AGU PUBLICATIONS

Geophysical Research Letters

RESEARCH LETTER

10.1002/2015GL064935

Key Points:

- AMHG is a mathematical construct arising from rating curve convergence
- Discharge estimation from remotely sensed images is explained by this finding
- AMHG is reinterpreted as an index of fluvial self-similarity

Supporting Information:

- Text S1 and Figures S1–S3

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Citation:
Gleason, C. J., and J. Wang (2015), Theoretical basis for at-many-stations hydraulic geometry, *Geophys. Res. Lett.*, *42*, 7107–7114, doi:10.1002/2015GL064935.

Received 12 JUN 2015
Accepted 6 AUG 2015
Accepted article online 30 JUL 2015
Published online 8 SEP 2015

Theoretical basis for at-many-stations hydraulic geometry

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Abstract At-many-stations hydraulic geometry (AMHG) is a recently discovered set of geomorphic relationships showing that the empirical parameters of at-a-station hydraulic geometry (AHG) are functionally related along a river. This empirical conclusion seemingly refutes previous decades of research defining AHG as spatially independent and site specific. Furthermore, AMHG was the centerpiece of an unprecedented recent methodology that successfully estimated river discharge solely from satellite imagery. Despite these important implications, AMHG has remained an empirical phenomenon without theoretical explanation. Here we provide the mathematical basis for AMHG, showing that it arises when independent AHG curves within a reach intersect near the same values of discharge and width, depth, or velocity. The strength of observed AMHG is determined by the degree of this convergence. Finally, we show that AMHG enables discharge estimation by defining a set of possible estimated discharges that often match true discharges and propose its future interpretation as a fluvial index.

1. Introduction

The empirical relationships known as hydraulic geometry (HG) are highly enigmatic but enormously important equations in fluvial geomorphology. *Leopold and Maddock* [1953] proposed HG after observing strong power law relationships between instantaneous river discharge (Q), width (w), mean depth (d), and mean velocity (v) at specific cross sections. They termed this phenomenon “at-a-station hydraulic geometry” (AHG), and also defined “downstream hydraulic geometry” (DHG) to describe similar trends between mean annual discharge and width, depth, and velocity among cross sections in a downstream direction along a river. Both AHG and DHG are formulated as

$$w = aQ^b \tag{1}$$

$$d = cQ^f \tag{2}$$

$$v = kQ^m \tag{3}$$

where a, b, c, f, k, and m are empirically fitted parameters. Note that equations (1) through (3) are unit sum constrained (i.e., $b+f+m=1$ and $ack=1$, as $Q=wdv$ by definition).

Despite objections to the use of a power law form for HG [e.g., *Knighton*, 1974, 1975; *Richards*, 1973; *Phillips and Harlin*, 1984; *Ferguson*, 1986], HG has been a robust field of study for geomorphologists ever since Leopold and Maddock’s original publication. Definitive treatment of AHG exponents was given by *Ferguson* [1986], who famously reduced them to “hydraulics and geometry”: Ferguson used simple flow laws to show that the exponents of AHG were completely dependent on cross-sectional channel geometry and that AHG should only take the form of a power law when dictated by such geometry. AHG coefficients have been largely ignored, though *Dingman* [2007] gave analytical expressions for both coefficients and exponents that fell short of a derivation from first principles.

AHG remains a vital tool for hydrologists and water managers as the principle means of estimating river discharge worldwide but has not been the subject of much theoretical research in the past decade [*Gleason*, 2015]. The recent discovery of at-many-stations hydraulic geometry (AMHG), however, has reintroduced interest in the theoretical basis of AHG. AMHG was proposed by Gleason and Smith in 2014 and holds that the paired coefficients and exponents of AHG (a and b, c and f, and k and m) from many cross sections of a given river reach are functionally related to one another, following a log-linear relationship (Figure 1, second and fourth columns). This rather surprising finding suggests that equations (1)–(3) are redundant

