DEVELOPMENT OF A SIMULATION MODEL
FOR THE STUDY OF ADVANCED CONTROL
CONCEPTS FOR ARTICULATED MECHANISMS

by

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A THESIS

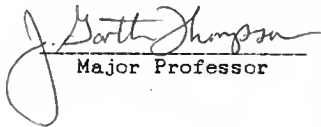submitted in partial fulfillment of the

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INTRODUCTION

     With the advancements in vision systems and terrain
preview algorithms, vehicles with on-board automatic
guidance systems are becoming of greater interest. Uses for
these types of vehicles range from monitoring hazardous
locations in plants, to military surveillance, to mobile
instrument platforms on distant planets. Walking robots are
of particular interest for these types of uses because of
their great mobility. The ability to climb over obstacles,
cross ditches, and manipulate objects with their legs are
arguments for the use of walking robots instead of wheeled
vehicles. It is interesting to note that there are many
forms of naturally occurring legged creatures, but no
naturally occurring wheeled creatures.

     The purpose of this work is to develop a tool
(simulation model) for the study of the application of
advanced control concepts to the guidance and control of
articulated devices, to develop and experiment with a
recursive spline path definition concept, and to explore a
novel feedforward control concept. The tool which is
developed is a simulation of a walking robot which has a
short cylindrical body with six, three jointed legs
positioned symmetrically about the body.

     The kinematic and inverse kinematic equations for the
legs are developed in Chapter 1. These equations are used

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in Chapter 2 in developing the dynamic model, in the simulation program to monitor the end positions of the non-supporting legs, and in developing the joint angle spline curves.

The dynamic model is developed in Chapter 2. In order to reduce the complexity of the mathematical model, the supporting legs are treated as kinematic linkages and the non-supporting legs are treated as if they were on an inertial reference frame. Lagrange's equations are used in writing the dynamics for the body and for the non-supporting legs. The nine constraint equations required for the body dynamics are obtained by requiring that the endpoints of the supporting legs remain at fixed locations in the world coordinate system.

The recursive polynomial splines used to define the motion of the body and of the joints of the non-supporting legs are developed in Chapter 3. A recursive formula is devised for computing the coefficients of the spline functions. This recursive formula eliminates the need to perform a matrix inversion when solving for the spline coefficients. The recursive nature of the algorithm provides a spline curve which passes through a large number of data points but uses only a few data points at a time. This concept is important for real time control applications where the desired path is generated in real time from sensory or operator inputs.

The control system used in the simulation is developed in Chapter 4. The use of polynomial spline functions to define the motion of the body and legs provides a command input signal which allows the use of a novel feedforward control concept. A study of the performance of this control concept is one of the objectives of this work. Also, a method is developed which allows the use of the body position and orientation in computing the joint torques of the supporting legs. The corrective body forces are computed using a control system with a novel feedforward feature. The supporting leg joint torques are calculated using the Lagrangian multipliers and constraint equations developed in Chapter 2. Calculating the supporting leg joint torques in this manner causes all of the supporting legs to act in unison in driving the body to its correct position.

The performance of the simulation is presented in Chapter 5. The body follows the position and orientation inputs with very little error. The joints of the non-supporting legs follow their inputs closely but not with the accuracy of the body, indicating the need for more development of the control system for the non-supporting legs.

I. KINEMATIC MODEL

The kinematic and inverse kinematic equations describing the legs of the walking robot are developed in this chapter. The inverse kinematic equations are used in the simulation program to compute the leg joint angles as functions of the end positions of the legs.

In order to avoid writing the kinematics independently for six legs, an imaginary joint (Joint 1) was placed at the center of the body. Rotations about the Z_o axis of $\theta_{1,j}$ radians corresponds to the position where leg j is attached to the body. While $\theta_{1,j}$ is shown as a variable, it is actually a fixed constant for each leg.

The kinematic model of the walking robot is shown in Figure 1.1. With reference to Figure 1.1, the following definitions describing the kinematic model are made,

X_o, Y_o, Z_o is the coordinate frame located at the center of mass of the body

$X_{i,j}$, $Y_{i,j}$, $Z_{i,j}$ is the coordinate system fixed on link i+1, j and centered at the joint between link i, j and link i+1, j

$\theta_{i,j}$ is the angular displacement of

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link i,j about $Z_{i-1,j}$

$l_{i,j}$ is the distance along the $X_{i-1,j}$ axis, from the origin of the $X_{i-1,j}$, $Y_{i-1,j}$, $Z_{i-1,j}$ coordinate system to the origin of the $X_{i,j}$, $Y_{i,j}$, $Z_{i,j}$ coordinate system

E_j is the endpoint of leg j
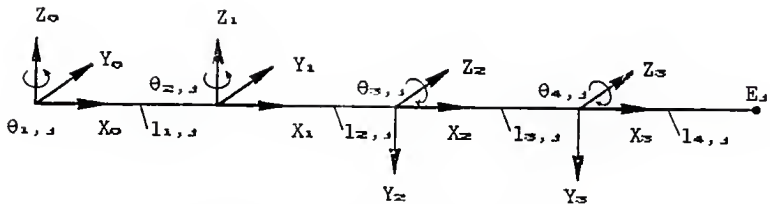
FIGURE 1.1   Kinematic Model
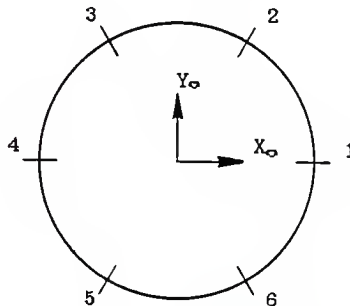
FIGURE 1.2   Leg Numbering System

The numbering system used for the legs is shown in Figure

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1.2. The view is looking down on the body from a position along the positive Z_o axis. The legs are spaced evenly at 60° intervals with leg 1 being located along the positive X_o axis.

The development of the kinematic and inverse kinematic equations follows the work of Paul [1]. Homogeneous transformations are used to describe the coordinate system of link n with respect to the coordinate system of link n-1. This transformation is called an **A** matrix and is referred to as A_n.

The **A** matrices are,

1.1 $\quad A_1 = \begin{bmatrix} C(\theta_1,_3) & -S(\theta_1,_3) & 0 & l_1,_3 C(\theta_1,_3) \\ S(\theta_1,_3) & C(\theta_1,_3) & 0 & l_1,_3 S(\theta_1,_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

1.2 $\quad A_2 = \begin{bmatrix} C(\theta_2,_3) & 0 & -S(\theta_2,_3) & l_2,_3 C(\theta_2,_3) \\ S(\theta_2,_3) & 0 & C(\theta_2,_3) & l_2,_3 S(\theta_2,_3) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

1.3 $\quad A_3 = \begin{bmatrix} C(\theta_3,_3) & -S(\theta_3,_3) & 0 & l_3,_3 C(\theta_3,_3) \\ S(\theta_3,_3) & C(\theta_3,_3) & 0 & l_3,_3 S(\theta_3,_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

1.4 $\quad A_4 = \begin{bmatrix} C(\theta_4,_3) & -S(\theta_4,_3) & 0 & l_4,_3 C(\theta_4,_3) \\ S(\theta_4,_3) & C(\theta_4,_3) & 0 & l_4,_3 S(\theta_4,_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

where $\quad C(\theta_n,_3) = \cos(\theta_n,_3)$

and $\quad S(\theta_n,_3) = \sin(\theta_n,_3)$

The transformation T_4, where,

1.5 $\quad T_4 = A_1 A_2 A_3 A_4$

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describes the end of link 4 with respect to the body (X_0, Y_0, Z_0) coordinate system. Doing this multiplication yields,

1.6

$$T_4(1,1) = C(\theta_{1,4} + \theta_{2,4})C(\theta_{3,4} + \theta_{4,4})$$

$$T_4(2,1) = S(\theta_{1,4} + \theta_{2,4})C(\theta_{3,4} + \theta_{4,4})$$

$$T_4(3,1) = -S(\theta_{3,4} + \theta_{4,4})$$

$$T_4(4,1) = 0$$

$$T_4(1,2) = -C(\theta_{1,4} + \theta_{2,4})S(\theta_{3,4} + \theta_{4,4})$$

$$T_4(2,2) = -S(\theta_{1,4} + \theta_{2,4})S(\theta_{3,4} + \theta_{4,4})$$

$$T_4(3,2) = -C(\theta_{3,4} + \theta_{4,4})$$

$$T_4(4,2) = 0$$

$$T_4(1,3) = -S(\theta_{1,4} + \theta_{2,4})$$

$$T_4(2,3) = C(\theta_{1,4} + \theta_{2,4})$$

$$T_4(3,3) = 0$$

$$T_4(4,3) = 0$$

$$T_4(1,4) = C(\theta_{1,4} + \theta_{2,4})[l_{4,4}C(\theta_{3,4} + \theta_{4,4}) + l_{3,4}C(\theta_{3,4}) + l_{2,4}] + l_{1,4}C(\theta_{1,4})$$

$$T_4(2,4) = S(\theta_{1,4} + \theta_{2,4})[l_{4,4}C(\theta_{3,4} + \theta_{4,4}) + l_{3,4}C(\theta_{3,4}) + l_{2,4}] + l_{1,4}S(\theta_{1,4})$$

$$T_4(3,4) = -l_{4,4}S(\theta_{3,4} + \theta_{4,4}) - l_{3,4}S(\theta_{3,4})$$

$$T_4(4,4) = 1$$

which has the form,

1.7
$$T_4 = \begin{bmatrix} N_x & O_x & A_x & E_{xo,4} \\ N_y & O_y & A_y & E_{yo,4} \\ N_z & O_z & A_z & E_{zo,4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $E_{xo,4}$, $E_{yo,4}$ and $E_{zo,4}$ specify the position of point

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E_J in the X_o, Y_o, Z_o coordinate system.

To obtain the inverse kinematics, Equation 1.5 is premultiplied by A_1^{-1} using the form of T_4 in Equation 1.7. Equating the $(1,4)$ and $(2,4)$ elements of the resulting matrices yields,

1.8　　$C(\theta_1,j)E_{xo,j}+S(\theta_1,j)E_{yo,j}-l_1,j =$

　　　　$C(\theta_2,j)[C(\theta_3,j+\theta_4,j)l_4,j+C(\theta_3,j)l_3,j+l_2,j]$

and,

1.9　　$-S(\theta_1,j)E_{xo,j}+C(\theta_1,j)E_{yo,j} =$

　　　　$S(\theta_2,j)[C(\theta_3,j+\theta_4,j)l_4,j+C(3,j)l_3,j+l_2,j]$

Solving for $C(\theta_2,j)$ and $S(\theta_2,j)$ yields,

1.10　　$C(\theta_2,j) = [C(\theta_1,j)E_{xo,j}+S(\theta_1,j)E_{yo,j}-l_1,j]$

　　　　　$/[C(\theta_3,j+\theta_4,j)l_4,j+C(\theta_3,j)l_3,j+l_2,j]$

and,

1.11　　$S(\theta_2,j) = [-S(\theta_1,j)E_{xo,j}+C(\theta_1,j)E_{yo,j}]$

　　　　　$/[C(\theta_3,j+\theta_4,j)l_4,j+C(\theta_3,j)l_3,j+l_2,j]$

Solving for θ_2,j,

1.12　　$\theta_2,j = \arctan\{[C(\theta_1;j)E_{yo,j}-S(\theta_1,j)E_{xo,j}]$

　　　　　$/[C(\theta_1,j)E_{xo,j}+S(\theta_1,j)E_{yo,j}-l_1,j]\}$

Equating elements $(1,4)$ and $(3,4)$ of T_4 in Equation 1.6 to the corresponding elements of T_4 in Equation 1.7 results in,

1.13　　$E_{xo,j} = C(\theta_1,j+\theta_2,j)[C(\theta_3,j+\theta_4,j)l_4,j+C(\theta_3,j)$

　　　　　$*l_3,j+l_2,j]+C(\theta_1,j)l_1,j$

and,

1.14　　$E_{xo,j} = -S(\theta_3,j+\theta_4,j)l_4,j-S(\theta_3,j)l_3,j$

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Rearranging and squaring Equation 1.13 yields,

1.15 $\quad K_{1,J}^2 = [C(\theta_{3,J}+\theta_{4,J})l_{4,J}+C(\theta_{3,J})l_{3,J}]^2$

where, $\quad K_{1,J} = [E_{xo,J}-C(\theta_{1,J}+\theta_{2,J})l_{2,J}$

$$-C(\theta_{1,J})l_{1,J}]/C(\theta_{1,J}+\theta_{2,J})$$

Multiplying Equation 1.14 by -1 and squaring yields,

1.16 $\quad E_{xo,J}^2 = [S(\theta_{3,J}+\theta_{4,J})l_{4,J}+S(\theta_{3,J})l_{3,J}]^2$

Adding Equations 1.15 and 1.16 and simplifying gives,

1.17 $\quad K_{1,J}^2+E_{xo,J}^2 = l_{4,J}^2+l_{3,J}^2+2C(\theta_{4,J})l_{4,J}l_{3,J}$

Solving for $C(\theta_{4,J})$,

1.18 $\quad C(\theta_{4,J}) = (K_{1,J}^2+E_{xo,J}^2-l_{4,J}^2-l_{3,J}^2)$

$$/(2l_{4,J}l_{3,J})$$

Then,

1.19 $\quad S(\theta_{4,J}) = sqrt[1-C^2(\theta_{4,J})]$

and,

1.20 $\quad \theta_{4,J} = arctan[S(\theta_{4,J})/C(\theta_{4,J})]$

To obtain $\theta_{3,J}$, the multiplication,

1.21 $\quad A_3^{-1}A_2^{-1}A_1^{-1}T_4 = A_4$

is performed using the form of T_4 in Equation 1.7. Equating
the (1,4) and (2,4) elements of the resulting matrices
gives,

1.22 $\quad C(\theta_{3,J})K_{2,J}+S(\theta_{3,J})(-E_{xo,J}) = C(\theta_{4,J})l_{4,J}$

$$+l_{3,J}$$

and,

1.23 $\quad C(\theta_{3,J})(-E_{xo,J})+S(\theta_{3,J})(-K_{2,J}) = S(4,J)l_{4,J}$

where, $\quad K_{2,J} = C(\theta_{1,J}+\theta_{2,J})E_{xo,J}+S(\theta_{1,J}+\theta_{2,J})E_{yo,J}$

$$-C(\theta_{2,J})l_{1,J}-l_{2,J}$$

In matrix form, Equations 1.22 and 1.23 become,

$$1.24 \quad \begin{bmatrix} K_2, & -E_{xo}, \\ -E_{xo}, & -K_2, \end{bmatrix} \begin{bmatrix} C(\theta_5,) \\ S(\theta_5,) \end{bmatrix} =$$

$$\begin{bmatrix} C(\theta_4,)l_4, +l_5, \\ S(\theta_4,)l_4, \end{bmatrix}$$

Cramer's rule is used to solve for $S(\theta_5,)$ and $C(\theta_5,)$.
Dividing $S(\theta_5,)$ by $C(\theta_5,)$ and taking the arctangent gives,

$$1.25 \quad \theta_5, = \arctan\{[S(\theta_4,)l_4, K_2,$$

$$+(C(\theta_4,)l_4, +l_5,)E_{xo},]$$

$$/[-(C(\theta_4,)l_4, +l_5,)K_2,$$

$$+S(\theta_4,)l_4, E_{xo},]\}$$

This completes the derivation of the kinematic and inverse kinematic equations. Equations from this chapter will be used for path planning and for deriving the dynamic equations of the walking robot.

II. DYNAMIC MODEL

The mathematical model describing the walking robot is developed in this chapter. The mathematical model is used in the simulation program to determine the path of the body and the legs of the robot when torques are applied at each joint.

The nature of the walking robot leads to nonlinear, highly coupled, and quite extensive equations of motion. In order to reduce the complexity of the equations and of the simulation, the following assumptions are made.

1. The body is heavy and slow moving compared to the legs.

2. The body maintains a near level attitude.

3. The body and all links are symmetric about their principle axes.

4. The body and all links are made of homogenous material with no concentrated masses.

5. Three legs are supporting the body at any given time (legs 2,4,and 6 or legs 1,3,and 5).

6. A selective surface is used for the robot to walk on. This surface has the following properties:

   a. When it is time to switch support from one set of legs to the other, the level of the surface is the same as the level of the

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endpoints of each of the legs that are
just becoming supporting legs.

b. The surface never interferes with the
motion of the non-supporting legs.

Assumption 1 allows for the supporting legs to be
treated as massless kinematic linkages. Assumptions 1 and 2
allow for the non-supporting legs to be treated as if they
were on an inertial reference frame. Assumptions 3 and 4
reduce the calculations needed for the moments of inertia of
the body and the links. Assumption 5 allows for instant-
aneous transfer of weight from one set of legs to the
other. Assumption 6 simplifies the simulation program in
two ways. First, 6.a allows for the transfer of body weight
to come at predetermined times, thus doing away with the
necessity of monitoring the end position of each leg.
Second, 6.b eliminates the need for constraints on the
possible positions of the non-supporting legs. While
assumption 6 is not entirely accurate for a walking robot,
it allows the simulation program to concentrate on the
dynamics of the walking robot and the application of spline
functions to guide the robot.

2-1. DYNAMICS OF BODY AND SUPPORTING LEGS

The origin of the body coordinate system (X_o, Y_o, Z_o) is
located in inertial world coordinates by the position vector
P,

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where, $P = \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix}$

The orientation of the body is defined by the Euler angles Φ, Θ, and Ψ; where Φ is a rotation about the z axis of the body, Θ is a rotation about the new x axis, and Ψ is a rotation about the new z axis.

In writing the dynamics for the body, P_x, P_y, P_z, Φ, Θ, Ψ, and the joint angles of the supporting legs are the independent generalized coordinates. This gives 15 generalized coordinates while the body has 6 degrees of freedom. The 9 constraint equations come from holding the endpoint, E_j, of each of the supporting legs at a fixed location in world coordinates.

Lagrange's equations are used to write the dynamics of the walking robot. Defining K as the kinetic energy of the system and P as the potential energy, the Lagrangian, L, is defined as,

2.1 $L = K - P$

From Greenwood [3], Lagrange's equations are written in the form,

2.2 $\dfrac{d}{dt} \left(\dfrac{\partial L}{\partial \dot{q}_i} \right) - \dfrac{\partial L}{\partial q_i} = \sum\limits_{m=1}^{9} \lambda_m a_{m,i} + T_i$,

$i = 1, 2, \ldots, 15$ and,

$\sum\limits_{i=1}^{15} a_{m,i} \dot{q}_i + a_{m,t} = 0$, $m = 1, 2, \ldots, 9$

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where,

q_i is the ith generalized coordinate,

λ_m are Lagrangian multipliers,

T_i is the force applied at the ith generalized coordinate,

$$a_{m,i} = \frac{\partial B_m}{\partial q_i},$$

$$a_{m,\tau} = \frac{\partial B_m}{\partial t},$$

B_m is the mth constraint equation.

The generalized coordinates are assigned as follows,

$q_1 = P_x$	$q_6 = \psi$	$q_{11} = \theta_{3,q}$
$q_2 = P_y$	$q_7 = \theta_{2,p}$	$q_{12} = \theta_{4,q}$
$q_3 = P_z$	$q_8 = \theta_{3,p}$	$q_{13} = \theta_{2,r}$
$q_4 = \phi$	$q_9 = \theta_{4,p}$	$q_{14} = \theta_{3,r}$
$q_5 = \theta$	$q_{10} = \theta_{2,q}$	$q_{15} = \theta_{4,r}$

with p, q, and r being the supporting legs.

The angular velocities about the principle body axes are defined as,

$\omega_{x,o}$ = angular velocity about the X_o axis

$\omega_{y,o}$ = angular velocity about the Y_o axis

$\omega_{z,o}$ = angular velocity about the Z_o axis.

From Syman[2], the components of the angular velocity vector are written as functions of the body orientation angles as follows,

2.3 $\omega_{x,o} = \dot{\theta}C(\psi) + \dot{\phi}S(\theta)S(\psi)$

2.4 $\omega_{y,o} = -\dot{\theta}S(\psi) + \dot{\phi}S(\theta)C(\psi)$

2.5 $\quad \omega_{x,o} = \dot{\psi}+\dot{\phi}C(\theta)$.

The kinetic energy of the body is,

2.6 $\quad K = (1/2)Mb(\dot{P}_x{}^2+\dot{P}_y{}^2+\dot{P}_z{}^2)$

$\quad\quad\quad +(1/2)[I_{x,o}(\omega_x,o)^2+I_{y,o}(\omega_y,o)^2$

$\quad\quad\quad +I_{z,o}(\omega_z,o)^2]$.

where,

$\quad\quad$ Mb = mass of the body,

$\quad\quad$ $I_{x,o}$, $I_{y,o}$, and $I_{z,o}$ are the moments of

$\quad\quad$ inertia about the X_o, Y_o, and Z_o axis.

$\quad$ The potential energy of the body is,

2.7 $\quad P = MbGP_z$

where G is the acceleration due to gravity.

$\quad$ The Lagrangian is,

2.8 $\quad L = (1/2)Mb(\dot{P}_x{}^2+\dot{P}_y{}^2+\dot{P}_z{}^2-2GP_z)$

$\quad\quad\quad +(1/2)[I_{x,o}(\omega_x,o)^2+I_{y,o}(\omega_y,o)^2$

$\quad\quad\quad +I_{z,o}(\omega_z,o)^2]$

Taking the required derivatives of L gives,

2.9 $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{P}_x}\right) = Mb\ddot{P}_x$

2.10 $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{P}_y}\right) = Mb\ddot{P}_y$

2.11 $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{P}_z}\right) = Mb\ddot{P}_z$

2.12 $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) = \ddot{\phi}[S^2(\theta)S^2(\psi)I_{x,o}+S^2(\theta)C^2(\psi)I_{y,o}$

$\quad\quad\quad +C^2(\theta)I_{z,o}]+\ddot{\theta}[S(\theta)C(\psi)S(\psi)I_{x,o}$

$\quad\quad\quad -S(\theta)C(\psi)S(\psi)I_{y,o}]+\ddot{\psi}C(\theta)I_{z,o}$

$\quad\quad\quad +2\dot{\phi}\dot{\theta}[C(\theta)S(\theta)S^2(\psi)I_{x,o}$

$\quad\quad\quad +C(\theta)S(\theta)C^2(\psi)I_{y,o}-C(\theta)S(\theta)I_{z,o}]$

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$$+2\dot{\phi}\dot{\psi}[S^2(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})]$$

$$+\dot{\theta}^2[C(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})]$$

$$+\dot{\theta}\dot{\psi}[S(\theta)(C^2(\psi)-S^2(\psi))(I_{x,o}-I_{y,o})$$

$$-S(\theta)I_{z,o}]$$

2.13 $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}}\right) = \ddot{\phi}S(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})$

$$+\ddot{\theta}[C^2(\psi)I_{x,o}+S^2(\psi)I_{y,o}]$$

$$+\dot{\phi}\dot{\theta}C(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})$$

$$+\dot{\phi}\dot{\psi}S(\theta)(C^2(\psi)-S^2(\psi))(I_{x,o}-I_{y,o})$$

$$-2\dot{\theta}\dot{\psi}C(\psi)S(\psi)(I_{x,o}-I_{y,o})$$

2.14 $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\psi}}\right) = \ddot{\phi}C(\theta)I_{z,o}+\ddot{\psi}I_{z,o}-\dot{\phi}\dot{\theta}S(\theta)I_{z,o}$

2.15 $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}_{2,j}}\right) = 0$

2.16 $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}_{3,j}}\right) = 0$ $\Biggr\}$ $j = p,q,r$

2.17 $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}_{4,j}}\right) = 0$

2.18 $\dfrac{\partial L}{\partial P_x} = 0$

2.19 $\dfrac{\partial L}{\partial P_y} = 0$

2.20 $\dfrac{\partial L}{\partial P_z} = -MbG$

2.21 $\dfrac{\partial L}{\partial\phi} = 0$

2.22 $\dfrac{\partial L}{\partial\theta} = \dot{\phi}^2C(\theta)S(\theta)[S^2(\psi)I_{x,o}+C^2(\psi)I_{y,o}$

$$-I_{z,o}]+\dot{\phi}\dot{\theta}C(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})$$

$$-\dot{\phi}\dot{\psi}S(\theta)I_{z,o}$$

2.23 $\dfrac{\partial L}{\partial\psi} = \dot{\phi}^2S^2(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})$

$$+\dot{\phi}\dot{\theta}S(\theta)(C^2(\psi)-S^2(\psi))(I_{x,o}-I_{y,o})$$

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$$-\dot{\theta}^2 C(\psi)S(\psi)(I_{\kappa,o}-I_{\gamma o})$$

2.24 $\quad \dfrac{\partial L}{\partial \theta_{z,j}} = 0$

2.25 $\quad \dfrac{\partial L}{\partial \theta_{3,j}} = 0 \qquad \Big\} \; j = p,q,r$

2.26 $\quad \dfrac{\partial L}{\partial \theta_{4,j}} = 0$

The constraint equations are developed upon the assumption that the ends of the supporting legs stay at a fixed location in the world coordinate system. From Chapter 1, the position of E_j in the X_o, Y_o, Z_o system is,

2.27 $\quad E_{\kappa o,j} = C(\theta_{1,j}+\theta_{z,j})[C(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$\qquad\qquad +C(\theta_{3,j})l_{3,j}+l_{z,j}]+C(\theta_{1,j})l_{1,j}$

2.28 $\quad E_{\gamma o,j} = S(\theta_{1,j}+\theta_{z,j})[C(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$\qquad\qquad +C(\theta_{3,j})l_{3,j}+l_{z,j}]+S(\theta_{1,j})l_{1,j}$

2.29 $\quad E_{\kappa o,j} = -S(\theta_{3,j}+\theta_{4,j})l_{4,j}-S(\theta_{3,j})l_{3,j}$

The position of E_j in world coordinates is,

2.30 $\quad \begin{bmatrix} E_{\kappa w,j} \\ E_{\gamma w,j} \\ E_{\kappa w,j} \\ 1 \end{bmatrix} = \begin{bmatrix} \text{Euler} \\ \text{rotation} \\ \text{and} \\ \text{translation} \end{bmatrix} \begin{bmatrix} E_{\kappa o,j} \\ E_{\gamma o,j} \\ E_{\kappa o,j} \\ 1 \end{bmatrix}$

where,

2.31 $\quad \begin{bmatrix} \text{Euler} \\ \text{rotation} \\ \text{and} \\ \text{translation} \end{bmatrix} =$

$$\begin{bmatrix} C(\phi)C(\psi)-S(\phi)C(\theta)S(\psi) & -C(\phi)S(\psi)-S(\phi)C(\theta)C(\psi) & S(\phi)S(\theta) & P_{\kappa} \\ S(\phi)C(\psi)+C(\phi)C(\theta)S(\psi) & -S(\phi)S(\psi)+C(\phi)C(\theta)C(\psi) & -C(\phi)S(\theta) & P_{\gamma} \\ S(\theta)S(\psi) & S(\theta)C(\psi) & C(\theta) & P_{\kappa} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Performing the multiplication in Equation 2.30 gives,

2.32 $E_{xw,j} = C(\Phi)C(\psi+\theta_{1,j}+\theta_{z,j})[C(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$+C(\theta_{3,j})l_{3,j}+l_{z,j}]+C(\Phi)C(\psi+\theta_{1,j})l_{1,j}$

$-S(\Phi)C(\theta)S(\psi+\theta_{1,j}+\theta_{z,j})$

$*[C(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$+C(\theta_{3,j})l_{3,j}+l_{z,j}]$

$-S(\Phi)C(\theta)S(\psi+\theta_{1,j})l_{1,j}$

$-S(\Phi)S(\theta)[S(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$+S(\theta_{3,j})l_{3,j}]+P_x$

2.33 $E_{yw,j} = S(\Phi)C(\psi+\theta_{1,j}+\theta_{z,j})[C(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$+C(\theta_{3,j})l_{3,j}+l_{z,j}]+S(\Phi)C(\psi+\theta_{1,j})l_{1,j}$

$+C(\Phi)C(\theta)S(\psi+\theta_{1,j}+\theta_{z,j})$

$*[C(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$+C(\theta_{3,j})l_{3,j}+l_{z,j}]$

$+C(\Phi)C(\theta)S(\psi+\theta_{1,j})l_{1,j}$

$+C(\Phi)S(\theta)[S(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$+S(\theta_{3,j})l_{3,j}]+P_y$

2.34 $E_{zw,j} = S(\theta)S(\psi+\theta_{1,j}+\theta_{z,j})[C(\theta_{3,j}+\theta_{4,j})l_{4,j}$

$+C(\theta_{3,j})l_{3,j}+l_{z,j}]+S(\theta)S(\psi+\theta_{1,j})l_{1,j}$

$-C(\theta)[S(\theta_{3,j}+\theta_{4,j})l_{4,j}+S(\theta_{3,j})l_{3,j}]$

$+P_z$

The constraint equations, which are of the form,

2.35 $B_m = 0$,

are written in terms of the support legs p, q, and r.

