] hold in our situation as well.*

We denote by $\mathbf{M}^{sp}_{\overline{Q},\gamma}$ the space of seminilpotent representations of dimension γ. Then by [6, Th. 1.15], the space of seminilpotent representations of Π_Q of dimension γ, $\mathbf{M}^{sp}_{\Pi_Q,\gamma} \subset \mathbf{M}^{sp}_{\overline{Q},\gamma}$, is a Lagrangian subvariety of $\mathbf{M}_{\overline{Q},\gamma}$.

Let

$$\mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l} = \{x \in \mathbf{M}^{sp}_{\Pi_Q,\gamma} | \mathrm{codim}(\bigoplus_{j\neq i, a:j\to i \text{in} \overline{Q}} \mathrm{Im} x_a) = l\}.$$

Then $\mathbf{M}^{sp}_{\Pi_Q,\gamma} = \bigcup_{i\in I, l\geqslant 1} \mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l}$ by the seminilpotency condition. There is a one to one correspondence of the sets of irreducible components (see [6, Prop.1.14])

$$\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l}) \xrightarrow{\sim} \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma-le_i,i,0}) \times \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i}), \tag{4.1}$$

where $e_i = (\delta_{ij})_{j\in I}$. For any vertex i, we have $\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}) = \bigsqcup_{l\geqslant 0} \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l})$. Now let us discuss case by case.

1) If $i \in I^{re}$ then $\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i})$ consists of only one element, namely the zero representation. We denote by $Z_{i,l}$ the only element in $\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,le_i})$.

2) If $i \in I^{im}$, then there are two cases.

a) If the number of loops $\omega_i = 1$, then $\mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q, le_i})$ is parametrized by $\mathfrak{C}_{i,l} = \{c = (c_k)\}$, the set of partitions of l (i.e., $\sum_k c_k = l$, $c_k > 0, \forall k$, and $c_{k+1} \geqslant c_k$).

b) If $\omega_i > 1$, then it is parametrized by the set of compositions also denoted by $\mathfrak{C}_{i,l}$ (i.e., $\sum_k c_k = l$, $c_k > 0, \forall k$).

We put $|c| = \sum_k c_k$ for $c \in \mathfrak{C}_{i,l}$, and denote by $Z_{i,c} \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q, le_i})$ the irreducible component corresponding to c. Let $Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma})$, then there exists $i \in I$ and $l \geqslant 1$ such that $Z \bigcap \mathbf{M}^{sp}_{\Pi_Q,\gamma,i,l}$ is dense in Z. We denote by $\varepsilon_i(Z)$ the corresponding partition or composition if $i \in I^{im}$, and $\varepsilon_i(Z) = l$ if $i \in I^{re}$, via the one to one correspondence (4.1).

Now let $\mathscr{M}_\gamma$ be the $\mathbb{Q}$-vector space of constructible functions $f : \mathbf{M}^{sp}_{\Pi_Q,\gamma} \to \mathbb{Q}$ which are constant on any $\mathbf{G}_\gamma$-orbit, and $\mathscr{M} = \bigoplus_\gamma \mathscr{M}_\gamma$. Then one can define a product $*$ on $\mathscr{M}$ in the way which is analogous to the definition of Lusztig for nilpotent case in [46, Section 12].

More precisely, let us denote by $\mathbf{M}^{sp}_{\Pi_Q,V}$ the space of seminilpotent representations of Π_Q with I-graded vector space V, and $\mathscr{M}_V$ the $\mathbb{Q}$-vector space of constructible functions $f : \mathbf{M}^{sp}_{\Pi_Q,V} \to \mathbb{Q}$ constant on any $\mathbf{G}_\gamma$-orbit. Let V_1, V_2 and V be I-graded vector spaces of dimensions γ_1, γ_2 and $\gamma = \gamma_1 + \gamma_2$ respectively, and $f_i \in \mathscr{M}_{V_i}, i = 1, 2$. Then $f_1 * f_2 \in \mathscr{M}_V$ is defined using the diagram

$$\mathbf{M}^{sp}_{\Pi_Q,V_1} \times \mathbf{M}^{sp}_{\Pi_Q,V_2} \xleftarrow{\;p_1\;} \mathbf{F}' \xrightarrow{\;p_2\;} \mathbf{F}'' \xrightarrow{\;p_3\;} \mathbf{M}^{sp}_{\Pi_Q,V}$$

where the notations are as follows:

- F'' is the variety of pairs (x, U) with $x \in \mathbf{M}^{sp}_{\Pi_Q,V}$ and U an x-stable I-graded subspace of V with dimension γ_2;
- F' is the variety of quadruples (x, U, R'', R') where $(x, U) \in F''$, $R'' : V_2 \xrightarrow{\sim} U$ and $R' : V_1 \xrightarrow{\sim} V/U$;
- The map $p_1(x, U, R'', R') = (x_1, x_2)$ where $xR' = R'x_1$ and $xR'' = R''x_2$,

- $p_2(x, U, R'', R') = (x, U)$,

- $p_3(x, U) = x$.

Note that p_2 is a $\mathbf{G}_{V_1} \times \mathbf{G}_{V_2}$-principal bundle and p_3 is proper. Let $f(x_1, x_2) = f_1(x_1)f_2(x_2)$, then there is a unique function $f_3 \in \mathscr{M}_{F''}$ such that $p_1^* f = p_2^* f_3$. Finally, define $f_1 * f_2 = (p_3)_!(f_3)$. By identifying the vector spaces $\mathscr{M}_V$ for various V with $\mathscr{M}_\gamma$ in a coherent way ($\dim(V) = \gamma$), we define the product $*$ on $\mathscr{M}$, making it an associative $\mathbb{Q}$-algebra.

One can also reformulate this product using the diagram of stacks

$$\mathbf{M}_{\Pi_Q,\gamma_2}/\mathbf{G}_{\gamma_2} \times \mathbf{M}_{\Pi_Q,\gamma_1}/\mathbf{G}_{\gamma_1} \leftarrow \mathbf{M}_{\Pi_Q,\gamma_2,\gamma_1}/\mathbf{G}_{\gamma_2,\gamma_1} \rightarrow \mathbf{M}_{\Pi_Q,\gamma}/\mathbf{G}_\gamma.$$

We denote by $1_{i,c}$ (resp. $1_{i,l}$) the characteristic function of $Z_{i,c}$ (resp. $Z_{i,l}$), and $\mathscr{M}_0 \subseteq \mathscr{M}$ the subalgebra generated by $1_{i,(l)}$ and $1_{i,1}$. For any $Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma})$ and $f \in \mathscr{M}_\gamma$, let $\rho_Z(f) = c$ if $Z \bigcap f^{-1}(c)$ is open dense in Z.

Theorem 4.2.3. *(see [6, Prop. 1.18]) For any $Z \in \mathrm{Irr}(\boldsymbol{M}^{sp}_{\Pi_Q,\gamma})$ there exists $f_Z \in \mathscr{M}_{0,\gamma} = \mathscr{M}_0 \cap \mathscr{M}_\gamma$ such that $\rho_Z(f_Z) = 1$, and $\rho_{Z'}(f_Z) = 0$ for $Z' \neq Z$.*

4.3 Generalized quantum group

We recall some definitions and facts about generalized quantum group introduced in [6].

Let $(\bullet, \bullet)$ be the symmetric Euler form on $\mathbb{Z}^I$ defined by $(i, j) = 2\delta_{ij} - a_{ij} - a_{ji}$, and $(\iota, j) = l(i, j)$ if $\iota = (i, l) \in I_\infty = (I^{re} \times \{1\}) \bigcup (I^{im} \times \mathbb{N}_{\geqslant 1})$ and $j \in I$.

Definition 4.3.1. *Let F be the $\mathbb{Q}(v)$-algebra generated by $(E_\iota)_{\iota \in I_\infty}$, $\mathbb{N}^I$-graded by $|E_\iota| = li$ for $\iota = (i, l)$. If $A \subseteq \mathbb{N}^I$, then let $\mathrm{F}[A] = \{E \in \mathrm{F} | |E| \in A\}$.*

For any $\gamma = (\gamma^i)_{i \in I} \in \mathbb{Z}^I$, let $\mathrm{ht}(\gamma) = \sum_i \gamma^i$ be its height, and $v_\gamma = \prod_i v_i^{\gamma^i}$, where $v_i = v^{(i,i)/2}$. We endow F with a coproduct $\delta(E_{i,l}) = \sum\limits_{l_1+l_2=l} v_i^{l_1 l_2} E_{i,l_1} E_{i,l_2}$, where $E_{i,0} = 1$. Then for any family $(v_\iota)_{\iota \in I_\infty} \subseteq \mathbb{Q}(v)$, there is a bilinear form $\{\bullet, \bullet\}$ on F such that

- $\{E, E'\} = 0$ if $|E| \neq |E'|$,
- $\{E_\iota, E_\iota\} = v_\iota$, $\forall \iota \in I_\infty$,
- $\{EE', E''\} = \{E \otimes E', \delta(E'')\}$, $\forall E, E', E'' \in \mathrm{F}$.

It turns out that $\sum\limits_{l_1+l_2=-(\iota,j)+1} (-1)^{l_1} \frac{E_{j,1}^{l_1}}{l_1!} E_\iota \frac{E_{j,1}^{l_2}}{l_2!}$ is in the radical of $\{\bullet, \bullet\}$.

