Problem solving in physics:

Undergraduates' framing, procedures, and decision making

by

Bahar Modir

B.S., Arak University, 2006

M.S., University of Tehran, 2011

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Physics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2017

Abstract

In this dissertation I will start with the broad research question of what does problem solving in upper division physics look like? My focus in this study is on students' problem solving in physics theory courses. Some mathematical formalisms are common across all physics core courses such as using the process of separation of variables, doing Taylor series, or using the orthogonality properties of mathematical functions to set terms equal to zero. However, there are slight differences in their use of these mathematical formalisms across different courses, possibly because of how students map different physical systems to these processes. Thus, my first main research question aims to answer how students perform these recurring processes across upper division physics courses.

I break this broad question into three particular research questions: What knowledge pieces do students use to make connections between physics and procedural math? How do students use their knowledge pieces coherently to provide reasoning strategies in estimation problems? How do students look ahead into the problem to read the information out of the physical scenario to align their use of math in physics?

Building on the previous body of the literature, I will use the theory family of Knowledge in Pieces and provide evidence to expand this theoretical foundation. I will compare my study with previous studies and provide suggestions on how to generalize these theory expansions for future use. My experimental data mostly come from video-based classroom data. Students in groups of 2-4 students solve in-class problems in quantum mechanics and electromagnetic fields 1 courses collaboratively. In addition, I will analyze clinical interviews to demonstrate how a single case study student plays an epistemic game to estimate the total energy in a hurricane.

My second research question is more focused on a particular instructional context. How do students frame problem solving in quantum mechanics? I will lay out a new theoretical

framework based in epistemic framing that separates the problem solving space into four frames divided along two axes. The first axis models students' framing in math and physics, expanded through the second axis of conceptual problem solving and algorithmic problem solving. I use this framework to show how students navigate problem solving. Lastly, I will use this developed framework to interpret existing difficulties in quantum mechanics.

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Major Professor
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Acknowledgments

From the first day of attending K-State I felt love and support which I am extremely thankful for. I would like to thank all the people, without their help this work was not possible.

I would like to thank my dear advisor, Dr. Eleanor Sayre, for teaching me the alphabet of the Physics Education Research field and her enormous guidance, support, and advise. I could not have chosen a better advisor. My advisor for me is a role model of a successful professional woman in the STEM field which I enjoyed working with.

I would like to thank Dr. Brett DePaola for all his guidance and support in providing me an open opportunity to develop an online course as a teaching assistant. I would like to thank the rest of my committee members: my outside chair Dr. Michael Herman, Dr. James Laverty, and Dr. Bruce Babin for all their support and feedback towards completion of my dissertation.

It was an honor for me to become part of the Physics Education Research Group at K-State and conduct my PhD study. I would like to thank Dr. Dean Zollman for his encouraging advise and feedback on my work.

I would like to thank Erfan Saydanzad, Javad Robatjazi, and Neda Dadashzadeh for all their help, support, and kindness over the last four years. I would like to thank Dr. Sanjay Rebello and Dr. Paul Irving for their support and kind guidance during the initial years of my PhD.

I would like to thank my current and past colleagues on the fourth floor for always being there for me and ready for productive discussions: Nandana Weliweriya, Dina Zohrabi-Alaee, Tra Huynh, John D. Thompson, Ben Archibeque, Deepa Chari, Hai Dong Nguyen and Elise Agra. I would also like to thank the wonderful staff of the physics department. My special thanks to Kim Coy for always being kind and helpful to me.

Finally, I would like to thank my husband for giving me love over the past four years and encouraging me to complete my PhD and pursue my future career.

Dedication

To my parents and grandparents.

Chapter 1

Introduction

Problem solving as an essential part of physics employs different kinds of knowledge and skills. Mathematics as the language of physics has an important role in understanding the causal physical relation in form of mathematical symbols. The process of converting a physical representation to a mathematical representation to do math on it is called mathematization[4]. Doing math is different from mathematization procedure.

Researchers have broadly investigated how students create and understand equations to describe a physical behavior[5], and what are the expectation of students in using equations in physics problem solving that cause them to play particular epistemic games, which may result in a mathematical error[6].

In this study my broad research question is how students do problem solving in upper division physics. For this purpose, I am not interested in just the final correctness of students' answers, but I am interested in the detail of students processes in problem solving in the domain knowledge of math-in-physics. My ultimate goal in this dissertation is to build better theoretical lenses to explain how students perform processes in physics problem solving.

My focus in this study is on students' problem solving in physics theory courses. There are certain mathematical formalisms that are in common across all physics core courses such as using the process of separation of variables, doing Taylor series, or using the orthogonality properties of mathematical functions to set terms equal to zero. The slight difference among the use of these mathematical formalisms in different courses such as quantum mechanics vs electromagnetic fields course is due to how students map the physical system to these processes; make judgements, make decisions; and do several readouts in the moment of the problem solving to describe the features of the physical system. These common formalism can be interpreted as a substrate that connects the courses to each other (see figure1.1). Thus it is important to investigate how students do procedures while manipulating these mathematical formalisms in physics courses, which forms my first research question. Among the physics theory courses my focus is on quantum mechanics and electromagnetism fields 1 course. Separately, in an interview setting I investigate how a student solve a fermi problem, which is an estimation problem.

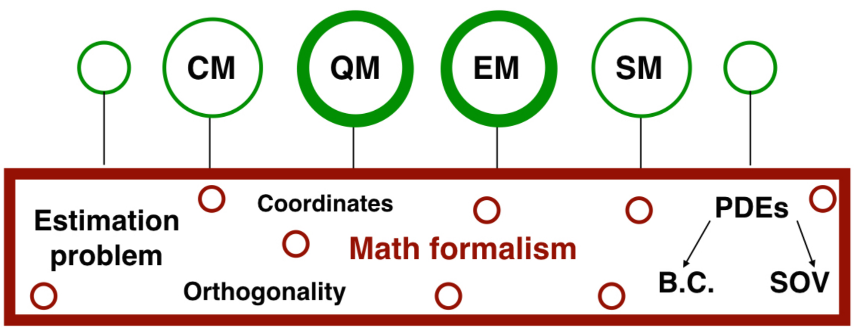


Figure 1.1: *Schematic graph of physics courses in upper division that are connected via a substrate indicating common mathematical formalisms. This dissertation investigates students use of partial differential equations (PDEs) across two core courses of quantum mechanics (QM), and electromagnetism fields I (EM) course. Separately in an interview setting this dissertation models one case study student solving a fermi problem.*

I break my first main research question into three leading research questions. What are the knowledge pieces that students use to make connection between physics and procedural math? How do students use their knowledge pieces coherently to provide reasoning strategies in estimation problems? Lastly, how students look ahead into the problem to read the information out of the physical scenario to align their use of math in physics? I respond to these three questions in the domain knowledge of math-in-physics.

Previous researchers have examined theoretical constructs to understand how students provide intuitive reasoning[7], understand equations[5], make mathematical errors[6], and activate procedural resources to show mathematical actions[8].

I select the family of Knowledge-in-Pieces (KiP) theories to describe student's problem solving behavior. Phenomenological primitives[7] (p-prims) from the pieces theory are atomistic units of students' reasoning used to provide an intuitive sense of how physical mechanisms work. Under the same family of theories, symbolic forms[5] describe how students use their p-prims along with mathematical equations to conceptualize a physical situation. Epistemic game (e-game) is another cognitive structure[9] that describes how students use mathematical tools[6]. Students based on their expectation on how to solve a problem can enter an epistemic game. The structure of an epistemic game is determined by its entry conditions and the moves played in the game. The knowledge base used in association with the moves could be p-prims, symbolic forms, or interpretative devices[6].

Researchers use different theories to allow them to interpret student's thinking process. I am interested to see how students use their knowledge of math and physics to solve a problem. I use the knowledge in pieces (KiP) theory to model student's problem solving activities. This theory allows to view students different ways of thinking and context dependent aspects of problem solving.

The resources theory[10] under the umbrella of KiP, models students thinking in terms of the fine grain pieces of knowledge that are called resources. The KiP theory accounts for context dependent ways of student's thinking. Context can be internal or external. Students as a result of their expectation which is an internal factor can activate particular ideas in a problem solving situation. On the other hand, students can activate different ideas as the result of reading information out of an external context such as the problem statement. In addition, both of these contexts can be present in student's problem solving activity. Activation of one piece of idea may cue the activation of other ideas. The resources theory at a fine grain level focuses on how students link these small bits of their ideas to form an argument or a reasoning. Over time students can provide different reasonings that can be very different in nature. Thus investigating student's dynamic of problem solving may reveal more information about how they are thinking over time.

One of the theories under the umbrella of KiP theory is called epistemic game[6](e-game). Epistemic game has an ontological part and a structural part. Activated resources during problem solving form the knowledge base of an e-game. By reading the problem statement students set a goal to enter a particular e-game, follow several moves to find an answer and exit the e-game. Using the e-game theory lens, we can find information about how students shift among the moves, what is the frequency of shifts between two moves, or how interplay of another move can increase the frequency of the shifts. Thus e-games provide a representation that makes it possible to observe the dynamic of students problem solving. Different students can play the variation of the same game. However, there are instances in problem solving situations that student may get "stuck" and in order to get "un-stuck" they may undergo a sharp change in the nature of their activity that can not be captured through an e-game with strict structure of moves. Thus the e-game construct with a particular goal that guides to enter a specific game does not capture the variations in students goals during a problem solving. In order to see the variation of students thinking in a more broader sense I use the epistemological framing theory[11], which is a more flexible construct that allows me to view changes in students epistemological views in the use of their knowledge of math and physics and the way they choose to conduct algorithmic or conceptual problem solving.

In Chapter 3, I provide evidence on how students within one group unpack the procedural resource of separation of variables to its constituents. I identify a new procedural resource called "bringing out", which has an important role in this unpacking process. My data is drawn from classroom videos of upper-division quantum mechanics problem solving. I analyze an example of students' use of separation of variables to solve a partial differential equation for a free particle problem. The "bringing out" resource helps students separate the time part from the space part of the wave function in the course of solving the time-dependent Schrödinger equation. I also discuss the conceptual resources that students activate and how they relate to the procedural resources.

In Chapter 4, as part of a study into students problem solving behaviors, We asked an upper-division physics student to solve an estimation problem in a clinical interview. I use the theory lens of resources framework and epistemic games to describe students' problem solving moves. I present a new epistemic game, which is called estimation epistemic game (e-e-game). In the estimation epistemic game students break the larger problem into a series of smaller, tractable problems. Within each sub-problem, they try to remember a method for solving the problem, and use estimation and reasoning abilities to justify their answers. I demonstrate how a single case study student plays the game to estimate the total energy in a hurricane.

In Chapter 5, I use coordination class theory to describe how students connect physical scenarios with mathematical insight. Within coordination class theory, students read information out of problem statements, connecting the specifics of the problem with generalized conceptual schemata (the "coordination class") in a causal net. While previous research using coordination classes has focused on identifying particular coordination classes or details of the causal net, my research focuses on an extended readout strategy, which we call "looking ahead". To characterize the mechanism of looking ahead, we study students' problem solving with separation of variables and Taylor series expansions. When students look ahead in a problem, their mathematical and physical insight can help them avoid time consuming calculations. I discuss the structure of looking ahead and illustrate it with video-based classroom data.

After discussing students procedures in problem solving I ask my second main research question which is more comprehensive. How students frame problem solving in a senior level quantum mechanics course. Students at this stage of their education are at the "journeyman-level" [12], that have access to many ideas and toolboxes to create different paths and ways of thinking to solve a problem. Thus, it is important to investigate what is the mechanism for students to control over which part of their knowledge to activate and how navigate their knowledge use to work in a situation. In Chapter 7, I layout a new theoretical framework called "two coordinate axis framework" that separates the problem solving space into four frames. The first axis models students' framing in math and physics, expanded through the second axis of conceptual problem solving and algorithmic problem solving.

Students difficulties in quantum mechanics may be the result of unproductive framing and not fundamental inability to solve the problems or misconceptions about physics content. I observed groups of students solving quantum mechanics problems in an upper-division physics course. Using the lens of the epistemological framing, I investigated four frames in my observational data: algorithmic math, conceptual math, algorithmic physics, and conceptual physics. I discuss the characteristics of each frame as well as causes for transitions between different frames, arguing that productive problem solving may occur in any frame as long as students transition appropriately between frames. My work extends framing theory on how students frame discussions in upper-division physics courses.

In Chapter 8, using the theoretical lens of epistemological framing, I apply the "two coordinate axis framework" to seek an underlying structure to the long lists of published difficulties that span many topics in quantum mechanics. Mapping descriptions of published difficulties into errors in epistemological framing and resource use, we analyzed descriptions of students' problem solving to find their frames, and compared students' framing to the framing (and frame shifting) required by problem statements. I found three categories of error: mismatches between students' framing and problem statement framing; inappropriate or absent shifting between frames; and insufficient resource activation within an appropriate frame. Given this framework, I can predict the kinds of difficulties that will emerge for a given problem in quantum mechanics, yielding a possible deeper structure to student difficulties. Lastly, I use this developed framework to frame difficulties in quantum mechanics.

Finally, in Chapters 9 and 10, I discuss the conclusion and suggestions for future works. The schematic overview of my dissertation is sketched in figure1.2. I use a methaphorical representation to show that students across physics courses use common mathematical tools. The mathematical formalisms are like a substrate that ties different upper division physics course. However, students might need to apply different strategies or conduct different processes in order to map the features of different physical systems to their mathematical representation in each course. In order to investigate different ways of mapping the knowledge of physics to mathematical formalisms I use theory family of "knowledge in pieces", which metaphorically can be interpreted as a light that shines on my observational data and allows me to interpret students different problem solving activities. We collected data from two upper division physics core courses of quantum mechanics (QM) and electromagnetism fields I (EM) course. My two main research questions are "how students do procedures in problem solving?" and "how students frame problem solving?" For the first research question I analyze data from both physics theory courses of quantum mechanics course and electromagnetic fields 1 course.

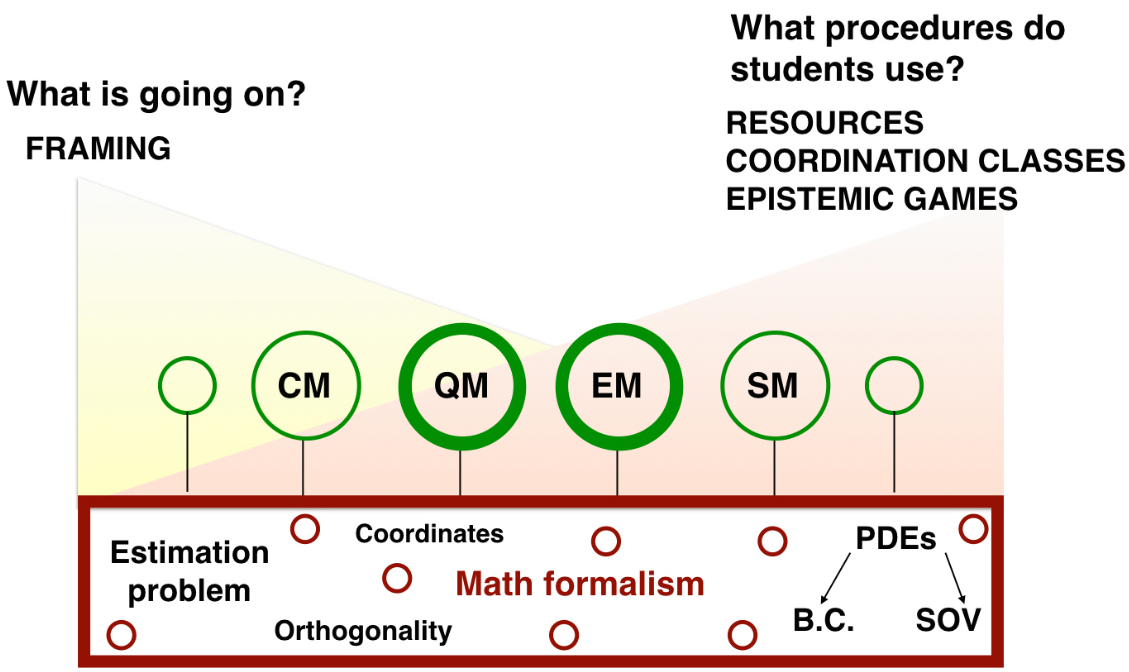


Figure 1.2: *Schematic graph of my project overview. This dissertation models student's problem solving. The theory of knowledge in pieces is used to answer two main research questions of how students frame problem solving situations and how students conduct processes while solving a problem.*

Chapter 2

Literature Review Overview

Over the last 50 years[13] Physics education researchers have conducted research to support student learning and teaching strategies. Application of these results into K-12 education and on up to upper division and graduate physics consequently open avenues for further research. Researchers have developed different theories in physics education research to investigate how students learn physics, how students provide reasoning, and how students do physics problem solving. In order to answer these broad questions, researchers identify what knowledge students have, what are the constituents of their knowledge[7], how students employ their knowledge[4] and how that knowledge develops under physics instructions[14]. Different cognitive theories have accounted for understanding the structure of students' knowledge. In this chapter, I review two big families of theories that have tried to model students' thinking in physics. The first theory is based on a misconception view which investigates students' conceptual understanding in physics. The second big family of theory is a manifold view that aims to understand in a more structural sense what are the elements of knowledge that come together in a situation to provide a reasoning.

Starting in the late 1970s until the 1990s, researchers mostly focused on students difficulties with conceptual understanding in physics. In this view these difficulties form stable units of thinking which have to be confronted and replaced by new conceptions. In the 1990s the role of context was investigated more which led to the idea that students reasoning were small atomistic pieces, which were called phenomenological primitives[7] (p-prims). In the late 1990s and the early 2000s, researchers developed the resources theory that student's reasoning are formed from larger pieces of thoughts which could be reusable and link together in different ways. Here I reference a resource letter which was published in 1999 by McDermott and Redish[15] that aggregates the work and theory on researched-based guidelines of many of the research articles published between the 1980s and late 1990s. This is a good reference that includes the studies done on student conceptual understanding across different topics and early research on epistemological beliefs of students about learning physics. Researchers have used resources theory to explain conceptual change[16], mechanistic reasoning[17], and along with other theories to serve as the knowledge base of epistemic games[6], which is a structure of moves that students play to solve a problem. Many of these resources studies have become published and updated in 2013 by Redish and Sayre[18] as the resource framework bibliography that covers research from the early 1990s and on up to 2013.

2.1 Unitary view

I use two terms of "unitary" and "manifold" views[19], to explain students reasoning from the view of two theoretical constructs of "misconceptions" and "pieces".

In a misconceptions view, students apply an incorrect model of a concept across a wide range of situations independent of the context[20;21]. The core of conceptual understanding occurs by confronting the incorrect conception, and replacing it with a new concept. Several studies using the unitary view have investigated student difficulty in understanding various topics such as the concept of velocity in one dimension[22], difficulties in connecting graphs and physics[23], work-energy and impulse-momentum theorems[24], understanding of light as an EM wave[25], single-slit diffraction and double-slit interference[26], difficulties with torque[27], and the first law of thermodynamics[28].

Studies that aim to explore students conceptual understanding use certain methods[29]. The first step is to identify students misconceptions across certain topics, then surveys[30] and multiple choice questions are designed based on students difficulties, which become administered to a large population of students to find the prevalence pattern of those difficulties. Many studies have used these results to improve teaching strategies[31–34], and design research-based tutorials[35–38] to fix students' understanding in a certain topic. The improvement in student conceptual understanding are measured[39] through assessment tools such as the Force Concept inventory[40] (FCI) and the Force and Motion Conceptual Evaluation[41] (FMCE) in introductory mechanics, and the Brief Electricity and Magnetism Assessment[42] (BEMA) in introductory electricity and magnetism.

This unitary view of students' reasoning guides our attention as researchers toward the identification of topics with which students have difficulties. However, focusing on identification of large scale difficulty concepts can be at the cost of missing students' epistemological changes[43], because a difficulties view predicts a stable model of thinking that is repeatable, and does not account for sudden or contextual changes in the nature of student reasoning.

A study by McCloskey[21] shows how students use their knowledge to explain the motion of the ball. Consider the following example of one person throws a ball upward into the air, what forces act on the ball? Student's incorrect notion involves an internal force that acts on the ball from the hand. This force fights against gravity to keep the ball in motion. As the ball goes up the force or the impetus gradually dies away and the ball slows down. The ball stops at the peak of the trajectory due to the impetus force and the gravity being in equilibrium. Then the gravity overcomes the internal force and causes the ball to fall down. However, a physicist, in the absence of air resistance force, only identifies the force of gravity as acting on the ball. As the ball is tossed into the air, the ball is not in contact with the hand, thus there could not be any force from the hand acting on the ball. However, the hand at the beginning of the motion gives the ball momentum which decreases and gets zero at the peak of the trajectory.

McCloskey considers that these incorrect answer patterns, which are consistent among students, form a naive "Impetus theory". The impetus theory asserts that an object set into the motion has an internal impetus to keep the object into the motion and the impetus gradually dissipates as the result of external influences. The misconception of the motion implies force is not consistent with the first law of Newton that asserts an object in a constant speed under the absence of an external forces maintains the constant speed forever. The impetus theory is a naive theory of motion which is different from the experts view in physics. diSessa in contrast, does not think that the "impetus theory" has the systematicity which a theory should possess. diSessa thinks that the misconception view cannot explain students intuitive physics knowledge. He proposes that students intuitive ideas cannot lead to developed misconceptions and instead, students' reasonings are fragmented and even contradictory to each other. In the rest of this chapter I explain the knowledge in pieces approach and reanalyze the ball in motion example from the phenomenological primitives[7] (p-prims) approach by diSessa.

2.2 Manifold view

An alternative view to a unitary difficulties view is a manifold "knowledge in pieces" view. In this view of student reasoning, student thinking is conceptualized as being highly context dependent and composed of small, reusable elements of knowledge and reasoning called "pieces". These pieces are not themselves correct or incorrect, but the ways in which students put them together to solve problems may be either. By focusing on the pieces of student reasoning and how they fit together, this view of student reasoning foregrounds the seeds of productive reasoning and not just incorrect answers. Theories in this family include phenomenological primitives[7] (p-prims), resources[10], and symbolic forms[44]. In contrast to a unitary view where students' misconceptions are in contradiction to expert view in physics, a manifold view explains how students can develop their use of p-prims to finally get in alignment with the way experts use their physical intuitions.

2.2.1 Phenomenological primitives (p-prims)

Previous research has modeled students' ideas as small chunk of resources called phenomenological primitives (p-prims)[7] that are used in their intuitive reasoning. diSessa[7] argues that students' misconceptions are not theories, because they do not show the features of the theories, instead these are primitive reasonings that are context dependent and do not have the systematicities that scientific theories own. diSessa under the knowledge in pieces (KiP) view introduced the idea of p-prims, which are atomistic pieces of reasoning that are abstracted from experience. People by living in this world from the early age of childhood experience the physical world to throw objects, pull objects, etc, which lead to primitive sense of mechanisms for how things work. The word phenomenological reflects this abstraction and the word primitive shows that people simply use p-prims without further justification because they think this is the way how things are and that's it. P-prims are neither inherently right or wrong, and their correctness is determined by the context. For example, the p-prim of "closer means stronger" is correctly activated in the context of proximity to a fire. As we get closer to the fire the effect is stronger and we feel warmer.

If we ask the question of why the earth is warmer in the summer than it is in the winter, students might say because the earth is naturally closer to the sun in the summer thus it is warmer. Still the same p-prim of closer means stronger is activated, however the activation here does not lead to the correct conclusion as the earth's axis or rotation angle is responsible for this seasonal weather change. Hammer[10] argues that students do not possess robust wrong notions called misconceptions, but these types of reasoning could become activated right in the spot within the context. This context of sun-earth by considering the source and the proximity to the source have a higher cueing priority for the p-prim of "closer is stronger" over other p-prims.

P-prims[7] can be refined to be used in an expert way. For example, in constraint phenomena such as when a ball in motion hits a wall, the motion gets blocked and the ball can come to rest. One can simply explain this situation by activating the blocking p-prim and simply explains that the ball stops because the wall blocked the motion. After physics instruction as students learn to explain the same phenomena in terms of force, they might think of this situation as an interaction between the ball and the wall, and think of the wall as an agent that can apply a force on the ball. This way of thinking is more likely to activate the force agency p-prim. Thus the probability of the blocking p-prim decreases in favor of increase in the activation of the force agency p-prim. P-prims with high level of priority gradually become the central ideas in students thoughts. The high frequency of activation of a p-prim over other p-prims indicates the higher cueing priority for that central p-prim. If a p-prim stays activated after becoming activated then the reliability priority for that p-prim is high. If a p-prim stay activated after leading to activation of other collection of p-prims, then that p-prim has a high reliability priority with respect to that particular context of activation[7;44].

2.2.2 Manifold vs Unitary view in one example

Hammer[19], from the language of diSessa[7] explains how the knowledge in pieces approach can account for the contradictory explanations in students responses to the ball problem, which is not understandable through the impetus theory approach. Students based on the misconception of the motion requires force reason that an impetus force which acts in the direction of the motion, fights against the downward gravitational force and keeps the ball in an upward motion. Based on the impetus theory, the internal force gradually dissipates and comes to zero at the peak, since the ball stops and is not in motion at its peak. However, at the peak of the motion students use another line of reasoning that the impetus force and the gravitational force are in equilibrium. This notion is not consistent with the impetus theory which states motion requires force and the impetus gradually dissipates as the object slows down. Hammer argues that under the robust misconception view it is difficult to think why students suddenly change their notion and even use them in contradictions. diSessa's[7] account of context dependent p-prims explains that students break this problem into different contexts and each context cue a different p-prim.

In the first mini context, from where the ball is tossed before it gets to its peak students have activated the p-prim of maintaining agency. In order to maintain an effect, which in this case is the motion of the ball, an agent must keep acting on the ball (object). In the mini context of the ball in its peak, students have activated the balancing p-prim, which implies an upward influence is in balance with a downward influence.

2.2.3 Symbolic forms

The use of p-prims are not limited only in providing conceptual reasonings. Students can activate p-prims along with their mathematical reasonings to explain how a physical system work both in a qualitative and a quantitative manner. Sherin[44] uses the idea of p-prim as a sense of mechanism to investigate how students in the context of problem solving conceptually understand equations in algebra-physics. He then argues that not only p-prims are active in students intuitive reasonings, but also forms can be considered as part of this intuitive understanding. He develops the theoretical construct of symbolic forms consist of a conceptual schema and a mathematical form. Sherin argues that physical understanding can be tuned with mathematical understanding, such that activation of a p-prim could cue the activation of a mathematical form.

2.2.4 Resources

Under the KiP theory family, resources are other elements of thoughts that can come together to make a reasoning. Resources are not atomistic such as p-prims[7]. Resources are reusable thoughts that can be broad and even have an internal structure. Resources different than p-prims do not only encompass the content knowledge and conceptual understanding. Resources can have different types: epistemological[45], mathematical[6], conceptual[46], or procedural[1]. P-prims are intuitive sense of reasoning that become activate to explain how things work. Epistemological resources form the beliefs as to how to activate knowledge. For example, knowledge can be accessed from an authority source such as textbooks. Conceptual resources deal with understanding concepts, such as *coordinate systems*[14]. Mathematical reasonings such as symbolic templates similar to p-prims can be abstracted from experience with mathematical forms. Symbolic templates and p-prims can become reinforced in activation and form symbolic forms[44].

Researchers using the resources framework try to answer questions such as what are the ways that students put their knowledge together, how students link their ideas, how can students change part of their reasoning, how can students unpack their reasoning[1;47], how resources develop over time[14] or more particularly, what are the resources associated with thinking algorithmically, conceptually or computationally.

Resources theory has been used broadly with other theories such as framing[48;49], epistemic games[6], or conceptual change theory[50]. Resources are small, reusable elements of student reasoning, that may or may not have an internal structure. Resources are neither wrong nor right, and are context dependent[14;19].

Resources also can vary in size. Some resources such as primitives do not have an internal structure. Other resources such as *coordinate systems* can have an internal structure accessible by the user. Although we can keep track of the frequency and the length of p-prim activation (Section 2.2.1), but the nature of p-prims can not change. The internal structure of p-prims may not be explorable by the user[14]. Whereas, larger resources such as epistemological, conceptual[50;51], and procedural[47] could have an internal structure by connecting to other ideas.

There are two technical terms that are used to categorize if a resource has an internal structure or not[14]. Solid resources that are durable can be used as individual resources. These resources have become solid as the result of frequent use in several contexts of activation. Thus the internal structure of the resources become less accessible, and the user can use that resource on its own. On the other hand, resources that are being created for students for the first time by the instructor or a group mate are less solid and more plastic, thus students need to activate several resources in close association to each other to learn the meaning of a new conceptual resource to describe a physical system, or learn how a mathematical procedure such as *separation*[47] works to separate variables.

I am interested in a particular kind of mathematical reasoning used by students that are procedural resources and could be larger in size. In Chapter 3, using the resource framework I investigate how students by activating several procedural resources can perform the processes of separation of variable to separate the time dependent Schrödinger equation. Students based on their expertise can use one resource without using the details, or they can unpack a resource as the collection of other resources.

By expansion of resources theory with newly identified resources, the connection between

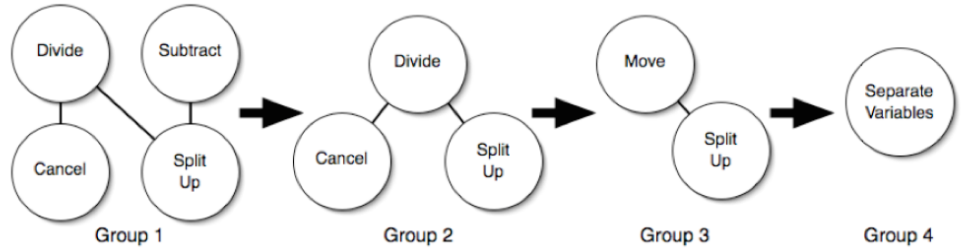


Figure 2.1: *A model of the formation of the procedural mathematical resource Separate Variables. Figure originally from*[1] *link. This graph shows the procedural resources that each group activated while separating variables as they solve an air resistance problem in an intermediate mechanics class.*

mathematics and physics becomes more clear. Researchers have identified mathematical actions that met the resources criteria, which are procedural resources. Students use procedural resources[47] to perform actions such as separating variables to solve partial differential equations (PDEs). Students for showing a procedural resource, instead of using an algebraic representation, use hand gestures to metaphorically show the mathematical action[52]. Wittmann et al.,[47] investigated procedural resources such as *separation*, finding that their internal structure includes several resources: *grouping, division* or *multiplication, moving* and *subtraction*, explorable by students. The level of structure can vary among students with different level of expertise. Students can use *separation* as a single resource, or they can activate additional resources to successfully separate variables on both sides of an equation. In an earlier study, the same group of researchers[1] investigate different ways of separating variables within four groups. Interestingly, each group uses the *separation* resource differently. The authors imposed an order on these patterns of activations to say perhaps there is a developmental trend, from the fourth group that uses *separation* as an individual resource, to the first group that uses the *separation* resource as the most unpacked, and accesses to its constituents (see Fig. 2.1). In Chapter 3, using the resources theory I provide evidence that this developmental trend can happen within one group as students activate and link resources to fully unpack the separation of variables resource.

Following the general characteristic of KiP theories, resources are neither right nor wrong, and it is in the connection to other ideas and resources that leads to the right or wrong conclusion.

2.2.5 Epistemic games

By zooming out our view of resource use we can investigate what are the patterns of resource activation in students problem solving in physics, which I discuss in Chapter 4, as a case study student plays an epistemic game (e-game) to solve an estimation-based problem. Students by using different resources can play epistemic games. An e-game consists of rules, strategies, and actions that can be taken at different points in the game ("moves"). Epistemic games have a knowledge base and a structural part. The knowledge base of e-games can be formed by p-prims[7], symbolic forms[44], and interpretive devices[53]. The structural part is the entering of the game and the moves taken in the game. The nature of activities done in different moves are different. If two moves happen to always follow each other it is likely that they both belong to the same epistemic game[6].

The "list making" game is one of the simplest e-games[9]. In the list-making game, the goal is to make a list. The basic moves are remove, split, or add an item to the list[9]. Tuminaro and Redish[6] identified six different descriptive e-games specific to physics. Chen et al.,[54] identified an answer-making epistemic game (AMEG) in students solutions to conceptual problems. Epistemological frames reveal students ways of thinking and expectations. Students' expectation about problem solving in physics can play an important role in what e-game select to play. In addition, other environmental factors such as interaction with the problem statement or certain keywords in the problem statement might cue students to activate particular part of their knowledge or skills, and enter a specific game that uses those resources as its knowledge base. Students' judgement as when to exit an e-game is based on their expectation whether the answer is sufficient or not.

In Chapter 4, I use the Resources Framework[10] to present an analysis of a single student's thinking at the micro-level. I am particularly interested in her problem solving moves and how she activates p-prims[7], mathematical[55], and procedural resources[47]. Students' epistemic stances are manifold and context-sensitive[19]. By applying e-games to estimation-type problems, we can describe students ways of thinking and their knowledge and developmental strategies for finding a solution in context-dependent and manifold ways. We choose this student's work as an exemplary for solving this kind of problem, and analyze it in detail[6]. Through e-games and the Resources Framework, we can describe students' tacit expectations[6] about how to solve physics problems. Students can take different pathways towards completion of a problem solving situation. The collection of qualitative and quantitative resources and reasoning abilities that emerges during the problem solving process forms the knowledge base component of an e-game[55]. A single student can take different paths depending on the relative difficulty of the problem and which resources they activate[54]. Different games may share some moves; however, the set of moves and structure of an individual game is unique.

Tuminaro identifies six games and categorizes these e-games based on two main epistemological framing, that introductory-level students may posses to solve a physics problem[6]. Students can frame problem solving as rote manipulation of equations as an input output tool, or in contrast, students may frame problem solving as a sense making activity. Tuminaro et al., then split the sense making frame into two other frames of quantitative and qualitative sense making frames. In the rote memorization frame Tuminaro[6] identifies two games: Recursive Plug-and-Chug and Transliteration to Mathematics. In both of these games students do not provide any conceptual understanding of the equations that they use. Mostly they recall an equation that contains the target in it and insert the known quantities to find the unknown target quantity. If more than one unknown is in the problem they play the e-game to recall an equation that contains the sub-target and then plug it back to the main equation. If students could not find the sub-target then they might get stuck.