2.36 $B_1 = [C(\Phi)C(\psi+\theta_{1,p}+\theta_{z,p})$

$-S(\Phi)C(\theta)S(\psi+\theta_{1,p}+\theta_{z,p})]$

$*[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{z,p}]$

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$$+C(\Phi)C(\psi+\theta_{1,p})l_{1,p}$$

$$-S(\Phi)[C(\theta)S(\psi+\theta_{1,p})l_{1,p}$$

$$+S(\theta)S(\theta_{3,p}+\theta_{4,p})l_{4,p}$$

$$+S(\theta)S(\theta_{3,p})l_{3,p}]+P_x-E_{xw,p}$$

2.37 $$B_2 = [C(\Phi)C(\psi+\theta_{1,q}+\theta_{2,q})$$

$$-S(\Phi)C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})]$$

$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$

$$+C(\Phi)C(\psi+\theta_{1,q})l_{1,q}$$

$$-S(\Phi)[C(\theta)S(\psi+\theta_{1,q})l_{1,q}$$

$$+S(\theta)S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$+S(\theta)S(\theta_{3,q})l_{3,q}]+P_x-E_{xw,q}$$

2.38 $$B_3 = [C(\Phi)C(\psi+\theta_{1,r}+\theta_{2,r})$$

$$-S(\Phi)C(\theta)S(\psi+\theta_{1,r}+\theta_{2,r})]$$

$$*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}]$$

$$+C(\Phi)C(\psi+\theta_{1,r})l_{1,r}$$

$$-S(\Phi)[C(\theta)S(\psi+\theta_{1,r})l_{1,r}$$

$$+S(\theta)S(\theta_{3,r}+\theta_{4,r})l_{4,r}$$

$$+S(\theta)S(\theta_{3,r})l_{3,r}]+P_x-E_{xw,r}$$

2.39 $$B_4 = [S(\Phi)C(\psi+\theta_{1,p}+\theta_{2,p})$$

$$+C(\Phi)C(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})]$$

$$*[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$$

$$+S(\Phi)C(\psi+\theta_{1,p})l_{1,p}$$

$$+C(\Phi)[C(\theta)S(\psi+\theta_{1,p})l_{1,p}$$

$$+S(\theta)S(\theta_{3,p}+\theta_{4,p})l_{4,p}$$

$$+S(\theta)S(\theta_{3,p})l_{3,p}]+P_y-E_{yw,p}$$

2.40 $$B_5 = [S(\Phi)C(\psi+\theta_{1,q}+\theta_{2,q})$$

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$$+C(\phi)C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})]$$

$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$

$$+S(\phi)C(\psi+\theta_{1,q})l_{1,q}$$

$$+C(\phi)[C(\theta)S(\psi+\theta_{1,q})l_{1,q}$$

$$+S(\theta)S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$+S(\theta)S(\theta_{3,q})l_{3,q}]+P_y-E_{yw,q}$$

2.41 $B_6 = [S(\phi)C(\psi+\theta_{1,r}+\theta_{2,r})$

$$+C(\phi)C(\theta)S(\psi+\theta_{1,r}+\theta_{2,r})]$$

$$*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}]$$

$$+S(\phi)C(\psi+\theta_{1,r})l_{1,r}$$

$$+C(\phi)[C(\theta)S(\psi+\theta_{1,r})l_{1,r}$$

$$+S(\theta)S(\theta_{3,r}+\theta_{4,r})l_{4,r}$$

$$+S(\theta)S(\theta_{3,r})l_{3,r}]+P_y-E_{yw,r}$$

2.42 $B_7 = S(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})$

$$*[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$$

$$+S(\theta)S(\psi+\theta_{1,p})l_{1,p}-C(\theta)S(\theta_{3,p}+\theta_{4,p})l_{4,p}$$

$$-C(\theta)S(\theta_{3,p})l_{3,p}+P_x-E_{xw,p}$$

2.43 $B_8 = S(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})$

$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$

$$+S(\theta)S(\psi+\theta_{1,q})l_{1,q}-C(\theta)S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$-C(\theta)S(\theta_{3,q})l_{3,q}+P_x-E_{xw,q}$$

2.44 $B_9 = S(\theta)S(\psi+\theta_{1,r}+\theta_{2,r})$

$$*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}]$$

$$+S(\theta)S(\psi+\theta_{1,r})l_{1,r}-C(\theta)S(\theta_{3,r}+\theta_{4,r})l_{4,r}$$

$$-C(\theta)S(\theta_{3,r})l_{3,r}+P_x-E_{xw,r}$$

The $a_{m,i}$ are derived from the constraint equations.

$$2.45 \quad a_{1,1} = 1$$

$$a_{1,2} = 0$$

$$a_{1,3} = 0$$

$$\begin{aligned}
a_{1,4} = \;& [-S(\phi)C(\psi+\theta_{1,p}+\theta_{2,p}) \\
& -C(\phi)C(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})] \\
& *[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}] \\
& -S(\phi)C(\psi+\theta_{1,p})l_{1,p} \\
& -C(\phi)C(\theta)S(\psi+\theta_{1,p})l_{1,p} \\
& -C(\phi)S(\theta)S(\theta_{3,p}+\theta_{4,p})l_{4,p} \\
& -C(\phi)S(\theta)S(\theta_{3,p})l_{3,p}
\end{aligned}$$

$$\begin{aligned}
a_{1,5} = \;& S(\phi)S(\theta)S(\psi+\theta_{1,p}+\theta_{2,p}) \\
& *[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p})+l_{2,p}] \\
& +S(\phi)S(\theta)S(\psi+\theta_{1,p})l_{1,p} \\
& -S(\phi)C(\theta)S(\theta_{3,p}+\theta_{4,p})l_{4,p} \\
& -S(\phi)C(\theta)S(\theta_{3,p})l_{3,p}
\end{aligned}$$

$$\begin{aligned}
a_{1,6} = \;& [-C(\phi)S(\psi+\theta_{1,p}+\theta_{2,p}) \\
& -S(\phi)C(\theta)C(\psi+\theta_{1,p}+\theta_{2,p})] \\
& *[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}] \\
& -C(\phi)S(\psi+\theta_{1,p})l_{1,p} \\
& -S(\phi)C(\theta)C(\psi+\theta_{1,p})l_{1,p}
\end{aligned}$$

$$\begin{aligned}
a_{1,7} = \;& [-C(\phi)S(\psi+\theta_{1,p}+\theta_{2,p}) \\
& -S(\phi)C(\theta)C(\psi+\theta_{1,p}+\theta_{2,p})] \\
& *[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]
\end{aligned}$$

$$\begin{aligned}
a_{1,8} = \;& [C(\phi)C(\psi+\theta_{1,p}+\theta_{2,p}) \\
& -S(\phi)C(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})] \\
& *[-S(\theta_{3,p}+\theta_{4,p})l_{4,p}-S(\theta_{3,p})l_{3,p}]
\end{aligned}$$

$$-S(\phi)S(\theta)[C(\theta_{3,p}+\theta_{4,p})l_{4,p}$$
$$+C(\theta_{3,p})l_{3,p}]$$

$$a_{1,7} = [-C(\phi)C(\psi+\theta_{1,p}+\theta_{2,p})$$
$$+S(\phi)C(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})]$$
$$*S(\theta_{3,p}+\theta_{4,p})l_{4,p}$$
$$-S(\phi)S(\theta)C(\theta_{3,p}+\theta_{4,p})l_{4,p}$$

$$a_{1,10} = 0$$

$$a_{1,11} = 0$$

$$a_{1,12} = 0$$

$$a_{1,13} = 0$$

$$a_{1,14} = 0$$

$$a_{1,15} = 0$$

$$a_{1,t} = 0$$

$$a_{2,1} = 1$$

$$a_{2,1} = 0$$

$$a_{2,3} = 0$$

$$a_{2,4} = [-S(\phi)C(\psi+\theta_{1,q}+\theta_{2,q})$$
$$-C(\phi)C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})]$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$
$$-S(\phi)C(\psi+\theta_{1,q})l_{1,q}$$
$$-C(\phi)C(\theta)S(\psi+\theta_{1,q})l_{1,q}$$
$$-C(\phi)S(\theta)S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$
$$-C(\phi)S(\theta)S(\theta_{3,q})l_{3,q}$$

$$a_{2,5} = S(\phi)S(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$
$$+S(\phi)S(\theta)S(\psi+\theta_{1,q})l_{1,q}$$

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$$-S(\phi)C(\theta)S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$-S(\phi)C(\theta)S(\theta_{3,q})l_{3,q}$$

$$a_{2,6} = [-C(\phi)S(\psi+\theta_{1,q}+\theta_{2,q})$$

$$-S(\phi)C(\theta)C(\psi+\theta_{1,q}+\theta_{2,q})]$$

$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$

$$-C(\phi)S(\psi+\theta_{1,q})l_{1,q}$$

$$-S(\phi)C(\theta)C(\psi+\theta_{1,q})l_{1,q}$$

$$a_{2,7} = 0$$

$$a_{2,8} = 0$$

$$a_{2,9} = 0$$

$$a_{2,10} = [-C(\phi)S(\psi+\theta_{1,q}+\theta_{2,q})$$

$$-S(\phi)C(\theta)C(\psi+\theta_{1,q}+\theta_{2,q})]$$

$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$

$$a_{2,11} = [C(\phi)C(\psi+\theta_{1,q}+\theta_{2,q})$$

$$-S(\phi)C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})]$$

$$*[-S(\theta_{3,q}+\theta_{4,q})l_{4,q}-S(\theta_{3,q})l_{3,q}]$$

$$-S(\phi)S(\theta)[C(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$+C(\theta_{3,q})l_{3,q}]$$

$$a_{2,12} = [-C(\phi)C(\psi+\theta_{1,q}+\theta_{2,q})$$

$$+S(\phi)C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})]$$

$$*S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$-S(\phi)S(\theta)C(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$a_{2,13} = 0$$

$$a_{2,14} = 0$$

$$a_{2,15} = 0$$

$$a_{2,t} = 0$$

$$a_{3,1} = 1$$

$$a_{3,2} = 0$$

$$a_{3,3} = 0$$

$$
\begin{aligned}
a_{3,4} = &\ [-S(\phi)C(\psi+\theta_{1,r}+\theta_{2,r}) \\
&-C(\phi)C(\theta)S(\psi+\theta_{1,r}+\theta_{2,r})] \\
&*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}] \\
&-S(\phi)C(\psi+\theta_{1,r})l_{1,r} \\
&-C(\phi)C(\theta)S(\psi+\theta_{1,r})l_{1,r} \\
&-C(\phi)S(\theta)S(\theta_{3,r}+\theta_{4,r})l_{4,r} \\
&-C(\phi)S(\theta)S(\theta_{3,r})l_{3,r}
\end{aligned}
$$

$$
\begin{aligned}
a_{3,5} = &\ S(\phi)S(\theta)S(\psi+\theta_{1,r}+\theta_{2,r}) \\
&*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}] \\
&+S(\phi)S(\theta)S(\psi+\theta_{1,r})l_{1,r} \\
&-S(\phi)C(\theta)S(\theta_{3,r}+\theta_{4,r})l_{4,r} \\
&-S(\phi)C(\theta)S(\theta_{3,r})l_{3,r}
\end{aligned}
$$

$$
\begin{aligned}
a_{3,6} = &\ [-C(\phi)S(\psi+\theta_{1,r}+\theta_{2,r}) \\
&-S(\phi)C(\theta)C(\psi+\theta_{1,r}+\theta_{2,r})] \\
&*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}] \\
&-C(\phi)S(\psi+\theta_{1,r})l_{1,r} \\
&-S(\phi)C(\theta)C(\psi+\theta_{1,r})l_{1,r}
\end{aligned}
$$

$$a_{3,7} = 0$$

$$a_{3,8} = 0$$

$$a_{3,9} = 0$$

$$a_{3,10} = 0$$

$$a_{3,11} = 0$$

$$a_{3,12} = 0$$

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$$a_{3,13} = [-C(\phi)S(\psi+\theta_{1,r}+\theta_{2,r})$$
$$-S(\phi)C(\theta)C(\psi+\theta_{1,r}+\theta_{2,r})]$$
$$*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}]$$

$$a_{3,14} = [C(\phi)C(\psi+\theta_{1,r}+\theta_{2,r})$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,r}+\theta_{2,r})]$$
$$*[-S(\theta_{3,r}+\theta_{4,r})l_{4,r}-S(\theta_{3,r})l_{3,r}]$$
$$-S(\phi)S(\theta)[C(\theta_{3,r}+\theta_{4,r})l_{4,r}$$
$$+C(\theta_{3,r})l_{3,r}]$$

$$a_{3,15} = [-C(\phi)C(\psi+\theta_{1,r}+\theta_{2,r})$$
$$+S(\phi)C(\theta)S(\psi+\theta_{1,r}+\theta_{2,r})]$$
$$*S(\theta_{3,r}+\theta_{4,r})l_{4,r}$$
$$-S(\phi)S(\theta)C(\theta_{3,r}+\theta_{4,r})l_{4,r}$$

$$a_{3,t} = 0$$

$$a_{4,1} = 0$$

$$a_{4,2} = 1$$

$$a_{4,3} = 0$$

$$a_{4,4} = [C(\phi)C(\psi+\theta_{1,p}+\theta_{2,p})$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})]$$
$$*[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$$
$$+C(\phi)C(\psi+\theta_{1,p})l_{1,p}$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,p})l_{1,p}$$
$$-S(\phi)S(\theta)S(\theta_{3,p}+\theta_{4,p})l_{4,p}$$
$$-S(\phi)S(\theta)S(\theta_{3,p})l_{3,p}$$

$$a_{4,5} = -C(\phi)S(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})$$
$$*[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$$
$$-C(\phi)S(\theta)S(\psi+\theta_{1,p})l_{1,p}$$

$$+C(\phi)C(\theta)S(\theta_{3,p}+\theta_{4,p})l_{4,p}$$

$$+C(\phi)C(\theta)S(\theta_{3,p})l_{3,p}$$

$$a_{4,6} = [-S(\phi)S(\psi+\theta_{1,p}+\theta_{2,p})$$

$$+C(\phi)C(\theta)C(\psi+\theta_{1,p}+\theta_{2,p})]$$

$$*[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$$

$$-S(\phi)S(\psi+\theta_{1,p})l_{1,p}$$

$$+C(\phi)C(\theta)C(\psi+\theta_{1,p})l_{1,p}$$

$$a_{4,7} = [-S(\phi)S(\psi+\theta_{1,p}+\theta_{2,p})$$

$$+C(\phi)C(\theta)C(\psi+\theta_{1,p}+\theta_{2,p})]$$

$$*[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$$

$$a_{4,8} = [S(\phi)C(\psi+\theta_{1,p}+\theta_{2,p})$$

$$+C(\phi)C(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})]$$

$$*[-S(\theta_{3,p}+\theta_{4,p})l_{4,p}-S(\theta_{3,p})l_{3,p}]$$

$$+C(\phi)S(\theta)C(\theta_{3,p}+\theta_{4,p})l_{4,p}$$

$$+C(\phi)S(\theta)C(\theta_{3,p})l_{3,p}$$

$$a_{4,9} = -[S(\phi)C(\psi+\theta_{1,p}+\theta_{2,p})$$

$$+C(\phi)C(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})]$$

$$*S(\theta_{3,p}+\theta_{4,p})l_{4,p}$$

$$+C(\phi)S(\theta)C(\theta_{3,p}+\theta_{4,p})l_{4,p}$$

$$a_{4,10} = 0$$

$$a_{4,11} = 0$$

$$a_{4,12} = 0$$

$$a_{4,13} = 0$$

$$a_{4,14} = 0$$

$$a_{4,15} = 0$$

$$a_{4,t} = 0$$

$$a_{5,1} = 0$$

$$a_{5,2} = 1$$

$$a_{5,3} = 0$$

$$a_{5,4} = [C(\phi)C(\psi+\theta_{1,q}+\theta_{2,q})$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})]$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$
$$+C(\phi)C(\psi+\theta_{1,q})l_{1,q}$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,q})l_{1,q}$$
$$-S(\phi)S(\theta)S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$
$$-S(\phi)S(\theta)S(\theta_{3,q})l_{3,q}$$

$$a_{5,5} = -C(\phi)S(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$
$$-C(\phi)S(\theta)S(\psi+\theta_{1,q})l_{1,q}$$
$$+C(\phi)C(\theta)S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$
$$+C(\phi)C(\theta)S(\theta_{3,q})l_{3,q}$$

$$a_{5,6} = [-S(\phi)S(\psi+\theta_{1,q}+\theta_{2,q})$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,q}+\theta_{2,q})]$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$
$$-S(\phi)S(\psi+\theta_{1,q})l_{1,q}$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,q})l_{1,q}$$

$$a_{5,7} = 0$$

$$a_{5,8} = 0$$

$$a_{5,9} = 0$$

$$a_{5,10} = [-S(\phi)S(\psi+\theta_{1,q}+\theta_{2,q})$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,q}+\theta_{2,q})]$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$

$$a_{5,11} = [S(\phi)C(\psi+\theta_{1,q}+\theta_{2,q})$$
$$+C(\phi)C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})]$$
$$*[-S(\theta_{3,q}+\theta_{4,q})l_{4,q}-S(\theta_{3,q})l_{3,q}]$$
$$+C(\phi)S(\theta)C(\theta_{3,q}+\theta_{4,q})l_{4,q}$$
$$+C(\phi)S(\theta)C(\theta_{3,q})l_{3,q}$$

$$a_{5,12} = -[S(\phi)C(\psi+\theta_{1,q}+\theta_{2,q})$$
$$+C(\phi)C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})]$$
$$*S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$
$$+C(\phi)S(\theta)C(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$a_{5,13} = 0$$

$$a_{5,14} = 0$$

$$a_{5,15} = 0$$

$$a_{5,t} = 0$$

$$a_{6,1} = 0$$

$$a_{6,2} = 1$$

$$a_{6,3} = 0$$

$$a_{6,4} = [C(\phi)C(\psi+\theta_{1,r}+\theta_{2,r})$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,r}+\theta_{2,r})]$$
$$*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}]$$
$$+C(\phi)C(\psi+\theta_{1,r})l_{1,r}$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,r})l_{1,r}$$
$$-S(\phi)S(\theta)S(\theta_{3,r}+\theta_{4,r})l_{4,r}$$
$$-S(\phi)S(\theta)S(\theta_{3,r})l_{3,r}$$

$$a_{6,5} = -C(\phi)S(\theta)S(\psi+\theta_{1,r}+\theta_{2,r})$$
$$*[C(\theta_{3,r}+\theta_{4,r})l_{4,r}+C(\theta_{3,r})l_{3,r}+l_{2,r}]$$
$$-C(\phi)S(\theta)S(\psi+\theta_{1,r})l_{1,r}$$

$$+C(\phi)C(\theta)S(\theta_3,_r+\theta_4,_r)l_4,_r$$

$$+C(\phi)C(\theta)S(\theta_3,_r)l_3,_r$$

$$a_{6,6} = [-S(\phi)S(\psi+\theta_1,_r+\theta_2,_r)$$

$$+C(\phi)C(\theta)C(\psi+\theta_1,_r+\theta_2,_r)]$$

$$*[C(\theta_3,_r+\theta_4,_r)l_4,_r+C(\theta_3,_r)l_3,_r+l_2,_r]$$

$$-S(\phi)S(\psi+\theta_1,_r)l_1,_r$$

$$+C(\phi)C(\theta)C(\psi+\theta_1,_r)l_1,_r$$

$$a_{6,7} = 0$$

$$a_{6,8} = 0$$

$$a_{6,9} = 0$$

$$a_{6,10} = 0$$

$$a_{6,11} = 0$$

$$a_{6,12} = 0$$

$$a_{6,13} = [-S(\phi)S(\psi+\theta_1,_r+\theta_2,_r)$$

$$+C(\phi)C(\theta)C(\psi+\theta_1,_r+\theta_2,_r)]$$

$$*[C(\theta_3,_r+\theta_4,_r)l_4,_r+C(\theta_3,_r)l_3,_r+l_2,_r]$$

$$a_{6,14} = [S(\phi)C(\psi+\theta_1,_r+\theta_2,_r)$$

$$+C(\phi)C(\theta)S(\psi+\theta_1,_r+\theta_2,_r)]$$

$$*[-S(\theta_3,_r+\theta_4,_r)l_4,_r-S(\theta_3,_r)l_3,_r]$$

$$+C(\phi)S(\theta)C(\theta_3,_r+\theta_4,_r)l_4,_r$$

$$+C(\phi)S(\theta)C(\theta_3,_r)l_3,_r$$

$$a_{6,15} = -[S(\phi)C(\psi+\theta_1,_r+\theta_2,_r)$$

$$+C(\phi)C(\theta)S(\psi+\theta_1,_r+\theta_2,_r)]$$

$$*S(\theta_3,_r+\theta_4,_r)l_4,_r$$

$$+C(\phi)S(\theta)C(\theta_3,_r+\theta_4,_r)l_4,_r$$

$$a_{6,t} = 0$$

$a_{7,1} = 0$

$a_{7,2} = 0$

$a_{7,3} = 1$

$a_{7,4} = 0$

$a_{7,5} = C(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})$
$\quad *[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$
$\quad +C(\theta)S(\psi+\theta_{1,p})l_{1,p}+S(\theta)S(\theta_{3,p}+\theta_{4,p})l_{4,p}$
$\quad +S(\theta)S(\theta_{3,p})l_{3,p}$

$a_{7,6} = S(\theta)C(\psi+\theta_{1,p}+\theta_{2,p})$
$\quad *[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$
$\quad +S(\theta)C(\psi+\theta_{1,p})l_{1,p}$

$a_{7,7} = S(\theta)C(\psi+\theta_{1,p}+\theta_{2,p})$
$\quad *[C(\theta_{3,p}+\theta_{4,p})l_{4,p}+C(\theta_{3,p})l_{3,p}+l_{2,p}]$

$a_{7,8} = S(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})$
$\quad *[-S(\theta_{3,p}+\theta_{4,p})l_{4,p}-S(\theta_{3,p})l_{3,p}]$
$\quad -C(\theta)C(\theta_{3,p}+\theta_{4,p})l_{4,p}-C(\theta)C(\theta_{3,p})l_{3,p}$

$a_{7,9} = -S(\theta)S(\psi+\theta_{1,p}+\theta_{2,p})S(\theta_{3,p}+\theta_{4,p})l_{4,p}$
$\quad -C(\theta)C(\theta_{3,p}+\theta_{4,p})l_{4,p}$

$a_{7,10} = 0$

$a_{7,11} = 0$

$a_{7,12} = 0$

$a_{7,13} = 0$

$a_{7,14} = 0$

$a_{7,15} = 0$

$a_{7,t} = 0$

$a_{8,1} = 0$

30

$$a_{6,2} = 0$$

$$a_{6,3} = 1$$

$$a_{6,4} = 0$$

$$a_{6,5} = C(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$
$$+C(\theta)S(\psi+\theta_{1,q})l_{1,q}+S(\theta)S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$
$$+S(\theta)S(\theta_{3,q})l_{3,q}$$

$$a_{6,6} = S(\theta)C(\psi+\theta_{1,q}+\theta_{2,q})$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$
$$+S(\theta)C(\psi+\theta_{1,q})l_{1,q}$$

$$a_{6,7} = 0$$

$$a_{6,8} = 0$$

$$a_{6,9} = 0$$

$$a_{6,10} = S(\theta)C(\psi+\theta_{1,q}+\theta_{2,q})$$
$$*[C(\theta_{3,q}+\theta_{4,q})l_{4,q}+C(\theta_{3,q})l_{3,q}+l_{2,q}]$$

$$a_{6,11} = S(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})$$
$$*[-S(\theta_{3,q}+\theta_{4,q})l_{4,q}-S(\theta_{3,q})l_{3,q}]$$
$$-C(\theta)C(\theta_{3,q}+\theta_{4,q})l_{4,q}-C(\theta)C(\theta_{3,q})l_{3,q}$$

$$a_{6,12} = -S(\theta)S(\psi+\theta_{1,q}+\theta_{2,q})S(\theta_{3,q}+\theta_{4,q})l_{4,q}$$
$$-C(\theta)C(\theta_{3,q}+\theta_{4,q})l_{4,q}$$

$$a_{6,13} = 0$$

$$a_{6,14} = 0$$

$$a_{6,15} = 0$$

$$a_{6,t} = 0$$

$$a_{7,1} = 0$$

$$a_{7,2} = 0$$

$$a_{7,3} = 1$$

$$a_{7,4} = 0$$

$$a_{7,5} = C(\theta)S(\psi+\theta_1,_r+\theta_2,_r)$$
$$*[C(\theta_3,_r+\theta_4,_r)l_4,_r+C(\theta_3,_r)l_3,_r+l_2,_r]$$
$$+C(\theta)S(\psi+\theta_1,_r)l_1,_r+S(\theta)S(\theta_3,_r+\theta_4,_r)l_4,_r$$
$$+S(\theta)S(\theta_3,_r)l_3,_r$$

$$a_{7,6} = S(\theta)C(\psi+\theta_1,_r+\theta_2,_r)$$
$$*[C(\theta_3,_r+\theta_4,_r)l_4,_r+C(\theta_3,_r)l_3,_r+l_2,_r]$$
$$+S(\theta)C(\psi+\theta_1,_r)l_1,_r$$

$$a_{7,7} = 0$$

$$a_{7,8} = 0$$

$$a_{7,9} = 0$$

$$a_{7,10} = 0$$

$$a_{7,11} = 0$$

$$a_{7,12} = 0$$

$$a_{7,13} = S(\theta)C(\psi+\theta_1,_r+\theta_2,_r)$$
$$*[C(\theta_3,_r+\theta_4,_r)l_4,_r+C(\theta_3,_r)l_3,_r+l_2,_r]$$

$$a_{7,14} = S(\theta)S(\psi+\theta_1,_r+\theta_2,_r)$$
$$*[-S(\theta_3,_r+\theta_4,_r)l_4,_r-S(\theta_3,_r)l_3,_r]$$
$$-C(\theta)C(\theta_3,_r+\theta_4,_r)l_4,_r-C(\theta)C(\theta_3,_r)l_3,_r$$

$$a_{7,15} = -S(\theta)S(\psi+\theta_1,_r+\theta_2,_r)S(\theta_3,_r+\theta_4,_r)l_4,_r$$
$$-C(\theta)C(\theta_3,_r+\theta_4,_r)l_4,_r$$

$$a_{7,t} = 0$$

The complete set of Lagrange's equations are,

$$2.46 \qquad Mb\ddot{P}_x-\sum_{m=1}^{9}\lambda_m a_{m,1} = 0$$

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2.47 $M_b \ddot{P}_v - \sum\limits_{m=1}^{9} \lambda_m a_{m,2} = 0$

2.48 $M_b \ddot{P}_z + M_b G - \sum\limits_{m=1}^{9} \lambda_m a_{m,3} = 0$

2.49 $\ddot{\phi}[S^2(\theta)(S^2(\psi)I_{x,o} + C^2(\psi)I_{y,o}) + C^2(\theta)I_{z,o}]$

$+ \ddot{\theta} S(\theta) C(\psi) S(\psi)(I_{x,o} - I_{y,o})$

$+ \ddot{\psi} C(\theta) I_{z,o}$

$+ 2\dot{\phi}\dot{\theta} C(\theta) S(\theta)[S^2(\psi)I_{x,o} + C^2(\psi)I_{y,o} - I_{z,o}]$

$+ 2\dot{\phi}\dot{\psi} S^2(\theta) C(\psi) S(\psi)(I_{x,o} - I_{y,o})$

$+ \dot{\theta}^2 C(\theta) C(\psi) S(\psi)(I_{x,o} - I_{y,o})$

$+ \dot{\theta}\dot{\psi} S(\theta)[(C^2(\psi) - S^2(\psi))(I_{x,o} - I_{y,o}) - I_{z,o}]$

$- \sum\limits_{m=1}^{9} \lambda_m a_{m,4} = 0$

2.50 $\ddot{\phi} S(\theta) C(\psi) S(\psi)(I_{x,o} - I_{y,o})$

$+ \ddot{\theta}(C^2(\psi)I_{x,o} + S^2(\psi)I_{y,o})$

$+ \dot{\phi}\dot{\psi} S(\theta)[(C^2(\psi) - S^2(\psi))(I_{x,o} - I_{y,o}) + I_{z,o}]$

$- 2\dot{\theta}\dot{\psi} C(\psi) S(\psi)(I_{x,o} - I_{y,o})$

$- \dot{\phi}^2 C(\theta) S(\theta)[S^2(\psi)I_{x,o} + C^2(\psi)I_{y,o} - I_{z,o}]$

$- \sum\limits_{m=1}^{9} \lambda_m a_{m,5} = 0$

2.51 $\ddot{\phi} C(\theta) I_{z,o} + \ddot{\psi} I_{z,o} - \dot{\phi}^2 S^2(\theta) C(\psi) S(\psi)(I_{x,o} - I_{y,o})$

$- \dot{\phi}\dot{\theta} S(\theta)[(C^2(\psi) - S^2(\psi))(I_{x,o} - I_{y,o}) + I_{z,o}]$

$+ \dot{\theta}^2 C(\psi) S(\psi)(I_{x,o} - I_{y,o})$

$- \sum\limits_{m=1}^{9} \lambda_m a_{m,6} = 0$

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2.52 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,7} = T_7$

2.53 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,8} = T_8$

2.54 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,9} = T_9$

2.55 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,10} = T_{10}$

2.56 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,11} = T_{11}$

2.57 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,12} = T_{12}$

2.58 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,13} = T_{13}$

2.59 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,14} = T_{14}$

2.60 $\quad -\sum\limits_{m=1}^{9} \lambda_m a_{m,15} = T_{15}$

2.61 $\quad \sum\limits_{i=1}^{15} a_{1,i} \dot{q}_i + a_{1,e} = 0$

2.62 $\quad \sum\limits_{i=1}^{15} a_{2,i} \dot{q}_i + a_{2,e} = 0$

2.63 $\quad \sum\limits_{i=1}^{15} a_{3,i} \dot{q}_i + a_{3,e} = 0$

2.64 $\quad \sum\limits_{i=1}^{15} a_{4,i} \dot{q}_i + a_{4,e} = 0$

$$2.65 \quad \sum_{i=1}^{15} a_{5,i}\dot{q}_i + a_{5,t} = 0$$

$$2.66 \quad \sum_{i=1}^{15} a_{6,i}\dot{q}_i + a_{6,t} = 0$$

$$2.67 \quad \sum_{i=1}^{15} a_{7,i}\dot{q}_i + a_{7,t} = 0$$

$$2.68 \quad \sum_{i=1}^{15} a_{8,i}\dot{q}_i + a_{8,t} = 0$$

$$2.69 \quad \sum_{i=1}^{15} a_{9,i}\dot{q}_i + a_{9,t} = 0$$

where,

T_7 is the torque forcing $\theta_{2,p}$

T_8 is the torque forcing $\theta_{3,p}$

T_9 is the torque forcing $\theta_{4,p}$

T_{10} is the torque forcing $\theta_{2,q}$

T_{11} is the torque forcing $\theta_{3,q}$

T_{12} is the torque forcing $\theta_{4,q}$

T_{13} is the torque forcing $\theta_{2,r}$

T_{14} is the torque forcing $\theta_{3,r}$

T_{15} is the torque forcing $\theta_{4,r}$

Rearranging Equations 2.52 thru 2.60 and leaving out the zero terms yields

$$2.70 \quad \begin{bmatrix} a_{1,7} & a_{4,7} & a_{7,7} \\ a_{1,8} & a_{4,8} & a_{7,8} \\ a_{1,9} & a_{4,9} & a_{7,9} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_4 \\ \lambda_7 \end{bmatrix} = \begin{bmatrix} -T_7 \\ -T_8 \\ -T_9 \end{bmatrix}$$

