

MATHEMATICS KNOWLEDGE FOR TEACHING OF ELEMENTARY AND SECONDARY
TEACHERS WITH REGARDS TO DIVISION BY FRACTIONS

by

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B. S. Ed., Pittsburg State University, 2000

M. S., Pittsburg State University, 2002

AN ABSTRACT OF A DISSERTATION

Submitted in partial fulfillment of the Requirements for the degree

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Curriculum and Instruction

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Abstract

This study examined pedagogical content knowledge (PCK) and Mathematics knowledge for teaching (MKT) from a perspective that blends existing definitions, questions, and methodologies into a unique method of collecting and analyzing data. Many studies on MKT have been done using qualitative methodology, usually with the researcher interviewing or testing individual teachers and analyzing that data using qualitative methods. A smaller number of studies have attempted to measure MKT using a quantitative approach, often times involving paper pencil tests with multiple choice and some open ended questions. Current research is also heavily weighted towards pre-service elementary teachers in the area of MKT for division by fractions (Depaepe et al, 2013). Although it may be true many pre-service elementary teachers have difficulties with division by fractions (Li & Kulm, 2008; Ma, 1999; Tirosh, 2000), we do not know if these problems persist for in-service elementary teachers or with secondary teachers at any stage.

This study used a survey created by the researcher as a synthesis of existing questions from research on MKT with regards to division by fractions. This survey was delivered through an online format. The qualitative data in this research was then coded into quantitative data using a rubric developed by the researcher producing MKT scores that could be analyzed using statistical methods and generalized to a larger population. This study examined the Mathematical Knowledge for Teaching held by elementary, middle, and secondary in-service teachers in the domain of division by fractions. In particular this research asked if there was a relationship between the educational background, training, and experience of teachers and their MKT.

Multiple analysis including independent t-tests, independent one-way ANOVAs, and Multiple regression analyses revealed that Middle school teachers and teaching at a middle school were significant predictors of increased MKT scores when compared with other groups of teachers. The type of teaching license, the type of degree held, total experience, and grade level experience were all found to have no significant relationship to MKT scores.

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Approved by:

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Chapter 1- Introduction

George Bernard Shaw wrote, “Those who can, do; those who can’t, teach” (1903). This oft repeated quip questioned the competencies of teachers in their prospective fields. Are teachers competent and knowledgeable in their prospective fields? In the area of mathematics instruction, the question is whether mathematics teachers are skilled and knowledgeable in doing mathematics. A second and related question posed by Shulman (1986) relates to knowledge teachers require that is specific to the profession of teaching. In other words, what is the type of mathematical knowledge that relates specifically to the task of teaching that is not needed outside the realm of teaching? When teaching mathematics, what knowledge is essential for instruction? The answers to these questions support the idea that teaching mathematics requires a blend of pedagogical and mathematical knowledge.

In the last three decades, an area of mathematics education research has grown up around the idea that there exists a body of knowledge unique to teachers of mathematics (Ball, 2000; Davis & Renert 2012; Depaepe, Verschaffel & Kelchtermans, 2013; Hill, Schilling & Ball, 2004; Shulman, 1986). Shulman (1986) referred to it as “Pedagogical Content Knowledge” or PCK and defined it as “the particular form of content knowledge that embodies the aspects of content most germane to its teach-ability.” More recently, Hill et al, (2004) have applied Shulman’s theoretical construct of pedagogical content knowledge to the domain of mathematics education with their conception of “Mathematics Knowledge for Teaching” (MKT).

A broad and diverse field of studies now exists related to PCK and MKT. Studies throughout the last three decades have revealed some areas of concern regarding the MKT of some teachers. Several studies have demonstrated deficiencies in elementary teachers’ MKT

with regards to conceptual depth in the operations, reasoning and proof, word problems, multiple representations, and other topics (Eli, Mohr-Schroeder & Lee, 2013, Isik & Kar, 2012, Simon, 1993, Behr, Harel, Post, & Lesh, 1992). Less is known about secondary teachers. One hypothesis is grounded in observations about the current practice of teacher preparation. Secondary usually obtain a major in mathematics and minor in pedagogy. Lo and Luo, (2012) purposely excluded mathematics or science majors from their study. They reasoned that the results for teachers' PCK would be skewed because secondary teachers were required to take many college-level mathematics courses. Pre-service secondary teachers often take most of their classes in a mathematics department and receive instruction from professors who specialize in mathematical content. These professors and instructors may or may not have expertise in the area of pedagogy. Studies comparing the MKT of elementary and secondary teachers are rare (Depaepe et al, 2013). Conversely, elementary teachers major in education and often have a limited number of content courses in the areas of reading, science, music, and mathematics. These courses tend to focus more on the methods of instruction in the context of reading, science, mathematics, etc... These courses are taught by professors who specialize on pedagogy, often from an education department. In other cases they may be taught by someone who is a professor in the particular area of content. Some pre-service teachers receive training in methods for teaching mathematics from both mathematics and education professors. It is unknown whether professors and instructors in education and mathematics departments have significantly different knowledge of mathematics and pedagogy. It is also unknown whether the differences in training for elementary, middle, and secondary teachers influence their mathematical knowledge as it relates to teaching. This research examined the relationship between the different

educational backgrounds of elementary and secondary teachers and their MKT with regard to division by fractions.

Overview of the Study

The specific domain of study in this research was related the Mathematical Knowledge for Teaching (MKT) with division by fractions. Students and teachers in elementary schools consistently have difficulty understanding the concept of division by fractions (Behr et al, 1992; Simon, 1993; Tirosh, 2000; Zhou, Peverly, & Xin 2006). Little research of a similar kind exists for secondary teachers. Do secondary and middle school teachers have similar MKT, or does their extensive mathematical training create a deeper and fuller MKT when it comes to division by fractions? This research examined the Mathematical Knowledge for Teaching held by elementary, middle, and secondary teachers in the domain of division by fractions. In particular this research asked if there was a relationship between the educational background, training, and experience of teachers and their MKT. This research also examined in-service teachers in response to the large body of existing research on pre-service teachers' MKT (Depaepe et al, 2013; Hill et al, 2005, Hill et al, 2004; Lo & Luo, 2012). This research asked whether teaching experience had any effect on MKT for elementary, middle, and/or secondary teachers.

Statement of the Problem

Current research is heavily weighted towards pre-service elementary teachers in the area of MKT for division by fractions (Depaepe et al, 2013). Part of the reason could stem from the fact that division, fractions, and division by fractions are first taught at the elementary grades. Although it may be true many pre-service elementary teachers have difficulties with division by fractions (Li & Kulm, 2008; Ma, 1999; Tirosh, 2000), we do not know if these problems persist

for in-service elementary teachers or with secondary teachers at any stage. How rare or prevalent is a deep, conceptual, or profound understanding of mathematics at the secondary level? Ma (1999) compared and contrasted the knowledge of division by fractions in her comparison of elementary teachers in the United States and China. Very few examples exist in the research that examined secondary teachers MKT. Of those studies that do examine secondary teachers MKT, few of them have been done with U.S. teachers (Depaepe et al, 2013; Kraus et al, 2008). There is research that supports the idea that teachers who have taken more courses in the subject matter they are teaching tend to have students with higher levels of achievement (Chiang, 1996; Chaney, 1995; Monk, 1994). This research looked at several indicators of a teacher's preparation that may have an influence on their MKT.

Research Questions

The research questions in this study linked a broad and diverse field of research on MKT and division by fractions for elementary teachers with a more diverse population. The inclusion of middle and secondary teachers, as well as selecting from in-service rather than pre-service teachers, allowed the researcher to ask the following questions:

1. Is there a difference in MKT (Division by Fractions) between teachers when grouped in the following ways?
 - a. Elementary(4-5), Middle (6-8), and Secondary (9-12)
 - b. Elementary(4-8) and Secondary (9-12)
 - c. Those who hold a degree in mathematics and those who do not.
 - d. Type of license: Mathematics, Elementary, Early-Late Childhood, Other

2. Which of the following variables are the greatest predictors of MKT (Division by Fractions)?
 - a. Elementary, Middle, or High School
 - b. Type of License
 - c. Grade levels licensed to teach
 - d. Grade level Experience
 - e. Total Experience

Purpose of the Study

The purpose of this study was to examine the MKT of teachers across a wide range of grade levels, experience, educational background, and areas of instruction. The particular strand of MKT was in regard to division by fractions. The researcher investigated if there were significant predictors of MKT among the individual independent variables of type of license, grade levels taught, years experience, educational background, or some underlying factor(s) composed of a combination of those variables.

The study extended existing research in the following ways:

1. The researcher tested an instrument that needed empirical verification in its validity and reliability. The instrument in this research was composed of questions created by Ball (1988) that have been used by other researchers for investigating teacher knowledge with regards to division by fractions (Ma, 1999). Original questions composed by the researcher were created using existing definitions from the literature.
2. This instrument was an attempt to synthesize definitions from existing research in a way that quantifies the theoretical construct of pedagogical content knowledge as defined by

Shulman (1986), addressed by Ma (1999), and further defined as Mathematics Knowledge for Teaching by Hill, Schilling and Ball (2000, 2004, 2008). Each of these researchers has taken steps to move MKT from a theoretical heuristic towards an empirically measurable quantity. This instrument is composed of questions that measure MKT using various aspects of existing research in one narrow context for a broad spectrum of teachers.

3. The researcher was unique in the comparison of elementary, middle, and secondary teachers of the predominant sampling frame of pre-service elementary teachers used in previous research. This sample allowed for comparisons within and between groups with varying experiences, licenses, degrees, and diverse backgrounds.

Limitations

This research was a part of a doctoral dissertation. The principal researcher in the study was a full time secondary high school teacher within the district where the research occurred. Several limitations affected this research. The researcher was limited in financial resources with which to fund incentives for participation. The timeline for the research was the spring semester of the 2015-2016 academic school year. Future research should be done on a larger scale among a more diverse sampling frame. This study was confined to a single district because of the limited time and resources necessary to collect and analyze data from a broader sample space. This research was quantitative in design. The researcher had no face to face contact with most of the participants, nor the opportunity to develop a depth of understanding of individual teachers' MKT. The greatest limitation and threat to external validity was the method of sampling. This research failed to achieve random sampling which is one of the essential assumptions of inferential statistics. Although this research did not meet the requirements and assumptions

necessary for inferential statistical analysis, this was a result of a lack of resources more than a methodological or design flaw. It is extremely significant to point out that a study performed under an almost identical design with a larger randomly sampled population would allow for inferential statistical analysis that could be generalized to a larger population. This benefit should not be underestimated as researchers have been attempting to develop large scale instruments and study designs for measuring MKT of teachers on a large scale (Ball, Thames & Phelps, 2008, Hill, Rowan & Ball, 2005).

Delimitations

This research was framed using a cognitive theoretical perspective of MKT rather than an experiential perspective which views MKT as situated within the context of the act of teaching. This perspective focuses the attention of the researcher on that knowledge that can be measured directly through a test administered to the participant. The instrument was self administered through online data collection software through the researcher's university.

The questions used in the instrument could have been administered face to face by a researcher which would have had different advantages and disadvantages. When administering the instrument in person the participants would have had the opportunity to ask for clarification of questions, draw diagrams, and demonstrate teaching that would be more difficult to convey in the online open-ended questionnaire. The researcher would also have had the opportunity to probe any of the participant's responses for clarification.

Lastly, content knowledge questions would have been more accurately assessed when participants are responding to the researcher in person rather than on their own where they may

have the opportunity to seek help when they are unable to answer questions. All of these advantages provide the researcher with a greater depth of understanding.

Although the advantages of observation are significant, they also come at a cost. The amount of time and resources required to contact, schedule, and perform multiple interviews with a large number of participants was prohibitive. Most importantly, the in-depth qualitative interview process has often been prohibitive of research involving large samples that could produce generalizable results.

This research chose instead to deliver the questions in the format of an online survey. This drastically reduces the time commitment required of the researcher and participants. The number of participants can be increased greatly and data collection and analysis becomes much more straightforward, though still complex. The type of data is very narrowly focused and coded into quantitative parts that could be analyzed using statistical models that fit the questions of the researcher.

Definitions

1. **MKT – Mathematics Knowledge for Teaching.** For the purposes of this study, MKT is a composite of the most useful forms of representations or examples of division by fractions, procedural and conceptual explanations that make it comprehensible to others, an understanding of what makes the learning division by fractions easy or difficult, the preconceptions and misconceptions that students have about division by fractions, where students will have trouble, the reasons they will have trouble, and how to help students work through their difficulties.

- a. Procedurally Accurate – An algorithm or process is correctly performed in solving a division by fractions problem.
 - b. Correct procedural explanation- The process for solving a particular division by fractions problem can be given that will produce a correct answer to any division by fractions problem in general.
 - c. Representation – This is the ability to create a story problem correctly representing the mathematical problem in all its parts, divisor, dividend, quotient, and operation.
 - d. Error Analysis- Error analysis involves several aspects. Detecting errors, isolating the specific step where errors are occurring, correcting errors in student understanding and procedures concerning division by fractions.
 - e. Procedural flexibility -The teacher can accurately perform a division by fractions problem using more than one procedure or model.
 - f. Alternate methods for explaining the problem- The teacher can create multiple representations for a division by fractions problem.
 - g. Correct justification for “invert and multiply” – The teacher can explain using properties of multiplication, division, and fractions why the algorithm of “invert and multiply” works with division by fractions.
2. Experience – years teaching mathematics
 3. Teacher License – The content areas and grade levels that teachers are licensed by the state to teach
 4. School – Teachers will identify as teaching in an Elementary, Middle, or High School. It is possible a teacher may teach in more than one.

5. Methods of Mathematics instruction course – Most teachers have one course on the methods of mathematics instruction. This is generally taught by someone in the mathematics or education departments.
6. Degree – This is the bachelor’s degree of each teacher.

Summary

This research addressed how the MKT of elementary teachers for division by fractions differs from that of their peers at the middle and secondary level within the same district. Researchers have conjectured about how teacher knowledge affects instruction (Hill et al, (2005); National Mathematics Advisory Panel, 2008; Ma, 1999). This researcher examined the MKT of teachers at grade levels where students learn, use, and apply division by fractions. By investigating the knowledge of teachers across a diverse sample of experience, licensure, and grade level assignment, the researcher had a cross section of the MKT of an entire school district’s faculty for one specific mathematical concept. This study served to inform mathematics educators in two ways. First, variables or factors that are found to positively or negatively correlate with MKT for division by fractions can inform school districts, administrators, Department chairs, community members, and individual teachers of the distribution of knowledge for teaching division by fractions within their district. Second, the results of this study can be used to inspire larger scale, highly generalizable results about the MKT of teachers on a larger scale. These studies could confirm or disprove untested hypotheses about what influences a teacher’s mathematical knowledge for teaching division by fractions.

Chapter 2- Review of Literature

Introduction

For much of history, those who were considered the most knowledgeable in any field were given the task of teaching. Shulman (1986) gave a historical review of teaching as the foundation for the concept of his PCK. In his review of history, Shulman points to ancient thinkers like Aristotle and historical accounts of medieval universities as examples of those people and institutions that historically have seen no distinction between knowledge and the ability to teach. In more recent history, mathematics education and research has been varied in its focus on content and pedagogy. During the late 19th century, certification exams for teachers were often ninety percent or more content focused with very little emphasis on pedagogical knowledge. Somewhere in the mid-twentieth century, the focus of research and teacher preparation shifted towards the processes of what teachers were doing without regard to what teachers knew (Hill, Schilling, & Ball, 2004). This shift toward a new process-product line of research shifted its focus more towards instruction over curriculum knowledge (Hill et al, 2004; Shulman, 1986).

These process-product studies led Shulman (1986) to refer to content knowledge the “missing paradigm” and Hill, Schilling and Ball (2004) to conclude that this type of research virtually ignored content. Teaching was examined without regard to content knowledge. This research had great value in that it examined effective and ineffective methods of practice by comparing specific practices within the work of teaching. The underlying assumption behind the results of these types of research was that some methods are better than others, *given the content knowledge of the teacher is the same*. As researchers learned more about the process of teaching,

other researchers like Shulman (1986), Ball (1990, 2000), Ma (1999), and Hill et al (2004, 2005) wonder about what Shulman called the “missing paradigm” of content knowledge. As process product research grew, the balance and focus of teacher preparation and certification changed. At the time of Shulman’s 1986 article and address, some teacher exams, especially for elementary teachers, focused exclusively on pedagogy with little emphasis on content knowledge (Shulman, 1986). This was a time when the pendulum had swung away from a focus on content knowledge to things like time on task, anatomy of a lesson, activating prior knowledge, discovery learning, posting clear student learning objectives, and cooperative learning in many forms (me-we-two–you, think-pair-share, shoulder partners, etc...)

Shulman’s work did a great deal to push the pendulum back towards a balance of content and pedagogical knowledge. Ball (2008) gave a description of the far reaching impact of Shulman’s PCK. Mathematics knowledge for teaching (MKT) is the knowledge of mathematics especially as it relates to pedagogy. It is the application of Shulman’s (1986) theoretical construct of Pedagogical Content knowledge in the area of mathematics. MKT made strides towards defining PCK in the domain of teaching and giving it a theoretical grounding for empirical verification and measurement. Researchers have investigated the connection between the teacher’s content knowledge and their students’ achievement (Rowan, Chian, & Miller, 1997, Hill, Rowan, Ball, 2005). Rowan, Chian, and Miller (1997) demonstrated a positive correlation between teacher performance on one item of a teacher survey and students score on a mathematics achievement test. Hill, Rowan, and Ball (2005) conducted a large scale study with teachers and students assessing teacher content knowledge in mathematics and student performance. Teacher content knowledge showed a significant and positive correlation with student achievement. Defining and operationally measuring teacher knowledge has been

attempted using a variety of perspectives, methods, instruments, and analyses with a diverse but skewed population (Depaepe, Verschaffel, Kelchtermans, 2013). Most research into MKT of teachers is done with pre-service elementary or middle level teachers. Research on MKT has looked at various mathematical concepts and topics, but one area that has persisted as a topic of study for three decades is division by fractions (Ball, 1988, 1990; Behr, Harel, Post, & Lesh, 1992; Ma, 1999; Bulgar, 2003; Li & Smith, 2007; Lo & Luo, 2012; Özel, 2013). Students and teachers alike continue to have difficulty understanding division by fractions (Ma, 1999; Isik & Kar, 2012). When teachers have a shallow and often inadequate understanding of any mathematical concept it seems to follow quite logically that their students' understanding is limited. The nature of MKT involves questions of how it develops, how it can be measured, how does it change over time, and what factors contribute to its development.