In contrast, two e-games of Mapping Meaning to Mathematics and Mapping Mathematics to Meaning both have the element of conceptual understanding that may constituent one or two of the moves in the whole game. The resources activated in these games are intuitive mathematical reasoning, interpretive devices[53] or symbolic forms[44]. The most difficult game for students to play is Mapping Meaning to Mathematics. The hard move for students is to map the physical representation into a mathematical representation that indicates the physical causal relation in the mathematical entities. The next move which is mathematical manipulation is easy for students. The knowledge base of both Mapping games are the same, but the structure of moves are different. In Mapping Mathematics to Meaning students start with identifying equations including the target and the relation of the target with other physics quantities. Up to this point the structure of the Mapping Mathematics to Meaning e-game is exactly the same to the Recursive Plug and Chug game. What makes these two games different is the expectation of the students in problem solving. In Mapping Mathematics to Meaning after setting up the mathematical basis, students in the next move check to see if the relation between physical quantities make sense or not. Whereas, in Recursive Plug and Chug game students plug the physical quantities into the equations without understanding the equation or thinking conceptually about the physical situation.

In e-games of Physical Mechanism Game and Pictorial Analysis, students frame the problem solving as qualitative sense making. These games rely on intuitive and conceptual reasoning of students to generate a sufficient answer and do not include any mathematical manipulations. Students in Pictorial Analysis game choose an external representation such as drawing a free body diagram to expand their conceptual understanding of the situation.

The AMEG[54] was developed based on students response to multiple choice survey, testing students' conceptual understanding of Archimedes' principle. AMEG has two main paths which are called justification path (AJ) and reasoning path (RA). In the AJ path students reach to an early answer in the problem solving process drawn from their existing physics knowledge, followed by justification. In contrast, in the RA path students reach to an answer after providing a conceptual reasoning, which may lead to some mathematical calculations to finally find an answer.

Students playing the AJ path enter the game by remembering an answer or provide an intuitive answer followed by a brief justification. Students exit the game quickly through the AJ path, since they are in an answer making frame and think that their answer is sufficient. If students do not remember an answer, they enter the RA path to provide a conceptual reasoning that helps them to get to an answer. If the conceptual reasoning does not lead to a sufficient solution, students can do math to find an answer. In contrast to AJ path, students in the RA path of the AMEG, frame problem solving as a sense making activity.

The AMEG with two flexible paths encompasses the four e-games by Tuminaro with the sense making frames. Playing the RA path followed by doing math to give a sufficient answer is comparable to the Mapping Meaning to Mathematics e-game, as both of them start with a conceptual story and map to a fruitful mathematical manipulation. If students play the RA path and use multiple representations such a drawing a free body diagram to get to the sufficient answer, then this is comparable to playing the Pictorial Analysis game. The Mapping Mathematics to Meaning is comparable to students doing math in an iterative cycle to finally get to a justified answer or a reasoning path. If students through the AJ path provide an intuitive answer followed by justification, this is similar to playing the Physical Mechanism Game.

2.2.6 Coordination class

The Coordination class is a particular kind of knowledge that allows to identify a type of information in the world. Not all the information is relevant. A coordination class is a specific concept that is usually hard to understand and even mediated by other coordination classes. For example, to explain the concept of force, one might read the information of mass and acceleration of a moving object and then uses the Newton's second law equation to make an inference that the force is mass times acceleration. The tools that allow to identify specific information, for example about the coordination classes of force or velocity, from the world is called readout strategies. Reading the hidden information out of the many situations available in the world specifies the function of the coordination class[56;57].Causal net is where the readouts and primed knowledge come together. I review research done on coordination class and their contribution to this theory. Research has broadly used coordination classes in physics education[58–60], mathematical education[61] and computer science education[62], since physics and mathematical constructs are good candidates to be specified as coordination classes. This is evident from students' possession of generalized knowledge structure about some of the mathematical and physics constructs.

This theory has been previously used to explain students' understanding of concepts across fifth graders, high school class rooms[60] and introductory level physics students[58]. The coordination class theory[58] has a perceptual part that is to read the information out of this world that shows the characteristics of the concept. The post perceptual part includes the reasoning strategies that turn the readouts to function as the concept.

The strategy that is used to turn the readouts to coordination classes is called concept projection. Not every projection of ideas determine a coordination class. A concept is a coordination class if the projection of information in different contexts leads to the same aligned information. In addition, the range of the contexts that one can apply the coordination class is called the span of that concept. Both span and alignment of a concept determines the power of the coordination class theory.[61]

Many researchers have determined students difficulties with these two constructs in their learning process. Students have lack of span if they can apply a concept projection in one context, but are not able to use a similar projection in a new context to determine the same information. Students have lack of alignment if they are unable to use their strategies to find the similar information in different contexts. In Chapter 5, I investigate the role of a particular kind of readout, which we call "looking ahead", which is an extended readout strategy. "Looking ahead" accounts for how to read out relevant information that cue the activation of generalized knowledge in our mind, make decisions and narrow down our knowledge use to describe a particular physical system.

diSessa et al[58] use the coordination class lenses and interviewed a student, called J, that is asked about the coordination class of force. J reveals several difficulties while coordinating different ideas. J does not seem to appreciate force as a coordination characteristic of this concept. The authors discuss that J's difficulty was mostly in her causal net. Instead of making correct inferences with the $F = ma$ equation, J rejected and substituted the equation with her naive ideas. J did not have difficulty with her readout strategy as she was able to read the information of a moving car or perceive the weight of an object. The causal net was considered as the core of learning a coordination class.

Wagner[61], investigates a student's understanding of the expected value concept. Wagner used the knowledge in pieces view to explain how a student, called Maria, in a new context, activated fine-grained elements of her knowledge to make a new concept projection and increased the span of her understanding of the expectation value. Wagner's study explains how incremental knowledge elements account for context variations. Variation in context helped Maria to read out other aspects of a problem as being relevant information and developed her rule of "the larger your sample the closer you get to the expectation value". Wagner mentioned that Maria did not have problem with alignment and her systematic growth of resources in a new context increased the span of application of her rule. Wagner considered this incremental growth in the systematization of Maria's concept projection as "transfer in pieces". In Chapter 5, within one context of a particular physical scenario, I show how students might use multiple readout strategies to refine their generalized knowledge in the problem.

Thaden-Koch et al[63], interviewed students to ask about the realism of motion in three different animations, among different trajectories with one ball and two ball. the features of each track was affecting the observations and thus students' judgements. Students did not show characteristics of a coordination class related to velocity, change in velocity, or energy conservation law in a systematic way as experts do. But, students coordinated these ideas into their readouts and causal nets. The causal nets were defined as students expectations about the realism of the motion. This study shows that a motion was realistic to the students if the readouts and their expectations based on those readouts were consistent. The readouts were context dependent, and adding one ball changed the readouts and thus the expectation of students about the realistic motion. The selected readouts in the context of two balls was the relative velocity. Whereas, the selected readouts in the context of one ball was the absolute velocity. Since there is no defined coordination classes, thus this study is limited to how giving feedback to students can help them to revise their readouts and causal nets, or how their causal net can lead to a contrary expectation.

Wittmann[64] interviewed introductory level students about the wave motion and used a coordination class view to analyze students' readouts and causal net. The salient readout in this study was related to picking the peak of the wave pulse to represent the motion. Consistent with this readout, students provided several reasoning resources to describe the object-like motion of the wave.

Parnafes[60] Interviewed middle and high school students to examine their understanding of the harmonic oscillation motion via the lens of coordination class theory. Student initially explored the physical oscillators and then engaged in computer-based simulation. The computer-based simulation provided a new context that students can learn the scientific coordination classes of velocity and period of the oscillatory system via refined perceptual readouts guided by the causal net. This study shows the dynamic of how an intuitive notion of "fastness" can evolve to a more structural scientific concept of velocity as it is used by experts.

Levrini et al[65] examined students ideas about the coordination class of proper time. The mathematics involved was at the algebra based level. Initially students, undergo a misalignment as their definition regarding that the frame of reference is uncertain. The instructor then provides more definition about proper time which increases the span of students understanding of the proper time, but does not help with the misalignment problem. Finally, in the third situation students can get into alignment. This study contributed to the coordination class theory by introducing the articulated alignment. This is a stronger form of alignment as students seek for reasons that how the different concept projections in different situations are aligned with each other.

Barth-Cohen et al[57] have discussed the role of different contexts in aligning the coordination classes for ninth grade students participating in an earth science class. Students use the energy theater (ET)[66] activity to develop an embodied model to understand the mechanism that keeps the energy of the earth in a steady state. The authors argue that in a classroom setting there can be different contexts such as *scientific context*, *classroom context*, and *student context*. The first notion of context is similar to what the last two studies have used to examine the span or alignment of a scientific concept. Furthermore, the authors also identify the context of classroom, which is the interpretation of the whole class from the conceptual model. For the third context the authors break the classroom context into each individual student's context to further account for the lack of span or alignment of each student.

Most of the studies discussed have contributed to the theory of coordination classes by focusing on the causal net as the core of learning, or discussing the increased span and alignment of a concept. In Chapter 5, I introduce how students "looking ahead" into the problem, which emphasizes on the role of an extended readout strategy. In addition, all of the studies have been conducted in the context of middle, high school, and introductory-based level physics. My study in Chapter 5, extends the use of coordination class theory to upper-level problem solving.

2.2.7 Epistemological framing

In section 2.2.5, I briefly discussed that students play different e-games based on their expectations in problem solving. This expectation can be conscious or unconscious[6]. Students can proceed to solve a problem based upon a preconceived expectation about the nature of problem solving in physics[6], or more particularly as an interaction effect between the problem statement and students' thinking. For example, certain keywords in the problem statement can cue students to frame the problem in a particular way such as algorithmically or conceptually. Students' framing of a problem can also changes by other external factors such as instructors. Framing is the expectation of "what's going on here?"[11]. Framing has different aspects[67]. For example, in a classroom setting students can frame the situation socially as ("Who will I interact with and how?"), students can frame the situation as what material are they going to be using, for example at an open book exam. As an education researcher I am interested in epistemological aspect of framing. Two students can have the same social component of framing, but have different epistemological framings. Students beliefs about learning could be based on sense making or rote memorizations[19]. The mechanism that allows control of which subset of resources are activated locally in a given context is epistemological framing[68]. Framing shows the nature of students' knowledge that emerges from a coherent set of fine-grained resources which coherently and locally work together in a situation[19]. Students' epistemological expectation has an important role in their learning science[19;45;69–72] As I discussed earlier in this chapter researcher have developed several concept inventories to assess conceptual understanding of students after instructions. Interestingly, research shows that students epistemological beliefs may remain unchanged despite improvement in students conceptual understanding[73;74]. Several surveys have been developed on students' beliefs about science[73;75–77]. For instruction, epistemologically-aware tutorials at the introductory level[78] has been shown to outperform difficulties-based tutorials[38] in student understanding of Newton's Third Law[79;80].

Frames can explain the structure of our expectations in different situations. Thus if students notice an error, this indicates something unexpected happening in-the-moment of their problem solving. By using epistemological framing lens researchers can investigate how students are going about to solve their errors, if they have got "stuck" in their solution how they are going to navigate frames to get "un-stuck". As I explain in more detail in Chapter 7, that observational data is a proper candidate to see sharp changes in students' behaviors as they face unseen conditions. We discuss several examples from quantum mechanics data in Chapter 7 to explain students' frame transitions.

2.3 Summary

In this section, I compare and contrast each piece of Knowledge in pieces (KiP) family theory, and give an overview of how different theories of KiP work together, and how researchers might choose one particular theory over the other. P-prims can account for the basic level of students' intuition that how things work[7]. Students sense of mechanism can develop by connecting to mathematical forms[44]. diSessa[7] argues that p-prims on their own do not posses sufficient systematicity to describe a complicated physical system. Thus he introduces coordination classes to explain how students learn complicated concepts[58]. If we are interested to investigate how students learn a concept for the first time, resources are good candidate to analyze the process in which students link their ideas to each other. By the time when the link between the resources strengthen they can compile to a unit. One of the properties of a resource that can keep track of these changes over time is the "plasticity" property of a resource. A particular problem solving context can prime the same patterns of activation, which over time solidifies the connections among resources, or incremental activation of new ideas account for the new contextual features in the problem[14]. Resources have worked with other theories to explains students' reasoning. Resources can be used as the knowledge base of an e-game. An e-game illustrates the pattern of knowledge activation along different moves. E-games can be a good theoretical lens for analyzing introductory level problem solving. But upper-level students problem solving strategies are not limited to strict moves, or may not follow a particular pattern of activation due to spontaneous in-the-moment unseen condition. Frames answer the question "what's going on", while resources are recruited into each frame.

Chapter 3

Unpacking separation of variables in partial differential equations in quantum mechanics

3.1 Introduction

As part of a larger project to investigate how upper-division students solve mathematically-intense problems, we are interested in how students use mathematics to solve partial differential equations (PDEs) and perform partial derivatives (PDs) via the method of separation of variables (SOV). PDEs are endemic in upper-division physics theory courses, from the Maxwell relations in thermodynamics, to Maxwell's equations in electromagnetism, and the Schrödinger equation in quantum mechanics. Previous research on student understanding of PDEs has focused primarily on thermodynamics[81;82]. PDEs are used to model many systems across STEM disciplines, from predator-prey models in biology to materials stress in mechanical engineering, and reaction rates in chemistry.

To understand how students solve PDEs and perform PDs, we conduct a qualitative case study on a group of students. This piece of our data with rich information, and being plausible theory driven motivated us to investigate what are the elements of students' thinking as they solve the time dependent Schrödinger equation (TDSE) in quantum mechanics. We use the resources framework to provide us a fine grain lens[10].

In this chapter using the resources framework I provide evidence that within one group students unpack their reasoning to perform the process of separation of variables.

In this study, within the class of procedural resources, I present a specific procedural resource – *bringing out* – which students use to solve PDEs and perform PDs. Whereas, previous study has only identified the procedural resources to separate an air resistance problem via a first order ordinary differential equation (ODEs)[47], I have identified both procedural and conceptual resources that come together to separate the TDSE. Though the work I present here is in quantum mechanics, I believe *bringing out* is important to students' understanding of PDEs. This topic is crosscutting in physics and is broadly applicable in STEM. Students use the words "carries out" or "comes out" when performing this resource. I call this resource, *bringing out* to show the actions verbalized by the students.

In this study, I present two examples from the same group of students enrolled in an upper-level quantum mechanics classroom. I analyze two in-class problem-solving sessions, one near the beginning of the course and one two months later. The first analysis is in more detail as the heart of this discussion emerges in the *bringing out* resource (3.3.3), and to show this existence we build resource graphs[50] using and unpacking the *separation of variables* resource in this context (3.3.1 and 3.3.2).

3.2 Context

We collected video data from one semester of a senior-level undergraduate quantum mechanics course. The class was a mixture of traditional lecture and in-class problem solving sessions. We recorded three different groups throughout the semester. Students worked in groups of 2-3 to solve the questions on table-based whiteboards, with a wide angle camera installed above. The instructor controlled the length of each problem solving interlude, generally 2-5 minutes. The textbook used was Griffiths' Introduction to Quantum Mechanics[83]. To determine evidence of students' use of mathematical actions in physics, we closely

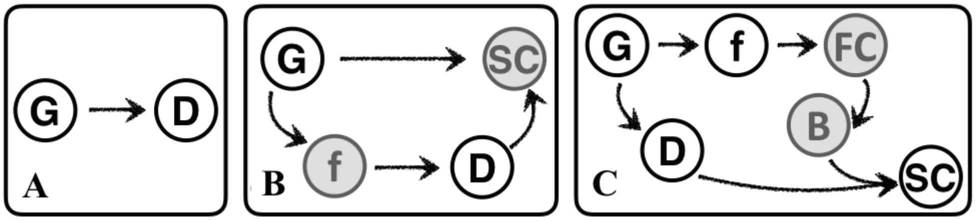


Figure 3.1: *Phases of unpacking separation of variables. A: initial unpacking with two procedural resources activated. B: intermediate unpacking with two conceptual resources activated. C: final unpacking with one procedural resource and one conceptual resource activated. The gray circles show the added resources in steps B and C. Key in Table 4.1*

Table 3.1: *Resources in unpacked separation of variables*

Abbr	Name	Type
G	*grouping*	procedural
D	*dividing*	procedural
f	*function*	conceptual
SC	*separation constant*	conceptual
FC	*functions-as-constants*	conceptual
B	*bringing out*	procedural

observed students' discourse, hand gesture and whiteboard writing as they worked on a problem. Our data analysis focuses on two students, "Alex" and "Eric" work together to obtain the time independent Schrödinger equation (TISE) for the space part, starting with the TDSE, with a zero potential term.

The TDSE ($H\Psi(\mathbf{r},t) = i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$), is a partial differential equation that can describe the time evolution of any physical system with different potentials and boundary conditions. The easiest problem to consider is the free particle in one dimension of space x, where the Hamiltonian has only the kinetic energy term, and the partial differential equation depends on two variables of x and t ($\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(\mathbf{r},t) = i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t}$). One way to solve a partial differential equation is via the method of SOV. The SOV condition is to assume that the total wave function is a product of independent one variable wave functions ($\Psi(\mathbf{r},t)$=$\psi(\mathbf{r})f(t)$). By substituting the total wave function into the TDSE, the partial derivatives can turn into ordinary derivatives and after a couple of mathematical procedures the equation becomes separable. The separated spatial part is in the form of $H\psi(\mathbf{r}) = E\psi(\mathbf{r})$. In this study, we focus on students solving the early steps of PDEs, via the SOV.

3.3 First interaction

Eric has difficulty in performing SOV to solve the TDSE and seeks for explanation from Alex. In contrast, Alex treats some of the procedures as trivial, and he does not explicitly mention those steps. However, Alex is mindful that they are necessary steps and brings them into play, when Eric asks him questions to explain some of the skipped steps. In this study, I will treat the interaction between Alex and Eric as a case of "unpacking" resources. As Eric seeks for more elaboration, Alex gradually unpacks his unit of SOV (see Fig. 3.1). This gives us an opportunity to identify the resources that students bring into play.

3.3.1 Grouping and dividing

On his first pass through the problem, Eric starts to write the TDSE in the form of an differential equation acting on ψ. As Eric writes the TDSE, Alex changes the small ψ symbol to the $\Psi(x,t)$. Their TDSE is missing a factor of $\frac{-1}{2m}$. Eric asserts that the spatial solution will be sine waves, saying dismissively "Oh well this is, like just a sine wave, something like that." At this point, he does not go through all the steps of solving the ordinary differential equation. Alex interrupts Eric's solution train midway to use a more formal separation of variables procedure by saying, "so we can have them as separable ψ, Capital Ψ is $\psi(x)$ times function of time."

1 **Alex** This is only a function of time [pointing to the left side of the TDSE].

2 **Alex and Eric** And that's only a function of x [both pointing to the right side of the TDSE].

3 **Alex** Well it will be, when we divide by capital Ψ [pointing to $\Psi(x,t){=}\psi(x)f(t)$]. So we get, so we get that $\frac{f'}{f} = -i\hbar\frac{\psi''}{\psi}$, once you divide through.

Eric readily goes along with and elaborates on Alex's bid to use separation of variables more formally. In lines 1 and 2, together they point to each side of the equation separately to group the space functions on the left side of the equation and the time functions on the right.

In his last statement, Alex references if $\Psi(x,t)$ is the product of two $\psi(x)f(t)$, the PDE is separable by dividing both sides by Ψ. At this point in their joint explanation, they are using two procedural resources to build the separation of variable resource: *grouping* terms by variable, and *dividing* both sides by the original function Ψ (see Fig. 3.1A). As shown in table 4.1, the procedural resources: *grouping* and *dividing* are previously identified[47] to show how students activate procedural thoughts that form mathematical manipulations.

3.3.2 Functions and separation constants

After seven seconds of silence, Eric points to the $\Psi(x,t)$ in the TDSE and says "oh ψ is double primed", it is not clear in the video if he is talking about $\Psi(x,t)$ or $\psi(x)$.

Alex further unpacks the *separation of variables* procedural resource. Alex points to each side of the TDSE and verbalizes what are forming the left side, and what are forming the right side of the equation. In line 4, Alex points to the TDSE to highlight the functions and the derivations in terms of $\psi(x)$ and $f(t)$ functions. We interpret Alex's explicitly unpacking of the functional relationship between Ψ, ψ, and f as activating the *function* resource and inserting it into the *separation of variables* procedure between *grouping* and *dividing*.

4 **Alex** Because this is ψ'' times f [pointing to the left side of the TDSE] and this is f' times ψ [pointing to the right side of the TDSE] and then [unintelligible] divide through by capital Ψ [pointing to the whole TDSE with gesture of hands with motion]...and you get this [pointing to the separated equation in line 3].

5 **Eric** Ok. Ok.

Alex continues immediately with explaining that each side of the separated equation ($\frac{f'}{f}$ $= -i\hbar\frac{\psi''}{\psi}$) "equals k...some constant". This is the first mention of a separation constant in the interaction. We interpret this as the activation and insertion of another resource into the graph for the *separation of variables* procedure (see Fig. 3.1B). In this part of the conversation Alex unpacked *separation of variables* more, inserting two more resources: *function* in the form of ψ and f, and *separation constant* in the form of k. These two resources are conceptual, not procedural, and serve to explain how *grouping* and *dividing*

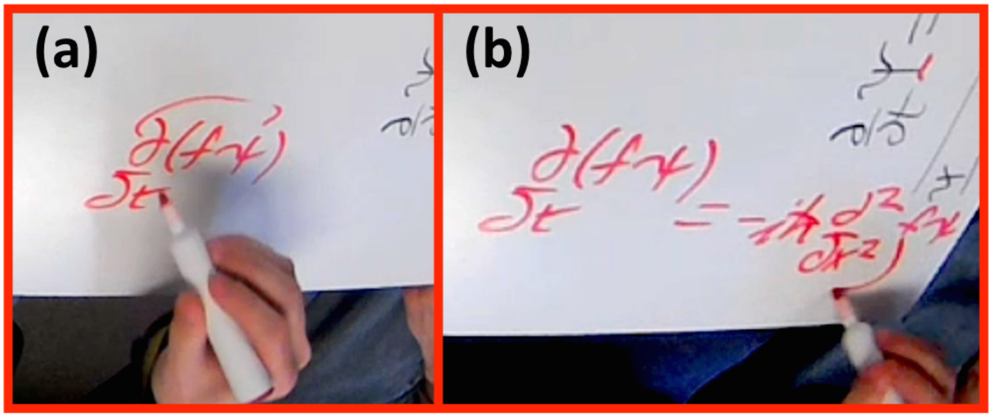


Figure 3.2: *Showing the bringing out hand gesture.* (*a*) *Bringing out the space function* ψ *from the time partial derivative.* (*b*) *Bringing out the time function* f *from the space partial derivative*

are connected. Alex does not further explain the procedures; what is held as a constant in each partial derivative, and with respect to what variable the derivation is taken.

3.3.3 Bringing out functions as constants

Eric is still a bit confused about how dividing both sides of the TDSE by $\Psi(x,t)$ results in the final separated solution, written by Alex in line 3. Eric again returns to the TDSE and points to the $\Psi(x,t)$ on one side of the TDSE, where Alex had pointed out to that location earlier to substitute the $\Psi(x,t)$ as a product of two functions. He points to the right side of the unseparated TDSE and asks: "How did you…like…get that…a step back?". Alex activates new ideas to elaborate on a missing step before performing the division step.

To explain the partial derivative, Alex uses a "bringing out" motion, drawing it on the shared whiteboard (see Fig. 3.2). At this point the assumption of the $\Psi(x,t) = \psi(x)f(t)$ makes sense to Eric as being consistent with the separated partial derivatives of space and time on each side of the TDSE (lines 7 and 9). Eric continues Alex's reasoning, remarking:

6 **Alex** We have $\frac{d}{dt}(f\psi)$, and then the ψ comes out [drawing a path showing the motion of *bringing out*], so we just have ψ partial derivative with respect to t.

7 **Eric** [surprised] Oh…We assume it looks like this [$\Psi(x,t) = \psi(x)f(t)$], because recognize that…form [TDSE].

8 **Alex** Yeah. Yeah. We assume it's separable. The assumption we are making is that it [Ψ] can be separated...And is function of x times function of t.

9 **Eric** Which is kind of implied because you have this time [partial time derivative] on one side and this [partial space derivative], and there is nothing else.

10 **Alex** Otherwise it's unsolvable [laughing]... So we do this, and then on the other side we have $-i\hbar\frac{\partial^2}{\partial x^2}f\psi$, and then that [$f$] comes out [while drawing the *bringing out motion*].

11 **Eric** Ok. Ok.

12 **Alex** You then divide by $f\psi$.

Alex on the bottom of the board writes the left side of the TDSE again to explain the procedure of "bringing out" resource (see Fig. 3.2a). Alex's use of word "comes out", in line 6, indicates a mathematical metaphor as if the term is able to move and comes out of the parentheses. He also uses a hand gesture at the same time to display the specific path connecting the source (inside the parentheses) to the destination which is behind the partial derivative.

Alex continues writing the equation (see Fig. 3.2(b)) explicitly as $\frac{\partial(f\psi)}{\partial t} = -i\hbar\frac{\partial^2}{\partial x^2}f\psi$. He temporarily treats the f as a constant to bring it out of the partial derivative on the right, saying "then that [f] comes out". Eric gives voice to Alex's treatment of f as a constant, confirming that "Ok. I see, because f... because that's [f is] a constant, ok I gotcha... that makes sense."

Alex, by using gestures, displays the mathematical metaphor, that a function can be treated as a constant and can come out of the partial derivative, which makes sense to Eric. Wittmann et al,[47] showed that students struggle to group terms properly on each side of the equation. In our data, instead Eric struggled with the idea of *functions-as-constants*. The procedural resource of *bringing out* made it easier to conceptually understand the only reason that one could bring the function out of the partial derivative is if it was a constant. I interpret this continued conversation as recruiting two more resources to unpack *separation of variables*: the conceptual *functions-as-constants* resource allows f to act like a constant in light of the partial space derivative, and therefore Alex can use the procedural *bringing out* resource to bring it outside the derivative. From this point, Eric can continue with *dividing* and *separation constant* to complete the problem. The fully unpacked resource graph for *separation of variables* is in Figure 3.1C.

3.4 Second interaction

Interestingly, about two months later, Alex and Eric become group mates again. In the first interaction, the potential was zero; in this interaction, the potential is a function of position and the problem is now in three dimensions, not one. Once again, the instructor wants the students to find the TISE in three dimensions for the space part.

Eric and Alex start the episode by discussing if the use of separation of variables is a good choice or not. They quickly run through the reasoning in Figure 3.1C, using the same hand gesture to show how functions can be treated as constants and come out of the derivatives.

Alex and Eric start the discussion with a valid assumption, that the potential and the Laplacian do not "vary" with time. Alex then uses a hand gesture indicating the path of *bringing out* resource and says "so all carries out, it's separable". Then Alex writes down the TDSE, with the time partial derivative on the left side of the equation, and the space partial derivative on the right side of the equation, followed by substitution of the $\psi(r)f(t)$ as Ψ into the equation. Then Alex uses a hand gesture to point to the $f(t)$, on the right side, and says "$f(t)$ carries out and then you can divide it over". Alex reuses the *bringing out* resource to treat the function $f(t)$ as a constant that can come out of the partial space derivatives. Alex, similar to the first interaction (line 12), talks about the *division* procedure after *bringing out*. Alex divides both sides by $\Psi(x,t)$ and cancels the constant functions of each side of the TDSE. In the next line, Alex writes, that $fun(t) = fun(r)$ both equal to a constant.

The first time that Alex says "it all carries out", the word "all" is ambiguous as what it refers to. When Alex uses the same word "carries out" for the $f(t)$ as it can come out of the partial space derivative, then we infer that the first "carries out" also refers to the potential and the Laplacian as they can come out of the partial time derivative. Alex two times verbally reuses the *bringing out* resource by saying "carries out" and showing a hand gesture indicating a path in the air.

Both Eric and Alex activate resources as they move forward through their solution. In this episode, Eric does not ask for further clarification in the problem solving steps, and even notices and corrects an Alex's error during the problem solving. Alex does not adopt the pedagogical tone he used in the prior interaction. Eric does not ask for further clarification in the problem solving steps, and even notices and corrects an Alex's error during the problem solving. They do not back and forth between different parts of their solution, or do not rewrite part of their solution on the whiteboard to elaborate on that.

3.5 Discussion

In this study I showed that Alex unpacks the resource of *separation of variables* in three steps. This unpacking process is in response to Eric's request for further elaboration. Whereas, previous research[1] had predicted that there is an internal structure to the resource of *separation of variables* in one imposed path, but they did not show how each step of unpacking happens. Because each phase occurred in an independent group. The authors mention that along this developmental path, gestures become important and the language of students become more procedural. However, the authors still can not explain that why one group have decided to further unpack their *separation of variables* procedure and why one group uses *separation of variables* as an entity.

In the current study, I distinguished different phases of unpacking by Alex in response to continuous clarification seeking questions of Eric. I showed how Alex identified two procedural resources in the first phase and then unpacked the *separation of variables* resource to activate two more conceptual resource that are aimed to explain how procedures in the first phase are connected. Alex in the last phase of unpacking, activates a new procedural resource, along with a new conceptual resource. Alex even in the first step was able to write down the separated equation. However, the procedural resource of *separation of variables* makes sense to Eric at the third phase of unpacking.

As we look across all of Alex's unpacking actions, we notice that his *separation of variables* procedure is decidedly non-linear. Initially, Alex uses *separation of variables* as an entity which is compressed (see Fig. 3.1A). He starts with skipping the procedure altogether, preferring to assert a solution by just one action of *division* (line 3). In response to Eric's ongoing confusion about how to separate variables, Alex unpacks his procedural resources, iteratively adding more conceptual and procedural resources to make explicit the parts of *separation of variables*. He expands to include two resources, then further details the link between them to describe two more. When Eric is still confused, Alex eventually resorts to *bringing out* and *functions-as-constants* to unpack *separation of variables* all the way. The ultimate structure of Alex's *separation of variables* in Figure. 3.1C, which makes sense to Eric, includes three conceptual resources and three procedural resources. By comparing the procedural resources activated by Alex to the procedural resources that are used by students in the air resistance problem[1], we can see that both studies have the resource of *divide*, the resource of *grouping* is also comparable to the resources of *split up* as it was responsible to put all the velocity terms on one side and all the space terms on the other side of the equation. However, the resource of *bringing out* in my study is a new procedural resource that has as important role in this unpacking process. In addition I also analyzed the role of conceptual resources that Alex activated to explain how procedural resources link.

Part of this nonlinear procedure of unpacking by Alex could be due to Eric's partial questions, such as Eric pointing to part of the unseparated equation and asking "how did you get from here to here?". On the other hand, it may be true that Alex in response to Eric is not able to unpack the *separation of variables* procedure all at once. Because he is used to using the *separation of variables* resource as an individual solid resource, with an internal structure which is not readily accessible by Alex.

More broadly, Thompson et al.[84] investigated student's difficulties with mixed partial derivatives. They reported one common difficulty of students with application of the product rule in deriving the mixed second partial derivatives of thermal expansion, and thermal compressibility coefficients with respect to pressure, and temperature respectively. Students exhibit a variety of errors, factoring out some functions inappropriately and treating others as constants too often. The simpler, unmixed partial derivatives in quantum mechanics better illustrate effective use of *bringing out* and *functions-as-constants*, and allow for better probes of the structure of students' *separation of variables* resource, and thus how students can productively perform PDs and solve PDEs.

In this study I identified a new procedural resource with a kinesthetic basis[52]. I investigated students' use of the "bringing out" resource in quantum mechanics to convert the Schrödinger partial differential equation to non-partial equations via separation of variables. The separated terms of partial derivatives, with respect to time and space, in the TDSE may cue students to group the equation based on two separated terms. However, not all of the students, similar to "Eric" in this study, are aware that this is only correct if the wave function is separated into a function of only space and a function of only time. Using the "bringing out" resource with gestural components is an efficient way for showing this elimination. I also showed that this resource is reusable as students after two months activate it again to solve a more general form of the TDSE successfully. The use of this resource is not limited only in partial differential equations in quantum mechanics; *bringing out* is an integral part of solving PDEs.

Using the resources framework, I provided evidence of how unpacking of the procedural resource of *separation of variables* is possible within one group of students. Researcher[85] have used other cognitive structures such as epistemic games to show how larger size procedural resources can be used as moves of an e-game when solving integrals to apply the boundary conditions. The moves in the Meaning Mapping to Mathematics epistemic game[6] can show how students apply boundary condition and map it to their use of mathematics. In Chapter 5, I will describe a problem that requires students to think about boundary conditions. In Chapter 5, I will provide evidence that how "looking ahead" at the early stage of problem solving can help with using math and physics insightfully.

In this chapter, by describing the unpacking process of the *separation of variables* resource I provided evidence that how resources procedurally and conceptually link to each other. In the next chapter, I use the theory of epistemic games to zoom out the resource activation of students, and provide evidence of pattern of activation of students' resources as a function of time.

Chapter 4

Learning about the Energy of a Hurricane System through an Estimation Epistemic Game

[1] Solving problems plays a major role in studying physics. Various researchers have developed theories and strategies to study students' problem solving in different contexts. Estimation problems are a type of Fermi questions[87] whose exact solutions are difficult or even impossible to measure. Solving Fermi questions requires intuition, mathematics, reasoning, and the skill to break down complex problems into discrete solvable parts. Estimation problems do not have a single exact route towards solution and that makes them ideal for studying students' problem solving decisions. We are interested in understanding the mechanism of the problem solving approach through the lens of epistemic games (e-games): an activation of patterns of activities that can be associated with sets of cognitive resources[6].

In this study, I identify a new e-game: the "estimation epistemic game" (e-e-game). In it, the student activates sets of resources and applies an estimation approach to evaluate possible solutions and produce knowledge and arguments.