Definition 4.3.2. *Let $\widetilde{\mathcal{U}}^+$ be the quotient of F by the ideal generated by the above element and the commutators $[E_{i,l}, E_{i,k}]$ for $\omega_i = 1$. Then $\{\bullet, \bullet\}$ is well-defined on $\widetilde{\mathcal{U}}^+$. Let $\mathcal{U}^+$ be the quotient of $\widetilde{\mathcal{U}}^+$ by the radical of $\{\bullet, \bullet\}$.*

Theorem 4.3.3. *(see [6, Th. 3.34]) There is an isomorphism of algebras*

$$\begin{aligned} \phi: \ & \mathcal{U}^+_{v=1} \to \mathscr{M}_0, \\ & E_{i,(l)} \mapsto 1_{i,(l)}, \quad i \in I^{im}, \\ & E_{i,1} \mapsto 1_{i,1}, \quad i \in I^{re}. \end{aligned}$$

Definition 4.3.4. *The semicanonical basis of $\mathcal{U}^+_{v=1}$ is $\phi^{-1}(\{f_Z | Z \in \mathrm{Irr}(\boldsymbol{M}^{sp}_{\Pi_Q})\})$.*

4.4 Semicanonical basis of $\mathcal{H}^0$

We have already seen that for an appropriate subspace $\mathbf{M}^{sp}_{\overline{Q},\gamma} \subset \mathbf{M}_{\overline{Q},\gamma}$, the degree 0 part $\mathcal{H}^0 \subset \mathcal{H}$ is a subalgebra of COHA. In particular, we can take $\mathbf{M}^{sp}_{\overline{Q},\gamma}$ to be the space of seminilpotent representations of $\overline{Q}$. Then $\mathbf{M}^{sp}_{\Pi_Q,\gamma}$ is the space of seminilpotent representations in $\mathbf{M}_{\Pi_Q,\gamma}$, and $\dim(\mathbf{M}^{sp}_{\Pi_Q,\gamma}/\mathbf{G}_\gamma) = -\chi_Q(\gamma,\gamma)$, so the classes of irreducible components $\{[Z] | Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma})\}$ lie in $\mathcal{H}^0$. In fact, these classes form a basis of $\mathcal{H}^0$ by the following theorem.

Theorem 4.4.1. *Let X be a scheme with top dimensional irreducible components $\{C^k\}$, and a connected algebraic group G acts on it. Then $H^{2top}_{c,G}(X)$ has a basis one to one corresponding to $\{C^k\}$, where top is the dimension of the stack X/G.*

Proof. Choose an embedding of groups $G \hookrightarrow GL(n, \mathbb{C})$. Let $fr(n, N)$ be the space of n-tuples of linearly independent vectors in $\mathbb{C}^N$ for $N \geqslant n$. Then $X \times fr(n, N)$ has irreducible components $\{C^k \times fr(n, N)\}$. Thus

$$X \times_G fr(n, N) = (X \times fr(n, N))/G$$

has irreducible components $\{\overline{C^k}\}$ one to one corresponding to $\{C^k\}$ since G is irreducible. Then the Borel-Moore homology $H^{BM}_{2\bullet}(X \times_G fr(n, N))$ has a basis $\{[\overline{C^k}]\}$, where

$$\bullet = \dim(X) + \dim(fr(n, N)) - \dim G,$$

implying that

$$H^{2\bullet}_c(X \times_G fr(n, N))^\vee = H^{BM}_{2\bullet}(X \times_G fr(n, N))$$

has basis one to one corresponding to $\{C^k\}$ (For details of Borel-Moore homology, see [11, Section 2.6]). Then

$$H^{2top}_{c,G}(X) = \lim_{N \to \infty} H^{2\bullet}_c(X \times_G fr(n, N)) \otimes \mathbb{T}^{-\dim fr(n,N)}$$

has basis one to one corresponding to $\{C^k\}$, where $top = \bullet - \dim(fr(n, N)) = \dim(X/G)$. $\square$

Definition 4.4.2. *We call the basis defined above the semicanonical basis of the subalgebra* $\mathcal{H}^0$*.*

Given an element $\mathcal{F}$ in $\mathscr{D}^b(X)$ with constructible cohomology, and $x \in X$, the function $\chi(\mathcal{F})(x) = \chi(\mathcal{F}_x) = \sum_i (-1)^i \dim(H^i(\mathcal{F}_x))$ is constructible. Moreover, the standard operations (pullback, pushforward, etc.) in $\mathscr{D}^b(X)$ and the corresponding operations on constructible functions are compatible.

Recall the family of constructible functions $\{f_Z | Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q})\}$. Then $U_Z = f_Z^{-1}(1)$ is constructible. Let $f_{Z,N}$ be the characteristic function of $\overline{(U_Z, \mathbf{G}_\gamma)}_N$, and $\mathbb{Q}_{Z,N}$ be the constant sheaf on $\overline{(U_Z, \mathbf{G}_\gamma)}_N$. Since the operations on constructible functions and constructible sheaves agree, there is an isomorphism of algebras $\Psi : \mathcal{H}^0 \to \mathscr{M}_0^{op}, [Z] \mapsto f_Z$. It is obtained by taking the dual of compactly supported cohomology and passing to the limit.

Furthermore, notice that $\mathcal{H}^0 \simeq (\mathcal{U}_{v=1}^+)^{op}$, and that Lusztig's product $*$ is opposite to the product of COHA.

The semicanonical basis of $\mathcal{H}^0$ is compatible with a certain filtration. More precisely, we have the following result.

Theorem 4.4.3. *Fix $d = (d_i) \in \mathbb{Z}^I_{\geqslant 0}$. Then the subspace spanned by*

$$\{[Z] | \exists i, s.t. |\varepsilon_i(Z)| \geqslant d_i\}$$

coincides with $\sum_{i\in I, |c|=d_i} \mathcal{H}^0[Z_{i,c}]$, where $Z_{i,c} \in \mathrm{Irr}(\boldsymbol{M}^{sp}_{\Pi_Q, le_i})$ is the irreducible component corresponding to c (defined in Section 2.3), and $c = l$ if $i \in I^{re}$.

Proof. By definitions, $\sum_{i\in I, |c|=d_i} \mathcal{H}^0[Z_{i,c}]$ is contained in the subspace spanned by

$$\{[Z] | \exists i, s.t. |\varepsilon_i(Z)| \geqslant d_i\}.$$

To prove the reverse inclusion it suffices to show that for any $i \in I$, $\gamma \in \mathbb{Z}^I_{\geqslant 0}$, and $[Z] \in \mathcal{H}^0$ such that $Z \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q, \gamma})$ and $|\varepsilon_i(Z)| = l$, we have $[Z] \in \sum_{|c|=l} \mathcal{H}^0[Z_{i,c}]$. We use descending induction on $l \leqslant \gamma^i$. For above Z, we have $\gamma - le_i \in \mathbb{N}^I$, and by the proof of [6, Pro. 1.18], there exists a unique $Z' \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q, \gamma - le_i})$ and $Z_{i,c} \in \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q, le_i})$ such that $|\varepsilon_i(Z')| = 0$ and

$$[Z'][Z_{i,c}] = Z + \sum_{|\varepsilon_i(\widetilde{Z})| > l} a_{\widetilde{Z}} [\widetilde{Z}]$$

for some $a_{\widetilde{Z}} \in \mathbb{Q}$. By applying the induction hypothesis to $\widetilde{Z}$ we have that the subspace spanned by

$$\{[Z] | \exists i, s.t. |\varepsilon_i(Z)| \geqslant d_i\}$$

is contained in $\sum_{i\in I,|c|=d_i} \mathcal{H}^0[Z_{i,c}]$. Thus the two subspaces coincide. □

The dual of representations of Π_Q induces a bijection

$$* : \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}) \to \mathrm{Irr}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}),$$

$$Z \mapsto Z^*,$$

thus an antiautomorphism of $\mathcal{H}^0$. Then the dual of the above theorem holds:

Theorem 4.4.4. *The subspace spanned by*

$$\{[Z] | \exists i, s.t. |\varepsilon_i(Z^*)| \geqslant d_i\}$$

coincides with $\sum_{i\in I,|c|=d_i} [Z_{i,c}]\mathcal{H}^0$.

4.5 COHA as a shuffle algebra

The critical COHA of any quiver with potential (Q, W) is a shuffle algebra according to [12, Sec. 4], thus induces a shuffle algebra structure on the COHA of a preprojective algebra. To be precise, for a dimension vector $\gamma \in \mathbb{Z}^I_{\geqslant 0}$, let

$$\mathbf{T}_\gamma := \prod_{i\in I} (\mathbb{C}^*)^{\gamma^i} \subset \mathbf{G}_\gamma$$

be a maximal torus, and consider the $\mathbf{T}_\gamma$-equivariant critical cohomology with compact support $H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)$, and its dual $H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee$. Both of them admit an action of the product of symmetric groups $\mathrm{Sym}_\gamma := \prod_{i\in I} \mathrm{Sym}_{\gamma^i}$. Recall that there is a $H^\bullet_{\mathbf{T}_\gamma}(\mathrm{pt}, \mathbb{Q})$-

module structure on $H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee$. Furthermore,

$$H^\bullet_{\mathbf{G}_\gamma}(\mathrm{pt},\mathbb{Q}) = H^\bullet_{\mathbf{T}_\gamma}(\mathrm{pt},\mathbb{Q})^{\mathrm{Sym}_\gamma} = \bigotimes_{i\in I}\mathbb{C}[x_{i,1},\dots,x_{i,\gamma^i}]^{\mathrm{Sym}_{\gamma^i}}.$$

Definition 4.5.1. *Fix two dimension vectors γ_1 and γ_2, we denote*

$$\mathfrak{C}(Q,\gamma_1,\gamma_2) = \prod_{i\in I}\prod_{m=1}^{\gamma_1^i}\prod_{m'=1}^{\gamma_2^i}(x^{(2)}_{i,m'} - x^{(1)}_{i,m}),$$

where $x^{(1)}_{i,m} \in H^\bullet_{\boldsymbol{G}_{\gamma_1}}(\mathrm{pt},\mathbb{Q})$ and $x^{(2)}_{i,m'} \in H^\bullet_{\boldsymbol{G}_{\gamma_2}}(\mathrm{pt},\mathbb{Q})$.

Proposition 4.5.2. *(see [12, Prop. 4.3]) There are natural maps*

$$H^{\bullet,crit}_{c,\boldsymbol{T}_\gamma}(\boldsymbol{M}^{sp}_{Q,\gamma}, W_\gamma)^{\mathrm{Sym}_\gamma}\otimes\mathbb{T}^{\sum_{i\in I}((\gamma^i)^2-\gamma^i)} \xrightarrow{\sim} H^{\bullet,crit}_{c,\boldsymbol{G}_\gamma}(\boldsymbol{M}^{sp}_{Q,\gamma}, W_\gamma)$$

which are isomorphisms.