Pedagogical Content Knowledge

The idea of pedagogical content knowledge was first defined in theory, which in turn led to more detailed operational definitions proceeding from research in specific settings. Shulman defined pedagogical content knowledge as, “the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations-in a word, the ways of representing and formulating the subject that make it comprehensible to others” (p. 9, 1986). Besides representation which is focused on the content and curriculum, there are also aspects related to the student. Shulman continues, “Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (p. 9-10, 1986). Teachers are familiar with where students will have trouble, the reasons they will have

trouble, and how to help students work through their difficulties. Research in this area is essential to understanding what knowledge is needed for teaching.

We are gathering an ever growing body of knowledge about the misconceptions of students and about the instructional conditions necessary to overcome and transform those initial conceptions. Such research-based knowledge, an important component of the pedagogical understanding of subject matter, should be included at the heart of our definition of needed pedagogical knowledge. (Shulman, 1986, pg. 9).

Prior to Shulman's 1986 address, only one article in a peer reviewed journal came up in a search for "pedagogical content knowledge". Since then 13,859 works have cited Shulman's work and a search for "pedagogical content knowledge" yields 3 621 results (Google Scholar). Once the theoretical foundation for PCK was laid, researchers in the field of mathematics education began to frame their studies on teacher knowledge using Shulman's conception, and constructed their own categories of knowledge that related specifically to mathematics instruction.

Mathematics Knowledge for Teaching

Deborah Lowenberg Ball built on Shulman's work from the very beginning of her career. Just two years after Shulman's PCK, her 1988 doctoral dissertation was titled, "Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education". Throughout the last few decades she has been central to the development of the concept of Mathematics Knowledge for teaching (Ball, 1988, 1990, 2000, Hill, Schilling & Ball, 2004, Hill, Rowan & Ball, 2005, Ball, Thames & Phelps, 2008). In applying each aspect of PCK to mathematics instruction, Ball, Thames, and Phelps, (2008) developed the individual parts that

made up mathematics knowledge for teaching. Figure 1 below shows the parts of MKT as developed and defined by Ball and her colleagues (Ball et al, 2008).

Figure 1. Domains of Mathematical Knowledge for Teaching

Subject Matter Knowledge	Pedagogical Content Knowledge
Common content knowledge (CCK)	Knowledge of content and students (KCS)
Specialized content knowledge (SCK)	Knowledge of content and teaching (KCT)
Horizon content knowledge	Knowledge of content and curriculum

In her 1988 work, Ball developed questions aimed to get at elementary teachers' knowledge of division by fractions. Those items have been a part of further studies (Leung & Park, 2002) and were included in Ma's (1999) comparative study of Chinese and U.S. elementary teachers. Shortly after Ma's study, Ball (2000) concluded there was a divide and fragmentation between teaching and learning of content and pedagogy, i.e. curriculum and instruction. Ball was supporting the earlier call by Shulman to view knowledge of teaching and content knowledge as one body of connected knowledge. Throughout the last century there arose a divide between content and pedagogy (Kraus et al, 2008, Chinnappan, 2005). The purpose of current study pointed to the need to study pedagogy and content together as recommended by Shulman (1986), and Ball (Ball, 1988, 1990, 2000, Hill, Schilling & Ball, 2004, Hill, Rowan & Ball, 2005, Ball, Thames & Phelps, 2008).

The particular context in which this research examines the MKT of teachers is taken from one specific area studied extensively in mathematics education: division by fractions (Ball, 1990, Behr, Harel, Post, & Lesh, 1992, Simon, 1993, Ma, 1999, Tirosh, 2000, Leung & Park 2002,

Bulgar, 2003, Li & Kulm, 2008). Results of these studies were discussed in the researcher's section on Division by fractions. One exemplary study is mentioned below.

Ma (1999) looked at four different areas in her study of teachers MKT. She included subtraction with regrouping, multi-digit number multiplication, division by fractions, and the relationship between perimeter and area. This research overlapped with Ma's by using one of the same questions she used in the area of division by fractions. The questions that Ma used in her study were developed by Ball (1988) a decade earlier. This research was in part an attempt to achieve repeatability and generalizability by examining how previously obtained results compared to a similar study performed with a different population of teachers. Ma looked only at elementary teachers, while this research examined elementary, middle, and secondary teachers. This research viewed MKT from a cognitive perspective like Ma, but the methodology was quantitative instead of the qualitative approach Ma used. Figure 2 below is a concept map showing Ma's (1999) analysis of important pieces in the knowledge package of understanding division with fractions

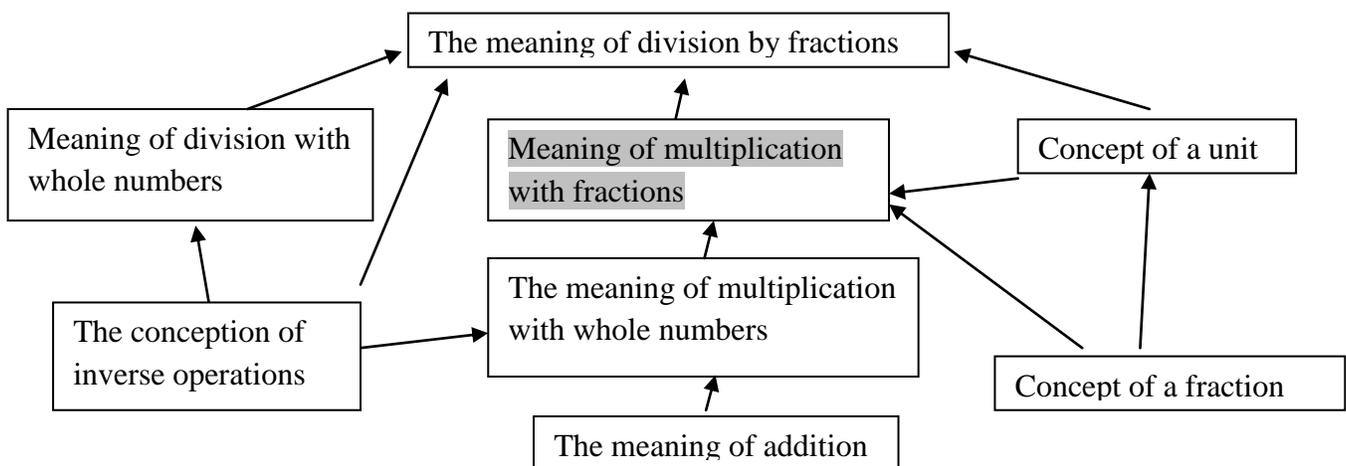


Figure 2. Division by fractions concept map

Qualitative and Quantitative Methods of Research

Research in mathematics education and more specifically, mathematics knowledge for teaching, varies on several key aspects. The assumptions of the researcher, the methods of argumentation and reasoning, and the methods of collecting, analyzing, and generalizing data are among the most important (Yore, 2003). The quantitative research processes are a type of inductive reasoning that leads to forming and testing conjectures. The arguments in these quantitative studies use probability to argue that the evidence for a given theory is not likely to have occurred due to chance. The argument of the quantitative researcher rests not on the evidence alone, but also on the assumptions, and world view of the researcher (Lester & Dylan, 2000). The quantitative studies seek to test and measure the conjectures and theories that often arise from complementary qualitative studies. Qualitative studies help researchers formulate hypotheses and develop theories that can be empirically tested. Research questions that can be measured empirically through quantitative studies seek to verify conjectures. The results of the quantitative studies can then be used to support existing theories or create new questions and hypotheses. This research used existing theories about mathematics knowledge for teaching test the conjectures formed by the researcher from the review of existing literature.

Most studies that have been done can be grouped in several different ways. Sample population, theoretical perspective, methodology, instrument, and purpose are important differences that separate research. Two broad categories are those that are divided by theoretical perspectives. The first common theoretical perspective is a cognitive view of MKT. Researchers with a cognitive view of teacher knowledge tend to measure teacher knowledge through paper and pencil type tests. Tests are composed of questions designed to measure both content knowledge and pedagogical knowledge that combine to form MKT. These researchers

believe that a teacher's knowledge of mathematics for teaching can be measured directly from the teacher using some form of test. Only 10 of the 60 studies analyzed by Depaepe, Verschaffel, and Kelchtermans (2013) used a test or questionnaire with large samples. Cognitive theorists were involved in both quantitative and qualitative research. Those studies that were qualitative in methodology and cognitive in theory tended to use interviews like Ma (1999). The second broad category of research operates with a perspective that sees MKT as situated within the experience of teaching. These types of research require that researchers collect data from teaching episodes, interviews, and meeting observations. Common approaches include video and audio recordings of lessons and interviews with teachers reflecting on lessons that were observed. Researchers with this perspective tend to believe that MKT must be measured within the context of instruction. Depaepe, Verschaffel, and Kelchtermans note that 20 of the 60 studies were of this type. Although there are some observational quantitative studies, these tend to be more qualitative in methodology (Depaepe, Verschaffel & Kelchtermans, 2013).

Qualitative Research

Depaepe, Verschaffel, and Kelchtermans, (2013) examined each of 60 research articles looking for systematic similarities and differences that address two research questions: How is PCK conceptualized in empirical mathematics educational research, and How is PCK investigated in empirical mathematics educational research? How is it measured? There was a divide between differing theoretical perspectives, which could be grouped together into two general groups. A cognitive view of PCK sees a teacher's mathematics knowledge for teaching as something that can be measured directly through some sort of test or examination. This perspective often resulted in methodology involving large scale, quantitative, paper and pencil teacher tests. The second group of perspectives had a more situated contextual view of PCK.

These perspectives believed that the teacher's mathematical knowledge for teaching or MKT could only be measured in context. Studies done from this perspective often involved small scale, qualitative, observations and interview studies. This research started with a perspective that teacher knowledge can be measured directly using an instrument created by the researcher.

When it comes to examining teachers' MKT, there seemed to be a preference for qualitative methodology. These qualitative methodologies, as well as those with quantitative methodologies, could be divided into two more specific groups based on their theoretical perspective of whether MKT can be measured directly from the teacher removed from the context of the classroom, or if MKT must be observed and measured in action within the context of instruction. A cognitive perspective examines the teacher's knowledge in isolation from the classroom. These studies viewed MKT as something that can be measured from the teacher directly using some sort of test Interviews, surveys, and questionnaires are some of the ways in which MKT is assessed. (Özel, 2013, Chapman, 2007, Chinnappan, 2005, Tirosh, 2000). This research shared a cognitive perspective and examined teacher knowledge using questions very similar to those found in qualitative interview settings.

Studies that viewed MKT as something to be observed in action generally used recorded observations and transcriptions of classroom instruction and interactions. Observers paid close attention to the ways in which teachers and students interact in the classroom. These studies had data that is extremely varied and rich. Video and audio recordings, transcripts of conversations, student work examples, student tasks, teacher led discussions, and anything produced by a teacher or student that could be recorded were often used in these studies (Thompson, 1984, Kinach, 2002, Yackel, 2002).

Quantitative Research

Quantitative research has been in the minority when it comes to measuring MKT. One of the main reasons is that Shulman (1986) gave a broad but powerful definition of PCK that could be applied in virtually any field of education. Researchers like Hill, Schilling, and Ball, (2004), Martin and Harel (1989), Rowan, Chian, and Miller (1997) have attempted to develop methodologies for quantifying PCK and MKT in mathematics education research. The most common methods involve coding open ended responses to tests designed to measure the different aspects of MKT and multiple choice items constructed with the same purpose in mind. Hill, Schilling, and Ball, (2004) empirically measured teacher content knowledge through a multiple choice instrument. Researchers believed that there was a gap in research measurement and instrumentation in regards to measuring teacher content and pedagogical knowledge. Their work was aimed at filling that gap. The authors of the study developed multiple-choice items aimed at measuring MKT. Although this research suggested that content knowledge CK and PCK were separate distinguishable factors, there are studies that show that as CK and PCK increase and deepen they tend to combine into one inseparable body of knowledge (Kraus et al, 2008). In their analysis, researchers concluded that most items were, on average, too easy. This purely quantitative, cognitive research was in need of support through investigations involving classroom observation as a measure of the validity of the results. Additionally, the findings from this study are contingent on the assumption that teachers MKT is somewhat related to the domain of content. These findings should be replicated, both through studies similar to the one reported here and also through the use of multiple methods, including interviews and observations of classroom instruction.

Chapman (2007) examined the use of specific tasks for pre-service teacher as a means of deepening understanding of arithmetic operations that have been shown to be lacking in the research. This research explored a means of addressing recent findings that elementary teachers have shallow mathematical content knowledge. The author cited diverse perspectives including Dewey (1916), Vygotsky's (1978) social/interactive perspective on learning, and Bruner's (1966) cognitive model with his stages of enactive, iconic, and symbolic mode. Participants initially held knowledge of arithmetic operations that was procedural in nature and generally lacked depth. Chapman focused on the use of learning tasks as a means of deepening teacher and student understanding of arithmetic operations. This research fit into a category of studies aimed at improving rather than describing MKT at the pre-service stage.

The methodology for designing the instrument in this study was borrowed from Manizade and Mason's (2011) development of an instrument using Delphi methodology. Instruments designed to measure MKT have generally been developed by researchers to fit the needs of their research purpose and questions. As with any newly developed instrument, there is a need to establish empirical validity and reliability. Many instruments lack a theoretical methodology for the development and revision of items. Delphi methodology stemmed from a cognitive approach that measured PCK cognitively with a paper pencil test apart from situated observations or interviews. Researchers contacted 20 experts in order to form a diverse panel made up of math education researchers, educators, and leaders. An instrument was developed by the researchers that consisted of 10 multiple part open-ended questions regarding PCK situated in hypothetical teaching situations. Each of the 20 participants reviewed the items individually and recommended changes with relation to several categories. The items were rated 1-5, and any additional comments were included. Researchers collected and analyzed the results. Results

were returned to the experts for another round of revision. This process would continue until there was consensus on all items. It only took 3 rounds in this study. Manizade and Mason (2011) intentionally consolidated existing definitions for MKT in the research and clearly defined for the study what MKT was. The initial definition was knowledge of: connections among big ideas, learning theories describing students' developmental capabilities, students' common misconceptions and subject specific difficulties, and knowledge of useful representations and appropriate instructional techniques. This methodology was implemented by the researcher in the design stage of the instrument for measuring MKT with regard to division by fractions.

Mixed Methods Research

A few studies dealing with PCK in mathematics, or MKT used a mixture of quantitative and qualitative methods. The most common blend was a test or survey of a sample with interviews performed for a small subset of the sample population. Five of the 60 studies analyzed by Depaepe, Verschaffel, and Kelchtermans (2013) used mixed methods. Chinnappan and Lawson (2005) are an example of this type of research. They provided a framework for describing and analyzing teacher content knowledge in geometry and content knowledge for teaching. Little detail had been given at the time to how exactly to analyze and describe the content knowledge and content knowledge for teaching. If these types of knowledge were important, then there existed a need for a clear and consistent approach to analysis. Interviews were used to create concept or schema maps representing the teachers' knowledge. Open ended interviews were given, and the responses were coded and categorized to fit into schema maps. The participants' responses were measured quantitatively by the number of defining features, related features, applications or representations, and other features. Schema maps were created

by connecting concepts that were discussed in interviews by the participants. Scoring and analysis of each participant's schema map demonstrate where each teacher has a depth of understanding by the number of connections and related concepts in their knowledge. Observational studies could be done involving this sort of analysis that would give a deeper and more detailed understanding of teacher knowledge.

Ball, Thames, and Phelps (2008) approached a practice-based theory of content knowledge for teaching built on Shulman's (1986) notion of pedagogical content knowledge. This approach differed from that of the cognitive theorists. Researchers studied actual mathematics teaching and identified mathematical knowledge for teaching as it presented itself in the classroom. This research made the argument that pedagogical content knowledge (PCK) and mathematics knowledge for teaching (MKT) lack a clear and usable definition. Although there was a broad use of the terms, there seemed to be a lack of specificity and concreteness to their meaning. Researchers approached the problem through extensive qualitative analyses of teaching practice. The results of these qualitative studies produced theory and hypotheses useful in the design of measures of mathematical knowledge for teaching. Researchers gave a practice based definition that was more usable and precise than existing notions of Mathematical Knowledge for teaching. Several questions remained at the end of their study. Two of those questions align very closely with the purpose and questions of this research. "Do different preparation programs have different effects on pedagogical content knowledge? Can the categories of knowledge be defined and measured more precisely?" (p. 405, Ball, Thames, & Phelps, 2008). This research examined each of the questions directly or indirectly.

Mathematics knowledge for teaching has been the topic of international study. Researchers have done studies similar to those of Hill et al (2004, 2005), and Ma (1999) in

European and Asian countries including Turkey (Özel, 2013, Isik & Kar, 2012), Taiwan (Lo & Luo 2012), China (Li & Huang, 2008), and Germany (Kraus, Bruner, Kunter, Baumert, Neubrand, Blum, & Jordan 2008). Often the setting and context of the study is quite different than in the United States. In Germany, Kraus et al (2008) examined pedagogical content knowledge and content knowledge of secondary mathematics teachers. The population of teachers was not taken from a system of schools like in the U.S., but from a national school system that is divided into tracks. Gymnasium schools were academic track schools, while non-gymnasium schools were devoted to training students for non academic careers. The research questions were, “Can PCK and CK be distinguished empirically?”, and “Does the mean level of knowledge on the two scales and their connectedness depend upon the level of expertise?” Cognitive theory situated the research to be measured directly from the teachers. A large scale study was done using a paper and pencil test to measure teacher knowledge. 198 German secondary mathematics teachers from gymnasium and non-gymnasium schools participated in the study. A paper pencil test was developed by the researchers composed of items that measured CK with 13 items and PCK using 10 items. All items were open ended. Teachers were given the test by a researcher on an individual basis. The procedures for the test resulted in the test being completed in about 2 hours. No resources were allowed to be used by the teachers. Scores were tallied by the researchers. Confirmatory factor analyses were used. PCK and CK were indistinguishable empirically for Gymnasium teachers while PCK and CK were separate bodies for the non-gymnasium teachers. Gymnasium teachers scored higher on both CK and PCK than Non-gymnasium teachers. Researchers noted that it was essential that items be “grade appropriate and mathematically unambiguous, and produce data that did not favor any single view of teaching” (p. 203, Kraus et al, 2008). Using this new PCK assessment tool to assess the

knowledge of middle school teachers in geometry will allow further investigation of teacher knowledge in mathematics education in relation to student cognition and student outcomes.