[1]This analysis was published in the Proceedings of the 2014 Physics Education Research Conference as[86] link

4.1 Methodology

We interviewed "Ava", an upper-division physics major, using a talk-aloud protocol. She solved the following problem: "Estimate the total energy in a typical hurricane system. State explicitly the assumptions that you make, explain your reasoning, and assess your result." Ava had access to unit conversions and useful constants. Ava plays the game over about 14 minutes. Our focus for identifying the new e-e-game played by Ava is based on the observation of specific types of declarative and procedural resources activated in association with specific moves. We used micro-genetic analysis as a comprehensive way to examine moment-by-moment conceptual changes of Ava's learning activity. This method analyzes students' discourse and physical activities within a short period of time[88]. In order to observe moves in the macro level of the game, we need to identify and enlarge fine-grained information underlying sub-moves through the lens of a micro-analysis method[88]. The smallest observable time scales we have seen in the activation of resources were on the order of one second. Two researchers independently coded Ava's work, reaching 100 % agreement after discussion. Five additional researchers offered feedback several times along the process. Based on the micro-genetic analysis of the student's behavior, our observations, and evidence from previous studies, we propose that Ava is entering a previously un-articulated e-game, which consists of several basic moves.

4.2 Game Structure

The structure of an e-game consists of two components: the entry and exit conditions, and the moves. If Ava takes a particular path through the activated resources and refers to that specific path several times, we can call this path a part of an e-e-game or a move. There are six basic moves in this game (see Fig. 1.): 1. problematize, 2. propose method, 3. what to remember, 4. see if parts are enough, 5. pure calculations and 6. evaluations. Although there may be overlap in the moves and the activated resources of the e-e-game and the other e-games (detailed in discussion section), However, e-games to date have not been applied to estimation-type problems. Within the e-e-game, the ultimate goal is to solve a problem through estimation. However, we observe that Ava is unable to solve the problem at a glance and defines several immediate goals by breaking down the main question into several smaller estimation problems. She benefits from the fourth move, tied to the estimation aspect of the problem, to estimate a numerical answer for all of the physical sub-targets and brings into play the combination of the other moves.

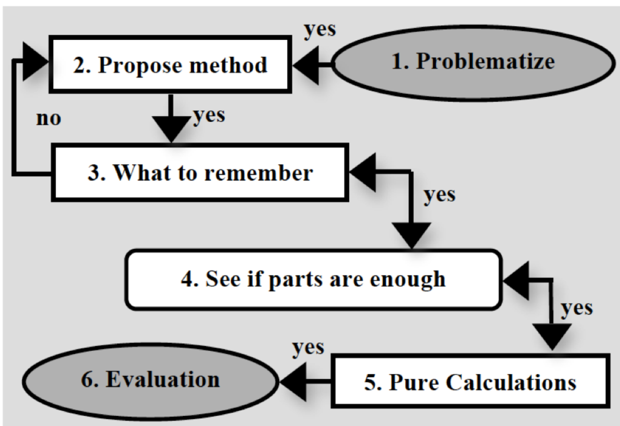


Figure 4.1: *Schematic graph of the estimation epistemic game (e-e-game) that Ava plays. The elliptical shapes indicate entry conditions of the games which are in grey color. Ava uses her estimation-based reasonings in move number 4.*

4.2.1 Problematize

In this case study, Ava enters the game by reading the first sentence of the question. She finds the question hard and begins to activate resources based on her prior informal knowledge about hurricanes, to obtain a general idea about the nature of the problem.

Ava: (00:08) Oh this sounds horrid! So what is a hurricane? A hurricane is mostly air that's moving in some sort of rotational system. Suppose... um... I suppose it's gonna (sic) lot to do with more high and low pressure areas. She starts her game with a quick, intuitive response and problematizes the hurricane system by describing a "rotational system" and "high and low pressure areas."

She starts her game with a quick, intuitive response and problematizes the hurricane system by describing a "rotational system" and "high and low pressure areas."

4.2.2 Propose Method

After she problematizes the question, Ava begins the second move, which is selecting a method. In this section, Ava starts with choosing the rotational energy formula, which is resulted from her expectations of the physics concepts in the question.

Ava: (00:52) Ok. I'm gonna start with just finding the rotational energy of the hurricane system. Because I think that sounds like something I can do (laughing) pretty reasonably. Ah, ok . . . so rotational energy

To calculate the inertia of a hurricane system, Ava begins to activate the physics quantity of inertia of a point mass and links it to other mathematical resources.

4.2.3 What To Remember

In this part Ava tabulates the facts, concepts, and equations by activating declarative resources and asking herself whether she can move forward. However, she doesn't remember enough about rotational systems to move forward with this method. She returns to the prior move, proposing a new method: examining the kinetic energy of the system.

Ava: (04:50) Energy equals $\frac{1}{2}mv^2$. So the mass is going to be the density times the volume.

While she records the related equation consistent with her method, new resources come to Ava's mind. This allows her to define new sub-targets and use correlative strategies to assemble and assess if the parts are sufficient to confer a reasonable estimation. Then she proceeds through the game and connects different parts of the problem.

4.2.4 See If Parts Are Enough

Ava activates her procedural resources by asking questions about how to solve different aspects of the problem: a process of mechanistic reasoning. In this step her resources become more developed by supporting them through reasonable estimations, examinations, measurements, and scales. Ava utilizes her conversion sheet to quickly determine that the density of water is greater than air. Within this short 'micro moment' she attempts to find a reason to neglect the effect of the water density by considering that the droplets of water consume less space than air.

Ava: (05:12) We will say it's probably mostly air. Water is significantly higher density but water droplets are small compared to the total volume of air. Maybe I should lowball one of these estimations and go on the other side of my other estimates. (pause) Ok ... how big is our hurricane? They usually look like this (showing with hands) around Florida - about that big. Let's see just a (sic) order of magnitude ... How big should this radius be? I'm assuming it's a cylinder because that sounds reasonable.

Ava uses her balancing resources ("lowball one ... go on the other side") to argue that she doesn't need to take into account the density of water. Then she activates her size resources to help her make an estimation to relate the size of a typical hurricane to the size of Florida. She decides that the shape of a typical hurricane is a cylinder ("I assume ..."). We infer that she chooses this shape ("that sounds reasonable") because she wants to perform some possible quick calculations ("how big should this radius be?"). We infer she has linked the intuition to her geometrical resources and specified the volume as a cylinder.

4.2.5 Pure Calculations

Now it's time to conduct her math calculations. At this point, by getting close to the end of the game, she increases her speed. Her affect becomes markedly more positive, and she enjoys this part of the game.

4.2.6 Evaluations

Before exiting the game, Ava does a kind of evaluation of her number by checking the units and considering her solution has some errors and might be different from the actual energy in a typical hurricane. In contrast to her earlier work, which is rife with sense-making and mechanistic reasoning, here Ava bemoans her inability to make sense of her number. She exits the game after evaluating her answer.

Ava: (15:40) We will say, 10^{15} joules for a typical hurricane system. I don't even know if that makes sense. I don't have a good way to check that.

4.3 Patterns of Moves

Ava is unable to solve the estimation problem at a glance; she breaks down the main question into several smaller estimation problems to generate heuristics sufficient for her to navigate sub-problems and arrive at a temporary goal. Ava is now in a position to use the fifth move (pure calculation) to combine the sub-targets and find the main answer. She switches several times between her third and fourth moves, mapping physical concepts and plugging estimate-based numbers into equations. She stops alternating when she has figured out all of the unknown physical quantities (Fig. 2). In the initial eight minutes of problem solving, Ava has longer, deep thinking periods while mostly inter-playing between the third and fourth moves. In the second time duration of six minutes of the game, she is using the third, fourth, and fifth moves, during which the rate of the changes in moves has increased thus speeding up the game.

In Table 1, we have indicated the total time that has been spent in each of these moves. Most of her time has been spent in the fourth move within which she does most of her estimations. She refers back to move 3 the largest number of times (34 times), each time for a brief period before returning to move 4 (in the first 8 minutes) or moves 4 and 5 (in the last 7 minutes).

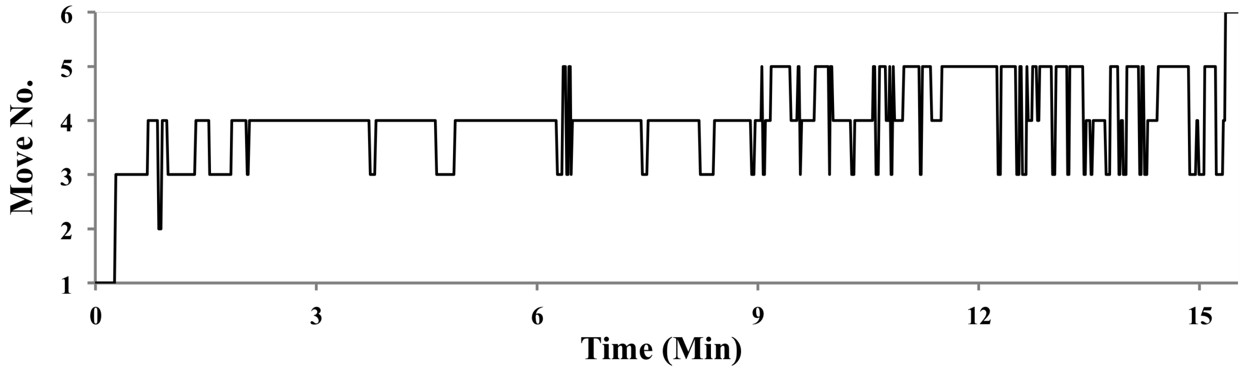


Figure 4.2: *Ava plays the e-e-game. Notice that initially she spends more time in each move, in the first 8 minutes she shifts mostly between move 3 and move 4, but around minute 10, she starts to change rapidly between three moves of 3, 4, and 5.*

Table 4.1: *Times and recall numbers allocated to different moves of the e-e-game.*

Move Number	3	4	5
Minutes in move	1.86	5.41	2.29
# transitions to here	34	27	26

4.4 Discussion and Implications

In this section first I will compare and contrast the relationship of e-e-game to other e-games identified by Tuminaro[6] and Chen[54]. Then I will compare and contrast all of the e-games together.

The e-e-game has some moves that are both similar to and different from the e-games identified by Tuminaro and Redish[6], and the AMEG identified by Chen et al[54]. The fourth move is similar to the justification path of the AMEG, in which students try to build a justification for their remembered or intuitive answers. The third move of this game is comparable to the second part of the Mapping Mathematics to Meaning e-game. In both, students relate the target of the problem statement to other physical quantities by plugging them into an equation, but it is different from the third part of the Mapping Mathematics to Meaning e-game which is telling a story to use mathematics in accordance to the physical system. Another likeness is with the sub-part of the third step in the Recursive Plug-and-Chug e-game in identifying other unknown sub-target quantities rather than the target quantity; however, in the Recursive Plug-and-Chug e-game, students achieve a numerical answer without making sense out of that answer. This differs from the reasoning conducted in the fourth move of this game.

The e-e-game played by Ava with specific paths and moves might be an artifact derived from the estimation-based question. Given our data, we can't distinguish between Ava's specific game-playing details in this context and a more generalized characterization of the e-e-game; however, preliminary analysis of other players in similar questions suggests that she is not unusual in her approach. Her experience as an upper-level undergraduate student would likely affect her choice to enter a flexible e-e-game, with several sub-moves to activate a variety of specific relevant resources available to her. There may be a large overlap in students' activation of resources within both physical and mathematical domains. However, the order of activation in a specific game could lead to different paths resulting in numerous starting conditions, moves, and ending outcomes. For future studies, we are interested in investigating in a larger scope other outcomes of the e-e-games that attempt to solve the same class of estimation problems.

Ava uses different knowledge resources of p-prims such as balancing, intuitive reasoning, or mathematical resources related to understanding geometry that forms the knowledge base of e-e-game. Students playing the AMEG, or the students in Tuminaro's games that have the conceptual reasoning as part of their moves also use these resources to provide reasoning. The AMEG with two different paths allows students to play a more flexible game compared to the e-games identified by Tuminaro. Since each path in the AMEG allows students with different expectations to enter the same game, solve the problem and get to the final exit condition which is finding a sufficient answer. Usually, students in the AMEG choose an effortless path, such as justification path, to find an answer in the minimum amount of time. But if students do not remember an intuitive answer they could switch to the reasoning path and finally find an answer. In addition AMEG captures students shift to mathematical manipulation move to support students' conceptual reasoning. So students by playing the AMEG are less likely to get stuck as compared to students in Tuminaro's study with more structural moves. Similar to the AMEG, the e-e-game also provides different moves, that students could progress in the problem. As evidenced by the pattern of Ava's moves in figure 4.2, Ava coordinates among her different knowledge pieces and mathematical tools to find a sufficient answer. Ava continuously switches among moves 3, 4, and 5 to proceed into the game. It is hard to say if playing the e-e-game implies an answer making or sense making frame. There are many obvious instances when Ava does sense making by providing a mechanistic story for validity of her estimations. In other instances, during the game Ava just remembers an intuitive answer without further justification, and mostly relying on the fact that she knows these numbers based on"watching the weather channel". However, in her last move when Ava tries to check if her numerical answer makes sense or not that is an indication of an attempt to a sense making problem solving. Thus students could change their epistemological frames during problem solving, which is not easy to see in the strict move structures of e-games. More particularly, the e-games in Tuminaro's study are developed to capture students' problem solving behaviors in introductory level. Ava's move patterns provides cue that upper-division students can navigate among the moves more flexible compared to introductory-level students. Thus for studying how upper-division students frame problem solving I will choose the theoretical construct of epistemological framing. This leads to my second main research question which I will discuss in chapters 6 through 8.

4.5 Conclusion

An e-game can be described as a list of activated procedural linked resources. By investigating the behavior of Ava at a finely grained level as she attempts to solve an estimation-based problem, we found that she enters into a previously unarticulated e-game by defining several sub-estimation problems. She activates different types of resources and maps them directly to each sub-problem situation, then combines the individual pieces of the sub-problems to find the final answer. Ava supports and links different resources mainly by using estimations in the form of an e-e-game. Physicists often use estimations to make a large number of difficult-to-solve problems tractable. It is worthwhile for students to solve estimation physics problems to develop practical, logical, and reasoning skills. Via e-e-games, students are able to apply their intuition and accessible knowledge in taking the first crucial steps of solving problems as a physicist.

Chapter 5

"Looking ahead" as an extended readout strategy in EM

Feil et al[89] showed that experts and novices with more relevant knowledge noticed the featural differences added to the problem when it caused a change to the physical system. But they didn't explain any mechanism that accounts for a relation between reading featural specifics of the problem and the physical meaning.

diSessa introduces the cognitive mechanism that describes knowledge can be perceived based on the sensory elements, or via reasoning tools. Concepts can become activated by either of the causal net or the the sensory information. Researchers[58–60] have broadly studied the interaction between the reasoning tools called p-prims and the concepts called coordination classes. diSessa calls the sensory tools for perceiving information as read out strategies. I think that the sensory tools under instruction can also become as important as the causal tools. In this study my goal is to investigate the role of an extended read out strategy and its interaction with concepts of coordination classes.

Coordination class theory explains not only how knowledge and causal net interact. But, also how reading out information can prime a particular kind of knowledge. Our generalized knowledge can interact with the problem statement via a reading out strategy. This is especially important in upper level physics courses that students need to solve exactly a physical system. In introductory level physics mostly the readouts are perceptual and simple. For example, to determine the information about the velocity of a moving car. But, in upper level physics problem solving the physical system is more complicated and require extended readout strategies[58]. Another main difference is that upper division physics relies more heavily on mathematics, thus students need to have an extended readout strategy to make sure that their math and physics agree with each other and describe the same physical system.

In this study I am interested to see how students in particular moments of their problem solving look ahead and readout specific information from the problem statement to map those information to the generalized knowledge that they have about math and physics. This is important to upper division courses where mathematics is critical to understanding the subject and also the problems are simplified versions of real world situations that have many physical features to be extracted from the problem statement and prime the relevant generalized knowledge in students' mind.

I use the coordination class lenses to explain the connection between the generalized knowledge in students' mind and the written information. Students' at this stage of their education are at the "journeyman-level"[49], that have access to many ideas and toolboxes. During a long problem solving session, student might frequently activate new ideas and it matters that they have an extended readout strategy to control over the activation and application of their knowledge.

5.1 Context

We collected video data from two classes, in Fall 2013 and Fall 2015, of an upper-level undergraduate Electromagnetic Fields 1 course. This is a junior-level course, the demographic of the class is mostly physics majors and minors. 10 % of the population of the class are females, and about 10 % of the students are educators. Careful observation of students' discourse of in-class group problem solving can provide clues for determining students' moment of looking ahead. The class is taught using Griffith's textbook, "Introduction to Electrodynamics"[90]. The course starts with the electric field in vacuums, and how charge distributions create fields, the effect of fields on charge distribution on and it moves to magnetic fields created by currents and electrodynamics and electromagnetic waves. The materials finishes up with the discussion of fields and materials. It meets for four 50-minute sessions each week. During class, lecture is interspersed with 3-4 group problem solving episodes. At the beginning of class the instructor introduces the topic and explains the relevant concepts. Each class has about 20 students divided into groups of 3-4 students solve problems collaboratively on shared table-based whiteboards. Each session follows one of three most common scenarios in the classroom: multiple problem solving session, extended problem solving session or tutorial session. Multiple problem solving involves students spending the majority of time working on two to three problems about a similar topic. An extended problem solving session typically describes students working on one longer problem. In a tutorial session, students solve tutorial problems in groups. These tutorials are developed by the instructor or taken from the PER group at University of Colorado, Boulder.

In most of the sessions students work on the tutorial problems. Most problem-solving sessions last about 5-10 minutes. Sometimes the instructor pauses the class problem solving to provide a feedback. Sometimes the instructor just asks a question in the lecture part and wants the students to think about it, and then they talk to the people near them. Students are remarkably collaborative, usually working together for the entire duration of each problem-solving session. In our data set, we observed approximately about three problem-solving sessions per class. During the problem solving session the instructor walks around the class, reaches each table individually and provides feedback on the processes.

The groups in this class are somewhat fluid, and students may form different groups on different days. Students occasionally recruit others from nearby groups to help them solve problems. The instructor does not explicitly tell students where to sit or with whom to work (other than with "people near you").

5.2 Methodology for video data

We are interested to see how students look ahead into the problem to solve the differential equations via the method of separation of variables, and doing the Taylor series expansions. In learning environments such as group problem solving in upper division contexts, one way to interpret the high level of interactions within group members is to carefully analyze the discourse and gestures of each member of the group. Ethnography provides an opportunity to understand the detail of students' discourse, behaviors, and any useful information while they are investigating a phenomena[91]. One of the methods for data collection in ethnography studies in through video recording of activities. This becomes more important in science by providing multiple researchers an opportunity to view and analyze the videos[92]. Previous researchers in education have used ethnography to study the culture of classroom activities[93] or even in more engaging learning environments, such as advanced physics laboratory[94]. Our goal was to develop a theoretical lens to enable us to explain how students look ahead into the solution during their in-class problem solving sessions. Another method that we choose to analyze our video data is interaction analysis[95]. Many learning activities such as group problem solving could be proceeded as segments of events. Each event has a structure that could become evident to the interaction analyst, as the participants do a series of meaningful activities and interactions. There could be an order to the kind of interactions in an event that have the same nature. Events can have starting and ending points. Sometimes moving from one segment to the next segment of an event is identifiable as a transition[95]. In this study, we are interested to see how the nature of students' readouts are different with regard to the generalized content knowledge that they map to. This is important, since if we know the nature of readouts then we know what content knowledge they are associated with. In our case, since the problem has three dimensions, whenever a generalized knowledge is cued, then students maintain in the sub-event till read all the information about the same general knowledge across all the three dimensions in the problem, which even make it more evident that students are reading specifics of the problem to fill the structure of the same generalized idea.

Selection of a video segment in the data is based on what guiding questions as a researcher we have[92]. For example, if we are interested in a larger time scale event then we might look at the class event and determine what happens before the problem solving sessions as well. In our study, usually, at the beginning of the class the instructor announces the upcoming events or distributes the graded homework. For this study we are not interested in those social activities and instead, our focus is on sub-events in the problem solving sessions of SOV and Taylor series. Our guiding research question is "how students are looking ahead?". The assumption of interaction analysis is that knowledge and practice activate, as the member of a social community interact with the external world. Thus, carefully analyzing the interactions among group members can reveal information about how they activate their knowledge to view the external world which is out of their mind[95]. By choosing interaction analysis as a method we can determine which part of this world is more salient to the students. This is important because, knowing what information students are reading out is associated with which coordination class students have activated. We started watching the videos from Fall 2013 and Fall 2015, that students are solving SOV. We noticed that students look at the graph in the problem statement and do multiple readouts. Thus, we became interested in these readouts as sub-events. When students were solving separation of variables we observed that at different times during the problem solving process students refer to the problem statement to discuss the features of the physical system, make a decision, and then again continue with their calculations. In contrast, we also observed students who relied on doing just heavily algebraic steps, either took a long time to complete, or students "got stuck" because, they did not know how to proceed. In our preliminary analysis of the students' group problem-solving activities, we observed that when students are doing the Taylor series expansion before doing their calculations. They conduct several sense-making activities and ask questions from their group mates that "what is the variable here", or "what is the constant here", or they might ask a more sophisticated question that "what is the specific functional form to take the derivative with respect to?".

From the tradition of progressive refinement of hypotheses[96], we set out to refine our observations through close interrogation of the video data. We started with selecting episode for close analysis based on their duration (longer is better), readout richness (more readout is better), and technical quality (more visible and audible are better). We reflected on these episodes, to understand how students looking ahead into their solution. Through recurring watching and examination of the details of the selected episodes, we sought to capture changes when students are setting up their differential equations (in SOV), or setting up their specific functional forms (In Taylor series), and then return to the problem statement to readout specific information on the graph. Then they return again to their equations and choose a sign that matches the specifics of the physical system to the equation (in SOV), or make a choice of with respect to what variable to take the derivative (in Taylor series). From the perspective of coordination class theory, we argue that the kind of activities that students are doing is to readout specific information and then they can apply their processes on. Thus, we concluded that students by making some specific readouts are consciously considering their general schemata and by making readouts are filling the specifics into it. We characterized the mechanism of looking ahead as an extended readout strategy that can help students to map the specifics of the problem to the generalized knowledge in their mind. Looking ahead into the solution makes the future of the problem easier and helps with decision making and insightful connection of math and physics.

Usually, 3-4 students pair in one group to work on a problem. Watching videos several times help us to see the overlapping moments in students speech. We also have episodes, when the instructor arrives and notices that students have not looked ahead into their solution, and the instructor encourages students to move a step backward and read the specifics of the problem to fix part of their solution. Thus the interactions even increase in the presence of the instructor and tracking all the interactions in the light of video observations become possible.

5.3 Generalized knowledge used by the students

Students use the method of separation of variables to find the voltage of a box, and use the Taylor series to expand the potential of different charge distributions at a point far away.

We are interested to see how students think insightfully use their math in physics problem solving. Students at upper division usually have a general schemata (coordination class) of their math knowledge or about a concept. However, students need to read information out of the physical situation to fill the specifics of that generalized coordination class in order to find the an answer.

In this research we have expanded the coordination class theory by defining a mechanism, called "looking ahead", that shows how students can read information out of the problem statement to fill the specifics of their general schemata and provide a causal net. (see Figure 5.1). The generalized ideas that students in the context of SOV and Taylor series expansion activate are about functions, functional forms such as sinusoids or exponentials and the ideas that they posses to solve second order simple homogenous ordinary differential equations. These problems are long and are heavily math dependent, thus students several times need to back and forth between the problem statement and their solution in order to readout all the relevant information and map it to their general conceptual schemata (the "coordination class").

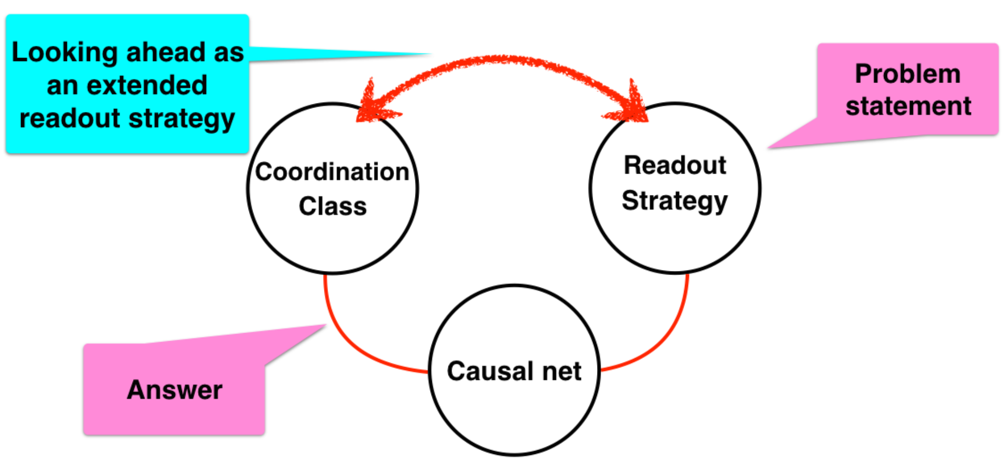


Figure 5.1: *Expanded coordination class theory with the emphasis on the role of an extended readout strategy. The mechanism of the looking ahead explains the interaction effect between the problem statement and the coordination class, which are the generalized knowledge that students possess.*

5.3.1 Physics of Separation of variables

In order to find if the potential varies in x, y, and z directions. One can read the information from the boundary conditions of the problem. Since whatever the solution will be it has to satisfy the boundary condition. Across the x axis, the potential is non zero at $x = a$, and the potential is zero and at $x = 0$. So the potential can look like a parabola shape. Mathematically this means, that the solution looks like combination of exponential functions, $Ae^{kx} + Be^{-kx}$.

In the y direction, the potential is zero at $y = 0$, and goes to zero at $y = b$ and is nonzero in the middle. So the potential can look like sinusoid. Mathematically, this means, that the solution looks like combination of sines and cosine functions, $Csin(ky) + Dcos(ky)$.

In the z direction, if we imagine slices of the box across the z axis, no matter where on the z axis, the potential is going to always look the same, which means the potential is a constant. mathematically, the solution will a constant coefficient such as "E".

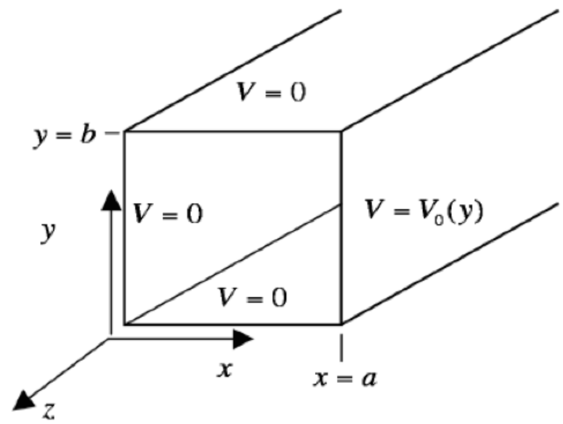
In addition to this geometrical argument one can solve the Laplacian equation. This problem in a formal sense asks to solve the Laplace's equation. By choosing the natural coordinate of a physical system, which is cartesian, one can expand the Laplace's equation via the method of separation of variables (SOV). One way to solve a partial differential equation is to break it into a series of independent one variable equations and solve each of those separated equations. This method is called SOV. The SOV condition is to assume that the general solution is a product of independent one variable solutions. By substituting the general solution in to the Laplace's equation, the partial derivatives can turn into ordinary derivatives and after a couple of mathematical procedures the equation becomes separable and results into two second order differential equations in terms of variables x and variables y. Since the variation of the potential in x is independent of the variation of the potential in y, then each equation has to be equal to a constant which differs in a minus sign. Usually, the separation constant is in the form of $\pm \mathrm{k}^2$.

In clips 1 and 2, students activate their generalized knowledge about functions and functional forms by using extended readout strategies. In clip 3, students use looking ahead to make a decision that where the sign goes and which solution belongs to x and which to y?

Within a very long , rectangular, hollow pipe, there are no electric charges. The walls of this pipe are kept at known voltage.

Three of the walls are grounded: $V(x=0,y,z)=0$; $V(x,y=0,z)=0$; $V(x,y=b,z)=0$

The fourth was maintains a potential that varies with y: $V(x=a,y,z)=V_0(y)$ which will



be specified later.

Figure 5.2: *Tutorial 5: Separation of variables, given to the students during the group problem solving activity in the Electromagnetism field 1 course The problem asks to find the potential inside and outside of the infinite square hollow pipe.*

5.3.2 Typographic note

Before we present data, here is a brief typographic note. In these interactions, students very frequently speak the names of mathematical symbols. We could have typeset their words as if they were equations or as if they were the names of isolated symbols. Equations are more compact – importing algebra to Europe caused a scientific revolution – but they lose some of the nuance of students' speech. Isolated symbol names, on the other hand, tend to be difficult to follow in text in a way that they are not difficult to follow in speech, especially as oftentimes students write as they speak. We have chosen a middle path, seeking to maximize clarity for the reader.

Additionally, we typeset a comma for brief pauses, a period for longer ones, and ellipses (. . .) for the longest ones. Stage directions are denoted by parentheses. Should we omit or alter some students' speech for clarity, changed words are denoted by square brackets and omitted ones by ellipses in square brackets ([. . .]).

5.3.3 Clip 1: Looking ahead in separation of variables

Students are going to solve tutorial 5 designed in University of Colorado - Boulder, "SEPARATION OF VARIABLES" (see Figure 5.2). The first part of the tutorial asks to determine if the potential varies with x, y, and z. In this clip, the discussion is mostly between Joe and Jack. In the first 4 lines, they start to talk about if the potential varies with z, also they talk a little bit about the potential variation in x and y directions.

1 Jack So z. . . there is no z right!

2 Dan Ahh. . .

3 Jack So z is everywhere. It does vary with y.

4 Joe Yeah z is constant.

5 Joe It does vary with y and y. . . actually, no it just varies with y. . . I think that wall will make it vary with x.

6 Jack Yeah.

7 Joe I think with that distribution.

8 Jack Eh. . . from here to here

9 Joe You will get like. . .

10 Jack Oh. . . it will be. . . dependent on y, right!

In the first line, Jack mentions, that "there is no z" which means there is no dependence on z. In line 3, Jack backs up his argument by saying z is everywhere, which is a piece of information extracted from the graph. Joe in line 4, also confirms that his coordination class of function in this case results to a constant for the potential on the z axis.

In lines 5 and 6, Jack and Joe continue with thinking about variation of the potential in the y axis. Joe thinks that is the only variation that the potential has. But interestingly, he reads further information from the nonzero wall $V = V_0(y)$ and assumes that the potential also varies with x. Joe is done with determining all the variations across three axes. and now he wants to know about the distributions and the functional forms as he states in lines 7 and 9. Interestingly, Jack more explicitly is reading out information across the y axis (line 8), which leads to his conclusion that the potential does vary with y (line 10).

In this clip, several instances of looking ahead was going on to activate students ideas about the functions and matching the specifics of the problem to those ideas. By the end of this clip, they are ready to see what does the functional distribution for these variations look like.

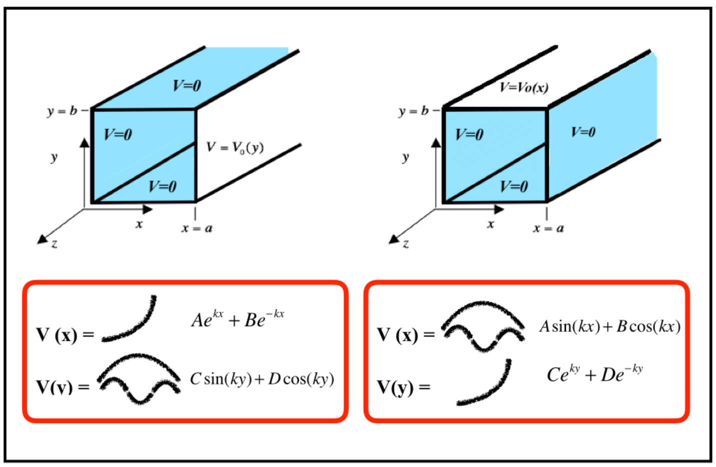


Figure 5.3: *Two physical systems with different boundary conditions and swapped mathematical solutions. For the physical system on the left, the exponential function belongs to x and the sinusoid function belongs to y. For the physical system on the right, the sinusoid function belongs to x and the exponential function belongs to y.*

5.3.4 Clip 2: Looking ahead to determine sines

11 Jack It looks like. . . I don't know. . . it looks like sine. though and cosine

12 Joe Ahhh. . .

13 Jack Because, this goes zero to zero on y.

14 Joe Yeah.

In this clip, Jack and Joe continue to find the functional forms for their solution in the y direction and back up their argument with looking ahead into the boundary conditions and associating the functional form of sinusoidal with specifics of the problem. Jack points to the wall at $y = 0$, and read out that it is at zero potential and "goes" to zero again on the wall at $y = b$.

Students by looking ahead are able to connect the specifics of the problem with their generalized conceptual knowledge that they have about functions and function forms.

5.3.5 Clip 3: Looking ahead and decision making

Students in this clip of the this video episode have set their coupled simple homogeneous second order ordinary differential equations for potentials in x and y axes, that are symbolized by capital X and Y. Mathematically, the solutions to these two differential equations are the same and just differ in a minus sign, but geometrically they differ and belongs to two different physical scenarios. So, students need to make decision, that which solution belongs to x and which solution belongs to y? and how can students mathematically show which solutions belong where? And that's what Jack and Tom, in this clip, are going to argue about. The heart of this argument and the next episode is where does the minus sign goes.

15 Jack Okay …So X is gonna be the exponential, and Y is gonna be be the sinusoid.

16 Tom : Oh yes.

17 Dan They could both be exponentials.

18 Jack Well that's what she is saying here.

19 Dan Right!…like your exponentials for your Y would be imaginary.

20 Tom Yeah you have complex exponentials for Y, it's exactly solving this equation. I hate jumping ahead and say hey oh it's sine.

21 Jack Well, the only reason that you could go to sine here is that because the potential is zero at the end [unintelligible].

In line 15, Jack's initial claim is that the potential "X" along the x axis is an exponential function and the potential "Y" along the "y" axis is a sinusoidal function. But, Tom in line 20, combats with Jack's idea and mentions that he hates jumping ahead and guess the functional form for the potential, instead he prefers to "exactly" solve the differential equation and find the answer. Jack then in line 21, backs up his claim that "the only reason" that allows him to guess the solution is by looking ahead in the problem and reading out information from the problem statement. The next step in the problem is to solve the coupled differential equations by applying the boundary conditions in a formal sense. As Jack solves the problem, he several times reads out the specific of the problem. Initially he look ahead into the solution to get an idea that if the potential varies along any of the three axis. He then reads out further information from the boundary condition to find the functional forms for the variation of the potential in the x and y axis. As Jack sets up his second order differential equations still he refers to the boundary condition of the physical system to make sure that his mathematical solution makes sense physically. In the future of this problem, students are going to apply the boundary condition in a formal sense and mathematically solve the differential equations since Jack has already looked ahead into the problem to make these geometrical reasonings he can productively apply the boundary condition in a formal way to his mathematical solution and find all the unknown coefficients. In the next example we will show how students by not looking ahead into the solution can become overwhelmed in the mathematical procedures that forget to check for the consistency between their mathematical representations and the specifics of the physics problem in hand.