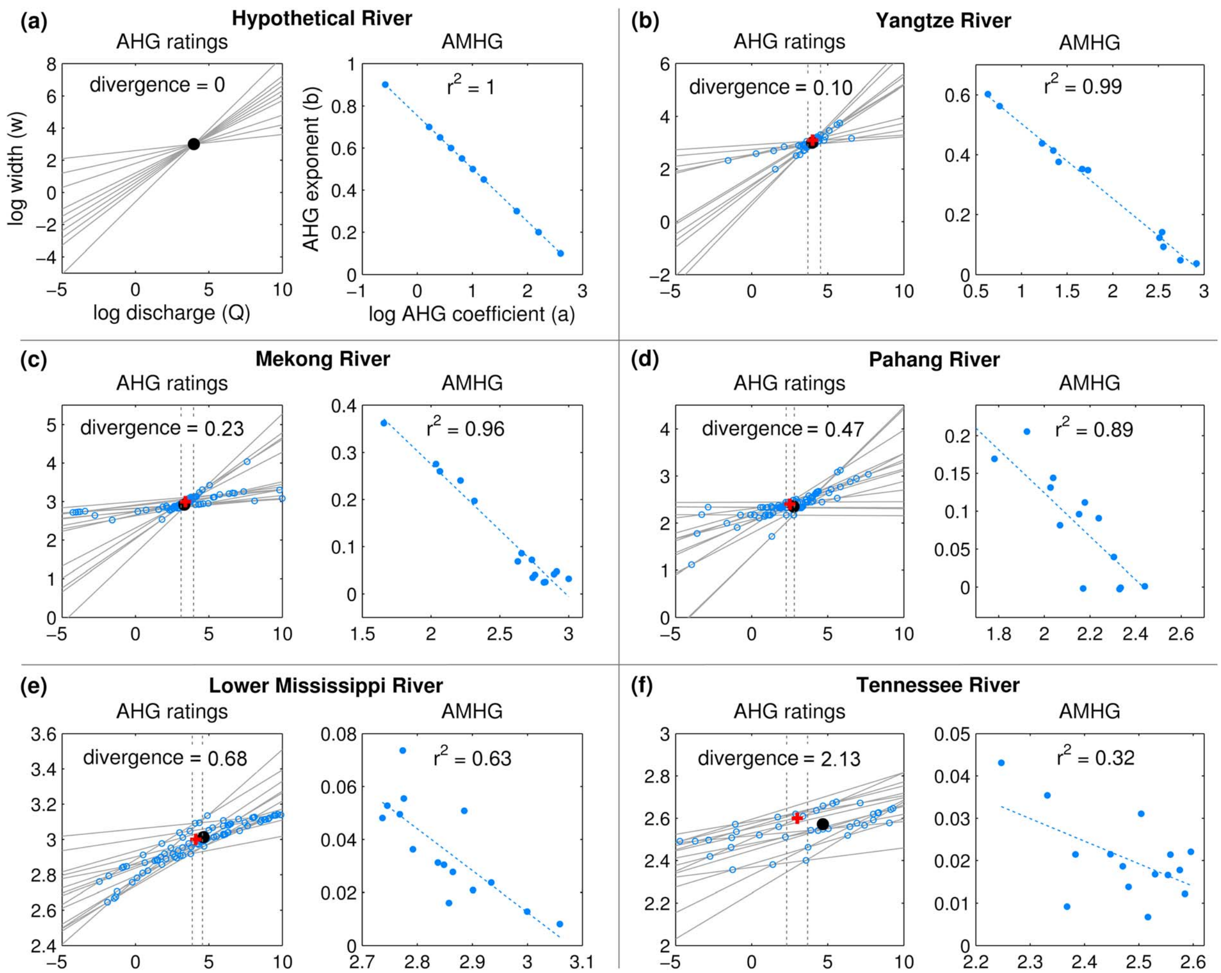

Figure 1. The mathematical basis for AMHG. Intersection of individual AHG rating curves gives rise to AMHG and (a–f) show AHG curves (grey), (Q_{c_w}, w_c) as approximated by the median center of rating curve intersections (black circles), AHG intersections (blue circles), $\left(\text{mode}\left(\overline{\log(Q_x)}\right), \text{mode}\left(\overline{\log(w_x)}\right)\right)$ (red crosses), and observed Q range (dashed lines). In Figure 1a AHG curves intersect at exactly (Q_{c_w}, w_c), yielding a perfect AMHG. Figures 1b–1f give examples of rating curve intersection for 5/50 rivers in this study as divergence increases. Congruence of red crosses to (Q_{c_w}, w_c) indicates AMHG can be given by these spatial modes for rivers with strong AMHG.

formulations, as, e.g., b is shown to be a function of a, and would then seem to suggest that AHG is not site specific as previously theorized [*Ferguson*, 1986; *Phillips*, 1990] but is instead dependent on the AHG of other cross sections in a given river reach. Gleason and Smith defined width AMHG as

$$F = a_{x1,x2,..xn}E^{b_{x1,x2,...xn}} \quad (4)$$

where the subscripts x_1, x_2,…x_n refer to spatially indexed cross sections along a river, a and b are the classic AHG parameters at each cross section, and E and F are river-specific constants. Thus, AMHG also suggests that there are two previously unknown hydraulic constants (E and F) that control the AHG of each cross section in a reach: another highly provocative idea.