Let

$$\mathcal{T}^{crit}_\gamma := (H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee)^{\mathrm{Sym}_\gamma}\otimes\mathbb{T}^{-\chi_Q(\gamma,\gamma)+\sum_{i\in I}((\gamma^i)^2-\gamma^i)},$$

and

$$\mathcal{T}^{crit} = \bigoplus_{\gamma\in\mathbb{Z}^I_{\geqslant 0}}\mathcal{T}^{crit}_\gamma.$$

The product on the space $\mathcal{T}^{crit}$ is defined as follows.

First consider $H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee$.

- $H^{\bullet,crit}_{c,\mathbf{T}_{\gamma_1}}(\mathbf{M}^{sp}_{Q,\gamma_1}, W_{\gamma_1})^\vee\otimes H^{\bullet,crit}_{c,\mathbf{T}_{\gamma_2}}(\mathbf{M}^{sp}_{Q,\gamma_2}, W_{\gamma_2})^\vee = H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma_1}\times\mathbf{M}^{sp}_{Q,\gamma_2}, W_{\gamma_1}\boxplus W_{\gamma_2})^\vee$. This is the Thom–Sebastiani isomorphism.
- $H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma_1}\times\mathbf{M}^{sp}_{Q,\gamma_2}, W_{\gamma_1}\boxplus W_{\gamma_2})^\vee \to H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma_1,\gamma_2}, W_{\gamma_1,\gamma_2})^\vee\otimes\mathbb{T}^{\sum_{i,j\in I}a_{ij}\gamma_2^i\gamma_1^j}$ is the pullback associated to the affine fibration $\mathbf{M}_{Q,\gamma_1,\gamma_2}\to\mathbf{M}_{Q,\gamma_1}\times\mathbf{M}_{Q,\gamma_2}$.

- $H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma_1,\gamma_2}, W_{\gamma_1,\gamma_2})^\vee \to H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee$ is the pushforward induced by the inclusion $\mathbf{M}_{Q,\gamma_1,\gamma_2} \to \mathbf{M}_{Q,\gamma}$.

- $H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee \to H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee[\mathfrak{C}(Q,\gamma_1,\gamma_2)^{-1}] \otimes \mathbb{T}^{-\gamma_1\cdot\gamma_2}$ is the division by $\mathfrak{C}(Q,\gamma_1,\gamma_2)$.

All the above maps are $\mathrm{Sym}_{\gamma_1} \times \mathrm{Sym}_{\gamma_2}$-equivariant. By restricting to invariant parts, composing the above maps and taking sum over all the shuffles of (γ_1,γ_2) into γ, we get a map

$$(H^{\bullet,crit}_{c,\mathbf{T}_{\gamma_1}}(\mathbf{M}^{sp}_{Q,\gamma_1}, W_{\gamma_1})^\vee)^{\mathrm{Sym}_{\gamma_1}} \otimes (H^{\bullet,crit}_{c,\mathbf{T}_{\gamma_2}}(\mathbf{M}^{sp}_{Q,\gamma_2}, W_{\gamma_2})^\vee)^{\mathrm{Sym}_{\gamma_2}} \to (H^{\bullet,crit}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee_L)^{\mathrm{Sym}_\gamma} \otimes \mathbb{T}^c,$$

where the subscript L means $\pi_*\mathfrak{C}(Q,\gamma_1,\gamma_2)$ is formally inverted for every shuffle π.

Proposition 4.5.3. *(see [12, Cor. 4.7])*

The above map factors through $(H^{\bullet,crit}_{c,\boldsymbol{T}_\gamma}(\boldsymbol{M}^{sp}_{Q,\gamma}, W_\gamma)^\vee)^{\mathrm{Sym}_\gamma}$, and induces an associative multiplication ($\boldsymbol{T}$-equivariant multiplication) on $\mathcal{T}^{crit}$.

Proposition 4.5.4. *(see [12, Cor. 4.8]) The algebra $\mathcal{T}^{crit}$ is isomorphic to the critical COHA of (Q,W) defined in Chapter 3.*

Given a quiver Q, we apply the above definition to the triple quiver with potential $(\widehat{Q}, W)$. Using dimensional reduction, we obtain an associative algebra ($\mathbf{T}$-equivariant COHA of the preprojective algebra Π_Q)

$$\mathcal{T} = \bigoplus_{\gamma \in \mathbb{Z}^I_{\geqslant 0}} \mathcal{T}_\gamma,$$

where

$$\mathcal{T}_\gamma = (H^{\bullet}_{c,\mathbf{T}_\gamma}(\mathbf{M}^{sp}_{\Pi_Q,\gamma}, \mathbb{Q})^\vee)^{\mathrm{Sym}_\gamma} \otimes \mathbb{T}^{-\chi_Q(\gamma,\gamma)+\sum_{i\in I}((\gamma^i)^2-\gamma^i)}.$$

It is straightforward to see

Theorem 4.5.5. *Given a quiver Q, the $\boldsymbol{T}$-equivariant COHA $\mathcal{T}$ of its preprojective algebra Π_Q is isomorphic to the COHA $\mathcal{H}$ of Π_Q.*

Chapter 5

2 Calabi-Yau categories and quivers

In this chapter we first recall the definition of ind-constructible Calabi-Yau categories. Then we will prove that the equivalence classes of a certain class of 2 Calabi-Yau categories are in one-to-one correspondence with a certain type of quivers. This is an analog of the statement of 3 Calabi-Yau case in [42, Sec. 8].

5.1 Calabi-Yau categories

We give a basic introduction of ind-constructible Calabi-Yau categories following [42].

5.1.1 Ind-constructible categories

Let $\mathbf{k}$ be a field with $\overline{\mathrm{k}}$ its algebraic closure.

Definition 5.1.1. *Let S be a variety over* k*, i.e., a reduced separated scheme of finite type over* k*. A subset $X \subset S(\mathrm{k})$ is called constructible over* k *if it belongs to the Boolean algebra generated by $\overline{\mathrm{k}}$-points of open (equivalently closed) subschemes of S.*

In other words, a constructible set is the union of a finite collection of $\overline{\mathrm{k}}$-points of disjoint locally closed subvarieties $(S_i \subset S)_i$.

The category $\mathcal{CON}_{\mathbf{k}}$ of constructible sets over $\mathbf{k}$ has objects (X, S) as above. The morphisms $Hom_{\mathcal{CON}_{\mathbf{k}}}((X_1, S_1), (X_2, S_2))$ is defined to be the set of maps $f : X_1 \to X_2$ such that there exists a decomposition of X_1 into the finite disjoint union of $\overline{\mathrm{k}}$-points of varieties $(S_i \subset S_1)_i$ so that the restriction of f to each $S_i(\overline{\mathrm{k}})$ is a morphism of schemes $S_i \to S_2$.

Definition 5.1.2. *An ind-constructible set over* k *is given by a chain of embeddings of constructible sets* $X := (X_1 \to X_2 \to X_3 \to \cdots)$. *A morphism of ind-constructible sets* $X := (X_1 \to X_2 \to X_3 \to \cdots)$ *and* $Y := (Y_1 \to Y_2 \to Y_3 \to \cdots)$ *is defined as* $g : \cup_i X_i(\overline{\mathrm{k}}) \to \cup_i Y_i(\overline{\mathrm{k}})$, *such that for any* i *there is an* n_i *so that* $g|_{X_i(\overline{\mathrm{k}})} : X_i(\overline{\mathrm{k}}) \to Y_{n_i}(\overline{\mathrm{k}})$ *comes from a constructible map.*

We have the following ind-constructible version of the notion of an A_∞-category:

Definition 5.1.3. *An ind-constructible* A_∞*-category over* k *consists of the data:*

1) The set of objects

$$\mathcal{M} = Ob(\mathcal{C}) = \sqcup_{i \in I} X_i,$$

which is an ind-constructible set over k.

2) The bundles of morphisms of degree n, which is a collection of ind-constructible vector bundles

$$\mathcal{HOM}^n \to \mathcal{M} \times \mathcal{M}, n \in \mathbb{Z}.$$

The restriction $\mathcal{HOM}^n \to X_i \times X_j$ *is a finite-dimensional constructible vector bundle for any* $n \in \mathbb{Z}, i, j \in I$, *and there exists a constant* $C_{i,j}$ *such that* $\mathcal{HOM}^n \to X_i \times X_j$ *is a zero bundle for* $n \leqslant C_{i,j}$.

3) The higher composition maps, which are ind-constructible morphisms of ind-constructible bundles

$$m_n : p_{1,2}^* \mathcal{HOM}^{l_1} \otimes \cdots \otimes p_{n,n+1}^* \mathcal{HOM}^{l_n} \to p_{1,n+1}^* \mathcal{HOM}^{l_1 + \cdots + l_n + 2 - n},$$

for $n \geqslant 1, l_1, \ldots, l_n \in \mathbb{Z}$. *Here* $p_{i,i+1}$ *and* $p_{1,n+1}$ *denote the natural projections from* $\mathcal{M}^{n+1}$ *to* $\mathcal{M}^2$.

The above data satisfy the axioms:

A1) *Higher associativity property for* $m_n, n \geqslant 1$ *in the sense of* A_∞*-categories.*

A2) *(weak unit) There exists a constructible section* s *of the ind-constructible bundle* $\mathcal{HOM}^0_{|diag} \to \mathcal{M}$ *such that the image of* s *belongs to the kernel of* m_1*, and gives rise to the identity morphisms in* $\mathbb{Z}$*-graded* $\overline{\mathrm{k}}$*-linear category* $H^\bullet(\mathcal{C}(\overline{\mathrm{k}}))$.