5.3.6 Instructor encourages looking ahead

In this episode, students initially do a few readouts to figure out, that the potential does not vary with z. Then they continue with separating the Laplace's equation mathematically and get to the two second order ordinary differential equations. However, they do not do any look ahead into the problem statement and just pick up the signs of the separation constant randomly for each differential equation. So they get swapped solutions. Instead of having exponential functions as the solution of the potential in the x axis, they find the sinusoidal functions as the solution for the potential in the x axis and vice versa. However, this matters physically since the swapped solution describes another physical system which is shown in Figure 5.3. The next step in the problem solving procedure is to apply the boundary condition in a formal sense to the solutions. The instructor arrives at this moment of their problem solving.

22 Instructor How are things over here guys?

23 Nick We think we are going alright. I guess, I speak for myself.

24 Instructor One of these are sine and cosines, and one of these are exponentials.

25 Nick Uhum …

26 Lee Yes

27 Instructor Which one is which?

28 Nick X which is negative is sine and cosine.

29 Instructor Okay! Does that make sense physically?

30 Nick How can you tell whether it makes sense physically?

31 Instructor Well you can look at the problem.

In this episode, the students provide a solution, which is not describing the physical system in the problem. Instead, their solution describes another physical system with a subtle difference in the boundary condition. This episodes shows that students could have a strong mathematical base, but not having looked ahead into their solution can cause a hard time to the future of their problem. If the students were going to apply the boundary conditions to the swapped solutions, they were going to have a hard time mathematically to simplify the solutions and find the same final answers. In line 30, Nick asks an interesting question that "how can we tell" if a mathematical solution matches with the physical characteristics of the problem. The instructor uses the term "particle in the box" problem as a more familiar example to help students to read out some specifics in this problem similar to the features of the particle in the box problem. In line 33, Chris by looking ahead into the problem statement can readout the salient information that could help him to decide what would the solution along the x axis look like.

32 Instructor One of these as you go across is gonna look like the particle in a box that get's to be sine and cosine... and one of them as you go across is gonna look like not particle in the box

33 Chris I think the X looks like exponential

34 Instructor The X look like exponential?

35 Chris Yes. Because, this is (wall at $V = V_0(y)$) ... like, this corner is constant, so as you move away from this it should go down.

36 Instructor And as we go from top to bottom here, this is zero and this is zero, so sine and cosine makes a lot of sense in the vertical part.

37 Nick Yeah...

The students without the instructor even talking about the signs notice, that in order for their mathematics be consistent with the physics of the system, they just need to switch the signs in their solutions. They also realize that with the new swapped solution the next step is going to become easier.

38 Nick Okay. We should move our negative... is this what you are saying.

39 Chris Yes, it makes life much more easier!

40 Nick I'm just gonna change all my Y's to X's

41 Chris Yeah…

42 Nick Yeah, that was easier.

5.4 Physics of Taylor series expansion

Students in this class use the Taylor series expansion in several physical contexts to expand the potential. For example, to find the potential of an arbitrary charge distribution, the potential of a ring of charge on the z axis, and the potential of a line of charge on the z axis. The first two problems are solved during extended problem solving sessions. The third problem is taken from the Tutorial 6: Multipole expansion designed in University of Colorado - Boulder. Students need to first find the final answer of the potential and then decide around what point they want to expand their solution. Students need to readout information from the boundary condition to help them choose the expanding point, determine the functional form for the potential, and determine what other factors are constant and thus the derivative does not act on them.

5.4.1 How do students looking ahead when doing Taylor series expansion for a uniform charge line

In this episode, students are solving the tutorial 6: Multipole expansion designed in University of Colorado - Boulder. Students start the problem with finding the charge that a differential element of dz carries. Students can find the mathematical representation of the distance between each small piece of charge distribution on the charge line, and the point of interest through several readouts. Finally, by taking couple of algorithmic mathematical steps students can calculate the contribution of each piece of charge and find the potential of a line of charge, $V(z)$ on the z axis, as $V(z) = \frac{\lambda}{4\pi\epsilon_o} \ln(1 + \frac{d}{z})$. Our focus in this study is on the next step, where students expand the potential $V(z)$ into a Taylor series. Before starting to write the first few terms in the Taylor series, student have to find the specific functional relationship which is part of the following conversation among Dan, Lee and Adam.

43 Dan Are we taking...ehh...the function...ehh...$\ln(1+\epsilon_o)$ here as the function?

44 Lee You are saying that ϵ is equal to...

45 Adam ϵ is just an approximate of...just be d...

46 Dan d over z

47Adam With a large z greater than d then epsilon is very small

Students at this point in their conversation are not writing anything in their tutorials. Instead, they refer to their generalized knowledge that they have about the functional form $f(x_o+\epsilon)$ in Taylor series around the point x_o. Dan in line 43, introduce what the functional form looks like. It is not clear to Lee that what is ϵ. Both Adam and Dan explain in lines 46 and 47, that ϵ is $\frac{d}{z}$. Adam backs up this choice by saying that z is large compared to d and reasons that ϵ is "very small" which makes it a proper option to expand the potential around it. In this episode, it is not clear if Adam is reading out the distance z and comparing it to the distance d, or if he is just making a conclusion that ϵ is small due to z being a greater number compared to d. Part of this confusion could be due to the fact that the problem statement provides a hint that $z >> d$. In the next example, students find the potential and expand their solution for a problem, without given an explicit hint. In the next problem solving session students use more explicit readouts to make their choices.

5.4.2 How do students looking ahead when doing Taylor series expansion for a uniform charge ring

In this problem solving session, the instructor wants the students to find the potential of a ring of charge everywhere in the 3D space. The class has already found the electric field for the same charge distribution on the z axis in the first tutorial. The instructor as a hint suggests, that students start from that point to find the potential on the z axis and then expand the potential around a point at infinity. Students in this group, use the computer on their table to have the Wolfram calculate the integral of the potential of a ring of charge for them. They also use Wolfram to give them the Taylor expansion of the same function around several different expanding points. In this episode, we focus on how they look ahead into the problem and use their readout strategy to pick up one of the expansion functions.

48 Ed We need to expand this one (potential of the uniform ring of charge) ... we need to expand this one ... [unintelligible] we are not gonna expand this... is there a ... does wolfram do the expansion for this

49 Larry Yeah

50 Ed (Matt scrolls down the page with the mouse) Down ... down ... down ... scroll it down

51 Matt It's not scrolling... there is an expansion

52 Ed Maybe we just gonna use that ... we'll take the first two terms of that... wait is this ...

53 Matt This one is about zero (Matt looking at the screen)... we are expanding about zero... right! (Matt looking at Ed)

54 Ed Yes, because the ... we're ... P is in infinity (Ed uses hand gesture to show the point P is far away)

55 Matt I like, that they (Wolfram) put a squared on every thing ... that's silly!

56 Ed I like the bottom one. what's the... what is x equal to [unintelligible]?

57 Matt That one is about z equals infinity.

58 Ed Well z does go to infinity, ϵ being d over z... goes to zero.

59 Matt Oh... Ok. Ok.

The conversation is mostly between Ed and Matt. The Wolfram provides the expansion around two points. The first expansion is around point $z = 0$. The second expansion, which

Ed refers to it, in line 56 as "bottom one", is around the point $z = \infty$. Matt in line 53, mentions that the first expansion is around zero, and confirms that in their problem also they want to expand around zero. Ed then uses hand gestures to show that since the observation point "p" is going to be far away then the expansion point will also be around point zero. However, Ed is not talking about the variable z as being equal to zero and he is talking about the ϵ as being equal to zero. Ed in line 58, further clarifies the relation between variable ϵ, and variable z, which is "ϵ being d over z". At this point it makes sense to Matt that the second option is a proper choice, as it provides an approximate potential on z axis far away at ∞. The instructor then asks students to use the general form of multiple expansion series far away form the charge distribution and match the terms in the Taylor expansion and the multiple expansion to find the coefficients in the multiple expansions term by term. By plugging back the found coefficients into the multiple expansion, the potential everywhere in the 3D space will be found.

5.5 Conclusion and Implications

Sherin[44] considers that expert intuition is a combination of conceptual understanding which is the sense of mechanism and symbolic forms. Activation of mathematical form can give rise to the activation of the conceptual reasoning and vice versa. Sherin argues that the activation of a conceptual schema can strongly affect the activation of a particular mathematical form. Thus activation of both of these knowledge elements happen almost at the same time. And does not explain how any of these kinds of knowledge become activated first and then prime the other part consequently.

In the misconception studies[20;21] students' in the moment problem solving strategies are limited to elicit, confront and replace the old incorrect concept with a new concept. In contrast, in a manifold view[7;10] many ideas can be activated in a coherent pattern in the moment of problem solving to account for a conceptual reasoning, which provides more flexibility in problem solving. A particular problem solving context can prime the same patterns of activation, which over time solidifies the connections among resources, or incremental activation of new ideas account for the new contextual features in the problem [14]. In a manifold view many ideas can be activated in a pattern in the moment of problem solving and account for a conceptual reasoning, which provides more flexibility in problem solving. Studies on epistemological framings [49;97–99] accounts for changes in the nature of students activities. For example, students by getting "stuck" in the moment of problem solving might shift to more productive ways of thinking to get "unstuck".

In the moment problem solving allows for different variabilities in students' moves and processes which become emergent in their specific knowledge activation. The focus of most of the studies mentioned in the unitary and manifold view is that for providing a scientific reasoning in problem solving, students need to replace their knowledge [20;21], activate new pieces of their knowledge

In this study, I investigated how students by reading salient information from the problem statement can use that information productively to solve the problem. Practicing readout strategies will help students to read the specifics of different problems and notice subtle but important differences in the physical systems. For future research this study can be used as a source to see how students by practicing looking ahead into the problem can better transfer their learning across different physical scenarios.

Perception of the problem could affect the deep structure of the problem. Naive readers perceive the problem based on the surface features of the problem. For example, which topic the problem belongs to. Whereas, expert problem solvers perceive the deep structure of the problem based on which problem solving schemata (underlying principle) they associate with. [100]. However, in upper division problem solving students in addition to the perceptual reading out strategies need extended reading out strategies to map their generalized knowledges of physics and math to the specifics of the problem in hand.

This study has both research-based and instructional implications. We have expanded the use of coordination class theory beyond the contexts of high school classes and introductory-level courses, and have used the theory in the upper division courses. This research contributes to the coordination class by focusing on the role of an extended readout strategy, which we call "looking ahead". At the theory level, this is important as it accounts that the core of learning can happen due to an extended readout strategy and not just as a relation between the coordination classes that students posses and the causal net that they provide. The context of upper-division physics problem solving better illustrate this newly found relation between the causal net and the coordination classes.

Our results can help the instructors to encourage students to look ahead into the problem. This lessens students' algebraic calculations, and provides more sense making opportunities to students' discussions. We believe that the result of this study can also be used across other theory coursework in physics, as well as in the lab environments, where students deal with design problems[101] that require extended readout strategies to make decisions and do troubleshooting.

In addition, as we discussed in one of the examples in section 5.3.6, students might posses a strong mathematical background to that extent that become overwhelmed in their calculations and miss to check the consistency of their solution with regard to the physics of the situation. Looking ahead as a controlling strategy can encourage students to insightfully connect their mathematical and physical insights, which may help students to expand the use of their mathematics knowledge in the context of physics.

5.6 Summary

In the last three chapters, I reviewed both literature reviews and my studies to show how students map their knowledge of math into physics problem solving. Using different theories in family of "knowledge in pieces" view, I provided evidence of three different ways on analysis that how students can do procedures in problem solving. From the resource theory[10] perspective students can activate different ideas of conceptual[50], mathematical[6], or procedural[47] and link different ideas together to work in a situation. In Chapter 3, I discussed a new procedural resource that helped "Eric" to understand the procedure of unpacking[?] resource. The decision of at which size resources can be used depend on the users. "Alex" was able to separate the TDSE by just two procedural resources of *grouping* and *division*. Whereas for "Eric" it took three procedural resources linked with three conceptual resources to understand how to separate variables.

Epistemic games are cognitive tools that allow researchers to view activation patterns of students' resources over time to provide reasoning strategies[55]. As I showed in Chapter 4, "Ava" by frequently navigating across three moves in her estimation epistemic game was able to find a sufficient answer. I also discussed that the strict structure of epistemic games make it hard to see how students use their expectations in problem solving. This is especially important in upper division students where student can have different expectations about problem solving at different times. As evidenced by "Ava", she showed instances of indicating both answer making and sense making problem solving. These two epistemological expectations have been broadly discussed as the frames of previously identified epistemic games[54;55].

In Chapters 6 through 8, I will discuss how upper division students frame problem solving I will choose the theoretical construct of epistemological framing. This leads to my second main research question which is how students frame problem solving in physics.

In this chapter, I investigated how students by using the mechanism of "looking ahead" can read out resources from the problem statement. This is especially important in long problems with boundary conditions that are heavily dependent on math. Previous research has showed how students apply the boundary condition toward the end of the problem. For example after separating variables[85] students set up the integrals and at that point in the problem students start to think about the physical situation to insert the appropriate limits of the integral. In this chapter, I showed how students using the mechanism of "looking ahead" read the boundary information at the beginning of the problem solving. "looking ahead" into the problem can help students to insightfully solve their partial differential equations in accordance to the physical system.

Chapter 6

Students' Understanding in Quantum Mechanics

In this chapter I will review the previous research on students understanding in quantum mechanics from two main views of difficulty and epistemological framing. I will also discuss the concept inventories, research-based assessments, and surveys that have been developed mostly based on the difficulty research in quantum mechanics.

6.1 Difficulties framework

The difficulty framework aims to extract students' difficulties across various topics in quantum mechanics[102] over the last 20 years. Researchers refer to identified difficulties as universal patterns, since they occur across a wide range of student populations despite varying academic backgrounds[103]. Students have difficulty with distinguishing between the Hilbert space and the 3D space in the context of Stern Gerlach experiment. Students have various difficulties with the time dependence aspect of quantum mechanics quantities such as time dependency of expectation values[104], time dependency of probability density[2], time dependency of energy measurements[2;105], time dependency of wave functions[104], and *Time independent Schrödinger equation is most fundamental*. Each of these broad topics breaks into several subtopics of difficulties that form a long list of students' difficulties in quantum mechanics. In addition, students' understanding on topics of Dirac notations [106], Tunneling effect [46;107], role of the Hamiltonian [104], stationary states [108], addition of angular momentum [109] have been identified.

Students responses to multiple response questions are sensitive and dependent of the context and phrasing of the problem. Since the distractors are chosen based on the previously extracted common difficulties in a certain topic [110]. The goal of the difficulty studies mostly focuses on students' conceptual understanding with less emphasis on students use of mathematical formalisms, or procedures during problem solving. Several research-based assessments have been developed based on the extracted difficulties in quantum mechanics. However, since the nature of the difficulties only focus on final wrong answers, thus the assessments are not able to evaluate processes in quantum mechanics problem solving.

6.2 Quantum mechanics research-based conceptual assessments

Based on the discussed difficulties many tutorials [103], simulations [111], research based assessments [112] and concept inventories [113], visualization instrument [114] are created to help with reform-based instruction [115] and as a result increase in students' understanding of quantum mechanics. Unlike designing classical mechanics concept inventories such as FCI [40], or FMCE [41], designing conceptual questions to extract students quantum mechanics difficulties is challenging. One of the reasons is that different text books and faculties have different approaches. While teaching quantum mechanic, some faculty rely mainly on the derivation of mathematical formalism, whereas other faculties more expand the physical meaning of the formalism related to the quantum mechanics phenomenons. Textbooks also vary in the extent of use of mathematical formalisms in quantum mechanics. This aspect can affect the validity of quantum mechanics assessments, since it is hard to design universal questions with a broad consensus among the faculties over important topics that have to be covered in a quantum mechanics course. There are several quantum mechanics research-based concept inventories and surveys at different course levels of sophomore, junior and senior and graduate.

The Quantum Mechanics Concepts Inventory (QMCI)[113] is a very narrow test with only 9 questions, which mostly covers students' understanding of the topics of quantum tunneling through a potential energy barriers at sophomore level and junior level. Most of these surveys are designed based on the students' difficulties with conceptual understanding of quantum mechanics without explicit mathematical manipulations. The Quantum Mechanics Conceptual Survey (QMCS)[116] is developed to assess students understanding of quantum mechanics at sophomore level and junior level. This survey only has 12 questions, which covers the most common concepts based on faculty consensus to be taught in a modern physics course. The QMCS is not a valid choice for examining graduate students' understanding of quantum mechanics as the materials covered in graduate-level quantum mechanics course is different from an undergraduate-level quantum mechanics course. The reliability aspect of the tests focuses on the wording of the questions to be away from any ambiguity and implies what the problem statement is intended to.

Marshman[110] developed Quantum Mechanics Formalism and Postulates Survey (QMFPS), which focuses on the formalisms and postulates in quantum mechanics on several topics such as measurement, wave function, eigenstates of physical observables, spin angular momentum, Dirac notation and time dependence aspect of quantum mechanics quantities. The author validated the survey by getting feedback from faculties to make sure the assessment covers the materials intended to be covered in a junior/senior level quantum mechanics course. The distractors in each survey question was based on students' common difficulties in a certain topic[3;108].

Quantum Mechanics Survey (QMS)[117] is an assessment tool at junior and graduate level. The test has 31 multiple-option questions. This test mostly covers students' conceptual understanding in one dimensional quantum mechanics materials. Graduate students' poor results on this test indicate that they lack a sufficient qualitative understanding of quantum mechanics phenomenas despite having sufficient quantitative skills.

Based on the nature of the difficulty problems it is hard to find out what are the different productive ways that students might get an answer correct and what are the good seeds of students reasonings even if they get an incorrect final answer. Since I am interested in the good seeds of students reasoning thus my focus is on what are students reasoning in problem solving and how they interpret the problem solving situation as to "what's going on here?", what kind of knowledge students activate, and how do students pick up strategies. I use the lens of epistemological framing to encompass all of these questions to one comprehensive question that "how do students frame problem solving?"

6.3 Epistemological framing theory

Epistemological framing is a window to individuals implicit state of thinking. Epistemological frames reveal students'[68;118] ways of thinking and expectations. They govern which ideas students link together and utilize to solve problems. Students' epistemological framing is highly context sensitive. Being in the appropriate frame and shifting between frames are determining factors in students' success[49;98;119]. Productive problem solving requires both an appropriate frame[119] and appropriate shifting between frames[120].

Scherr et al[119], mention that epistemological framing can be evidenced by students' consistent behaviors and speech. They found that students in a collaborative group problem solving while working on tutorials can frame activities in four different ways. They applied a behavioral methodology and focused on students behavior and speech.

Scherr et al[119] identified four behavioral clusters associated with students epistemological framings. In the work sheet cluster (blue behavioral cluster), students faces are neutral. Students' bodies lean forward. Their eye contacts are mostly on the worksheet. Their communication is brief, concomitant to a low tone of voice and minimal discourse amongst each other, the tone of their voice is low and mostly limited to a quick checking with each other. They don't use gestures except for making brief comments on each others' works.

In contrast, students in a discussion cluster (green behavioral cluster), seat with their bodies straight such that they can make eye contacts with each other. Their voice is clear and suggestive to the rest of the group. Students express their ideas and discuss for example why they think the Newton's second law is not consistent with their intuitions. Thus they might provide different sensory evidences or get involve in a chain of reasonings in order to convince each other.

In the TA frame (red behavioral cluster), students seat straight. Student in this cluster remain quiet and mostly listen to the TA. It is possible that one member of the group even remains in a worksheet frame while focusing on the sheet and writing notes making minimal effort to make eye contact and listen to the TA. If the TA asks students on a topic which has been previously discussed by the members of the group then the group switch to the reduced version of the discussion mode by responding to the TA.

Students could also be in an unstable frame that usually does not last as long as the other three frames. Students in this frame look around the room and their gaze in not fixated, students laugh by having a joking voice. The authors called this frame yellow behavioral cluster.

Irving et al[98] discuss that students could be in a joking frame but also having a productive physics discussion among themselves or in the presence of the TA. Thus the discrete frame category was not suggestive of the richness of their data in Irving's et al study, thus they defined a two axes framework. Axes of silly vs serious and narrow vs expansive. Students in an expansive framing in addition to being engaged in a conceptual discussion can also be connecting their ideas to real world situations, using gestures and also using different representations such as narrative with kinesthetic. In chapter 3, from a finer grain-size perspective of resources I showed that Alex by using the gesture of "bringing out" was able to explain to Eric the notion of function as a constant in taking a partial derivative.

In Irving study their data was not suggestive of a TA frame, but the TA was considered as a facilitator to encourage students either in an expansive frame or a narrow frame. They showed that TA in the moment of problem solving can several times encourage students to switch frames. For example the TA might encourage student to focus on something in their solution or in the problem statement which is a narrow frame. On the other hand, the TA could ask open questions[121] that encourage students to frame the situation as more expansive such as "how do you know that is correct?" in response to such a question students might provide evidence from their experience, intuition or connection with real world situation, which are indicator of an expansive frame. Students could be in a silly frame either having a narrow or an expansive discussion even in the presence of the TA. This framework was able to capture the detail of about 85% of their data, which was collected in the weekly homework help sessions (HHS) of an intermediate mechanics course.

Bing et al[49] discussed students use of math in the context of upper-division courses such as quantum mechanics. They analyzed students' arguments by considering the kind of knowledge that students use to justify their reasoning (e.g. the students' warrants[122]). Students could provide facts from authority, or they might provide some plausible calculations to support their reasoning. On the other hand, students might consider the physical system in the problem to map the meaning into informative mathematical forms, or use the mathematical consistency in two similar physics or math scenarios.

Some students, despite having strong numerical tools or knowledge, still "get stuck" in certain problem solving situations[123]. Bing et al[49] from the lens of four epistemological frames provided evidence that students getting stuck in the process of problem solving is not due to lack of knowledge. The frame within which students apply their knowledge to work with, might not include certain information that require other aspects of their knowledge. For example, the authors discuss a group of students, that get stuck after finding the result of their integral is undefined. This confusion continues until one of the group members finds the source of the error by shifting from the calculation frame to a physical frame. This transition allows the student to notice that the limits of the integral in the problem is not infinity ("Hey, it's not infinity to infinity... We only have to integrate over the square well!")[49].

In addition students could shift in their epistemological framing not just during problem solving but in a broader sense in their epistemology toward learning quantum mechanics. In this study by Dini et al[124] a case study student "Baily" in an interview setting tries to explain the quantum effect of Aharanov-Bohm (AB). "Bailey" explains that in contrast to classical mechanics, here mathematics is mostly responsible to explain the phenomena. The authors compare this students view as similar to playing the e-game of Mapping Mathematics to Meaning. On the other hand, in the second problem which is related to scattering effect in quantum mechanics, "Baily" by getting inspiration from classical mechanics is able to build a conceptual story for the situation followed by mathematical manipulations. Dini et al, interpret this inconsistency view of the student in the role of mathematics in quantum mechanics as a productive epistemological shift.

Students can have multiple strategies in problem solving but they will pick one based on the internal and external factors and go for it. Internal factors such as students' expectation [68] about a situation can affect students choice of use of their tools and pieces of their knowledge. I will discuss this aspect of framing in more detail in Eric's example of frame shift in chapter 7 (Section 7.6.1). There are other external factors that can affect framing as well. Other external factors such as a TA or a group mate can affect one's framing. A TA can nudge students toward a more productive frame [120] in their solution or discussion among each other. Scherr et al [119], identified a frame called TA, which characterizes students' behaviors in the presence of a TA. Irving et al [98] have investigated the role of the TA more broadly as times when students are nudged toward more expansive frame or more narrow frames depending on the in-the-moment problem solving state of the group. A disagreement across students group members can also cause the group to change their epistemological framing. [49;120]

One of the most important external factors that students interact with even before being affected by other humans such as group mates or the instructor is the problem statement. This interaction effect between the problem statement and students' ideas can affect students framing. I will talk about this interaction effect in more depth in chapter 8 of my dissertation. For each study, I will discuss the research and instructional implications and then suggest some open questions.

Chapter 7

Students' epistemological framing in quantum mechanics problem solving

7.1 Introduction

[1]For students to be successful quantum mechanics problem solvers, it is insufficient to think about only the features of the physical system. They also need to coordinate different representations by thinking conceptually about the mathematical representations that satisfy the physical system, evaluate the algorithmic steps, and reflect upon their work. Unsurprisingly, students often have trouble unifying these ideas during problem solving.

Researchers in student understanding of quantum mechanics have used "difficulties" theory to understand student reasoning (e.g.[2;104;105]), which forms long lists of difficulties that span many topics in quantum mechanics. However, we posit that these disparate difficulties can be unified through the lens of epistemological framing[118], and errors in transitions between frames[98]. Epistemological frames reveal students'[68;118] ways of thinking and expectations. They govern which ideas students link together and utilize to solve problems. Careful observation of student behaviors, gaze, and discourse can provide clues for determining students' epistemological frames. Productive problem solving requires both an appropriate

[1]This chapter was submitted and accepted to the Physical Review Physics Education Research Journal as[97] link.

frame[119] and appropriate transitions between frames[99].

Some students, despite having strong numerical tools or skills, still “get stuck” in certain problem solving situations[123]. This happens particularly in upper-division courses such as quantum mechanics, where mathematics is critical to understanding the subject. Quantum mechanics is a great choice for this study, because students are trying to coordinate difficult, often counter-intuitive concepts and complicated, often novel mathematical formalism.

Bing et al[49] identified four epistemological frames to aid in understanding the role of math as a reasoning tool as opposed to a numerical tool. They analyzed students’ thinking while the students translate physical ideas into informative mathematical forms, or compare a mathematical structure in two similar physics or math scenarios. Though Bing et al identified “Physical Mapping”, and “Math Consistency” frames, their “Calculation” frame is biased toward the use of formal math, independent of physical sense making. However, they did not further differentiate between trivial math calculations and conceptual math reasoning.

On the other hand, Kuo et al[125] differentiated between the use of equations as an input-output calculator template, instead of attending to the conceptual meaning embedded in the equation to create shortcuts. They referred to “cognitive elements”[44] to capture students’ understanding of equations while they blend their reasoning with symbolic forms and create a shortcut to interpret the situation. They concluded that successful problem solvers are able to make a decision as to which tools they bring into play for an efficient understanding of the problem situation.

Earlier studies identified other possible avenues that students may follow to obtain a correct solution by using conceptual physics and algorithmic math as numerical tools. These problem solving studies often focused on the differences between experts and novices[126–128]. A novice adopts an inverse strategy by simply attending to the goal of the problem, recalling and manipulating an equation that contains the unknown quantities. In contrast, an expert moves forward based on having a representation for the situation, and then choosing the relevant principles[126]. However, this study is limited because experts’ expertise far exceeds the difficulty of the end-of-chapter problems, and so such a study can not show the heuristics of expert-like problem solvers.

Heller et al[127] designed context-rich problems to challenge introductory students beyond end-of-chapter exercises. This method requires students to make sense of the physical system, and justify what strategies to adopt as experts do. Their work assesses how students initially translate the problem statement into a visual representation in order to help them to adopt a proper strategy for determining the implicit unknown physical quantities. The strategy would allow students to translate their physical representation into a mathematical representation in order to do the algorithmic steps, and find the unknown quantity to make sense of their solution. While this problem solving strategy was initially quite prescriptive in the nature and order of the problem solving steps, later research has permitted a less-linear structure to problem solving.

Building on this work, Caballero et al[129] worked to explain the common difficulties of upper-division students in problem solving, focusing on four steps in mathematical tool use: activation, construction, execution, and reflection. These steps could be completed in any order, and solutions may vary among students and problems. One student could stay mostly in the execution phase to process the algorithmic steps. Another student might evaluate the solution by staying in the construction phase and skipping the execution elements in favor of conceptual steps, or one could bring into play both components of execution and construction. Students' use of certain steps in this theory does not necessarily imply difficulties with the missing component of their problem solving process. This could become important when the problem statement of the question nudges students toward the use of one of these four components more than the others.

Broadly speaking, these three research traditions – research into student difficulties in quantum mechanics, research into epistemological framing, and research into student problem solving – suggest several approaches for understanding how students understand quantum mechanics problems. One approach may consider the initial physical understanding of the problem as more critical, with less emphasis on the mathematical manipulations, whereas other approaches may consider equally a close relationship between math and physics, or focus on the conceptual meaning of math in reasoning. Our present study integrates these three approaches to capture the various facets of students' epistemological framing during problem solving at the upper division. Our theoretical framework[2] takes up the idea of epistemic frames to explain student problem solving without prescribed steps. Our model framework aims to show, that difficulties are an interaction effect between question asked and students ideas, which suggests that there is an underlying structure to the difficulties in quantum mechanics.

In this dissertation we develop a theoretical framework[3]which models students' framing in math and physics, expanded through the algorithmic and conceptual space of students' problem solving. I investigate four frames: algorithmic math, conceptual math, algorithmic physics, and conceptual physics, looking for moments where students' problem solving is impeded because they are in an unproductive frame. I applied this theoretical framework to observational data from quantum mechanics classes in which students solve typical quantum problems in pairs and small groups. Our purpose is to illustrate our theory, not to exhaustively show the prevalence of specific frames or to catalog the methods by which students may transition between them.

7.2 Context

We video recorded the class meetings of one semester of a senior-level quantum mechanics class. The class is taught using Griffith's Introduction to Quantum Mechanics[130] using a wavefunctions-first topic order. It meets for four 50-minute sessions each week. During class, lecture is interspersed with small group problem solving. Groups of 2-3 students solve problems collaboratively on shared table-based whiteboards. Most problem-solving sessions last 2-5 minutes, though they can be as long as 15 minutes for more difficult problems.

[2]It is a particular linguistic difficulty of research on students' epistemological frames that the theory used to describe them is a "framework". In this dissertation I follow the convention that a "framework" is something that researchers use, while a frame is something that humans (in our case, students) use. A framework is an appropriate technical term for a set of connected theoretical statements (e.g. "Resources Framework") Students – humans – frame ideas, have epistemological frames, and participate in framing activities. There are some subtle differences between these three forms of "frame", but all of them are related to the idea "how you know what's going on".

[3]This chapter was submitted to Physical Review Physics Education Research Journal as[97]

Students are remarkably collaborative, usually working together for the entire duration of each problem-solving session. In our data set, we see about one problem-solving session per class, though this decreases in frequency near the end of the semester.

Generally, these problem-solving sessions begin when the professor halts the lecture to ask the students to attempt to solve a problem related to their current topic, or to introduce a new topic. Occasionally, they also arise when students initiate a class discussion and the professor decides to assign a problem to gauge their understanding.

The groups in this class are somewhat fluid, and students may form different groups on different days. Students occasionally recruit others from nearby groups to help them solve problems. The instructor does not explicitly tell students where to sit or with whom to work (other than "people near you"). Generally speaking, students work in pairs or threes; occasionally fours.

7.3 Methodology for video data

In learning environments such as group problem solving in upper-division contexts, one way to interpret the high level of interactions within group members is to carefully analyze the discourse and gestures of each member of the group. Ethnography provides an opportunity to understand the detail of students' discourse, behaviors, as well as capture useful information while they are investigating a phenomena[91]. One of the methods for data collection in ethnography studies in through video recording of activities. This becomes more important by providing multiple researchers an opportunity to view and analyze the videos[92]. Previous researchers in education have used ethnography to study the culture of classroom activities[93] or in more engaging learning environments, such as advanced physics laboratory[94]. Our goal was to develop a theoretical lens to enable us to explain problem solving within various topics in quantum mechanics.

We divide class into episodes of problem solving and episodes of lecture, discarding episodes of lecture because they don't help us understand student reasoning. The problem solving episodes have distinct boundaries: they start with the professor explicitly asking students to begin working on their table-based whiteboards and end when the professor either asks for answers or begins explaining the answer.

In my preliminary analysis of the students' group problem-solving activities, I observed that some aspects of the data represent a conceptual approach and other aspects represent an algorithmic approach. I also noticed students' use of conceptual physics and algorithmic math. This distinction is consistent with the ACER[129] and framing[49] literature on problem solving from upper-division physics classes, showing how students' understanding of physical systems maps to algorithmic representations. However, neither theoretical framework adequately captured the richness of our data, prompting us to take further steps to interpret our data set. From the tradition of progressive refinement of hypotheses[131], we set out to refine our observations through close interrogation of the video data.

We started with selecting episodes for close analysis based on their duration (longer is better), conceptual richness (more complex is better), and technical quality (more visible and audible are better). We reflected on these episodes, seeking to answer "what's going on?" for each of them. Through repeated watching and examining the details of the selected episodes, we sought to capture changes in students' discussion or behavior that might indicate a shift in the students' problem solving processes. We began to focus on instances where students "got stuck" in their problem solving processes. This momentary impasse prompted them to try a different kind of reasoning until suddenly they were able to get "un-stuck". We examined the interactions immediately preceding and following the unsticking moments to look for regularities in unsticking behavior.

We developed a preliminary theoretical framework[132] which mapped student behavior onto three discrete frames: conceptual physics, conceptual math, and algorithmic math. The two math frames – concordant with research in mathematics education on concepts and processes[133;134] – suggested that we expand our ideas to look for the "missing" physics frame: algorithmic physics.

Concurrently, we grew troubled with the idea of discrete frames. Sometimes, students seemed exceptionally "mathy", operating without regard to any sense of physical meaning. It is possible, however, to blend conceptual ideas from both math and physics domains, or to move fluidly and rapidly between conceptual and algorithmic thinking. We reframed our ideas into two orthogonal axes: conceptual to algorithmic and math to physics, defining a coordinate plane in which students' problem solving roams. In pursuit of evidence to refine this two coordinate-axis framework, we delved again into our observational data, seeking examples of all four quadrants and transitions among them.