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$$2.71 \quad \begin{bmatrix} a_{2,10} & a_{5,10} & a_{6,10} \\ a_{2,11} & a_{5,11} & a_{6,11} \\ a_{2,12} & a_{5,12} & a_{6,12} \end{bmatrix} \begin{bmatrix} \lambda_2 \\ \lambda_5 \\ \lambda_6 \end{bmatrix} = \begin{bmatrix} -T_{10} \\ -T_{11} \\ -T_{12} \end{bmatrix}$$

$$2.72 \quad \begin{bmatrix} a_{3,13} & a_{6,13} & a_{7,13} \\ a_{3,14} & a_{6,14} & a_{7,14} \\ a_{3,15} & a_{6,15} & a_{7,15} \end{bmatrix} \begin{bmatrix} \lambda_3 \\ \lambda_6 \\ \lambda_7 \end{bmatrix} = \begin{bmatrix} -T_{13} \\ -T_{14} \\ -T_{15} \end{bmatrix}$$

Solving for the λ's gives,

$$
\begin{aligned}
2.73 \quad \lambda_1 = [&-T_7(a_{4,8}a_{7,9}-a_{4,9}a_{7,8}) \\
&+T_8(a_{4,7}a_{7,9}-a_{4,9}a_{7,7}) \\
&-T_9(a_{4,7}a_{7,8}-a_{4,8}a_{7,7})]/\text{DEN1}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_4 = [&T_7(a_{1,8}a_{7,9}-a_{1,9}a_{7,8}) \\
&-T_8(a_{1,7}a_{7,9}-a_{1,9}a_{7,7}) \\
&+T_9(a_{1,7}a_{7,8}-a_{1,8}a_{7,7})]/\text{DEN1}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_7 = [&-T_7(a_{1,8}a_{4,9}-a_{1,9}a_{4,8}) \\
&+T_8(a_{1,7}a_{4,9}-a_{1,9}a_{4,7}) \\
&-T_9(a_{1,7}a_{4,8}-a_{1,8}a_{4,7})]/\text{DEN1}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_2 = [&-T_{10}(a_{5,11}a_{6,12}-a_{5,12}a_{6,11}) \\
&+T_{11}(a_{5,10}a_{6,12}-a_{5,12}a_{6,10}) \\
&-T_{12}(a_{5,10}a_{6,11}-a_{5,11}a_{6,10})]/\text{DEN2}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_5 = [&T_{10}(a_{2,11}a_{6,12}-a_{2,12}a_{6,11}) \\
&-T_{11}(a_{2,10}a_{6,12}-a_{2,12}a_{6,10}) \\
&+T_{12}(a_{2,10}a_{6,11}-a_{2,11}a_{6,10})]/\text{DEN2}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_6 = [&-T_{10}(a_{2,11}a_{5,12}-a_{2,12}a_{5,11}) \\
&+T_{11}(a_{2,10}a_{5,12}-a_{2,12}a_{5,10}) \\
&-T_{12}(a_{2,10}a_{5,11}-a_{2,11}a_{5,10})]/\text{DEN2}
\end{aligned}
$$

$$
\begin{aligned}
\lambda_3 = [&-T_{13}(a_{6,14}a_{7,15}-a_{6,15}a_{7,14}) \\
&+T_{14}(a_{6,13}a_{7,15}-a_{6,15}a_{7,13}) \\
&-T_{15}(a_{6,13}a_{7,14}-a_{6,14}a_{7,13})]/\text{DEN3}
\end{aligned}
$$

$$\lambda_6 = [T_{13}(a_5,_{14}a_7,_{16}-a_5,_{16}a_7,_{14})$$

$$-T_{14}(a_5,_{13}a_7,_{16}-a_5,_{16}a_7,_{13})$$

$$+T_{16}(a_5,_{13}a_7,_{14}-a_5,_{14}a_7,_{13})]/DEN3$$

$$\lambda_7 = [-T_{13}(a_5,_{14}a_6,_{16}-a_5,_{16}a_6,_{14})$$

$$+T_{14}(a_5,_{13}a_6,_{16}-a_5,_{16}a_6,_{13})$$

$$-T_{16}(a_5,_{13}a_6,_{14}-a_5,_{14}a_6,_{13})]/DEN3$$

where,

$$DEN1 = a_1,_7(a_4,_6a_7,_7-a_4,_7a_7,_6)$$

$$-a_4,_7(a_1,_6a_7,_7-a_1,_7a_7,_6)$$

$$+a_7,_7(a_1,_6a_4,_7-a_1,_7a_4,_6)$$

$$DEN2 = a_2,_{10}(a_6,_{11}a_6,_{12}-a_6,_{12}a_6,_{11})$$

$$-a_6,_{10}(a_2,_{11}a_6,_{12}-a_2,_{12}a_6,_{11})$$

$$+a_6,_{10}(a_2,_{11}a_6,_{12}-a_2,_{12}a_6,_{11})$$

$$DEN3 = a_5,_{13}(a_6,_{14}a_7,_{16}-a_6,_{16}a_7,_{14})$$

$$-a_6,_{13}(a_5,_{14}a_7,_{16}-a_5,_{16}a_7,_{14})$$

$$+a_7,_{13}(a_5,_{14}a_6,_{16}-a_5,_{16}a_6,_{14})$$

Writing Equations 2.46 thru 2.51 in matrix form gives,

2.74
$$\left[\quad \underline{M} \quad \right]
\begin{bmatrix} \ddot{P}_x \\ \ddot{P}_y \\ \ddot{P}_z \\ \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix}
+
\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6 \end{bmatrix}
+
\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\underline{M}$ is the 6 by 6 mass matrix.

The non zero elements of $\underline{M}$ are,

2.75 $m_{1,1} = Mb$

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$$m_{2,2} = Mb$$

$$m_{3,3} = Mb$$

$$m_{4,4} = S^2(\theta)[S^2(\psi)I_{x,o}+C^2(\psi)I_{y,o}]+C^2(\theta)I_{z,o}$$

$$m_{4,5} = S(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})$$

$$m_{4,6} = C(\theta)I_{z,o}$$

$$m_{5,4} = m_{4,5}$$

$$m_{5,5} = C^2(\psi)I_{x,o}+S^2(\psi)I_{y,o}$$

$$m_{6,4} = m_{4,6}$$

$$m_{6,6} = I_{z,o}$$

The elements of the **G** and **Q** vectors are,

2.76
$$G_1 = 0$$

$$G_2 = 0$$

$$G_3 = 0$$

$$\begin{aligned}
G_4 = {} &2\dot{\phi}\dot{\theta}C(\theta)S(\theta)[S^2(\psi)I_{x,o}+C^2(\psi)I_{y,o}-I_{z,o}]\\
&+2\dot{\phi}\dot{\psi}S^2(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})\\
&+\dot{\theta}^2C(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})\\
&+\dot{\theta}\dot{\psi}S(\theta)[(C^2(\psi)-S^2(\psi))(I_{x,o}-I_{y,o})-I_{z,o}]
\end{aligned}$$

$$\begin{aligned}
G_5 = {} &-\dot{\phi}^2C(\theta)S(\theta)[S^2(\psi)I_{x,o}+C^2(\psi)I_{y,o}-I_{z,o}]\\
&+\dot{\phi}\dot{\psi}S(\theta)[(C^2(\psi)-S^2(\psi))(I_{x,o}-I_{y,o})+I_{z,o}]\\
&-2\dot{\theta}\dot{\phi}C(\psi)S(\psi)(I_{x,o}-I_{y,o})
\end{aligned}$$

$$\begin{aligned}
G_6 = {} &-\dot{\phi}^2S^2(\theta)C(\psi)S(\psi)(I_{x,o}-I_{y,o})\\
&-\dot{\phi}\dot{\theta}S(\theta)[(C^2(\psi)-S^2(\psi))(I_{x,o}-I_{y,o})+I_{z,o}]\\
&+\dot{\theta}^2C(\psi)S(\psi)(I_{x,o}-I_{y,o})
\end{aligned}$$

2.77
$$Q_1 = -\sum_{m=1}^{9} \lambda_m a_{m,1}$$

$$Q_2 = -\sum_{m=1}^{9} \lambda_m a_{m,2}$$

$$Q_3 = M_bG - \sum_{m=1}^{9} \lambda_m a_{m,3}$$

$$Q_4 = -\sum_{m=1}^{9} \lambda_m a_{m,4}$$

$$Q_5 = -\sum_{m=1}^{9} \lambda_m a_{m,5}$$

$$Q_6 = -\sum_{m=1}^{9} \lambda_m a_{m,6}$$

Solving Equation 2.74 for the second derivative terms gives,

2.78
$$
\begin{bmatrix} \ddot{P}_x \\ \ddot{P}_y \\ \ddot{P}_z \\ \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix}
= -\underline{M}^{-1}
\begin{bmatrix}
\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ G_6 \end{bmatrix}
+
\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6 \end{bmatrix}
\end{bmatrix}
$$

The non-zero elements of $\underline{M}^{-1}$ are,

$$m^{-1}_{1,1} = 1/m_{1,1}$$

$$m^{-1}_{2,2} = 1/m_{1,1}$$

$$m^{-1}_{3,3} = 1/m_{1,1}$$

$$m^{-1}_{4,4} = m_{5,5}m_{6,6}/DET$$

$$m^{-1}_{4,5} = -m_{4,5}m_{6,6}/DET$$

$$m^{-1}_{4,6} = -m_{4,6}m_{5,5}/DET$$

$$m^{-1}_{5,4} = m^{-1}_{4,5}$$

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$$m^{-1}_{5,5} = (m_{4,4}m_{6,6} - m_{4,6}^2)/\text{DET}$$

$$m^{-1}_{5,6} = m_{4,5}m_{4,6}/\text{DET}$$

$$m^{-1}_{6,4} = m^{-1}_{4,6}$$

$$m^{-1}_{6,5} = m^{-1}_{5,6}$$

$$m^{-1}_{6,6} = (m_{4,4}m_{5,5} - m_{4,5}^2)/\text{DET}$$

where,

$$\text{DET} = m_{4,4}m_{5,5}m_{6,6} - m_{4,5}^2 m_{6,6} - m_{4,6}^2 m_{5,5}$$

The state variables are defined as,

2.80

$y_1 = P_x$	$y_5 = \theta$	$y_9 = \dot{P}_z$
$y_2 = P_y$	$y_6 = \psi$	$y_{10} = \dot{\phi}$
$y_3 = P_z$	$y_7 = \dot{P}_x$	$y_{11} = \dot{\theta}$
$y_4 = \phi$	$y_8 = \dot{P}_y$	$y_{12} = \dot{\psi}$

The state equations for the body in final form are,

2.81

$$\dot{y}_1 = y_7$$

$$\dot{y}_2 = y_8$$

$$\dot{y}_3 = y_9$$

$$\dot{y}_4 = y_{10}$$

$$\dot{y}_5 = y_{11}$$

$$\dot{y}_6 = y_{12}$$

$$\dot{y}_7 = (-Q_1 - G_1)/m_{1,1}$$

$$\dot{y}_8 = (-Q_2 - G_2)/m_{1,1}$$

$$\dot{y}_9 = (-Q_3 - G_3)/m_{1,1}$$

$$\dot{y}_{10} = [m_{5,5}m_{6,6}(-Q_4 - G_4) - m_{4,5}m_{6,6}(-Q_5 - G_5)$$
$$- m_{4,6}m_{5,5}(-Q_6 - G_6)]/\text{DET}$$

$$\dot{y}_{11} = [-m_{4,5}m_{6,6}(-Q_4 - G_4)$$
$$+ (m_{4,4}m_{6,6} - m_{4,6}^2)(-Q_5 - G_5)$$

$$+ \underline{m}_{4,8} \underline{m}_{4,6} (-Q_6 - G_6)] / DET$$

$$\dot{y}_{12} = [-\underline{m}_{4,6} \underline{m}_{6,8} (-Q_4 - G_4) + \underline{m}_{4,8} \underline{m}_{4,6} (-Q_5 - G_5)$$

$$+ (\underline{m}_{4,4} \underline{m}_{6,8} - \underline{m}_{4,8}^2)(-Q_6 - G_6)] / DET$$

Rearranging Equations 2.61 thru 2.69 and leaving out the zero terms gives,

2.82
$$\begin{bmatrix} a_{1,7} & a_{1,8} & a_{1,7} \\ a_{4,7} & a_{4,8} & a_{4,7} \\ a_{7,7} & a_{7,8} & a_{7,7} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{2,p} \\ \dot{\theta}_{3,p} \\ \dot{\theta}_{4,p} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_4 \\ C_7 \end{bmatrix}$$

2.83
$$\begin{bmatrix} a_{2,10} & a_{2,11} & a_{2,12} \\ a_{5,10} & a_{5,11} & a_{5,12} \\ a_{8,10} & a_{8,11} & a_{8,12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{2,q} \\ \dot{\theta}_{3,q} \\ \dot{\theta}_{4,q} \end{bmatrix} = \begin{bmatrix} C_2 \\ C_5 \\ C_8 \end{bmatrix}$$

2.84
$$\begin{bmatrix} a_{3,13} & a_{3,14} & a_{3,15} \\ a_{6,13} & a_{6,14} & a_{6,15} \\ a_{7,13} & a_{7,14} & a_{7,15} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{2,r} \\ \dot{\theta}_{3,r} \\ \dot{\theta}_{4,r} \end{bmatrix} = \begin{bmatrix} C_3 \\ C_6 \\ C_7 \end{bmatrix}$$

where,

$$C_1 = -(\dot{P}_x + a_{1,4}\dot{\phi} + a_{1,5}\dot{\theta} + a_{1,6}\dot{\psi})$$

$$C_2 = -(\dot{P}_x + a_{2,4}\dot{\phi} + a_{2,5}\dot{\theta} + a_{2,6}\dot{\psi})$$

$$C_3 = -(\dot{P}_x + a_{3,4}\dot{\phi} + a_{3,5}\dot{\theta} + a_{3,6}\dot{\psi})$$

$$C_4 = -(\dot{P}_y + a_{4,4}\dot{\phi} + a_{4,5}\dot{\theta} + a_{4,6}\dot{\psi})$$

$$C_5 = -(\dot{P}_y + a_{5,4}\dot{\phi} + a_{5,5}\dot{\theta} + a_{5,6}\dot{\psi})$$

$$C_6 = -(\dot{P}_y + a_{6,4}\dot{\phi} + a_{6,5}\dot{\theta} + a_{6,6}\dot{\psi})$$

$$C_7 = -(\dot{P}_z + a_{7,5}\dot{\theta} + a_{7,6}\dot{\psi})$$

$$C_8 = -(\dot{P}_z + a_{8,5}\dot{\theta} + a_{8,6}\dot{\psi})$$

$$C_9 = -(\dot{P}_z + a_{9,5}\dot{\theta} + a_{9,6}\dot{\psi})$$

Solving for the $\dot{\theta}_{1,j}$ terms gives,

2.85 $\quad \dot{\theta}_{2,p} = [C_1(a_{4,5}a_{7,7} - a_{4,7}a_{7,5})$

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$$-C_4(a_1, {}_{6}a_7, {}_{7}-a_1, {}_{7}a_7, {}_{6})$$

$$+C_7(a_1, {}_{6}a_4, {}_{7}-a_1, {}_{7}a_4, {}_{6})]/DEN1$$

$$\dot{\theta}_{3,p} = [-C_1(a_4, {}_{7}a_7, {}_{7}-a_4, {}_{7}a_7, {}_{7})$$

$$+C_4(a_1, {}_{7}a_7, {}_{7}-a_1, {}_{7}a_7, {}_{7})$$

$$-C_7(a_1, {}_{7}a_4, {}_{7}-a_1, {}_{7}a_4, {}_{7})]/DEN1$$

$$\dot{\theta}_{4,p} = [C_1(a_4, {}_{7}a_7, {}_{6}-a_4, {}_{6}a_7, {}_{7})$$

$$-C_4(a_1, {}_{7}a_7, {}_{6}-a_1, {}_{6}a_7, {}_{7})$$

$$+C_7(a_1, {}_{7}a_4, {}_{6}-a_1, {}_{6}a_4, {}_{7})]/DEN1$$

$$\dot{\theta}_{2,q} = [C_2(a_6, {}_{11}a_6, {}_{12}-a_6, {}_{12}a_6, {}_{11})$$

$$-C_6(a_2, {}_{11}a_6, {}_{12}-a_2, {}_{12}a_6, {}_{11})$$

$$+C_6(a_2, {}_{11}a_6, {}_{12}-a_2, {}_{12}a_6, {}_{11})]/DEN2$$

$$\dot{\theta}_{3,q} = [-C_2(a_6, {}_{10}a_6, {}_{12}-a_6, {}_{12}a_6, {}_{10})$$

$$+C_6(a_2, {}_{10}a_6, {}_{12}-a_2, {}_{12}a_6, {}_{10})$$

$$-C_6(a_2, {}_{10}a_6, {}_{12}-a_2, {}_{12}a_6, {}_{10})]/DEN2$$

$$\dot{\theta}_{4,q} = [C_2(a_6, {}_{10}a_6, {}_{11}-a_6, {}_{11}a_6, {}_{10})$$

$$-C_6(a_2, {}_{10}a_6, {}_{11}-a_2, {}_{11}a_6, {}_{10})$$

$$+C_6(a_2, {}_{10}a_6, {}_{11}-a_2, {}_{11}a_6, {}_{10})]/DEN2$$

$$\dot{\theta}_{2,r} = [C_3(a_6, {}_{14}a_7, {}_{15}-a_6, {}_{15}a_7, {}_{14})$$

$$-C_6(a_3, {}_{14}a_7, {}_{15}-a_3, {}_{15}a_7, {}_{14})$$

$$+C_7(a_3, {}_{14}a_6, {}_{15}-a_3, {}_{15}a_6, {}_{14})]/DEN3$$

$$\dot{\theta}_{3,r} = [-C_3(a_6, {}_{13}a_7, {}_{15}-a_6, {}_{15}a_7, {}_{13})$$

$$+C_6(a_3, {}_{13}a_7, {}_{15}-a_3, {}_{15}a_7, {}_{13})$$

$$-C_7(a_3, {}_{13}a_6, {}_{15}-a_3, {}_{15}a_6, {}_{13})]/DEN3$$

$$\dot{\theta}_{4,r} = [C_3(a_6, {}_{13}a_7, {}_{14}-a_6, {}_{14}a_7, {}_{13})$$

$$-C_6(a_3, {}_{13}a_7, {}_{14}-a_3, {}_{14}a_7, {}_{13})$$

$$+C_7(a_3, {}_{13}a_6, {}_{14}-a_3, {}_{14}a_6, {}_{13})]/DEN3$$

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where DEN1, DEN2, and DEN3 are given with Equations 2.73.

The joint angles for the supporting legs are computed using information from Chapter 1 and Chapter 2. The end position of each of the supporting legs in body coordinates is,

$$
2.86 \quad \begin{bmatrix} E_{xo,j} \\ E_{yo,j} \\ E_{zo,j} \\ 1 \end{bmatrix} = \begin{bmatrix} \text{Euler} \\ \text{rotation} \\ \text{and} \\ \text{translation} \end{bmatrix}^{-1} \begin{bmatrix} E_{xw,j} \\ E_{yw,j} \\ E_{zw,j} \\ 1 \end{bmatrix}
$$

The joint angles are computed using Equations 1.12, 1.20, and 1.25.

This concludes the development of the dynamic equations for the body and supporting legs of the walking robot.

2-2. DYNAMICS OF NON-SUPPORTING LEGS

As stated at the beginning of this chapter, the non-supporting legs are treated as if they were on a inertial reference frame. This allows the non-supporting legs to be treated as independent, three degree of freedom, robotic arms.

Lagrange's equations for leg j are,

$$
2.87 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{i,j}}\right) - \frac{\partial L}{\partial \theta_{i,j}} = T_{i,j}; \quad i=2,\ 3,\ 4
$$

In order to write the kinetic and potential energies of the non-supporting legs, the following definitions are made,

1. $r_{i,j,x}$, $r_{i,j,y}$, and $r_{i,j,z}$ are the x, y, z components of the center of mass of link i,

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leg j, in the X_0, Y_0, Z_0 coordinate system.

2. $lm_{i,j}$ is the distance from the origin of the i-1 coordinate system to the center of mass of link i on leg j.

3. $\omega_{i,j,x}$, $\omega_{i,j,y}$, and $\omega_{i,j,z}$ are the angular velocities about the X_i, Y_i, and Z_i axes of link i+1 on leg j.

4. $I_{i,j,x}$, $I_{i,j,y}$, and $I_{i,j,z}$ are the moments of inertia about the principle axes of link i on leg j.

5. $m_{i,j}$ is the mass of link i on leg j.

Calculating the values for r gives,

2.88

$$r_{2,j,x} = C(\theta_{1,j}+\theta_{2,j})lm_{2,j}+C(\theta_{1,j})l_{1,j}$$

$$r_{2,j,y} = S(\theta_{1,j}+\theta_{2,j})lm_{2,j}+S(\theta_{1,j})l_{1,j}$$

$$r_{2,j,z} = 0$$

$$r_{3,j,x} = C(\theta_{1,j}+\theta_{2,j})C(\theta_{3,j})lm_{3,j}$$
$$+C(\theta_{1,j}+\theta_{2,j})l_{2,j}+C(\theta_{1,j})l_{1,j}$$

$$r_{3,j,y} = S(\theta_{1,j}+\theta_{2,j})C(\theta_{3,j})lm_{3,j}$$
$$+S(\theta_{1,j}+\theta_{2,j})l_{2,j}+S(\theta_{1,j})l_{1,j}$$

$$r_{3,j,z} = -S(\theta_{3,j})lm_{3,j}$$

$$r_{4,j,x} = C(\theta_{1,j}+\theta_{2,j})[C(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$
$$+C(\theta_{3,j})l_{3,j}+l_{2,j}]+C(\theta_{1,j})l_{1,j}$$

$$r_{4,j,y} = S(\theta_{1,j}+\theta_{2,j})[C(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$
$$+C(\theta_{3,j})l_{3,j}+l_{2,j}]+S(\theta_{1,j})l_{1,j}$$

$$r_{4,j,z} = -S(\theta_{3,j}+\theta_{4,j})lm_{4,j}-S(\theta_{3,j})l_{3,j}$$

Taking the time derivatives of the r's yields (note $\theta_{1,j}$ =

const.)

2.89 $\dot{r}_{2,J,H} = -\dot{\theta}_{2,J}S(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\dot{r}_{2,J,V} = \dot{\theta}_{2,J}C(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\dot{r}_{2,J,Z} = 0$

$\dot{r}_{3,J,H} = -\dot{\theta}_{2,J}S(\theta_{1,J}+\theta_{2,J})[C(\theta_{3,J})lm_{3,J}+l_{2,J}]$
$\qquad -\dot{\theta}_{3,J}C(\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})lm_{3,J}$

$\dot{r}_{3,J,V} = \dot{\theta}_{2,J}C(\theta_{1,J}+\theta_{2,J})[C(\theta_{3,J})lm_{3,J}+l_{2,J}]$
$\qquad -\dot{\theta}_{3,J}S(\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})lm_{3,J}$

$\dot{r}_{3,J,Z} = -\dot{\theta}_{3,J}C(\theta_{3,J})lm_{3,J}$

$\dot{r}_{4,J,H} = -\dot{\theta}_{2,J}S(\theta_{1,J}+\theta_{2,J})[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$
$\qquad +C(\theta_{3,J})l_{3,J}+l_{2,J}]$
$\qquad -\dot{\theta}_{3,J}C(\theta_{1,J}+\theta_{2,J})[S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$
$\qquad +S(\theta_{3,J})l_{3,J}]$
$\qquad -\dot{\theta}_{4,J}C(\theta_{1,J}+\theta_{2,J})S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$

$\dot{r}_{4,J,V} = \dot{\theta}_{2,J}C(\theta_{1,J}+\theta_{2,J})[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$
$\qquad +C(\theta_{3,J})l_{3,J}+l_{2,J}]$
$\qquad -\dot{\theta}_{3,J}S(\theta_{1,J}+\theta_{2,J})[S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$
$\qquad +S(\theta_{3,J})l_{3,J}]$
$\qquad -\dot{\theta}_{4,J}S(\theta_{1,J}+\theta_{2,J})S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$

$\dot{r}_{4,J,Z} = -\dot{\theta}_{3,J}[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}+C(\theta_{3,J})l_{3,J}]$
$\qquad -\dot{\theta}_{4,J}C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$

Calculating the ω's gives

2.90 $\omega_{1,J,H} = 0$

$\omega_{1,J,V} = 0$

$\omega_{1,J,Z} = \dot{\theta}_{2,J}$

$\omega_{2,J,H} = -\dot{\theta}_{2,J}S(\theta_{3,J})$

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$$\omega_{2,j,y} = -\dot{\theta}_{2,j}C(\theta_{3,j})$$

$$\omega_{2,j,z} = \dot{\theta}_{3,j}$$

$$\omega_{3,j,x} = -\dot{\theta}_{2,j}S(\theta_{3,j}+\theta_{4,j})$$

$$\omega_{3,j,y} = -\dot{\theta}_{2,j}C(\theta_{3,j}+\theta_{4,j})$$

$$\omega_{3,j,z} = \dot{\theta}_{3,j}+\dot{\theta}_{4,j}$$

The kinetic energy of leg j is,

2.91
$$K = \frac{1}{2}\sum_{i=2}^{4}\{m_{i,j}[(\dot{r}_{i,j,x})^2+(\dot{r}_{i,j,y})^2+(\dot{r}_{i,j,z})^2]$$
$$+[I_{i,j,x}(\omega_{i-1,j,x})^2$$
$$+I_{i,j,y}(\omega_{i-1,j,y})^2$$
$$+I_{i,j,z}(\omega_{i-1,j,z})^2]\}$$

The potential energy is,

2.92
$$P = -m_{3,j}GS(\theta_{3,j})lm_{3,j}-m_{4,j}G[S(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$
$$+S(\theta_{3,j})l_{3,j}]$$

and the Lagrangian is,

$$L=K-P$$

Taking the required derivatives of L gives,

2.93
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{2,j}}\right) = \ddot{\theta}_{2,j}\{m_{2,j}lm_{2,j}^2$$
$$+m_{3,j}[C(\theta_{3,j})lm_{3,j}+lm_{2,j}]^2$$
$$+m_{4,j}[C(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$
$$+C(\theta_{3,j})l_{3,j}+l_{2,j}]^2$$
$$+I_{2,j,z}+I_{3,j,x}S^2(\theta_{3,j})$$
$$+I_{3,j,y}C^2(\theta_{3,j})$$
$$+I_{4,j,x}S^2(\theta_{3,j}+\theta_{4,j})$$
$$+I_{4,j,y}C^2(\theta_{3,j}+\theta_{4,j})\}$$
$$+2\dot{\theta}_{2,j}\dot{\theta}_{3,j}\{-S(\theta_{3,j})lm_{3,j}m_{3,j}$$

$$*(C(\theta_{3,J})lm_{3,J}+l_{2,J})$$
$$-m_{4,J}[C(\theta_{3,J}+\theta_{4,J})S(\theta_{3,J}+\theta_{4,J})$$
$$*lm_{4,J}{}^2+C(\theta_{3,J})S(\theta_{3,J})l_{3,J}{}^2$$
$$+S(2\theta_{3,J}+\theta_{4,J})l_{3,J}lm_{4,J}$$
$$+S(\theta_{3,J}+\theta_{4,J})l_{2,J}lm_{4,J}$$
$$+S(\theta_{3,J})l_{2,J}l_{3,J}]$$
$$+C(\theta_{3,J})S(\theta_{3,J})(I_{3,J,\kappa}-I_{3,J,v})$$
$$+C(\theta_{3,J}+\theta_{4,J})S(\theta_{3,J}+\theta_{4,J})$$
$$*(I_{4,J,\kappa}-I_{4,J,v})\}$$
$$+2\dot{\theta}_{2,J}\dot{\theta}_{4,J}\{-m_{4,J}S(\theta_{3,J}+\theta_{4,J})$$
$$*lm_{4,J}[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{2,J}]$$
$$+C(\theta_{3,J}+\theta_{4,J})S(\theta_{3,J}+\theta_{4,J})$$
$$*(I_{4,J,\kappa}-I_{4,J,v})\}$$

2.94
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{3,J}}\right)=\ddot{\theta}_{3,J}[m_{3,J}lm_{3,J}{}^2$$
$$+m_{4,J}(lm_{4,J}{}^2+2C(\theta_{4,J})l_{3,J}lm_{4,J}$$
$$+l_{3,J}{}^2)+I_{3,J,z}+I_{4,J,z}]$$
$$+\ddot{\theta}_{4,J}[m_{4,J}(lm_{4,J}{}^2$$
$$+C(\theta_{4,J})l_{3,J}lm_{4,J})+I_{4,J,z}]$$
$$-2\dot{\theta}_{3,J}\dot{\theta}_{4,J}S(\theta_{4,J})l_{3,J}lm_{4,J}m_{4,J}$$
$$-\dot{\theta}_{4,J}{}^2S(\theta_{4,J})l_{3,J}lm_{4,J}m_{4,J}$$