Depaepe et al (2013) performed a recent analysis of research that was very beneficial in the collection and review of research relating to this study. The conceptualization of PCK in empirical mathematics educational research and the methods of investigation in empirical mathematics educational research were examined. Shulman introduced the PCK as a theoretical construct and Ball has given it an empirical construct in MKT. Both are lacking in empirical evidence supporting their existence and importance. This was a systematic examination of existing research on "pedagogical content knowledge" and mathematics. 60 peer-reviewed empirical studies were examined. The two research questions were addressed by examining each of the 60 research articles looking for systematic similarities and differences that address the research questions. Each article was analyzed by how they defined/conceptualized PCK and also the methodology in each study. Sample size, qualitative or quantitative, country of origin, all were examined. Half of the studies were in line with Schulman's conceptualization, about one fourth used Ball's MKT, many did not define what they meant by PCK. There was a divide between a cognitive view of PCK that affected methodology towards large scale, quantitative, paper and pencil teacher tests and the situated contextual view of PCK that affected methodology towards small scale, qualitative, observations and interview studies. Researchers pointed out strengths and weaknesses of both types of studies. There was an imbalance between studies on the development of PCK which represented a large majority, and the link between PCK and student performance, a minority.

What have been some different ways of measuring or conceptualizing this type of knowledge? Manizade, A. G., Mason, M. M., (2011) contend that instruments are scarce and

still lacking empirical validity and reliability. Many instruments lack a theoretical methodology for the development and revision of items. Kraus, Bruner, Kunter, Baumert, Neubrand, Blum, and Jordan (2008) studied secondary teachers in Germany at different schools. They found that PCK and CK were indistinguishable empirically for Gymnasium teachers while PCK and CK were separate bodies for the non-gymnasium teachers. Gymnasium teachers scored higher on both CK and PCK than non-gymnasium teachers. These studies revealed uncertainty when it comes to what MKT is and whether or not it can be measured directly. In a review of research from a decade ago, Ball, Thames, and Phelps (2008) concluded, “Although the term pedagogical content knowledge is widely used, its potential has been only thinly developed. Many seem to assume that its nature and content are obvious. Yet what is meant by pedagogical content knowledge is underspecified. The term has lacked definition and empirical foundation, limiting its usefulness” (p. 389, Ball et al, 2008). Here the researchers made it clear that there was a need to connect differing methodologies and underlying theoretical perspectives through clarification and definition of mathematical knowledge for teaching. In a study four years earlier in Hill, Schilling, and Ball (2004) conducted a study that was meant as a trial to empirically measuring teacher content knowledge through a multiple choice instrument. One result of that study was a finding that the multiple choice items in the instrument provided the most information at abilities below the average teacher, that is, items were, on average, too easy. Multiple choice items are often chosen because they are simple to score. One disadvantage is that they may overestimate teacher knowledge by providing the answer to each question along with a few alternatives. This study eliminates this possibility by using only open-ended questions. Teachers must create their responses rather than correctly choose responses that have been created in advance. This study

also contains items that have a broad range of scoring options aimed at creating an instrument that is not too easy, but has the ability to measure deep teacher knowledge.

Sampling populations

Much of the research on MKT has been done in the area of elementary teachers, both pre-service and in-service, (Chapman, 2007). In their analysis Depaepe, Verschaffel, and Kelchtermans identified 29 studies of elementary teachers, 24 studies involving secondary teachers, but only 4 studies that examined both elementary and secondary teachers (Meredith, 1993; Sorto et al., 2009; Von Minden et al., 1998). Those four studies included only pre-service secondary teachers. Özel, S. (2013) examined 10 in-service middle school teachers from three different schools in Turkey. Six of the teachers were graduated from primary education departments whereas the rest were graduated from secondary school science and mathematics departments. Lo & Luo (2012) purposely excluded mathematics or science majors from this study to avoid skewing the results because they were required to take many college-level mathematics courses. Kraus et al (2008) compared 198 in-service secondary mathematics teachers at gymnasium and non-gymnasium schools in Germany. Ball (1990) examined 10 pre-service elementary and 9 pre-eservice secondary teachers. The elementary students had an average of 2 math courses since high school, whereas the secondary teachers had an average of 9 math classes in college. There was no comparison found of in-service teachers' MKT at the elementary, middle, and secondary levels in any study.

The emphasis on pre-service teachers in the research literature provided evidence that in-service teachers MKT has been neglected. This combined with a tendency mentioned above to focus mainly on elementary teachers, often to the exclusion of middle and secondary teachers.

In those studies that did examine middle and secondary teachers' MKT they did not include a comparison with a population of peers at the elementary level. There was a need to examine what relationship existed between elementary, middle, and secondary teachers' MKT.

Theoretical Perspective

Constructivism

This research was viewed through the perspective of a constructivist theory, modeled most closely by Jerome Bruner's view of constructivism. Bruner (2007) believed that learning was a continuous process and children could learn any topic if it was appropriately matched in complexity with their age and development. He viewed experience and prior knowledge as the main limitations of students' learning and understanding. Bruner identified the transition that takes place in a child's ability to understand something concretely with physical objects, symbolically through representations of the objects, and lastly at a purely symbolic level. Constructivist theory supports certain aspects of MKT. Teachers must be aware of the necessary knowledge pieces and skills that will be useful to gain an understanding of division by fractions.

Discovery

Discovery learning is a central part of the cognitive learning theory. As Bruner (1995) describes, "any average teacher of mathematics can do much to aid his or her pupils to the discovery of mathematical ideas for themselves. Probably we do violence to the subtlety of such technique by labeling it simply the 'method of discovery,' for it is certainly more than one method" (Bruner, 1995, p. 49). The idea of discovery is that information cannot be "put into" a child, but instead each child must construct their own meaning out of existing knowledge as they discover new concepts. When students learn through discovering solutions to problems the

teacher's ability to create representations of problems becomes an important quality. Problems set in familiar content that students can manipulate give students the ability to discover the properties of fraction division. If teachers possess only a shallow or procedural understanding of division by fractions they will be limited in their ability to create broad and diverse learning scenarios where students can encounter and work with division by fractions.

Stages of learning

Bruner believes that the learning mathematics tends to pass through three stages. The Enactive stage happens when the student experiences a real physical relationship between different quantities. As that relationship is studied in various forms and variations, it is then moved into the iconic stage. In the Iconic stage other relationships are considered using only pictures of the physical objects and relationships which allows greater depth and breadth to be explored. Finally, the physical objects can be removed and the relationships are examined using only symbols, like with variables in algebra. There need not be a physical reference point in this stage. In a cognitive learning theory students repeatedly go through these stages as they acquire new information and construct new learning. These stages can often be seen clearly in the classroom, and yet there are times when they are glaringly absent. When teaching division by fractions, teachers with knowledge of a procedure like "invert and multiply" may not be able to create a physical representation of a problem like $1\frac{3}{4}$ divided by $\frac{1}{2}$. How much better would it be if the students could go through the stages of enactive, iconic, and representational. A teacher could bring in a container with $1\frac{3}{4}$ cups of flour. The teacher also had measuring cups of different fractions of a cup, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$. The teacher could allow students to physically test how many $\frac{1}{2}$ cups of flour there are in $1\frac{3}{4}$ cups. Without the enactive and iconic stages of learning teachers can attempt something similar to building a home starting with the second floor.

Spiraling

Spiral curriculum is Bruner's preferred way of organizing content to be taught because it follows from how students represent content from the enactive to the symbolic stage. It also is a direct application of the theory that students can learn anything given the right context and support. Bruner would first introduce new content in an interactive mode of representation using physical objects. Then later he would come back to the same content but reintroduce it in an iconic mode using images to represent the content. Finally he would present the content a third time but this time he would do so using a symbolic mode of representation. As the students are able to understand concepts in the symbolic form, the teacher can then repeat the process with greater depth, complexity, and/or connections to other related concepts. With division of fractions there are so many skills and knowledge pieces that could be spiraled in for review and support. Properties of fractions, division with whole numbers, inverse operations, multiplicative inverses, multiplicative identities, and any other skill or knowledge piece that is used in division by fractions can be reviewed and referenced when teaching division by fractions.

Scaffolding

Bruner, like Vygotsky, emphasized the social nature of learning, citing that other people should help a child develop skills through the process of scaffolding. The term scaffolding first appeared in the literature when Wood, Bruner and Ross described how tutors' interacted with preschooler to help them solve a block reconstruction problem (Wood et al., 1976). The concept of scaffolding is very similar to Vygotsky's notion of the zone of proximal development, and it not uncommon for the terms to be used interchangeably. *Scaffolding* involves helpful, structured interaction between an adult and a child with the aim of helping the child achieve a specific goal. Scaffolding refers to the steps taken to reduce the degrees of freedom in carrying out some task

so that the child can concentrate on the difficult skill she is in the process of acquiring (Bruner, 1978, p. 19). As the name indicates, these structures are meant to be temporary aids that allow the student to construct meaning dependently until they reach a point of independence and the structures can be removed. Scaffolding is dependent upon several parts of MKT. A teacher needs to be able to look at a student who is having difficulties and assess several things. What are they not understanding, and why do they not understand it? This is error analysis. What knowledge is essential for the student in order to correct their wrong thinking? This is a deep conceptual knowledge of the mathematics. What are the most useful and illuminating representations and strategies that can help the student do the mathematics? This is a flexible ability to use multiple representations and solution strategies.

Culture

More recently, Bruner has acknowledged culture as more of the context for learning, instead of a part of the curriculum. Culture is spoken about as a toolkit, “An example of the toolkit is the language that is commonly used in a particular cultural tradition; it includes not only grammar and vocabulary but also such things as knowledge, beliefs, and values shared by the people of the culture” (Takaya, 2008, p.2). What a student learns and how they interact with other students and teachers all takes place in the context of various cultures that are represented in the classroom. Posner (2008) cites “studies indicate that children often experience difficulty in classrooms that are organized according to assumptions about the use of time, space, language, and instructional strategies that are different from those in their homes” (Posner, 2008, p.129). The greater the diversity within a classroom, the more culture must be considered as a factor influencing learning. How helpful would cooking examples be to a student who never saw their parent cook or bake using measuring cups? Knowledge of the students and their own

cultural circumstances reveals the need for a teacher to present children a variety of contexts where division by fractions occurs and a variety of strategies for solving these types of problems.

Intuition

In an examination of Bruner and his cognitive learning theory, Takaya (2008) says that Bruner saw in school, “a totally different and unnatural mode of thinking is imposed upon children, that is, the kind of thinking that needs to be clear, distinct, informed, and explicit from the beginning, and it turns children off from engaging in thought. He says that intuition “is less rigorous with respect to proof, more visual or iconic, more oriented to the whole problem than to particular parts, less verbalized with respect to justification, and based upon a confidence in one’s ability to operate with insufficient data” (Takaya, 2008, p. 8-9) This is especially true in division by fractions. The procedure of inverting and multiplying is hardly intuitive. Although the generalization is not intuitive, it is not hard for students to understand that when you ask how many half dollars there are in a five dollar bill the answer is ten, or two times five. Often the intuitive questioning is skipped in order to present students with the “quickest” and most “efficient” strategy that has been predetermined by other mathematicians who had the opportunity to try different methods and choose for themselves the “best” method.

Quantitative measurement and coding

There have been a few efforts at measuring MKT on a large scale with quantitative measures and instruments (Baumert et al., 2010, Kraus et al., 2008, Hill, Schilling, & Ball, 2004). Hill, Schilling, & Ball (2004) incorporated multiple choice questions that were piloted with elementary teachers enrolled in number and operations institutes. Teachers were not recruited on the basis of their mathematical knowledge or other characteristics. In 2001 the authors wrote, and later pilot tested, numerous multiple-choice items intended to represent the

mathematical knowledge used in teaching elementary mathematics. Results suggested knowledge of content and knowledge of students and content were distinguishable factors. As mentioned earlier, these multiple choice items were found to be more reliable in terms of lower standard errors for teachers' scores that were .5 to 2.0 standard deviations below the average. Standard errors were higher indicating less reliable scores for teachers scoring above the average. This was counted as a first step in the measures development process. Additionally, teachers' knowledge of mathematics for teaching was found to be at least partly domain specific rather than simply related to a general factor such as overall intelligence, mathematical ability, or teaching ability.

Kraus et al (2008) conducted a study with from a different but related perspective that had many similarities to the current research. This study, also referenced earlier, examined a sample of 198 German secondary mathematics teachers from gymnasium and nongymnasium schools. A test was composed of items that measured CK with 13 items and PCK using 10 items. The items were designed to test three aspects of PCK, ability to identify multiple solution paths, ability to recognize students' misconceptions, difficulties, and solution strategies, and knowledge of different representations and explanations of standard problems. These items were submitted to a panel of experts to ensure the items were actually measuring the content accurately as a check of the content validity. Confirmatory factor analyses were used to produce an assessment of teachers' PCK in one area of middle school geometry. All items were open ended Teachers were given the test by a researcher on an individual basis. This difficulty of the items was an improvement on Hill, Schilling, and Ball (2004). When pilot testing was done with students in advanced mathematics courses and science teachers they were mostly unsolvable. Researchers

concluded that the measurement of PCK through an instrument of this type needed to be mathematically unambiguous, and allow strategies from differing views of teaching.

One of the great advantages and strengths of the study above by Kraus et al (2008), is the amount of necessary resources. Because the researchers were able to use such a large sample, their results were highly generalizable. Unfortunately, it is difficult to generalize the results to U.S. teachers because the school system is so different in Germany. The separation into schools of different tracks creates a different system that is not comparable to public education in the U.S. The amount of time and resources invested in this study can also be a great disadvantage. Administering the test required approximately two hours of time from both the participant and the researcher. With a sample of approximately 200 teachers that amounts to almost 800 man hours for data collection alone. This is only possible with a team of researchers with a large amount of funding.

Division by Fractions

Elementary Teachers

Ball, D. (1990) examined prospective elementary and secondary teachers' understanding of Division. It is from this study that Ma (1999) gets her questions about division by fractions. 19 pre-service teachers, 10 elementary and 9 secondary, were interviewed. Three different mathematical contexts were employed to examine prospective teachers' knowledge of division: division with fractions, division by zero, and division with algebraic equations. In each case, the teacher candidates were asked to explain or to generate representations. 17 of 19 participants could correctly divide with fractions, but only 5 of the 19 could generate a story that represented the fraction division problem. Knowledge was very limited, procedural, and compartmentalized.

All but one participant answered questions with regard to the specific calculation without any conceptual connection among division by fractions, division by zero, or division with variables in solving equations.

Simon, M. A. (1993) examined pre-service elementary teachers. The participants were prospective elementary teachers from a large state university. Volunteers were solicited from the required methods course and participants were randomly selected from the list of volunteers. Five open ended questions assessed two aspects of prospective elementary teachers' knowledge of division. 70 % of the US prospective elementary teachers in his study were not able to create an appropriate division word problem for $\frac{3}{4}$ divided by $\frac{1}{4}$.

Secondary Teachers

In their analysis of how PCK has been conceptualized in mathematics education research, Depaepe et al (2013) searched multiple databases with the terms, “Pedagogical Content Knowledge” and “mathematics”. The search resulted in 811 results which were narrowed to 60 using the following criteria. The articles came from peer-reviewed journals, they were written in English, they dealt strictly with PCK in the area of mathematics instruction, and they collected empirical data of a qualitative or quantitative nature. Eleven of the 60 studies examined the domain of fractions, decimals, and percentages, with 9 studies using elementary teachers, two studies using secondary teachers, and no studies examining both elementary and secondary teachers in the area of fractions or division.

Depaepe et al report three of the sixty studies they found on PCK in mathematics that compared teachers from a “mixed” population. Meredith (1993) examined the attitudes of 12 pre-service teachers. She suggests investigating the prior learning of these groups as a possible

cause for differences in PCK. Sorto, Marshall, Luschei, and Carnoy (2009) examined teachers in elementary and middle grades in Panama and Costa Rica. The research questions were concerned with differences between teachers from the two different countries, not between different grade levels. Von Minden, Walls, and Nardi, (1998) analyzed the PCK of a sample of 3 university-level mathematicians, 3 university-level mathematics-methods educators, 3 high school math teachers (Grades 9-12), 3 middle school teachers (Grades 5-8), and 3 elementary school teachers (Grades 1-4). Four tasks were used to create concept maps. Each participant visually represented the connectedness of eleven mathematical concepts by placing concepts they viewed as closely connected closer together and concepts they viewed as not connected as further apart. Participants also drew lines connected concepts they saw as related. These concept maps analyzed by measuring all the inter concept distances. All the distances were averaged for the participants in each group and Pearson correlation coefficients were calculated for all 66 inter concept distance mean scores. Their results indicated that when each group was compared with the others the university mathematicians, methods educators, and high school math teachers all had the weakest agreement with elementary school teachers. There was no significant difference in the agreement between middle school teachers and the other groups. Elementary teachers had their strongest agreement with high school teachers and their weakest agreement with university mathematicians.

Chapter 3-Methodology

Overview of the Study

This study combined elements of existing research study methodologies in a combination that fit the purpose of the study and allowed the researcher to answer the research questions in alignment with the theoretical perspective of the researcher. The design of the study could be viewed as a hybrid of several previous studies. The theoretical definition for Pedagogical Content Knowledge PCK was taken from Shulman (1986). This definition was refined to the domain of mathematics by Hill, Schilling, and Ball (2004). Almost every part of the design of this study was borrowed from one or more previous studies. The process for designing the instrument (Manizade, & Mason, 2011), the questions on the instrument (Ball, 1990, Ma 1999), the theoretical definitions (Shulman, 1986, Hill, Schilling, & Ball, (2004), the independent and dependent variables (Hill, Rowan, & Ball, 2005), and the data analysis (Hill, Rowan, & Ball, 2005, Hill, Schilling, & Ball, 2004, Ball, 1990) have all been used in similar fashion in previous studies. The existing research contained a wealth of knowledge on what MKT is, how to measure it, and what we know about the MKT of certain populations. This research examined MKT for division of fractions from a perspective that differed from other studies in the following ways:

1. The researcher used an online survey that teachers could take using any computer or smart phone.
2. The survey consisted of open ended questions that previously have been used in an interview setting for qualitative research.