After several more iterative cycles of analysis and refinement, we reached a stable point where new episodes did not change the theoretical framework or our application of it. Operating with the newly-stable framework, two independent raters came to consensus on every episode; two additional raters checked a selection of episodes with agreement of $> 90\%$. We selected episodes for analysis based on frequency of students' discussion regarding concepts and processes, as well as displays and features of potential frame transitions. We categorized episodes with conceptually rich discussions and frame negotiations as strong examples, and established inter-rater reliability about the content of the episodes and regarding which episodes strongly or weakly evidenced frame transitions.

We also identified very weak examples when it was hard to find evidence of students' framing from the group discussion. This could happen due to noisy or garbled audio, or when students were writing on part of the whiteboard that was not in the view of the camera, or in general the raters did not have enough information to determine students' framing.

Once I identified students' frames, we looked for transitions in those frames to help us to interpret the dynamic of students' problem solving behaviors or identify the impasse students reach when they fail to notice certain factors that could have triggered a transition to a more appropriate state.

I acknowledge the existence of other frames that could describe students' behavior while they are engaged with other kinds of activities in a classroom e.g. turning in home work to the instructor, taking break within solving several parts of a long problem, or discussing upcoming social events[120]. While these other frames can be important for problem solving more broadly[98], in this study our focus is on investigating students' topical discussions during problem solving sessions which last about 2-5 minutes.

Epistemological frames are context dependent[68]. For example, by walking into a restaurant relevant resources consistent with behaving in the situation are activated to read a menu, order food, pay the tip, etc. However, in the setting of the restaurant we don't access our resources for behaving in a library. Students' perceptions of the problem context affect their framing of what subset of their knowledge to activate. In an interactive class environments such as group problem solving the instructor's framing can also affect the students framing of the situation[98;119;120]. Even within group problem solving, other students' framing can affect an individual's framing as well. The instructor can nudge students to frame the problem more conceptually by asking about the physics of the situation, or more algorithmically by asking about formulae[120].

7.4 Theoretical framework

Our theoretical framework consists of two axes: an algorithmic versus conceptual axis, and a math versus physics axis (Figure 7.1).

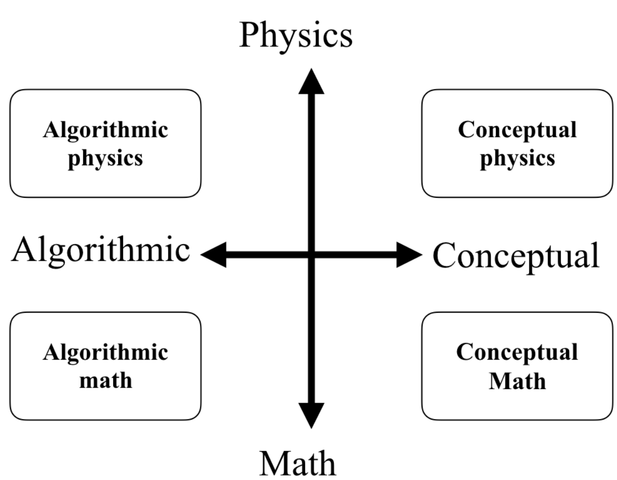


Figure 7.1: *Math-physics-algorithmic-conceptual theoretical framework. The horizontal axis indicates algorithmic and conceptual directions. The vertical axis represents the math versus physics directions. This framework gives four frames as labeled in the figure.*

The two axes divide different aspects of students' problem solving into four regions: algorithmic math, conceptual math, algorithmic physics, and conceptual physics. It is important to note that none of these frames are inherently "good", "bad", or even universally useful. At different times, different frames may be productively used to solve problems in physics, and often more complicated problems require multiple transitions between frames.

7.4.1 Algorithmic and conceptual math

In algorithmic frames, students are focused on following a known series of steps to solve a problem. Since the result of each step is used to process the following step, students are focused on their task to prevent errors. They tend to be in a writing mode, and have less discussion as compared to conceptual frames. In their discussions, they tend to focus on error-checking ("what did you get for part c?" or minutiae of their steps ("you are missing a minus sign").

When the problem statement requires explicit algorithmic calculations to find an answer, students enter an algorithmic math frame to spend a considerable amount of time setting up a series of algebra-based steps to evaluate integrals, take derivatives and simplify their solutions by dividing or multiplying a term on both sides of their solution.

Algorithmic math can be a quick and powerful problem-solving mindset as they may take several fast steps over a long period of time. However, without other quadrants it is quite difficult to check whether or not the solution makes sense. This frame leaves students with a narrow[98;135] discussion mostly to check the signs, or to alert each other of the missing symbols, while they are focused in their numerical calculations. For these features, we consider the "just math" frame[136] is an example of students' prolonged use of the algorithmic math frame. Narrow framing which focuses only on the task at hand[98;135] is not exclusive to algorithmic frames, but it is common within them.

In contrast, students in a conceptual math frame use a conceptual approach to understand the mathematics. They reason based on general properties of a class of information in math. This could help them to apply practical ideas about the behavior of the mathematical functions, and determine the result of an operation without actually computing it. For example, by knowing that sine functions are independent of each other, and discussing the orthogonality properties of the sine functions, one can shortcut the integration of product of two sine functions of different periods, preventing the use of many trigonometric identities, and simply "see" that the integral equals zero. Creating a "shortcut" solution[125] to the problem reduces the procedures and lessens the writing. Concurrently, discussing mathematical problems conceptually gives students more opportunities for sense-making discussions with other members of the group[98;119]. This kind of thinking is generally more expansive[137], as students connect general cases of mathematics to the specifics in this problem or bring in connections to other problems. Attention to conceptual mathematics is a large part of numeracy, and as such is an important part of learning mathematics[134] and physics[125], especially at the upper-division[14;138].

7.4.2 Algorithmic and conceptual physics

Just as I find algorithmic and conceptual frames in math, I find them in physics as well. Students in a conceptual physics frame try to think in terms of the features of the physical system and might coordinate between different representations such as graphical, geometric or gestural to visualize the physical system. They coordinate different physical laws and concepts to explain the situation. I provide examples from the context of Electromagnetic fields course as motivating examples to show the broader phenomena. In the next section, I will provide several quantum mechanics examples from our own data.

For example, students might argue that the total charge on a spherical shell whose surface charge density σ is proportional to $\sin(2\theta)$ (where θ is the azimuthal angle) is equal to zero because the northern hemisphere is positive while the southern is negative, and those two halves must be equal and opposite. In this case, students use conceptual reasoning to map charges to a sphere, employing balancing resources to come to the conclusion that the net charge equals zero. If one moves to algorithmic math frame to write the integral of the charge density over the surface area, $\int \sigma dA \propto \int \sin(2\theta) r^2 \sin(\theta) d\theta d\phi$, then it's possible, of course, to solve this problem algorithmically (using trigonometric substitutions) or conceptually without reference to physical systems (via the orthogonality of sine functions). In either the conceptual math or conceptual physics frames, discussing the problem plan in the conceptual physics frame can make later algorithmic calculations easier.

Thinking conceptually about the underlying physics of the situation encourages students to create connections to real word situations and other classes of problems as well.

For example, to estimate the far distant electric field of a uniformly charged disk, one method is to expand the solution by mostly engaging in algebra to get the answer. Or one can visualize that far from the disk a continuous charge looks similar to a point charge, and by knowing the electric field of a point charge, the leading terms in the solution can be guessed. In each case, students are engaged in an activity to find an answer, but the nature of the activities are different. The latter case is more expansive, as students are open to make connection between the current situation and another class of problem.

In contrast, students in the algorithmic physics frame tend to recall equations, facts, and properties of physical quantities without conceptual justifications. They use math as a tool to adjust equations via a series of algebra-based steps to relate physics quantities to each other, or to check the correctness of the physical quantities in the problem. For example, by doing dimensional analysis students can check the correctness of their answer. Just as with algorithmic math, students tend to frame their work narrowly in algorithmic physics and focus on following procedures to find answers. Someone who applies normalization conditions for wave functions by rote, for example, is operating algorithmically.

It's important to note that framing problems algorithmically can be fast. An expert doesn't need to engage in extensive conceptual thinking about the steps of a trivial problem; she can just solve it.

7.4.3 Continua vs. categories

A careful reader might be concerned because I started this section by claiming that there are two axes, implying a continuous distribution of possible framings, yet continued by identifying four frames which appear to be discrete. We chose the axes for theory-driven reasons: it's possible that students' framing exists on a continuum between very mathy and very physicsy, or very algorithmic to very conceptual, and discrete frames cannot capture this sense. We kept it for practical reasons: on occasion, students appear to move fluidly and rapidly among frames, and there's not enough evidence to assign them a single, quasi-stable frame before they move to the next. I am interested in quasi-stable frames because we want to study how students transition between frames, and it is practically very difficult to find transitions between frames without first identifying (quasi-)stable frames. We use the words mathy and physicsy to denote students' framings which are more in the math direction or which are more in the physics direction respectively.

By using axes, we hope to capture a sense of directionality from more mathy to more physicsy and more conceptual to more algorithmic. I do not imply that these axes constitute a formal metric or scale. While some prior work in student framing of problem solving in physics has used discrete frames (e.g.[49;119]), other work has used continua in the same way (e.g.[98], building on[139]).

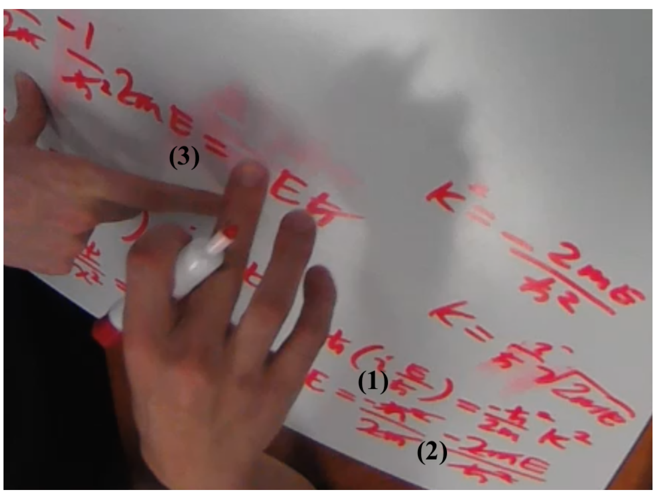


Figure 7.2: *Group problem solving in algorithmic math frame, as trying to find the wave function for the one dimensional free particle.*

7.5 Illustrative episodes

In this section, I present four brief episodes which illustrate the four quadrants in our framework. Before the examples, I divert into a brief review of the quantum mechanics of

$$K^2 = -\frac{2mE}{\hbar^2} \qquad K = \frac{i}{\hbar}\sqrt{2mE} \qquad E = \frac{-\hbar^{\cancel{2}}}{\cancel{2m}}\,\frac{-\cancel{2m}E}{\cancel{\hbar^2}}$$

$$i\hbar\left(i\frac{E}{\hbar}\right) = -\frac{\hbar^2}{2m}K^2 \qquad \frac{-1}{\cancel{\hbar^2}}2mE = -E\cancel{\hbar}$$

Figure 7.3: *Diagram of students' solution in algorithmic math frame.*

free particles and a typographic note on how I present transcript.

7.5.1 Physics of free particles

Most of the examples in this section are chosen from the same physics context of the free particle system, so I review the physics of this system briefly for the reader. In quantum mechanics, the free particle is characterized by a zero potential energy, thus the Hamiltonian has just one term, the kinetic energy. This problem is a good candidate for understanding basic properties of the wave function and the Schrödinger equation. Since the Hamiltonian is in form of $p^2/2m$, the eigenfunction solution can be considered as a plane wave, which can be expanded in terms of sinusoidal wave functions. However, for one particle the wave function with a determined momentum is not normalized over all the space. Thus a linear combination of all solutions is considered as a normalizable wave function. This is physically interpreted as a traveling wave pocket.

Because the Hamiltonian has only the kinetic energy term, the time independent Schrödinger (TISE) equation results in a homogeneous second-order differential equation. In order to write the eigenvalue equation in terms of a differential equation students might need to recall some relations from algorithmic physics. Solving and finding the eignenfunctions of the equation leaves room for either algorithmic math calculations, or conceptual math discussions.

7.5.2 Typographic note

Before I present data, here is a brief typographic note. In these interactions, students very frequently speak the names of mathematical symbols. I could have typeset their words as if they were equations or as if they were the names of isolated symbols. Equations are more compact – importing algebra to Europe caused a scientific revolution – but they lose some of the nuance of students' speech. Isolated symbol names, on the other hand, tend to be difficult to follow in text in a way that they are not difficult to follow in speech, especially as oftentimes students write as they speak. I have chosen a middle path, seeking to maximize clarity for the reader.

Additionally, I typeset a comma for brief pauses, a period for longer ones, and ellipses (. . .) for the longest ones. Stage directions are denoted by parentheses. Should I omit or alter some students' speech for clarity, changed words are denoted by square brackets and omitted ones by ellipses in square brackets ([. . .]).

7.5.3 Episode: Algorithmic math

In this example a group of three students are solving the Schrödinger equation to find the wave function of a free particle. They treat the space part and the time part separately. They start with the TISE for the space part. Guess a solution in the form of e^{kx}, and substitute it into the TISE to find the constant k. Adam[4] and Emma work together quickly to solve the problem, while their third groupmate (Eric) stays silent.

Figure 7.2 shows a snapshot of their whiteboard during group problem solving, taken while Emma is pointing to the both sides of their written equation to review the taken algorithmic steps in search for the missing sign. The numbers indicated on the figure show the order of the students' actions and narrations and reference the numbers in the transcript below. Figure 7.3 shows a transcript of their writing on the whiteboard.

In this problem, Adam takes the derivative of the time solution of the Schrödinger equation, and replaces it into the equation using the whiteboard in front of both students. He

[4] All student names are pseudonyms.

continues to replace the factors that the group has manipulated earlier in their solution and setting both sides of the equation equal to each other to verify if their solution satisfies the Schrödinger equation.

Adam Minus E equal

Emma Minus $\hbar$ squares over two. (1)

Adam Minus $2mE$ over $\hbar$ ... squared ... Boom ... Boom ... Boom (while canceling the same quantities from two sides of the equation) (2)

Emma Cancel, cancel, cancel, and we are off by a negative (2)

Adam Yeah [unintelligible] sign

Emma With a negative up here, because these two are negative.

Adam Yeah that is true

Emma So something happened here (pointing to the two sides of their equation) (3)

Adam Or we lost a sign (3)

Emma and Adam use short sentences and talk quickly to be able to proceed to the next step of their algorithmic evaluation. They speak primarily of mathematical terms and operations, and do not talk explicitly or extensively about the physical quantities these symbols represent. At the end, they come up with an extra negative sign in one side of their solution. After reviewing their solution, Adam removes a negative sign in the earlier line of his solution, which he thinks is extra, but he does not further discuss the reason behind his decision.

At this point of their problem solving session, neither Adam nor Emma try further to make a transition to another frame to resolve thier error. Adam just in a low voice points to the power of the time phase factor exponential and says (oh wait this $[e^{i\frac{E}{\hbar}t}]$ gonna be a negative). Emma says ("but I think we can start normalizing...), then she starts to normalize their wave function.

In this brief moment of problem solving session the group is in an algorithmic math frame. Students are operating in algorithmic math frame because that is all they are talking about and the way they are talking is running algorithmically through the math.

Prior to this episode, Adam by making a transition from algorithmic math to algorithmic physics is able to resolve the cause of their group error and then again he returns to algorithmic math to continue the solution. (discussed in section 7.6.3)

The group does not spend further time to find the "dropped negative" since students are toward the end of their problem solving session and are asked to normalize the wave function, which is discussed in the next episode.

7.5.4 Episode: Conceptual math

In this example, the same group is working on normalization of the free particle wave function by considering the general solution Ψ as the sum of two functions of Ae^{ikx} and Be^{-ikx}. They initially set the algorithmic steps, take the modulus square of the wave function ($\mid \Psi^*\Psi \mid$) and insert the limits of the integral.

$$\int_{-\infty}^{\infty} \mid \Psi^*\Psi \mid dx = \int_{-\infty}^{\infty} \mid A \mid^2 + \mid B \mid^2 + AB^* e^{-2ikx} + A^* B e^{2ikx} dx \tag{7.1}$$

$$= x(\mid A \mid^2 + \mid B \mid^2)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} AB^* e^{-2ikx} dx + \int_{-\infty}^{\infty} A^* B e^{2ikx} dx \tag{7.2}$$

$$= AB^* \tfrac{1}{-2ikx} e^{-2ikx}\Big|_{-\infty}^{\infty} + A^* B \tfrac{1}{2ikx} e^{-2ikx}\Big|_{-\infty}^{\infty} \tag{7.3}$$

There are a few errors in the students' solution which will not affect their conceptual discussion later in the episode. Emma leaves just one differential elements of length (dx) for the whole expression right after the last term in the integrand (Equation 7.1). However, Emma is mindful of her incorrect notation, as she will take the integral of each term of the integrand with respect to x in the following lines. The other error takes place after taking the integral of the e^{2ikx}: Emma leaves an extra negative sign in the power of the exponential, and none of the group members notice the extra sign for the second exponential integral (Equation 7.3). Eric notes that the result of the constant integrals are infinite; Emma will not further discuss and skip those terms in the last line of the solution.

They end up with the variable x in both the denominator and the power of the exponential function in the numerator of the fraction (Equation 7.3). I acknowledge that the frame of students is mixed ahead of time while they are setting up their integrals algoritmically. But after this setup they switch to a purely conceptual reasoning and start their discussion again.

Without evaluating the integral numerically, they realize that the answer of the integrals might be infinite. They start in a conceptual conversation in the math context by arguing based on words and properties of the wave functions rather than working out equations to justify their answer as being finite.

Emma We don't have to worry, this [the exponential term in the numerator] is gonna blow up faster than this [the denominator], right!

Eric They both blow up

Emma Yeah, But one blows up faster and that matters

Adam Definitely the exponential (points to numerator)

Emma Yeah...

Emma has a discussion about which term "blows up" faster than the other. Emma gives more evidence of her conceptual understanding of the behavior of the two functions in the second exponential integral, when Eric says "but both terms blow up". She then compares the decay rate of the functions as an important factor that "matters". Emma conveys her generalized expectation[140] of the situation by saying "We don't have to worry".

Although Emma only uses the term "blow up" briefly, there is a conceptual meaning embedded in this phrase which shows her understanding of the situation by comparing the rate. Adam seems to agree with Emma when pointing to the exponential function as blowing up faster.

7.5.5 Episode: Conceptual physics

The Instructor asks the students to solve the Schrödinger equation to find the wave function for the free particle. Robert begins to write down the time independent Schrödinger equation ($\hat{H}\psi = E\psi$). At this point Alex states that the equation written by Robert is time independent. On the other hand, Robert seems certain that the wave function is time independent. Robert pauses writing, and both of the students start a conceptual discussion before continuing any algorithmic manipulations. Their arguments are based upon reasoning and discussion rather than working out specific equations:

Robert I think. . . .

Alex That's time independent. . .

Robert Yeah. . . Why do we need time? hmm?

Alex Hmm. . . Because the wave function might have it.

Robert If there is no force then. . . um. . . why would anything about the wave function change over time?

Alex Because the wave function might depend on time. . . from its initial condition.

Robert I don't think it did. At least. . . [unintelligible].

Alex Oh, okay.

Robert refers to the properties of the physical system to reason that the wave function is time independent because there is no force acting on it, whereas Alex has doubts about how the initial condition can affect the evolution of the system over time. However, Alex does not have enough evidence to justify his reasoning.

In this episode, both students are in a conceptual physics frame, justifying their reasoning (albeit briefly) with arguments about physical quantities instead of mathematics or procedures, and bringing in more expansive reasoning. Alex thinks in terms of the feature of the problem by mentioning the initial condition. Thinking about the features of this problem also helps Robert to set the force equal to zero. This helps him to think more deeply about the underlying concepts and justify a zero change of the wave function over time.

7.5.6 Episode: Algorithmic physics

In contrast to the prior example, here Robert and Alex shift into a more algorithmic frame. Robert continues to rearrange the equation of the Hamiltonian into kinetic and potential energy. He then sets the value of the potential energy equal to zero, and continue to recall the physical equations for the momentum based on the velocity and substitute them into the equations.

Robert (writing math as he speaks) Whole definition is... $T + U$... Zero (crosses out U) is, uh 1/2... mv squared. This is p... $\frac{1}{2}pv$... and v equals $\dot{x}$ So H is $1/2p\dot{x}$

Alex You can have that H equals... or T equals $\frac{p^2}{2m}$. It's skipping all of this.

Robert goes through multiple steps of algebra to remember the other physics equations in order to relate the kinetic energy to the momentum. However, Alex directly recalls the equation of the kinetic energy in terms of the momentum. Alex's framing is distinguished from algorithmic mathematics because he's not performing mathematical manipulations, merely recalling general physics formulae. Robert is also in a recall mode, as evidenced by the words which start his observation: "whole definition is". Both of them together implicitly agree that the goal of this part of the interaction is to lay out physical laws using mathematical formalism, not to discuss the applicability of those laws or derive them from first principles, as evidenced by Alex's comment that they can "[skip] all of" Robert's more elaborate efforts.

7.6 Accounting for frame transitions

The idea of the math-physics-algorithmic-conceptual framework is itself a development in how I model student thinking about math in physics contexts. However, only the briefest of problems (Heller & Heller's "exercises" [141]) require only one frame to solve them. To better model longer problems in upper-division physics, I must look at how students transition between frames in the course of problem solving. Frame transitions – or inability to transition – in students' problem solving illuminates the connections among ideas and procedures in longer problems.

I identify transitions by first identifying preceding and following frames. The transition, definitionally, occurs between two different frames. Broadly speaking, I notice that the timing of transitions is relatively short (on the order of a few seconds, less than length of a few turns at talk).

7.6.1 Example: Conceptual math to algorithmic math

In the following example, the group transitions from a conceptual math frame to an algorithmic frame, which is a move from an expansive to a narrow frame.

In the previous class session, students discussed that the probability density of a stationary state is time independent. Immediately prior to this example, the instructor asks the class to work in groups and find if the probability density of a superposition of two stationary states (Ψ_1 and Ψ_2) is time dependent or independent.

Eric decides to talk through the solution to the problem with his group. His preference is to start in the conceptual math frame by comparing this problem to previous problems, and his initial conclusion is that the probability density is time independent.

Eric I think… Cause when you do the, um, absolute value, you have to multiply by the complex conjugate, so I'm pretty sure that e thing [complex exponential part of the wave function] will just go to one, cause you'll replace that with… that e to the minus blah blah blah with e to the plus blah blah blah, and then when you multiply the… 1 over, you know, x over x. That's what I'm thinking.

I believe Eric is in the conceptual math frame because he uses reasoning based on the behavior of the exponential function and complex conjugate to determine the "form" of the answer; namely, that the complex conjugate causes complex exponential terms to drop out when multiplied together. He isn't working on an algorithmic solution; he's arguing from the nature of these functions that his solution is reasonable. After he outlines the reasoning behind his conclusion, he begins working this problem out to check his answer.

After about two minutes the instructor mentions that the answer is time dependent, which confuses Eric, who proclaims his violated expectation loudly.

Eric It is time dependent? Why? (While the instructor is explaining, he works on his paper) There's cross terms! Stupid... (smacks himself on forehead) ugh... That's why. Okay. Ugh, so stupid.

(From Eric's tone of voice, I interpret that Eric uses "stupid" to mean that his reasoning was thoughtless, not that he is personally stupid.)

The instructor's answer violates Eric's previous conclusion, prompting him to shift to another frame to explain the new answer. He realizes that his conceptual shortcut that exponentials will cancel with each other caused him to make a mistake. By viewing the problem algorithmically, Eric is able to review his work and determine what went wrong. He tries to find an answer for his question by transitioning to the algorithmic math frame and noticing that the "cross terms" are non-zero in this case.

7.6.2 Example: Algorithmic physics to algorithmic math

The next example illustrates a transition from algorithmic physics to algorithmic math. Both of these frames are narrow and actions in them are taken rapidly and frequently.

In this session the instructor asks the students to find the wave function of the free particle by solving the Schrödinger equation. Emma begins to write down time dependent Schrödinger equation (TDSE). She tries to remember where to put $\hbar$ in the time dependent side of the equation and asks Eric if he remembers. Eric then recalls and writes the TDSE on the whiteboard ($\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t}$). After they both finish writing the equations, Emma compares them.

Emma This is what I have, good we agree. (very quickly reviewing the facts) F= 0, V=0, ... Separable.

Emma is in an algorithmic physics frame, recalling equations and matching them term-by-term in preparation for solving the problem using a known procedure: separation of variables.

However, Eric is in a different frame. He bids to begin their problem solving by thinking about the physical system:

Eric So this is like the infinite [square well], except for we don't have boundaries.

Eric's comment compares the current problem to a well-understood system and provides opportunities for further thinking about their current system. Here Eric is using reasoning about a physical system by analogy to a previous problem, which is indicative of conceptual physics thinking. However, Emma does not take up Eric's bid to use the conceptual physics frame.

After Eric's comment, Emma asks Adam to join their group. Adam's involvement transitions the group into the algorithmic math frame, picking up Emma's earlier work on algorithmic physics to state known equations.

Adam begins by checking that the conditions for this problem satisfy the time independent Schrödinger equation. He spends a short time in algorithmic math frame to justify that the energy in the spatial part of the Schrödinger equation could be anything. This makes him ready to solve the bulk of the problem algorithmically and in a math frame via separation of variables.

Adam leads the group through his solution, which takes about five minutes. Adam has already taken a course on partial differential equations from the math department, and he feels very comfortable with this mathematical procedure. Adam's process begins in algorithmic math with finding the general form for Ψ and the separation constant K. Adam uses the letter λ as a known constant in the solution and explains it's relation to the separation constant K.

$$\hat{H}\psi = E\psi \tag{7.4}$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = E\psi \tag{7.5}$$

$$k^2 = \frac{-2mE}{\hbar^2} \tag{7.6}$$

$$\psi'' = \frac{-2mE}{\hbar^2}\psi \tag{7.7}$$

$$k = \frac{i}{\hbar}\sqrt{2mE} \tag{7.8}$$

$$\psi = Ae^{\lambda x} + Be^{-\lambda x} \tag{7.9}$$

$$\psi = Ae^{\frac{i}{\hbar}\sqrt{2mE}} + Be^{-\frac{i}{\hbar}\sqrt{2mE}} \tag{7.10}$$

Adam and Emma K squared equals minus E...

Emma (interjecting) and then we do the e to the λ sign and then we find λ

Adam and Emma K squared equals minus E…

Emma (interjecting) and then we do the e to the sign and then we nd …So now we want to find out what λ is

Adam It is just square root of K (Adam points to equation 8). No wait it's K, if we define the square [of] K by that…(Adam points to equation 6)

Emma It is k…yeah

Adam So our space part is just e to the i, square root of $2mE$ over $\hbar$…(Adam writes equation 10)

Emma Just write K…Just write K…

Adam Plus B to the…

Emma Why would you not just write as K.

Eric Because this is the real name…I don't [unintelligible] (Adam finishes writing equation 10)

The students appear to stay in algorithmic math after Adam transitions them there: there is no discussion about how functions behave, what the physical system looks like, the effects on the final solution, or whether this makes sense. The students are purely focused on how to define K, the separation constant, and its relationship to λ, a known constant in the solution. This episode shows that algorithmic math can be a quick and powerful problem-solving mindset, but without other quadrants it is quite difficult to check if solutions makes sense.

7.6.3 Example: Algorithmic math to algorithmic physics

The group in the previous example continued their calculations to find out the solution of the wave function for the space part and time part. They find that the space part is equal to $Ae^{kx} + Be^{-kx}$. and the solution of the wave function for the time part as $e^{-i(\mu/\hbar)t}$. The next step in their algorithmic calculations is to find how μ is related to k. They multiply both functions and substitute the "whole thing" into the TDSE. Earlier in their solution, they have found k^2 as $\frac{-2mE}{\hbar^2}$. Part of this calculation happens while they are writing on the part of the board that is not visible in the camera. As they are taking the derivatives with respect to time and space, they forget a sign, which they will not notice it until later in their problem solving session (Section 7.5.3).

Adam So just μ equals $\hbar^2$ over $2m$ times k^2 [$\mu = (\frac{\hbar^2}{2m})k^2$].

Eric What's μ?

Emma μ was our constant from when we were doing this part (pointing to the time derivative part of the TDSE).

Eric For time?

Emma and Adam Yeah.

Emma Because we have $e^{-i(\mu/\hbar)t}$.

While Emma is explaining to Eric where the coefficient μ comes from, Adam plugs in the value of k^2. However, he makes an error in the denominator, and only writes $\hbar$ instead of $\hbar^2$. Emma and Adam continue with algorithmic simplifications. This mistake causes their final μ to have an extra coefficient of $\hbar$.

Adam So, boom... boom.

Emma cancel ... cancel... cancel ... cancel. We get ... $\hbar$E

Adam Hah ... (writes $-E\hbar$ and taps his finger on the board)

Emma Is that wrong?

Adam ... Yeah.

Emma Because you have the $\hbar$ ...

Adam ... Joules (pointing to the E) ... joules-second (pointing to the $\hbar$). Yeah I don't know if that's right.

Emma and Eric start algorithmic checking on the other side of the board, while Adam is silent after checking units by doing dimensional analysis. At this point, we can hardly hear the conversation between Emma and Eric, since the instructor has paused the problem solving session and is giving feedback to the class. The group ignores the instructor's explanation and continue to work quietly on their own.

Emma Oh wait ... wait. Isn't that ... ?

Emma No $\hbar$ is here (pointing to the $e^{-i(\mu/\hbar)t}$ to explain something to Eric).

Emma The $\hbar$ is here (pointing to the $\hbar$ in the time solution), and that will cancel with this one (pointing to the extra $\hbar$ in the final answer)

Emma And then we will have ...

Adam Oh wait a minute ... hold on ... hold on ...

Adam Wait ... I know what I did ... I did not square this (pointing to the $\hbar$ in the denominator).

Adam adds the missing $\hbar$ power to his solution, and cancels it with the remaining $\hbar$ in the final answer.

Adam So it's just minus E ... Yes ... (while raising his fist in a triumphant gesture).

In the previous episode we showed Adam's participation caused the whole group to shift from algorithmic physics to algorithmic math. In this episode only Adam shifts. He transitions from algorithmic math to algorithmic physics to find the source of his error by thinking in terms of units of the physical quantities. In contrast to Emma and Eric, Adam does not recheck his derivatives. Instead he checks units to make sense of his answer as a physics quantity and not just a symbolic answer. After finding the source of his error he continues the simplifications in the algorithmic math frame to find the final answer for μ. This episode shows that by coordinating multiple frames students can better monitor their calculation process, saving time and/or making sense of their final answers.

7.7 Discussion

In this study I identified the state of students' thinking associated with four discrete frames including algorithmic math, conceptual math, conceptual physics and algorithmic physics. I presented several examples of students' group problems solving switching frames to productively and correctly solve a problem.

While upper-division students are generally facile at problem solving, on occasion they get stuck. By observing students' behaviors I noticed moments that students change the nature of their activities to make a decision that affects the future of their problem solving, to find the source of an error in their solution, or to get "un-stuck".

Epistemological framing is a window to individual's implicit state of thinking. This internal state can alter as a result of interaction with external artifacts in the environment such as the instructor's framing[98;99]. "Eric's"(Section 7.6.1) shift from conceptual math to algorithmic math is responsive to the instructor's correct answer to the class. In group problem solving, shifts can also be internal to the group: when members of a group disagree, one student might cause the group to shift to another frame[99]. Even in individual problem solving, students may shift to another frame in the ordinary course of solving a problem.

Epistemic games have been previously used for studying problem solving behaviors at the introductory level[123]. However, at the upper division the strict move structure of these introductory e-games breaks down, and it may be more productive to look at which frames students operate in[98;99;142].

In a similar contrast, Sherin[44] compares the conceptual schemata associated with symbolic forms with Larkin's[126] "principled-based schemata view". He explains that the goal of his schemata is conceptual understanding and the goal of the latter schemata is step-based problem solving. However, these two views are again situated at the introductory level where conceptual mathematics is rare. In our framework, both conceptual understanding and algorithmic thinking can be mathematical or physical, allowing for greater freedom in modeling upper-division student thinking. As evidenced by "Eric" (Section 7.6.1), conceptual thinking in not the only productive aspect of thinking about physics.

"Eric" (Section 7.6.1) switched from conceptual math to algorithmic math, I do not mean to imply that algorithmic math is universally more productive than conceptual math. Rather, what counts as productive framing depends strongly on the problem context, and different frames may be productive at different times within a problem. Students' difficulties in quantum mechanics – such as thinking that the probability density is time independent for a superposition of two stationary states, as this student does – may simply be the result of unproductive framing and not fundamental inability to solve these problems or conceptual "difficulties"[2;104;105]. Modeling students' problem solving as movement in the math-physics-algorithmic-conceptual plane allows for a richer description of students' problem solving behavior than mere difficulty identification, even as difficulty identification may more exactly specify the particular confusion or incorrect reasoning students exhibit.

There is another external factor that is more important in influencing students' framing in a problem solving context, even before being affected by other humans such as group mates or the instructor. As soon as students read a problem, the problem statement framing interacts with the students' framing. In the next work, I will use this theoretical framework to categorize students' framing and then analyze their responses as an artifact of the problem statement and not just due to the final correct answer or correct reasoning path.

7.8 Conclusion

In this study my goal was to examine students' problem solving behaviors in the often-messy setting of the classroom. I'm particularly interested in how students solve problems collaboratively in groups, and especially in the ways they connect math and physics reasoning to solve upper-division problems.

I identified four epistemological frames: algorithmic math, conceptual math, conceptual physics and algorithmic physics. I presented several examples of students' group problems solving switching frames to productively and correctly solve a problem. This framework divides possible student errors into three different categories as displacement error, transition error and content error. The displacement error reveals students unproductive frame of the situation. Content error shows what pieces of knowledge have to be activated to understand all the ideas incorporated in the problem frame. The last error is transition, where students have ideas in different mental spaces but do not coordinate them.

This framework developed as a result of analysis of spontaneous and natural moments of in-class activities during one semester. There was no constraint in the problem solving sessions of the class, except, that the time duration of the problem solving session was limited. However, still students had enough time to illustrate natural moments of problem solving, to become creative, to get enough engaged with the problem to "get stuck" and then "un-stuck", or become so deeply engaged in the group problem solving to ignore the instructor's comment for several minutes after he has already paused the problem solving session. This kind of problem solving is more ecologically valid than the problem solving in individual clinical interviews[143], and thus as a field we should attend more carefully to it.