An ind-constructible A_∞-category $\mathcal{C}$ gives rise to a collection of ind-constructible bundles over $Ob(\mathcal{C}) \times Ob(\mathcal{C})$ given by

$$\mathcal{EXT}^i := H^i(\mathcal{HOM}^\bullet), i \in \mathbb{Z},$$

whose fiber over a pair of objects (E, F) is

$$Ext^i(E, F) := H^i(\mathcal{HOM}^\bullet_{E,F}, m_1).$$

A3) *(local regularity) There exists a family of schemes* (S_i) *of finite type over* k*, a collection of algebraic* k*-vector bundles* $HOM^n_i, n \in \mathbb{Z}$ *over* $S_i \times S_i$ *for all* i*, and ind-constructible identifications*

$$\sqcup_i S_i(\overline{\mathrm{k}}) \simeq \mathcal{M}, HOM^n_i \simeq \mathcal{HOM}^n_{|S_i \times S_i}, n \in \mathbb{Z},$$

such that all higher compositions $m_n, n \geqslant 2$ *are morphisms of algebraic vector bundles considered for objects in* S_i *for any given* i*.*

The basic example of an ind-constructible A_∞-category is the category $Perf(A)$ of perfect A-modules, where A is an A_∞-algebra over $\mathbf{k}$ with finite dimensional cohomology.

Definition 5.1.4. *An ind-constructible* A_∞*-category is called minimal on the diagonal if the restriction of* m_1 *to the diagonal* $\Delta \subset \mathcal{M} \times \mathcal{M}$ *is trivial.*

Any ind-constructible A_∞-category is equivalent to one which is minimal on the diagonal.

Remark 5.1.5. *One can define the property of an ind-constructible weakly unital A_∞-category $\mathcal{C}$ to be triangulated using the notion of a functor between two ind-constructible A_∞-categories. See [42, Sec. 3.1].*

5.1.2 Ind-constructible Calabi-Yau categories

Assume that the field **k** has characteristic zero.

Definition 5.1.6. *A Calabi-Yau category of dimension d is a weakly unital* k*-linear triangulated A_∞-category $\mathcal{C}$, such that the $\mathbb{Z}$-graded vector space $Hom^\bullet(E,F) = \oplus_{n\in\mathbb{Z}} Hom^n(E,F)$ is finite-dimensional for any objects E and F. This implies that $Ext^\bullet(E,F)$ is also finite-dimensional. Moreover, we have the following data:*

- *A non-degenerate pairing*

$$(\bullet,\bullet) : Hom^\bullet(E,F) \otimes Hom^\bullet(F,E) \to \mathrm{k}[-d],$$

  which is symmetric with respect to interchaging E and F.

- *A polylinear $\mathbb{Z}/N\mathbb{Z}$-invariant map*

$$W_N : \otimes_{1\leqslant i\leqslant N}(Hom^\bullet(E_i, E_{i+1})[1]) \to \mathrm{k}[3-d],$$

  for any $N \geqslant 2$ and objects $E_1 = E_{N+1}, \ldots, E_N$.

- *The above maps are compatible in the sense of*

$$W_N(a_1,\ldots,a_N) = (m_{N-1}(a_1,\ldots,a_{N-1}), a_N).$$

The collection $(W_N)_{N\leqslant 2}$ is called the potential of $\mathcal{C}$.

In the following sections we will consider $d = 2$ case, namely, 2-dimensional Calabi-Yau categories.

5.2 Correspondence between quivers and 2CY categories

In Section 8 of [42], M. Kontsevich and Y. Soibelman proved that the equivalence classes of a certain type of 3-dimensional Calabi-Yau categories are in one-to-one correspondence with the gauge equivalence classes of quivers with minimal potential (Q, W). This section gives an analogue in 2-dimensional Calabi-Yau case. We assume that $\mathbf{k}$ is a field of characteristic zero.

Theorem 5.2.1. *Let $\mathcal{C}$ be an ind-constructible 2-dimensional* k*-linear Calabi-Yau category generated by a finite collection $\mathcal{E} = \{E_i\}_{i \in I}$ of generators satisfying*

- $Ext^0(E_i, E_i) = \mathrm{k} \cdot id_{E_i}$,
- $Ext^0(E_i, E_j) = 0, \forall i \neq j$,
- $Ext^{<0}(E_i, E_j) = 0, \forall i, j$.

The equivalence classes of such categories with respect to A_∞-transformations preserving the Calabi-Yau structure and $\mathcal{E}$, are in one-to-one correspondence with finite symmetric quivers with even number of loops at each vertex.

Proof. Let's denoted by $\mathscr{A}$ the set of equivalence classes of such 2 Calabi-Yau categories, and $\mathscr{B}$ the set of finite symmetric quivers with even number of loops at each vertex.

Given such a category $\mathcal{C}$, we associate a quiver Q whose vertices $\{i\}_{i \in I}$ are in one-to-one correspondence with $\mathcal{E} = \{E_i\}_{i \in I}$, and the number of arrows from i to j is equal to $\dim Ext^1(E_i, E_j)$. Since $\mathcal{C}$ is 2 Calabi-Yau, we have

$$\dim Ext^1(E_i, E_j) = \dim Ext^1(E_i, E_j)^\vee = \dim Ext^1(E_j, E_i),$$

so Q is symmetric. The supersymmetric non-degenerate pairing on $Ext^{\bullet}(E_i, E_i)$ leads to a symplectic pairing on $Ext^1(E_i, E_i)$, thus $\dim Ext^1(E_i, E_i)$ is even, which means that the number of loops at each vertex is even. This construction defines a map $\Phi : \mathscr{A} \to \mathscr{B}$.

To prove that Φ is a bijection, we consider a category $\mathcal{C}$ with single generator E, and a quiver Q with single vertex for simplicity. The general case can be proved in a similar way.

Let Q be a quiver with one vertex and $|J| = 2n$ loops, where J is the set of loops. We will construct a 2 Calabi-Yau category with one generator E, such that $2n = \dim\ Ext^1(E, E)$. Assuming that such a category exists, we will find an explicit formula for the potential on $A = Hom^{\bullet}(E, E)$. Let's consider the graded vector space

$$Ext^{\bullet}(E, E)[1] = Ext^0(E, E)[1] \oplus Ext^1(E, E) \oplus Ext^2(E, E)[-1] = \mathbf{k}[1] \oplus \mathbf{k}^{2n} \oplus \mathbf{k}[-1].$$

We introduce graded coordinates on $Ext^{\bullet}(E, E)[1]$:

a) the coordinate α of degree 1 on $Ext^0(E, E)[1]$,

b) the coordinate β of degree -1 on $Ext^2(E, E)[-1]$,

c) the coordinates $x_i, \xi_i, i = 1, ..., n$ of degree 0 on $Ext^1(E, E) = Ext^1(E, E)^{\vee}$.

The Calabi-Yau structure gives rise to the minimal potential $W = W(\alpha, x_i, \xi_i, \beta)$, which is a series of cyclic words on the space $Ext^{\bullet}(E, E)[1]$. Furthermore, A defines a non-commutative formal pointed graded manifold endowed with a symplectic structure (c.f. [42]). The potential W satisfies the equation $\{W, W\} = 0$, where $\{\bullet, \bullet\}$ is the corresponding Poisson bracket.

We need to construct the formal series W of degree 1 in cyclic words on the graded vector space $\mathbf{k}[1] \oplus \mathbf{k}^{2n} \oplus \mathbf{k}[-1]$, satisfying $\{W, W\} = 0$ with respect to the Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i}](f, g) + [\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta}](f, g).$$

Let

$$W_{can} = \alpha^2\beta + \sum_{i=1}^{n}(\alpha x_i\xi_i - \alpha\xi_i x_i).$$

This potential makes $Ext^{\bullet}(E,E)$ into a 2 Calabi-Yau algebra with associative product and the unit. The multiplications are as follows: the multiplication of $Ext^0(E,E)$ and the other components is scalar product, and is a non-degenerate bilinear form on the components

$$Ext^1(E,E) \otimes Ext^1(E,E) \to Ext^2(E,E) \simeq \mathbf{k}.$$

In addition,

$$\begin{aligned}
\{W_{can}, W_{can}\} &= \sum_{i=1}^{n}[\tfrac{\partial W_{can}}{\partial x_i}, \tfrac{\partial W_{can}}{\partial \xi_i}] + [\tfrac{\partial W_{can}}{\partial \alpha}, \tfrac{\partial W_{can}}{\partial \beta}] \\
&= \sum_{i=1}^{n}(\xi_i\alpha - \alpha\xi_i)(\alpha x_i - x_i\alpha) - (\alpha x_i - x_i\alpha)(\xi_i\alpha - \alpha\xi_i) \\
&\quad + (\alpha\beta + \beta\alpha + \sum_{j=1}^{n}(x_j\xi_j - \xi_j x_j))\alpha^2 - \alpha^2(\alpha\beta + \beta\alpha + \sum_{k=1}^{n}(x_k\xi_k - \xi_k x_k)) \\
&= 0
\end{aligned}$$

The above construction from Q to $\mathcal{C}$ shows that Φ is a surjection.