3. Responses entered by the participant were coded according to a rubric developed by quantifying commonly used variables. Definitions and scoring methods were gathered from existing research (Ball, 2000; Ball et al, 2008; Depaepe, 2013; Manizade & Mason, 2011; National Mathematics Advisory Panel, 2008; Shulman, 1986; Ma 1999).
4. In-service teachers of students in grades 4-12 formed a sample population distinct from many previous studies that have focused on predominantly pre-service elementary teachers (Depaepe et al, 2013).
5. Inferential statistical analyses were used that have the power to generalize results to a larger population. In this case the sample could be generalized only to the population of the two school districts involved in the study.

Design of the Study

Many elements of the design of this research were original. The researcher designed an instrument very similar to those used in small scale qualitative studies. A scoring rubric was created that aligned with previous research about what constitutes MKT. The MKT instrument was composed of two parts. Part one contained nine open-ended questions on the topic of teaching division by fractions, and part two which was composed of six questions designed to collect independent variables about the participant that have been linked to MKT. The questions were scored on a numeric scale for the sole purpose of quantifying the theoretical construct of MKT and its respective elements. The survey was delivered through an online format. This was unique. In a search for “pedagogical content knowledge”, and “online survey” no results were found that attempted to measure classroom teacher’s pedagogical content knowledge through an online survey. Archambault and Barnett (2010) measured technological pedagogical content

knowledge (TPACK) through an online survey with a population of online teachers for virtual schools. This study, along with Shih and Fan's (2008) meta-analysis of studies comparing web-based surveys to mail surveys found the response rate to be significantly lower for web-based surveys that contacted participants through email and provided a link to a web-based survey. Most of the existing studies of PCK or MKT involved in-person interviews, paper pencil tests administered by the researchers, or some combination of the two (Depaepe, Verschaffel, & Kelchtermans, 2013, Özel, 2013, Eli, Mohr-Schroeder, Lee, 2013, Lo, & Luo, 2012, Isik, & Kar, 2012, Manizade, & Mason, 2011, Baumert et al., 2010, Wilkins, 2008, Kraus et al., 2008, Li & Smith, 2007, Hill, Schilling, & Ball, 2004, Simon, 1993, Ball, 1990). This research also examined in-service teachers across grade levels 4-12. The large majority of studies of MKT have been done with pre-service elementary teachers. Although this research was limited in its scope and sample size, it was intended to serve as a foundation for larger more generalizable studies that would address a broader population.

The independent variables collected by the researcher had a high likelihood of collinearity. The grade level teachers are licensed to teach, the type of school in which they teach (Elementary, Middle, or High School), the years of experience teaching each grade level, and the department that provided the instruction for each teacher's methods for teaching course are all likely to correlate strongly with one of two factors: elementary or secondary teachers. Teachers who are licensed to teach elementary grades were likely to have taught at elementary schools. Teachers in elementary schools were likely to have more experience teaching elementary grade students. Teachers who were licensed to teach elementary students were likely to have elementary degrees with methods teachers from their university's school of education. A

Factor analysis was performed to see if each of these variables are measuring different aspects of some larger factor, most likely two factors: elementary or secondary teachers.

The dependent variables were hypothesized to measure aspects of the same underlying factor, Mathematics Knowledge for Teaching. This assumption was tested through a confirmatory factor analysis. The researcher had two competing hypotheses: 1) All nine items would correlate to form one factor, i.e. MKT, or 2) Three distinct factors could emerge as different aspects of MKT, Conceptual Understanding and Accuracy (Knowledge), Error Analysis of student thinking (Analysis), and Ability to use multiple representations (Representation).

There were three main levels of analysis of the data. The first two levels of analysis were independent factor analyses done on the independent and dependent variables. This allowed the researcher to isolate the main factors that were represented by several different variables. The second factor analysis was done on the nine items of the MKT instrument. Multiple items were intended to measure different aspects of MKT. Content knowledge was measured in the ability to accurately solve and explain a division by fractions problem. Knowledge of student errors was a second measure addressed by multiple items. The ability to create multiple representations and solve a problem using multiple solution pathways was a third measure. In each case the factor analysis of the dependent variables addressed how many related factors exist in the data on MKT and whether all the variables contribute to one composite body of knowledge, MKT. Once we had the respective variables combined into factors, the third level of analysis was performed using a MANOVA if MKT is determined to be more than one distinct factor and the data meets assumptions for parametric tests. If MKT is one continuous factor then there are two types of analysis that could be done depending upon whether or not the assumptions for parametric tests have been met. If assumptions for parametric tests were met an independent t-test compared the

MKT of elementary and secondary teachers. When the assumptions were not met, a Mann-Whitney test was run. The data was also analyzed using three predictor variables including total experience and grade level experience in addition to teacher type. This was done using a Multiple Regression analysis. The participants were examined with respect to each of the three variables individually, and then compared while controlling for each variable. This was done to determine the interaction between the variables by comparing their significance and effect sizes individually and taken together. This study did not meet the assumptions required for generalization of inferential statistics, but this is not due to the design, but the resources and access to a sample population. This research was a test run of a methodology that would be more powerful with a larger random sample. The analysis was chosen to fit not only the design of the study, but the research questions being asked. This study examined two questions:

1. Is there a difference in MKT (Division by Fractions) between teachers when grouped in the following ways?
 - a. Elementary(4-5), Middle (6-8), and Secondary (9-12)
 - b. Elementary(4-8) and Secondary (9-12)
 - c. Those who hold a degree in mathematics and those who do not.
 - d. Type of license: Mathematics, Elementary, Early-Late Childhood, Other
2. Which of the following variables are the greatest predictors of MKT (Division by Fractions)?
 - a. Elementary, Middle, or High School
 - b. Type of License
 - c. Grade levels licensed to teach
 - d. Grade level Experience

- e. Total Experience

Independent variables

- a) Years Experience
- b) Degree attained
- c) Grade levels licensed to teach
- d) Type of license (Mathematics, Elementary, other)
- e) Experience teaching each grade level
- f) Present school assignment (Elementary, Middle, Secondary, or mixture)
- g) Department of teaching for methods course (Education, Mathematics, or both)

Dependent Variables

a) MKT of Division by Fractions

- a. Procedurally Accurate
- b. Correct procedural explanation
- c. Ability to create a story problem correctly representing the mathematical problem
- d. Error Analysis (procedural or conceptual)
- e. Alternate methods of performing the operation (procedures)
- f. Alternate methods for explaining the problem (models, including the measurement model, partitive model, factors and product model...)
- g. Correct justification for “invert and multiply”

Each of the independent and dependent variables will be measured through a survey administered online to individual teachers. This survey was created by the researcher using elements of existing research blended with items created by the researcher.

Population and Sample

Population

This study investigates teachers within two districts similar in size and geographic location at different grade levels in order to compare the CK and MKT of teachers across grade level. Looking at teachers within two districts roughly the same size and geographic location will control for differences within subjects due to selection with regards to geographic location and size of school. At the same time this will limit the generalizability to only teachers from similarly sized districts.

Sampling

A convenience sample would be taken from a sampling of approximately 53 K-12 teachers made up of 32 fourth and fifth grade teachers, 9 sixth, seventh, and eighth grade teachers, and 12 High School teachers from one 5A school district in northeast Kansas. A secondary school district was considered by the researcher for inclusion because of a participation rate lower than originally expected in the pilot testing. A sample of teachers in this district was comparably composed of 53 K-12 teachers made up of 38 fourth and fifth grade teachers, 6 sixth, seventh, and eighth grade teachers, and 9 High School teachers from this 5A school district also in northeast Kansas.

In order to investigate the feasibility of a sufficient sample, the researcher collected public information about the potential participants. A spreadsheet was created with the names of teachers who teach math to students in grades 4-12 in two districts in northeast Kansas. The topic of Division of Fractions is usually introduced and taught in upper elementary. This led the researcher to eliminate early elementary teachers from the population and focus on teachers of grades 4-12. Another variable of interest that does not correlate exactly with the grade taught is the type of license and degrees held by each teacher. For example, a middle school teacher could have an elementary license K-6, K-9, or a Math License 5-8, or 6-12.

Each district page had a link to a separate page for each school in the district. In one district there was one high school, one middle school, six elementary schools, and two alternative schools. In the other district there was one high school, one middle school, and four elementary schools. The home page of each school had a staff directory. The staff directory was very helpful in that it listed all the staff at that school with first and last name, title, and email address. The title would list what grade or subject matter they taught. For the elementary schools anyone listed as a fourth or fifth grade teacher was included. All middle school teachers were included who taught math. The author of this study was teaching mathematics full-time at one of the high schools in this study. All the teachers in the sample space were then looked up on the page: <https://online.ksde.org/TLL/SearchLicense.aspx>. The search results included all teaching endorsements on their license as well as when they completed any degrees. Teacher degrees with dates of completion, as well as all endorsements for each teacher were recorded. This information was used to theorize the different subgroups that existed within the sample population.

Sampling techniques

The researcher contacted district officials to gain permission for the research and access to the sample population. A district official from each district checked and verified the initial spreadsheet of teachers to ensure teachers taught mathematics to students in grades 4-12. In one district, the school official forwarded all emails from the researcher to a mailing list composed of potential participants. In the second district, the official gave the researcher permission to email the teachers directly. An initial email was sent out that briefly described the purpose of the study, the time commitment required to participate in the study and any potential costs and benefits of the study. The email included a link that would direct teachers to the MKT survey. The size of the sampling frame and the resources of the researcher prohibited distribution of any financial incentive to potential participants. Incentives only to those who participated in the study were also impractical as they would require the researcher to know the identity of respondents.

Data Collection

MKT Instrument

The instrument in this study was designed from a cognitive theory perspective that holds that teacher knowledge can be measured directly through questioning in the form of an interview, questionnaire, survey, test, or other method of direct examination. This contrasts theorists who view PCK and MKT as situated within the context of classroom interactions. The theories are not competing or contradictory, but rather complementary in that they view the same question through different lenses, and as such, ask different questions and give different answers.

A process known as Delphi methodology as described by Manizade and Mason (2011) was used in development of a new instrument for measuring MKT. This instrument was created

by the researcher and then referred to a panel of experts for review. These experts were professors at four different universities in Kansas. Each expert has a Ph. D in either Mathematics, or Mathematics Education. Two of the experts were members of the committee evaluating this researcher as a part of the dissertation process at Kansas State University. Two experts were faculty at Fort Hays State University who were recommended by the faculty researcher advising the researcher for this study. One is a professor at Pittsburg State University who teaches the secondary methods courses in the math department.

The expert panel committed to review and recommend changes to any items on the instrument in order to better assess teachers' MKT as defined by the researcher. Experts on the panel were asked to rate each item for face validity on a scale of 1–5 (1 being a poorly designed item and 5 being a very well designed item based on the definition of MKT as given by the researcher). Additional comments or questions for the researcher were requested.

The researcher sent out an email to the panel of experts describing the purpose of the research and how Mathematics Knowledge for Teaching was defined for this study. The email included a definition of Mathematics Knowledge for Teaching that was divided between two components. These components were: Knowledge of content and teaching, and Knowledge of content and students.

Knowledge of Content and Teaching -

- correct and complete solutions
- create a story problem correctly representing the mathematical problem
- multiple representations
- alternate methods or procedures for performing the operation
- able to justify “invert and multiply”

Knowledge of Content and Students -

- student misconceptions
- common mistakes
- alternate methods for explaining the problem (measurement model, partitive model, factors and product model...)
- able to diagnose and correct student errors

Each expert on the panel was sent a copy of the instrument with directions to review and recommend changes to an instrument designed to be given to mathematics teachers for grades 4-12. The purpose of the instrument was described as measuring teachers' mathematical knowledge for teaching, or MKT with regards to the very narrow strand of division by fractions. MKT was defined as having two main aspects with several corresponding subcomponents. First, knowledge of content and teaching was defined as the ability to: give correct and complete solutions, create a story problem correctly representing the mathematical problem, create multiple representations, explain alternate methods or procedures for performing the operation, and the ability to justify algorithms like "invert and multiply". Second, knowledge of content and students was defined as the ability to: identify student misconceptions, common mistakes, and diagnose and correct student errors. The panel of experts was given a timeline to respond with comments, questions, and feedback. After all the experts responded, the feedback was organized into a single document with all comments, questions, and feedback grouped by question. Changes were made to the wording of several items in order to clarify the meaning and purpose of each question. The survey was divided into two parts. Part one asked questions designed to

measure the participant's MKT with regard to division by fractions, and part two asked questions about the participant's educational background and experience.

The first two items in part one were unchanged in order to maintain consistency with previous studies that used the same questions (Ball, 1990, Ma, 1999, Leung & Park, 2002). In other items wording was changed to clarify or focus the intent of the question. More than one expert pointed out that the original survey referred back to previous problems and questions, but in the online survey only one question could be viewed at a time. This caused the researcher to restate any problem or question that was referenced so that each item could be understood without looking at previous questions or responses. One item was moved in the order of questions to prevent the participant from learning how to answer the question from previous items on the survey.

Most of the items in part two were modified for clarity. This was to ensure that teachers from any background and teaching situation would understand how to answer questions about their current teaching assignment, the type of license they hold, their experience teaching each grade level, and the department in which they took their methods of mathematics instruction course.

This process was repeated once more. After the experts reviewed the revised instrument along with all the feedback from the panel they were given the opportunity to make additional comments, suggestions, or questions. After the second round all experts were satisfied with the changes made to the instrument.

After pilot testing, one additional change was made to the instrument. One of the experts had mentioned the order of an item asking for alternate methods of solving the problem. This expert pointed out that learning can take place in a survey when participants learn from items at

the beginning of the instrument which affects their responses on items later in the survey. The researcher noticed some participants that didn't have a successful strategy at the beginning of the survey were using a strategy that was part of a question about how to respond to a student. For this reason, the item that included a strategy that could be used by the participant to solve the problem was moved after the item that was asking the participant for alternate strategies.

The following is a discussion of each question in the instrument. The origin of each question, the relevance of each question, the specific aspect of MKT the question is attempting to measure, and the connection of each question to existing research is included.

The first two questions come directly from existing research. In her comparative analysis of elementary teachers in China and the United States, Ma (1999) used the following questions:

1. "People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?"

$$1\frac{3}{4} \div \frac{1}{2} =$$

2. Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2} = ?$?"

The preceding two questions are the first two questions on the instrument. The first question will measure content knowledge and procedural accuracy. It will be open ended so there will likely be a variety of responses including correct solution and correct reasoning, incorrect solution and correct reasoning, correct solution and incorrect reasoning, incorrect solution and incorrect

reasoning, and incomplete solution or incomplete. The second question from Ma's research measures the teacher's ability to accurately choose a situation that represents this mathematical problem and correctly communicate that representation. The third and fourth items measure the teacher's ability to recognize and correct student errors. The third item contains incorrect student reasoning, while the fourth contains correct reasoning. The teacher is asked to comment on what they would say about the validity the student's thinking in each question. The fifth item is the last of the error analysis items. It gives a student's story problem that was created to represent $1\frac{3}{4} \div \frac{1}{2}$. There is an error in the student's story and the teacher is asked what they would say to the student about the correctness of their story problem. The sixth item asks teachers to list pieces of knowledge they believe are important for understanding the concept of division by fractions. This comes from existing research that teachers with greater MKT make more connections between related concepts and skills. The seventh item asks the teacher to give an alternate problem solving pathway for a student who didn't understand the first method. This item measured the teacher's ability to solve a problem more than one way. The eighth item asks the teacher to identify any common student errors that might arise when trying to solve a problem like the one given. This relates to the teacher's knowledge of the student and common misconceptions. The ninth and last item relates to justifying the invert and multiply rule for fraction division. The teacher is asked to justify why inverting and multiplying is a valid procedure for division with fractions. This item is testing the depth and conceptual knowledge of the teacher.

Four elements of MKT are highlighted in this instrument. Procedural accuracy, error analysis, accurate and meaningful representation of the problem, and procedural flexibility are all addressed in this instrument.

Procedural accuracy. A teacher should be able to accurately divide by fractions, and justify their work with an explanation that is free from error and describes the procedure they used to come to the correct definition.

Error Analysis. Error analysis will be coded by examining several aspects of the response to an incorrect student solution: First, is the teacher able to detect the error in student reasoning? If so, that is a point of MKT. Second, can the teacher isolate and correct the student's calculation/procedural error? If so, that demonstrates MKT. Third, is the teacher able to identify conceptual problems that may lead to such an error? Points will be given for conceptual misunderstandings that would result in the student making the given error. Fourth, is the teacher able to recommend a course of action to clarify student misconceptions? MKT involves knowing how to remedy misconceptions and clarify important concepts.

Representation. Accurate Representation of the problem was measured by asking the teacher to give a story that illustrates the mathematical problem under discussion. A story that accurately represents the mathematical problem is an indicator of MKT. If the teacher is able to produce stories that represent the problem from different perspectives that would demonstrate a deeper, more extensive MKT. With the division of fractions problem, Ma (1999) identified at least three different types of stories/models. The first is the product and factors model which asks, "What factor times $\frac{1}{2}$ is equal to $1\frac{3}{4}$?" The second is the partitive model which asks, "How many halves are in $1\frac{3}{4}$?" A third model asks, "What is $1\frac{3}{4}$ half of?"

Procedural flexibility. How many different procedures can the teacher use? Invert and multiply is very common, but there are more ways to solve the problem. The problem can be solved visually by creating $1\frac{3}{4}$ wholes and then dividing the whole into halves and the $\frac{3}{4}$ into 1

$\frac{1}{2}$ halves. The dividend can be converted into an improper fraction and then the numerator of the dividend can be divided by the numerator of the divisor. Likewise the denominator of the dividend can be divided by the denominator of the divisor.

The instrument was designed to incorporate many aspects of existing research in order to create the opportunity to look for patterns and make comparisons between this research and existing research. This instrument intentionally contained items identical to previous research, but the methodology and analysis are different. Ma (1999) used some of the same questions in her interviews with Chinese and American elementary school teachers. She was able to question and probe her participants for a deeper than surface level understanding. Hill, Schilling and Ball and Ball have sought to measure MKT through similar means, but have also done work to create a cognitive instrument composed of multiple choice items. Krauss et al (2008) constructed a 34 item open ended instrument which was designed to measure MKT on three different scales; knowledge of mathematical tasks (Task) 13, knowledge of student misconceptions and difficulties (Student) 11, and knowledge of mathematics-specific instructional strategies (Instruction) 10.