Beyond its novelty as a geomorphic phenomenon, AMHG has also been invoked in an important application: remote sensing of river discharge. *Gleason and Smith* [2014] demonstrated that AMHG could be approximated from multiple observations of cross-sectional river widths. With this approximation in hand, they demonstrated that useful estimates of river discharge could be made solely from repeated satellite imagery. Since AMHG posits a relationship between a and b, these are reduced to a single parameter (if AMHG is known), thus simplifying the AHG system from $2n+1$ unknowns per n cross sections to $n+1$ unknowns in a mass conserved reach (as Q is unknown but constant between cross sections, w is observed via remote sensing, and a and b are linked by AMHG). This system is sufficiently simple for unconstrained minimization of flow residuals, thereby solving for Q by iteratively testing *a/b* pairs constrained by AMHG at different cross sections until Q is conserved among all cross sections (Gleason and Smith used a genetic algorithm for this purpose). *Gleason et al.* [2014] followed this initial demonstration of discharge retrieval with an updated methodology and a thorough sensitivity analysis for 34 rivers worldwide and found continued satisfactory performance for most river morphologies. Despite this successful application of AMHG, its underlying geomorphic principles have not yet been described, and its startling assertion that AHG parameters are strongly linked in space has not yet been fully explained.

In this letter, we find that AMHG is a mathematical construct arising from the use of power laws at a station that also requires certain underlying geomorphic criteria to be met and show that E and F to correspond to previously known hydraulic quantities. These findings counter the notion (furnished by empirical analysis of AMHG) that AHG is a redundant formulation and are consistent with previous decades of HG research indicating that AHG is a site-specific expression of local hydraulics and geometry. Our analysis also shows that the proposed remotely sensed proxy for AMHG given by *Gleason and Smith* [2014] is purely coincidental, and we propose a new proxy for future use. Despite these findings, the utility of AMHG for remotely sensed discharge estimation is unaffected: our findings show that the mathematical construct of AMHG actually *enables* successful discharge estimation. Finally, we propose an interpretation of AMHG as a fluvial similarity index.

2. Data

The derivation of AMHG given below is confirmed using width and discharge data from 50 river reaches in diverse physiographic and climate settings. 34 of these reaches are the same as those in *Gleason et al.* [2014], where discharge is given by a gauge (supplied by the U.S. Geological Survey or Global Runoff Data Center) and ~15 cross-sectional widths are manually digitized from 7 to 19 Landsat TM images in a ~10 km reach including or immediately abutting the gauge. Width and discharge data for the other 16 reaches were generated from hydrodynamic models. In these data, flow is imposed on measured bathymetry and river hydraulic quantities at every cross section are solved by finite element analysis. These data are generally 1 year simulations with daily output of width and discharge, and the number of cross sections per river reaches ranges from 40 to 500 with a spatial scale of 50–400 km. These 50 data sets allow for robust validation of our analysis and provide context for our conclusions. Further descriptions of the selected river reaches are provided in section S1.1.

3. Theoretical Basis for AMHG

We propose that AMHG arises when individual rating curves for each cross section (x) in a given river reach converge at the same value of width and discharge. This is seen in the following mathematical analysis:

Rewriting equation (4) as

$$b_x = -\frac{\log(a_x)}{\log(E)} + \frac{\log(F)}{\log(E)} \quad , \tag{5}$$

Gleason and Smith noted that width AMHG has the same mathematical construction as width AHG (rewriting equation (1))

$$b_x = -\frac{\log(a_x)}{\log(Q)} + \frac{\log(w)}{\log(Q)}, \tag{6}$$

except that E and F in AMHG are constants while Q and w in AHG are variables. If we suppose there is a unique w-Q pair shared by all cross-sectional rating curves, then $E = \log(Q_{c\_w})$ and $F = \log(w_c)$, where w_c and Q_{c_w} are the common width and discharge values, which are now treated as constants. Furthermore, if a cross section's AHG includes w_c and Q_{c_w}, equation (1) can be written as