Instructionally, this framework is a useful tool for instructors to assess and facilitate different moments of problem solving sessions in their classroom settings. Some students like "Eric"(Section 7.6.1), are self reflective and get "un-stuck" by noticing the missing parts of their solution. However, not all of the students might act as "Eric" does. Using this framework may help instructors notice when students are stuck because of unproductive framing, and give them tools to nudge students into a more productive frame. Instructors can tip students into different frames[98;144] or gently nudge students to use additional resources[104;145] to resolve content errors.

Chapter 8

Framing difficulties in quantum mechanics

8.1 Introduction

[1]. Researchers in student understanding of quantum mechanics have used a "difficulties" framework to understand student reasoning identifying long lists of difficulties which span many topics in quantum mechanics. The goal of research in quantum difficulties is to determine common, repeatable incorrect patterns of students' reasoning[2;102;104;105]. Researchers refer to identified difficulties as universal patterns, since they occur across a wide range of student populations despite varying academic backgrounds[103].

Although the realms of quantum and classical mechanics are different – the classical world is simpler and more intuitive than the quantum world – researchers have long considered the possibility of difficulties in quantum mechanics being analogous to the misconceptions in classical mechanics[20]. This similarity is due to both persistent misconceptions or difficulties in students' reasoning[108], and students not having enough preparation with the formalism of quantum mechanics[106].

Research has detailed lists of student difficulties in determining the time dependency

[1]This chapter was submitted to the Physical Review Physics Education Research Journal as [146]

of stationary, superposed, and degenerate eigenfunctions[2]; the effect of time dependency of different physical systems on the probability densities[2]; energy measurements of a quantum mechanical system[105]; concepts of the time dependent Schrödinger equation[105] (TDSE); and the role of Hamiltonian physics in determining energy.[105]

However, we posit that these disparate difficulties can be unified through the lens of epistemological framing[11], errors in frame transitions[98], and errors in the content of a frame (e.g. with the Resources Framework[10]). This paper presents a secondary analysis of published difficulties in quantum mechanics through the lens of epistemological framing.

My goal in this study is to reanalyze students' difficulties in quantum mechanics. We apply a set of frames previously developed by our research team[97;99], to a long list of published difficulties in quantum mechanics in order to find an underlying structure to them. After developing our theoretical lens on our own video-based data, we turned to the published literature on student difficulties in quantum mechanics to seek an underlying structure to students' difficulties.

8.2 Theoretical framework

8.2.1 Difficulties

In a misconceptions view, students apply an incorrect model of a concept across a wide range of situations independent of the context[20;21]. The core of conceptual understanding occurs by confronting the incorrect conception, and replacing the new concept. This unitary view of students' reasoning guides our attention as researchers toward the identification of topics with which students have difficulties at the cost of missing students' epistemological changes[43] because a difficulties view predicts a stable model of thinking that is repeatable, and does not account for sudden or contextual changes in the nature of student reasoning.

A large number of students showing the same wrong answer to the same question implies a widespread difficulty in a certain topic; if the same difficulty presents across multiple questions or over time, it is robust. There have been many difficulties identified in quantum mechanics over the last 20 years across many different topics. For example, there are several sub-topic difficulties reported related to the topic of time dependence of the wave function: incorrect belief that the time evolution of a wave function is always via an overall phase factor of the type $e^{\frac{-iEt}{\hbar}}$; inability to differentiate between $e^{\frac{-iHt}{\hbar}}$ and $e^{\frac{-iEt}{\hbar}}$; and belief that for a time-independent Hamiltonian, the wave function does not depend on time[104].

Research into student difficulties is often focused on eliciting them in regular ways (possibly also involving the development of research-based conceptual assessments[147]), developing curricula to ameliorate or replace them, and iteratively improving the curricula. Common methods for the identification and documentation of difficulties are outlined in section 8.3.1; this paper is not concerned with the curriculum development or evaluation aspects of difficulties research.

8.2.2 Manifold views

An alternative view to a unitary difficulties view is a manifold "knowledge in pieces" view. In this view of student reasoning, we conceptualize student thinking as being highly context dependent and composed of small, reusable elements of knowledge and reasoning called "pieces". These pieces are not themselves correct or incorrect, but the ways in which students put them together to solve problems may be. By focusing on the pieces of student reasoning and how they fit together, this view of student reasoning foregrounds the seeds of productive student reasoning and not just incorrect answers. Theories in this family include phenomenological primitives[7] (p-prims), resources[10], and symbolic forms[44].

8.2.3 Epistemological framing

A strong thread of research using knowledge in pieces is to investigate students' epistemologies. Epistemological resources[69] connect to conceptual[10] and procedural[1] resources in networks[16] to help students solve problems.

The mechanism that allows control of which subset of resources are activated locally in a given context is epistemological framing[68]. Framing shows the nature of students' knowledge that emerges from a coherent set of fine-grained resources which coherently and locally work together in a situation [19].

Epistemological frames reveal students' [11;68] ways of thinking and expectations. They govern which ideas students link together and utilize to solve problems. Students' epistemological framing is highly context sensitive. Being in the appropriate frame and shifting between frames are determining factors in students' success [49;98;119]. Productive problem solving requires both an appropriate frame [119] and appropriate shifting between frames [99]. Careful observation of student behaviors, gaze, and discourse can provide clues for determining students' epistemological frames.

In our prior work, we developed a set of four inter-related frames around the idea of math-in-physics [97;99]. We applied it to observational data to model students' framing in math and physics during in-class problem solving in two upper-division courses, quantum mechanics [97] and electromagnetic fields (E&M) [99]. Briefly, our math-in-physics frames capture students' framing in math and physics, expanded through the algorithmic and conceptual space of students' problem solving. The four frames are: algorithmic math, conceptual math, algorithmic physics, and conceptual physics. Using this set of frames, we looked for moments where students' problem solving is impeded because they are in an unproductive frame.

This paper applies the math-in-physics frames to secondary analysis of quantum mechanics difficulties. We present a mapping of over thirty quantum mechanics difficulties from the literature [2;3;104;105] to our math-in-physics frames. This secondary analysis provides a deeper underlying structure to the reported difficulties and demonstrates the broad applicability of these frames to many kinds of quantum mechanics student data. However, this work is limited to the amount of information that is provided in the difficulty papers. In section 8.8, we have defined a category called "not enough information" for those examples that we do not have enough evidence to map the difficulty into our categorizations.

8.2.4 Interactions between problem statements and responses

Students' reasoning comes as responses to specific questions, and those questions strongly influence their framing. We examined problem statements for what frame(s) they initially promote. For example, consider these two problems:

- Using the time-independent Schrödinger equation (TISE), calculate the changes to E_0, the ground state, as the given well shrinks from L wide to $L/2$ wide.
- What happens to the energy of the ground state when a finite square well gets narrower?

The first encourages students to think mathematically ("calculate") and algorithmically (by hinting at a procedure). The second is more conceptual, specifying neither numbers nor procedures.

A major element of difficulties research is to carefully craft problem statements so as to best elicit student difficulties. To honor this careful work, our secondary analysis of difficulties considers difficulties as they are paired to problem statements.

8.3 Methodology

8.3.1 Difficulty identification

Because this paper reinterprets existing datasets using new theory, we first review where the data come from and how they were originally analyzed using a difficulties framework.

Researchers in difficulty studies have multiple methods for data collection, both quantitative and qualitative. The population of students are from advanced undergraduate students and first year graduate students from several different US universities. Students are administered a written test, usually at the beginning of the semester[104], or after relevant instructions[2;105]. Some students also participate in think aloud interviews intended to both develop the test and discover common responses to it. Data analysis on the interviews and written responses extracts common difficulties despite the differences in the students' backgrounds. The results of the analysis from both sources of interviews and tests are consistent; several cycles of test development and administration adjust the questions to best elicit student difficulties and ensure validity and reliability.

The original data in two papers[2;105] were collected at the University of Washington (UW), where undergraduate students are required to take between one to three quantum mechanics courses. The first course (sophomore-level) covers the first five chapters of McIntyre's textbook[148], and the second and third courses (Junior-level) cover all the materials of Griffiths' textbook[130]. Students were given a pretest in a written form before relevant tutorial instruction, but after lecture instruction. In some of the tasks a variation of the questions were given to the students.

In the third paper[104], survey data of first-year graduate students were collected from seven different universities. Researchers also conducted interviews with fifteen students at the University of Pittsburgh.

Both research groups at the University of Pittsburgh and the University of Washington have long histories of difficulties research in quantum mechanics and other physics subjects, and their expertise in developing questions, developing tests and curricula, and identifying difficulties is second to none. We chose their papers for secondary analysis because they represent the best that difficulties research in quantum mechanics has to offer.

8.3.2 Mapping students' difficulties to framing

We posit that many student difficulties in quantum mechanics may be due to unproductive framing in problem solving, because students' current frame may not help them with actual problem solving, because students find themselves temporarily unable to shift to a more productive frame, or because they cannot activate productive resources within their current frame. To investigate this postulate, we conducted a secondary analysis of published student difficulties in quantum mechanics.

We mapped descriptions of published difficulties into errors in epistemological framing and resource use. We considered the problem statement as the "jumping off" point for students' framing, reasoning that students' initial problem framing is probably strongly influenced by framing in the problem statement. From published descriptions of student responses – including their written responses, where available – we identified students' response frames and compared them to the frame of the problems to categorize errors.

Because this is a secondary analysis, we take the difficulty as the unit of analysis, not an individual students' response. This is a practical choice on our part, as some authors do not identify the frequency of each difficulty they identify and we did not have access to all of the descriptions of students' problem solving. The numbers reported for the error rates indicate how many difficulties fall in each of these categories and do not indicate how many students have the difficulties under a category. This kind of analysis is strange in the knowledge-in-pieces research tradition, as it severely hampers us from looking at what students do that is correct or productive; difficulties-focused research does not report productive ideas, only incorrect ones.

8.3.3 Methodology for secondary analysis

Selection criteria

We gathered published works which describe student difficulties in quantum mechanics from *Physical Review Special Topics – Physics Education Research, Physical Review – Physics Education Research*, and the *American Journal of Physics.* We identified four papers and thirty six student difficulties in quantum mechanics.

From these papers, we sought difficulties in which the authors had sufficiently described their problem statements (or instructor interactions) for us to determine initial problem framing, excluding those difficulties whose problem statements were omitted, or where variations on a problem statement were alluded to but not presented. Twenty eight difficulties survived these selection criteria.

Coding

We examined students' responses with respect to the features four frames in our math-in-physics set and coded the student framing present. We started with student responses to problems that matched our qualitative data[97]. Descriptions of student responses – and the resulting difficulties identified by researchers – matched our observational qualitative data well. Emboldened, we extended our coding of student responses to difficulties not present in our qualitative data. As much as possible, we investigated students' statements (or equations, on occasion) to identify the nature of their reasoning. For example, a response which is just a piece of an equation suggests that the student used an algorithmic frame to generate their response, whereas a response which coordinates energy and probability descriptions comes from a conceptual physics frame.

Some problem statements, particularly multipart problems, require students to start in one frame and shift to another one (for example, see section 8.4.1). In those cases, we coded for which sequence(s) of frames would yield correct answers.

Through intensive discussion among multiple researchers, we coded for which frame(s) a problem statement promoted, and which frame(s) were evident in students' reasoning. For some difficulties, students' responses or the descriptions of students' reasoning did not contain enough detail to figure out students' framing. Our goals in these discussions were to come to agreement about our inferences of student reasoning. As our discussion reached consensus and our codebook stabilized, two independent raters coded both the rating of the problem statements and the ratings of the students responses (or the descriptions of students' reasoning), with an agreement rate of $> 90\%$ for both kinds of coding.

Error type determination

Once problem statements were coded for frames promoted and student responses were coded for frames used, we classified students' difficulties into three categories:

- Transition error: when a problem statement requires shifting between frames, and students are unable to make that transition.
- Displacement error: when a problem statement promotes one frame, but students' reasoning puts them in another frame.

- Content error: when students appear to be framing the problem correctly, but are not activating appropriate resources to solve it.

Limitations

Many student responses to these questions are correct, and our secondary analysis of student difficulties cannot capture those responses. This is a fundamental limitation of difficulties-based research: it seeks to describe the ways students are wrong, not the ways that their responses are reasonable.

Some difficulties could have arisen because of multiple types of error. This is a limitation of secondary analysis – we don't have full reports of student reasoning – and of the survey-style free-response data on which many of the original difficulties are based. For this reason, we classified some difficulties as arising from multiple error types. With sufficiently detailed data, we believe that each difficulty-displaying student response can be classified into a single error type.

Additionally, some surveys were multiple choice. While the original researchers based the choices on common student reasoning, and that reasoning could have showed evidence of student framing, the multiple-choice answers themselves are often insufficiently detailed to determine students' framing. As much as possible, we coded researchers descriptions of student difficulties, but sometimes we simply did not have enough information.

In the follow sections, we show examples of each kind of error, arguing from our data and from published difficulties that difficulties can be categorized by framing error type. Within each type, we tabulate published difficulties. Because some problems require transitions between frames and some do not, we classify difficulties first by the kind of problem they come from and second by the kind of errors they produce.

8.4 Transition error

Transition errors occur when a problem statement expects students to shift between two frames, and the students either do not shift, or shift into an unproductive frame. In this section, we first motivate the idea of transition errors through extended analysis of one example, then tabulate all difficulties which exhibit transition errors.

8.4.1 Transition error example

The first example illustrates a transition error which arises from interpreting a graph of wave function vs position[2] (Figure 8.1). The problem asked students to explain if the probability of finding the particles within a marked region depends on time or not.

The probability density depends on time if the modulus square of the wave function depends on time. The wave functions are given at time $t = 0$. The authors mention that the problem requires students to think about the time dependent phase of each term in the superposition wave function. This encourages students to frame the question as thinking about what it means to be in a superposition of states, what are the energies of each term in the superposition, and how the system evolves over time. Additionally, the problem statement includes multiple representations for students to conceptually coordinate. This way of framing the problem suggests conceptual physics as the initial frame.

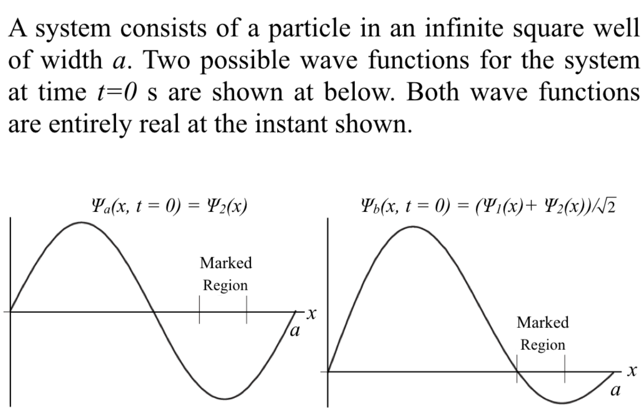


Figure 8.1: *Does the probability of finding a particle in the marked region depend on time? On this problem, student difficulties display transition error. Figure originally from*[2].

Students may start this problem by thinking of the time evolution operator, which is determined by the Hamiltonian of the system. After students recognize the correct time phase factors, they need to coordinate mathematical representations to show how the phase factors determine the time dependence of the probability density. Our set of math-in-physics frames has two possible transitions from conceptual physics into mathematics frames:

Algorithmic math In algorithmic math, a student would manipulate the modulus square of the superposed wave function explicitly and algorithmically, finding that the time dependence of the pure terms falls out, and the time dependence of the cross terms persists.

Conceptual math In conceptual math, a student initially could use a conceptual mathematical shortcut: the exponential term multiplied by its complex conjugate sets the product equal to one. However, this solution leads to neglecting the role of the cross terms.

Because the problem starts in a conceptual physics frame, it may be easier or more appealing to transition first into conceptual math than algorithmic math; in our observational data, this is the transition we observed.[97].

Emigh et al.[2] describe student reasoning in response to the same task:

Student While it it true that the general wave function is of the form $\sqrt{\frac{1}{2}}\phi_1 e^{\frac{-iE_1 t}{\hbar}} + \sqrt{\frac{1}{2}}\phi_2 e^{\frac{-iE_2 t}{\hbar}}$ again the function we are interested in is $P(x) = \mid \phi \mid^2$ which loses its time dependence.

The first part of this statement shows that the student has correctly used the ideas of the problem statement frame to note the different energies of each energy eigenstate in the wave function. The second part of the statement suggests that the student coordinates the physics and conceptual math to recall that the probability density is the modulus square of the wave function. However, the student would not do any further algorithmic calculations, instead arguing that the probability "loses the time dependence". This is congruent with conceptual math reasoning above.

$10-20\%$ of Emigh et al's students (N=416) applied the same kind of reasoning to argue that the "time drops out" or "the probability is squared and the time won't matter"[2]. These arguments indicate that students don't feel a need to actually do the math because their conceptual math frame has solved and justified their time dependence answer. While Emigh et al interpret these responses as a difficulty – students' "tendency to treat all wave functions as having a single phase" – we interpret it as an example of error in frame transition.

8.4.2 Difficulties which exhibit transition errors

Table 8.1: *Difficulties that exhibit transition error only. These difficulties are labeled "T" for transition errors; the ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.*

No.	Name	Ref.
T1	Mathematical representations of non stationary state wave functions	[104]
T2	Belief that the wave function is time dependent	[2]

We found two published difficulties for which students exhibit only transition errors (Table 8.1).

In the first difficulty *non-stationary state wave functions* (T1) in Table 8.1, the students were asked if different wave functions: $A\sin^3(\pi x/a)$, $A[\sqrt{\frac{2}{5}}\sin(\pi x/a) + \sqrt{\frac{3}{5}}\sin(2\pi x/a)]$ and $Ae^{-(\frac{(x-\frac{a}{2})}{a})^2}$ can be proper candidates for an infinite square well of width a with boundaries at $x = 0$ and $x = a$. This problem requires student to start from a conceptual physics frame, to extract the boundary condition information, and readout that the potential is infinity at the boundary conditions, and thus the wave function has to become zero to satisfy the continuity of the wave function at the boundaries. This problem requires transition as students may need to plug the values of the boundary conditions, do some algorithmic steps and figure out if the solution satisfies the boundaries of the problem. On this problem students perform two error categories of displacement and transition errors.

Some students considered that just the linear combination $A[\sqrt{\frac{2}{5}}\sin(\pi x/a)+\sqrt{\frac{3}{5}}\sin(2\pi x/a)]$, or a pure sinusoidal wave function are allowed, but the $A\sin^3(\pi x/a)$ is not allowed and "only simple sines or cosines are allowed" as proper wave functions. Some other students mentioned that for a particle in a box, only the wave functions in the form of $A\sin(n\pi x/a)$ are allowed and the $Ae^{-(\frac{(x-\frac{a}{2})}{a})^2}$ wave function is only allowed for a simple harmonic oscillator. We consider that students with this type of responses are not in the frame of the problem as they are not thinking about the characteristics of the boundary conditions. Instead, they just recall what the solutions for physical systems of a particle in a box or a harmonic oscillator look like. They assert that they know how the answer will look, having worked out the problem before. So, while the students might not necessarily attempt the algorithmic processes to arrive at this conclusion during the interview, they are relying on the fact that they have done this calculations before and can recall the conclusion. This is an appeal to having previously worked through the problem of a particle in a box, which fits into the algorithmic physics frame. We categorized students with this type of responses as having a displacement error.

Another type of incorrect response suggests that many of the students thought two conditions must be satisfied 1) wave functions are smooth, single valued, and satisfy the boundary condition of the physical system, and 2) wave functions can be written as a superposition of stationary states, satisfying the time independent Schrödinger equation (TISE). A typical response of the students looks like:

Student $A \sin^3(\pi x/a)$ satisfies b.c. but does not satisfy Schrödinger equation that is, it cannot represent a particle wave. The second one is a solution to S.E. it is a particle wave. The third does not satisfy b.c.

The author mentions that students do not note that even the superposition wave function does not satisfy the TISE. We think that this student is in the frame of the problem since they match the boundary conditions with each wave function to see if they satisfy the boundaries of the physical system. However, we do not have further information that how students are working on this problem, as part of their solution are they taking some algorithmic steps to match the boundary condition or they are just reasoning verbally.

We think this student can activate ideas regarding expansion of the function $\sin^3(\pi x/a)$ based on the energy stationary states by making a transition to the conceptual mathematics frame.

The second difficulty is that students believe the wave function is time independent because it satisfies the TISE. Students who generate these responses provide a mathematical basis for their answer. The author mentions that students think that the superposition wave function satisfies the TISE.

Student [both wave functions] satisfy the time independent Schrödinger equation so Ψ_1 and Ψ_2 do not have time dependence.

Although the solution to the TISE does not depend on time, the TISE solution is incomplete because this problem is time-dependent. We categorized this difficulty as a transition error, as students need to shift to conceptual frames (either conceptual physics or conceptual mathematics) to complete the problem. Shifting to conceptual physics may lead them to think in terms of the independent eigenfunctions of space and time; shifting to conceptual math may lead them to think about missing orthogonal functions.

8.5 Displacement error

Displacement errors arise when students are meant to be in one frame, but are instead operating in another. In this section, we first reinterpret a student's response the extended example problem in the prior section as if the student exhibited a displacement error, then tabulate difficulties which exhibit displacement errors for problems with and without transitions.

8.5.1 Displacement error example

For the same task as shown above in Figure 8.1, other difficulties are possible. For example, this student writes the time dependence of the wave function instead of finding the probability, an incomplete answer:

$$\sqrt{\frac{1}{2}}e^{-iEt/\hbar}(\psi_1 + \psi_2) \tag{8.1}$$

This short answer segment suggests that the student is in an algorithmic frame; there is no other information about student reasoning (such as narration or a graph provided by the student). This student has not picked up the conceptual framing intended by the problem statement. Starting from the conceptual physics frame could enable the student to think conceptually about superposition of wave functions and the different energy terms instead of a single time dependent phase. The authors do not provide the percentage of students that answered in this way. However, they mention that the *tendency to consider just a single phase wave function* for a superposition state is very common. 25% of their students ($N = 223$) on a final exam showed the same difficulty on a version of the same task. Students were given the time dependence wave function, and they were asked about the "time dependence of the probability of a particular outcome of a position measurement".

We classify this difficulty as a displacement error: the student is in the wrong frame initially, and does not transition to a more productive frame.

8.5.2 Difficulties which exhibit displacement errors

In problems which require frame shifting (like the problem in Figure 8.1), we found five difficulties which exhibit only displacement error (Table 8.2).

On problem from Emigh study[2] (Problem in Figure 8.1),the difficulty *Confusion between the time dependence of the potential and other quantities* (Ds2) in table 8.2 shows that students correspond the time dependence of one quantity to another such that both of physics quantities obtain the same time evolution. Between 5%-20% of the students in their data (N=416) have provided this type of reasoning.

Student The wave function is time independent. Thus, its probability density does not change. If the wave function is time dependent, then [its] probability density would change in time too.

This student does not calculate the module square either via an algorithmic mathematical frame, or a conceptual mathematical frame. Neither does the student think conceptually about the different energy eigenvalues of each term in the superposition. This student is not in the frame of the problem, which is conceptual physics. Instead, the student is in the algorithmic physics frame. These errors occur mostly when students just recall a fact about a physical quantity and they do not justify it. Usually in these cases, students do not interact with the problem statement to extract information out of it, because they think the recalled fact is unconditionally true in any context or physical system.

For the stationary state wave function on the same problem, about 5% of the students think that the stationary state wave function is time independent.

Student This is a stationary state so the wave function will not evolve with time.

This piece of data suggests that this student is not initially in the conceptual frame of the problem. The student might have been previously derived that some properties, which is the probability density is time independent for a stationary state. However, they do not accurately remember the conclusion. The incorrect notion that the wave function is time independent rather than the probability density further implies that the student is in an algorithmic physics frame and tries to recall a fact about stationary states.

In simpler problems that do not require frame shifting, we find two difficulties which exhibit displacement error (Table 8.3). The difficulty *that the time evolution of a wave function is always via an overall phase factor of the type* $e^{\frac{-iEt}{\hbar}}$ (Dn1) in table 8.3 shows that students are performing displacement error. The problem asks students to find the time dependent wave function $\Psi(x,t)$ for a system in an initial state of superpositions of the ground state and first excited states, $\Psi(x,t=0) = \sqrt{\frac{2}{7}}\phi_1(x) + \sqrt{\frac{5}{7}}\phi_2(x)$. The equations of the eigenfunctions and the eigenvalues are given in the problem statement. We frame this problem as algorithmic physics as it requires to recall the time phase factor and follow simple algorithmic steps to assign the readout energy eigenvalues from the problem statement into the time phase factor for each term and write the time dependent wave function in terms of ϕ_1 and ϕ_2. About one third of the students ($N = 202$) in this study wrote:

Student $\Psi(x,t) = \psi(x,0)e^{\frac{-iEt}{\hbar}}$

This piece of response suggests that the student is not in the frame of the problem statement. The student does not read the information regarding the energy eigenvalues from the given equations in the problem statement, and does not attempt to write the answer in terms of ϕ_1 and ϕ_2. The frame of the student is algorithmic physics and thus we categorized this difficulty as a displacement error.

Table 8.2: *Difficulties which exhibit displacement error in problems that require frame shifting. These difficulties are labeled "D" for displacement errors and "s" because their problems require shifting; the ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.*

No.	Name	Ref.
Ds1	Incorrect belief that $H\psi = E\psi$ holds for any possible wave function ψ	3
Ds2	Confusion between the time dependence of wave functions and probability density	2
Ds3	Belief that for a time-independent Hamiltonian, the wave function does not depend on time	104
Ds4	Tendency to associate the time dependence of energy measurements with properties of stationary states	105
Ds5	Tendency to treat superposition as having a multiple distinct phases	2

Table 8.3: *Difficulties that exhibit only displacement error in simpler problems which do not require frame shifting; and "n" because their problems require no shifting. These difficulties are labeled "D" for displacement errors; the ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.*

No.	Name	Ref.
Dn1	Incorrect belief that the time evolution of a wave function is always via an overall phase factor of the type $e^{\frac{-iEt}{\hbar}}$	104
Dn2	Difficulties related to outside knowledge in energy measurements	105

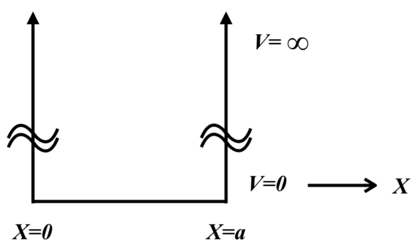
8.6 Content error

A third kind of error occurs when students are in the appropriate frame for a problem statement, but have not activated enough of (or the correct) resources to complete the problem. We term this kind of error “content error”. In this section, we illustrate content error with one example difficulty, then tabulate content errors.

8.6.1 Content error example

To illustrate content error, we draw an example from Singh[3]. This example comes from the interview data of first-year graduate students. For this problem, students are given the problem in Figure 8.2, to calculate expectation value of the superposition of the ground state and first excited stationary state of the system.

The wave function of an electron in a one dimensional infinite square well of width a at time $t=0$ is given by $\Psi(x,0)=\sqrt{2/7}\ \phi_1(x)+\sqrt{5/7}\ \phi_2(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are the ground state and first excited stationary state of the system. ($\phi_n(x)=\sqrt{2/a}\ sin(n\Pi x/a)$, $E_n=n^2\Pi^2\hbar^2/(2ma^2)$ where n=1,2,3. . .)



1. Write down the time dependent wave function.
2. You measure the energy of an electron at time $t = 0$. Write down the possible values of the energy and the probability of measuring each
3. Calculate the expectation value of the energy in the state (x, t) above.

Figure 8.2: *Calculate the expectation value of the energy in the state $\psi(x, t)$. On this problem, student difficulties display content error. Figure originally from*[3].

Although 67% of the students were able to answer part (2) correctly, only 39% were able to answer part (3) correctly, and many of the students were not able to use the information to apply in part (3). Instead, students explicitly calculated the integrals of the expectation value.

We analyze their description of a student’s response to part (3). The problem statement starts students in an algorithmic frame (directing them to "calculate"). The frame is algorithmic physics, rather than algorithmic math, because the students must first start by recalling some facts and equations about expectation values and wave functions.

The student writes down the TISE as $\hat{H}\phi_n = E_n$ without ϕ_n on the right hand side of the equation, but correctly writing ϕ_n as the sum of ϕ_1 and ϕ_2 on the left hand side. This is an appropriate initial framing to this problem, but it's missing a key piece of content. This mistake results in an incorrect answer in terms of ϕ_1 and ϕ_2. At this point, the student is not confused that his answer does not make sense because he is unaware of his error. The interviewer points to the part of the solution with the missing element, but the student is still unable to find his mistake. Finally, the interviewer explicitly gives the right TISE, $\hat{H}\phi_n = E_n\phi_n$ to the student.

The student can then review the math conceptually in his solution by applying the orthonormality properties of the eigenstates, simplify the integration, and get a correct answer. It seems that all he needs is a correct TISE, and he is able to frame the problem appropriately and continue to a successful solution. We does not consider this example as a case of a simple typographic error on the student's part, because the instructor several times notifies the student about his error, but the student believes that his written TISE is fine. After finding the correct answer, student is able to reflect on his answer, and even conceptually reason about the expectation value.

The interviewer continues: she asks the student if he can think of the response to part (3) in terms of the response of part (2). The student responds "Oh yes . . . I never thought of it this way. . . I can just multiply the probability of measuring a particular energy with that energy and add them up to get the expectation value because expectation value is the average value." The interviewer's intervention to explicitly connect parts (2) and (3) prompts the student to think more physically in terms of the underlying concept of expectation value. He relates it to the parameter of the physical system such as energy eigenvalues, and probability of measuring each.

We categorized this difficulty as a content error because the student is in the frame of the problem and is not able to find the correct answer until the interviewer provides more content.

8.6.2 Difficulties which exhibit content error

Among problems which require transitions, we classified four difficulties as content error only (Table 8.4).

For Cs2, the authors[2] provided a student's reasoning:

Student Since the wave equation will gain a $e^{\frac{-iE_2 t}{\hbar}}$ term to represent its evolution as time goes on, the probability of finding the particle in the marked area will decrease [...] since the square of its wave equation will decrease as well.

This student is in the same frame as the problem statement, which is conceptual physics, as discussed in section 8.4.1. The only difference is that the student's response is with regard to the stationary state wave function. The student has determined the energy of the stationary state E_2, and knows how to perform the appropriate calculations to find the probability density. However, his exponential term has a (negative) real power instead of an imaginary one. We interpret this as a content error: he has activated incorrect resources and reasoned from them.

Table 8.4: *Difficulties that exhibit only content error in problems which require shifting frames. These difficulties are labeled "C" for content errors; and "s" because their problems require shifting; the ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.*

No.	Name	Ref.
Cs1	Tendency to treat wave function for bound systems as traveling waves	2
Cs2	Tendency to treat time-dependent phase factors as decaying exponentials	2
Cs3	Difficulties related to outside knowledge in energy measurements	105

In simpler problems, we found an additional six difficulties (Table 8.5).

The first difficulty, *differentiating between the Hamiltonian and energy* (Cn8, Table 8.4), occurs when students mis-apply the energy eigenstate instead of the Hamiltonian operator in the time evolution operator[104]. The problem asks students to find the time dependent wave function $\Psi(x,t)$ for a system in an initial state of superpositions of the ground state and first excited states, $\Psi(x,t=0) = \sqrt{\frac{2}{7}}\phi_1(x) + \sqrt{\frac{5}{7}}\phi_2(x)$. The equations of the eigenfunctions and the eigenvalues are given in the problem statement. We frame this problem as algorithmic physics as it requires to recall the time phase factor and follow simple algorithmic steps to assign the readout energy eigenvalues from the problem statement into the time phase factor for each term and write the time dependent wave function in terms of ϕ_1 and ϕ_2. The authors mention that students write an intermediate state for $\Psi(x,t)$:

$$\Psi(x,t) = \Psi(x,t=0)e^{-iEt/\hbar} = \sqrt{\frac{2}{7}}\phi_1(x)e^{-iE_1t/\hbar} + \sqrt{\frac{5}{7}}\phi_2(x)e^{-iE_2t/\hbar}$$

Since the student proceeded from an intermediate state, we presume that the student does not attempt to re-derive the relationship between a space portion and time portion of a wave function. This student is in the frame of the problem by reading out the energy eigenvalues E_1 and E_2 and assigning each energy into the time phase factors. However, the intermediate step does not convey any algorithmic process or physical meaning and can not lead to the final step.

We classify this problem as a content error, since the solution misses to acknowledge the symbol H as the Hamiltonian operator, and the symbol E as the energy eigenvalue of the system. This is evidenced from the data, that further probe by the interviewer revealed the difficulty differentiating between the Hamiltonian operator and its eigenvalue.

The difficulty *vectors in real space - Hilbert space* (Cn7) in Table 8.2 indicates that students have difficulty to differ vector in real 3D space from vector in Hilbert space, such that students may not be able to distinguish between the 3D space describing the gradient of the magnetic field in the z direction, and the 2D Hilbert space for describing a spin-$\frac{1}{2}$ particle. The question is about the Stern-Gerlach experiment. "a beam of electrons propagating along the y direction into the page in spin state $\frac{(\uparrow_z + \downarrow_z)}{\sqrt{2}}$ is sent through a Stern-Gerlach apparatus (SGA) with a vertical magnetic field gradient. Sketch the electron cloud pattern that you expect to see on a distant phosphor screen in the x-z plane. Explain your reasoning." Due to the magnetic field gradient in the z direction, the beam of electrons will experience a force and become deviated. However, electrons due to having an intrinsic angular momentum, which is their spins, split only into two directions along the z axis and form two spots on the screen. The frame of this question is conceptual physics, because it encourages students to think about "what's going on" in this physical apparatus. The problem statement requires different readouts about the direction of the magnetic filed gradient or the direction of the electron beam. Students are asked to use graphical representation and justify their reasoning. Only 41% of the student (N= 202) answered correctly and the rest of the students predicted that there will be only a single spot on the screen. A typical response of the students looks like:

Student All of the electrons that come out of the SGA will be spin down with expectation value $\frac{-\hbar}{2}$ because the field gradient is in $-z$ direction.