2.95
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_{4,J}}\right)=\ddot{\theta}_{3,J}[lm_{4,J}{}^2m_{4,J}$$
$$+C(\theta_{4,J})l_{3,J}lm_{4,J}m_{4,J}+I_{4,J,z}]$$
$$+\ddot{\theta}_{4,J}[lm_{4,J}{}^2m_{4,J}+I_{4,J,z}]$$
$$-\dot{\theta}_{3,J}\dot{\theta}_{4,J}S(\theta_{4,J})l_{3,J}lm_{4,J}m_{4,J}$$

$$2.96 \quad \frac{\partial L}{\partial \dot{\theta}_{2,\text{J}}} = 0$$

$$
\begin{aligned}
2.97 \quad \frac{\partial L}{\partial \theta_{3,\text{J}}} = \;& \dot{\theta}_{2,\text{J}}{}^2\{-m_{3,\text{J}}[C(\theta_{3,\text{J}})S(\theta_{3,\text{J}})lm_{3,\text{J}}{}^2 \\
& +S(\theta_{3,\text{J}})l_{2,\text{J}}lm_{3,\text{J}}] \\
& -m_{4,\text{J}}[C(\theta_{3,\text{J}}+\theta_{4,\text{J}})S(\theta_{3,\text{J}}+\theta_{4,\text{J}})lm_{4,\text{J}}{}^2 \\
& +S(2\theta_{3,\text{J}}+\theta_{4,\text{J}})l_{3,\text{J}}lm_{4,\text{J}}+S(\theta_{3,\text{J}}+\theta_{4,\text{J}}) \\
& *l_{2,\text{J}}lm_{4,\text{J}}+C(\theta_{3,\text{J}})S(3,\text{J})l_{3,\text{J}}{}^2 \\
& +S(\theta_{3,\text{J}})l_{2,\text{J}}l_{3,\text{J}}] \\
& +C(\theta_{3,\text{J}})S(3,\text{J})(I_{3,\text{J},\kappa}-I_{3,\text{J},\nu}) \\
& +C(\theta_{3,\text{J}}+\theta_{4,\text{J}})S(\theta_{3,\text{J}}+\theta_{4,\text{J}}) \\
& *(I_{4,\text{J},\kappa}-I_{4,\text{J},\nu})\} \\
& +m_{3,\text{J}}GC(\theta_{3,\text{J}})lm_{3,\text{J}} \\
& +m_{4,\text{J}}G[C(\theta_{3,\text{J}}+\theta_{4,\text{J}})lm_{4,\text{J}}+C(\theta_{3,\text{J}})l_{3,\text{J}}]
\end{aligned}
$$

$$
\begin{aligned}
2.98 \quad \frac{\partial L}{\partial \theta_{4,\text{J}}} = \;& \dot{\theta}_{2,\text{J}}{}^2 S(\theta_{3,\text{J}}+\theta_{4,\text{J}})\{-m_{4,\text{J}}[C(\theta_{3,\text{J}}+\theta_{4,\text{J}}) \\
& *lm_{4,\text{J}}{}^2+C(\theta_{3,\text{J}})l_{3,\text{J}}lm_{4,\text{J}}+l_{2,\text{J}}lm_{4,\text{J}}] \\
& +C(\theta_{3,\text{J}}+\theta_{4,\text{J}})(I_{4,\text{J},\kappa}-I_{4,\text{J},\nu})\} \\
& +\dot{\theta}_{3,\text{J}}{}^2 m_{4,\text{J}}S(\theta_{4,\text{J}})l_{3,\text{J}}lm_{4,\text{J}} \\
& +\dot{\theta}_{3,\text{J}}\dot{\theta}_{4,\text{J}}m_{4,\text{J}}S(\theta_{4,\text{J}})l_{3,\text{J}}lm_{4,\text{J}} \\
& +m_{4,\text{J}}GC(\theta_{3,\text{J}}+\theta_{4,\text{J}})lm_{4,\text{J}}
\end{aligned}
$$

Lagrange's equations for the non-supporting legs are written as,

$$
\begin{aligned}
2.99 \quad T_{2,\text{J}} = \;& \ddot{\theta}_{2,\text{J}}\{m_{2,\text{J}}lm_{2,\text{J}}{}^2 \\
& +m_{3,\text{J}}[C(\theta_{3,\text{J}})lm_{3,\text{J}}+l_{2,\text{J}}]^2 \\
& +m_{4,\text{J}}[C(\theta_{3,\text{J}}+\theta_{4,\text{J}})lm_{4,\text{J}} \\
& +C(\theta_{3,\text{J}})l_{3,\text{J}}+l_{2,\text{J}}]^2 \\
& +I_{2,\text{J},\kappa}+I_{3,\text{J},\kappa}S^2(\theta_{3,\text{J}})+I_{3,\text{J},\nu}C^2(\theta_{3,\text{J}})
\end{aligned}
$$

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$$+I_{4,1,x}S^2(\theta_{3,1}+\theta_{4,1})$$

$$+I_{4,1,y}C^2(\theta_{3,1}+\theta_{4,1})\}$$

$$+2\dot{\theta}_{2,1}\dot{\theta}_{3,1}\{-m_{3,1}S(\theta_{3,1})lm_{3,1}$$

$$*[C(\theta_{3,1})lm_{3,1}+l_{2,1}]$$

$$-m_{4,1}[C(\theta_{3,1}+\theta_{4,1})S(\theta_{3,1}+\theta_{4,1})lm_{4,1}{}^2$$

$$+C(\theta_{3,1})S(\theta_{3,1})l_{3,1}{}^2$$

$$+S(2_{3,1}+\theta_{4,1})l_{3,1}lm_{4,1}$$

$$+S(\theta_{3,1}+\theta_{4,1})l_{2,1}lm_{4,1}+S(\theta_{3,1})l_{2,1}l_{3,1}]$$

$$+C(\theta_{3,1})S(\theta_{3,1})(I_{3,1,x}-I_{3,1,y})$$

$$+C(\theta_{3,1}+\theta_{4,1})S(\theta_{3,1}+\theta_{4,1})$$

$$*(I_{4,1,x}-I_{4,1,y})\}$$

$$+2\dot{\theta}_{2,1}\dot{\theta}_{4,1}\{-m_{4,1}S(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$

$$*[C(\theta_{3,1}+\theta_{4,1})lm_{4,1}+C(\theta_{3,1})l_{3,1}+l_{2,1}]$$

$$+C(\theta_{3,1}+\theta_{4,1})S(\theta_{3,1}+\theta_{4,1})$$

$$*(I_{4,1,x}-I_{4,1,y})\}$$

2.100 $T_{3,1} = \ddot{\theta}_{3,1}\{m_{3,1}lm_{3,1}{}^2+m_{4,1}[lm_{4,1}{}^2$

$$+2C(\theta_{4,1})l_{3,1}lm_{4,1}$$

$$+l_{3,1}{}^2]+I_{3,1,z}+I_{4,1,z}\}$$

$$+\ddot{\theta}_{4,1}\{m_{4,1}lm_{4,1}[lm_{4,1}+C(\theta_{4,1})l_{3,1}]$$

$$+I_{4,1,z}\}+\dot{\theta}_{2,1}{}^2\{m_{3,1}S(\theta_{3,1})lm_{3,1}$$

$$*[C(\theta_{3,1})lm_{3,1}+l_{2,1}]$$

$$+m_{4,1}[C(\theta_{3,1}+\theta_{4,1})S(\theta_{3,1}+\theta_{4,1})lm_{4,1}{}^2$$

$$+S(2\theta_{3,1}+\theta_{4,1})l_{3,1}lm_{4,1}+S(\theta_{3,1}+\theta_{4,1})$$

$$*l_{2,1}lm_{4,1}+C(\theta_{3,1})S(\theta_{3,1})l_{3,1}{}^2$$

$$+S(\theta_{3,1})l_{2,1}l_{3,1}]-C(\theta_{3,1})S(\theta_{3,1})$$

$$*(I_{3,1,x}-I_{3,1,y})-C(\theta_{3,1}+\theta_{4,1})$$

$$*S(\theta_3,_4+\theta_4,_4)(I_4,_4,_x-I_4,_4,_y)\}$$

$$+2\dot{\theta}_3,_4\dot{\theta}_4,_4[-m_4,_4S(\theta_4,_4)l_3,_4lm_4,_4]$$

$$-\dot{\theta}_4,_4{}^2[m_4,_4S(\theta_4,_4)l_3,_4lm_4,_4]$$

$$-m_3,_4GC(\theta_3,_4)lm_3,_4$$

$$-m_4,_4G[C(\theta_3,_4+\theta_4,_4)lm_4,_4+C(\theta_3,_4)l_3,_4]$$

2.101 $$T_4,_4 = \ddot{\theta}_3,_4\{m_4,_4lm_4,_4[lm_4,_4+C(\theta_4,_4)l_3,_4]$$

$$+I_4,_4,_z\}+\ddot{\theta}_4,_4[m_4,_4lm_4,_4{}^2+I_4,_4,_z]$$

$$+\dot{\theta}_2,_4{}^2S(\theta_3,_4+\theta_4,_4)\{m_4,_4lm_4,_4$$

$$*[C(\theta_3,_4+\theta_4,_4)lm_4,_4+C(\theta_3,_4)l_3,_4+l_2,_4]$$

$$-C(\theta_3,_4+\theta_4,_4)(I_4,_4,_x-I_4,_4,_y)\}$$

$$+\dot{\theta}_3,_4{}^2[-m_4,_4S(\theta_4,_4)l_3,_4lm_4,_4]$$

$$+2\dot{\theta}_3,_4\dot{\theta}_4,_4[-m_4,_4S(\theta_4,_4)l_3,_4lm_4,_4]$$

$$-m_4,_4GC(\theta_3,_4+\theta_4,_4)lm_4,_4$$

Writing equations 2.99 thru 2.101 in matrix form gives,

2.102
$$\left[\begin{array}{c} \underline{M} \end{array}\right]\begin{bmatrix} \ddot{\theta}_2,_4 \\ \ddot{\theta}_3,_4 \\ \ddot{\theta}_4,_4 \end{bmatrix} = \begin{bmatrix} T_2,_4 \\ T_3,_4 \\ T_4,_4 \end{bmatrix} - \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

Where the elements of $\underline{M}$, Q, and G are,

2.103 $$m_1,_1 = m_2,_4lm_2,_4{}^2+m_3,_4[C(\theta_3,_4)lm_3,_4+l_2,_4]^2$$

$$+m_4,_4[C(\theta_3,_4+\theta_4,_4)lm_4,_4$$

$$+C(\theta_3,_4)l_3,_4+l_2,_4]^2$$

$$+I_2,_4,_z+S^2(\theta_3,_4)I_3,_4,_x+C^2(\theta_3,_4)I_3,_4,_y$$

$$+S^2(\theta_3,_4+\theta_4,_4)I_4,_4,_x$$

$$+C^2(\theta_3,_4+\theta_4,_4)I_4,_4,_y$$

$$m_1,_2 = 0$$

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$$m_{1,3} = 0$$

$$m_{2,1} = 0$$

$$m_{2,2} = m_{3,i}lm_{3,i}^2 + m_{4,i}[lm_{4,i}^2$$
$$+ 2C(\theta_{4,i})l_{3,i}lm_{4,i} + l_{3,i}^2] + I_{3,i,z} + I_{4,i,z}$$

$$m_{2,3} = m_{4,i}lm_{4,i}[lm_{4,i} + C(\theta_{4,i})l_{3,i}] + I_{4,i,z}$$

$$m_{3,1} = 0$$

$$m_{3,2} = m_{2,3}$$

$$m_{3,3} = m_{4,i}lm_{4,i}^2 + I_{4,i,z}$$

2.104
$$G_1 = 2\dot\theta_{2,i}\dot\theta_{3,i}\{-m_{3,i}S(\theta_{3,i})lm_{3,i}$$
$$*[C(\theta_{3,i})lm_{3,i}+l_{2,i}]-m_{4,i}[C(\theta_{3,i}+\theta_{4,i})$$
$$*S(\theta_{3,i}+\theta_{4,i})lm_{4,i}^2+C(\theta_{3,i})S(\theta_{3,i})l_{3,i}^2$$
$$+S(2\theta_{3,i}+\theta_{4,i})l_{3,i}lm_{4,i}+S(\theta_{3,i}+\theta_{4,i})$$
$$*l_{2,i}lm_{4,i}+S(\theta_{3,i})l_{2,i}l_{3,i}]+C(\theta_{3,i})$$
$$*S(\theta_{3,i})(I_{3,i,x}-I_{3,i,y})+C(\theta_{3,i}+\theta_{4,i})$$
$$*S(\theta_{3,i}+\theta_{4,i})(I_{4,i,x}-I_{4,i,y})\}$$
$$+2\dot\theta_{2,i}\dot\theta_{4,i}\{-m_{4,i}S(\theta_{3,i}+\theta_{4,i})lm_{4,i}$$
$$*[C(\theta_{3,i}+\theta_{4,i})lm_{4,i}+C(\theta_{3,i})l_{3,i}+l_{2,i}]$$
$$+C(\theta_{3,i}+\theta_{4,i})S(\theta_{3,i}+\theta_{4,i})$$
$$*(I_{4,i,x}-I_{4,i,y})\}$$

$$G_2 = 2\dot\theta_{3,i}\dot\theta_{4,i}[-m_{4,i}S(\theta_{4,i})l_{3,i}lm_{4,i}]$$
$$-\dot\theta_{4,i}^2[m_{4,i}S(\theta_{4,i})l_{3,i}lm_{4,i}]$$

$$G_3 = 2\dot\theta_{3,i}\dot\theta_{4,i}[-m_{4,i}S(\theta_{4,i})l_{3,i}lm_{4,i}]$$

2.105
$$Q_1 = 0$$

$$Q_2 = \dot\theta_{2,i}^2\{m_{3,i}S(\theta_{3,i})lm_{3,i}[C(\theta_{3,i})lm_{3,i}+l_{2,i}]$$
$$+m_{4,i}[C(\theta_{3,i}+\theta_{4,i})S(\theta_{3,i}+\theta_{4,i})lm_{4,i}^2$$
$$+S(2\theta_{3,i}+\theta_{4,i})l_{3,i}lm_{4,i}$$

$$+S(\theta_{3,J}+\theta_{4,J})l_{2,J}lm_{4,J}$$

$$+C(\theta_{3,J})S(\theta_{3,J})l_{3,J}{}^2+S(\theta_{3,J})l_{2,J}l_{3,J}]$$

$$-C(\theta_{3,J})S(\theta_{3,J})(I_{3,J,\kappa}-I_{3,J,\nu})$$

$$-C(\theta_{3,J}+\theta_{4,J})S(\theta_{3,J}+\theta_{4,J})(I_{4,J,\kappa}-I_{4,J,\nu})\}$$

$$-m_{3,J}GC(\theta_{3,J})lm_{3,J}-m_{4,J}G[C(\theta_{3,J}+\theta_{4,J})$$

$$*lm_{4,J}+C(\theta_{3,J})l_{3,J}]$$

$$Q_3 = \dot{\theta}_{2,J}{}^2 S(\theta_{3,J}+\theta_{4,J})\{m_{4,J}lm_{4,J}$$

$$*[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}+C(\theta_{3,J})l_{3,J}+l_{2,J}]$$

$$-C(\theta_{3,J}+\theta_{4,J})(I_{4,J,\kappa}-I_{4,J,\nu})\}$$

$$+\dot{\theta}_{3,J}{}^2[-m_{4,J}S(\theta_{4,J})l_{3,J}lm_{4,J}]$$

$$-m_{4,J}GC(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$

Solving Equation 2.102 for the $\ddot{\theta}_{1,J}$ terms gives,

$$2.106 \quad \begin{bmatrix} \ddot{\theta}_{2,J} \\ \ddot{\theta}_{3,J} \\ \ddot{\theta}_{4,J} \end{bmatrix} = \underline{M}^{-1} \left(\begin{bmatrix} T_{2,J} \\ T_{3,J} \\ T_{4,J} \end{bmatrix} - \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \right)$$

where the elements of $\underline{M}^{-1}$ are,

$$2.107 \quad m^{-1}{}_{1,1} = 1/m_{1,1}$$

$$m^{-1}{}_{1,2} = 0$$

$$m^{-1}{}_{1,3} = 0$$

$$m^{-1}{}_{2,1} = 0$$

$$m^{-1}{}_{2,2} = m_{3,3}/DET$$

$$m^{-1}{}_{2,3} = -m_{2,3}/DET$$

$$m^{-1}{}_{3,1} = 0$$

$$m^{-1}{}_{3,2} = m^{-1}{}_{2,3}$$

$$m^{-1}{}_{3,3} = m_{2,2}/DET$$

where,

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$$DET = m_{3,3}m_{2,2} - m_{2,3}^2$$

The state variables are defined as,

2.108 $y_1 = \theta_{2,3}$ $y_4 = \dot{\theta}_{2,3}$

        $y_2 = \theta_{3,3}$ $y_5 = \dot{\theta}_{3,3}$

        $y_3 = \theta_{4,3}$ $y_6 = \dot{\theta}_{4,3}$

The state equations in final form are,

2.109 $\dot{y}_1 = y_4$

        $\dot{y}_2 = y_5$

        $\dot{y}_3 = y_6$

        $\dot{y}_4 = (T_{2,3} - Q_1 - G_1)/m_{1,1}$

        $\dot{y}_5 = [m_{3,3}(T_{3,3} - Q_2 - G_2) - m_{2,3}(T_{4,3} - Q_3 - G_3)]/DET$

        $\dot{y}_6 = [m_{2,2}(T_{4,3} - Q_3 - G_3) - m_{2,3}(T_{3,3} - Q_2 - G_2)]/DET$

This concludes the development of the dynamics for the non-supporting legs of the walking robot.

2-3.   PHYSICAL PROPERTIES

In this section the physical properties of the walking robot are developed. This walking robot is intended to simulate a small, lightweight, remotely controlled system which has some type of terrain preview information available.

The body is a cylinder 18.0 inches in diameter, 6.0 inches thick, and weighs 50lbs. Using G = 32.2 ft/s², the following body constants are calcuated,

2.110   Mb = 1.55 slug

        $I_{xo} = 0.250$ slug ft²

        $I_{yo} = 0.250$ slug ft²

$$I_{xo} = 0.436 \text{ slug ft}^2$$

Link 2,j measures 2.5 inches by 2.5 inches by 3.0 inches and has a density of 0.05 lb/in³. This density is intended to represent a strong aluminum structure with no drive motors located on the link. The constants for link 2,j are calculated as,

2.111 $m_{2,j} = 0.0291$ slugs

$I_{2,j,x} = 0.000211$ slug ft²

$I_{2,j,y} = 0.000257$ slug ft²

$I_{2,j,z} = 0.000257$ slug ft²

$lm_{2,j} = 0.125$ ft

$l_{2,j} = 0.25$ ft

Link 3,j measures 2.0 inches by 2.0 inches by 18.0 inches and has a density of 0.100 lb/in³. This density again represents a strong aluminum structure but with the addition of drive motors for joints 3 and 4. The constants for link 3,j are calculated as,

2.112 $m_{3,j} = 0.224$ slugs

$I_{3,j,x} = 0.00104$ slug ft²

$I_{3,j,y} = 0.0425$ slug ft²

$I_{3,j,z} = 0.0425$ slug ft²

$lm_{3,j} = 0.75$ ft

$l_{3,j} = 1.5$ ft

Link 4,j measures 1.5 inches by 1.5 inches by 24 inches and has a density of 0.05 lb/in³. This density was chosen for the same reason as that of link 2,j. The

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constants of link 4,j are calculated as,

2.113 $m_{4,j} = 0.0839$ slugs

$I_{4,j,x} = 0.000218$ slugs ft^2

$I_{4,j,y} = 0.0281$ slug ft^2

$I_{4,j,z} = 0.0281$ slug ft^2

$lm_{4,j} = 1.0$ ft

$l_{4,j} = 2.0$ ft

This ends the development of the dynamic equations and physical constants for the walking robot. The material developed in this chapter will be used in the simulation program and in Chapter 4 in estimating joint inertias for the control system.

III. RECURSIVE SPLINE PATH CONTROL

In prior chapters the kinematic and dynamic equations for a six legged walking robot were developed. In the next two chapters control concepts will be developed. In order to control the walking robot it is necessary to define a desired path along which it is to walk. The path must be defined in the terms of all six coordinates (P_x, P_y, P_z, ϕ, θ, ψ) which define the position and orientation of the body. A parametric representation where the six coordinates are specified as functions of a parameter t is convenient and natural if the parameter is related to time. A functional form which satisfies the needs of this work is the uniform natural B-spline function. B-spline functions have the following properties which make them particularly appealing for the present application:

1. A B-spline function is invariant under linear transformations.

2. A Uniform B-spline function of degree m is m-2 differentiable.

3. A B-spline function has local support.

The second property defines the smoothness of the path and is used in a novel feedforward control concept which will be developed in the next chapter. The third property is the basis for the development of a recursive algorithm which

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permits the path to be defined in real time. The recursive
spline algorithm provides the ability to use vision or other
sensory inputs or operator commands to specify the desired
path as it is being traversed. Finally, the use of B-spline
functions provides a smooth and graceful path with a
relatively small number of data points.

   In addition to defining the desired path of the body
of the walking robot it is necessary to define the desired
motion of the non-supporting legs. The recursive B-spline
function is used for this purpose also.

   3-1. B-SPLINE CURVE FITTING

   The degree of the spline function determines the
number of continuous derivatives and the extent of the
support. A fourth degree (cubic) spline will be used in
this work. The cubic B-spline function on the interval
$t_k \leq t \leq t_k+1$ is given by,

$$3.1 \qquad Q_k(t) = \sum_{i=-1}^{2} a_{k+i} N_{k+i}[(t-t_{k+i})/Dt]$$

where the following definitions are made relative to Figures
3.1 and 3.2

   3.2 q_k is the value of the data at $t = t_k$
               q'_k is the slope of the curve at $t = t_k$
               N_k is the B-Spline $(N(t))$ basis function
                   centered over t_k
               a_k is the weighting factor that multiplies N_k
               Q_k is the cubic polynomial defined over the

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interval $t_k \leq t \leq t_{k+1}$

n+1 is the number of data points (knots)

Dt is the interval $t_k - t_{k-1}$
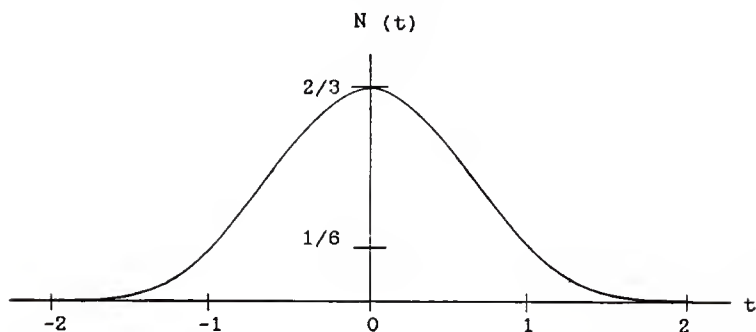
FIGURE 3.1   B-Spline Basis Function
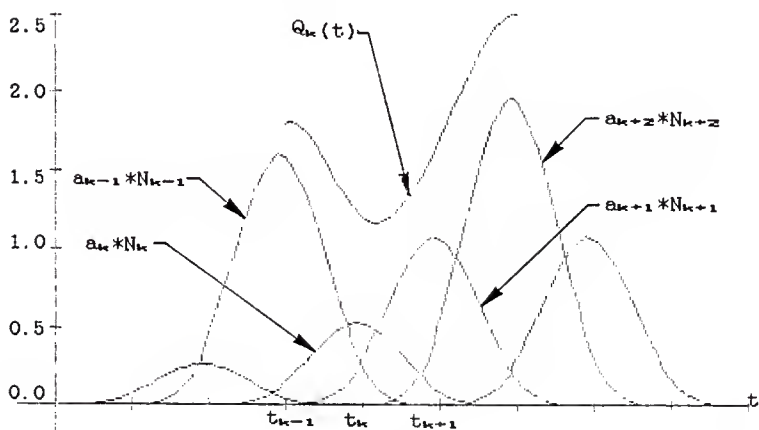
FIGURE 3.2   B-Spline Curve Development

The B-spline basis function is shown in Figure 3.1 and is defined analytically by the equations,

$$
3.3 \qquad N(t) =
\begin{cases}
(t+2)^3/6 & -2 \leq t \leq -1 \\[4pt]
[-3(t+2)^3+12(t+2)^2 \\
\quad -12(t+2)+4]/6 & -1 \leq t \leq 0 \\[4pt]
N(-t) & 0 \leq t \leq 2
\end{cases}
$$

The values of the spline basis function and its first derivative evaluated at the knots are

3.4 t	$N(t)$	$dN(t)/dt$
-2	0	0
-1	1/6	1/2
0	2/3	0
1	1/6	-1/2
2	0	0

The method of fitting the spline function through a set of data points is now described. The data points are aligned with the knots so that each data point is equal to the weighted sum of the spline basis function evaluated at the knots, thus

$$
3.5 \qquad q_0 = a_{-1}(1/6)+a_0(2/3)+a_1(1/6)
$$
$$
q_1 = a_0(1/6)+a_1(2/3)+a_2(1/6)
$$
$$
\vdots \qquad \qquad \vdots
$$
$$
q_n = a_{n-1}(1/6)+a_n(2/3)+a_{n+1}(1/6)
$$

This results in n+1 equations in terms of the n+3 weights a_i $(i = -1, 0, \ldots, n+1)$. Two additional equations are required. These may be obtained by specifying the slopes at t_0 and t_n, thus

$$
3.6 \qquad q'_0 = -1/2a_{-1}+0a_0+1/2a_1
$$

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$$q'_n = -1/2a_{n-1} + 0a_n + 1/2a_{n+1}$$

Combining Equations 3.5 and 3.6 and writing them in matrix form gives,

3.7
$$
\begin{bmatrix}
-1/2 & 0 & 1/2 & & & & \\
1/6 & 2/3 & 1/6 & & & & \\
& 1/6 & 2/3 & 1/6 & & & \\
& & \cdot & \cdot & \cdot & \cdot & \\
& & & \cdot & \cdot & \cdot & \cdot \\
& & & & 1/6 & 2/3 & 1/6 \\
& & & & -1/2 & 0 & 1/2
\end{bmatrix}
\begin{bmatrix}
a_{-1} \\
a_o \\
a_1 \\
\vdots \\
a_{n-1} \\
a_n \\
a_{n+1}
\end{bmatrix}
=
\begin{bmatrix}
q'_o \\
q_o \\
q_1 \\
\vdots \\
q_{n-1} \\
q_n \\
q'_n
\end{bmatrix}
$$

Solving the equation corresponding to the first row of Equation 3.7 for a_{-1} yields,

3.8 $a_{-1} = -2q'_o + a_1$

Substituting Equation 3.8 into the equation corresponding to the second row of Equation 3.7 and solving for a_o yields,

3.9 $a_o = (3/2)q'_o + (1/2)q'_o - (1/2)a_1$

Subsituting into each successive row yields,

3.10 $a_k = \sum\limits_{j=1}^{k-1} P_j q_{k-j} + R_k q_o + S_k q'_o + T_k a_1$, $k \geq 2$

where,

3.11 $P_1 = 6$

$P_2 = -24$

$P_j = -4P_{j-1} - P_{j-2}$, $j \geq 3$

3.12 $R_2 = -3/2$

$R_3 = 12/2$

$R_j = -4R_{j-1} - R_{j-2}$, $j \geq 4$

3.13 $S_2 = -1/2$

$S_3 = 4/2$

$S_j = -4S_{j-1} - S_{j-2}$, $j \geq 4$

3.14 $T_2 = -7/2$

$T_3 = 26/2$

$T_j = -4T_{j-1} - T_{j-2}, \quad j \geq 4$

Writing a_{n+1} in the form of Equation 3.10 gives,

3.15 $a_{n+1} = \sum\limits_{j=1}^{n} P_j q_{n+1-j} + R_{n+1} q_0 + S_{n+1} q'_0$

$+ T_{n+1} a_1$

Using the last row of Equation 3.7, a_{n+1} can also be written as,

3.16 $a_{n+1} = 2q'_n + a_{n-1}$

where,

$a_{n-1} = \sum\limits_{j=1}^{n-2} P_j q_{n-1-j} + R_{n-1} q_0 + S_{n-1} q'_0$

$+ T_{n-1} a_1$

Equating Equations 3.15 and 3.16 and solving for a_1 yields,

3.17 $a_1 = [\sum\limits_{j=1}^{n} P_j q_{n+1-j} - \sum\limits_{j=1}^{n-2} P_j q_{n-1-j}$

$+ (R_{n+1} - R_{n-1}) q_0 + (S_{n+1} - S_{n-1})$

$* q'_0 - 2q'_n] / (T_{n-1} - T_{n+1}) \quad , \quad n \geq 3$

The a_k's can be solved by computing the four series in Equations 3.11 thru 3.14, then solving for a_1 using Equation 3.17. Next, a_{-1} and a_0 are solved by using Equations 3.8 and 3.9, and a_2 thru a_{n+1} are solved using the formula,

3.18 $a_k = 6q_{k-1} - 4a_{k-1} - a_{k-2}$

3-2.   RECURSIVE PATH PROPAGATION

The development of the method for computing the cubic spline in Section 3-1 is for fitting a curve through a set of n+1 data points. In general, all of the data points must be known and the slopes at the beginning and end of the curve must also be known.