$$w_c = aQ_{c\_w}^b, \tag{7}$$

and, if any two cross sections share w_c and Q_{c_w}, then we can solve equation (7) for w_c at each cross section and equate the two expressions:

$$b_1 \log(Q_{c\_w}) + \log(a_1) = b_2 \log(Q_{c\_w}) + \log(a_2). \tag{8}$$

Solving equation (8) for $1/\log(Q_{c\_w})$ gives

$$\frac{1}{\log(Q_{c\_w})} = \frac{b_1 - b_2}{\log(a_2) - \log(a_1)}. \tag{9}$$

Finally, the slope of AMHG defined by two cross sections is given empirically (see Figure 1) as

$$\text{AMHG slope} = \frac{\Delta b}{\Delta \log(a)} \tag{10}$$

It is easily seen that equations (9) and (10) are equivalent (with a change in sign), showing that if a pair of cross sections each contain w_c and Q_{c_w}, the slope of their AMHG (the relationship between b and $\log(a)$ for those two cross sections) is equivalent to $-\frac{1}{\log(Q_{c\_w})}$, and using this value as E in equation (4) yields an AMHG intercept of $\frac{\log(w_c)}{\log(Q_{c\_w})}$. *Gleason* [2015] gave a similar mathematical analysis to equations (8)–(10) but stopped short of deriving AMHG and did not give Q_{c_w} or w_c.

Therefore, a river will exhibit a perfect AMHG (with $r^2 = 1$) when individual AHG rating curves in a river reach converge exactly at (Q_{c_w}, w_c), as the slope between any two points on the AMHG curve will be exactly equal to equation (9) (Figure 1a). Note that the existence of the width AMHG does not mean that discharge is conserved among all cross sections, nor does it mean that observed widths are equal: if width AMHG is observed all cross sections simply intersect near the same width-discharge tuple. While all of the reaches in this study are mass conserved and many are short (~10 km), AMHG will be observed at any scale if rating curves reliably intersect. The presence of strong AMHG for longer, nonmass conserved rivers in *Gleason and Smith* [2014] is an indication that this rating convergence can happen over large distances, and rating convergence can happen in any reach where changes in discharge remain in similar orders of magnitude and are thus similar in log space (note ranges of Q in Figures 1b–1f). The mathematical basis for the width AMHG is easily applied to the depth and velocity AMHGs, yielding our proposed formulations for AMHG:

$$w_c = a_{x1,x2,..xn} Q_{c\_w}{}^{b_{x1,x2,...xn}} \tag{11}$$

$$d_c = c_{x1,x2,..xn} Q_{c\_d}{}^{f_{x1,x2,...xn}} \tag{12}$$

$$v_c = k_{x1,x2,..xn} Q_{c\_v}{}^{m_{x1,x2,...xn}} \tag{13}$$

This analysis shows that AMHG does not contradict previous decades of AHG research as previously suggested by empirical analysis. Instead, the functional relation between AHG parameters posited by AMHG is shown to arise as a result of local AHG curves converging at the same hydraulic quantities, and E and F are not new river hydraulic parameters but rather $\log(Q_{c\_w})$ and $\log(w_c)$, respectively. Thus, AMHG states AHG is only a de facto redundant formulation when AMHG is strong and only when multiple cross sections are considered, and it is this redundancy that enables successful remote sensing of river discharge (discussed in sections 5 and 6). Leopold and Maddock's choice to fit a power law to AHG data, and the subsequent adoption of this formulation despite strong evidence of its serendipity [e.g., *Richards*, 1973; *Ferguson*, 1986], has had lasting and unintended consequences far beyond their original intent: by assuming these power laws hold at a station, AMHG, and a new way of viewing fluvial morphology, is enabled.

4. AMHG as Geomorphic Index

The mean AMHG r^2 for the 50 rivers in this study is 0.82, which suggests that rating curve convergence is widespread in our data and raises the questions: *why* do rating curves tend to converge at (Q_{c_w}, w_c), and to *what hydraulic quantities* do (Q_{c_w}, w_c) correspond? We propose that rating curves intersect and give rise to AMHG if two conditions are met. First, AHG exponents (b, f, m) need to be sufficiently variable to ensure that rating curves indeed cross: the more similar AHG exponents, the more parallel the rating curves and less likely they are to intersect (Figures 1e and 1f). This variability also increases the likelihood that AHG curves will cross within the range of observed discharge data (vertical lines in Figure 1), an important factor in successful discharge estimation (see section 6). Second, we propose that for rivers that exhibit strong AMHG, Q_{c_w}, Q_{c_d}, Q_{c_v}, w_c, d_c, and v_c are given in practice by the spatial mode of the time mean of each of these cross-sectional quantities. This is because ordinary least squares regression (OLS) is typically used to calculate the linear AHG