This student is thinking conceptually by reading out information about the direction of the magnetic field from the problem statement and connecting that to the idea about spin$-\frac{1}{2}$ and thinking that this measurement has only one outcome and thus the expectation value is $\frac{-\hbar}{2}$. However, the student needs to more carefully readout from the problem statement that the state of the system before the measurement is in the combination of two sates of spin up $\uparrow_z$, and spin down $\downarrow_z$, and the state of the system is not just prepared in one state of down $\downarrow_z$ to stay unchanged after the measurement. We categorize this problem as a content error since the student's reasoning is missing some content that is preventing to get a correct answer. However, the description of the student in not enough that we could identify which exact content is missing from this piece of reasoning. It could be helpful to think about what does it mean for a beam of electrons before passing through an SGA to be in the combination of states spin up and spin down, or to think what does it mean for an electron to have an intrinsic angular momentum which is the spin.

The difficulty *interpretation of a subsequent energy measurement* (Cn1 and Cn2) is mostly limited to a content error. Students are able to use the ideas of the problem statement and operate in the frame of the question but are activating the wrong resources to productively and correctly solve the problem.

For example, a student is asked about the outcomes of an energy measurement after a following measurement on the system of a particle in an infinite square well in an initial state $\Psi(x,0) = 0.6\Psi_1(x,0) + 0.8i\Psi_2(x,0)$. Part A of the question asks "What value or values would a measurement of the energy yield?" And part B of the question asks what would be the result of the second energy measurement after time t_2. The student in response to part B states[105]:

Student The particle is described by a wave function with elements in both eigenstates. Although a measurement of energy collapses it to one, the possibility of the other still exists, so a second measurement could get the other E.

The frame of this question is conceptual physics which requires to think about the idea that repeating an energy measurement does not change the state of the system and yields the same result as the first measurement, since the system is already collapsed to one of the energy eigenstate and is isolated from its surroundings.

The student has activated several ideas about energy measurement on a physical system in a superposition wave function. In the first part of the response. The student acknowledges that a particle "is described by a wave function with elements in both eigenstates", and also "a measurement of energy collapses it to one". These are both correct ideas about two content areas in this problem. With this being the case, it may be difficult to understand why the student arrives at the wrong answer despite seeming to have all correct ideas about the system. For us, the key is in the student's usage of the word "although".

This student acknowledges the fact, that when a system collapses it has only one energy, but the probability of other energy, "E", "still exists" and associates this possibility with the second measurement on the system. Whereas, this is only true before doing any measurement on the system. For a system in a superposition state, if we prepare a system n times in the exact same way and each time make a measurement on the system, we would exactly know how many times the energy measurement yields E_1, and how many times the energy measurement yields E_2. But as soon as we make an energy measurement the system collapses into one of the energy eigenstates and repeating an energy measurement yields the same result as the first measurement.

This student uses the word "although" to put these two ideas in opposition to each other. The student somehow decides that his knowledge of state collapse is not applicable here and a measurement possibly yield multiple possible energy values. This student needs to activate more content about what information the problem statement provides about the system before and after an energy measurement and repeating an energy measurement on the system.

In the study by Singh et al.[3;104] they showed that students have difficulty with the *time development of the wave function after measurement of an observable* (Cn6). Students were asked about the wave function a long time after measurement of energy E_2 for an electron in an infinite square well. Some of the students stated similar to this response that, "If you are talking about what happens at the instant you measure the energy, the wave function will be ϕ_2, but if you wait long enough it will go back to the state before the measurement." The first part of the response suggests that the student is able to correctly relate the measured energy eigenvalue to the associated eigenstate of the system ϕ_2, by activating the resources of an instant measurement. However, the student does not further investigate the idea that long after the measurement a phase will be added to the eigenstate, which does not change the state of the system to any other combination of eigenstates, or the system will not "go back to the state before the measurement."

8.7 Difficulties where more than one error type is possible

For some difficulties, multiple error types are possible. Additional details of student reasoning could resolve these ambiguities, but these details are either not gathered (survey data) or not available to us (interview data) as secondary analysts.

Table 8.5: *Difficulties that exhibit content error in simpler problems. These difficulties are labeled "C" for content errors; and "n" because their problems require no shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.*

No.	Name	Ref.
Cn1	Determining the outcomes of a subsequent energy measurement	105
Cn2	Difficulties with the possible outcomes of a measurement	104
Cn3	Failure to recognize that the time evolution of an isolated system is determined by the Schrödinger equation: "Decaying reasoning"	105
Cn4	Belief that the wave function will return to its initial state	2
Cn5	Failure to recognize that the time evolution of an isolated system is determined by the Schrödinger equation: "Diffusion reasoning"	104
Cn6	Difficulties with time development of the wave function after measurement of an observable	3
Cn7	Difficulties in distinguishing between vectors in real space and Hilbert space	3
Cn8	Inability to differentiate between $e^{\frac{-iHt}{\hbar}}$ and $e^{\frac{-iEt}{\hbar}}$	104

Table 8.6: *Difficulties that exhibit both displacement and content error in problems which require transitions. These difficulties are labeled "DC" for displacement and content errors; and "s" because their problems require shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.*

No.	Name	Ref.
DCs1	Confusion between the time dependence of energy measurements and other quantities	2
DCs2	Belief that the wave function will spread out over time	2

The first difficulty with the *energy measurement probability - time dependence* (DCs1) in table 8.6 indicates that students could exhibit either displacement or content error. The task asks about the time dependence aspect of energy probability measurements of a particle in a QM harmonics oscillator system in the initial state, $\psi = \frac{i}{\sqrt{3}}\psi_0 - \frac{\sqrt{2}}{\sqrt{3}}\psi_1$ Displacement error occurs when students associate the time dependence of the probabilities of energy measurements to the time independent properties of the "probability density" or the "wave function". In these types of answers students usually recall some properties of the physical quantities with out any justification, since students think their reasoning is correct the way they are.

Student It [the energy probability] depends on the probability density. If it's time independent then no, if time dependent then yes.

The content error occurs when students are able to determine some of the features of the physical system by being in the frame of the question; however, they are not considering all aspects of the problem context. As in the example mentioned in the study by Emigh et al.[2], the student by stating, that "A linear combo of stationary states is not stationary" is mindful that a superposition of eigenstates is not a stationary state. The student is also able to differentiate between the energy levels of each eigenstate and gives a presentation of the system by stating "The system will oscillate around E_0 and E_1". This response was also provided to the same task as mentioned above.

Table 8.7: *Difficulties that exhibit both displacement and transition errors in problems which require frame shifting. These difficulties are labeled "DT" for displacement and transition errors; and "s" because their problems require shifting. The ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.*

No.	Name	Ref.
DTs1	Tendency to treat all the wave functions as having a single phase	[2]
DTs2	Finding the probability of an energy measurement from the wave function	[2]

Table 8.7 indicates, that students' difficulties with considering just a *single phase wave function* (DTs1) can be mapped as a displacement error or a transition error. A displacement error indicates that student has not attended to the frame of the question to blend the information effectively with the corresponding concepts in the task. This is similar to the described example in section 8.5 from the Emigh et al. study[2]. The other possible error is when students frame the task properly and are able to coordinate between frames; however, they fail to productively transition between frames to remove all the barriers (Section 8.4.1).

The *the real and imaginary parts of the wave function*(CTs1 in table 8.8) difficulty shows that some students establish a conceptual discussion in a math frame to relate the real and imaginary parts of the wave function in the complex plane. However, viewing the problem as purely conceptual (e.g. in the conceptual math frame) prevents students to note other related ideas in the problem statement. Shifting to the algorithmic math frame could help the

Table 8.8: *Difficulty that exhibits both content and transition errors in a problem which requires frame shifting. This difficulty is labeled "CT" for content and transition errors, and "s" because its problem requires shifting.*

No.	Name	Ref.
CTs1	Tendency to misinterpret the real and imaginary components of the wave function	2

student to recall other related facts to successfully solve the problem. Alternately, moving to the conceptual physics might help the students to reframe the unrelated ideas of their framed representation consistent with the goal of the problem.

Table 8.9: *Difficulties that exhibit all types of error in problems which require frame shifting. This difficulties is labeled "DCT" because it may involve all three types of error, and "s" because the problem require shifting.*

No.	Name	Ref.
DCTs1	Interpreting the meaning of expectation value	3

Table 8.9, shows that students' can exhibit different types of errors in *interpreting the meaning of the expectation value* (DCT1). As been discussed in the third example (Section 8.6), the task required students to start with recalling physics relations to calculate the expectation value. The student experiencing content error will activate an unstructured piece of his knowledge related to the TISE, which will be corrected by the interviewer[3], we already discussed this example in section 8.6. The displacement error occurs when the student is outside of the problem statement frame (algorithmic physics), and writes down just a mathematical expression $\Psi \hat{H} \Psi$, which lacks blended information from the physical space.

An example of transition error is when the student is able to blend the physical meaning of the probability of the energy values to the related coefficients, and translate the problem into procedural steps. However, the student might leave an extra coefficient ($\frac{1}{2}$) in the final solution, $\frac{\frac{2}{7}E_1+\frac{5}{7}E_2}{2}$, which can be removed by reviewing the solution and thinking purely conceptually about the quantity of the expectation value[3]. For this example only the final answer of the student was provided in the paper and no further narration of the student was available which leaves uncertainty in our analysis.

Table 8.10: *Difficulties that exhibit both displacement and content errors in simpler problems. This difficulty is labeled "DC" for displacement and content errors and "n" because the problem does not require shifting.*

No.	Name	Ref.
DCn1	Time dependence of expectation values-distinguishing between stationary states and other eigenstates, and recognizing their properties	104

Table 8.10 shows students' difficulties with calculation of time dependence expectation value in the context of Larmor precession for problems that does not require transition between frames. Difficulties with *recognizing the special properties of stationary states* could result in a content error, as students similar to this case state, that for a stationary state the commutation of the Hamiltonian and the operator Q is nonzero, thus "its expectation value must depend on time" [104].

Student Since $\hat{S}_x$ does not commute with $\hat{H}$, its expectation value must depend on time.

Although the student is able to apply the Ehrenfest's theorem correctly, the student does not note that being in a stationary state changes the Hamiltonian in the time dependence phase factor from an operator to a number, which commutes with the operator Q. In addition, difficulties with *distinguishing between stationary states and eigenstates of operators other than energy* could result to a displacement error as students think that "if a system is initially in an eigenstate of $\hat{S}_x$, then only the expectation value of S_x will not depend on time." [104]

8.8 Not enough information

Because of the limitations of secondary analysis and because some original data were drawn from surveys with limited details of students' reasoning, there were times when our research team came to consensus that there was not enough information to determine students' probable framing. (Difficulties on problems for which the problem statement wasn't reported in enough detail are excluded from our analysis altogether.)

As an example of a reported difficulty without enough information, we consider an example from Passante et al.[105]. A particle is in the ground state of an infinite square well.At a known time, a perturbation is added to the system, which acts for a known duration and then is removed from the system. The problem asks about the probability of the energy measurement outcomes before, during, and after the perturbation.

Passante et al.[105] mention that students exhibit a difficulty in relating *the role of the Hamiltonian in determining the possible energy values* before, during, and after the perturbation. The correct answer requires students to consider that before the perturbation the energy is unchanged. During the time that perturbation is applied, the energy measurement of the system yields the energy eigenvalues of the new Hamiltonian.

Nearly 30% (N=40) of the junior-level students exhibit the difficulty that during the perturbation the energy of the initial wave function is still measurable. This incorrect response was also common for 40% (N=31) of the graduate students that were given a version of this problem. Some of the students had similar reasoning to this: "[the system] should still be E_2 – the wave will take time to conform."

We don't know what students are thinking here: there's simply not enough context to know whether students are in the appropriate frame but not activating the right resources (content error) or if they are in an inappropriate frame (e.g. algorithmic physics or conceptual math where conceptual physics is anticipated, as might be expected in a displacement error) or if this is small portion of their reasoning and they exhibit transition errors in the longer problem.

We found nine difficulties for which there was insufficient information reported to determine framing error types (Table 8.11)

8.9 Error rates

Figure 8.3 shows all the possible ways that descriptions of published difficulties can be mapped into errors in framing and resource use. Each number in the Figure 8.3 refers to the number of difficulties in that error category.

Table 8.11: *Difficulty topics for problems without enough information. These difficulties are with any letter because we don't know which kind(s) of error they may represent; the ordering of the difficulties in the table is for labeling purposes only and does not represent a hierarchy.*

No.	Name	Ref.
1	Belief that the wave function is time independent	2
2	Confusion between coordinate and Hilbert space	2
3	Difficulty interpreting the meaning of expectation value	104
4	Difficulties related to outside knowledge	105
5	Failure to recognize the role of the Hamiltonian in determining the possible energy values	105
6	Failure to recognize that the time evolution of an isolated system is determined by the Schrödinger equation- "Revival reasoning"	105
7	Confusion between the time dependence of probabilities of energy measurements and other quantities	2
8	Belief that the wave function will spread out over time	2
9	Confusion between the time dependence of wave functions and probability density	2

This figure shows that all of the error categories and combination of them are populated with some difficulty topics. Given our set of math-in-physics frames we can predict the kinds of difficulties that will emerge for a given problem in quantum mechanics, which can provide a possible deeper structure to student's cognitive process.

Figure 8.4, and Figure 8.5 give an overview of the occurrence of the three error types – displacement, transition, and content – among all the topics. Figure 8.4 shows that displacement error is the most frequent among problem statements which require transition. This suggests that when students solve problems which require additional steps, they are more likely to begin from an inappropriate frame. In contrast for simpler questions that do not require transition, content errors are the most frequent. On these problems, the problem statement is more likely to cue students into an appropriate frame, but students are less likely to finish successfully.

A great deal of effort goes into designing and testing questions which will reveal or cause student difficulties, so it's possible that these error rates are an artifact of the kinds of questions most likely to produce difficulty-like responses.

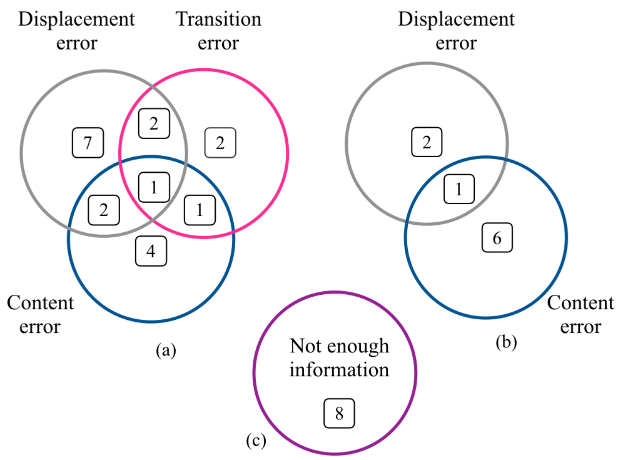


Figure 8.3: *The number of difficulties mapped to error categories, a) for questions that require transition, b) for questions that do not require transition, and c) unspecified due not having enough information*

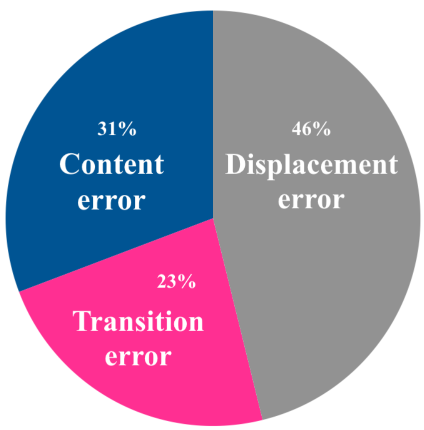


Figure 8.4: *Displacement, transition and content error categories of difficulty topics for questions that require transition, with the displacement as the most common type of error.*

8.10 Conclusion and Implications

The goal of this paper is to reinterpret research on student difficulties in quantum mechanics through the lens of epistemological framing, particularly using the set of four math-in-physics frames which we have previously applied to our observational classroom data of student problem solving in quantum mechanics and electromagnetic theory. We seek an underlying structure to the kinds of difficulties other researchers have identified.

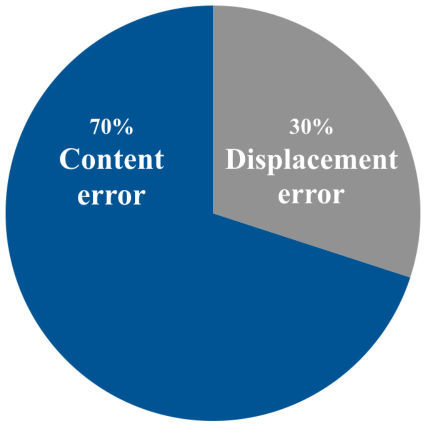


Figure 8.5: *Displacement and content error categories of difficulty topics for questions that do not require transition with the content as the most common type of error.*

This framework splits the underlying thought process of student into three different categories as displacement error, transition error, and content error. The displacement error reveals students' unproductive frame of the situation. Content error shows what pieces of knowledge have to be activated to understand all the ideas incorporated in the problem frame. Transition error shoes that students are able to activate resources in one frame, but cannot shift to another frame to continue with productive problem solving.

This framework reveals different levels of cognitive process of students thinking, which has instructional implications. Instructors' awareness of student's error categories may help them scaffold students' reasoning better, as instructors can tip students into different frames[98;144] or gently nudge students to use additional resources[104;145] to resolve content errors.

Our analysis of secondary data is hampered by the very nature of that data: we don't have access to full student reasoning because sometimes primary sources don't report it, and sometimes the nature of their survey data precludes them from collecting it. We expect that, were sufficiently detailed data available, all of the student reasoning attributed to difficulties in quantum mechanics could be analyzed using these frames.

This work has implications for both future research and for instruction. For research, it is an open question as to whether these four frames – conceptual physics, conceptual math, algorithmic physics, and algorithmic math – constitute the optimal basis set for epistemological frames in student understanding of quantum mechanics. However, they form a more compact basis set than is possible (let alone extant) with difficulties.

For instruction, epistemologically-aware tutorials at the introductory level[78] has been shown to outperform difficulties-based tutorials[38] in student understanding of Newton's Third Law[79;80]. More broadly, supporting students' epistemologies in the classroom may have far-reaching implications for retention and persistence[70;149;150]. Development work in quantum mechanics at the upper-division is exclusively in a difficulties-based mode, though some epistemologically-aware work has occurred at the introductory level in quantum mechanics[151]. Curriculum developers could take up framing as a guiding theoretical framework for development at the upper-division. This is particularly interesting for quantum mechanics because the conceptual content is epistemologically difficult for students.

Chapter 9

Conclusion

The goal of this dissertation is to build better theories capturing the richness of reasoning and problem solving activities as students mathematize a physics problem. I used the family of knowledge in pieces (KiP) theory to account for student's different ways of thinking and context dependent aspects of their problem solving situations. Under the KiP theory, resources account for students fine-grain level of ideas. I analyzed an interaction between two students to provide evidence that a fine-grain level resource size can also have an internal structure. I start my dissertation with a resource analysis to show that the internal structure of a resource may or may not be accessible by a user. In Chapter 3, under the lens of resources, I provided evidence from one case study group to show the process of unpacking the resource of separation of variables (SOV). As part of this unpacking I identified a new procedural resource called "bringing out" that was helpful to student understanding of the idea of *functions-as-constants* while taking a partial derivative.

"Alex" was able to separate variables via activating only two procedural resources of "grouping" and "dividing". However, it took three procedural resources and three conceptual resources for "Eric" to understand the process of SOV. "Alex" did not need to access the internal structure of SOV because he was able to use it as a solid resource. Perhaps there was a time when the resource of SOV was more plastic for "Alex" and he was referring to its internal structure while using it. By the time the link between resources strengthens and students can use them as a compiled unit[14] Black et al[1], had predicted the SOV resource progression within four different groups. Our work is complementary to the previous work as shows this unpacking process within one group which better explains the link between resources and their activation patterns in different phases of unpacking processes.

Resource analysis can reveal important information of how fine-grained level ideas become activated and connect to each other to form reasoning. For viewing the dynamic of problem solving moves in a broader time scale I use the theory construct of epistemic games to provide evidence of how the resources of students come together to make larger scale moves and progress toward a goal. In Chapter 4, I identified a new e-game played by Ave to solve an estimation physics problem called estimation epistemic game (e-e-game). Previous research by Tuminaro et al[6], identified six e-games that students might enter to use their knowledge of math in physics. They also identified three frames of rote memorization, quantitative, and qualitative sense making associated with their e-games. Chen et al[54] also identified an answer making game where students can either enter a game to find an answer or enter a game relying mostly on sense making reasoning. I observed that Ave, while playing the e-e-game, can show evidence of framing the situation via answer making or sense making. However, the structure of e-games with strict moves makes it hard to see the freedom of students in creating different solution pathways. I also show that students shift among various moves. By viewing the dynamic of student shifts among the moves we can find useful information about students thought process. Ava's game is mainly divided into two parts in the first 8 minutes, Ava bounces back and forth between two moves. She spends more time to use her estimation reasoning strategies to find a justified answer for the physics quantities, which indicates more conceptual and deeper way of thinking. In the last 6 minutes of the game the speed of the game increases. With inclusion of the calculation move the frequency of the shifts among the three moves increases. One reason could be that when students do calculation as part of their problem solving activity, they take several short steps over a long period of time.

In Chapters 3 and 4, I investigate how resources come together to build arguments, and how these arguments interact with each other in the forms of moves during a problem solving session. I conclude that the latter gives a more dynamic view of students problem solving structure. The former gives a more static view, as it focuses on how individual resources are activated and how the links between resources become broken and other resources are added to show small but efficient changes in understanding. In chapter 5, I consider the role of the problem context more specifically in resource activation by defining the mechanism of "looking ahead". This is especially important if the problem statement includes some information about a physical system. The mechanism of "looking ahead" explains the interaction effect between the problem statement and the students ideas. Thus I use the theory of coordination class and expand on the interaction of the coordination class and the readout strategy. "Looking ahead" is an extended readout strategy that helps students to cue their generalized knowledge in a situation initially or during different moments of the problem solving activity. Reading out and activation of all the coordination classes that are necessary for a problem is difficult to complete at once. Thus by "looking ahead" into the problem, students in the moment of the problem solving can insightfully use their knowledge of math in physics.

In Chapters 3-5, I investigate the differences and similarities among student processes in upper division problem solving. For example "Alex" was able to separate the variables by only using two procedural resources of "grouping" and "dividing". On the other hand, "Eric" was able to understand the process of separation of variable only after three procedural and three conceptual resources in relation to each other have been activated. The study of "Eric" and "Alex" was focused only on the separation of the time dependent Schrödinger equation without applying the boundary conditions. Thus the nature of students problem solving was mostly algorithmic. In Chapter 5, I investigated a harder problem which students have to solve the Laplacian equation by applying the boundary conditions. Thus students, not only need to do the separation of variables process to solve the differential equations, but they need to conduct all these processes in accordance with the physical characteristics of the problem. One mechanism that helped "Jack" with his decision to solve the separated second order simple homogenous ordinary differential equations was an extended readout strategy, which I called it "looking ahead". Jack by reading the information out of the problem statement was able to insightfully read the physics information from the problem statement and blend it with his knowledge that he had about solving partial differential equations. Although the reading out strategy has been previously compared to a mechanism that can affect student's framing[50], I explicitly use the epistemological framing construct to account for both internal and external factors that can affect student's framing during a problem solving activity. In addition, epistemological framing is a more flexible theoretical construct that can account for student's unseen condition during the problem solving activities. Under the lens of epistemological framing, in Chapter 7, I developed a framework which suggests different ways that students can use their knowledge of math in physics. This math-in-physics framework divides the space of problem solving into four frames of conceptual physics, algorithmic physics, conceptual math, and algorithmic math which is novel compared to other upper division studies[49] of students use of math in physics. Tuminaro[6] uses e-games to identify student mathematical errors and differences between expert and novice behaviors in problem solving.

None of these frames in my study are universally more productive than the others. Rather, what counts as productive framing depends strongly on the problem context and different frames may be productive at different times within a problem.

In Chapter 8, using the same framework, I analyzed the interaction effect between the problem statement and student ideas in quantum mechanics difficulties to find an underlying structure to this long list of difficulties[2;104;105]. This framework divides possible student errors into three different categories as displacement error, transition error and content error. The displacement error reveals students unproductive frame of the situation. Content error demonstrates what pieces of knowledge have to be activated in order to understand all the ideas incorporated in the problem frame. The last error, transition, is where students activate resources in one frame, but cannot shift to another frame to continue with productive problem solving.

Across all of these studies I have investigated the manifold and context dependent aspects of students problem solving in physics by entering to the domain of pieces. Within the domain of pieces I showed that a piece of resource such as a procedural resource can consist of other resources that form the internal structure of a resource. I used the epistemic games[6] (e-games) theoretical lens to examine the association patterns of resource activation across different moves during the estimation epistemic game (e-e-game). Although e-games show the activation pattern, but the strict move structure of the e-games breaks down as the result of unseen condition in student upper division problem solving. This was also evidenced by instances of mixed framing in Ava's epistemic play. Thus I used the epistemological framing to show how students navigate problem solving.

I was interested in how students generalized ideas in mathematical formalisms can become activated in accordance with physical situations. In Chapter 5, I illustrated the looking ahead mechanism as students read out the resources out of the problem statement context, which helps students to map the specifics of the problem to their generalized ideas and understanding. Thus the generalized ideas in students minds interact with the problem statement via an extended readout strategy. In Chapter 8, I also discussed this context dependent aspect using the epistemological framing. I defined the difficulties as an interaction effect between the problem statement frame and students frame. By frame, I mean students expectation of "what's going on?" in the problem; whereas, the frame of the problem is for what frame(s) they initially promote. In Chapters 5 and 8, I expanded the theory of KiP to better illustrate the context dependent aspect of upper division students' knowledge of math in physics.

In Chapters 4 and 7, I demonstrated the manifold aspect of the KiP theory. I discussed Ava's moves and the resources that she uses as her reasoning strategies. I observed that Ava frequently activates her resources to map physical concepts and plug estimate-based numbers into equations and finally using pure calculations to find the answer.

As part of students mathematical thinking in physics, previous studies have identified e-games to define students knowledge activation in an organized manner. Research suggests that experts are more likely to play Mapping games; either Mapping Meaning to Mathematics or Mapping Mathematics to Meaning[6]. Under the lens of blending theory, researcher have emphasized the role of conceptual schema in understanding mathematics and equations[125].

But neither of these four developed frames in my study are universally more productive.

Rather, what counts as productive framing depends strongly on the problem context, and different frames may be more or less productive at different times within a problem.

"Eric" navigation from conceptual math to algorithmic math enables his to see a new part of the problem that his expectation about problem solving initially blocked his attention to view that new piece of information.

Interestingly, "Eric" solved this problem mostly by navigating between mathematical frames, along the algorithmic and conceptual problem solving axis.

On the other hand, "Adam" coordinated his knowledge of math in physics, but not conceptually. He made a transition from algorithmic math to algorithmic physics and back to continue his solution.

This does not imply at all that Adam is not able to think conceptually, but external factors such as the problem statement could affect students framing to solve a problem. This laid out framework best describes our data as it occurred in the natural and spontaneous moments of in-class activities.

Chapter 10

Future work

I discuss the research and instructional implications of each of my studies as they open new avenues to both research and research-based instruction in upper division physics courses.

The research implication of my resource study is to investigate the role of “bringing out” resource across other physics courses such as Thermodynamics, and to see if students mostly use the same set of resources in unpacking their separation of variables (SOV) resource. Possibly due to the nature of the mixed partial differential equation[84] in Thermodynamic topics students may activate other procedural or conceptual resources. If so, it will be interesting to identify how resources in the context of mixed partial differential equations are linked to each other. The instructional implication of this study is that by becoming more familiar with these procedural and conceptual resources, teachers can better guide student reasoning. Viewing students reasoning from the fine-grained perspective of resources can help instructors to view the detail of students reasoning. Instructors can improve students understanding of separation of variable method by using these procedural and conceptual resources when teaching in math method or quantum mechanics courses.

My open questions for this study are: From the point of view of resources, how do resourced become packed and used by a user such as ”Alex” as a unit? If other students were in the group working with “Alex” and “Eric” how would the pattern of unpacking of

SOV change?

For the epistemic games research that I have done, I suggest that there might be variations to the e-e-game. It will be interesting to investigate the variations of the e-e-games to be played by other populations of students in a physics course that actually uses estimation-based problems as part of its curriculum.

My open question is “under what condition might students be able to shortcut some of the moves in the e-e-game?” If there exist other variations of the e-e-games how students might shift from one game to another? Would the underlying epistemological frame that describe the variations of the games be the same?

One of the areas in research that focuses on students attention to extract the salient information on the screen is visual attention[152]. The mechanism of “looking ahead” can be applied to those research areas to investigate how experts use several extended readout strategies in the course of solving a problem, and how those cue the experts concept activation and reasoning strategies. “Looking ahead” into the problem can help students to coordinate between new pieces of information available in the problem statement and the generalized knowledge that they possess.

One of the other areas in physics that require students to read the information from the screen and make decision and justification is in the lab environments. Future research can investigate the role of an extended readout strategy (“looking ahead”) in the lab environments. Students during the lab experiments may several times read the information from the devices and experiment to reduce their error or check for consistency between their experimental and theoretical results. Instructors by encouraging students to look ahead can help students to consistently use their knowledge of math in physics, which will leave more room for more sense making discussions.

For the quantum study of my dissertation, my first goal was to develop a framework by doing an existence proof. One way to expand this work for future use is to do the prevalence proof using this same framework. This framework is best applicable in upper division physics theory courses that require students to broadly use their knowledge of math in physics. Especially in group problem solving sessions by talking to each other, students provide an opportunity to the researchers to view students different ways of framing a problem. Using this framework along with observational-based methodologies can yield bright results. Researchers from the same research group[120] (Mathematization group) have done a prevalence proof in a tutorial-based course of electromagnetic fields 1 course. In this course, during the problem solving sessions, the instructor is present and gives feedback to the students. Researchers found that typically the instructor nudges students toward conceptual physics frame. Their data suggested that towards the end of the course, students were more likely to choose conceptual physics frame as an option for thinking about the problem. This study also can be conducted in homework group problem solving sessions in the presence of a TA or a learning assistant.

For instructional future work, by becoming familiar with this framework, instructors can more carefully design questions for use in their discussion or problem solving sessions. For example, designing and providing questions that have more than one pathway to answer, can trigger an expansive discussion in the class environment. Instructors based on their learning goals can either design problems that have multiple solutions, or design problems that encourage students toward a more particular way of thinking. Problems can be designed in such a way that dominantly trigger conceptual physics reasoning, conceptual mathematical reasoning, or algorithmic manipulations in students minds. This framework can also be used in the future to identify the frame of problems at the end of the chapters of upper division textbooks to see if they are consistent with the approach of the faculty that teaches the textbook or not. For example, if the approach of the instructor mostly relies on quantitative mathematical formalisms, but the problems at the end of the chapter of the textbook encourage more conceptual thinking, then this gap between students conceptual and algorithmic thinking has to become filled by other means of teaching. It is interesting to investigate if familiarity of students with characteristics of this framework can affect students' overall epistemology about learning quantum mechanics and their self-reflections on their solutions.

I spent roughly half of my dissertation talking about students processes in problem solving in physics. My first research question is how can I measure if students are coming to understanding while doing these processes. These process measures are necessarily grounded in qualitative research. One of the ways to expand this process measures as a future work is to connect them to outcome measures. For example, one way is to design survey questions that focus on the packed form of procedures that students have gone through in their in-class problem solving sessions. The way that “Alex” used the separation of variable method was very compact compared to procedural and unpacked way of using separation of variable by “Eric”. As instructors we hope our students after the course instruction and going through all the procedures of learning new things that towards the end of the semester students behave more like “Alex” and are able to use their knowledge as a solid unit[153]. By connecting outcome measures to process measures we can use outcome measures to guess what the process must have been; because even though the process measure is important, they are time expensive to measure.

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Appendix A

TRANSCRIPTS

A.1 Chapter 3

A.1.1 First interaction

Eric: Oh well this is...like just a sine waves...something like that.

Alex: Eh...We have separating

Eric: Wait where is that i

Alex: Yes that's just [Unintelligible]

Alex: So we can have them as separable ψ (pointing to the ψ in the equation) Capital Ψ is ψ of x times function of time.

Eric: Ok...Oh I see, I see.

Alex: Ψ of x (Alex starts to write the form of the capital $\Psi(x, t)$ as a product of two independent functions.)

Eric: Oh yeah so then these two each must equal a constant because...right.

Alex: This is only a function of time...(pointing to the left side of the TDSE)

Alex and Eric: (Eric becomes unison with Alex) And that's only a function of x (both pointing to the right side of the TDSE)

Alex: Well it will be when we divide by capital Ψ...so we get...so we get that $\frac{f'}{f}$ equals $-ih\frac{\psi''}{\psi}$...once you divide through (pointing to the whole TDSE).

Eric: Oh psi is double primed...

Alex: Because this is ψ'' times f and this is f' times ψ (pointing to each side of the TDSE) and then [unintelligible] divide through by capital Ψ (pointing to the whole TDES with gesture of hands with motion)...and then you get this (pointing to his previously separated written solution)

Eric: Okey okey

Alex: So then this equals k...some constant

This is with respect to x?

Eric: How did you like, get to that [unintelligible] you start (pointing to the right side of the unseparated TDSE)

Alex: We have $\frac{d}{dt}(f\psi)$...and then the ψ comes out (while drawing a path showing the motion of coming out), so we just have psi partial derivative with respect to t (Eric surprisingly says "oh...)

Eric: We assume it's like this because...

Alex: We assume it's separable.

Eric: Because we recognize that form.

Alex: Yeah... The assumption we are making is that it can be separated and do a function of x times a function of time.

Eric: Ok... Which is kind of implies because you have this time on one side and this ... and there is nothing else.

Alex: Yeah otherwise it will be really unsolvable (laughing).

Eric: Ok... Ok

Alex So we do this, and then on the other side we have $-i\hbar\frac{\partial^2}{\partial x^2}f\psi$ and then that (f) comes out (while drawing the bringing out motion).

Eric: Ok. Ok.

Alex: You then divide by $f\psi$.

Eric: Ok. I see, because f…because that's a constant, because of the…I gotcha. That makes sense.

A.1.2 Second interaction

Eric: The potential doesn't vary with time

Alex: The laplacian doesn't vary with time, so all caries out (using a hand motion), it's separable. let's write it down.

Eric: It just start with the…like

Alex: i …$\hbar$ partial ψ (writing)

Eric: Yeah start with the Schrödinger

Alex: And then the ψ of x and $f(t)$ with respect to t (writing)

Alex and Eric: And then equals

Eric: Minus $\hbar$ squared

Alex: Minus $\hbar$ squared…Laplacian of ψ (writing)

Eric: Over $2m$

Alex: Over $2m$ (writing)

Alex: ψ of x f of t plus V of r …I don't remember if its r or t.