MKT Coding Rubric and Instrument Reliability

In order to check the inter rater reliability for the instrument a pilot test was performed. Two professors at a large university provided the researcher with the contact information for pre-service teachers. These students came from two different classes, one was made up of pre-service elementary teachers and the other group was pre-service secondary teachers of mathematics. 31 students were invited to participate in this pilot study. Students were told that participation was completely voluntary and had no impact on their standing in their class. Out of

the 31 students asked to participate, eleven responded to the researcher by email consenting to participate. The researcher then distributed the survey to these eleven students and asked them to complete the survey within one week of the initial invitation to participate. Because some students responded later than others, the timeline for participation was extended. The researcher ended the survey after one week. Upon first inspection, it appeared that nine of eleven participants had completed the survey, but after reviewing their responses, two participants terminated the survey after answering the first question without looking at the rest of the survey. These two surveys were not included in the analysis. The initial coding was then performed with the seven remaining fully completed surveys.

The entire process was repeated, but the second time it was done with in-service teachers at the elementary, middle, and high school level. These teachers were referred to the researcher by one of the experts on the panel that reviewed the instrument. Seven in-service teachers completed the survey and the results were analyzed along with one additional pre-service teacher who had volunteered to participate after the first round had ended. Surveys were coded using an updated rubric with the researcher and another rater scoring the survey responses independently.

This instrument was designed to quantify MKT for teacher based on open-ended responses. This requires that the researcher read and code the data according to a coding scheme or rubric. The researcher developed a coding rubric for each item based on existing methods used in previous research (Ma, 1999), or on qualitative heuristics that are defined differently by researchers. The researcher used coding that was consistent with similar methods wherever possible. For example, when Ma (1999) analyzed the question used in item 1, she divided the responses into 5 categories: 1) correct algorithm, complete answer, 2) correct algorithm, incomplete answer, 3) Incomplete algorithm, unsure, incomplete answer 4) Fragmentary memory

of the algorithm, no answer, 5) Wrong strategy, no answer. These response categories were divided into two parts for the purposes of coding and quantifying the responses. The correctness of the answer was coded 0 if incorrect and 1 if correct, while the explanation of the process, i.e. the algorithm was scored as a 2 if it would work for all division by fraction problems, 1 if it worked in this situation but not every division by fractions problem, and 0 if the explanation was missing, incomplete, or incorrect.

The purpose for conducting a pilot study was to investigate and verify the instrument's reliability. One of the experts that participated in the panel reviewing the instrument items agreed to score a sample of the responses from the pilot study. The researcher spoke with this expert via video conference about the coding process and instrument rubric. The researcher and expert discussed how to code each item. This was done before administering the survey. Once the responses from the pilot study were collected, the researcher and the expert colleague scored each survey according to the rubric independently. Every survey that was completed was scored because of the small number of responses. The scores on each item for each participant were then analyzed.

There are several statistical measures of reliability when comparing two raters who are coding data. Each measure is designed with specific criterion and assumptions. The researcher examined a variety of tests in selecting the appropriate method for comparing the ratings of the two coders and determining the inter-rater reliability. Cronbach's (2004) alpha was selected and measured at each stage of pilot testing analysis. Initial pilot testing took place with two rounds of different participants. In the first round, seven pre-service teachers were surveyed, in the second round seven in-service teachers and one pre-service teacher were surveyed.

Preliminary analysis of rater reliability indicated the need to revise the scoring rubric for three items. Table 3-1 displays the number of items on which the raters agreed as well as the number of items in which the raters disagreed by at least one point. Item one was divided into two parts which resulted in 10 items total.

Table 3-1
Rater agreement by question number – Round 1

	Difference in scores										Item
	between raters										
	1a	1b	2	3	4	5	6	7	8	9	Total
0	100%	86%	86%	57%	100%	86%	86%	29%	14%	29%	67%
1	0%	0%	14%	29%	0%	14%	14%	29%	86%	71%	26%
2	0%	14%	0%	14%	0%	0%	0%	14%	0%	0%	4%
3	0%	0%	0%	0%	0%	0%	0%	14%	0%	0%	1.4%
4	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
5	0%	0%	0%	0%	0%	0%	0%	14%	0%	0%	1.4%
α	1.00	.77	.91	.63	1.00	.91	.91	.86	-.46	.30	.84

There was a large amount of disagreement for items seven, eight, and nine. The rubric was discussed and then revised by the lead researcher. The same data was then rescored on item seven after clarification of the language in the rubric. Items 8 and 9 had several items scored differently, but a consensus was reached without any change to the wording of the rubric.

There was very little improvement in rater agreement, so the researchers discussed why they were giving each rating and what language in the rubric could be clarified. Examples of what would constitute a score of each value were discussed. Item seven asked participants to give any additional methods for solving the original division by fractions problem. The original and revised rubric for item seven required the scorer to compare the response for item seven to

the participant's response to item one. This was not something the second rater had realized. After clarifying this point, there was much greater agreement about the clarity of the rubric. This revised rubric was used in the second round of the pilot study and had a higher rate of agreement. The results are given below in Table 3-2.

Table 3-2
Rater agreement by question number – Round 2

	Difference in scores										Item
	between raters										
	1a	1b	2	3	4	5	6	7	8	9	Total
0	100%	75%	75%	87.5%	62.5%	62.5%	75%	75%	100%	62.5%	77.5%
1	0%	25%	25%	12.5%	0%	37.5%	25%	12.5%	0%	37.5%	17.5%
2	0%	0%	0%	0%	37.5%	0%	0%	12.5%	0%	0%	5%
α	1.00	.91	.94	.77	.69	.85	.64	.92	1.00	.62	.92

Each item that was rated differently was discussed and a consensus was reached. The results of the second round of pilot testing show items 1a, 3, 7, 8, and 9 increased the number of items in perfect agreement. Items 1b, 2, 4, 5, and 6 had a decrease in the percentage of items in perfect agreement, while the total variance from round 1 to round 2 decreased from 1-5 points in round 1 to 1 or 2 points in round 2. One observation that may contribute to this decrease in agreement is the difference in sample populations between the two rounds of pilot testing. The first round was done with pre-service teachers who were university students, and the second round was done with in-service teachers. Cronbach's alpha was calculated at each stage to measure the inter-rater reliability. Alpha increased at each stage with round one $\alpha = .84$, and round two after discussing and clarifying items on the scoring rubric $\alpha = .92$.

Summary

This research blends examines existing research questions from new combinations of methodologies and perspectives. All questions in the MKT survey are either identical to questions from previous and well known studies, (Leung & Park, 2002; Ma, 1999; Zhou et al, 2006), or are grounded in theoretical constructs found in existing research (Ball, 1988,2000; Hill et al, 2004; Ma, 1999; Manizade & Mason, 2011; Shulman, 1986 Tirosh, 2000). The design and methodology of the study was unique in multiple ways. The delivery of the MKT instrument was through an on-line survey. The open-ended responses to questions from existing research were coded for quantitative analysis using a rubric created by the researcher. The design of the research allowed for the possibility of data collection on a large scale that could be generalized to larger populations. Finally, the sample population for this research was unique in the comparison of in-service teachers for grades 4-12.

Appendices

Appendix 1 – Mathematics Knowledge for Teaching Survey

Appendix 2 – Invitation to participate in the study

Appendix 3 – Expert Panel Feedback

Chapter 4: Results

Overview

The purpose of this study was to examine the MKT of teachers across a wide range of grade levels, experience, educational background, and areas of instruction in the domain of division by fractions. The researcher investigated if there were significant predictors of MKT among the individual independent variables of type of license, grade levels taught, years experience, educational background, or some underlying factor(s) composed of a combination of those variables. This research synthesized interview questions from existing small scale qualitative studies with an online survey format and a scoring rubric developed by the researcher to enable the researcher to quantify responses previously analyzed from a primarily qualitative methodology.

This chapter begins with a description the survey sample. The total sample size, as well as the sample sizes of individual subgroups of the sample, played an important part in determining which statistical tests were valid and to what extent they could be generalized. This research began with a small sampling frame which was expanded during the data collection process as the result of several factors. The analyses performed in this study were determined largely by whether the data supported certain hypotheses made by the researcher. The researcher hypothesized there may be some underlying factor(s) composed of a combination of the individual independent variables of school workplace, type of license, grade levels licensed to teach, and educational background. This was tested through a principal component analysis of the independent variables. Educational background included what department provided instruction for their methods of instruction course, whether the teacher held a degree in mathematics, and whether the teacher held a Master's degree. Experience variables included the

type of school where the teacher worked and the years experience teaching at each level (elementary, middle, and secondary). The type of teaching license was also considered in the Factor Analysis. The researcher was asking if these variables form two (Elementary and Secondary) or three (Elementary, Middle, and Secondary) distinct groups? The sample included teachers with very diverse combinations of these independent variables. This produced a lack of clearly defined factors arising from these variables.

The chapter continues with an analysis of the nine items which were assigned ten scores. Item one received two scores, one score for the accuracy of the answer and another score for the explanation of how to find the answer. The researcher conducts similar tests to determine if MKT can be described as one composite variable, does it exist as several distinct factors, or do the ten scores on the nine items correlate at all. Analysis revealed that MKT could be viewed as one variable, but the data was not suitable for identifying factors within the data.

Once the independent and dependent variables were analyzed, the researcher investigated whether or not the data met the assumptions required for parametric tests. Four main assumptions were checked including the normality of the mean MKT scores within each group, the homogeneity of the variance of groups that were being compared, the use of MKT scores as interval data, and the independence of scores among participants. The results showed that the data did meet the assumptions necessary for parametric tests.

Research question one was concerned with whether there was a difference in MKT for division by fractions for teachers grouped in four different ways. Two of those groupings had two categories. Dividing the teachers as Elementary (4-8) and Secondary (9-12) and those teachers who hold a degree in mathematics and those who do not. To answer these two comparisons independent t-tests were conducted. The other two groupings had three categories.

Elementary, Middle, and Secondary school teachers, and those holding an elementary license, a mathematics license, and those holding both an elementary and a mathematics license. These two groupings were compared using one-way independent ANOVAs.

The final research question was concerned with which of the variables was the greatest predictor of MKT for division by fractions. This researcher sought to answer this question using a multiple regression analysis. With a sample size of 29, the regression analysis would only be suitable for comparing two or three predictors. The researcher split this question into two smaller comparisons. The two multiple regressions were run and the results are included.

Response Rates

During the two rounds of pilot testing the researcher recorded the response rate at each stage. The first level of pilot testing was done with pre-service teachers at a large Midwestern university. These students came from two different classes, one was made up of pre-service elementary teachers and the other group was pre-service secondary teachers of mathematics. 31 students were invited to participate in this pilot study. Out of the 31 students asked to participate, eleven responded to the researcher by email consenting to participate, nine participants answered only the first question, and seven students completed the surveys. This resulted in a response rate of 23%. The entire process was repeated, but the second time it was done with in-service teachers at the elementary, middle, and high school level. 34 teachers were invited to participate in this round. Seven in-service teachers completed the survey along with one additional pre-service teacher who had volunteered to participate after the first round had ended. This increased the response rate for round one to 26%, and round two had a rate of 21%. This is consistent with the findings of Shih and Fan (2008) that university students are more respondent to web surveys when compared with mail surveys, while teachers tend to be more

responsive to mail surveys in comparison to web surveys. This response rate can be compared with Archambault and Barnett's (2010) response rate of 33% (596 responses to 1795 emails) in their large scale study of technological pedagogical content knowledge (TPACK) through an online survey with a population of online teachers for virtual schools. In their analysis of 39 comparative studies that incorporated mail and web surveys, Shih and Fan (2008) found the mean response rate of online surveys to be 11% lower than that of mail surveys with mean response rates of 34% and 45% respectively. For these reasons, I anticipated a response rate of 20-30%.

Dropout Rates

The sampling frame for the two combined districts consisted of 106 teachers, made up of 70 fourth and fifth grade teachers, 15 sixth, seventh, and eighth grade teachers, and 21 high school teachers. The response rates by number of questions completed and the percentages of the sampling frame are given in Table 4-1.

Table 4-1

Response Rate by Question

Action Taken	n	%
Clicked on Survey link	67	63%
Answered Question 1	53	50%
Answered Question 2	44	42%
Answered Question 3	38	36%
Answered Question 4	36	34%
Answered Question 5	36	34%
Answered Question 6	34	32%
Answered Question 7	33	31%
Answered Question 8	31	29%
Completed the survey	29	27%

The dropout rate was sharp at first and then decreased as participants answered each additional question. Of the 38 people who opened the survey but did not complete it, 23 (61%) did not answer the second question. On the other hand, of those participants who completed question 4, 29 of 36 (81%) completed the entire survey.

The two districts in this study had a very similar distribution of faculty with multiple elementary schools in different locations in the district and a single middle school and high school for the entire district. This led to an imbalanced number of each teacher type in the sampling frame. The actual sample was more balanced in the number of elementary, middle, and secondary teachers. The response rate by teacher type is given in Table 4-2.

Table 4-2

Sample Composition by Elementary, Middle, and Secondary Level

Type of Teacher	N	Percent of Sampling Frame	n	Percent of sample
Elementary	70	66%	13	45%
Middle	15	14%	9	31%
Secondary	21	20%	7	24%

Note. N is the number of teachers in the population, n is the number of teachers in the sample.

Response rates by teacher type are given in Table 4-3. Elementary teachers made up the majority of the sample, but were still the least likely to respond when compared to secondary teachers who were almost twice as likely to respond, and middle school teachers who were five times as likely to respond. In order to increase the representativeness of the sample and make inferences about a larger population, each subgroup of the sample could be weighted so that it was not over or under representing any particular group.

Table 4-3

Response Rate by Teacher Type

Type of Teacher	N	n	Response Rate
Elementary	70	13	19%
Middle	15	9	60%
Secondary	21	7	33%

Note. N is the number of teachers in the population, n is the number of teachers in the sample.

Coding

The next step was coding the data. Coding the first question resulted in further refinement of the rubric. The first component the instrument is designed to measure is whether teachers can correctly divide by fractions. The original rubric coded responses to the first question as correct or incorrect. Survey responses at each stage of pilot testing and data collection resulted in responses that correctly described how to find the correct answer, but did not explicitly give a correct answer. The wording of the question was identical to questions used

in multiple studies (Ball, 1988, 1990; Li & Huang, 2006; Leung & Park, 2002; Ma, 1999; Zhou, Peverly & Xin, 2006). This wording was used in the context of interviews. Question 1 reads:

“ People seem to have different approaches to solving problems involving division with fractions. *How would you solve a problem like this one?* (Please be as detailed as possible.)

$$1\frac{3}{4} \div \frac{1}{2} =$$

This question was scored as two separate items. The first score is based on the correctness of the answer given, and the second score is based on the explanation of how they arrived at their score. The wording of the question could be interpreted as asking exclusively about the process without regard for the actual solution to the specific problem. This seemed to be the case for several participants who gave a correct explanation of how they would solve the problem without actually solving it. Three responses are given below where the participant described how they would solve the problem without actually solving it.

Response 1

“I would convert to an improper fraction on the first one. Then I would change it to multiply by the reciprocal of the second one. I would reduce top and bottom if possible and the multiply straight across.”

Response 2

“Change the mixed number into an improper fraction. Then do the steps keep, change, flip. Keep the improper fraction, change the division operation into multiplication and flip the second fraction from 1/2 to 2/1. Multiply across then simplify.”

Response 3

“Change to improper fractions, find reciprocal of 1/2, then multiply across.”

These responses led the researcher to broaden the rubric scores from two possible scores to four possible scores. While the previous three responses do not give a correct answer, they also do not give an incorrect answer. Two additional score possibilities were added:

3 points - Correct solution given of $3\frac{1}{2}$

2 points – *No solution given, but the explanation would produce the correct result of $3\frac{1}{2}$.*

1 point – *No solution given, the explanation contains no errors, but does not explicitly state each step necessary to reach a correct and simplified answer of $3\frac{1}{2}$.*

0 points - Incorrect solution or no solution given, and the explanation is not sufficient to guarantee a correct result of $3\frac{1}{2}$.

Responses 1 and 2 received scores of 2 points because they did not contain the correct answer, but their explanation would produce the correct answer if performed. Response 3 received a score of 1 point because it did not include the solution, contained no errors, but if the process were followed it would produce the answer of $14/4$. This does not address the simplification of the answer.

Rater Reliability

Once the data was coded according to the revised rubric a sample of nine participants' responses were coded by a second rater. Scores were compared by percent agreement and using Cronbach's alpha. Cronbach's alpha was roughly equivalent to the first round of pilot testing at $\alpha = .86$. The results for item agreement are included below in Table 4-4. Raters did not discuss or review the rubric with one another prior to scoring this sample. The second round of pilot testing had the highest rates of agreement and reliability. Raters discussed the rubric at length after round one and scored round two immediately after their discussion. This leads the

researcher to hypothesize that if the study was replicated another rater would achieve acceptable results, but discussion of the rubric between raters would likely decrease measurement error and increase rater reliability.

Table 4-4
Rater agreement by survey item – original rubric

Difference in scores given	Percent of MKT survey item scores at each level of agreement										
	1a	1b	2	3	4	5	6	7	8	9	Total
0	78%	78%	89%	33%	89%	78%	55%	78%	56%	67%	70%
1	22%	22%	0%	56%	11%	22%	22%	11%	44%	33%	24%
2	0%	0%	11%	11%	0%	0%	22%	11%	0%	0%	6%
α	.92	.71	.78	.48	.94	-.29	.44	.90	.85	-.44	.86

Although the inter-rater reliability looked very good at this point, subsequent analysis of the MKT instrument and rubric revealed a problem. The consistency of the items was low which could occur for a variety of reasons. One reason could be that there weren't enough participants for alpha to accurately describe the consistency between items. Another could be that the items do not measure the same thing, and therefore they would not have a high alpha for internal consistency. A third possibility seemed to be more likely. Cronbach's alpha is calculated using the sum of the item variances divided by the total variance in the scores. If the individual items are scored on too narrow of a scale, then there is not enough descriptive space to accurately measure or "capture" the true variability of the participants. On examination of the 10 scored items on the scale, the researcher noted that five of the ten items had only three possible scores, three had four possible scores, one had a range of five possible scores, and one of the ten items gave a score for each additional problem solving strategy and could receive a score of six or

more. The researcher read through the description of scores for each item. Several item score descriptions had multiple parts that could be split into two different scores, thereby increasing the scale's variability. An example is given below in Table 4-5 from the scoring rubric for item four. The original rubric is compared with the expanded rubric.