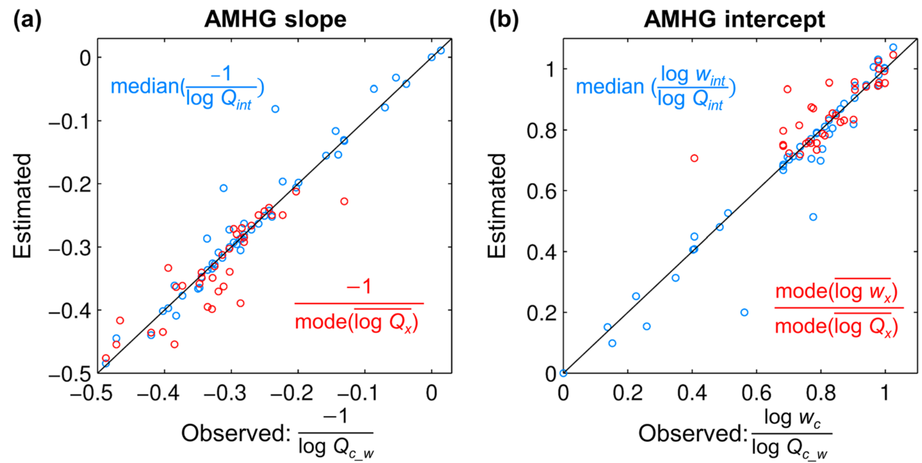

Figure 2. Approximation of the slope and intercept of AMHG. Observed AMHG slope and intercept (given by Q_{c_w}, w_c) are approximated in two ways: by using median values of the intersections of rating curves (Q_{int}, w_{int}, blue) and by the spatial modes of observed mean Q and w across x cross sections (red). Approximation using AHG intersections is highly accurate but less applicable in practice as prior knowledge of multiple AHG rating curves is required. Approximation via spatial hydraulic modes indicates that AMHG can be practically defined in terms of these readily quantified hydraulic variables (fitting slopes 0.84 and 0.73, respectively) for rivers with strong AMHG (rivers shown here have AMHG $r^2 > 0.80$).

for each cross section in log space. In OLS (and more generally), the intercept of a linear relation is given by definition as $\alpha = \overline{y} - \beta\,\overline{x}$, therefore a in equation (1) is given as

$$\log(a) = \overline{\log(w)} - b\,\overline{\log(Q)}. \quad (14)$$

Equation (14) provides a w-Q pair mathematically *required* to appear in any rating curve. Since AMHG arises because of a shared (Q_{c_w}, w_c) among cross sections, if $\overline{\log(Q)}$ and $\overline{\log(w)}$ are similar among cross sections, then these mathematically required at a station quantities become the point of intersection for AHG curves and therefore w_c and Q_{c_w}. Thus, in cases where AMHG is strong, we propose a definition of (Q_{c_w}, w_c) as

$$Q_{c\_w} = \text{mode}\left(\overline{\log(Q_x)}\right) \quad (15)$$

$$w_c = \text{mode}\left(\overline{\log(w_x)}\right), \quad (16)$$

validated empirically in Figure 2 (x again refers to spatially indexed cross sections). Note that taking the mode of these continuous variables is problematic, so we first round $\overline{\log(Q)}$ and $\overline{\log(w)}$ at each cross section to the nearest tenth before taking the mode.

Following these results, we propose that the linearity (r^2) of AMHG should be interpreted as a geomorphic index indicating the degree of convergence of AHG curves, the cross sectional geometric variability, and the hydraulic self-similarity of a given river reach. The divergence from a perfect AMHG is calculated as the median distance from individual rating curve intersections to (Q_{c_w}, w_c) relative to Q_{c_w}, and explains the strength of AMHG (Figure 1). While AMHG is a mathematical construct, AHG curves can only intersect at the same values of Q_{c_w}, Q_{c_d}, Q_{c_v}, w_c, d_c, or v_c when AHG exponents are sufficiently variable. Additionally, in practice we define $Q_{c\_w} = \text{mode}\left(\overline{\log(Q_x)}\right)$ and $w_c = \text{mode}\left(\overline{\log(w_x)}\right)$, so if a river's width AMHG is strong, it indicates that the time mean widths and discharges at each cross section throughout that reach are similar (Figure 2).