Eric: But you can do the generalized form

Alex: What do you mean... f of t carries out

Eric: Yeah, yeah, yeah

Alex: And then you divide over and then this is not a function of t so you can divide [Unintelligible]...

A.2 Chapter 4

Ava: 0.00 Estimate the total energy in a typical hurricane system. State explicitly the assumptions that you make, explain your reasoning, assess your result? ...Estimate the total energy in a typical hurricane system (Reading)

Ava: 0.08: Oh this sounds horrid

Ava: 0.16 So What is a hurricane? Hurricane is mostly air that's moving in some sort of rotational system.

Ava: 0.38: Suppose emmm...I suppose its gonna lot to do with more high and low pressure areas

Ava: 0.43: But maybe that's a causing rotation so I don't have to count it seperately.

Ava: 0.52 Uummm

Ava: 0.52: Ok. I'm gonna start with just finding rotational energy of the hurricane system.

Ava: 0.56: Because I think that Sounds like something I can do laughing pretty reasonably.

Ava: 01.0 Ok so so I think...rotational energy delay for 3 seconds writing formulate of energy

Ava: 1:08 Ok...So rotational energy that's ahh!!

Ava: 1:22 Its one half M, R squared if you just got a point mass. If only Integrate something to get that for the point mass.

Ava: 1:30 But I think . . . How doing the integrate. Its been to long since I've done math.

Ava: 1:37 Ummm . . . let's say If you just have a point mass.

Ava: 1:49 Let's say. . .

Ava: 1:56 We can even don't do it that way

Ava: 2:03 I suppose I would be able to find the mass of a particular point because I do have the density of air. I'm sure there is a lot of water being [Unintelligible] around, but more air.

Ava: 2:21 Em I can estimate the typical.

Ava: 2:27 Oh I have a foot to meter conversion that sounds cool laughing

Ava: 2:33 Ahh I suppose like has to estimate the speed ahh

Ava: 2:41 Air movement like typical hurricane

Ava: 2:43 Ehh. . . I don't really know if there is any sort. . . if it is since like rotating properly.

Ava: 2:57 How a hurricane works I don't know maybe I can just say that.

Ava: 3:04 Ehh its all moving at some speed.

Ava: 3:12 Like obviously ok so if you have got a really squirrel hurricane at the center you don't have any rotation for some reason it like the eye of the storm and presumably out of the edge its goes to zero too.

Ava: 3:26 And I think its like the steepest right there not emm. . . like steepest the air moves the fastest then like the wall

Ava: 3:35 Everything I learned about hurricanes comes through magic school bus so this is hard.

Ava: 3:41 Emm so its obviously not on the moving like constant speed I've no idea with that.

Ava: 3:48 Distribution would be like some maybe I gonna guess like a constant about with the average speed of the hurricane I think it should be and then look at the total volume.

Ava: 4:00 Because I don't know if I can make a better estimated than that. Given my lack of knowledge of about hurricanes.

Ava: 4:08 Lets do that ok

Ava: 4:10 So lets see ... A high wind speed I feel like it's maybe around 50 miles an hour.

Ava: 4:20 I feel like I'm mostly just playing that numbers out of no where based on some intuition coming from watching the weather channel.

Ava: 4:33 So well I don't if that is high wind speed or high hurricane wind speed... So ok we've got some total volume of wind its all going at 50 miles-per-hour never together

Ava: 4:50 Energy equals $\frac{1}{2}mv^2$. So the mass is going to be the density times the volume of ahh... I should need more explicit

Ava: 5:06 Velocity versus volume, but, what ever this is physics.

Ava: 5:12 Emm ... emm so we said its probably mostly air although, ok lets look at the density of air which is the density of water. Water is significantly higher density.

Ava: 5:35 So maybe I should be taken into account because there is a lot of water than the air, its not negligible at that point.

Ava: 5:45 There is probably really saturated if its gonna be a hurricane.

Ava: 5:53 Emmm... its not a just a little thing... air ahhh

Interviewer: 6:05 What are you thinking?

Ava: 6:07 If I should just put the difference or or make some approximation. But water droplets are small compared to the total volume of air. There should be.

Ava: 6:24 Maybe I should lowball one of this estimations and go up and go on the other side of my other estimates

Ava: 6:32 Ok we are gonna with the density of air. I think that's we are gonna do here

Ava: 6:39 What gonna do here

Ava: 6:42 Ok the total volume. How big is our hurricane, so I'm thinking of might average weather map where using might conveniently drawn hurricane they usually look like this around florida a pallet? that big.

Ava: 7:06 So how big is florida. How big is U.S.have no idea.

Ava: 7:12 Emmm... probably the wind probably isn't going to be really as fast as toward as its edges maybe I should make a more conservative estimate there.

Ava: 7:24 Emm let's see just a order of magnitude how big should this radius be .

Ava: 7:36 I assuming it's a cylinder because that sounds reasonable

Ava: 7:46 Maybe What is it what is a good distance on that scale I can use to estimate.

Ava: 7:54 I think its probably ridiculous if just I think how big florida should be maybe how big is 100s of mile how big is 1000 of miles

Ava: Ok if I draw it from here to Missouri I put it like 500 miles 500 miles is pretty big then so maybe a 100 miles

Ava: 8:24 Maybe not the radius is 100 miles... Maybe diameter of 100miles Whatever I'll write out the formula

Ava: 8:36 So it would be, the height times pie R squared emmm... eheh so we are gonna say that the height is gonna be maybe a mile tall I don't think that's has an absurd height what else do planes fly at?

Ava: 9:12 Emmm like 100 feet oh yeah I in miles good I forgot to my metric intuitions so.

Ava: 9:21 Thank you for this helpful conversions. i.6 Km so we would put that 1 KM.height. Sure.

Ava: 9:31 Because that would be 1 Km times pie is three cool!

Ava: 9:43 ahh and then by radius minimum emm there maybe a 100 miles in radius?

Ava: 10:02 I don't know. sure Laughing. A 100 miles sure make that. 200 Km squared Ok now I'm gonna make an estimate of this velocity emm…right we were I was up we were talking I was talking

Ava: 10:33 That like 50 miles an hour seems alright for for a wind speed I have been lowballing these (Laughing)

Ava: 10:44 Maybe I should go big or go home and choose very high speed winds

Ava: 10:48 emmm…I don't think that's a reasonable estimation maybe like 80 miles an hour that should be 40 sec/sec by this conversion so we make that we just make that 50 meter/sec. Cool that's my speed

Ava: 11:22 ok emm Cool I guess I can start multiplying this up so got my factor got of a half. Times this density for 1 Kg/meter cubed spelling error…

Ava: 11:41 Emmm times ok 200 Km squared I just do this. I'm gonna multiply take care by estimate here Put 2 zeroes ok so 40000 …So 1 Km times 3 times 40000 Km squared times 50 meter/sec. Ok so that the half in so it's a 1 kg/meter cubed times

Ava: 12.48 emm…I don't need that one still 3 half 20000, sixty thousand km cubed times 50 meter per seconds.

Ava: 13:05 Ok emm…units make sense ok cool! Ahh then we got…I kg/meter cubed 60000 6 times 1, 2, 3, 4 cubed, 50 meters/sec.

Ava: 13:59 Then I'm gonna wanna put some sort of conversion from Km cubes to meter cubes.

Ava: 14:09 Emm ohh I can do this. . . that's 3 times over would be ten to the nine meters cubed equals 1 Km cubed.

Ava: 14:28 Ok so then we want the KM on bottom here to . . . 1 Km cubed ten to the ninth meters cubed cool!. . . Emm then we want to have got. Dude, This is huge I don't know if it is reasonable.

Ava: 14:55 Oh well six times 5 times ten my all powers of ten plus forth plus 10 to the one plus 10 to the nine because the pluses are in the exponential

Ava: 15:19 So that would be tenth to the fourteenth.

Ava: 15:23 And Then we can cancel out meter cubed meter cubed and then we got a Kg meter/sec which is the units that we would like I think its, that a joule that's unit of energy.

Ava: 15:40 Emm . . . so 6 times 5, 30 at add another order of magnitude we will say, tenth to the fifteen joule.

Ava: 15:53 For a typical hurricane system.

Ava: 15:55 ok ah

Ava: 15:60 I don't even know if that make sense, I don't have a good way to check that. I don't have any sort of intuition on amounts of energy

Interviewer: Ok emm . . . Could you talk more about what you are saying with the velocities in that picture.

Ava: Ok emm em. . . so if we look at a hurricane like this the extent of the hurricane I know that its not all of the wind in the volume of the hurricane that's moving at this speed because that is not making any sense it's not like this the edge of my hurricane and the wind here is going at what did I say a 100 miles an hour and this is going at zero. . . Because Hurricane system cant be like that

Ava: 16:58 So it kind of sort of distribution in there and I know that there is an eye of the hurricane.

Ava: 17:06 I remember that from, whatever weather was a topic in school.

Ava: 17:12 And the speeds there is zero Like its totally calm... and then right next look at the wall of the hurricane is extra fast there So The Speed mostly huge here, zero here and then decreasing some fashion probably approaching zero or close to zero.

Ava: 17:34 So its gotta be something like that.

Ava: 17:38 Yeah maybe there was a complete ridiculous estimate saying that the whole volume is going at that speed.

Ava: 17:45 Since its probably, When we are talking about this sort of hurricane like 50 mile an hour they probably just talking about that area right next to the wall of the hurricane.

Ava: 17:55 They probably are not talking about the edges...

Ava: 17:58 But I can't think I'm not probably make a much better estimate there in terms of.

Ava: 18:12 I can make a guess of that function but it seems like a much harder problem perhaps my other errors in not knowing how big a hurricane is would be far far greater than my ability not to be able to identify a good function there. Ok.

A.3 Chapter 5

A.3.1 Clips 1-2

Jack: : So Laplacian.

Dan: Yeah

Jack: Is that equal zero?

Dan: Well ehh, the equation is equal to zero.

Jack: So x duble prime yz plus y double prime xz

Jack: So z, there is no z right!

Dan: Ahhhhh

Jack: So z is everywhere. It does vary with y.

Joe: Yeah z is constant

Joe: It does vary with y and y, actually no it just varies with y, I think that wall will makes it vary with x.

Jack: Yeah

Joe: I think with that distribution.

Jack: Eh from here to here,

Joe: You will get like,

Jack: Oh a it will be, dependent on y, right! Joe: then you get

Jack: then you get zero at x equal 0

Joe: Would you get a parabola shape.

Joe: Yeah!

Jack: It looks like, I don't know. it looks like sine though and cosine

Joe: Ahhh...

Jack: Because this goes zero to zero.

Joe: Yeah

Jack: On y

Joe: This one side a constant

Jack: Yeah

Jack: We are quite over here brother.

Tom: Yeah I'm jumping ahead a little bit

Joe: So that one goes to zero because there is no z dependent

Jack: Yes

Dan: Z goes to zero

Joe: Z double prime, there is no z dependence.

Dan: Gotcha you

Joe: It's a constant.

Jack: x double prime yz plus y double prime equal to zero

Joe: And then you guys get rid of the $z?s$

Jack: z (double prime) is cancelled

Dan: Divided over by xyz

Joe: Can you set z equal 0

Dan: Yes the equation is set equal to zero

Tom: Which parts you guys are working on

Jack: Then what we were looking at here was, there is no z dependence.

Jack: Like this top wall is equal to zero, side wall is zero, bottom is zero, and this here is V of y

Jack: So this part here varies with y

Dan: Divide your original equation by xyz, so you get x double prime for x plus y double prime over y plus z double prime over z

Tom: So how is that V is z independent.

Joe: z is constant, so z double prime goes away.

Tom: How is z a constant?

Joe: There is no dependence on z, there is only one wall, and all down the wall has a constant charge, and other three are equal to zero. No matter where you go in z, it looks the same.

Tom: So it says that it's constant with zero, because I am interpreting this as

Dan: It says fourth wall maintains a potential that varies with y and that?s it. So it doesn't vary with x, because x is the same, it doesn't vary with z, so z has to be a constant.

Instructor: Potential does not vary in x?

Jack: It does vary is x

Dan: Well it does vary in$x$, yeah, I mean on the wall. On this wall it's different than this wall.

Joe: But is doesn't vary at all in z

Instructor: Right! potential does not vary in z. So the function Z of z, what is that equal to? some unknown constant or some variable.

Dan: Constant of z

Instructor: Some constant

Dan: Yes

Instructor: Is that constant equal to zero?

Dan: : No

Instructor: Could it, I don't think it will be equal to zero.

Jack: Oh no! you are right, you are right.

Dan: : Then we will get potential be zero

Instructor: Then your potential will be zero, and that's, that?s definitely not zero, definitely something not zero.

Jack: But that's ... sounds right yeah! (pointing to the equation)

Instructor: Because the second derivative of a constant is equal to zero

Jack: Yes

Dan: So we this thing.

Tom: So we can just say a is equal to a ... are we finding potential inside or just on the walls

Dan and Jack: Inside

Tom: So still we have to treat x as variables.

Tom: I thought my xyz are constant but I was wrong.

Dan: When do we pick up our... like the k?

Jack: Coming up, when you set your BC.

Jack: Well at here. (showing the 2nd page)

Dan: I'm at that point.

Jack: These both equal to a constant.

Jack: Because it is a 2nd derivative, it's probably c squared. what ever k squared.

Jack: Makes sense, so this gives minus c squared, this is c squared (k squared).

Tom: You are saying all that equal to zero.

Jack: Because that's what the equation says.

Tom: Yeah

Tom: So you just got those to separate them out, just as we did in quantum. Divide by z.

Jack: Yeah z cancels on both sides.

Joe: So that means, that are trying to solve x, this side is constant and we are trying to solve for x, and this side is constant trying to solve for y.

Tom: . . . minus (thinking out aloud)

Jack: Right!

Joe: So we have to get negative k squared something.

Jack: This thing, I thing is negative k squared (Y equation) and this thing (X equation) is k squared.

Joe: And then you have the Helmholtz equations.

Jack: And then you have the Helmholtz equations was easy to solve.

Joe: Well I think what we need to do here, I'm not sure about [Unintelligible] we know it's gonna be either sine and cosine or exponentials.

Joe: Yeah

Dan: I think how we did it, how we solved it first time, it was x equals to e to the like i something.

A.3.2 Clip 3

Jack: Okay. So X is gonna be exponential. and Y be sinusoid.

Tom: Oh yes.

Dan: Well they can both be exponentials.

Jack: That's what she is saying here.

Dan: Like your exponential for the Y will be imaginary. Yeah you have complex exponentials for Y.

Tom: It's exactly solving this equation. I hate jumping ahead and say hey oh it's sine.

Jack: Well the only reason that you go to sine here is that because potential is zero.

Tom: No Jack, do not apply the B.C.s yet.

Tom: What did we learn last semester about being rigorous man. We spent lot's of hours to doing this.

Jack: It's true

Dan: So it will be sine of y or or sine of something else.

Jack: Go ahead

Jack: It would be e to the [Unintelligible]

Tom: It would be e to the iky

Dan: I mean we are not doing. We are just guessing y equal sine

Tom: That's cheating. But you could say sin of kx

Joe: [Unintelligible] Every body was happy to apply it rigorously.

Dan: That habit.

Tom: This way is always right.

Joe: We were trying to do in quantum and...

Tom: I'm gonna call this λ, x in general,

Joe: Y is a function e to the iky

Jack: Is e to the iky, some, that would be like A e to the iky plus B e to the minus iky. Is that what you are doing?

Tom: I just called my A and used α and β.

Joe: And then e to the negative kx.

Tom: Now it will be positive.

Dan: So we need random constants in front of our variables.

Jack: Yeah yeah. you will be taking those away by plugging your B.C.

Joe: yeah I skipped those steps.

Jack: That's okay.

Joe: So A e to the kx plus B e to the $-kx$

Jack: Yeah

Jack: [Unintelligible] With the imaginary

Tom: You are trying to jump ahead. You can't include I at the very beginning. So solve...

Dan: What do you mean you can't include I at the very beginning.

Tom: You can't assume that it's complex. Math will work out. So set y e to the λ, y

Dan: λ being k?

Tom: You are gonna solve for that.

Joe: But then we have to do two terms in y also

Tom: Because you will take the square root and that gives you the plus and minus. So I subtract these and find out λ square equals to minus k squared. So that implies λ has to be equal to our $k\ldots$ plus or minus

Joe: So we will get that

Dan: Yeah it's plus or minus.

Joe: It's too [Unintelligible]

Jack: [Unintelligible]

Tom: So I'm solving for the B.C. generally, do not apply the B.C. yet.

Joe: I think you have to leave it as A to the e what ever to the B to the e whatever

Tom: That's what I'm thinking. Right! so I have got these A and B, and I used α and β

Joe: Right! so

Dan: Why do we have two terms.

Tom: So any time you have a second order differential equation, you are gonna have two solutions.

Dan: Ok

Tom: If I have a third order I have a three solutions.

Tom: And how I get my two solutions is, I get down to this step and my exponentials cancels out right. So I get λ squared k squared. Well if I square root. I will get plus and minus k.

Dan: Ok makes sense.

Tom: And the same thing happens here, but over here we have that negative sign, so we find that λ squared is equal to negative k squared.

Dan: Yeah

Tom: So λ equals plus or minus square root of negative k, which is plus or minus ik

Dan: Right!

Tom: That's what I'm talking. We don't assume that k is complex from the beginning but the math will tell us [Unintelligible]

Tom: We actually did a lot of this.

Tom: This time I actually think about it. Joe: this was like a raw in a test in quantum.

Tom: I mean this was like part of the course is if you could not solve this differential equations in a matter of 2 seconds, you are not gonna finish the test.

Dan: Are we going to convert this back into sine and cosines.

Tom: I haven't yet, because it asks me to solve the differential equations generally. That's the general solution.

Dan: Ok

Tom: Now when I start the B.C. that's when I want to know sine and cosine from the B.C.

A.3.3 Instructor encourages Looking ahead

Instructor How are things over here guys?

Nick We think we are going alright. I guess, I speak for myself.

Instructor One of these are sine and cosines, and one of these are exponentials.

Nick Uhum =...

Lee Yes

Instructor Which one is which?

Nick X which is negative is sine and cosine.

Instructor Okay! Does that make sense physically?

Nick How can you tell whether it makes sense physically?

Instructor Well you can look at the problem.

Instructor One of these as you go across is gonna look like the particle in a box that get's to be sine and cosine. . . and one of them as you go across is gonna look like not particle in the box

Chris I think the X looks like exponential

Instructor The X look like exponential?

Chris Yes. Because, this is (wall at $V = V_0(y)$) . . . like, this corner is constant, so as you move away from this it should go down.

Instructor And as we go from top to bottom here, this is zero and this is zero, so sine and cosine makes a lot of sense in the vertical part.

Nick Yeah. . .

Nick Okay. We should move our negative. . . is this what you are saying.

Chris Yes, it makes life much more easier!

Nick I'm just gonna change all my Y's to X's

Chris Yeah. . .

Nick Yeah, that was easier.

A.3.4 Taylor series for a line of charge

43 Dan Are we taking...ehh...the function...ehh...ln$(1 + \epsilon_o)$ here as the function?

44 Lee You are saying that ϵ is equal to...

45 Adam ϵ is just an approximate of...just be d...

46 Dan d over z

47Adam With a large z greater than d then epsilon is very small

A.3.5 Taylor series expansion for a uniform charge ring

Ed So we need to expand this function, that's a constant.

Matt Are we be able to say a_l or b_l are zero at this point.

Ed a_l equals to zero 0 because at infinity we can't have that r^l.

Ed The first one is just one, so you are doing b_1 over r squared plus, it's not over r squared. So b over...over r. And then the next term, over r squared times cos...theta.

Ed Yeah but theta is zero

Matt Oh so they are all one. Yeah they are all one.

Instructor Yeah they are all one.

Matt Em..., since we are on the z axis all the P_l terms are one.

Ed Yes because cosine theta is one

Instructor It is true that cosine of zero is one, and that gives us P_l of one, all of the P_l terms are now one. That's the property of the Legendre series. $P_l(1)$ is one no matter.

Ed Oh yeah that's what basically what we are saying.

Matt So r is z, we have to change all r to z in Legendre polynomials.

Ed We need to expand this one (potential of the uniform ring of charge) ... we need to expand this one ... [unintelligible] we are not gonna expand this... is there a ... does wolfram do the expansion for this

Larry Yeah

Ed (Matt scrolls down the page with the mouse) Down ... down ... down ... scroll it down

Matt It's not scrolling... there is an expansion

Ed Maybe we just gonna use that ... we'll take the first two terms of that... wait is this ...

Matt This one is about zero (Matt looking at the screen)... we are expanding about zero... right! (Matt looking at Ed)

Ed Yes, because the ... we're ... P is in infinity (Ed uses hand gesture to show the point P is far away)

Matt I like, that they (Wolfram) put a squared on every thing ... that's silly!

Ed I like the bottom one. what's the... what is x equal to [unintelligible]?

Matt That one is about z equals infinity.

Ed Well z does go to infinity, ϵ being d over z... goes to zero.

Matt Oh... Ok. Ok.

Ed I can't read the second term is that z cubed. so we have to have quadrupole term... because b_1 is zero right! is it possible we do not have a dipole term, but have a quadruple term.

Ed So when we expand the Legendre polynomial we get the first few terms. But we expand this (the potential), there is no dipole. then that means the b_1 term is equal to zero.

Ed Is that ok that the dipole term is equal to zero, but the quadrupole comes back?

Instructor Yes it is ok for some terms to be zero, and some terms to not be zero. And there should be some kind of pattern to it. But is doesn't have be once, there could monopole term, and no dipole term. So if you are doing a pure point charge,

Instructor and Matt Just be a monopole and nothing else. . . If it has no net charge then it should not have a monopole, but depending on the separation you might get a quadrupole.

Matt Well a pure dipole the separation is infinitesimal right!

Instructor So for that you should only get a dipole term but if you have, but you have some physical dipole separation then you should get a quadrupole.

Instructor Find the coefficients?

Matt So these are all zs, b_0 so is equal minus 1, b_1 is equals a squared. no no no. b_1 is equal to zero.

Ed b_1 is equal to zero. b_0 is equal to, did you already solved b_0?

Matt b_0 is equal to minus one.

Ed Why?

Matt Because we have our taylor here, oh wait it's minus one times all the crap. Sorry I forget about it. . . I forgot about the constants.

Ed 2 *lambada*. . .

Larry You can simplify 2, , *lambada* to

Matt b, to the ϵ_0 is it a to the fourth.

Matt b_0 should with all the constant minus a over 2 ϵ_0, d, yeah I think so.

Matt b_1 is zero, b_2 is going to, let's see. So a squared over 2 times a, λ.

Larry I think it's a squared [unintelligible]

Matt No it's a squared because you wanna match up with the powers of z nod.

Ed And the expansion is b_0, is a cubed?

Matt No it's a squared

Matt That's a two

Ed Oh then it is a cubed. You have an a term that is a constant. Well it is a squared over two times the constant but yes

Matt So you end up with a cubed.

Larry How many terms are we supposed to have?

Matt Enough to make a pattern. I like to have at least, three non zero terms.

Ed So what is the answer. I don't know what is the answer. oh I get it.

Matt So the answer is we solved for b_l and then we plug it back in.

Ed Yeah gotcha you

Larry So there are only even terms

Matt That is true. I guess.

Larry So b_1 is zero and b_3 is also zero. only even terms.

Ed you are right. Is it all minus?

Matt No they are not all minus

Larry It alternates

Ed I lost my minus signs then. So it's plus minus sign...

Matt So b_3 is zero. . . I like to have three non zero terms to make a pattern, besides we have seven taylor series. terms.

Ed What is our third taylor series terms

Larry Minus three over fourth eight

Matt So then our first three non zero terms are minus a, λ over two. . . right!

Larry Yeah I think so.

Ed What was the b_4 term?

A.4 Chapter 7

A.4.1 Adam, Emma, and Eric' group

Emma Is it $\hbar$ over r? Is it i $\hbar$ I can't remember.

Emma i $hbar$, d, dt equals . . . is it $\hbar$ squared over $2m$ (writing $\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t}$)

Emma $2m$ negative

Emma This is what I have, good we agree. (very quickly reviewing the facts) $F = 0$, $V = 0$, . . . Separable.

Eric So this is like the infinite [square well], except for we don't have boundaries.

Emma Are you part of our group?

Adam I can be

Emma Ok

Eric Awesome!

Emma Can you finish writing the answer since you . . .

Adam So to satisfy the time independent Schrödinger equation, right

Emma Yeah...so write the $\hat{H}$

Adam Ok so we know that H *psi*, E, *psi* right! (writing $\hat{H}\psi = E\psi$), So we know that minus $\hbar squaredover$2m,dψ, d x is equal to E, ψ (writing $\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi = E\psi$)

Emma Right!

Adam But...Ok this can be anything...right!, so we have ψ with respect to x, is $2mE$ over $\hbar$ squared (writing $\psi'' = \frac{-2mE}{\hbar^2}\psi$)

Emma Over $\hbar$ squared

Adam The minus sign (writing $\psi'' = \sqrt{\frac{-2mE}{\hbar^2}\psi}$)

Eric Oh ψ double primed is with respect to x ok

Adam Wait...Yeah I'm just saying

Eric So the primes will be space and the the dots will be time

Adam I should have wrote just $2mE$ here we go over $\hbar$ squared, right!...ψ. So we call this k and then k equals

Adam and Emma k squared equals

Emma k squared equals minus $2mE$

Adam Negative $2mE$ over $\hbar$ squared, right! (writing $k = \frac{i}{\hbar}\sqrt{2mE}$)

Emma And now we set it up as...

Eric Oh because of this]...

Emma Yes you don't have to write it up

Eric [Unintelligible] Yeah ok

Adam So k is gonna be i over $\hbar$ root of $2mE$ (writing $k = \frac{i}{\hbar}\sqrt{2mE}$)

Emma Aha

Adam Ok, anyway

Emma

Emma (interjecting) and then we do the e to the sign

Adam So psi is A e is

Emma To the λ

Eric Some constant

Adam To the ...

Emma You don't want to do the bit where we write λ and then we find λ

Emma e to the minus

Adam x, λ (writing $psi = Ae^{\lambda x} + Be^{-\lambda x}$)

Emma Oh, no... alright! so now we want to find out what λ is

Adam It is just square root of K (Adam points to equation 8). No wait it's K, if we define the square [of] K by that... (Adam points to equation 6)

Emma It is k... yeah

Adam So our space part is just e to the i, square root of $2mE$ over $\hbar$... (Adam writes equation 10)

Emma Just write K... Just write K...

Adam Plus B to the...

Emma Why would you not just write as K.

Eric Because this is the real name…I don't [unintelligible] (Adam finishes writing equation 10)

Emma You left x out (laughing)

Adam Oh thank you very much so this is the space part right, so all we need to do is multiple by the space part

Emma Which is just gonna be e to the i e [unintelligible] E over $\hbar$ or not E, E_n over $\hbar$ and there would be a constant here but there is a constant here so…so we don't care

Adam Let me see if I do this twice then I get this, now for the time part

Emma You only do it once

Adam Yeah but the time part if just here, you end up with i, $\hbar$, d, ψ over d, t divided by ψ or the ϕ, the time part equals a constant right!

Emma Yeah

Adam So ϕ has to be, ϕ time whatever

Emma E to the i

Adam What we wanna call this…What do we call this constant here?

Emma I don't know, I also normally call it k, which gets confusing

Adam μ, i, $\hbar$ right!, so then ϕ has to be e to the minus, i

Emma Constant over $\hbar$

Eric Joe, Joe, ha ha ha

Adam Oh so the time part is also real ha…I thing that's what I got

Emma Why did you put the minus down here?

Adam Well what I did was I wanna solve this equation and, right! so if I take a time derivative I better get μ, oh wait a minute, you are right!

Emma You don't want the i down here. You wanna leave it out

Adam I wanna change this to negative i.

Emma Yeah that's what you did

Adam Yeah cool! That's it. But this constant is related to k somehow.

Emma Yeah because its related to the energy. We wanna write this as, we wanna be as ψ, x, t equals... so we have the nonsense x must be e to the negative nonsense x, and then e to the negative i

Adam I think we should plug it in

Emma Constant t all my constant are always like k, It doesn't help, so it isn't like a good system.

Adam We didn't call this square root of k just k right!

Emma We call it nonsense

Adam This is and minus , with my k right here

Emma Yeah

Adam Ok good so we need to relate k and μ right! So you put that into the Schrödinger equation, so i, $\hbar$ d time, Hamiltonian times $\psi. Ok, so for time derivative you get \psi$ time this, umm, whatever this constant is right

Emma Right

Adam i, $\hbar$ because all this is constant,... time... you get minus μ over $\hbar$, time

Emma μ over $\hbar$ and then the exponential

Adam And then you get everything else back, so you get ψ again, so then the Hamiltonian

Emma You left out, wait, are you leaving out this part

Adam No I'm putting the whole thing into it, the entire equation.

Emma Oh Ok. I see

Adam Because still you end up with this whole deal right here,

Emma Right!

Adam So then for this you end up with kx, minus kx... oh wait the sign is changed right!

Emma Why do you want to change the signs?

Adam If I differentiate this twice, I just end up with $2k$, k squared in the front right!

Emma Yeah

Adam So all I get is minus $\hbar$ squared over $2m$, d over dx, but that's just ahh, k squared times ψ

Emma There is an i in the k, does that matter?

Eric Yeah that's ok , it should be...

Adam Yeah that's ok

Emma Because if you square the i, it will get negative.

Adam Yeah it will get minus

Eric You should having i, because the they are going to be sinusoid

Adam So this is..., we can just μ and k, so cancel the ψs

Emma Ahh, these $\hbar$s are gonna cancel, these is are gonna cancel.

Adam So just one, so just μ equals $\hbar$ squared over $2m$, times k squared [$\mu = (\frac{\hbar^2}{2m})k^2$].

Eric What's μ?

Emma μ was our constant from when we were doing this part (pointing to the time derivative part of the TDSE).

Eric For time?

Emma and Adam Yeah. When we were separating.

Emma Because we have $e^{-i(\mu/\hbar)t}$.

Adam So minus one over $\hbar$ $2mE$, boom, boom

Emma Cancel cancel cancel we get $\hbar$ E

Adam Huh?

Emma Is that wrong, because you have the $\hbar$

Adam Joule seconds is Joule, Yeah I don't know if that's write

Emma Oh wait wait because $\hbar$ is here, and that will cancel with this one, and then we'll have.

Eric It cancels out [unintelligible].

Emma Up here because we plugged the...

Adam Oh wait a minute because we... hold on hold on hold on, wait I know what I did. Emm... I did not square this. So its just E. its minus E. Yes ... (while raising his fist in a triumphant gesture).

Emma Minus is to the positive [unintelligible] is it comes out [unintelligible] negative with the negative

Adam Yeah I guess it doesn't matter

Emma But we didn't normalize it yet, we need to normalize this. Do you want to start normalizing?

Adam Not really, I will just write this out

Emma Are you writing this part

Emma So i, $\hbar$ times i E, ψ or don't write ψ, equals k squared

Adam Yeah I'm gonna just cancel

Adam Right! so this is...

Emma We get E equals minus [unintelligible] $\hbar$ squared, over $2m$, k squared (1)

Adam Minus $2mE$ over $\hbar$ ... squared ... Boom ... Boom ... Boom (while canceling the same quantities from two sides of the equation) (2)

Emma Cancel, cancel, cancel, and we are off by a negative (2)

Adam Yeah [unintelligible] sign

Emma With a negative up here, because these two are negative.

Adam Yeah that is true

Emma So something happened here (pointing to the two sides of their equation) (3)

Adam Or we lost a sign (3)

Emma But I think we can start normalizing, we should start normalize now.

Adam Oh wait is [unintelligible] has to be a negative

Emma You can keep looking at that work on that and Eric and I will normalize it

Emma What (laughing) You can just tell me what to write

Eric A star B e to the $2i$, kx

Emma Positive?

Eric Positive this time, and then integrated from minus infinity to infinity

Emma So obviously this is coming out . . . oh right! it will be form negative infinity to positive infinity

Emma AB, B star, e to the negative $2i$, kx, dx plus

Emma Did you figured out where the negatives comes [unintelligible] yet?

Adam No

Eric We are gonna have some infinities right here

Emma We will do it just from here

Eric So this part

Emma We won't worry about that yet

Eric infinity minus infinity

Emma So we can pull the AB to the star out, and then what's the integral of e to the negative $2i$, kx, dx, one over $2i$

Adam These are conjugate of each other, so the conjugate of an integral and the integral of a conjugate. . . That's interesting

Emma We don't have to worry, this [the exponential term in the numerator] is gonna blow up faster than this [the denominator], right!

Eric They both blow up

Emma Yeah, But one blows up faster and that matters

Adam Definitely the exponential (points to numerator), So goes to infinity this goes to zero [unintelligible] this goes to zero this goes to infinity

Emma Yeah…

Adam Same problem here

A.4.2 Alex and Robert's Group

Robert I think….

Alex That's time independent…

Robert Yeah…Why do we need time? hmm?

Alex Hmm…Because the wave function might have it.

Robert If there is no force then…um…why would anything about the wave function change over time?

Alex Because the wave function might depend on time…from its initial condition.

Robert I don't think it did. At least…[unintelligible].

Alex Oh, okay.

Robert (writing math as he speaks) Whole definition is… $T + U$…Zero (crosses out U) is, uh 1/2… mv squared. This is p… $\frac{1}{2}pv$…and v equals $\dot{x}$ So H is $1/2p\dot{x}$

Alex You can have that H equals…or T equals $\frac{p^2}{2m}$. It's skipping all of this.

Robert I couldn't remember what it was on the test, I went through it this way to get the…

A.4.3 Eric's episode

Eric I think…Cause when you do the, um, absolute value, you have to multiply by the complex conjugate, so I'm pretty sure that e thing [complex exponential part of the wave function] will just go to one, cause you'll replace that with…that e to the minus blah blah blah with e to the plus blah blah blah, and then when you multiply the... 1 over, you know, x over x. That's what I'm thinking.

Eric But I guess we could ... alright! Squared dx equal to the probability distribution I guess its just the same as the function [unintelligible] I should say ... anyway

Eric It is time dependent? Why? (While the instructor is explaining, he works on his paper) There's cross terms! Stupid... (smacks himself on forehead) ugh... That's why. Okay. Ugh, so stupid.