Finally, we need to check that Φ is an injection. The 2 Calabi-Yau algebras we are considering can be thought of as deformations of the 2 Calabi-Yau algebra $A_{can} = Ext^{\bullet}(E,E)$ corresponding to the potential W_{can}. The deformation theory of A_{can} is controlled by a differential graded Lie algebra (DGLA for short)

$$\mathfrak{g}_{can} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}^n_{can},$$

which is a DG Lie subalgebra of the DGLA

$$\widehat{\mathfrak{g}} = \prod_{k\geqslant 1} Cycl^k(A_{can}[1])^{\vee} = \bigoplus_{n\in\mathbb{Z}} \widehat{\mathfrak{g}}^n.$$

Here we write

$$\widehat{\mathfrak{g}}^n = \{W | coh.deg W = n\},$$

and

$$\mathfrak{g}^n_{can} = \{W \in \widehat{\mathfrak{g}}^n | cyc.deg W \geqslant n+2\},$$

where coh.deg means the cohomological degree of W, and cyc.deg means the number of letters $\alpha, x_i, \xi_i, \beta, i = 1, ..., n$ that W contains. In these DGLAs, the Lie bracket is given by the Poisson bracket and the differential is given by $d = \{W_{can}, \bullet\}$. The DGLA $\mathfrak{g}_{can}$ is a DG Lie subalgebra of $\widehat{\mathfrak{g}}$ since d increases both coh.deg and cyc.deg by 1. As vector spaces,

$$\widehat{\mathfrak{g}} = \mathfrak{g}_{can} \bigoplus \mathfrak{g},$$

where

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}^n,$$

and

$$\mathfrak{g}^n = \{W \in \widehat{\mathfrak{g}}^n | cyc.deg W < n+2\}.$$

For the same reason as $\mathfrak{g}_{can}$, we have that $\mathfrak{g}$ is also a DG Lie subalgebra of $\widehat{\mathfrak{g}}$. It follows that $\mathfrak{g}_{can}$ is a direct summand of the complex $\widehat{\mathfrak{g}}$. The latter is quasi isomorphic to the cyclic complex $CC_\bullet(A_{can})^\vee$. Let $A^+_{can} \subset A_{can}$ be the non-unital A_∞-subalgebra consisting of terms of positive cohomological degree. Then for cyclic homology,

$$HC_\bullet(A_{can}) \simeq HC_\bullet(A^+_{can}) \bigoplus HC_\bullet(\mathbf{k}).$$

In terms of dual complex $\widehat{\mathfrak{g}}$, this isomorphism means the decomposition into a direct sum of the space of cyclic series in variables $x_i, \xi_i, \beta, i = 1, ..., n$ (corresponds to $HC_\bullet(A^+_{can})^\vee$), and the one in variable α (corresponds to $HC_\bullet(\mathbf{k})^\vee$). We have that series in α don't contribute to the cohomology of $\mathfrak{g}_{can}$ since $\{W_{can}, \alpha\} = -\alpha^2$. Moreover, the cohomological degree of series in $x_i, \xi_i, \beta, i = 1, ..., n$ is non-positive. Hence $H^{\geqslant 1}(\mathfrak{g}_{can}) = 0$. In particular, $H^1(\mathfrak{g}_{can}) = 0$, which means that deformation of A_{can} is trivial. Thus, Φ is an injection. □

Thus the ind-constructible category $\mathcal{C}$ can be canonically reconstructed from its full subcategory consisting of the collection $\mathcal{E}$.

Chapter 6

2 Calabi-Yau categories and Donaldson-Thomas series

In Chapter 4 we discussed the semicanonical basis obtained as a result of the dimensional reduction from $3CY$ category to the $2CY$ category. In this chapter we are going to discuss Donaldson-Thomas series for $2CY$ categories. We will first review the notion of stability structures, and then define the motivic Hall algebra of a $2CY$ category $\mathcal{C}$. A map from this algebra to the quantum torus gives rise to the motivic Donaldson-Thomas series, which satisfy the Factorization Property. There is a conjecture about DT-invariants in Section 6.6. This theory appears in [58].

6.1 Stability structures

In this section we will follow [42, Sec. 3.4].

Let $\mathcal{C}$ be an ind-constructible weakly unital A_∞-category over a field $\mathbf{k}$ of arbitrary characteristic. Let $cl : Ob(\mathcal{C}) \to \Gamma \simeq \mathbb{Z}^n$ be a map of ind-constructible sets, such that the induced map $Ob(\mathcal{C})(\overline{\mathbf{k}}) \to \Gamma$ factors through a group homomorphism $cl_{\overline{\mathbf{k}}} : K_0(\mathcal{C}(\overline{\mathbf{k}})) \to \Gamma$. For any field extension $\mathbf{k}' \supset \mathbf{k}$ we obtain a homomorphism $cl_{\overline{\mathbf{k}}'} : K_0(\mathcal{C}(\overline{\mathbf{k}}')) \to \Gamma$.

If $\mathcal{C}$ is a Calabi-Yau category, then we assume that Γ is endowed with an integer-valued bilinear form $\langle \bullet, \bullet \rangle$, and the homomorphism $cl_{\overline{\mathbf{k}}}$ is compatible with $\langle \bullet, \bullet \rangle$ and the Euler form on $K_0(\mathcal{C}(\overline{\mathbf{k}}))$.

For ind-constructible triangulated A_∞-categories we have the following version of stability structure.

Definition 6.1.1. *A constructible stability structure on* $(\mathcal{C}, cl)$ *is giver by the following data:*

- *an ind-constructible subset*

$$\mathcal{C}^{ss} \subset Ob(\mathcal{C})$$

  consisting of semistable objects, and for each object it contains all the objects isomorphic to it,

- *an additive map*

$$Z : \Gamma \longrightarrow \mathbb{C}$$

  called the central charge, such that $Z(E) := Z(cl(E)) \neq 0$ *if* $E \in \mathcal{C}^{ss}$*,*

- *a choice of the branch of logarithm* $\mathrm{Log}Z(E) \in \mathbb{C}$ *for any* $E \in \mathcal{C}^{ss}$ *which is constructible as a function of* E*.*

These data satisfy the axioms

- *for all* $E \in \mathcal{C}^{ss}$ *and* $n \in \mathbb{Z}$ *we have* $E[n] \in \mathcal{C}^{ss}$*, and*

$$\mathrm{Arg}Z(E[n]) = \mathrm{Arg}Z(E) + n\pi,$$

  where $\mathrm{Arg}(E) \in \mathbb{R}$ *is the imaginary part of* $\mathrm{Log}Z(E)$*,*

- *for all* $E_1, E_2 \in \mathcal{C}^{ss}$ *with* $\mathrm{Arg}(E_1) > \mathrm{Arg}(E_2)$ *we have*

$$Ext^{\leqslant 0}_{\mathcal{C}}(E_1, E_2) = 0,$$

- *for any* $E \in \mathcal{C}$ *there is an* $n \geqslant 0$ *and a chain of morphisms*

$$0 = E_0 \to E_1 \to \cdots \to E_n = E$$

such that

$$F_i := Cone(E_{i-1} \to E_i), i = 1, \ldots, n$$

are semistable and $\mathrm{Arg}(F_1) > \cdots > \mathrm{Arg}(F_n)$,

- *for each* $\gamma \in \Gamma \setminus \{0\}$, *in* $\mathcal{C}^{ss}_\gamma \subset Ob(\mathcal{C})_\gamma$ *consisting of semistable objects* E *such that* $cl(E) = \gamma$ *and* $\mathrm{Arg}(E)$ *is fixed, the set of isomorphism classes is a constructible set,*

- *(Support Property) For a norm* $\|\cdot\|$ *on* $\Gamma_\mathbb{R} = \Gamma \otimes \mathbb{R}$, *there exists* $C > 0$ *such that* $\|E\| \leqslant C|Z(E)|$ *for all* $E \in \mathcal{C}^{ss}$.

Equivalently, one has the following data and axioms.

Definition 6.1.2. *A constructible stability structure on* $(\mathcal{C}, cl)$ *is given by the data:*

- *an additive map* $Z : \Gamma \to \mathbb{C}$,

- *for any bounded connected set* $I \subset \mathbb{R}$, *an ind-constructible subset*

$$\mathcal{P}(I) \subset Ob(\mathcal{C})(\overline{\mathbf{k}}),$$

such that if $E \in \mathcal{P}(I)$ *then all the isomorphic objects belong to* $\mathcal{P}(I)$.

These data satisfy the axioms

- *the zero object of the category* $\mathcal{C}(\overline{\mathbf{k}})$ *belongs to all* $\mathcal{P}(I)$,

- $\cup_{n \in \mathbb{Z}_{>0}} \mathcal{P}([-n, n]) = Ob(\mathcal{C})(\overline{\mathbf{k}})$,

- *if* $I_1 < I_2$, *i.e., every element of* I_1 *is strictly less than any element of* I_2, *then for any* $E_k \in \mathcal{P}(I_k), k = 1, 2$ *one has* $Ext^{\leqslant 0}(E_2, E_1) = 0$,

- $\mathcal{P}(I+1) = \mathcal{P}(I)[1]$, *where* $[1]$ *is the shift functor in* $\mathcal{C}(\overline{\mathbf{k}})$,

- *(Extension Property) if* $I = I_1 \sqcup I_2$ *and* $I_1 < I_2$, *then the ind-constructible set* $\mathcal{P}(I)$ *is isomorphic to the ind-constructible subset consisting of objects* $E \in Ob(\mathcal{C})(\overline{\mathbf{k}})$ *which are extensions* $E_2 \to E \to E_1$ *with* $E_k \in \mathcal{P}(I_k), k = 1, 2$,

- *if* I *is an interval of length strictly less than 1, and* $0 \neq E \in \mathbb{P}(I)$, *then* $Z(E)$ *belongs to the strict sector*

  $$V_I = \{z = re^{\pi i \varphi} \in \mathbb{C}^* | r > 0, \varphi \in I\},$$

- *there is a non-degenerate quadratic form* Q *on* $\Gamma_{\mathbb{R}}$ *such that* $Q_{|\mathrm{Ker} Z} < 0$, *and for an interval* I *with length strictly less than 1, the set*

  $$\{cl(E) \in \Gamma | E \in \mathcal{P}(I)\} \subset \Gamma$$

  belongs to the convex cone $C(V_I, Z, Q)$ *generated by the set*

  $$S(V_I, Z, Q) = \{\gamma \in \Gamma_{\mathbb{R}} \setminus \{0\} | Z(\gamma) \in V_I, Q(\gamma) \geqslant 0\},$$

- *if* I *has length strictly less than 1, and* $\gamma \in \Gamma$, *then the set*

  $$\{E \in \mathcal{P}(I) | cl(E) = \gamma\}$$

  is constructible.