Another interesting geomorphic consequence of our derivation of AMHG is that we can now predict the linearity of AMHG from width observations in a mass conserved reach. Since the strength of AMHG is controlled by the number of rating curves that intersect at or near (Q_{c_w}, w_c) within the range of observed Q values, the percentage of rating curve intersections falling within observed w and Q ranges defines an index (p_{int}) highly correlated to AMHG strength. If mass is conserved, and given observed widths, the topological relationships between rating curve intersections (and therefore p_{int}) remain unchanged for any arbitrary Q values, as linear AHG curves are simply stretched or compressed in the Q dimension in log space. Therefore, assuming an imposed minimum and maximum conserved Q at all stations and a vector of time variable widths for multiple cross sections, we can calculate synthetic rating curves solely for the purpose of determining AMHG strength that have the same p_{int} as true rating curves. We find that mass conserved rivers with a p_{int} of less than ~15% have AMHG r^2 of less than 0.80 (Figure S1). This has significant implications for remote sensing of discharge (see section 6), and for further discussion of p_{int}, please see section S1.2.

5. Gleason and Smith's Remotely Sensed Proxy for AMHG

In order to successfully estimate discharge from AMHG following *Gleason and Smith* [2014], its slope and intercept (defined by Q_{c_w} and w_c) must be known a priori in order to simplify the set of AHG equations for a reach. This is problematic for remote sensing of discharge, as in situ data collection is required to characterize AMHG that would render Gleason and Smith's AMHG technique redundant by providing AHG itself! However, *Gleason and Smith* [2014] proposed an empirical proxy for the slope of AMHG as the parameter y in equation (17) that reliably predicted true AMHG slope in their study. This approximation enabled remote sensing of discharge without a priori or in situ knowledge of AMHG, which was critical in their assertion that discharge could be estimated anywhere with sufficient remotely sensed imagery.

$$\max\left(w_{x1,x2,\ldots xn}\right) = p\left(\max\left(w_{x1,x2,\ldots xn}\right)^2 - \min\left(w_{x1,x2,\ldots xn}\right)^2\right)^y \tag{17}$$

Given that the slope of AMHG is shown to be $-\frac{1}{\log(Q_{c\_w})}$, we can only conclude that the congruence between the proxy and AMHG slope in *Gleason and Smith* [2014] is coincidental. This finding matches the poor congruence between the proxy and the slope of AMHG in many cases as found by *Gleason et al.* [2014]. However, Gleason and Smith's method of calculating AMHG intercept from its slope (by multiplying slope by the grand mean of all observed widths) was essentially correct, as calculating $\overline{\log(w_x)}$ in both time and space is similar to taking the spatial mode of time mean w per cross section (w_c) to generate AMHG intercept ($\frac{\log(w_c)}{\log(Q_{c\_w})}$).

Based on this finding, we cannot recommend *Gleason and Smith*'s [2014] slope proxy for future use, but we propose a different strategy for estimating a river's AMHG slope from remotely sensed data. First, we propose to simply use an a priori slope value of −0.3, which corresponds to a Q_{c_w} of 2150 $m^3\,s^{-1}$. *Gleason et al.*'s [2014] discharge estimation methodology includes tuning of AMHG as well as local AHG, and their recommended range of variation for AMHG slope was 0.1: when combined with our proposed proxy value of −0.3 this range covers 35/50 of the observed AMHG slopes in this study and corresponds to Q_{c_w} values ranging from 316 to 100000 $m^3\,s^{-1}$. In addition, 11/15 rivers not covered by the proposed range have observed AMHG slopes greater than −0.2, leading to a Q_{c_w} that is obviously outside the range of observed data (Q_{c_w} is 4 million $m^3\,s^{-1}$ in some cases, see Figure S3). These erroneous Q_{c_w} values also lead to p_{int} values less than 15%, and therefore, discharge estimation is not suitable for these rivers as rating curves do not converge (with weak AMHG predicted by p_{int}). Thus, our large proposed a priori AMHG range should account for the Q_{c_w} of most natural rivers observable by current satellite technology with moderate resolution, and heuristic optimization of both AMHG and local AHG has been shown to be effective at solving for Q even with this large range [*Gleason et al.*, 2014]. However, if users want a more exact initial slope value (before heuristic optimization), the spatial mode of $\overline{\log(Q_x)}$ should be estimated to yield Q_{c_w} by either hydrologic modeling (e.g., Variable Infiltration Capacity model [*Liang et al.*, 1994], Water Balance Model [*Vörösmarty et al.*, 1998], and Water Global Assessment and Prognosis [*Alcamo et al.*, 2003]) or empirical means [e.g., *Moody and Troutman*, 2002]. Refer to section S1.3 for more information on our proposed prior Q_{c_w} value.