In certain applications for walking robots, such as surveillance or exploration, the terrain or the path that the robot must take may not be known in advance. Desired path points may be obtained as the path is being traversed. This makes it necessary to modify the construction of the spline function so that a segment of the curve can be constructed using a few data points. Subsequent segments can be constructed as additional points become available. To begin the curve, points q_0 through q_k, and the slope q'_0 are used. By assuming the slope q'_k is zero, a spline curve can be drawn between t_0 and t_1. Since the curve is drawn only over one interval, a_{-1} through a_2 are the only a's that need to be calculated. Once the curve has reached t_1, a new spline for the interval t_1 through t_2 must be developed. This is done by dropping q_0 and adding q_{k+1} and shifting the q's by one. The beginning slope for the new segment is the slope at t_1 from the old segment and the ending slope is again assumed to be zero. By calculating a_{-1} through a_2 the curve can be extended from t_1 to t_2. This recursive procedure can be repeated for as many

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segments as required.

Figures 3.3 through 3.5 show a recursive spline curve, along with its first and second derivatives, through a set of randomly generated data points spaced at even intervals along the t axis. The curves in Figure 3.3 were constructed using three data points to draw each segment. The recursive spline is smooth and looks very much like the spline drawn using the entire set of data points (Figure 3.6). By observing the first derivative curve, abrupt changes in the slope can be seen which show up as discontinuities in the second derivative. Figure 3.4 is a recursive spline curve drawn through the same set of data points as Figure 3.3 but using four data points to calculate each segment. The spline and its first derivative are similar to that of Figure 3.3, but the discontinuities in the second derivative have almost disappeared. Using five points (Figure 3.5), there is no apparent discontinuity in the second derivative. Figure 3.6 is a spline, with its first and second derivatives, drawn through the complete set of data points using the method of section 3-1.

The spline curves that are drawn in a recursive fashion using only a few data points to construct each segment follow very closely the spline that is drawn using the entire set of data points. When four or more data points are used the discontinuities in the second derivative term are of no appreciable magnitude.
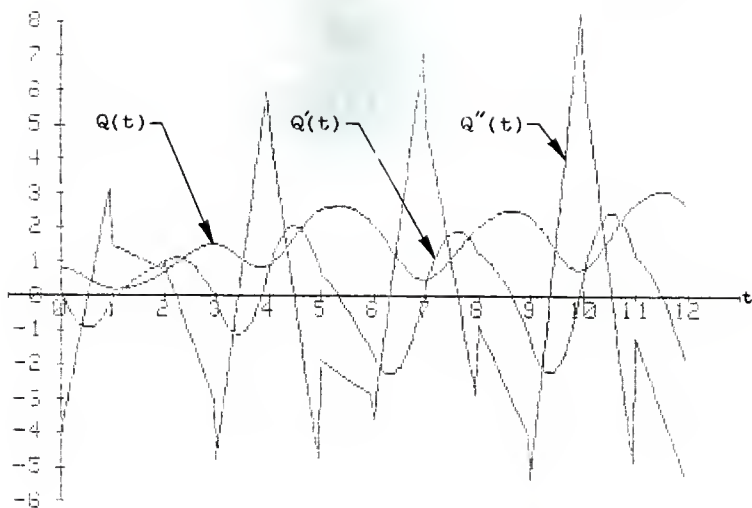
FIGURE 3.3   Recursive Spline Function Using

Three Data Points
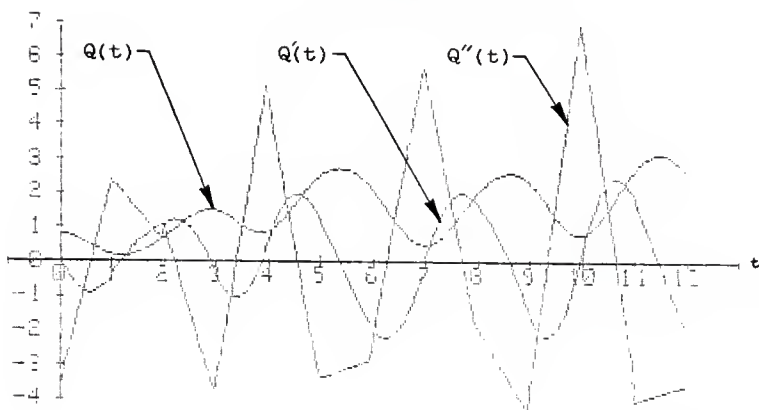
FIGURE 3.4   Recursive Spline Function Using

Four Data Points

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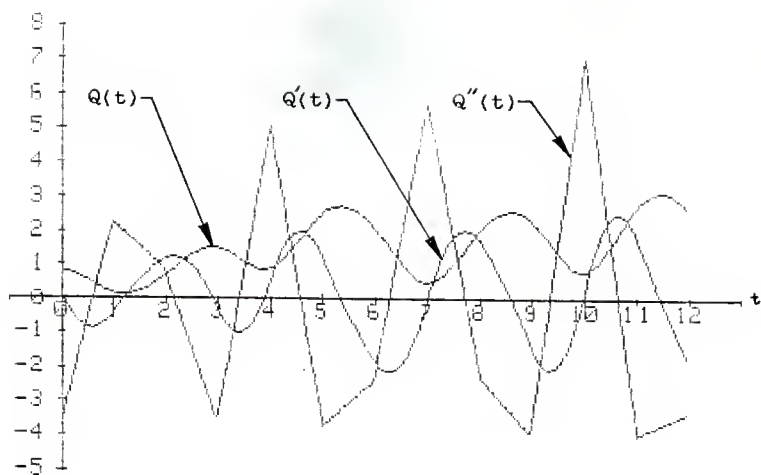
FIGURE 3.5 Recursive Spline Function Using

Five Data Points
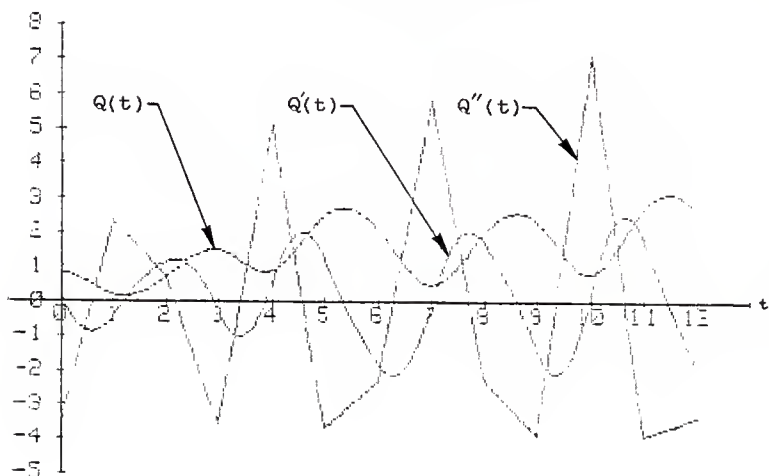
FIGURE 3.6 B-Spline Curve

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3-3.   APPLICATION

The data used to construct the path function for the walking robot are shown in table 3.1. All the data shown are with respect to the world coordinate system.

The data values for P_x, P_y, P_z, ϕ, θ, and ψ are converted directly into recursive spline functions and used to guide the body of the walking robot.

The data values describing the endpoints of the non-supporting legs are used along with the body position and orientation data values to create spline functions for the individual joint angles. By using the inverse kinematics of Chapter 1 and the Euler transformations of Chapter 2, the corresponding joint angle values are computed. These joint angle values are used to construct the recursive spline functions that control the individual joints.

For use in the simulation program, the recursive spline functions are written in the form,

$$Q_k(t) = K_3 t^3 + K_2 t^2 + K_1 t + K_0$$

where,

$$K_3 = (-a_{-1} + 3a_0 - 3a_{+1} + a_2)/6$$
$$K_2 = (3a_{-1} - 6a_0 + 3a_1)/6$$
$$K_1 = (-3a_{-1} + 3a_1)/6$$
$$K_0 = (a_{-1} + 4a_0 + a_1)/6$$
$$t = (t - t_k)/Dt$$

The recursive spline functions developed in this chapter will be used in Chapter 4 and in the simulation

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program to create the paths that guide the body and

non-supporting legs of the walking robot.

TABLE 3.1 Input Data Values

P_x	P_y	P_z	ϕ	θ	Time(s)
0.0	0.0	1.0	0.0	0.0	0
0.25	0.0	1.0	0.0	0.0	1
0.5	0.0	1.0	0.0	0.0	2
0.75	0.0	1.0	0.0	0.0	3
1.0	0.0	1.0	0.0	0.0	4
1.25	0.0	1.0	0.0	0.0	5
1.5	0.0	1.0	0.0	0.0	6
1.75	0.0	1.0	0.0	0.0	7
2.0	0.0	1.0	0.0	0.0	8
2.25	0.0	1.0	0.0	0.0	9
2.5	0.0	1.0	0.0	0.0	10
2.75	0.0	1.0	0.0	0.0	11
3.0	0.0	1.0	0.0	0.0	12
3.25	0.0	1.0	0.0	0.0	13
3.5	0.0	1.0	0.0	0.0	14
3.5	0.0	1.0	0.0	0.0	15
3.5	0.0	1.0	0.0	0.0	16

ψ	$E_{xw,1}$	$E_{yw,1}$	$E_{zw,1}$	$E_{xw,2}$	Time(s)
0.0	3.0	0.0	0.0	1.5	0
0.0	3.5	0.0	0.5	1.5	1
0.0	4.0	0.0	0.0	1.5	2
0.0	4.0	0.0	0.0	2.0	3
0.0	4.0	0.0	0.0	2.5	4
0.0	4.0	0.0	0.0	3.0	5
0.0	4.0	0.0	0.0	3.5	6
0.0	4.5	0.0	0.5	3.5	7
0.0	5.0	0.0	0.75	3.5	8
0.0	5.5	0.0	0.5	3.5	9
0.0	6.0	0.0	0.0	3.5	10
0.0	6.0	0.0	0.0	4.0	11
0.0	6.0	0.0	0.0	4.5	12
0.0	6.0	0.0	0.0	5.0	13
0.0	6.0	0.0	0.0	5.5	14
0.0	6.0	0.0	0.0	5.5	15
0.0	6.0	0.0	0.0	5.5	16

$E_{yw,2}$	$E_{zw,2}$	$E_{xw,3}$	$E_{yw,3}$	$E_{zw,3}$	Time(s)
2.598	0.0	-1.5	2.598	0.0	0
2.598	0.0	-1.0	2.598	0.5	1
2.598	0.0	-0.5	2.598	0.0	2

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2.598	0.5	-0.5	2.598	0.0	3
2.598	0.75	-0.5	2.598	0.0	4
2.598	0.5	-0.5	2.598	0.0	5
2.598	0.0	-0.5	2.598	0.0	6
2.598	0.0	0.0	2.598	0.5	7
2.598	0.0	0.5	2.598	0.7	8
2.598	0.0	1.0	2.598	0.5	9
2.598	0.0	1.5	2.598	0.0	10
2.598	0.5	1.5	2.598	0.0	11
2.598	0.75	1.5	2.598	0.0	12
2.598	0.5	1.5	2.598	0.0	13
2.598	0.0	1.5	2.598	0.0	14
2.598	0.0	1.5	2.598	0.0	15
2.598	0.0	1.5	2.598	0.0	16

$E_{xw,4}$	$E_{yw,4}$	$E_{zw,4}$	$E_{xw,5}$	$E_{yw,5}$	Time(s)
-3.0	0.0	0.0	-1.5	-2.598	0
-3.0	0.0	0.0	-1.0	-2.598	1
-3.0	0.0	0.0	-0.5	-2.598	2
-2.5	0.0	0.5	-0.5	-2.598	3
-2.0	0.0	0.75	-0.5	-2.598	4
-1.5	0.0	0.5	-0.5	-2.598	5
-1.0	0.0	0.0	-0.5	-2.598	6
-1.0	0.0	0.0	0.0	-2.598	7
-1.0	0.0	0.0	0.5	-2.598	8
-1.0	0.0	0.0	1.0	-2.598	9
-1.0	0.0	0.0	1.5	-2.598	10
-0.5	0.0	0.5	1.5	-2.598	11
0.0	0.0	0.75	1.5	-2.598	12
0.5	0.0	0.5	1.5	-2.598	13
1.0	0.0	0.0	1.5	-2.598	14
1.0	0.0	0.0	1.5	-2.598	15
1.0	0.0	0.0	1.5	-2.598	16

$E_{zw,5}$	$E_{xw,6}$	$E_{yw,6}$	$E_{zw,6}$	Time(s)
0.0	1.5	-2.598	0.0	0
0.5	1.5	-2.598	0.0	1
0.0	1.5	-2.598	0.0	2
0.0	2.0	-2.598	0.5	3
0.0	2.5	-2.598	0.75	4
0.0	3.0	-2.598	0.5	5
0.0	3.5	-2.598	0.0	6
0.5	3.5	-2.598	0.0	7
0.75	3.5	-2.598	0.0	8
0.5	3.5	-2.598	0.0	9
0.0	3.5	-2.598	0.0	10
0.0	4.0	-2.598	0.5	11
0.0	4.5	-2.598	0.75	12
0.0	5.0	-2.598	0.5	13

0.0	5.5	-2.598	0.0	14
0.0	5.5	-2.598	0.0	15
0.0	5.5	-2.598	0.0	16

IV. CONTROL SYSTEMS

The control systems used in the simulation of the walking robot are developed in this chapter. The use of polynomial spline functions for the input signals allow the use of a novel feedforward concept as well as feedback compensation, thus providing the systems a degree of anticipation of the control input.

4-1.   CONTROL OF THE BODY

In order to maintain coordinated control over the supporting legs of the walking robot, the position and orientation of the body are used in the feedback control system.   Once the required constraint forces are calculated by the control system, the Lagrangian multipliers of Chapter 2 are used to calculate the required joint torques of the supporting legs to correctly drive the body of the walking robot.

A simplified linear model of the body is used for developing the body control system.   Neglecting all nonlinear and coupling effects yields a model of the form Km/Js^2, where Km is the generalized gain constant and J is either the mass of the body (for P_x, P_y, and P_z) or the moment of inertia about a given axis (for Φ, θ, and ψ).

The feedforward, feedback control system that is used is shown in Figure 4.1. Kd is a derivative feedback gain, Gc
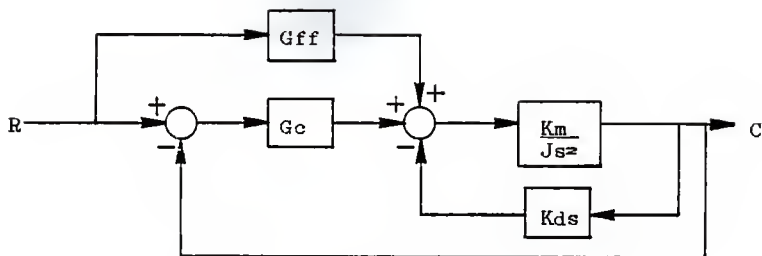
FIGURE 4.1   Feedforward Control System

is a series compensator, and Gff is the feedforward
controller. By simplifying the inner loop, the control
system can be shown in the form of Figure 4.2. The form of G
is,

4.1 $G = \dfrac{Km}{Js^2 + KmKds}$
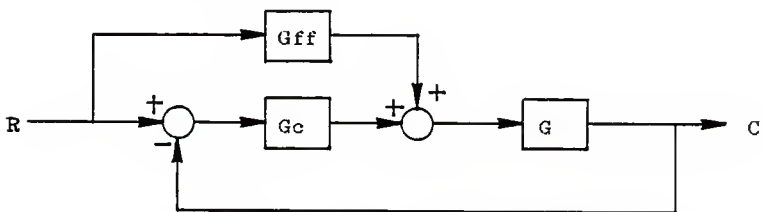
FIGURE 4.2   Simplified Feedforward Control

System

Letting Gff = 1/G and solving for C gives,

4.2 $C = R(1+GGc)/(1+GGc) = R$

Equation 4.2 indicates that the feedforward controller
produces a system where the output follows the input

regardless of the form of Gc. Since G is a simplification
of the walking robot's dynamics, Gff will not be the exact
compensator required for the output to exactly follow the
input. For this reason, Gc is used to gain better control
of the system when the control model does not exactly fit
the dynamic model.

Since there is derivative feedback in G, Gc is chosen
to be a proportional-integral compensator of the form,

4.3 $Gc = Kp + \dfrac{Ki}{s}$

In order to calculate the gains for the compensator,
the feedforward loop is removed yielding the control system
in Figure 4.3. The gains are picked to give good response
characteristics when G and Gff do not match the dynamic
model.
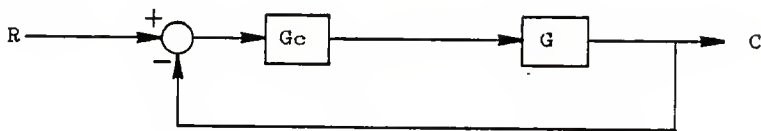
FIGURE 4.3   Feedback Control System

Solving for the characteristic equation of the system
in Figure 4.3 gives,

4.4 $s^3 + \dfrac{KmKds^2}{J} + \dfrac{KmKps}{J} + \dfrac{KmKi}{J} = 0$

Since the term Km/J appears with each of the gains Kd, Kp,
and Ki, the value of Km/J is picked so that the remaining
gains will have moderate values.

The roots of the characteristic equation are picked to be -1±i1, -5, resulting in a natural frequency of 1.414 r/s and a damping ratio of 0.7071 for the dominant roots. The Km/J constant is picked to be 0.4, resulting in the following gain values,

4.5   Kd = 17.5

4.6   Kp = 30.0

4.7   Ki = 25.0

For the body position variables, P_x, P_y, and P_z, the value for J is the mass of the body, 1.55 slugs, giving Km a value of 0.62 lb/ft.

The inertia that the body orientation angle Φ sees is $I_x,oC(\theta)+I_y,oS(\theta)$. Thus for Φ, $J = 0.436*C(\theta)+0.250*S(\theta)$ slug ft^2 and Km = 0.4*J ft lb / radian.

The inertia that the body orientation angle θ sees is I_x,o giving $J = 0.250$ slug ft^2 and Km = 0.1 ft lb / radian.

For the body orientation angle ψ, the inertia is I_z,o so $J = 0.436$ slug ft^2 and Km = 0.174 ft lb / radian.

In order to transform the required body forces into joint torques for the supporting legs the Lagrangian multipliers of Chapter 2 are used. T_1 thru T_6 are defined as the constraint forces on the body in the P_x, P_y, P_z, Φ, θ, and ψ directions respectively and are given as,

4.8 $T_i = \sum\limits_{m=1}^{9} \lambda_m a_{m,i}$ i=1,6

By letting $\lambda_1 = \lambda_2 + \lambda_3$, $\lambda_5 = \lambda_4 + \lambda_6$, $\lambda_9 = \lambda_7 + \lambda_8$, and the T_1 equal the required body forces, Equation 4.8 is reduced to a set of 6 linear equations with 6 unknowns (λ_2, λ_3, λ_4, λ_6, λ_7, λ_8). These λ's are obtained using a Gaussian elimination method.

By defining $T_{i,j}$ as the torque forcing $\theta_{i,j}$, Equations 2.52 through 2.60 can be rewritten as,

$$4.9 \qquad T_{2,p} = -\sum_{m=1}^{9} \lambda_m a_{m,7}$$

$$T_{3,p} = -\sum_{m=1}^{9} \lambda_m a_{m,8}$$

$$T_{4,p} = -\sum_{m=1}^{9} \lambda_m a_{m,9}$$

$$T_{2,q} = -\sum_{m=1}^{9} \lambda_m a_{m,10}$$

$$T_{3,q} = -\sum_{m=1}^{9} \lambda_m a_{m,11}$$

$$T_{4,q} = -\sum_{m=1}^{9} \lambda_m a_{m,12}$$

$$T_{2,r} = -\sum_{m=1}^{9} \lambda_m a_{m,13}$$

$$T_{3,r} = -\sum_{m=1}^{9} \lambda_m a_{m,14}$$

$$T_{4,r} = -\sum_{m=1}^{9} \lambda_m a_{m,15}$$

These supporting leg torques are used in the dynamic

equations to calculate the motion of the body.

4-2.   CONTROL OF THE NON-SUPPORTING LEGS

The control system used for each joint of the
non-supporting legs is of the same form as the system used
to control the body as shown in Figure 4.1.   The inputs to
these systems are spline functions describing the individual
joint trajectories and the feedback is from the individual
joints.

After experimenting with several values, the roots of
the characteristic equation (Equation 4.4) are picked to be
$-2\pm i2$, -8, resulting in a natural frequency of the dominant
roots of 2.828 r/s and a damping ratio of 0.7071.   The Km/J
constant is left at 0.4 resulting in the following values
for Kd, Kp, and Ki,

4.10   Kd = 30.0
        Kp = 100.0
        Ki = 160.0

The values of the moments of inertia about the joint
axes are computed at a median leg position.   The values of
the joint angles at this position are,

4.11 $\theta_{2,j} = 0.0$ rad
        $\theta_{3,j} = -0.601$ rad
        $\theta_{4,j} = 1.78$ rad

By defining $J_{1,j}$ as the moment of inertia about $\theta_{1,j}$,
the equations for moments of inertia are,

4.12 $J_{2,j} = [I_{2,j,x} + m_{2,j} lm_{2,j}^2] + [I_{3,j,y} C(\theta_{3,j})$

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$$+I_{3,j,\kappa} \, S(\theta_{3,j})$$
$$+m_{3,j}(l_{2,j}+lm_{3,j}C(\theta_{3,j}))^2]$$
$$+[I_{4,j,\nu} \, C(\theta_{3,j}+\theta_{4,j})$$
$$+I_{4,j,\kappa} \, S(\theta_{3,j}+\theta_{4,j})$$
$$+m_{4,j}(l_{2,j}+l_{3,j}C(\theta_{3,j})$$
$$+lm_{4,j}C(\theta_{3,j}+\theta_{4,j}))^2]$$
$$J_{3,j} = [I_{3,j,z}+m_{3,j}lm_{3,j}{}^2]$$
$$+[I_{4,j,z}+m_{4,j}(l_{3,j}{}^2+lm_{4,j}{}^2$$
$$-2l_{3,j}lm_{4,j}C(\,-\theta_{4,j}))]$$
$$J_{4,j} = I_{4,j,z}+m_{4,j}lm_{4,j}{}^2$$

Performing these operations gives,

4.13 $J_{2,j} = 0.511$ slug ft^2

        $J_{3,j} = 0.417$ slug ft^2

        $J_{4,j} = 0.112$ slug ft^2

The gain constant Km can be calculated for each joint. These calculations give,

4.14 $Km_{2,j} = 2.04$ ft lb / radian

        $Km_{3,j} = 0.167$ ft lb / radian

        $Km_{4,j} = 0.0448$ ft lb / radian

This completes the development of the control systems for the walking robot. The control systems developed in this chapter are used in the simulation program. The performance of that simulation are given in the next chapter.

V. RESULTS AND RECOMMENDATIONS

In Chapter 1 the kinematic and inverse kinematic
equations of a six legged walking robot were developed. In
Chapter 2 the dynamic equations describing the six degree of
freedom motion of the body with three supporting legs was
developed. Constraint equations were developed based on the
end points of the supporting legs remaining at a fixed
location in world coordinates. Dynamic equations were also
developed for the three non-supporting legs. In Chapter 3 a
recursive spline technique was developed for describing the
desired paths of the body position and orientation and the
joints of the non-supporting legs. In Chapter 4 control
systems were developed for the body and the joints of the
non-supporting legs using derivative feedback, proportional
plus integral series compensation, and a novel feedforward
compensation which takes advantage of the differentiability
of the spline input functions. All of these developments
have been integrated into a simulation of the walking
robot. The simulation is written in Fortran 77 and
implemented on a Vax 750 computer system. A documented
listing of the Fortran source code and input files is
included in Appendix 2. In this chapter the response of the
simulation will be presented and discussed. The objectives
of this work were to develop a tool for the study of the

application of advanced control concepts to articulated mechanisms and to initiate the study of the use of a recursive spline path planning and a novel feedforward control concept for these mechanisms. The simulation was successful in meeting these objectives. Recommendations for further study are given.

5-1. SIMULATION RESULTS

Figures 5.1 through 5.9 show the results of the simulation. The body position and orientation along with the joint angles of leg 5 are shown. Since all of the legs follow similar type paths, leg 5 was choosen as the representative leg. In Figures 5.1 through 5.6, the times at which the body weight is transferred from one set of legs to the other are marked by X's along the time axis.

Figure 5.1 shows the forward motion of the body, P_x, and the command input versus time. As may be observed, the body response gets ahead of its input by a very small amount at the beginning of the simulation and maintains an even distance until the body stops at 14 sec.

The side to side motion of the body is represented by P_y and is shown in Figure 5.2. The response remains very close to zero, the command input, with the largest deviation being about $2*10^{-6}$ ft.

The vertical position of the body, P_z, is shown in Figure 5.3. The largest deviation from the input is less than $2*10^{-6}$ ft. The body does not sag under the force of
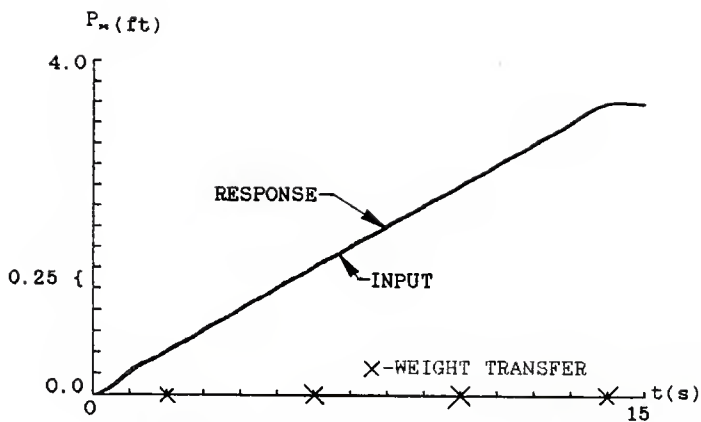
FIGURE 5.1 P_x-Spline input and response
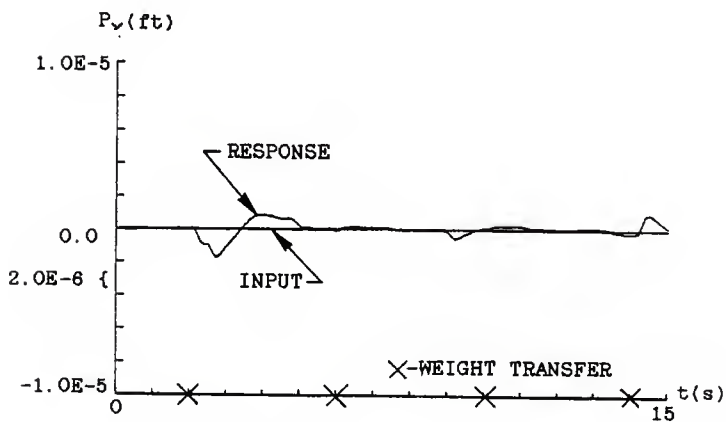
FIGURE 5.2 P_y-Spline input and response
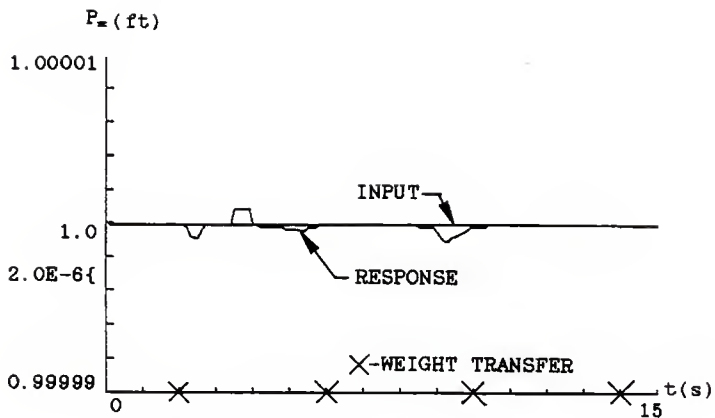
FIGURE 5.3 P_x-Spline input and response
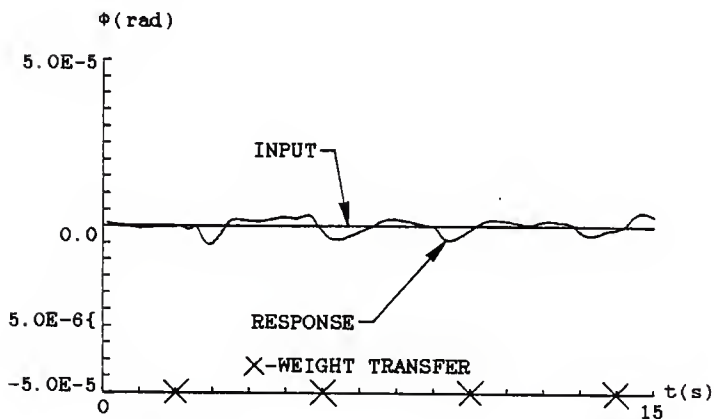
FIGURE 5.4 Φ-Spline input and response
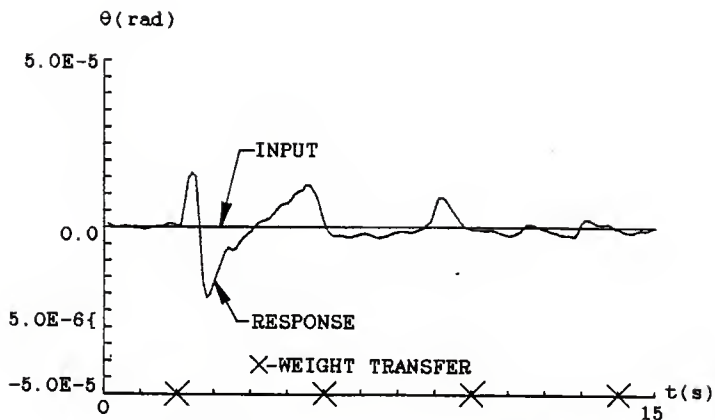
FIGURE 5.5 θ-Spline input and response
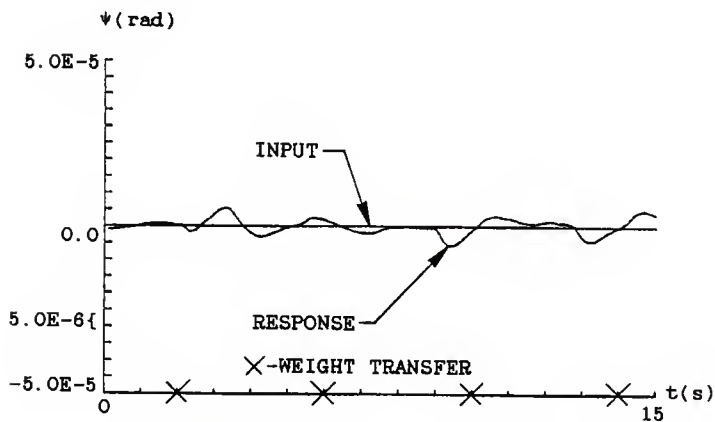
FIGURE 5.6 ψ-Spline input and response
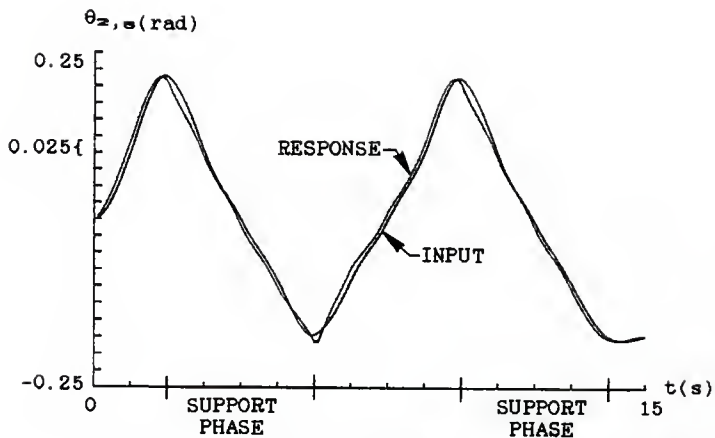
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FIGURE 5.7 $\theta_{2,s}$-Spline input and response