For a fixed category $\mathcal{C}$ and a class map cl, we denote the set of stability conditions $(Z, \mathcal{C}^{ss}, (\mathrm{Log} Z(E))_{E \in \mathcal{C}^{ss}})$ by $Stab(\mathcal{C}) := Stab(\mathcal{C}, cl)$.

Remark 6.1.3. *The space* $Stab(\mathcal{C})$ *can be endowed with a Hausdorff topology.*

Theorem 6.1.4. *The forgetting map*

$$Stab(\mathcal{C}) \to \mathbb{C}^n \simeq Hom(\Gamma, \mathbb{C}),$$
$$(Z, \mathcal{C}^{ss}, (\mathrm{Log}Z(E))_{E \in \mathcal{C}^{ss}}) \mapsto Z$$

is a local homeomorphism.

Now let's introduce a subcategory $\mathcal{C}_{\Delta,\mathrm{Log}}$ of $\mathcal{C}$. Let $\Delta \subset \mathbb{C}$ be a triangle with one vertex at the origin. We choose an branch of the function $z \mapsto \mathrm{Log}z$ for $z \in \Delta$, and denote by $\mathrm{Arg}(z)$ the corresponding argument function.

Definition 6.1.5. *The A_∞-subcategory $\mathcal{C}_{\Delta,\mathrm{Log}}$ of $\mathcal{C}$ is generated by the zero object $\mathbf{0}$, the semistable objects E with $Z(E) \in \Delta$, $\mathrm{Arg}(E) \in \mathrm{Arg}(\Delta)$, and the extensions F of such objects such that $Z(F) \in \Delta$.*

If $\Delta = V$ for a sector V, then we denote this subcategory by $\mathcal{C}_{V,\mathrm{Log}}$.

It turns out that $\mathcal{C}_{\Delta,\mathrm{Log}}$ is an ind-constructible category. In the language of the ind-constructible sets $\mathcal{P}(I)$ we have $Ob(\mathcal{C}_{V_I,\mathrm{Log}}) = \mathcal{P}(I)$ for some choice of the branch Log.

6.2 Motivic Hall algebras

In this section we will introduce motivic stack functions and the motivic Hall algebras following [42].

6.2.1 Motivic stack functions

Let X be a constructible set over a field $\mathbf{k}$ of characteristic zero, and G an affine algebraic group acting on X. In this section we are going to recall the definition of the abelian group of stack functions $Mot_{st}((X,G))$ following [42, Section 4] (see also [25] for a different exposition).

Let us consider the following 2-category of constructible stacks over $\mathbf{k}$.

1) The objects are pairs (X, G), where X is a constructible set, and G is an affine algebraic group acting on it.

2) The category of 1-morphisms $\mathrm{Hom}((X_1, G_1), (X_2, G_2))$ consists of pairs (Z, f), where

   - Z is a $G_1 \times G_2$-constructible set such that $\{e\} \times G_2$ acts freely on Z in such a way that we have the induced G_1-equivariant isomorphism $Z/G_2 \simeq X_1$,
   - $f : Z \to X_2$ is a $G_1 \times G_2$-equivariant map with trivial action of G_1 on X_2.

   Furthermore, objects of $\mathrm{Hom}((X_1, G_1), (X_2, G_2))$ form naturally a groupoid.

The 2-category of constructible stacks carries a direct sum operation induced by disjoint union of stacks

$$(X_1, G_1) \sqcup (X_2, G_2) = ((X_1 \times G_2 \sqcup X_2 \times G_1), G_1 \times G_2),$$

and a product induced by the Cartesian product

$$(X_1, G_1) \times (X_2, G_2) = (X_1 \times X_2, G_1 \times G_2).$$

After these preliminaries we have the following definition of motivic stack functions:

Definition 6.2.1. *The group of motivic stack functions $Mot_{st}((X, G))$ is the abelian group generated by isomorphism classes of 1-morphisms of stacks $[(Y, H) \to (X, G)]$ with the fixed target (X, G), subject to the relations*

- $[((Y_1, H_1) \sqcup (Y_2, H_2)) \to (X, G)] = [(Y_1, H_1) \to (X, G)] + [(Y_2, H_2) \to (X, G)]$,
- $[(Y_2, H) \to (X, G)] = [(Y_1 \times \mathbb{A}^d_k, H) \to (X, G)]$ *if $Y_2 \to Y_1$ is an H-equivariant constructible vector bundle of rank d.*

One can define the following operations of elements of $Mot_{st}((X, G))$ in the natural way. Let $(Z, f) \in \mathrm{Hom}((X_1, G_1), (X_2, G_2))$. Then we define

- pullback

  $f^* : Mot_{st}((X_2, G_2)) \to Mot_{st}((X_1, G_1))$,

  $[(Y, H) \to (X_2, G_2)] \mapsto [(Y, H) \times_{(X_2,G_2)} (X_1, G_1) \to (X_1, G_1)]$,

- pushforward

  $f_! : Mot_{st}((X_1, G_1)) \to Mot_{st}((X_2, G_2))$,

  $[(Z_1, f_1) : (Y, H) \to (X_1, G_1)] \mapsto [(Z, f) \circ (Z_1, f_1) : (Y, H) \to (X_2, G_2)]$,

- fiber product

  $\cdot : Mot_{st}((X, G)) \times Mot_{st}((X, G)) \to Mot_{st}((X, G))$,

  $[(Y_1, H_1) \to (X, G)] \cdot [(Y_2, H_2) \to (X, G)] \mapsto [(Y_1, H_1) \times_{(X,G)} (Y_2, H_2) \to (X, G)]$.

6.2.2 Motivic Hall algebras

Let's remind the notion of motivic Hall algebra of a certain type of categories.

Let $\mathcal{C}$ be an ind-constructible locally regular (e.g. locally Artin) triangulated A_∞-category over a field **k** (see [42]). Then the stack of objects admits a countable decomposition into the union of quotient stacks

$$\mathcal{O}b(\mathcal{C}) = \sqcup_{i \in I} (Y_i, GL(N_i)),$$

where Y_i is a reduced algebraic scheme acted on by the group $GL(N_i)$.

Definition 6.2.2. *(cf. [42]) The motivic Hall algebra $H(\mathcal{C})$ is the $Mot(Spec(\boldsymbol{k}))-$module*

$$\bigoplus_{i \in I} Mot_{st}(Y_i, GL(N_i))[\mathbb{L}^n, n < 0]$$

(i.e. we extend the direct sum of the groups of motivic stack functions by adding negative powers of the Lefschetz motive $\mathbb{L}$), endowed with the product defined below.

The product is defined as follows. Let us denote $\dim Ext^i(E, F)$ by $(E, F)_i$, and use the truncated Euler characteristic

$$(E,F)_{\leq N}=\sum_{i\leq N}(-1)^i(E,F)_i.$$

Let $[\pi_i : Y_i \to \mathcal{O}b(\mathcal{C})], i=1,2$ be two elements of $H(\mathcal{C})$, then for any $n\in\mathbb{Z}$ we have constructible sets

$$W_n=\{(y_1,y_2,\alpha)|y_i\in Y_i,\alpha\in Ext^1(\pi_2(y_2),\pi_1(y_1)),(\pi_2(y_2),\pi_1(y_1))_{\leq 0}=n\}.$$

Then

$$[tot((\pi_1\times\pi_2)^*(\mathcal{EXT}^1))\to\mathcal{O}b(\mathcal{C})]=\sum_{n\in\mathbb{Z}}[W_n\to\mathcal{O}b(\mathcal{C})].$$

Define the product

$$[Y_1\to\mathcal{O}b(\mathcal{C})]\cdot[Y_2\to\mathcal{O}b(\mathcal{C})]=\sum_{n\in\mathbb{Z}}[W_n\to\mathcal{O}b(\mathcal{C})]\mathbb{L}^{-n},$$

where the map $W_n\to\mathcal{O}b(\mathcal{C})$ is given by

$$(y_1,y_2,\alpha)\mapsto Cone(\alpha:\pi_2(y_2)[-1]\to\pi_1(y_1)).$$

Theorem 6.2.3. *(see* [42, Prop. 10]*) The algebra $H(\mathcal{C})$ is associative.*

For a constructible stability condition on $\mathcal{C}$ with an ind-constructible class map $cl : K_0(\mathcal{C})\to\Gamma$, a central charge $Z:\Gamma\to\mathbb{C}$, a strict sector $V\subset\mathbb{R}^2$ and a branch Log of the logarithm function on V, we have (see [42]) the category $\mathcal{C}_V:=\mathcal{C}_{V,\mathrm{Log}}$ generated by semistable objects with the central charge in V. Then we define the corresponding completed motivic Hall algebra

$$\widehat{H}(\mathcal{C}_V):=\prod_{\gamma\in(\Gamma\cap C(V,Z,Q))\cup\{0\}}H(\mathcal{C}_V\cap cl^{-1}(\gamma)).$$

It contains an invertible element

$$A_V^{Hall} = 1 + \cdots = \sum_{i\in I} \mathbf{1}_{(\mathcal{O}b(\mathcal{C}_V)\cap Y_i, GL(N_i))},$$

where $\mathbf{1}$ comes from the zero object. The element A_V corresponds (roughly) to the sum over all isomorphism classes of objects of $\mathcal{C}_V$, each counted with the weight given by the inverse to the motive of the group of automorphisms.

Theorem 6.2.4. *(see* [42, Prop. 11]*) The elements A_V^{Hall} satisfy the Factorization Property:*

$$A_V^{Hall} = A_{V_1}^{Hall} \cdot A_{V_2}^{Hall}$$

for a strict sector $V = V_1 \sqcup V_2$ (decomposition in the clockwise order).