Table 4-5

Original Rubric for Item 4

Score	Description
0	The teacher incorrectly accepts the student's story as correct.
1	The teacher recognizes the student is dividing by 2 instead of $\frac{1}{2}$ and states that the story is incorrect, but does not correct the illustration.
2	The teacher addresses the error in the student's reasoning <u>and</u> does one of the following: 1. Corrects the student's example so that it does illustrate the problem. 2. Gives an alternate way of stating the problem.

This original rubric did not have the ability to capture the differences that were seen in responses to this question. There were many responses that did not fit one of the three descriptions. Two additional scores were added to account for additional types of responses. The revised rubric for item four is given in Table 4-6 below.

Table 4-6

Revised and Expanded Rubric for Item 4

Score	Description
0	The teacher incorrectly accepts the student's story as correct.
1	The teacher makes a comment that neither accepts nor rejects the student's story.
2	The teacher states that the story is incorrect, but does not correct the illustration or identify the student error as division by 2.
3	The teacher recognizes the student is dividing by 2 instead of $\frac{1}{2}$ and states that the story is incorrect, but does not correct the illustration.
4	The teacher addresses the error in the student's reasoning and does one of the following: 1. Corrects the student's example so that it does illustrate the problem, or 2. Gives an alternate way of stating the problem correctly.

This revised and expanded rubric added 10 additional score values, increasing the original number of possible scores from 35 to 45. The use of a revised and expanded scale led the researcher to conduct another test of rater reliability using scores from seven different participants from the sample. Cronbach's alpha using the expanded 45 point scale was $\alpha = .82$. This was a drop from the reliability of $\alpha = .86$ with the 35 point rubric. The researcher and the independent rater discussed all differing scores and came to a consensus score for all items. The results for item agreement are included below in Table 4-7.

Table 4-7

Rater agreement by survey item-Expanded Rubric

	Difference in scores between raters		Percent of MKT survey item scores at each level of agreement								
	1a	1b	2	3	4	5	6	7	8	9	Total
0	100%	57%	86%	57%	57%	71%	29%	57%	56%	14%	59%
1	0%	43%	14%	29%	43%	0%	29%	29%	44%	71%	30%
2	0%	0%	0%	14%	0%	14%	29%	14%	0%	14%	9%
3	0%	0%	0%	0%	0%	14%	0%	0%	0%	0%	1%
5	0%	0%	0%	0%	0%	0%	14%	0%	0%	0%	1%
α	1.00	.55	.98	.75	.92	.62	.46	.94	.92	.40	..82

Mathematics Knowledge for Teaching - Composite Scores

Once the survey responses were scored for each participant and the rater reliability was checked, the item scores were analyzed to answer two main questions:

1. Do the composite scores for each participant reflect one construct, namely MKT with regards to division by fractions?
2. Do the scores for participants group together into individual subcategories of MKT (Procedural accuracy, error analysis, accurate and meaningful representation of the problem, and procedural flexibility)?

Cronbach's alpha for the nine items in the scale was $\alpha = .69$, and when the item scores were standardized this produced a slight increase to $\alpha = .71$. With an acceptable level of internal consistency a principal components analysis was done to investigate whether distinct factors exist within the scores on the scale. The Kaiser-Meyer-Olkin measure of sampling adequacy

was $KMO = .45$. Although this value suggested that the data was not suitable to be viewed as distinct and reliable factors, it is worthwhile to note that the sample size was well below the acceptable number needed to produce reliable factors. Field (2009) cites two different recommendations for sample size. Nunnally (1978) recommended having ten times as many participants as variables, and Kass and Tinsley recommended between 5 and 10 participants per variable. With the nine items on the MKT survey resulting in ten scores this would require a minimum sample size of 50-100 participants. The results of these two measures were taken together by the researcher as evidence that the scores on the MKT survey could be summed for a single composite score, but should not be viewed as representing smaller distinct factors.

Assumptions for parametric tests

The next question addressed by the researcher was whether or not the data met the assumptions for parametric tests. These four assumptions are: 1) Normally distributed data, 2) Homogeneity of variance, 3) Interval data, and 4) Independence. These assumptions were addressed for the sample as a whole as well as for each group. All assumptions were met in nine of the ten groups. Teachers with an elementary license only did not meet the assumption of normality. This resulted in a non-parametric test statistic for this comparison.

Normality

This assumption is that the distribution of the sample means is approximately normal. This was checked using two tests. The first examined if there was significant skewness or kurtosis in the distribution. The second was the non-parametric Kolmogorov-Smirnov test. None of groups of interest were found to have significant skewness or kurtosis. The MKT composite score for the sample, $D(29) = .17$, and for the group elementary licenses only, $D(15) =$

.23, $p < .05$, were significantly non-normal. The fact that the sample as a whole was significantly non-normal according to the K-S test was not of concern since the analyses in this study were more concerned with comparing groups within the sample than describing the sample as a whole. The non-normality of the group of teachers with an elementary license only was a cause for a closer look. ANOVA is a robust test, and these groups were normal in terms of kurtosis and skew. The non-normality displayed in the K-S test was not confirmed by the Shapiro-Wilk's test which was not significant. The size of the sample group was 15, which was over half the sample. It was the judgment of the researcher that there was not enough evidence to reject the null hypothesis that the sampling distribution of the mean for elementary license group was normal. The results are given in Table 4-8 below.

Table 4-8

Tests of Normality – Skewness, Kurtosis, and Kolmogorov-Smirnov

Group	Skewness		Kurtosis		Kolmogorov-Smirnov		
	Statistic	Sig.	Statistic	Sig.	Statistic	df	Sig.
Sample	-.18	.86	.14	.89	.17	29	.03*
Elementary School	-1.04	.30	-.73	.47	.20	13	.19
Middle School	1.21	.23	1.56	.12	.23	9	.17
High School	.82	.41	.70	.48	.25	7	.20**
Elementary (4-8)	.47	.64	.40	.69	.18	18	.12
Secondary (9-12)	.09	.93	-1.13	.26	.17	11	.20**
Mathematics Degree	1.04	.30	.04	.97	.22	8	.20**
Non Mathematics Degree	.15	.88	-.04	.97	.15	21	.20**
Elementary only	-.79	.43	-1.05	.29	.23	15	.03*
Mathematics only	-.76	.45	-.03	.98	.22	7	.20**
Elementary and Mathematics	.46	.65	.89	.37	.23	6	.20**

* $p < .05$, **This is a lower bound of the true significance.**Homogeneity of variance**

The second assumption that was tested was homogeneity of variance. Levene's test was run for the sample as a whole, and for each of the subgroups of the sample. None of the statistics were significant, so the null hypothesis that the variances were equal was not rejected. The results of Levene's statistic are included in Table 4-9 below.

Table 4-9

Homogeneity of Variance

Group	Statistic	<i>df</i> 1	<i>df</i> 1	Sig.
Elementary, Middle, and High School	1.13	2	26	.34
Elementary (4-8) and Secondary (9-12)	.05	1	27	.83
Mathematics Degree and Non Mathematics Degree	.73	1	27	.40
Elementary, Mathematics, and Both Elementary and Mathematics	.43	2	25	.65

* $p < .05$

For MKT scores, the variances were equal for Elementary, Middle, and High School teachers, $F(2,26) = 1.13$; Elementary teachers grades 4-8 and Secondary teachers grades 9-12, $F(1,27) = .05$; Teachers licensed in Elementary, mathematics, or both elementary and mathematics, $F(2,25) = .43$; and Teachers who held a mathematics degree and those who do not, $F(1,27) = .73$, *ns*.

Interval Data

This assumption is one that can be hard to define in the social sciences. Two of the nine items used on the survey were clearly interval data. Question five from the survey asked for related knowledge and skills necessary for an understanding division by fractions. The rubric gave a score of zero for no knowledge or skills identified, one point for one skill or knowledge package, two points for two, and so on. Question eight from the survey asked for common errors or misconceptions students have when dividing by fractions. The rubric gave a score of zero for no misconceptions identified, one point for one error or misconception, two points for two, and so on. Other items were not so concretely interval over ordinal. Welkowitz and Lea (2011) discuss the decision that social scientists must make when determining whether data is qualitative (nominal, categorical, ordinal) or quantitative (interval/ratio). They note that often

the only clear distinctions are at the extremes of a spectrum like color as a categorical variable, and weight as a ratio variable. Often in between those types of variables the distinction is less clear. Social scientists often deal with such variables that seem to be interval and treat them as such, even though they lack the clearly defined differences between scores seen in the physical sciences. This was the decision in this study.

Independence

This is the assumption that the data collected from each participant was independent. The 106 teachers in the sampling frame came from two districts with 53 teachers in each district. These teachers came from 14 different schools within those two districts. The small number of schools of different types resulted in some teachers in the sample coming from the same school. There could be correlation among participants by school that was not identified. This was due to the design of this study that protected the anonymity the participants because of the researcher's working relationship within each of the two districts in the study. Surveys were administered online to each participant. Each participant was directed to complete the survey using only their own knowledge without referring to any outside sources. Although there was no evidence of one participant influencing any other participant it was possible given the lack of any proctor for the survey.

Based on investigation in to the assumptions of parametric tests and the nature of the data collected the researcher was satisfied that the assumptions were met and the data could be analyzed using the parametric statistical tests.

Comparing MKT for Different Groups

The first research question asked if there was a significant difference in MKT for division by fractions between various groups. This question looked at differences that may exist between teachers when grouped four different ways. The question was:

1. Is there a difference in MKT (Division by Fractions) between teachers when grouped in the following ways?
 - a. Elementary(4-5), Middle (6-8), and Secondary (9-12)
 - b. Elementary(4-8) and Secondary (9-12)
 - c. Those who hold a degree in mathematics and those who do not.
 - d. Type of license: Mathematics, Elementary, Early-Late Childhood, Other

Each of these parts (a-d) was addressed by the researcher.

Elementary, Middle, and Secondary teachers

In order to compare the means of these three separate groups a one way independent ANOVA was performed. The three groups were teachers at elementary schools, middle schools, and high schools. The researcher chose to use stepwise ordering because this research was exploratory in nature and there was a lack of existing research or a theoretical basis for performing planned contrasts. The stepwise process chosen was backward which had greater power for detecting a small effect. When there is no correlation between the independent predictor variables the order of the entry of variables into the ANOVA doesn't matter. The groups in this analysis were mutually exclusive with each teacher teaching at only one of the three schools. This made the predictor variables uncorrelated. Descriptive statistics for the three groups are given in Table 4-10 below.

Table 4-10

MKT scores by teacher type

	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean	
					Lower Bound	Upper Bound
Elementary	13	16.62	5.92	1.64	13.04	20.20
Middle	9	23.11	4.70	1.57	19.50	26.73
High School	7	14.43	4.83	1.82	9.97	18.89
Total	29	18.10	6.23	1.57	15.73	20.47

There was a significant effect of the teacher type by school on MKT composite scores, $F(2,28) = 6.15$, $p < .01$. There was a significant quadratic trend, $F(1,28) = 12.23$, $p < .01$, $\omega = .51$, indicating a large effect that middle school teachers scored significantly higher than both elementary and secondary teachers. Both Gabriel's and Bonferroni's post hoc tests were conducted because the analysis was exploratory and the sample sizes of the groups were different. Bonferroni's test indicated that there was no significant difference between elementary and secondary teachers, $p = 1.00$, $d = 2.19$, but there were significant differences between middle school teachers and both elementary, $p = .03$, $d = 6.50$ and secondary teachers, $p = .01$, $d = 8.68$. Gabriel's test gave the same results of no significant difference between elementary and secondary teachers, $p = .76$, $d = 2.19$, but there were significant differences between middle school teachers and both elementary, $p = .03$, $d = 6.50$ and secondary teachers, $p = .01$, $d = 8.68$.

Elementary and Secondary teachers

In this analysis teachers were identified as secondary if they met at least two of three criteria. The three criteria were: teaching at a high school, holding a mathematics license, and holding a license to teach high school students. If teachers met one or fewer of these criteria

they were considered to be an elementary teacher. On average, elementary teachers received higher MKT scores ($M = 18.83$, $SE = 1.57$) than secondary teachers ($M = 16.91$, $SE = 1.68$). This difference was not significant $t(27) = .80$, $p > .05$; however, it did represent a small-sized effect $r = .15$.

This analysis required that the researcher assign middle school teachers to either the elementary or secondary group. In order to determine if a middle school teacher should be considered elementary or secondary the researcher examined two variables. If the middle school teacher had a mathematics license and were licensed to teach mathematics in the high school then they were considered secondary. If they did not have both of those qualities they were considered elementary. This divided the nine middle school teachers roughly in half with four being classified as secondary and the remaining five classified as elementary. Descriptive statistics for each group, elementary and secondary, are included in Table 4-11 below along with the descriptive statistics for each of the four subgroups: elementary (4-5), middle elementary, middle secondary, and secondary.

Table 4-11

MKT scores by teacher type

	N	Mean	Std. Deviation	Std. Error
Elementary	18	18.83	6.65	1.57
Elementary (4-5)	13	16.62	5.92	1.64
Middle Elementary	5	23.11	4.70	1.57
Secondary	11	16.91	5.58	1.68
Middle Secondary	4	21.25	4.11	2.06
Secondary (9-12)	7	14.43	4.83	1.82
Total	29	18.10	6.23	1.57

Mathematics and Non mathematics degree holders

In this analysis teachers were separated by whether or not they held a degree in mathematics. On average, teachers with a mathematics degree received higher MKT scores ($M = 18.75$, $SE = 1.91$) than teachers without a mathematics degree ($M = 17.86$, $SE = 1.45$). This difference was not significant $t(27) = -.34$, $p > .05$. The effect size was not significant $r = .07$.

Elementary license, Mathematics license, or Both

In order to compare the means of these three separate groups a one way independent ANOVA was performed. The three groups were teachers with an elementary license only, a mathematics license only, and those with both an elementary and a mathematics license. The researcher chose to perform an analysis consistent with the previous examination of teacher type because this research was exploratory in nature and there was a lack of existing research or a theoretical basis for performing planned contrasts. Descriptive statistics for the three groups are given in Table 4-12 below.

Table 4-12
MKT scores by license type

	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean	
					Lower Bound	Upper Bound
Elementary only	15	16.80	6.21	1.60	13.36	20.24
Mathematics only	7	18.00	5.35	2.02	13.05	22.95
Both Elementary and Mathematics	6	22.33	6.56	2.68	15.45	29.22
Total	28	18.29	6.26	1.18	15.86	20.72

In this analysis teachers with both an elementary and mathematics license had a higher mean score, $M = 22.33$, $S = 6.56$ when compared with teachers that held an elementary license, $M =$

16.80, $S = 6.21$, and teachers with a mathematics license, $M = 18.00$, $S = 5.35$. These differences were not statistically significant. The differences represent a small to medium effect size, $\omega = .23$.

Predictors of Mathematics Knowledge for Teaching

The second research question asked which variables were the greatest predictor(s) of MKT for division by fractions. This question looked at the relative importance of each variable in determining MKT. The question was:

Which of the following variables are the greatest predictors of MKT (Division by Fractions)?

- a. Elementary, Middle, or High School
- b. Type of License
- c. Grade levels licensed to teach
- d. Grade level Experience
- e. Total Experience

These five original predictors were not suitable for a multiple regression analysis of this sample size. An additional 40 to 60 participants would be required for a reliable analysis. The question was then split into two smaller questions. Grade level experience was actually three distinct variables. The researcher separated this variable into its own multiple regression analysis. The three variables were elementary experience, middle school experience, and high school experience. The analysis was meant to answer the question of whether experience in a particular grade range of mathematics teaching is a stronger predictor than another.

The remaining variables were examined to look for distinct factors. Two variables were chosen for the multiple regression analysis. The first variable was total experience which did not highly correlate with any other independent variable. The second variable was a combination of

three variables. If a teacher met at least two of three descriptions they were considered a middle school teacher, if they met one or less of the conditions they were considered an elementary or secondary. The three conditions were: does the teacher teach in a middle school, does the teacher hold a mathematics and elementary degree, is the teacher licensed to teach mathematics in middle school. The two multiple regressions were run and the results are included.

Grade Level Experience.

The first multiple regression investigated the influence of experience teaching different grade levels on MKT for division by fractions. Three predictor variables were entered. Years experience teaching grades 3-5, years experience teaching grades 6-8, and years experience teaching grades 9-12. $R^2 = .16$ meant that grade level experience for these three levels only accounted for 16% of the total variance of MKT scores. None of the predictor variables were significant. Beta values, standard errors, and their significance values are included below in Table 4-13.

Table 4-13

Multiple Regression – Grade level experience

	B	SE B	β
Constant	21.16	1.81	
Experience teaching grades 3-5	-0.39	.21	-.37
Experience teaching grades 6-8	-0.24	.18	-.25
Experience teaching grades 9-12	-0.21	.23	-.18

Note: $R^2 = .16$

* $p < .05$

Middle School Teachers and Total Experience.

The second multiple regression analysis investigated the influence of experience being a middle school teacher on MKT for division by fractions along with the teacher’s total experience in years. If a teacher had two of the following three qualities: taught in a middle school, held a

mathematics and elementary license, or were licensed to teach mathematics in middle school, they were considered a middle school teacher. If they met one or less of the conditions they were considered an elementary or secondary. $R^2 = .22$ indicated that being a middle school teacher and total experience accounted for 22% of the total variance of MKT scores. The test was significant with $F(2,26) = 3.59, p < .05$. Beta values, standard errors, and their significance values are included below in Table 4-14.