FIGURE 5.8 $\theta_{3,s}$-Spline input and response

$\theta_4,_5(rad)$

RESPONSE— ←INPUT

2.1

0.05 {

1.5

0   SUPPORT   SUPPORT   15   t(s)
         PHASE   PHASE
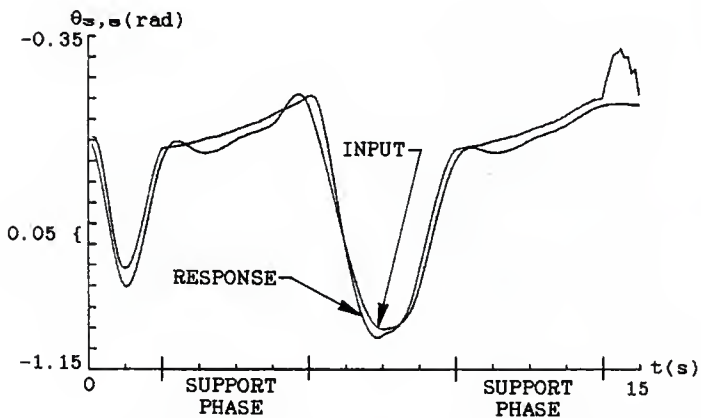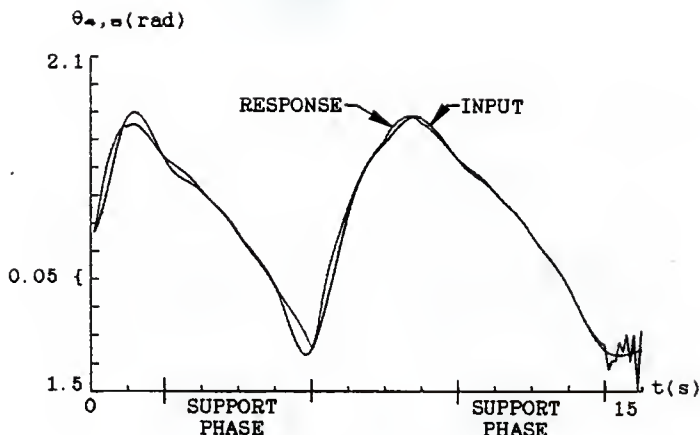
FIGURE 5.9 $\theta_4,_5$-Spline input and response

gravity at the beginning of the simulation because there is a corrective force added by the control system to compensate for the pull of gravity in the P_z direction.

The body orientation angles, Φ, θ, and ψ, are shown in Figures 5.4 through 5.6. Again the response follows the input very closely with the largest deviation being about $2.5*10^{-5}$ rad.

The joint angles of leg 5 are shown in Figures 5.7 through 5.9. During the periods marked SUPPORT PHASE the joints are being driven by the body control system. During the remainder of the simulation the joints are being controlled using the joint spline curves and the non-support leg control systems as described in Chapter 4. The control torques at $\theta_3,_5$ and $\theta_4,_5$ have corrective terms added to them

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to reduce the effects of gravity on the joints. These correction terms help in the smooth transition from the support phase to the non-support phase.

It may be observed from the figures that the response of the body is better than the response of the joint angles. This may be due to the fact that the model used in developing the control system represents the dynamic model of the body very closely (this is especially true at low velocities of the orientation angles Φ, θ, and Ψ). The model for control of the non-supporting legs was developed at a median leg position. Changes in inertia at different leg positions, torque coupling between joints, and switching between supporting and non-supporting phases produced some deviation of the joint responses from the inputs.

5-2   RECOMMENDATIONS FOR FURTHER STUDY

With the research tool provided by this study the possibilities for further work are extensive. Further work might focus on altering the physical system that was modeled, modifying or extending the system model, evaluating the recursive spline path planning algorithm, or improving the individual control strategies and systems. Some specific recommendations concerning this study are given in the following paragraphs.

As the speed of the walking robot is increased and as the control paths become more complicated, such as walking on inclines or over uneven terrain, there may be a need for

a more complete dynamic model. The beginning of the development of a more complete model is given in Appendix 1. In this model, Lagrange's equations are used to write the equations of motion for the entire walking robot, with all the masses and inertias of all the links being included in the model. The constraint equations are the same as those used in the simplified model, that is, the end points of the three supporting legs must remain at a fixed location in world coordinates. Other sets of constraints could be considered. Modeling slippage of the end point on a smooth surface or plastic deformation of the supporting surface are possibilities. The use of a Roll, Pitch, Yaw orientation system on the body rather than the Euler angles should be considered. There are two reasons for this recommendation. First, in order to go from a level body position to a small rotation about the Y_o axis, using Euler angles, requires step changes of $90°$ in the ϕ and ψ angles. The use of spline functions to create the paths for ϕ and ψ prohibit these step changes in their values. Second, when using the Euler orientation angles, the mass matrix, $\underline{M}$, in Equation 2.74 becomes singular when the body orientation angle θ is zero. This makes it necessary to use a second set of equations in the simulation program to solve for the body orientation when θ becomes very small.

The recursive spline path definition functions developed in Chapter 3 are for cubic splines with equally

spaced knots. The use of higher order splines might eliminate the discontinuity in the second derivative when using a small number of points to represent the curve. Also, some consideration to using unequally spaced knots might be given. The advantage of unequal knot spacing would come in areas where the knots need to be kept close together for more accurate path representation, but where a high speed is also required. For example, knots could be spaced at one second intervals for course paths and at 1/2 second intervals for more accurate paths.

The control systems used in the simulation program were not extensively analyzed. While the body control system worked extremely well for the motions considered some work needs to be done with the relationships between the Lagrangian multipliers to insure the torques at the joints of the supporting legs are well balanced. Also, the matrix formed from Equation 4.8 becomes singular when the body orientation angles become zero. This makes it necessary to add a line in the GAUSSX subroutine that does not allow the row divisor $A(i,j)$ to become zero. Evaluation of the performance of the body control system for other types of motion is suggested.

The non-supporting legs might be a good place to implement a model referenced adaptive control strategy. While many adaptive control algorithms are too lengthy to be used in real time control, an algorithm to adaptively adjust

the gains in the feedforward controller might be simple enough to realize real time implementation.

The addition of interactive input to permit dynamic control of the path of the walking robot would be a valuable extension to the work. The addition of interactive input would require the addition of graphics to display the motion of the body and legs. The addition of a hierarchical control structure which could receive sensory and command inputs and produce path planning data would simulate a more autonomous robot.

This thesis has presented the results of a project to develop a tool for the study of the application of advanced control concepts to articulated mechanisms. The study of two advanced control concepts has been initiated with interesting results. Many suggestions for useful continuation of the work have been presented.

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APPENDIX I

Full Dynamics

Appendix 1 is the partial development of the full

dynamic model for the walking robot described in Chapters 1

and 2. All the definitions concerning the physical model

are the same as those of Chapters 1 and 2. However, the

development in this appendix accounts for the masses and

inertias of all the links on all the legs and of the body

mass and inertia in one set of Lagrange's equations.

As in Chapter 2, the kinetic energy of the body, Kb,

is,

A1.1 $Kb = 1/2Mb(\dot{P}_x{}^2+\dot{P}_y{}^2+\dot{P}_z{}^2)+1/2(I_{x,o}\omega^2{}_{x,o}+$

$I_{y,o}\omega^2{}_{y,o}+I_{z,o}\omega^2{}_{z,o})$

Recalling that $\omega_{1,j,x}$, $\omega_{1,j,y}$, and $\omega_{1,j,z}$ are the

angular velocities of link i+1 about the X_1, Y_1, Z_1 axes,

the ω's are,

A1.2 $\omega_{1,j,x} = \dot{\theta}C(\psi+\theta_{1,j}+\theta_{2,j})+\dot{\phi}S(\theta)S(\psi+\theta_{1,j}+\theta_{2,j})$

$\omega_{1,j,y} = -\dot{\theta}S(\psi+\theta_{1,j}+\theta_{2,j})+\dot{\phi}S(\theta)C(\psi+\theta_{1,j}+\theta_{2,j})$

$\omega_{1,j,z} = \dot{\psi}+\dot{\phi}C(\theta)+\dot{\theta}_{2,j}$

$\omega_{2,j,x} = \dot{\theta}C(\psi+\theta_{1,j},+\theta_{2,j})C(\theta_{3,j})$

$+\dot{\phi}[S(\theta)S(\psi+\theta_{1,j}+\theta_{2,j})C(\theta_{3,j})$

$-C(\theta)S(\theta_{3,j})]$

$-\dot{\psi}S(\theta_{3,j})-\dot{\theta}_{2,j}S(\theta_{3,j})$

$\omega_{2,j,y} = -\dot{\theta}C(\psi+\theta_{1,j}+\theta_{2,j})S(\theta_{3,j})$

$-\dot{\phi}[S(\theta)S(\psi+\theta_{1,j}+\theta_{2,j})S(\theta_{3,j})$

$$+C(\theta)C(\theta_3,_J)]$$
$$-\dot{\psi}C(\theta_3,_J)-\dot{\theta}_2,_J C(\theta_3,_J)$$
$$\omega_2,_J,_x = -\dot{\theta}S(\psi+\theta_1,_J+\theta_2,_J)+\dot{\phi}S(\theta)C(\psi+\theta_1,_J+\theta_2,_J)$$
$$+\dot{\theta}_3,_J$$

$$\omega_3,_J,_x = \dot{\theta}C(\psi+\theta_1,_J+\theta_2,_J)C(\theta_3,_J+\theta_4,_J)$$
$$+\dot{\phi}[S(\theta)S(\psi+\theta_1,_J+\theta_2,_J)C(\theta_3,_J+\theta_4,_J)$$
$$-C(\theta)S(\theta_3,_J+\theta_4,_J)]-\dot{\psi}S(\theta_3,_J+\theta_4,_J)$$
$$-\dot{\theta}_2,_J S(\theta_3,_J+\theta_4,_J)$$

$$\omega_3,_J,_y = -\dot{\theta}C(\psi+\theta_1,_J+\theta_2,_J)S(\theta_3,_J+\theta_4,_J)$$
$$-\dot{\phi}[S(\theta)S(\psi+\theta_1,_J+\theta_2,_J)S(\theta_3,_J+\theta_4,_J)$$
$$+C(\theta)C(\theta_3,_J+\theta_4,_J)]-\dot{\psi}C(\theta_3,_J+\theta_4,_J)$$
$$-\dot{\theta}_2,_J C(\theta_3,_J+\theta_4,_J)$$

$$\omega_3,_J,_z = -\dot{\theta}S(\psi+\theta_1,_J+\theta_2,_J)+\dot{\phi}S(\theta)C(\psi+\theta_1,_J+\theta_2,_J)$$
$$+\dot{\theta}_3,_J+\dot{\theta}_4,_J$$

$R_1,_J,_x$, $R_1,_J,_y$, and $R_1,_J,_z$ are defined as the coordinates of the center of mass of link i, leg J in the world coordinate system (X_w, Y_w, Z_w). The values for the R's are given by the following equation,

A1.3
$$\begin{bmatrix} R_1,_J,_x \\ R_1,_J,_y \\ R_1,_J,_z \\ 1 \end{bmatrix} = \begin{bmatrix} Eular \\ rotation \\ and \\ translation \end{bmatrix} \begin{bmatrix} r_1,_J,_x \\ r_1,_J,_y \\ r_1,_J,_z \\ 1 \end{bmatrix}$$

Where the r's define the centers of mass of the links in the X_o, Y_o, Z_o coordinate system and are given in Equation 2.88.

Performing the multiplication in Equation A1.3 for each of the links of leg j gives,

A1.4 $R_{2,J,X} = C(\Phi)C(\Psi)C(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad +C(\Phi)C(\Psi)C(\theta_{1,J})l_{1,J}$

$\qquad -S(\Phi)C(\theta)S(\Psi)C(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad -S(\Phi)C(\theta)S(\Psi)C(\theta_{1,J})l_{1,J}$

$\qquad -C(\Phi)S(\Psi)S(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad -C(\Phi)S(\Psi)S(\theta_{1,J})l_{1,J}$

$\qquad -S(\Phi)C(\theta)C(\Psi)S(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad -S(\Phi)C(\theta)C(\Psi)S(\theta_{1,J})l_{1,J}+P_X$

$R_{2,J,Y} = S(\Phi)C(\Psi)C(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad +S(\Phi)C(\Psi)C(\theta_{1,J})l_{1,J}$

$\qquad +C(\Phi)C(\theta)S(\Psi)C(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad +C(\Phi)C(\theta)S(\Psi)C(\theta_{1,J})l_{1,J}$

$\qquad -S(\Phi)S(\Psi)S(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad -S(\Phi)S(\Psi)S(\theta_{1,J})l_{1,J}$

$\qquad +C(\Phi)C(\theta)C(\Psi)S(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad +C(\Phi)C(\theta)C(\Psi)S(\theta_{1,J})l_{1,J}+P_Y$

$R_{2,J,Z} = S(\theta)S(\Psi)C(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad +S(\theta)S(\Psi)C(\theta_{1,J})l_{1,J}$

$\qquad +S(\theta)C(\Psi)S(\theta_{1,J}+\theta_{2,J})lm_{2,J}$

$\qquad +S(\theta)C(\Psi)S(\theta_{1,J})l_{1,J}+P_Z$

$R_{3,J,X} = C(\Phi)C(\Psi)C(\theta_{1,J}+\theta_{2,J})l_{2,J}$

$\qquad +C(\Phi)C(\Psi)C(\theta_{1,J})l_{1,J}$

$\qquad -S(\Phi)C(\theta)S(\Psi)C(\theta_{1,J}+\theta_{2,J})l_{2,J}$

$\qquad -S(\Phi)C(\theta)S(\Psi)C(\theta_{1,J})l_{1,J}$

$\qquad -C(\Phi)S(\Psi)S(\theta_{1,J}+\theta_{2,J})l_{2,J}$

$\qquad -C(\Phi)S(\Psi)S(\theta_{1,J})l_{1,J}$

$$-S(\Phi)C(\theta)C(\psi)S(\theta_{1,J}+\theta_{2,J})lz,J$$

$$-S(\Phi)C(\theta)C(\psi)S(\theta_{1,J})l_{1,J}$$

$$+C(\Phi)C(\psi)C(\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})lm_{3,J}$$

$$-S(\Phi)C(\theta)S(\psi)C(\theta_{1,J}+\theta_{2,J})$$

$$*C(\theta_{3,J})lm_{3,J}-C(\Phi)S(\psi)S(\theta_{1,J}+\theta_{2,J})$$

$$*C(\theta_{3,J})lm_{3,J}-S(\Phi)C(\theta)C(\psi)$$

$$*S(\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})lm_{3,J}-S(\Phi)S(\theta)$$

$$*S(\theta_{3,J})lm_{3,J}+P_{\mathsf{M}}$$

$$R_{3,J,y} = S(\Phi)C(\psi)C(\theta_{1,J}+\theta_{2,J})lz,J$$

$$+S(\Phi)C(\psi)C(\theta_{1,J})l_{1,J}$$

$$+C(\Phi)C(\theta)S(\psi)C(\theta_{1,J}+\theta_{2,J})lz,J$$

$$+C(\Phi)C(\theta)S(\psi)C(\theta_{1,J})l_{1,J}$$

$$-S(\Phi)S(\psi)S(\theta_{1,J}+\theta_{2,J})lz,J$$

$$-S(\Phi)S(\psi)S(\theta_{1,J})l_{1,J}$$

$$+C(\Phi)C(\theta)C(\psi)S(\theta_{1,J}+\theta_{2,J})lz,J$$

$$+C(\Phi)C(\theta)C(\psi)S(\theta_{1,J})l_{1,J}$$

$$+S(\Phi)C(\psi)C(\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})lm_{3,J}$$

$$+C(\Phi)C(\theta)S(\psi)C(\theta_{1,J}+\theta_{2,J})$$

$$*C(\theta_{3,J})lm_{3,J}-S(\Phi)S(\psi)S(\theta_{1,J}+\theta_{2,J})$$

$$*C(\theta_{3,J})lm_{3,J}+C(\Phi)C(\theta)C(\psi)$$

$$*S(\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})lm_{3,J}+C(\Phi)S(\theta)$$

$$*S(\theta_{3,J})lm_{3,J}+P_{y}$$

$$R_{3,J,z} = S(\theta)S(\psi)C(\theta_{1,J}+\theta_{2,J})lz,J$$

$$+S(\theta)S(\psi)C(\theta_{1,J})l_{1,J}$$

$$+S(\theta)C(\psi)S(\theta_{1,J}+\theta_{2,J})lz,J$$

$$+S(\theta)C(\psi)S(\theta_{1,J})l_{1,J}$$

$$+S(\theta)S(\psi)C(\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})lm_{3,J}$$
$$+S(\theta)C(\psi)S(\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})lm_{3,J}$$
$$-C(\theta)S(\theta_{3,J})lm_{3,J}+P_x$$
$$R_{4,J,M} = C(\phi)C(\psi)C(\theta_{1,J}+\theta_{2,J})l_{2,J}$$
$$+C(\phi)C(\psi)C(\theta_{1,J})l_{1,J}$$
$$-S(\phi)C(\theta)S(\psi)C(\theta_{1,J}+\theta_{2,J})l_{2,J}$$
$$-S(\phi)C(\theta)S(\psi)C(\theta_{1,J})l_{1,J}$$
$$-C(\phi)S(\psi)S(\theta_{1,J}+\theta_{2,J})l_{2,J}$$
$$-C(\phi)S(\psi)S(\theta_{1,J})l_{1,J}$$
$$-S(\phi)C(\theta)C(\psi)S(\theta_{1,J}+\theta_{2,J})l_{2,J}$$
$$-S(\phi)C(\theta)C(\psi)S(\theta_{1,J})l_{1,J}$$
$$+C(\phi)C(\psi)C(\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})l_{3,J}$$
$$-S(\phi)C(\theta)S(\psi)C(\theta_{1,J}+\theta_{2,J})$$
$$*C(\theta_{3,J})l_{3,J}-C(\phi)S(\psi)S(\theta_{1,J}+\theta_{2,J})$$
$$*C(\theta_{3,J})l_{3,J}-S(\phi)C(\theta)C(\psi)$$
$$*S(\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})l_{3,J}-S(\phi)S(\theta)$$
$$*S(\theta_{3,J})l_{3,J}$$
$$+C(\phi)C(\psi)C(\theta_{1,J}+\theta_{2,J})$$
$$*C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$-S(\phi)C(\theta)S(\psi)C(\theta_{1,J}+\theta_{2,J})$$
$$*C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$-C(\phi)S(\psi)S(\theta_{1,J}+\theta_{2,J})$$
$$*C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$-S(\phi)C(\theta)C(\psi)S(\theta_{1,J}+\theta_{2,J})$$
$$*C(\theta_{1,J}+\theta_{4,J})lm_{4,J}$$
$$-S(\phi)S(\theta)S(\theta_{3,J}+\theta_{4,J})lm_{4,J}+P_x$$

$$R_{4,3,y} = S(\phi)C(\psi)C(\theta_{1,3}+\theta_{2,3})l_{2,3}$$
$$+S(\phi)C(\psi)C(\theta_{1,3})l_{1,3}$$
$$+C(\phi)C(\theta)S(\psi)C(\theta_{1,3}+\theta_{2,3})l_{2,3}$$
$$+C(\phi)C(\theta)S(\psi)C(\theta_{1,3})l_{1,3}$$
$$-S(\phi)S(\psi)S(\theta_{1,3}+\theta_{2,3})l_{2,3}$$
$$-S(\phi)S(\psi)S(\theta_{1,3})l_{1,3}$$
$$+C(\phi)C(\theta)C(\psi)S(\theta_{1,3}+\theta_{2,3})l_{2,3}$$
$$+C(\phi)C(\theta)C(\psi)S(\theta_{1,3})l_{1,3}$$
$$+S(\phi)C(\psi)C(\theta_{1,3}+\theta_{2,3})C(\theta_{3,3})l_{3,3}$$
$$+C(\phi)C(\theta)S(\psi)C(\theta_{1,3}+\theta_{2,3})$$
$$*C(\theta_{3,3})l_{3,3}-S(\phi)S(\psi)S(\theta_{1,3}+\theta_{2,3})$$
$$*C(\theta_{3,3})l_{3,3}+C(\phi)C(\theta)C(\psi)$$
$$*S(\theta_{1,3}+\theta_{2,3})C(\theta_{3,3})l_{3,3}+C(\phi)S(\theta)$$
$$*S(\theta_{3,3})l_{3,3}$$
$$+S(\phi)C(\psi)C(\theta_{1,3}+\theta_{2,3})$$
$$*C(\theta_{3,3}+\theta_{4,3})lm_{4,3}$$
$$+C(\phi)C(\theta)S(\psi)C(\theta_{1,3}+\theta_{2,3})$$
$$*C(\theta_{3,3}+\theta_{4,3})lm_{4,3}$$
$$-S(\phi)S(\psi)S(\theta_{1,3}+\theta_{2,3})$$
$$*C(\theta_{3,3}+\theta_{4,3})lm_{4,3}$$
$$+C(\phi)C(\theta)C(\psi)S(\theta_{1,3}+\theta_{2,3})$$
$$*C(\theta_{3,3}+\theta_{4,3})lm_{4,3}$$
$$+C(\phi)S(\theta)S(\theta_{3,3}+\theta_{4,3})lm_{4,3}+P_y$$
$$R_{4,3,x} = S(\theta)S(\psi)C(\theta_{1,3}+\theta_{2,3})l_{2,3}$$
$$+S(\theta)S(\psi)C(\theta_{1,3})l_{1,3}$$
$$+S(\theta)C(\psi)S(\theta_{1,3}+\theta_{2,3})l_{2,3}$$

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$$+S(\theta)C(\psi)S(\theta_{1,j})l_{1,j}$$

$$+S(\theta)S(\psi)C(\theta_{1,j}+\theta_{2,j})C(\theta_{3,j})l_{3,j}$$

$$+S(\theta)C(\psi)S(\theta_{1,j}+\theta_{2,j})C(\theta_{3,j})l_{3,j}$$

$$-C(\theta)S(\theta_{3,j})l_{3,j}$$

$$+S(\theta)S(\psi)C(\theta_{1,j}+\theta_{2,j})$$

$$*C(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$

$$+S(\theta)C(\psi)S(\theta_{1,j}+\theta_{2,j})$$

$$*C(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$

$$-C(\theta)S(\theta_{3,j}+\theta_{4,j})lm_{4,j}+P_z$$

Taking the time derivative of each of the R's gives,

A1.5
$$\dot{R}_{1,j,x} = \dot{\phi}a_{1,j,x}+\dot{\theta}b_{1,j,x}+\dot{\psi}c_{1,j,x}$$
$$+\dot{\theta}_{2,j}d_{1,j,x}+\dot{\theta}_{3,j}e_{1,j,x}+\dot{\theta}_{4,j}f_{1,j,x}+\dot{P}_x$$

$$\dot{R}_{1,j,y} = \dot{\phi}a_{1,j,y}+\dot{\theta}b_{1,j,y}+\dot{\psi}c_{1,j,y}$$
$$+\dot{\theta}_{2,j}d_{1,j,y}+\dot{\theta}_{3,j}e_{1,j,y}+\dot{\theta}_{4,j}f_{1,j,y}+\dot{P}_y$$

$$\dot{R}_{1,j,z} = \dot{\phi}a_{1,j,z}+\dot{\theta}b_{1,j,z}+\dot{\psi}c_{1,j,z}$$
$$+\dot{\theta}_{2,j}d_{1,j,z}+\dot{\theta}_{3,j}e_{1,j,z}+\dot{\theta}_{4,j}f_{1,j,z}+\dot{P}_z$$