Let's fix the following data:

(1) a triple $(\Gamma, \langle \bullet, \bullet \rangle, Q)$ consisting of a free abelian group Γ of finite rank endowed with a bilinear form $\langle \bullet, \bullet \rangle : \Gamma \otimes \Gamma \to \mathbb{Z}$, and a quadratic form Q on $\Gamma_{\mathbb{R}} = \Gamma \otimes \mathbb{R}$,

(2) an ind-constructible , $Gal(\overline{\mathbf{k}}/\mathbf{k})$-equivariant homomorphism

$$cl_{\overline{\mathbf{k}}} : K_0(\mathcal{C}(\overline{\mathbf{k}})) \longrightarrow \Gamma$$

compatible with the Euler form of $\mathcal{C}$ and the bilinear form $\langle \bullet, \bullet \rangle$,

(3) a constructible stability condition $\sigma \in Stab(\mathcal{C}, cl)$ compatible with the quadratic form Q in the sense that $Q|_{Ker(Z)} < 0$ and $Q(cl_{\overline{\mathbf{k}}}(E)) \geq 0$, $\forall E \in \mathcal{C}^{ss}(\overline{\mathbf{k}})$.

Given a commutative unital ring R containing an invertible symbol $\mathbb{L}^{\frac{1}{2}}$, we have

Definition 6.2.5. *The quantum torus $\mathcal{R}_{\Gamma,R}$ over R is an R-linear associative algebra*

$$\mathcal{R}_{\Gamma,R} := \bigoplus_{\gamma\in\Gamma} R \cdot \widehat{e}_\gamma,$$

where the generators $\widehat{e}_\gamma, \gamma \in \Gamma$ *satisfy the relations*

$$\begin{aligned} \widehat{e}_{\gamma_1}\widehat{e}_{\gamma_2} &= \mathbb{L}^{\frac{1}{2}\langle\gamma_1,\gamma_2\rangle}\widehat{e}_{\gamma_1+\gamma_2}, \\ \widehat{e}_0 &= 1 \end{aligned} \tag{6.1}$$

For any strict sector $V \subset \mathbb{R}^2$*, we define the quantum torus associated with* V *by*

$$\mathcal{R}_{V,R} := \prod_{\gamma \in \Gamma \cap C_0(V,Z,Q)} R \cdot \widehat{e}_\gamma,$$

where

$$C_0(V,Z,Q) := C(V,Z,Q) \cup \{0\},$$

and $C(V,Z,Q)$ *is the convex cone generated by*

$$S(V,Z,Q) = \{x \in \Gamma_\mathbb{R} \setminus \{0\} | Z(x) \in V, Q(x) \geq 0\}.$$

In the case when $\mathcal{C}$ is a $3CY$ category, one can define a homomorphism from the algebra $\widehat{H}(\mathcal{C}_V)$ to an appropriate motivic quantum torus (the word "motivic" here means that the coefficient ring R is a certain ring of motivic functions). This homomorphism was defined in [42] via the motivic Milnor fiber of the potential of the $3CY$ category. The notion of motivic DT-series was also introduced in the loc.cit.

It was later shown in [43] that in the case of quivers with potential one can define motivic DT-series differently, using equivariant critical cohomology (cf. our Chapter 3). In that case instead of the motivic Hall algebra one uses COHA.

6.3 A class of 2CY categories

Let us consider a class of 2-dimensional Calabi-Yau categories $\mathcal{C}$ which are:

1) Ind-constructible and locally ind-Artin in the sense of [42] (cf. Chapter 5).

2) Endowed with a constructible homomorphism of abelian groups (class map)

$$cl : K_0(\mathcal{C}) \longrightarrow \Gamma,$$

where $\Gamma \simeq \mathbb{Z}^I$ carries a symmetric integer-valued bilinear form $\langle \bullet, \bullet \rangle$, and the class map cl satisfies

$$\langle cl(E), cl(F) \rangle = \chi(E,F) := \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{dim} Ext^i(E,F).$$

3) Generated by a spherical collection $\mathcal{E} = (E_i)_{i \in I}$ in the sense of loc. cit. such that $cl(E_i) \in \Gamma_+ \simeq \mathbb{Z}^I_{\geq 0}$. This means that $Ext^\bullet(E_i, E_i) \simeq H^\bullet(S^2)$, and that $Ext^m(E_i, E_j)$ can be non-trivial for $m = 1$ only as long as $i \neq j$.

4) For any $\gamma \in \Gamma_+$ the stack $\mathcal{C}_\gamma(\mathcal{E})$ of objects F of the heart of the t- structure corresponding to $(E_i)_{i \in I}$ such that $cl(F) = \gamma$ is a countable disjoint union of Artin stacks of dimensions less or equal than $-\frac{1}{2}\langle \gamma, \gamma \rangle$.

5) For any strict sector $V \subset \mathbb{R}^2$ with the vertex at $(0,0)$, and a constructible stability central charge $Z : \Gamma \to \mathbb{C}$ such that $\mathrm{Im}(Z(E_i)) := Z(cl(E_i)) \in V, i \in I$, the stack of objects of the category $\mathcal{C}_V$ generated by semistable objects with central charges in V is a finite union of Artin stacks satisfying the inequality of 4) above.

With the category from our class one can associate a symmetric quiver as in Chapter 5. Similarly to [42, Sec. 8] one can prove a classification theorem for our categories in terms of Ginzburg algebras associated with quivers. Many $2CY$ categories which appear in "nature" belong to our class. For example, if Q is not an ADE quiver, then the derived category of finite-dimensional representations of Π_Q belongs to our class. Without any restrictions on Q one can construct a $2CY$ category as the category of dg-modules over the corresponding Ginzburg algebra.

6.4 Stability conditions and braid group action

Assume that $\mathcal{C}$ is a $2CY$ category from our class described in Section 6.3. We consider an open subset of the space $Stab(\mathcal{C})$ of stability conditions which is defined as

$$U := \prod_{i\in I}(Im\, z_i > 0),$$

i.e. it is a product of upper-half planes. A point $Z = (z_i)_{i\in I} \in U$ defines the central charge $Z : \Gamma := \mathbb{Z}^I \to \mathbb{C}$ which maps classes of spherical generators to the open upper-half plane (hence the stability condition is determined by Z and the t-structure in $\mathcal{C}$ generated by $(E_i)_{i\in I}$).

Recall that with every $i_0 \in I$ we can associate an autoequivalence of $\mathcal{C}$ (called *reflection functor*) by the formula

$$R_{E_{i_0}} : F \mapsto Cone(Ext^{\bullet}(E_{i_0}, F) \otimes F \to F).$$

Then $R_{E_{i_0}}(E_{i_0}) = E_{i_0}[-1]$, and $R_{E_{i_0}}(E_j), j \neq i_0$ is determined as the middle term in the extension

$$0 \to E_j \to R_{E_{i_0}}(E_j) \to E_{i_0} \otimes Ext^1(E_{i_0}, E_j) \to 0.$$

The inverse reflection functor $R^{-1}_{E_{i_0}}$ is given by

$$R^{-1}_{E_{i_0}}(E_{i_0}) = E_{i_{i_0}}[1],$$

$$0 \to E_{i_0} \otimes Ext^1(E_{i_0}, E_j) \to R^{-1}_{E_{i_0}}(E_j) \to E_j \to 0.$$

Reflection functors $R_{E_i}, i \in I$ generate a subgroup $Braid_{\mathcal{C}} \subset Aut(\mathcal{C})$, which induces an action on $Stab(\mathcal{C})$. The orbit $D := Braid_{\mathcal{C}}(U) \subset Stab(\mathcal{C})$ is the union of consecutive "chambers" obtained one from another one by reflection functor R_{E_j}. Such consecutive chambers have a common real codimension one boundary singled out by the condition $Im\, Z(E_j) = 0$.

Remark 6.4.1. *The group $Braid_{\mathcal{C}}$ plays a role of the braid group (or Weyl group) in the theory of Kac-Moody algebras. If we add also the group $\mathbb{Z}$ of shifts $F \mapsto F[n], n \in \mathbb{Z}$ then we obtain an affine version of the braid group $Braid_{\mathcal{C}} \times \mathbb{Z}$. In some examples $\mathbb{Z} \subset Braid_{\mathcal{C}}$.*

6.5 Motivic DT-series for 2CY categories

Let $\mathcal{C}$ be an ind-constructible locally regular $2CY$ category over $\mathbf{k}$. Let us fix

$$R = Mot(Spec(\mathbf{k}))[\mathbb{L}^{\frac{1}{2}}, \mathbb{L}^{-1}, [GL(n)]^{-1}_{n \geqslant 1}]$$

as the ground ring for the quantum torus $\mathcal{R}_{\Gamma,R}$. We will denote the latter by $\mathcal{R}_{\Gamma}$. It is a commutative algebra generated by the elements $\widehat{e}_{\gamma}, \gamma \in \Gamma$ such that

$$\begin{aligned} \widehat{e}_{\gamma_1+\gamma_2} &= \widehat{e}_{\gamma_1}\widehat{e}_{\gamma_2}, \\ \widehat{e}_0 &= 1. \end{aligned} \tag{6.2}$$

Let us also fix a stability condition on $\mathcal{C}$ with the central charge $Z : \Gamma \to \mathbb{C}$.

Definition 6.5.1. *The motivic weight $\omega \in Mot(\mathcal{O}b(\mathcal{C}))$ is defined by*

$$\omega(E) = \mathbb{L}^{\frac{1}{2}(\chi(E,E))}.$$

Then we proved the following result.

Proposition 6.5.2. *(see [58]) The map*

$$\begin{aligned} \Phi :\ & H(\mathcal{C}) \to \mathcal{R}_{\Gamma}, \\ & \nu \mapsto (\nu, \omega)\widehat{e}_{\gamma}, \nu \in H(\mathcal{C})_{\gamma} \end{aligned} \tag{6.3}$$

satisfies the condition

$$\Phi(\nu_1 \cdot \nu_2) = \Phi(\nu_1)\Phi(\nu_2)$$

for $\mathrm{Arg}(\gamma_1) > \mathrm{Arg}(\gamma_2)$*, where* $\nu_i \in H(\mathcal{C})_{\gamma_i}$*. (here* $(\bullet, \bullet)$ *is the pairing between motivic measures and motivic functions.)*

In other words, Φ can be written as

$$[\pi : Y \to \mathcal{O}b(\mathcal{C})] \mapsto \int_Y \mathbb{L}^{\frac{1}{2}\chi(\pi(y),\pi(y))} \widehat{e}_{cl(\pi(y))}.$$

Proof. It suffices to prove the theorem for

$$\nu_{E_i} = [\delta_{E_i} : pt \to \mathcal{O}b(\mathcal{C})],$$

where $\delta_{E_i}(pt) = E_i \in \mathcal{O}b(\mathcal{C})$. Recall that we denote $\dim Ext^i(E,F)$ by $(E,F)_i, i \in \mathbb{Z}$.