Table 4-14

Multiple Regression – Middle school teaching and total experience

	B	SE B	β
Constant	18.59	1.86	
Middle School teacher	5.11	2.39	.37*
Total teaching experience	-0.17	.117	-.25

Note: $R^2 = .22$

* $p < .05$

Chapter 5: Conclusions

The data collected in this sample allowed the researcher to answer the research questions using inferential statistical tests. Two statistically significant results were produced related to one teacher descriptor. Middle school teachers had significantly higher MKT scores when compared with elementary and high school teachers, and were a significant predictor of higher MKT scores. Other variables of interest were not significant in this study. Whether teachers taught at elementary or high schools, whether they held a mathematics degree, the type of teacher license, the amount of experience teaching particular grade ranges, and total experience made no statistically significant difference on teachers' MKT with regards to division by fractions.

Several aspects of this study were limited by the small sample size. MKT scores could not be analyzed for distinct factors. Multiple regression analyses were limited to only two or three predictor variables. Independent variables of school, license, degree, methods teacher, grade level experience were could not be analyzed for underlying factors. After examining the number of variables involved in each question along with the response rate, the study should be repeated with a sampling frame of about 400, roughly four times larger than this study. This sampling frame should be collected from a population that is more diverse geographically, and from different sized school districts. Increased diversity in the sampling frame will increase the external validity of the results.

Group Comparisons

Elementary, Middle, and High School Teachers.

This study examined different groups of in-service teachers and asked if there were significant differences in their MKT for division by fractions. The first of these groups to be examined was the type of teacher as determined by whether they taught in an elementary, middle, or high school. The results indicated there was a significant quadratic trend with a large effect size. This quadratic model showed that the MKT of teachers was lower for elementary and secondary teacher with a higher peak for middle school teachers. When comparing elementary and secondary teachers there was no significant difference in MKT scores. The large effect size was a reflection of large differences in mean scores for each group. Middle school teachers had a mean MKT score 39% higher than elementary school teachers, and 60% higher than high school teachers. This is a contribution to existing research which was lacking studies that directly compared elementary, middle, and high school teachers MKT.

Elementary and Secondary Teachers.

The original assumptions of the researcher were that middle school teachers shared more in common with elementary teachers than high school teachers in their MKT. This was in part due to the fact that an elementary license allowed teachers to teach at both an elementary school and a middle school. The elementary license also covers all subject areas, whereas a secondary degree is subject specific. Two of the nine middle school teachers held an elementary generalist teaching license and degree. The other seven were split among two types. Three had only a mathematics license and four had both a mathematics and elementary license. In the comparison of teachers as elementary or secondary, middle school teachers were assigned to be one or the other by examining the type of degree they held and what grade levels they were licensed to teach. If the middle school teacher had a mathematics license and were licensed to teach mathematics in the high school then they were considered secondary. If they did not have both

of those qualities they were considered elementary. This divided the nine middle school teachers roughly in half with four being classified as secondary and the remaining five classified as elementary. On average, elementary teachers received higher MKT scores than secondary teachers. This difference was not significant; however, it did represent a small-sized effect $r = .15$. When secondary and elementary teachers were compared without the middle school teachers included, there was not significant difference. When the middle school teachers, who scored significantly higher than both elementary and secondary teachers were split between these two groups, the average MKT score of both groups increased, but the two groups were not statistically significant.

Mathematics Degree

In this analysis teachers were separated by whether or not they held a degree in mathematics. On average, teachers with a mathematics degree received higher MKT scores than teachers without a mathematics degree. This difference was not significant. The effect size was not significant $r = .07$. This was counter to the assumptions of the researcher. The researcher assumed that teachers with degrees in mathematics would have higher MKT scores when compared with teachers who were “generalists” whose emphasis was pedagogy and who did not hold a mathematics degree. This was a narrow strand of mathematics. This topic was mainly taught in the middle school. Neither holding a mathematics degree, nor holding an elementary degree was what mattered in MKT when it came to teaching division by fractions.

License Type

Initially, the researcher wondered if this comparison would produce results similar to the comparison between elementary, middle, and secondary teachers. General observation of the data indicated that many of the middle school teachers held both elementary and mathematics

licenses, while elementary teachers tended to hold only an elementary license and high school teachers tended to hold only a mathematics license. Teachers with both an elementary and mathematics license had a higher mean score when compared with teachers that held either an elementary license or a mathematics license. The correspondence between groups was not so straightforward. One elementary and one high school teacher held both an elementary and mathematics license. The nine middle school teachers included two with elementary licenses, four with elementary and mathematics licenses, and three with mathematics licenses. The differences between teachers with one license and those with both represent a small to medium size effect that was not statistically significant.

Predictors of MKT

Grade level experience

The first multiple regression analysis investigated the influence of experience teaching different grade levels on MKT for division by fractions. Much of the existing research on MKT has been done with pre-service teachers. The researcher asked if the experience of teachers teaching particular grade levels would predict MKT. The researcher examined experience teaching in three different grade ranges: elementary 3-5, middle 6-8, and secondary 9-12. Grade level experience for these three levels only accounted for 16% of the total variance of MKT scores. None of the predictor variables were significant.

Middle School teachers and Total Experience

The second multiple regression analysis investigated the influence of experience being a middle school teacher on MKT for division by fractions along with the teacher's total experience in years. If a teacher had two of the following three qualities: taught in a middle school, held a mathematics and elementary license, or were licensed to teach mathematics in middle school,

they were considered a middle school teacher. If they met one or less of the conditions they were considered an elementary or secondary. Whether a teacher taught at a middle school and total experience accounted for 22% of the total variance of MKT scores. Total experience was not a significant predictor, while being a middle school teacher significantly predicted MKT scores. The distribution of total experience for the sample was positively skewed with least experienced quartile having four years or less experience, the second quartile having five to eight years experience, the third quartile having nine to 19.5 years experience, and the last quartile having 20 to 40 years experience.

The two analysis involving experience add to the existing research by giving an example where neither total experience nor experience teaching particular grade levels are significant predictors of MKT.

Implications for future research

This study served as an important step in a new way of measuring and analyzing MKT. The synthesis of interview questions from qualitative studies, theoretical definitions, an online delivery and data collection process, and a rubric designed to code qualitative responses into quantitative data were all tested in this study. Several stages of analysis revealed the need for a larger total sample size and larger samples for each subgroup. The researcher suggests a minimum sampling frame of 400-500 potential participants in a replication study to produce adequate sample sizes for instrument reliability, factor analysis, and satisfying the assumptions of the accompanying parametric tests. Research on MKT in different areas of mathematics would require similar foundational studies to create survey items and rubrics corresponding to the desired content. The rubric should be continually refined with a goal of achieving a scale that is as close as possible to interval data. The instrument itself could be modified, but that

would come with the disadvantage of losing all the validity and reliability that has been established with the current survey. The diversity of the sampling frame should be expanded for the purpose of increased generalizability.

Threats to validity

There were several threats to both the internal and external validity of this study. Threats to the external validity include selection error through a low response rate and a high dropout rate, the small sample size of the individual groups, and the lack of absolute interval scaling of the MKT scores used in the analysis. Over 50% of the sampling frame showed interest in the study by following the email link to the study, but only half of those who clicked on the link completed the survey. It is unclear whether there are any traits unique to this 25% that would cause them to differ significantly from the population. This was also a convenience sample. Participants and schools were not chosen at random. Creating a truly random sample would require greater participation and resources from districts involved in the study. Researchers using this methodology should continue to look to psychometric findings for solutions to providing measurements of knowledge that are interval data. Determining if there is selection bias is complex. The sample cannot be compared with individuals who choose not to participate in the study. One solution could be to compare the sample with the study population using any information available about the population of interest, i.e. experience, degree type, license type, etc...

Internal threats to validity included the inability to probe or clarify the meaning of questions and responses and a lack of controls to separate variables and their interactions. In an interview setting the researcher can and should probe the participant's responses to look for

deeper understanding of their knowledge. Likewise, when a participant requires clarification on a question they can always ask the researcher to restate the question in order to clarify what is being asked. This methodology prohibited those options. This threatened the measurement of MKT in several ways. The participant may have misunderstood the question and answered a question different from what the researcher intended. The researcher might also have misunderstood the response. A third possibility is that both the question and the answer could have been misunderstood.

Controlling for the interaction between independent variables was closely tied to the sample size of the individual subgroups. Examining participants under multiple conditions of independent variables quickly resulted in unacceptable samples sizes of only one or two participants. This prohibited the researcher from controlling for the interaction of different independent variables in and analyzing their respective contributions to the effect on MKT scores.

Closing thoughts

This study opened a new avenue of blending existing research perspective and methodologies for the purpose of gaining a unique perspective on pedagogical content knowledge in the teaching of mathematics. This study examined in-service teachers as opposed to more commonly studied pre-service teachers. The comparison of elementary, middle, and secondary teachers on the same strand of MKT was unique and produced significant relationships between the type of teacher and MKT. This study also demonstrated the feasibility of using an online survey coupled with a coding rubric to collect large scale generalizable quantitative data on MKT that has previously been confined to smaller scale qualitative studies,

or larger scale multiple choice paper pencil measures of MKT. This study and its underlying methodology allow researchers to examine teacher knowledge using a balance of previously existing methods and perspectives. Researchers can have a depth not addressed by previously existing quantitative measures like multiple choice instruments and the resources to produce larger samples and generalizable results not practical with smaller sampled interview studies. This balance requires a sacrifice on both types of methodologies. Quantitative researchers are required to examine participant responses in much greater detail and scoring rubrics and rater reliability become vital. Quantitative researchers are forced to sacrifice the depth of understanding that can be achieved through multiple face to face interviews. The results of this data analysis and the methodological process open up new roads of inquiry into what teachers know when it comes to teaching mathematics.

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Table 3-1

Rater agreement by question number – Round 1

	Difference in scores		Item								
	1a	1b	2	3	4	5	6	7	8	9	Total
0	100%	86%	86%	57%	100%	86%	86%	29%	14%	29%	67%
1	0%	0%	14%	29%	0%	14%	14%	29%	86%	71%	26%
2	0%	14%	0%	14%	0%	0%	0%	14%	0%	0%	4%
3	0%	0%	0%	0%	0%	0%	0%	14%	0%	0%	1.4%
4	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
5	0%	0%	0%	0%	0%	0%	0%	14%	0%	0%	1.4%
α	1.00	.77	.91	.63	1.00	.91	.91	.86	-.46	.30	.84

Table 3-2

Rater agreement by question number – Round 2

	Difference in scores										Item
	between raters										
	1a	1b	2	3	4	5	6	7	8	9	Total
0	100%	75%	75%	87.5%	62.5%	62.5%	75%	75%	100%	62.5%	77.5%
1	0%	25%	25%	12.5%	0%	37.5%	25%	12.5%	0%	37.5%	17.5%
2	0%	0%	0%	0%	37.5%	0%	0%	12.5%	0%	0%	5%
α	1.00	.91	.94	.77	.69	.85	.64	.92	1.00	.62	.92

Table 4-1

Response Rate by Question

Action Taken	n	%
Clicked on Survey link	67	63%
Answered Question 1	53	50%
Answered Question 2	44	42%
Answered Question 3	38	36%
Answered Question 4	36	34%
Answered Question 5	36	34%
Answered Question 6	34	32%
Answered Question 7	33	31%
Answered Question 8	31	29%
Completed the survey	29	27%

Table 4-2

Sample composition by school level

Type of Teacher	N	Percent of Sampling Frame	n	Percent of sample
Elementary	70	66%	13	45%
Middle	15	14%	9	31%
Secondary	21	20%	7	24%

Note. N is the number of teachers in the sampling frame, n is the number of teachers in the sample.

Table 4-3

Response Rate by Teacher Type

Type of Teacher	N	n	Response Rate
Elementary	70	13	19%
Middle	15	9	60%
Secondary	21	7	33%

Note. N is the number of teachers in the sampling frame, n is the number of teachers in the sample.

Table 4-4

Rater agreement by survey item – original rubric

Difference in scores given	Percent of MKT survey item scores at each level of agreement										
	1a	1b	2	3	4	5	6	7	8	9	Total
0	78%	78%	89%	33%	89%	78%	55%	78%	56%	67%	70%
1	22%	22%	0%	56%	11%	22%	22%	11%	44%	33%	24%
2	0%	0%	11%	11%	0%	0%	22%	11%	0%	0%	6%
α	.92	.71	.78	.48	.94	-.29	.44	.90	.85	-.44	.86

Table 4-5

Original rubric for item 4

Score	Description
0	The teacher incorrectly accepts the student's story as correct.
1	The teacher recognizes the student is dividing by 2 instead of $\frac{1}{2}$ and states that the story is incorrect, but does not correct the illustration.
2	The teacher addresses the error in the student's reasoning <u>and</u> does one of the following: 1. Corrects the student's example so that it does illustrate the problem. 2. Gives an alternate way of stating the problem.

Table 4-6

Revised and expanded rubric for item 4

Score	Description
0	The teacher incorrectly accepts the student's story as correct.
1	The teacher makes a comment that neither accepts nor rejects the student's story.
2	The teacher states that the story is incorrect, but does not correct the illustration or identify the student error as division by 2.
3	The teacher recognizes the student is dividing by 2 instead of $\frac{1}{2}$ and states that the story is incorrect, but does not correct the illustration.
4	The teacher addresses the error in the student's reasoning and does one of the following: 1. Corrects the student's example so that it does illustrate the problem, or 2. Gives an alternate way of stating the problem correctly.

Table 4-7

Rater agreement by survey item-Expanded Rubric

	Difference in scores between raters		Percent of MKT survey item scores at each level of agreement								
	1a	1b	2	3	4	5	6	7	8	9	Total
0	100%	57%	86%	57%	57%	71%	29%	57%	56%	14%	59%
1	0%	43%	14%	29%	43%	0%	29%	29%	44%	71%	30%
2	0%	0%	0%	14%	0%	14%	29%	14%	0%	14%	9%
3	0%	0%	0%	0%	0%	14%	0%	0%	0%	0%	1%
5	0%	0%	0%	0%	0%	0%	14%	0%	0%	0%	1%
α	1.00	.55	.98	.75	.92	.62	.46	.94	.92	.40	..82

Table 4-8

Tests of normality – skewness, kurtosis, and Kolmogorov-Smirnov

Group	Skewness		Kurtosis		Kolmogorov-Smirnov		
	Statistic	Sig.	Statistic	Sig.	Statistic	df	Sig.
Sample	-.18	.86	.14	.89	.17	29	.03*
Elementary School	-1.04	.30	-.73	.47	.20	13	.19
Middle School	1.21	.23	1.56	.12	.23	9	.17
High School	.82	.41	.70	.48	.25	7	.20**
Elementary (4-8)	.47	.64	.40	.69	.18	18	.12
Secondary (9-12)	.09	.93	-1.13	.26	.17	11	.20**
Mathematics Degree	1.04	.30	.04	.97	.22	8	.20**
Non Mathematics Degree	.15	.88	-.04	.97	.15	21	.20**
Elementary only	-.79	.43	-1.05	.29	.23	15	.03*
Mathematics only	-.76	.45	-.03	.98	.22	7	.20**
Elementary and Mathematics	.46	.65	.89	.37	.23	6	.20**

*p<.05, **This is a lower bound of the true significance.

Table 4-9

Homogeneity of variance

Group	Statistic	df1	df1	Sig.
Elementary, Middle, and High School	1.13	2	26	.34
Elementary (4-8) and Secondary (9-12)	.05	1	27	.83
Mathematics Degree and Non Mathematics Degree	.73	1	27	.40
Elementary, Mathematics, and Both Elementary and Mathematics	.43	2	25	.65

*p<.05

Table 4-10

MKT scores by teacher type(3)

	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean	
					Lower Bound	Upper Bound
Elementary	13	16.62	5.92	1.64	13.04	20.20
Middle	9	23.11	4.70	1.57	19.50	26.73
High School	7	14.43	4.83	1.82	9.97	18.89
Total	29	18.10	6.23	1.57	15.73	20.47

Table 4-11

MKT scores by teacher type (2)

	N	Mean	Std. Deviation	Std. Error
Elementary	18	18.83	6.65	1.57
Elementary (4-5)	13	16.62	5.92	1.64
Middle Elementary	5	23.11	4.70	1.57
Secondary	11	16.91	5.58	1.68
Middle Secondary	4	21.25	4.11	2.06
Secondary (9-12)	7	14.43	4.83	1.82
Total	29	18.10	6.23	1.57

Table 4-12

MKT scores by license type

	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean	
					Lower Bound	Upper Bound
Elementary only	15	16.80	6.21	1.60	13.36	20.24
Mathematics only	7	18.00	5.35	2.02	13.05	22.95
Both Elementary and Mathematics	6	22.33	6.56	2.68	15.45	29.22
Total	28	18.29	6.26	1.18	15.86	20.72

Table 4-13

Multiple regression – grade level experience

	B	SE B	β
Constant	21.16	1.81	
Experience teaching grades 3-5	-0.39	.21	-.37
Experience teaching grades 6-8	-0.24	.18	-.25
Experience teaching grades 9-12	-0.21	.23	-.18

Note: $R^2 = .16$

* $p < .05$

Table 4-14

Multiple regression – middle school teachers and total experience

	B	SE B	β
Constant	18.59	1.86	
Middle School teacher	5.11	2.39	.37*
Total teaching experience	-0.17	.117	-.25

Note: $R^2 = .22$

* $p < .05$

APPENDIX A

MATHEMATICS KNOWLEDGE FOR TEACHING SURVEY

The following questions pertain to teaching students about division by fractions. The questions are open-ended so that you have the freedom to answer as completely as possible. You are asked to complete each question without referencing any resources outside your own knowledge and expertise.

1. People seem to have different approaches to solving problems involving division with fractions. How would you solve a problem like this one?

(Please be as detailed as possible.)

$$1\frac{3}{4} \div \frac{1}{2} =$$

a) **Accuracy:**

3 points - A correct solution of $3\frac{1}{2}$ is given.

2 points – No solution is given, but the procedure described would produce the correct result of $3\frac{1}{2}$.

1 point – No solution is given. The explanation contains no errors and would reach a partially correct solution (like $14/4$), but does not explicitly state each step necessary to reach a correct and simplified answer of $3\frac{1}{2}$.

0 points - An incorrect solution or no solution is given, and the explanation is not sufficient to guarantee a correct result of $3\frac{1}{2}$.

b) **Explanation**

3 points – The reasoning is valid and the method or rational described would produce a correct and simplified answer for all division by fractions problems. (...multiply by the reciprocal and simplify by eliminating common factors)

2 points – The reasoning is valid, but the method would only work for some division by fraction problems or would not always produce a simplified answer. (...multiply by the reciprocal)

1 point – The explanation is incomplete with no errors. (...multiply by 2)

0 points - The explanation contains incorrect reasoning.

2. Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content.

What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$?

3 points – A division story problem is given with a dividend of $1\frac{3}{4}$, a divisor of $\frac{1}{2}$, and a quotient of $3\frac{1}{2}$. Each part of the problem has a meaningful representation within the context of the problem.

2 points – A division story problem is given with a dividend of $1\frac{3}{4}$, a divisor of $\frac{1}{2}$, and a quotient of $3\frac{1}{2}$. Each part of the problem has a meaningful representation within the context of the problem, but the quotient of $3\frac{1}{2}$ is in units that are integer or whole numbers, i.e. $3\frac{1}{2}$ people...

1 point – A division problem by $\frac{1}{2}$ is given in a context that could work, but it is unclear or unstated what the $1\frac{3}{4}$ and $\frac{1}{2}$ represent.

0 points – No problem is given, or the story problem represents something mathematically different

3. You would like a student to think about the reasonableness of their answer. Imagine a student in your class is solving the problem $1\frac{3}{4} \div \frac{1}{2}$. How would you answer a student who makes the following comment, "Since $1\frac{3}{4}$ is less than 2, our answer should be less than one, since 1 is half of 2"?

4 points – The teacher does all the following:

1. Rejects the student's reasoning.
2. Explains that the student's reasoning applies to division by 2, but not $\frac{1}{2}$.
3. Rephrases the estimation correctly.
 - a. Example: "Since there are 3 halves in $1\frac{1}{2}$ our answer should be greater than 3 since $1\frac{3}{4}$ is greater than $1\frac{1}{2}$."

3 points – The teacher rejects this method and does **one** of the following:

1. Explains that the student's reasoning applies to division by 2, but not $\frac{1}{2}$.
2. Rephrases the estimation correctly.

2 points – The teacher rejects this method, and attempts an explanation, but doesn't do either of the following:

1. Identify the student's error as division by 2 instead of $\frac{1}{2}$.
2. Rephrase the estimation correctly.

1 point – The teacher does not affirm student's response as correct, but does not explain what is wrong with the reasoning.

0 points – The teacher affirms the student's response as correct.

4. Suppose you asked your students to come up with a story to illustrate the problem above. One of your students told you the following:

You could be using a pie, a whole pie, one, and then you have three fourths of another pie and you have two people, how will you make sure that this gets divided evenly, so that each person gets an equal share?

What would you say to the student about their story?

4 points – The teacher addresses the error in the student’s reasoning and does one of the following:

1. Corrects the student’s example so that it does illustrate the problem.
2. Gives an alternate way of stating the problem correctly.

3 points – The teacher recognizes the student is dividing by 2 instead of $\frac{1}{2}$ and states that the story is incorrect, but does not correct the illustration.

2 points – The teacher states that the story is incorrect, but does not correct the illustration or identify the student error as dividing by 2.

1 point – The teacher makes a comment that neither accepts nor rejects the student's story.

0 points – The teacher incorrectly accepts the student’s story as correct.

5. What do you believe are the most important skills and knowledge that a student needs in order to understand and be proficient at division by fractions? Why are those skills and knowledge pieces important?

4 points – The teacher identifies more than 3 related skills or knowledge packages connected to division by fractions.

3 points – The teacher identifies 3 related skills or knowledge packages connected to division by fractions.

2 points – The teacher identifies 2 related skills or knowledge packages connected to division by fractions.

1 point– The teacher identifies 1 skill or knowledge package connected to division by fractions.

0 points– The teacher identifies no related knowledge or skills.

6. There are often multiple ways to solve a problem. Previously you described how you would solve the problem below.

$$1\frac{3}{4} \div \frac{1}{2}$$

Can you explain any different methods for solving this problem that may help a student who didn't understand the first method you used?

(Please describe as many different methods as possible)

Compare each method to the procedure from question 1. For each alternate method. Give a score based on the descriptions below. For multiple methods give a cumulative score. For example: 3 methods scoring 2, 1, and 2 = a cumulative score of 5

3 points – An alternate method is used that works for all division by fractions problems.

2 points – An alternate method is used that works for this example, but not all division by fractions problems. For example: Multiply by the denominator.

1 point- An additional method is given that is unique, but the reasoning is incomplete.

0 points – The method is not distinctly different from others previously mentioned, the reasoning is incorrect, or no additional procedure is given.

7. Suppose a student in class is trying to solve the problem $1\frac{3}{4} \div \frac{1}{2}$ using improper fractions. She thinks if you convert $1\frac{3}{4}$ to an improper fraction you could solve the problem by dividing the numerators, $7 \div 1 = 7$ and then dividing the denominators, $4 \div 2 = 2$ to find the quotient. What would you say to this student about her method?

4 points- The teacher affirms this method as valid, and points out it will only produce a simplified fraction when the numerator and denominator of the dividend are evenly divisible by the numerator and denominator of the divisor.

3 points- The teacher rejects this method because it does not always produce a simplified answer, or because the numerators and denominators will not always be divisible.

2 points – The teacher affirms the solution and the method as accurate, but makes no comment as to when this method will work.

1 point – The teacher makes no judgment about the method in general, but notices it works in this case.

0 points – The teacher rejects this method as invalid, and states it will not always get a correct answer. The teacher states they do not know if this method is valid.

8. Consider the original problem that involved division by fractions.

$$1 \frac{3}{4} \div \frac{1}{2}$$

Can you think of any errors in calculation or misconceptions about the concept of dividing by a fraction that are common among your students?

Please provide examples of common mistakes in calculation or misconceptions about division by fractions as a whole.

4 points – The teacher identifies more than 3 common errors or student misconceptions that are indeed false.

3 points – The teacher identifies 3 common errors or student misconceptions that are indeed false.

2 points – The teacher identifies 2 common errors or student misconceptions that are indeed false.

1 point – The teacher identifies 1 common error or misconception that is indeed false.

0 points – The teacher does not identify any misconceptions.

9. One common procedure for dividing by fractions is the rule, "Invert and multiply".

When teaching about division by fractions a student may ask, "Why is inverting and multiplying the same as dividing by a fraction?" How would you respond to this question?

3 points – The teacher verifies the procedure in both of the following ways:

1. In general instead of using specific examples
2. A conceptual explanation involving multiplication and division as inverses and/or reciprocals as the multiplicative inverse.

2 points – The teacher verifies the procedure in one of the following ways:

1. In general instead of using specific examples.
2. A conceptual explanation involving multiplication and division as inverses and/or reciprocals as the multiplicative inverse.

1 point – An incomplete explanation is given. Incomplete could mean any of the following:

1. The process cannot be supported with properties of basic arithmetic involving multiplication and division.
2. A description of what they would do, without actually doing it.
3. Demonstrating the procedure with specific examples.
4. Explaining that inverting and multiplying prevents complex fractions.

0 points – No explanation provided, or an incorrect explanation is given

Part 2

The following questions ask about your current teaching assignment.

1. Do you teach students in Elementary (4-5), Middle (6-8), or High School (9-12)?

Elementary School - 0

Middle School - 1

High School - 2

2. What teaching license do you currently hold? (Select all that apply)

Mathematics - 0

Elementary - 1

Early-Late Childhood - 2

Other – 3

3. What grade level(s) are you licensed to teach? (0 – not licensed, 1 – licensed)

K 1 2 3 4 5 6 7 8 9 10 11 12

4. Including this year, how many years experience to you have teaching students in each grade range? (1-40)

K-2

3-5

6-8

9-12

5. During your undergraduate program, which department provided faculty to teach your methods of teaching mathematics course?

Education - 0

Mathematics - 1

Education and Mathematics departments equally - 2

6. Including this year, how many years have you been teaching? (1-40)

7. What degree(s) do you hold?

Appendix B- Email letter of Invitation

April 25, 2016

Fellow Teachers,

Kansas State University in cooperation with the XXXXX school district is conducting a study about mathematics teachers' knowledge for teaching division by fractions. You have been identified as someone who teaches mathematics to students in grades 4-12. I would like to invite you to participate in this study by sharing your knowledge in an online survey.

The link below will direct you to an online survey consisting of 9 questions on the topic of teaching students about division by fractions. All responses will be kept confidential and anonymous. When you submit the survey your participation in the research will be complete.

Participation in this study is voluntary and you are under no obligation to participate. Your participation in this study will further current research in the area of teacher knowledge for elementary, middle, and high school teachers.

The results of this research will be shared with you in a follow up email after the completion of the study. Thank you in advance for your willingness to share your knowledge and expertise with us.

Link to the Mathematics Knowledge for Teaching survey:

https://kstate.qualtrics.com/SE/?SID=SV_cvUZIdC7XThE00R

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Appendix C – Expert Panel Feedback on MKT survey

Part 1

Original Question 1: “Do you currently teach at an elementary, middle, or high school?”

On the first page, first question, I suggest changing the wording to “What level do you currently teach?” This wording matches the wording of the questions that follow.

2. When you ask Elem, Middle, High.....2 things. First, what if a teacher is teaching 7-12? Will she/he be able to select both? Second, I would recommend to specify what you mean by Elem/Middle/High. Does Elem stop at grade 5 or 6? When does Middle begin.....etc. You might just add some clarifiers, such as Elementary (4-6), Middle (7-8), etc.

New Question 1: “Do you teach students in an Elementary (4-5), Middle (6-8), or High School (9-12)?”

Original Question 2: “What type of teaching license do you currently hold?”

On the first page, the second question, I suggest the wording “What teaching license do you currently hold?”

When you ask about the type of teaching license, it might be useful to add a qualifier that suggests that they should select all that apply. I’m thinking of those teachers that have multiple certifications.

The wording of the second question is a little awkward.

New Question 2: “What teaching license do you currently hold? (Select all that apply)”

Original Question 3: “What grade levels are you licensed to teach?”

When you ask what grade levels they are licensed to teach, are you interested in knowing in what subject area? For example if a teacher is licensed to teach K-12, would it be useful to your research to know if that is in Math, ELL, SPED, or another area? I’m thinking about my certifications in general and I am certified K-9 Elementary, 5-9 Math, 5-9 ELA, 6-12 Math, and K-12 ELL. I don’t know if it would be useful to separate your data out more specifically or not based on your research objectives.

New Question 3: “What grade levels are you licensed to teach mathematics?”

Original Question 4: “What grade level(s) do you currently teach?”

When you ask about current grade levels, would you also be interested in past grade levels taught as well or is that not in the scope of the research? So, I’m thinking about someone who might have taught 6th grade math, but also taught 7th, 8th, and some high school and would that maybe have an effect on content knowledge or understandings?

New Question 4: “Including this year, how many years experience do you have teaching students in each grade range?”

This question now has a dropdown menu for each grade level with response options that match question 6.

Original Question 5: “During your undergraduate program, were your methods of mathematics teaching courses taught by faculty from the education or mathematics department?”

On the first page, the fifth question, I suggest the wording “During your undergraduate program, which department provided faculty to teach your mathematics teaching methods courses?”

New Question 5: “During your undergraduate program, which department provided faculty to teach your methods of teaching mathematics course?”

Original Question 6: “Including this year, how many years have you been teaching?”

On the first page, last question, I like how you are clear that this year is to be included in years teaching.

No Change to Question 6

Part 2

Original Instructions: “The following questions pertain to teaching students about division by fractions. The questions are open-ended so that you have the freedom to answer as completely as possible. You are asked to complete each question without referencing any resources outside your own knowledge and expertise.”

Original Question 1: “People seem to have different approaches to solving problems involving division with fractions. How would you solve a problem like this one?”

$$1 \frac{3}{4} \div \frac{1}{2}$$

On Question 1, I suggest the wording “How would you teach a student to solve a problem like this?” However, this may not be the intent of the question, and in that case, your wording works better. If a change is made here, then a change to Question 7 may be needed so that the questions go together as well as your original question and Question 7 do already.

When you ask how they would solve a problem like this one...are you thinking they will give you a narrative description? Do you know if there are tools available if they wanted to highlight

a bar model or fraction strip somehow? Again that may not be possible with the survey tool, but would be cool.

You could take out “seem to”, so that it would read:

“People have different approaches to solving problems involving division with fractions. How would you solve a problem like this one?”

This question is kept the same to match the wording used by Ma (1999)

Original Question 2: “Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$?”

On Question 2, I suggest the wording “To make division of fractions meaningful to students, you would like to relate division of fractions to a real world application. What real world example would you give students for ?”

This question is kept the same to match the wording used by Ma (1999)

Original Question 3: “Sometimes teachers will ask students to think about whether or not the answer they calculated is reasonable. Imagine a student in your class is solving the problem $1\frac{3}{4} \div \frac{1}{2}$. How would you answer a student who makes the following comment, ‘Since $1\frac{3}{4}$ is less than 2, our answer should be less than one, since 1 is half of 2’?”

On Question 3, I suggest the wording “You would like a student to think about the reasonableness of his answer. Imagine?”

New Question 3: “You would like a student to think about the reasonableness of their answer. Imagine a student in your class is solving the problem $1\frac{3}{4} \div \frac{1}{2}$. How would you answer a student who makes the following comment, ‘Since $1\frac{3}{4}$ is less than 2, our answer should be less than one, since 1 is half of 2?’”

Original Question 4: “Suppose a student in class is trying to solve this problem using improper fractions. The student thinks if you converted $1\frac{3}{4}$ to an improper fraction you could solve the problem by dividing the numerators, $7 \div 1 = 7$ and then dividing the denominators, $4 \div 2 = 2$ to find the quotient. What would you say to the student about how appropriate their method is?”

On Question 4, since you are referring to a student (singular), make sure that the pronouns that follow match. For example, “The student thinks if he converted to an improper fraction, then he could solve the problem by dividing the numerators, and then dividing the denominators, to find the quotient. What would you say to the student about how appropriate his method is?”

On question 4, I know you are using the same fraction operation in each problem, but it might be worth restating the problem just so no one is lost after thinking through some of the others.

You ask the teachers, “What would you say to the student about how appropriate their method is?” I think the word “appropriate” might get in the way of what you are trying to figure out. The teachers might focus on that word and I’m not sure that will tell you what you want.

New Question 4 (moved to after the original question 7): “Suppose a student in class is trying to solve the problem $1\frac{3}{4} \div \frac{1}{2}$ using improper fractions. She thinks if you convert $1\frac{3}{4}$ to an improper fraction you could solve the problem by dividing the numerators, $7 \div 1 = 7$ and then dividing the denominators, $4 \div 2 = 2$ to find the quotient. What would you say to this student about her method?”

Original Question 5: “Suppose you asked your students to come up with a story to illustrate the problem above. One of your students told you the following:

You could be using a pie, a whole pie, one, and then you have three fourths of another pie and you have two people, how will you make sure that this gets divided evenly, so that each person gets an equal share?

What would you say to the student about their story?”

On Question 5, I suggest the wording “What would you say to the student about her story?” I included a feminine pronoun to balance with the masculine pronoun in Question 4. I prefer this to “he/she” references.

On question 5, there’s no problem above.

The teachers in the survey can’t go back to question 4 to refer to the question “above”, you may want to show it at the beginning of question 5 too.

3. On question #5, instead of saying "the problem above", I would clarify it with $1\frac{3}{4} / \frac{1}{2}$ again.

Instead of asking, “What would you say to the student about their story?” you could ask: What specific feedback would you give the student about their story? I would worry that they would say “good job”, or something simple instead of giving some specific feedback about the story and its mathematical relevance.

New Question 4: “Suppose you asked your students to come up with a story to illustrate the problem $1\frac{3}{4} \div \frac{1}{2}$. One of your students told you the following:

You could be using a pie, a whole pie, one, and then you have three fourths of another pie and you have two people, how will you make sure that this gets divided evenly, so that each person gets an equal share?

What would you say to this student about his story?”

Original Question 6: “What do you believe are the most important skills and knowledge that a student needs in order to understand and be proficient at division by fractions? Why are those skills and knowledge pieces important?”

On question 6, would it be helpful for them to specify what they are considering knowledge or what they are considering skills?

No change, now question 5

Original Question 7: “Previously you solved the problem below.

$$1\frac{3}{4} \div \frac{1}{2}$$

Can you think of any different ways to solve it that may help a student who didn’t understand the first method you used?”

On Question 7, if you use my suggestion in Question 1, then you may want to use wording such as “Previously you showed how you would teach a student to solve the problem below.”

When you ask how they would solve a problem like this one...are you thinking they will give you a narrative description? Do you know if there are tools available if they wanted to highlight a bar model or fraction strip somehow? Again that may not be possible with the survey tool, but would be cool.

Is there a reason that you put this question so far away from the question it is referring to? Sometimes in surveys “learning” can happen through the experience of answering questions. Are you hoping to see if some of the questions help them come up with other solution strategies? This would be very difficult to show, but it could have an impact on their response. It may be better to put this question right after the question it refers to.

New Question 6: “There are often multiple ways to solve a problem. Previously you solved the problem below.

$$1 \frac{3}{4} \div \frac{1}{2}$$

Can you explain any different methods for solving this problem that may help a student who didn't understand the first method you used? (Please describe as many different methods as possible)”

Original Question 8: “Consider the original problem that involved division by fractions.

$$1 \frac{3}{4} \div \frac{1}{2}$$

Can you think of any errors in calculation or misconceptions about the concept of dividing by a fraction that are common among your students?

(Please provide examples of common mistakes in calculation or misconceptions about division by fractions and possible reasons students may make these mistakes or have these misconceptions.)

Question 8 would not need to move because it is just referring to the question not the teachers' response to the question.

No change to Question 8, now question 7

Original Question 9: “One common procedure for dividing by fractions is the rule, “Invert and Multiply”. When teaching about division by fractions a student may ask, ‘Why is inverting and multiplying the same as dividing by a fraction?’ How would you respond to this question?”

No change to Question 9

Additional Comments:

Again, I think that you have a nice sequencing of questions that will make most mathematics teachers think very hard about something that has normally been procedural without much (if any) reference to meaning or real world applications.

I know I mentioned this on the 4th bullet, but if there were ways that math tools or diagramming would be possible (which I'm sure there isn't J) so that the respondents could represent their thinking, it could add to your understanding of their responses. I'm just imagining what a description versus a visual representation might do for your understanding as a researcher.

1. Will everyone who receives this survey currently be teaching? If so, ignore this point. If not, you may need to add other options to the demographic portion.