Where,

$$i = 2, 3, 4$$

and where,

$$a_{2,j,x} = -C(\phi)C(\theta)S(\psi+\theta_{1,j}+\theta_{2,j})lm_{2,j}$$
$$-S(\phi)C(\psi+\theta_{1,j}+\theta_{2,j})lm_{2,j}$$
$$-S(\phi)C(\psi+\theta_{1,j})l_{1,j}$$
$$-C(\phi)C(\theta)S(\psi+\theta_{1,j})l_{1,j}$$

$$b_{2,j,x} = S(\phi)S(\theta)S(\psi+\theta_{1,j}+\theta_{2,j})lm_{2,j}$$
$$+S(\phi)S(\theta)S(\psi+\theta_{1,j})l_{1,j}$$

$$c_{2,j,x} = -C(\phi)S(\psi+\theta_{1,j}+\theta_{2,j})lm_{2,j}$$

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$$-S(\phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$-C(\phi)S(\psi+\theta_{1,J})l_{1,J}$$
$$-S(\phi)C(\theta)C(\psi+\theta_{1,J})l_{1,J}$$

$$d_{2,J,M} = -C(\phi)S(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$-S(\phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$

$$e_{2,J,M} = 0$$

$$f_{2,J,M} = 0$$

$$a_{2,J,v} = C(\phi)C(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$+C(\phi)C(\psi+\theta_{1,J})l_{1,J}$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,J})l_{1,J}$$

$$b_{2,J,v} = -C(\phi)S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$-C(\phi)S(\theta)S(\psi+\theta_{1,J})l_{1,J}$$

$$c_{2,J,v} = -S(\phi)S(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$-S(\phi)S(\psi+\theta_{1,J})l_{1,J}$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,J})l_{1,J}$$

$$d_{2,J,v} = C(\phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$-S(\phi)S(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$

$$e_{2,J,v} = 0$$

$$f_{2,J,v} = 0$$

$$a_{2,J,x} = 0$$

$$b_{2,J,x} = C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$+C(\theta)S(\psi+\theta_{1,J})l_{1,J}$$

$$c_{2,J,x} = S(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$
$$+S(\theta)C(\psi+\theta_{1,J})l_{1,J}$$

$$d_{2,J,K} = S(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})lm_{2,J}$$

$$e_{2,J,K} = 0$$

$$f_{2,J,K} = 0$$

$$a_{3,J,K} = -S(\Phi)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*(C(\theta_{3,J})lm_{3,J}+l_{2,J})$$
$$-C(\Phi)C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*(C(\theta_{3,J})lm_{3,J}+l_{2,J})$$
$$-C(\Phi)S(\theta)S(\theta_{3,J})lm_{3,J}$$
$$-S(\Phi)C(\psi+\theta_{1,J})l_{1,J}$$
$$-C(\Phi)C(\theta)S(\psi+\theta_{1,J})l_{1,J}$$

$$b_{3,J,K} = S(\Phi)S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*(C(\theta_{3,J})lm_{3,J}+l_{2,J})$$
$$-S(\Phi)C(\theta)S(\theta_{3,J})lm_{3,J}$$
$$+S(\Phi)S(\theta)S(\psi+\theta_{1,J})l_{1,J}$$

$$c_{3,J,K} = -C(\Phi)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*(C(\theta_{3,J})lm_{3,J}+l_{2,J})$$
$$-S(\Phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*(C(\theta_{3,J})lm_{3,J}+l_{2,J})$$
$$-C(\Phi)S(\psi+\theta_{1,J})l_{1,J}$$
$$-S(\Phi)C(\theta)C(\psi+\theta_{1,J})l_{1,J}$$

$$d_{3,J,K} = -C(\Phi)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*(C(\theta_{3,J})lm_{3,J}+l_{2,J})$$
$$-S(\Phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*(C(\theta_{3,J})lm_{4,J}+l_{2,J})$$

$$e_{3,J,K} = -C(\Phi)C(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})lm_{3,J}$$
$$+S(\Phi)C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})lm_{3,J}$$

$$-S(\phi)S(\theta)C(\theta_{3,4})lm_{3,4}$$

$$f_{3,4,x} = 0$$

$$a_{3,4,y} = C(\phi)C(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*(C(\theta_{3,4})lm_{3,4}+l_{2,4})$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*(C(\theta_{3,4})lm_{3,4}+l_{2,4})$$
$$-S(\phi)S(\theta)S(\theta_{3,4})lm_{3,4}$$
$$+C(\phi)C(\psi+\theta_{1,4})l_{1,4}$$
$$-S(\phi)C(\theta)S(\psi+\theta_{1,4})l_{1,4}$$

$$b_{3,4,y} = -C(\phi)S(\theta)S(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*(C(\theta_{3,4})lm_{3,4}+l_{2,4})$$
$$+C(\phi)C(\theta)S(\theta_{3,4})lm_{3,4}$$
$$-C(\phi)S(\theta)S(\psi+\theta_{1,4})l_{1,4}$$

$$c_{3,4,y} = -S(\phi)S(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*(C(\theta_{3,4})lm_{3,4}+l_{2,4})$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*(C(\theta_{3,4})lm_{3,4}+l_{2,4})$$
$$-S(\phi)S(\psi+\theta_{1,4})l_{1,4}$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,4})l_{1,4}$$

$$d_{3,4,y} = -S(\phi)S(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*(C(\theta_{3,4})lm_{3,4}+l_{2,4})$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*(C(\theta_{3,4})lm_{3,4}+l_{2,4})$$

$$e_{3,4,y} = -S(\phi)C(\psi+\theta_{1,4}+\theta_{2,4})S(\theta_{3,4})lm_{3,4}$$
$$-C(\phi)C(\theta)S(\psi+\theta_{1,4}+\theta_{2,4})S(\theta_{3,4})lm_{3,4}$$
$$+C(\phi)S(\theta)C(\theta_{3,4})lm_{3,4}$$

$$f_{5,4,y} = 0$$

$$a_{5,4,x} = 0$$

$$b_{5,4,x} = C(\theta)S(\psi+\theta_{1,4}+\theta_{2,4})(C(\theta_{5,4})lm_{5,4}+l_{2,4})$$
$$+S(\theta)S(\theta_{5,4})lm_{5,4}+C(\theta)S(\psi+\theta_{1,4})l_{1,4}$$

$$c_{5,4,x} = S(\theta)C(\psi+\theta_{1,4}+\theta_{2,4})(C(\theta_{5,4})lm_{5,4}+l_{2,4})$$
$$+S(\theta)C(\psi+\theta_{1,4})l_{1,4}$$

$$d_{5,4,x} = S(\theta)C(\psi+\theta_{1,4}+\theta_{2,4})(C(\theta_{5,4})lm_{5,4}+l_{2,4})$$

$$e_{5,4,x} = -S(\theta)S(\psi+\theta_{1,4}+\theta_{2,4})S(\theta_{5,4})lm_{5,4}$$
$$-C(\theta)C(\theta_{5,4})lm_{5,4}$$

$$f_{5,4,x} = 0$$

$$a_{4,4,H} = -S(\Phi)C(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*[C(\theta_{5,4}+\theta_{4,4})lm_{4,4}$$
$$+C(\theta_{5,4})l_{5,4}+l_{2,4}]$$
$$-C(\Phi)C(\theta)S(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*[C(\theta_{5,4}+\theta_{4,4})lm_{4,4}$$
$$+C(\theta_{5,4})l_{5,4}+l_{2,4}]$$
$$-C(\Phi)S(\theta)[S(\theta_{5,4}+\theta_{4,4})lm_{4,4}$$
$$+S(\theta_{5,4})l_{5,4}]$$
$$-S(\Phi)C(\psi+\theta_{1,4})l_{1,4}$$
$$-C(\Phi)C(\theta)S(\psi+\theta_{1,4})l_{1,4}$$

$$b_{4,4,H} = S(\Phi)S(\theta)S(\psi+\theta_{1,4}+\theta_{2,4})$$
$$*[C(\theta_{5,4}+\theta_{4,4})lm_{4,4}$$
$$+C(\theta_{5,4})l_{5,4}+l_{2,4}]$$
$$-S(\Phi)C(\theta)[S(\theta_{5,4}+\theta_{4,4})lm_{4,4}$$
$$+S(\theta_{5,4})l_{5,4}]$$
$$+S(\Phi)S(\theta)S(\psi+\theta_{1,4})l_{1,4}$$

$$c_{4,J,M} = -C(\Phi)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{2,J}]$$
$$-S(\Phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{2,J}]$$
$$-C(\Phi)S(\psi+\theta_{1,J})l_{1,J}$$
$$-S(\Phi)C(\theta)C(\psi+\theta_{1,J})l_{1,J}$$

$$d_{4,J,M} = -C(\Phi)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{2,J}]$$
$$-S(\Phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{2,J}]$$

$$e_{4,J,M} = -C(\Phi)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+S(\theta_{3,J})l_{3,J}]$$
$$+S(\Phi)C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+S(\theta_{3,J})l_{3,J}]$$
$$-S(\Phi)S(\theta)[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}]$$

$$f_{4,J,M} = -C(\Phi)C(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+S(\Phi)C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$-S(\Phi)S(\theta)C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$

$$a_{4,1,v} = C(\Phi)C(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*[C(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$
$$+C(\theta_{3,1})l_{3,1}+l_{2,1}]$$
$$-S(\Phi)C(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*[C(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$
$$+C(\theta_{3,1})l_{3,1}+l_{2,1}]$$
$$-S(\Phi)S(\theta)[S(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$
$$+S(\theta_{3,1})l_{3,1}]$$
$$+C(\Phi)C(\psi+\theta_{1,1})l_{1,1}$$
$$-S(\Phi)C(\theta)S(\psi+\theta_{1,1})l_{1,1}$$

$$b_{4,1,v} = -C(\Phi)S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*[C(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$
$$+C(\theta_{3,1})l_{3,1}+l_{2,1}]$$
$$+C(\Phi)C(\theta)[S(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$
$$+S(\theta_{3,1})l_{3,1}]$$
$$-C(\Phi)S(\theta)S(\psi+\theta_{1,1})l_{1,1}$$

$$c_{4,1,v} = -S(\Phi)S(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*[C(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$
$$+C(\theta_{3,1})l_{3,1}+l_{2,1}]$$
$$+C(\Phi)C(\theta)C(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*[C(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$
$$+C(\theta_{3,1})l_{3,1}+l_{2,1}]$$
$$-S(\Phi)S(\psi+\theta_{1,1})l_{1,1}$$
$$+C(\Phi)C(\theta)C(\psi+\theta_{1,1})l_{1,1}$$

$$d_{4,1,v} = -S(\Phi)S(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*[C(\theta_{3,1}+\theta_{4,1})lm_{4,1}$$

$$+C(\theta_{3,J})l_{3,J}+l_{z,J}]$$
$$+C(\phi)C(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{z,J}]$$

$$e_{4,J,y} = -S(\phi)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+S(\theta_{3,J})l_{3,J}]$$
$$-C(\phi)C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*[S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+S(\theta_{3,J})l_{3,J}]$$
$$+C(\phi)S(\theta)[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}]$$

$$f_{4,J,y} = -S(\phi)C(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$-C(\phi)C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$*S(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\phi)S(\theta)C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$

$$a_{4,J,z} = 0$$

$$b_{4,J,z} = C(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{z,J}]$$
$$+S(\theta)[S(\theta_{3,J}+\theta_{4,J})lm_{4,J}+S(\theta_{3,J})l_{3,J}]$$
$$+C(\theta)S(\psi+\theta_{1,J})l_{1,J}$$

$$c_{4,J,z} = S(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{z,J}]$$
$$+S(\theta)C(\psi+\theta_{1,J})l_{1,J}$$

$$d_{4,J,z} = S(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})[C(\theta_{3,J}+\theta_{4,J})lm_{4,J}$$
$$+C(\theta_{3,J})l_{3,J}+l_{z,J}]$$

$$e_{4,j,z} = -S(\theta)S(\psi+\theta_{1,j}+\theta_{2,j})$$
$$*[S(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$
$$+S(\theta_{3,j})l_{3,j}]$$
$$-C(\theta)[C(\theta_{3,j}+\theta_{4,j})lm_{4,j}+C(\theta_{3,j})l_{3,j}]$$
$$f_{4,j,z} = -S(\theta)S(\psi+\theta_{1,j}+\theta_{2,j})S(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$
$$-C(\theta)C(\theta_{3,j}+\theta_{4,j})lm_{4,j}$$

The total kinetic energy, K, of the system is now written as,

$$A1.6 \quad K = 1/2Mb(\dot{P}_x{}^2+\dot{P}_y{}^2+\dot{P}_z{}^2)$$
$$+1/2(I_{x,o}\omega^2{}_{x,o}+I_{y,o}\omega^2{}_{y,o}+I_{z,o}\omega^2{}_{z,o})$$
$$+1/2\sum_{j=1}^{6}\sum_{i=2}^{4}\{m_{i,j}[(\dot{R}_{i,j,x})^2+(\dot{R}_{i,j,y})^2$$
$$+(\dot{R}_{i,j,z})^2]$$
$$+[I_{i,j,x}(\omega_{i-1,j,x})^2$$
$$+I_{i,j,y}(\omega_{i-1,j,y})^2$$
$$+I_{i,j,z}(\omega_{i-1,j,z})^2]\}$$

And the potential energy, P, is,

$$A1.7 \quad P = MbGP_z + \sum_{j=1}^{6}\sum_{i=2}^{4}M_{i,j}GR_{i,j,z}$$

The Lagrangian L is then,

$$A1.8 \quad L = 1/2Mb(\dot{P}_x{}^2+\dot{P}_y{}^2+\dot{P}_z{}^2-2GP_z)$$
$$+1/2(I_{x,o}\omega^2{}_{x,o}+I_{y,o}\omega^2{}_{y,o}+I_{z,o}\omega^2{}_{z,o})$$
$$+1/2\sum_{j=1}^{6}\sum_{i=2}^{4}\{m_{i,j}[(\dot{R}_{i,j,x})^2+(\dot{R}_{i,j,y})^2$$
$$+(\dot{R}_{i,j,z})^2-2GR_{i,j,z}]$$
$$+[I_{i,j,x}(\omega_{i-1,j,x})^2$$
$$+I_{i,j,y}(\omega_{i-1,j,y})^2$$

$$+I_{i,j,z}(\omega_{i-1,j,z})^2]\}$$

The generalized coordinates are defined as,

A1.9
$q_1 = P_x$	$q_9 = \theta_{4,1}$	$q_{17} = \theta_{z,4}$
$q_2 = P_y$	$q_{10} = \theta_{z,z}$	$q_{18} = \theta_{4,4}$
$q_3 = P_z$	$q_{11} = \theta_{3,z}$	$q_{19} = \theta_{z,5}$
$q_4 = \phi$	$q_{12} = \theta_{4,z}$	$q_{20} = \theta_{3,5}$
$q_5 = \theta$	$q_{13} = \theta_{z,3}$	$q_{21} = \theta_{4,5}$
$q_6 = \psi$	$q_{14} = \theta_{3,3}$	$q_{22} = \theta_{z,6}$
$q_7 = \theta_{z,1}$	$q_{15} = \theta_{4,3}$	$q_{23} = \theta_{3,6}$
$q_8 = \theta_{3,1}$	$q_{16} = \theta_{z,4}$	$q_{24} = \theta_{4,6}$

By defining p, q, and r as the support legs and requiring the endpoints of the support legs to stay at a fixed location in the world coordinate system, the constraint equations become identical to those in Section 2-1.

Lagranges equations then become,

A1.10
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{m=1}^{9} \lambda_m a_{m,i} + T_i$$

$i = 1, 2, \ldots, 24$ and,

$$\sum_{i=1}^{24} a_{m,i}\dot{q}_i + a_{m,t} = 0$$

$$m = 1, 2, \ldots, 9$$

Where q_i is defined in Equation A1.9 and λm, $a_{m,i}$, $a_{m,t}$, and T_i are defined in Section 2.1.

To begin taking the partial of L with respect to $\dot{q}_i$, L is broken down into components and the partials of those components with respect to $\dot{q}_i$ are taken.

A1.11
$$\frac{\partial P_z}{\partial \dot{P}_x} = 0 \qquad \frac{\partial \dot{P}_x^2}{\partial \dot{P}_x} = 2\dot{P}_x$$

$$\frac{\partial \dot{P}_y{}^2}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial \dot{P}_z{}^2}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial R_{2,\jmath,\varkappa}}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial R_{3,\jmath,\varkappa}}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial R_{4,\jmath,\varkappa}}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial (\dot{R}_{2,\jmath,\varkappa})^2}{\partial \dot{P}_\varkappa} = 2[\dot{P}_\varkappa + \dot{\phi} a_{2,\jmath,\varkappa} + \dot{\theta} b_{2,\jmath,\varkappa}$$
$$+ \dot{\psi} c_{2,\jmath,\varkappa} + \dot{\theta}_{2,\jmath} d_{2,\jmath,\varkappa}]$$

$$\frac{\partial (\dot{R}_{2,\jmath,y})^2}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial (\dot{R}_{2,\jmath,z})^2}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial (\dot{R}_{3,\jmath,\varkappa})^2}{\partial \dot{P}_\varkappa} = 2[\dot{P}_\varkappa + \dot{\phi} a_{3,\jmath,\varkappa} + \dot{\theta} b_{3,\jmath,\varkappa}$$
$$+ \dot{\psi} c_{3,\jmath,\varkappa} + \dot{\theta}_{2,\jmath} d_{3,\jmath,\varkappa} + \dot{\theta}_{3,\jmath} e_{3,\jmath,\varkappa}]$$

$$\frac{\partial (\dot{R}_{3,\jmath,y})^2}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial (\dot{R}_{3,\jmath,z})^2}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial (\dot{R}_{4,\jmath,\varkappa})^2}{\partial \dot{P}_\varkappa} = 2[\dot{P}_\varkappa + \dot{\phi} a_{4,\jmath,\varkappa} + \dot{\theta} b_{4,\jmath,\varkappa}$$
$$+ \dot{\psi} c_{4,\jmath,\varkappa} + \dot{\theta}_{2,\jmath} d_{4,\jmath,\varkappa} + \dot{\theta}_{3,\jmath}$$
$$* e_{4,\jmath,\varkappa} + \dot{\theta}_{4,\jmath} f_{4,\jmath,\varkappa}]$$

$$\frac{\partial (\dot{R}_{4,\jmath,y})^2}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial (\dot{R}_{4,\jmath,z})^2}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial \omega^2_{\varkappa,o}}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial \omega^2_{y,o}}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial \omega^2_{z,o}}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial (\omega_{1,\jmath,\varkappa})^2}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial (\omega_{1,\jmath,y})^2}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial (\omega_{1,\jmath,z})^2}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial (\omega_{2,\jmath,\varkappa})^2}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial (\omega_{2,\jmath,y})^2}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial (\omega_{2,\jmath,z})^2}{\partial \dot{P}_\varkappa} = 0 \qquad\qquad \frac{\partial (\omega_{3,\jmath,\varkappa})^2}{\partial \dot{P}_\varkappa} = 0$$

$$\frac{\partial(\omega_{3,i,y})^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial(\omega_{3,i,z})^2}{\partial \dot{P}_x} = 0$$

A1.12
$$\frac{\partial \dot{P}_x}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial \dot{P}_x{}^2}{\partial \dot{P}_y} = 0$$

$$\frac{\partial \dot{P}_y{}^2}{\partial \dot{P}_y} = 2\dot{P}_y \qquad\qquad \frac{\partial \dot{P}_z{}^2}{\partial \dot{P}_y} = 0$$

$$\frac{\partial R_{2,i,x}}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial R_{3,i,x}}{\partial \dot{P}_y} = 0$$

$$\frac{\partial R_{4,i,x}}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial(\dot{R}_{2,i,x})^2}{\partial \dot{P}_y} = 0$$

$$\frac{\partial(\dot{R}_{2,i,y})^2}{\partial \dot{P}_y} = 2[\dot{P}_y + \dot{\phi}a_{2,i,y} + \dot{\theta}b_{2,i,y} + \dot{\psi}c_{2,i,y} + \dot{\theta}_{2,i}d_{2,i,y}]$$

$$\frac{\partial(\dot{R}_{2,i,z})^2}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial(\dot{R}_{3,i,x})^2}{\partial \dot{P}_y} = 0$$

$$\frac{\partial(\dot{R}_{3,i,y})^2}{\partial \dot{P}_y} = 2[\dot{P}_y + \dot{\phi}a_{3,i,y} + \dot{\theta}b_{3,i,y} + \dot{\psi}c_{3,i,y} + \dot{\theta}_{2,i}d_{3,i,y} + \dot{\theta}_{3,i}e_{3,i,y}]$$

$$\frac{\partial(\dot{R}_{3,i,z})^2}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial(\dot{R}_{4,i,x})^2}{\partial \dot{P}_y} = 0$$

$$\frac{\partial(\dot{R}_{4,i,y})^2}{\partial \dot{P}_y} = 2[\dot{P}_y + \dot{\phi}a_{4,i,y} + \dot{\theta}b_{4,i,y} + \dot{\psi}c_{4,i,y} + \dot{\theta}_{2,i}d_{4,i,y} + \dot{\theta}_{3,i}$$
$$*e_{4,i,y} + \dot{\theta}_{4,i}f_{4,i,y}]$$

$$\frac{\partial(\dot{R}_{4,i,z})^2}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial\omega^2_{x,o}}{\partial \dot{P}_y} = 0$$

$$\frac{\partial\omega^2_{y,o}}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial\omega^2_{z,o}}{\partial \dot{P}_y} = 0$$

$$\frac{\partial(\omega_{1,i,x})^2}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial(\omega_{1,i,y})^2}{\partial \dot{P}_y} = 0$$

$$\frac{\partial(\omega_{1,i,z})^2}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial(\omega_{2,i,x})^2}{\partial \dot{P}_y} = 0$$

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$$\frac{\partial(\omega_{2,j,y})^2}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial(\omega_{2,j,z})^2}{\partial \dot{P}_y} = 0$$

$$\frac{\partial(\omega_{3,j,x})^2}{\partial \dot{P}_y} = 0 \qquad\qquad \frac{\partial(\omega_{3,j,y})^2}{\partial \dot{P}_y} = 0$$

$$\frac{\partial(\omega_{3,j,z})^2}{\partial \dot{P}_y} = 0$$

A1.13
$$\frac{\partial P_x}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial \dot{P}_x^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial \dot{P}_y^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial \dot{P}_x^2}{\partial \dot{P}_x} = 2\dot{P}_x$$

$$\frac{\partial R_{2,j,x}}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial R_{3,j,x}}{\partial \dot{P}_x} = 0$$

$$\frac{\partial R_{4,j,x}}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial(\dot{R}_{2,j,x})^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial(\dot{R}_{2,j,y})^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial(\dot{R}_{2,j,x})^2}{\partial \dot{P}_x} = 2[\dot{P}_x + \dot{\theta}b_{2,j,x} + \dot{\psi}c_{2,j,x} + \dot{\theta}_{2,j}d_{2,j,x}]$$

$$\frac{\partial(\dot{R}_{3,j,x})^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial(\dot{R}_{3,j,y})^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial(\dot{R}_{3,j,z})^2}{\partial \dot{P}_x} = 2[\dot{P}_x + \dot{\theta}b_{3,j,x} + \dot{\psi}c_{3,j,x} + \dot{\theta}_{2,j}d_{3,j,x}$$
$$+ \dot{\theta}_{3,j}e_{3,j,x}$$

$$\frac{\partial(\dot{R}_{4,j,x})^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial(\dot{R}_{4,j,y})^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial(\dot{R}_{4,j,x})^2}{\partial \dot{P}_x} = 2[\dot{P}_x + \dot{\theta}b_{4,j,x} + \dot{\psi}c_{4,j,x}$$
$$+ \dot{\theta}_{2,j}d_{4,j,x} + \dot{\theta}_{3,j}e_{3,j,x}$$
$$+ \dot{\theta}_{4,j}f_{4,j,x}]$$

$$\frac{\partial\omega_{x,0}^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial\omega_{y,0}^2}{\partial \dot{P}_x} = 0$$

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$$\frac{\partial \omega^2_{z,0}}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial (\omega_{1,1,x})^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial (\omega_{1,1,y})^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial (\omega_{1,1,z})^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial (\omega_{2,1,x})^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial (\omega_{2,1,y})^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial (\omega_{2,1,z})^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial (\omega_{3,1,x})^2}{\partial \dot{P}_x} = 0$$

$$\frac{\partial (\omega_{3,1,y})^2}{\partial \dot{P}_x} = 0 \qquad\qquad \frac{\partial (\omega_{3,1,z})^2}{\partial \dot{P}_x} = 0$$

A1.14
$$\frac{\partial P_x}{\partial \phi} = 0 \qquad\qquad \frac{\partial \dot{P}_x^2}{\partial \phi} = 0$$

$$\frac{\partial \dot{P}_y^2}{\partial \phi} = 0 \qquad\qquad \frac{\partial \dot{P}_z^2}{\partial \phi} = 0$$

$$\frac{\partial R_{2,1,x}}{\partial \phi} = 0 \qquad\qquad \frac{\partial R_{3,1,x}}{\partial \phi} = 0$$

$$\frac{\partial R_{4,1,x}}{\partial \dot{\phi}} = 0$$

$$\frac{\partial (\dot{R}_{2,1,x})^2}{\partial \dot{\phi}} = 2a_{2,1,x}[\dot{\phi}a_{2,1,x}+\dot{\theta}b_{2,1,x}$$
$$+\dot{\psi}c_{2,1,x}+\dot{\theta}_{2,1}d_{2,1,x}+\dot{P}_x]$$

$$\frac{\partial (\dot{R}_{2,1,y})^2}{\partial \dot{\phi}} = 2a_{2,1,y}[\dot{\phi}a_{2,1,y}+\dot{\theta}b_{2,1,y}$$
$$+\dot{\psi}c_{2,1,y}+\dot{\theta}_{2,1}d_{2,1,y}+\dot{P}_y]$$

$$\frac{\partial (\dot{R}_{2,1,z})^2}{\partial \dot{\phi}} = 0$$

$$\frac{\partial (\dot{R}_{3,1,x})^2}{\partial \dot{\phi}} = 2a_{3,1,x}[\dot{\phi}a_{3,1,x}+\dot{\theta}b_{3,1,x}$$
$$+\dot{\psi}c_{3,1,x}+\dot{\theta}_{2,1}d_{3,1,x}$$
$$+\dot{\theta}_{3,1}e_{3,1,x}+\dot{P}_x]$$

$$\frac{\partial (\dot{R}_{3,1,y})^2}{\partial \dot{\phi}} = 2a_{3,1,y}[\dot{\phi}a_{3,1,y}+\dot{\theta}b_{3,1,y}$$
$$+\dot{\psi}c_{3,1,y}+\dot{\theta}_{2,1}d_{3,1,y}$$
$$+\dot{\theta}_{3,1}e_{3,1,y}+\dot{P}_y]$$

$$\frac{\partial(\dot{R}_{5,1,x})^2}{\partial\phi} = 0$$

$$\frac{\partial(\dot{R}_{4,1,x})^2}{\partial\phi} = 2a_{4,1,x}[\dot{\phi}a_{4,1,x}+\dot{\theta}b_{4,1,x}$$
$$+\dot{\psi}c_{4,1,x}+\dot{\theta}_{2,1}d_{4,1,x}$$
$$+\dot{\theta}_{3,1}e_{4,1,x}+\dot{\theta}_{4,1}f_{4,1,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{4,1,y})^2}{\partial\phi} = 2a_{4,1,y}[\dot{\phi}a_{4,1,y}+\dot{\theta}b_{4,1,y}$$
$$+\dot{\psi}c_{3,1,y}+\dot{\theta}_{2,1}d_{4,1,y}$$
$$+\dot{\theta}_{3,1}e_{4,1,x}+\dot{\theta}_{4,1}f_{4,1,y}+\dot{P}_y]$$

$$\frac{\partial(\dot{R}_{4,1,z})^2}{\partial\phi} = 0$$

$$\frac{\partial\omega^2_{x,0}}{\partial\phi} = 2S(\theta)S(\psi)[\dot{\phi}S(\theta)S(\psi)+\dot{\theta}C(\psi)]$$

$$\frac{\partial\omega^2_{y,0}}{\partial\phi} = 2S(\theta)C(\psi)[\dot{\phi}S(\theta)C(\psi)-\dot{\theta}S(\psi)]$$

$$\frac{\partial\omega^2_{z,0}}{\partial\phi} = 2C(\theta)[\dot{\phi}C(\theta)+\dot{\psi}]$$

$$\frac{\partial(\omega_{1,1,x})^2}{\partial\phi} = 2S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*[\dot{\phi}S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})$$
$$+\dot{\theta}C(\psi+\theta_{1,1}+\theta_{2,1})]$$

$$\frac{\partial(\omega_{1,1,y})^2}{\partial\phi} = 2S(\theta)C(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*[\dot{\phi}S(\theta)C(\psi+\theta_{1,1}+\theta_{2,1})$$
$$-\dot{\theta}S(\psi+\theta_{1,1}+\theta_{2,1})]$$

$$\frac{\partial(\omega_{1,1,z})^2}{\partial\phi} = 2C(\theta)[\dot{\phi}C(\theta)+\dot{\psi}]$$

$$\frac{\partial(\omega_{2,1,x})^2}{\partial\phi} = 2[S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})C(\theta_{3,1})$$
$$-C(\theta)S(\theta_{3,1})]$$
$$*\{\dot{\phi}[S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*C(\theta_{3,1})-C(\theta)S(\theta_{3,1})]$$
$$+\dot{\theta}C(\psi+\theta_{1,1}+\theta_{2,1})C(\theta_{3,1})$$

$$-\dot{\psi}S(\theta_3,_4)-\dot{\theta}_2,_4 S(\theta_3,_4)\}$$

$$\frac{\partial(\omega_2,_4,_Y)^2}{\partial\phi} = 2[S(\theta)S(\psi+\theta_1,_4+\theta_2,_4)S(\theta_3,_4)$$
$$+C(\theta)C(\theta_3,_4)]$$
$$*\{\dot{\phi}[S(\theta)S(\psi+\theta_1,_4+\theta_2,_4)$$
$$*S(\theta_3,_4)+C(\theta)C(\theta_3,_4)]$$
$$+\dot{\theta}C(\psi+\theta_1,_4+\theta_2,_4)S(\theta_3,_4)$$
$$+\dot{\psi}C(\theta_3,_4)+\dot{\theta}_2,_4 C(\theta_3,_4)\}$$

$$\frac{\partial(\omega_2,_4,_z)^2}{\partial\phi} = 2S(\theta)C(\psi+\theta_1,_4+\theta_2,_4)$$
$$*[\dot{\phi}S(\theta)C(\psi+\theta_1,_4+\theta_2,_4)$$
$$-\dot{\theta}S(\psi+\theta_1,_4+\theta_2,_4)+\dot{\theta}_3,_4$$

$$\frac{\partial(\omega_3,_4,_x)^2}{\partial\phi} = 2[S(\theta)S(\psi+\theta_1,_4+\theta_2,_4)C(\theta_3,_4+\theta_4,_4)$$
$$-C(\theta)S(\theta_3,_4+\theta_4,_4)]$$
$$*\{\dot{\phi}[S(\theta)S(\psi+\theta_1,_4+\theta_2,_4)$$
$$*C(\theta_3,_4+\theta_4,_4)$$
$$-C(\theta)S(\theta_3,_4+\theta_4,_4)]$$
$$+\dot{\theta}C(\psi+\theta_1,_4+\theta_2,_4)C(\theta_3,_4+\theta_4,_4)$$
$$-\dot{\psi}S(\theta_3,_4+\theta_4,_4)-\dot{\theta}_2,_4 S(\theta_3,_4+\theta_4,_4)\}$$

$$\frac{\partial(\omega_3,_4,_Y)^2}{\partial\phi} = 2[S(\theta)S(\psi+\theta_1,_4+\theta_2,_4)S(\theta_3,_4+\theta_4,_4)$$
$$+C(\theta)C(\theta_3,_4+\theta_4,_4)]$$
$$*\{\dot{\phi}[S(\theta)S(\psi+\theta_1,_4+\theta_2,_4)$$
$$*S(\theta_3,_4+\theta_4,_4)$$
$$+C(\theta)C(\theta_3,_4+\theta_4,_4)]$$
$$+\dot{\theta}C(\psi+\theta_1,_4+\theta_2,_4)S(\theta_3,_4+\theta_4,_4)$$
$$+\dot{\psi}C(\theta_3,_4+\theta_4,_4)+\dot{\theta}_2,_4 C(\theta_3,_4+\theta_4,_4)\}$$

$$\frac{\partial(\omega_3,_4,_z)^2}{\partial\phi} = 2S(\theta)C(\psi+\theta_1,_4+\theta_2,_4)$$

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$$*[\dot{\phi}S(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$-\dot{\theta}S(\psi+\theta_{1,J}+\theta_{2,J})+\dot{\theta}_{3,J}+\dot{\theta}_{4,J}$$