We have $\Phi(\nu_{E_i}) = \mathbb{L}^{\frac{1}{2}\chi(E_i,E_i)} \widehat{e}_{\gamma_i}$, which implies that

$$\Phi(\nu_{E_1})\Phi(\nu_{E_2}) = \mathbb{L}^{\frac{1}{2}(\chi(E_1,E_1)+\chi(E_2,E_2))} \widehat{e}_{\gamma_1+\gamma_2}.$$

On the other hand,

$$\nu_{E_1} \cdot \nu_{E_2} = \mathbb{L}^{-(E_2,E_1)_{\leq 0}} [\pi_{21} : Ext^1(E_2,E_1) \to \mathcal{O}b(\mathcal{C})].$$

Then

$$\begin{aligned}
\Phi(\nu_{E_1} \cdot \nu_{E_2}) &= \mathbb{L}^{-(E_2,E_1)_{\leq 0}} \int_{\alpha \in Ext^1(E_2,E_1)} \mathbb{L}^{\frac{1}{2}\chi(E_\alpha,E_\alpha)} \widehat{e}_{\gamma_1+\gamma_2} \\
&= \mathbb{L}^{-(E_2,E_1)_{\leq 0}} \mathbb{L}^{\frac{1}{2}(\chi(E_1,E_1)+\chi(E_2,E_2)+\chi(E_1,E_2)+\chi(E_2,E_1))} \int_{\alpha \in Ext^1(E_2,E_1)} \widehat{e}_{\gamma_1+\gamma_2} \\
&= \mathbb{L}^{-(E_2,E_1)_{\leq 0}+\frac{1}{2}(\chi(E_1,E_1)+\chi(E_2,E_2))+\chi(E_2,E_1)} \mathbb{L}^{(E_2,E_1)_1} \widehat{e}_{\gamma_1+\gamma_2} \\
&= \mathbb{L}^{\frac{1}{2}(\chi(E_1,E_1)+\chi(E_2,E_2))+(E_2,E_1)_2} \widehat{e}_{\gamma_1+\gamma_2}.
\end{aligned}$$

If $\mathrm{Arg}(\gamma_1) > \mathrm{Arg}(\gamma_2)$, then $(E_2, E_1)_2 = (E_1, E_2)_0 = 0$. Thus

$$\Phi(\nu_{E_1} \cdot \nu_{E_2}) = \Phi(\nu_{E_1})\Phi(\nu_{E_2}).$$

□

Recall the categories $\mathcal{C}_V$ and set $V = l$ be a ray. For a generic central charge Z let us consider the generating function

$$A_l^{mot} = \sum_{[E], E \in Ob(\mathcal{C}_l)} \frac{\omega(E)\widehat{e}_{cl(E)}}{[Aut(E)]}$$

$$= \sum_{[E], E \in Ob(\mathcal{C}_l)} \mathbb{L}^{\frac{1}{2}(\chi(E,E))} \frac{t^{cl(E)}}{[Aut(E)]},$$

where $t = \widehat{e}_{\gamma_0}$ for a primitive γ_0 such that $Z(\gamma_0) \in l$ generates $Z(\Gamma) \cap l$, and $[Aut(E)]$ denotes the motive of the group of automorphisms of E. More invariantly, $A_l^{mot} = \Phi(A_l^{Hall})$ where $A_l^{Hall} \in H(\mathcal{C}_l)$ corresponds to the characteristic function of the stack of objects of the full subcategory $\mathcal{C}_l \subset \mathcal{C}$ generated by semistables E such that $Z(E) \in l$ (cf. loc.cit.).

Definition 6.5.3. *We call A_l^{mot} the motivic DT-series of $\mathcal{C}$ corresponding to the ray l.*

Suppose that $\mathcal{C}$ is associated with the preprojective algebra Π_Q. One can show that A_l^{mot} can be obtained from the motivic DT-series for the $3CY$ category associated with $(\widehat{Q}, W)$ by the reduction to $\mathcal{C}$. Similarly to A_l^{mot} we define A_V^{mot} for any strict sector V.

The Proposition 6.5.2 implies that the series A_V^{mot} is the (clockwise) product of A_l^{mot} over all rays $l \subset V$. This can be also derived from the dimensional reduction and the results of [42].

Corollary 6.5.4. *The collections of elements $A_V^{mot} = \Phi(A_V^{Hall})$ parametrized by strict sectors $V \subset \boldsymbol{R}^2$ with the vertex at the origin satisfies the Factorization Property: if a strict sector*

V is decomposed into a disjoint union $V = V_1 \sqcup V_2$ in the clockwise order, then

$$A_V^{mot} = A_{V_1}^{mot} A_{V_2}^{mot}.$$

Proposition 6.5.5. *(see [58]) Motivic DT-series A_V^{mot} is constant on each connected component of the space of stability conditions.*

Proof. Similarly to the case of $3CY$ categories, each element A_V^{mot} does not change when we move in the space of stability conditions on $\mathcal{C}$ in such a way that central charges of semistable object neither enter nor leave the sector V. But in the case of $2CY$ categories the Euler form is symmetric, hence the motivic quantum torus is commutative. It follows that the wall-crossing formulas from [42] are trivial. This implies the result. □

For a $2CY$ category form our class one can construct the corresponding $3CY$ category (see Introduction). We expect that the motivic DT-series arising in this situation are quantum admissible in the sense of [43] and can be described in terms of the corresponding COHA (the latter is expected to exist for quite general $3CY$ categories, see [66]).

Therefore, by analogy with the case of $3CY$ categories, we can define DT-invariants $\Omega(\gamma)$ in $2CY$ case using (quantum) admissibility (see [43], Section 6) of our DT-series by the formula:

$$A_V^{mot} = Sym\left(\sum_{n\geq 0} \mathbb{L}^n \sum_{\gamma\neq 0, Z(\gamma)\in V} \Omega(\gamma)\widehat{e}_\gamma\right) =$$
$$= Sym\left(\frac{\sum_{\gamma\neq 0, Z(\gamma)\in V} \Omega(\gamma)\widehat{e}_\gamma}{1-\mathbb{L}}\right).$$

By Proposition 6.5.5 our motivic DT-invariants $\Omega(\gamma)$ depend only on the connected component of $Stab(\mathcal{C})$ which contains Z. The Conjecture 6.6.1 (see next section) says that $\Omega(\gamma)$ is (essentially) the same as Kac polynomial $a_\gamma(\mathbb{L})$ (or the motivic DT-invariant of the corresponding $3CY$ category, see Introduction).

Let us fix the connected component in $Stab(\mathcal{C})$ which contains such central charge Z that for each spherical generator E_i of $\mathcal{C}$ we have $Z(E_i) = (0, ..., 1, ...0)$ (the only nontrivial element 1 at the i-th place). We will call the corresponding t-structure *standard*. We denote the corresponding motivic DT-invariants by $\Omega_{\mathcal{C}}^{mot}(\gamma)$.

6.6 Kac polynomial of a 2CY category

We can now introduce an analog of the Kac polynomial in the case of a $2CY$ category from our class following the ideas of [50].

Notice that the coefficient ring

$$Mot(Spec(\mathbf{k}))[\mathbb{L}^{\frac{1}{2}}, \mathbb{L}^{-1}, [GL(n)]^{-1}_{n\geqslant 1}]$$

of the quantum torus $\mathcal{R}_\Gamma$ has a $\lambda-$ring structure, which can be lifted to the quantum torus (which is commutative in the case of $2CY$ categories). Recall that for a λ-ring we can introduce the operation of symmetrization by the formula:

$$Sym(r) = \sum_{n\geqslant 0} Sym^n(r) = \sum_{n\geqslant 0}(-1)^n\lambda^n(-r) = (\sum_{n\geqslant 0}(-1)^n\lambda^n(r))^{-1}.$$

For any ray $l \subset \mathbb{H}_+$, where $\mathbb{H}_+$ is the upper half plane, we have the (quantum) admissible element A_l^{mot}.

Let $\mathcal{C}$ be a $2CY$ category from our class. We fix the standard t-structure. Recall the motivic DT-series A_l^{mot}.

Conjecture 6.6.1. *(see [58]) There exist elements*

$$a_\gamma^{mot}(\mathbb{L}) \in Mot(Spec(\boldsymbol{k}))[\mathbb{L}^{\frac{1}{2}}, \mathbb{L}^{-1}, [GL(n)]^{-1}_{n\geqslant 1}]$$

which are polynomials in $\mathbb{L}$ and such that the following formula holds in the (commutative) *motivic quantum torus:*

$$A_l^{mot} = Sym\left(\frac{\sum_{\gamma, Z(\gamma)\in l}(-a_\gamma^{mot}(\mathbb{L})\cdot \mathbb{L})\widehat{e}_\gamma}{1-\mathbb{L}}\right).$$

Furthermore, there exists a $3CY$ *category* $\mathcal{B}$ *such that the elements* $a^{mot}(\mathbb{L})$ *coincide with motivic* DT*-invariants with respect to some stability condition on* $\mathcal{B}$*.*

Some related results can be found in [10], [13] [23], and especially in [50]. In fact Theorem 5.1 from [50] establishes the Conjecture in the framework of quivers. More precisely, if $\mathcal{C}$ is the $2CY$ category associated with the preprojective algebra of a quiver, then for its standard t-structure the element $a_\gamma^{mot}(\mathbb{L})$ coincides with the Kac polynomial $a_\gamma(\mathbb{L})$ of the Kac-Moody algebra corresponding to the quiver.

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