6. AMHG and Remote Sensing of Discharge

The derivation of AMHG given here agrees with the discharge estimation procedure described by *Gleason et al.* [2014], save that their proposed remotely sensed proxy for AMHG be changed as recommended above. In this approach, discharge is calculated at any cross section by inverting equation (1) and solving for Q given a/b pairs (via AMHG-aided optimization) and w (measured from remotely sensed images). In cases where AMHG is strong and known a priori, this inversion is likely to be successful by defining a set of possible inverted Q values that match true Q, illustrated by the green box bounded by the range of observed w and Q in Figure 3a. This chance for success arises because a correctly characterized and strong AMHG gives $\text{mode}\left(\overline{\log(Q_x)}\right)$ and $\text{mode}\left(\overline{\log(w_x)}\right)$ per Figure 2. Thus, any inverted rating curve from equation (1) must

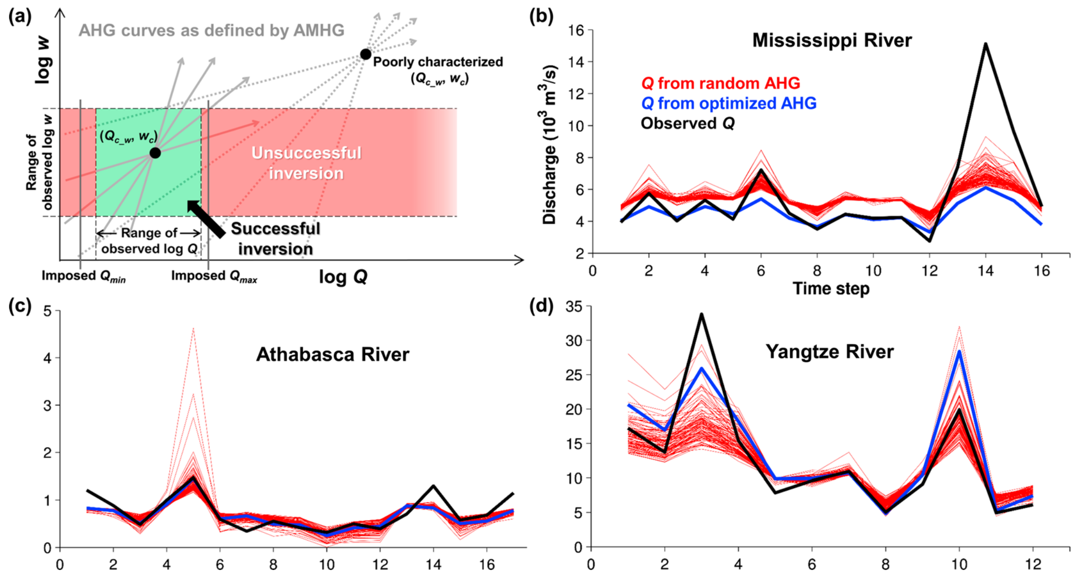

Figure 3. AMHG enables successful remotely sensed discharge estimation. So long as observed widths and the minimum and maximum discharge constraints of *Gleason and Smith* [2014] reasonably match the green region representing the observed Q range (a), forcing inverted AHG curves through (Q_{c_w}, w_c) within the green box leads to successful discharge estimation. Forcing rating curves outside the green region (when AMHG is poorly characterized) will give incorrect discharge inversion. Gleason and Smith anticipated this finding (b–d, reprinted from *Gleason and Smith* [2014]), as they showed that observed discharge (black), optimized discharge (blue), and hydrographs generated from random AHG parameters forced through (Q_{c_w}, w_c) (red) all match one another. The randomly seeded red hydrographs suggest knowledge of AMHG alone can calculate reasonably accurate hydrographs given a set of observed *w* values.

pass through these true hydraulic quantities, as seen in the AHG intersections in the green box in Figure 3a. Since inverted AHG curves are truncated both by observed widths and by imposed minimum and maximum discharge constraints (*Gleason and Smith* [2014] give "global" values of $Q_{\min}$ = minimum observed width × 0.5 m depth × 0.1 m/s velocity, $Q_{\max}$ = maximum observed width × 10 m depth × 5 m/s velocity), this truncated inversion region should include the green observed *Q* range in Figure 3a for most rivers. If this alignment occurs, then reach averaging over many different rating curves should yield a reach-averaged product contained within the range of observed *Q*. This is illustrated by the many reach-averaged hydrographs of Figures 3b–3d, where each red line represents a hydrograph generated by random AHG parameters and forced through (Q_{c_w}, w_c) as defined by AMHG. Almost all of these hydrographs are contained within the range of observed discharge as indicated by the range of the black lines in Figures 3b–3d. Similarly, discharge estimation success is unlikely given a weak AMHG where rating curve intersection does not occur or a poorly characterized AMHG that forces inverted rating curves though a point outside the green region in Figure 3a. Future users should calculate p_{int} using procedures described in section S1.2 to ensure that AMHG is strong before attempting to estimate discharge, and future work to refine the a priori estimate of AMHG would increase the accuracy of the method. Future users should also select realistic a priori discharge constraints when considering the scale of a particular study river.

Gleason et al. [2014] also found that braided, arid region, and low-*b* (where all cross-sectional AHG *b* exponents are less than 0.1) rivers were not well estimated by their methodology. Our results support these findings, as the mechanisms driving these poor estimations are controlled by issues with inversion of AHG power law exponents and not by AMHG (AMHG is also weak in low-*b* rivers as AHG exponents are not variable). These issues occur when inverting very low or very high (braided) *b* exponents or when there are order of magnitude changes in width or channel reorganization in flashy (arid) systems, and our results support the continued exclusion of these morphologies.

7. Conclusion

We have shown that AMHG is a consequence of imposing AHG power laws at a station: if AHG curves across all cross sections of a given river reach intersect near the same values of width and discharge, then a strong

AMHG *must* be observed. This analysis shows that AHG is not a redundant formulation and that AMHG does not result from previously unknown hydraulic constants. However, AMHG *is* a novel geomorphic phenomenon, but not in the sense that it repudiates AHG: the fact that the AMHG for the 50 rivers investigated here is so strong is an indication that rating curves often reliably intersect in these rivers. This gives rise to our interpretation of the strength of AMHG as an index of both geometric variability (to ensure rating curve intersection) and hydraulic self-similarity (so that mean w and Q are similar among cross sections, providing the intersection point Q_{c_w}, w_c). These conclusions suggest that the hydrologic and hydraulic drivers of AMHG are fertile ground for further study.

In addition, our analysis shows that *Gleason et al.*'s [2014] AMHG discharge estimation method is able to correctly estimate Q by first defining a region of possible Q inversions bounded by observed widths and minimum and maximum discharge constraints that matches true Q range. Users seeking to estimate discharge via AMHG should consider realistic minimum and maximum discharge constraints for their specific study rivers. Second, all inverted AHG rating curves are forced to pass through the true spatial mode of time mean Q (Q_{c_w}, defined by AMHG) within this region, thus usually ensuring correct matching of mean flow. Furthermore, we find that p_{int}, the percentage of rating curves intersecting within the range of observed data, is a critical a priori indicator of unsuccessful discharge estimation, and p_{int} should be greater than 15% if discharge estimation is to be attempted. AMHG must be remotely sensible to be useful for discharge estimation in ungauged basins: some prior estimate of AMHG is needed. We have shown that Gleason and Smith's previously given AMHG slope proxy is serendipitous and that it should not be used; we propose replacing it with a value of -0.3 or another a priori estimate of mode $(\overline{\log(Q_x)})$. Our results therefore suggest that AMHG-enabled discharge estimation remains a robust practice and that future work must focus on obtaining a reliable prior estimate of AMHG without relying on in situ data.

Acknowledgments
This research was supported by the NASA Remote Sensing Theory initiative (grant NNX12AB41G), NASA Surface Water Ocean Topography (SWOT) mission (grant NNX13AD88G), and NASA Earth and Space Sciences Fellowship NNX12AN32H. N.D. Smith provided a useful sounding board for the ideas in this paper, and L.C. Smith was instrumental in the initial publication of AMHG. This paper was greatly improved by review comments made by D. Yamazaki and one anonymous reviewer. M. Durand generously provided calibrated hydraulic model outputs. All data used in this paper are available upon request.

The Editor thanks Dai Yamazaki and an anonymous reviewer for their assistance in evaluating this paper.

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