A1.15

$$\frac{\partial P_x}{\partial \dot{\theta}} = 0 \qquad\qquad \frac{\partial \dot{P}_x^2}{\partial \dot{\theta}} = 0$$

$$\frac{\partial \dot{P}_y^2}{\partial \dot{\theta}} = 0 \qquad\qquad \frac{\partial \dot{P}_z^2}{\partial \dot{\theta}} = 0$$

$$\frac{\partial R_{2,J,x}}{\partial \dot{\theta}} = 0 \qquad\qquad \frac{\partial R_{3,J,x}}{\partial \dot{\theta}} = 0$$

$$\frac{\partial R_{4,J,x}}{\partial \dot{\theta}} = 0$$

$$\frac{\partial (\dot{R}_{2,J,x})^2}{\partial \dot{\theta}} = 2b_{2,J,x}[\dot{\theta}b_{2,J,x}+\dot{\phi}a_{2,J,x}$$
$$+\dot{\psi}c_{2,J,x}+\dot{\theta}_{2,J}d_{2,J,x}+\dot{P}_x]$$

$$\frac{\partial (\dot{R}_{2,J,y})^2}{\partial \dot{\theta}} = 2b_{2,J,y}[\dot{\theta}b_{2,J,y}+\dot{\phi}a_{2,J,y}$$
$$+\dot{\psi}c_{2,J,y}+\dot{\theta}_{2,J}d_{2,J,y}+\dot{P}_y]$$

$$\frac{\partial (\dot{R}_{2,J,z})^2}{\partial \dot{\theta}} = 2b_{2,J,z}[\dot{\theta}b_{2,J,z}+\dot{\psi}c_{2,J,z}$$
$$+\dot{\theta}_{2,J}d_{2,J,z}+\dot{P}_z]$$

$$\frac{\partial (\dot{R}_{3,J,x})^2}{\partial \dot{\theta}} = 2b_{3,J,x}[\dot{\theta}b_{3,J,x}+\dot{\phi}a_{3,J,x}$$
$$+\dot{\psi}c_{3,J,x}+\dot{\theta}_{2,J}d_{3,J,x}$$
$$+\dot{\theta}_{3,J}e_{4,J,x}+\dot{P}_x]$$

$$\frac{\partial (\dot{R}_{3,J,y})^2}{\partial \dot{\theta}} = 2b_{3,J,y}[\dot{\theta}b_{3,J,y}+\dot{\phi}a_{3,J,y}$$
$$+\dot{\psi}c_{3,J,y}+\dot{\theta}_{2,J}d_{3,J,y}$$
$$+\dot{\theta}_{3,J}e_{3,J,y}+\dot{P}_y]$$

$$\frac{\partial (\dot{R}_{3,J,z})^2}{\partial \dot{\theta}} = 2b_{3,J,z}[\dot{\theta}b_{3,J,z}+\dot{\psi}c_{3,J,z}$$
$$+\dot{\theta}_{2,J}d_{3,J,z}+\dot{\theta}_{3,J}e_{3,J,z}+\dot{P}_z]$$

$$\frac{\partial (\dot{R}_{4,J,x})^2}{\partial \dot{\theta}} = 2b_{4,J,x}[\dot{\theta}b_{4,J,x}+\dot{\phi}a_{4,J,x}$$
$$+\dot{\psi}c_{4,J,x}+\dot{\theta}_{2,J}d_{4,J,x}$$
$$+\dot{\theta}_{3,J}e_{4,J,x}+\dot{\theta}_{4,J}f_{4,J,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{4,J,Y})^2}{\partial\dot{\theta}} = 2b_{4,J,Y}[\dot{\theta}b_{4,J,Y}+\dot{\phi}a_{4,J,Y}$$
$$+\dot{\psi}c_{4,J,Y}+\dot{\theta}_{2,J}d_{4,J,Y}$$
$$+\dot{\theta}_{3,J}e_{4,J,Y}+\dot{\theta}_{4,J}f_{4,J,Y}+\dot{P}_Y]$$

$$\frac{\partial(\dot{R}_{4,J,Z})^2}{\partial\dot{\theta}} = 2b_{4,J,Z}[\dot{\theta}b_{4,J,Z}+\dot{\psi}c_{4,J,Z}$$
$$+\dot{\theta}_{2,J}d_{4,J,Z}+\dot{\theta}_{3,J}e_{4,J,Z}$$
$$+\dot{\theta}_{4,J}f_{4,J,Z}+\dot{P}_Z]$$

$$\frac{\partial\omega^2_{X,0}}{\partial\dot{\theta}} = 2C(\psi)[\dot{\theta}C(\psi)+\dot{\phi}S(\theta)S(\psi)]$$

$$\frac{\partial\omega^2_{Y,0}}{\partial\dot{\theta}} = 2S(\psi)[\dot{\theta}S(\psi)-\dot{\phi}S(\theta)C(\psi)]$$

$$\frac{\partial\omega^2_{Z,0}}{\partial\dot{\theta}} = 0$$

$$\frac{\partial(\omega_{1,J,X})^2}{\partial\dot{\theta}} = 2C(\psi+\theta_{1,J}+\theta_{2,J})[\dot{\theta}C(\psi+\theta_{1,J}+\theta_{2,J})$$
$$+\dot{\phi}S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})]$$

$$\frac{\partial(\omega_{1,J,Y})^2}{\partial\dot{\theta}} = 2S(\psi+\theta_{1,J}+\theta_{2,J})[\dot{\theta}S(\psi+\theta_{1,J}+\theta_{2,J})$$
$$+\dot{\phi}S(\theta)C(\psi+\theta_{1,J}+\theta_{2,J})]$$

$$\frac{\partial(\omega_{1,J,Z})^2}{\partial\dot{\theta}} = 0$$

$$\frac{\partial(\omega_{2,J,X})^2}{\partial\dot{\theta}} = 2C(\psi+\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})$$
$$*\{\dot{\theta}C(\psi+\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})$$
$$+\dot{\phi}[S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})$$
$$-C(\theta)S(\theta_{3,J})]$$
$$-\dot{\psi}S(\theta_{3,J})-\dot{\theta}_{2,J}S(\theta_{3,J})\}$$

$$\frac{\partial(\omega_{2,J,Y})^2}{\partial\dot{\theta}} = 2C(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})$$
$$*\{\dot{\theta}C(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})$$
$$+\dot{\phi}[S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})$$
$$+C(\theta)C(\theta_{3,J})]$$
$$+\dot{\psi}C(\theta_{3,J})+\dot{\theta}_{2,J}C(\theta_{3,J})\}$$

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$$\frac{\partial(\omega_{2,i,x})^2}{\partial\theta} = 2S(\psi+\theta_{1,i}+\theta_{2,i})[\dot{\theta}S(\psi+\theta_{1,i}+\theta_{2,i})$$
$$-\dot{\phi}S(\theta)C(\psi+\theta_{1,i}+\theta_{2,i})-\dot{\theta}_{3,i}]$$

$$\frac{\partial(\omega_{2,i,x})^2}{\partial\theta} = 2C(\psi+\theta_{1,i}+\theta_{2,i})C(\theta_{3,i}+\theta_{4,i})$$
$$*\{\dot{\theta}C(\psi+\theta_{1,i}+\theta_{2,i})C(\theta_{3,i}+\theta_{4,i})$$
$$+\dot{\phi}[S(\theta)S(\psi+\theta_{1,i}+\theta_{2,i})C(\theta_{3,i}+\theta_{4,i})$$
$$-C(\theta)S(\theta_{3,i}+\theta_{4,i})]$$
$$-\dot{\psi}S(\theta_{3,i}+\theta_{4,i})-\dot{\theta}_{2,i}S(\theta_{3,i}+\theta_{4,i})\}$$

$$\frac{\partial(\omega_{3,i,y})^2}{\partial\theta} = 2C(\psi+\theta_{1,i}+\theta_{2,i})S(\theta_{3,i}+\theta_{4,i})$$
$$*\{\dot{\theta}C(\psi+\theta_{1,i}+\theta_{2,i})S(\theta_{3,i}+\theta_{4,i})$$
$$+\dot{\phi}[S(\theta)S(\psi+\theta_{1,i}+\theta_{2,i})S(\theta_{3,i}+\theta_{4,i})$$
$$+C(\theta)C(\theta_{3,i}+\theta_{4,i})]$$
$$+\dot{\psi}C(\theta_{3,i}+\theta_{4,i})+\dot{\theta}_{2,i}C(\theta_{3,i}+\theta_{4,i})\}$$

$$\frac{\partial(\omega_{3,i,x})^2}{\partial\theta} = 2S(\psi+\theta_{1,i}+\theta_{2,i})[\dot{\theta}S(\psi+\theta_{1,i}+\theta_{2,i})$$
$$-\dot{\phi}S(\theta)C(\psi+\theta_{1,i}+\theta_{2,i})$$
$$-\dot{\theta}_{3,i}-\dot{\theta}_{4,i}$$

A1.16

$$\frac{\partial\dot{P}_x}{\partial\psi} = 0 \qquad\qquad \frac{\partial\dot{P}_x^2}{\partial\psi} = 0$$

$$\frac{\partial\dot{P}_y^2}{\partial\psi} = 0 \qquad\qquad \frac{\partial\dot{P}_x^2}{\partial\psi} = 0$$

$$\frac{\partial R_{2,i,x}}{\partial\psi} = 0 \qquad\qquad \frac{\partial R_{2,i,x}}{\partial\psi} = 0$$

$$\frac{\partial R_{4,i,x}}{\partial\psi} = 0$$

$$\frac{\partial(\dot{R}_{2,i,x})^2}{\partial\psi} = 2c_{2,i,x}[\dot{\psi}c_{2,i,x}+\dot{\phi}a_{2,i,x}$$
$$+\dot{\theta}b_{2,i,x}+\dot{\theta}_{2,i}d_{2,i,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{2,i,y})^2}{\partial\psi} = 2c_{2,i,y}[\dot{\psi}c_{2,i,y}+\dot{\phi}a_{2,i,y}$$
$$+\dot{\theta}b_{2,i,y}+\dot{\theta}_{2,i}d_{2,i,y}+\dot{P}_y]$$

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$$\frac{\partial (\dot{R}_{2,J,z})^2}{\partial \dot{\psi}} = 2c_{2,J,z}[\dot{\psi}c_{2,J,z} + \dot{\theta}b_{2,J,z} + \dot{\theta}_{2,J}d_{2,J,z} + \dot{P}_z]$$

$$\frac{\partial (\dot{R}_{3,J,x})^2}{\partial \dot{\psi}} = 2c_{3,J,x}[\dot{\psi}c_{3,J,x} + \dot{\phi}a_{3,J,x} + \dot{\theta}b_{3,J,x} + \dot{\theta}_{2,J}d_{3,J,x} + \dot{\theta}_{3,J}e_{3,J,x} + \dot{P}_x]$$

$$\frac{\partial (\dot{R}_{3,J,y})^2}{\partial \dot{\psi}} = 2c_{3,J,y}[\dot{\psi}c_{3,J,y} + \dot{\phi}a_{3,J,y} + \dot{\theta}b_{3,J,y} + \dot{\theta}_{2,J}d_{3,J,y} + \dot{\theta}_{3,J}e_{3,J,y} + \dot{P}_y]$$

$$\frac{\partial (\dot{R}_{3,J,z})^2}{\partial \dot{\psi}} = 2c_{3,J,z}[\dot{\psi}c_{3,J,z} + \dot{\theta}b_{3,J,z} + \dot{\theta}_{2,J}d_{3,J,z} + \dot{\theta}_{3,J}e_{3,J,z} + \dot{P}_z]$$

$$\frac{\partial (\dot{R}_{4,J,x})^2}{\partial \dot{\psi}} = 2c_{4,J,x}[\dot{\psi}c_{4,J,x} + \dot{\phi}a_{4,J,x} + \dot{\theta}b_{4,J,x} + \dot{\theta}_{2,J}d_{4,J,x} + \dot{\theta}_{3,J}e_{4,J,x} + \dot{\theta}_{4,J}f_{4,J,x} + \dot{P}_x]$$

$$\frac{\partial (\dot{R}_{4,J,y})^2}{\partial \dot{\psi}} = 2c_{4,J,y}[\dot{\psi}c_{4,J,y} + \dot{\phi}a_{4,J,y} + \dot{\theta}b_{4,J,y} + \dot{\theta}_{2,J}d_{4,J,y} + \dot{\theta}_{3,J}e_{4,J,x} + \dot{\theta}_{4,J}f_{4,J,y} + \dot{P}_y]$$

$$\frac{\partial (\dot{R}_{4,J,z})^2}{\partial \dot{\psi}} = 2c_{4,J,z}[\dot{\psi}c_{4,J,z} + \dot{\theta}b_{4,J,z} + \dot{\theta}_{2,J}d_{4,J,z} + \dot{\theta}_{3,J}e_{4,J,z} + \dot{\theta}_{4,J}f_{4,J,z} + \dot{P}_z]$$

$$\frac{\partial \omega^2_{x,0}}{\partial \dot{\psi}} = 0 \qquad\qquad \frac{\partial \omega^2_{y,0}}{\partial \dot{\psi}} = 0$$

$$\frac{\partial \omega^2_{z,0}}{\partial \dot{\psi}} = 2[\dot{\psi} + \dot{\phi}C(\theta)]$$

$$\frac{\partial (\omega_{1,J,x})^2}{\partial \dot{\psi}} = 0 \qquad\qquad \frac{\partial (\omega_{1,J,y})^2}{\partial \dot{\psi}} = 0$$

$$\frac{\partial(\omega_{1,1,x})^2}{\partial\dot\psi} = 2[\dot\psi+\dot\phi C(\theta)+\dot\theta_{2,1}]$$

$$\frac{\partial(\omega_{2,1,x})^2}{\partial\dot\psi} = 2S(\theta_3,_1)\{\dot\psi S(\theta_3,_1)$$
$$-\dot\phi[S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})C(\theta_3,_1)$$
$$-C(\theta)S(\theta_3,_1)]-\dot\theta C(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*C(\theta_3,_1)+\dot\theta_{2,1}S(\theta_3,_1)\}$$

$$\frac{\partial(\omega_{2,1,y})^2}{\partial\dot\psi} = 2C(\theta_3,_1)\{\dot\psi C(\theta_3,_1)$$
$$+\dot\phi[S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})S(\theta_3,_1)$$
$$+C(\theta)C(\theta_3,_1)]+\dot\theta C(\psi+\theta_{1,1}+\theta_{2,1})$$
$$*S(\theta_3,_1)+\dot\theta_{2,1}C(\theta_3,_1)\}$$

$$\frac{\partial(\omega_{2,1,z})^2}{\partial\dot\psi} = 0$$

$$\frac{\partial(\omega_{3,1,x})^2}{\partial\dot\psi} = 2S(\theta_3,_1+\theta_4,_1)\{\dot\psi S(\theta_3,_1+\theta_4,_1)$$
$$-\dot\phi[S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})C(\theta_3,_1+\theta_4,_1)$$
$$-C(\theta)S(\theta_3,_1+\theta_4,_1)]$$
$$-\dot\theta C(\psi+\theta_{1,1}+\theta_{2,1})C(_3,_1+\theta_4,_1)$$
$$+\dot\theta_{2,1}S(\theta_3,_1+\theta_4,_1)\}$$

$$\frac{\partial(\omega_{3,1,y})^2}{\partial\dot\psi} = 2C(\theta_3,_1+\theta_4,_1)\{\dot\psi C(\theta_3,_1+\theta_4,_1)$$
$$+\dot\phi[S(\theta)S(\psi+\theta_{1,1}+\theta_{2,1})S(\theta_3,_1+\theta_4,_1)$$
$$+C(\theta)C(\theta_3,_1+\theta_4,_1)]$$
$$+\dot\theta C(\psi+\theta_{1,1}+\theta_{2,1})S(\theta_3,_1+\theta_4,_1)$$
$$+\dot\theta_{2,1}C(\theta_3,_1+\theta_4,_1)\}$$

$$\frac{\partial(\omega_{3,1,z})^2}{\partial\dot\psi} = 0$$

A1.17

$$\frac{\partial\dot P_x}{\partial\dot\theta_{2,1}} = 0 \qquad\qquad \frac{\partial\dot P_x^2}{\partial\dot\theta_{2,1}} = 0$$

$$\frac{\partial\dot P_y^2}{\partial\dot\theta_{2,1}} = 0 \qquad\qquad \frac{\partial\dot P_z^2}{\partial\dot\theta_{2,1}} = 0$$

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$$\frac{\partial R_{2,j,x}}{\partial \theta_{2,j}} = 0 \qquad\qquad \frac{\partial R_{3,j,x}}{\partial \theta_{2,j}} = 0$$

$$\frac{\partial R_{4,j,x}}{\partial \theta_{2,j}} = 0$$

$$\frac{\partial(\dot{R}_{2,j,x})^2}{\partial \theta_{2,j}} = 2d_{2,j,x}[\dot{\theta}_{2,j}d_{2,j,x}+\dot{\phi}a_{2,j,x}$$
$$+\dot{\theta}b_{2,j,x}+\dot{\psi}c_{2,j,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{2,j,y})^2}{\partial \theta_{2,j}} = 2d_{2,j,y}[\dot{\theta}_{2,j}d_{2,j,y}+\dot{\phi}a_{2,j,y}$$
$$+\dot{\theta}b_{2,j,y}+\dot{\psi}c_{2,j,y}+\dot{P}_y]$$

$$\frac{\partial(\dot{R}_{2,j,z})^2}{\partial \theta_{2,j}} = 2d_{2,j,z}[\dot{\theta}_{2,j}d_{2,j,z}+\dot{\theta}b_{2,j,z}$$
$$+\dot{\psi}c_{2,j,z}+\dot{P}_z]$$

$$\frac{\partial(\dot{R}_{3,j,x})^2}{\partial \theta_{2,j}} = 2d_{3,j,x}[\dot{\theta}_{2,j}d_{3,j,x}+\dot{\phi}a_{3,j,x}$$
$$+\dot{\theta}b_{3,j,x}+\dot{\psi}c_{3,j,x}$$
$$+\dot{\theta}_{3,j}e_{3,j,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{3,j,y})^2}{\partial \theta_{2,j}} = 2d_{3,j,y}[\dot{\theta}_{2,j}d_{3,j,y}+\dot{\phi}a_{3,j,y}$$
$$+\dot{\theta}b_{3,j,y}+\dot{\psi}c_{3,j,y}$$
$$+\dot{\theta}_{3,j}e_{3,j,y}+\dot{P}_y]$$

$$\frac{\partial(\dot{R}_{3,j,z})^2}{\partial \theta_{2,j}} = 2d_{3,j,z}[\dot{\theta}_{2,j}d_{3,j,z}+\dot{\theta}b_{3,j,z}$$
$$+\dot{\psi}c_{3,j,z}+\dot{\theta}_{3,j}e_{3,j,z}+\dot{P}_z]$$

$$\frac{\partial(\dot{R}_{4,j,x})^2}{\partial \theta_{2,j}} = 2d_{4,j,x}[\dot{\theta}_{2,j}d_{4,j,x}+\dot{\phi}a_{4,j,x}$$
$$+\dot{\theta}b_{4,j,x}+\dot{\psi}c_{4,j,x}$$
$$+\dot{\theta}_{3,j}e_{4,j,x}+\dot{\theta}_{4,j}f_{4,j,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{4,j,y})^2}{\partial \theta_{2,j}} = 2d_{4,j,y}[\dot{\theta}_{2,j}d_{4,j,y}+\dot{\phi}a_{4,j,y}$$
$$+\dot{\theta}b_{4,j,y}+\dot{\psi}c_{4,j,y}$$
$$+\dot{\theta}_{3,j}e_{4,j,x}+\dot{\theta}_{4,j}f_{4,j,y}+\dot{P}_y]$$

$$\frac{\partial(\dot{R}_{4,j,z})^2}{\partial \theta_{2,j}} = 2d_{4,j,z}[\dot{\theta}_{2,j}d_{4,j,z}+\dot{\theta}b_{4,j,z}$$
$$+\dot{\psi}c_{4,j,z}+\dot{\theta}_{3,j}e_{4,j,z}$$
$$+\dot{\theta}_{4,j}f_{4,j,z}+\dot{P}_z]$$

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$$\frac{\partial \omega^2_{x,0}}{\partial \dot{\theta}_{2,J}} = 0 \qquad\qquad \frac{\partial \omega^2_{y,0}}{\partial \dot{\theta}_{2,J}} = 0$$

$$\frac{\partial \omega^2_{z,0}}{\partial \dot{\theta}_{2,J}} = 0 \qquad\qquad \frac{\partial (\omega_{1,J,x})^2}{\partial \dot{\theta}_{2,J}} = 0$$

$$\frac{\partial (\omega_{1,J,y})^2}{\partial \dot{\theta}_{2,J}} = 0$$

$$\frac{\partial (\omega_{1,J,z})^2}{\partial \dot{\theta}_{2,J}} = 2[\dot{\theta}_{2,J} + \dot{\phi}C(\theta) + \dot{\psi}]$$

$$\frac{\partial (\omega_{2,J,x})^2}{\partial \dot{\theta}_{2,J}} = 2S(\theta_{3,J})\{\dot{\theta}_{2,J}S(\theta_{3,J})$$
$$-\dot{\phi}[S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})$$
$$-C(\theta)S(\theta_{3,J})]$$
$$-\dot{\theta}C(\psi+\theta_{1,J}+\theta_{2,J})C(\theta_{3,J})$$
$$+\dot{\psi}S(\theta_{3,J})\}$$

$$\frac{\partial (\omega_{2,J,y})^2}{\partial \dot{\theta}_{2,J}} = 2C(\theta_{3,J})\{\dot{\theta}_{2,J}C(\theta_{3,J})$$
$$+\dot{\phi}[S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})$$
$$+C(\theta)C(\theta_{3,J})]$$
$$+\dot{\theta}C(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J})$$
$$+\dot{\psi}C(\theta_{3,J})\}$$

$$\frac{\partial (\omega_{2,J,z})^2}{\partial \dot{\theta}_{2,J}} = 0$$

$$\frac{\partial (\omega_{3,J,x})^2}{\partial \dot{\theta}_{2,J}} = 2S(\theta_{3,J}+\theta_{4,J})\{\dot{\theta}_{2,J}S(\theta_{3,J}+\theta_{4,J})$$
$$-\dot{\phi}[S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})C(\theta_{3,J}+\theta_{4,J})$$
$$-C(\theta)S(\theta_{3,J}+_{4,J})]$$
$$-\dot{\theta}C(\psi+\theta_{1,J}+\theta_{2,J})C(\theta_{3,J}+\theta_{4,J})$$
$$+\dot{\psi}S(\theta_{3,J}+\theta_{4,J})\}$$

$$\frac{\partial (\omega_{3,J,y})^2}{\partial \dot{\theta}_{2,J}} = 2C(\theta_{3,J}+\theta_{4,J})\{\dot{\theta}_{2,J}C(\theta_{3,J}+\theta_{4,J})$$
$$+\dot{\phi}[S(\theta)S(\psi+\theta_{1,J}+\theta_{2,J})S(\theta_{3,J}+\theta_{4,J})$$
$$+C(\theta)C(\theta_{3,J}+_{4,J})]$$

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$$+\dot{\theta}C(\psi+\theta_{1,j}+\theta_{2,j})S(\theta_{3,j}+\theta_{4,j})$$

$$+\dot{\psi}C(\theta_{3,j}+\theta_{4,j})\}$$

$$\frac{\partial(\omega_{3,j,z})^2}{\partial\dot{\theta}_{2,j}} = 0$$

A1.18

$$\frac{\partial\dot{P}_z}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial\dot{P}_x^2}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial\dot{P}_y^2}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial\dot{P}_z^2}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial R_{2,j,x}}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial R_{3,j,x}}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial R_{4,j,z}}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial(\dot{R}_{2,j,x})^2}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial(\dot{R}_{2,j,y})^2}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial(\dot{R}_{2,j,z})^2}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial(\dot{R}_{3,j,x})^2}{\partial\dot{\theta}_{3,j}} = 2e_{3,j,x}[\dot{\theta}_{3,j}e_{3,j,x}+\dot{\phi}a_{3,j,x}$$
$$+\dot{\theta}b_{3,j,x}+\dot{\psi}c_{3,j,x}$$
$$+\dot{\theta}_{2,j}d_{3,j,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{3,j,y})^2}{\partial\dot{\theta}_{3,j}} = 2e_{3,j,y}[\dot{\theta}_{3,j}e_{3,j,y}+\dot{\phi}a_{3,j,y}$$
$$+\dot{\theta}b_{3,j,y}+\dot{\psi}c_{3,j,y}$$
$$+\dot{\theta}_{2,j}d_{3,j,y}+\dot{P}_y]$$

$$\frac{\partial(\dot{R}_{3,j,z})^2}{\partial\dot{\theta}_{3,j}} = 2e_{3,j,z}[\dot{\theta}_{3,j}e_{3,j,z}+\dot{\theta}b_{3,j,z}$$
$$+\dot{\psi}c_{3,j,z}+\dot{\theta}_{2,j}d_{3,j,z}+\dot{P}_z]$$

$$\frac{\partial(\dot{R}_{4,j,x})^2}{\partial\dot{\theta}_{3,j}} = 2e_{4,j,x}[\dot{\theta}_{3,j}e_{4,j,x}+\dot{\phi}a_{4,j,x}$$
$$+\dot{\theta}b_{4,j,x}+\dot{\psi}c_{4,j,x}$$
$$+\dot{\theta}_{2,j}d_{4,j,x}+\dot{\theta}_{4,j}f_{4,j,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{4,j,y})^2}{\partial\dot{\theta}_{3,j}} = 2e_{4,j,y}[\dot{\theta}_{3,j}e_{4,j,y}+\dot{\phi}a_{4,j,y}$$
$$+\dot{\theta}b_{4,j,y}+\dot{\psi}c_{4,j,y}$$
$$+\dot{\theta}_{2,j}d_{4,j,y}+\dot{\theta}_{4,j}f_{4,j,y}+\dot{P}_y]$$

$$\frac{\partial(\dot{R}_{4,j,z})^2}{\partial\dot{\theta}_{3,j}} = 2e_{4,j,z}[\dot{\theta}_{3,j}e_{4,j,z}+\dot{\theta}b_{4,j,z}$$

$$+\dot{\psi}c_{4,j,z}+\dot{\theta}_{2,j}d_{4,j,z}$$

$$+\dot{\theta}_{4,j}f_{4,j,z}+\dot{P}_z]$$

$$\frac{\partial\omega^2_{x,o}}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial\omega^2_{y,o}}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial\omega^2_{z,o}}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial(\omega_{1,j,x})^2}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial(\omega_{1,j,y})^2}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial(\omega_{1,j,z})^2}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial(\omega_{2,j,x})^2}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial(\omega_{2,j,y})^2}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial(\omega_{2,j,z})^2}{\partial\dot{\theta}_{3,j}} = 2[\dot{\theta}_{3,j}+\dot{\phi}S(\theta)C(\psi+\theta_{1,j}+\theta_{2,j})$$

$$-\dot{\theta}S(\psi+\theta_{1,j}+\theta_{2,j})]$$

$$\frac{\partial(\omega_{3,j,x})^2}{\partial\dot{\theta}_{3,j}} = 0 \qquad\qquad \frac{\partial(\omega_{3,j,y})^2}{\partial\dot{\theta}_{3,j}} = 0$$

$$\frac{\partial(\omega_{3,j,z})^2}{\partial\dot{\theta}_{3,j}} = 2[\dot{\theta}_{3,j}+\dot{\phi}S(\theta)C(\psi+\theta_{1,j}+\theta_{2,j})$$

$$-\dot{\theta}S(\psi+\theta_{1,j}+\theta_{2,j})+\dot{\theta}_{4,j}]$$

A1.19
$$\frac{\partial P_z}{\partial\dot{\theta}_{4,j}} = 0 \qquad\qquad \frac{\partial\dot{P}_x^2}{\partial\dot{\theta}_{4,j}} = 0$$

$$\frac{\partial\dot{P}_y^2}{\partial\dot{\theta}_{4,j}} = 0 \qquad\qquad \frac{\partial\dot{P}_z^2}{\partial\dot{\theta}_{4,j}} = 0$$

$$\frac{\partial R_{2,j,z}}{\partial\dot{\theta}_{4,j}} = 0 \qquad\qquad \frac{\partial R_{3,j,z}}{\partial\dot{\theta}_{4,j}} = 0$$

$$\frac{\partial R_{4,j,z}}{\partial\dot{\theta}_{4,j}} = 0 \qquad\qquad \frac{\partial(\dot{R}_{2,j,x})^2}{\partial\dot{\theta}_{4,j}} = 0$$

$$\frac{\partial(\dot{R}_{2,j,y})^2}{\partial\dot{\theta}_{4,j}} = 0 \qquad\qquad \frac{\partial(\dot{R}_{2,j,z})^2}{\partial\dot{\theta}_{4,j}} = 0$$

$$\frac{\partial(\dot{R}_{3,j,x})^2}{\partial\dot{\theta}_{4,j}} = 0 \qquad\qquad \frac{\partial(\dot{R}_{3,j,y})^2}{\partial\dot{\theta}_{4,j}} = 0$$

$$\frac{\partial(\dot{R}_{3,1,x})^=}{\partial\dot{\theta}_{4,1}} = 0$$

$$\frac{\partial(\dot{R}_{4,1,x})^=}{\partial\dot{\theta}_{4,1}} = 2f_{4,1,x}[\dot{\theta}_{4,1}f_{4,1,x}+\dot{\phi}a_{4,1,x}$$
$$+\dot{\theta}b_{4,1,x}+\dot{\psi}c_{4,1,x}$$
$$+\dot{\theta}_{z,1}d_{4,1,x}+\dot{\theta}_{3,1}e_{4,1,x}+\dot{P}_x]$$

$$\frac{\partial(\dot{R}_{4,1,y})^=}{\partial\dot{\theta}_{4,1}} = 2f_{4,1,y}[\dot{\theta}_{4,1}f_{4,1,y}+\dot{\phi}a_{4,1,y}$$
$$+\dot{\theta}b_{4,1,y}+\dot{\psi}c_{4,1,y}$$
$$+\dot{\theta}_{z,1}d_{4,1,x}+\dot{\theta}_{3,1}e_{4,1,y}+\dot{P}_y]$$

$$\frac{\partial(\dot{R}_{4,1,z})^=}{\partial\dot{\theta}_{4,1}} = 2f_{4,1,z}[\dot{\theta}_{4,1}f_{4,1,z}+\dot{\theta}b_{4,1,z}$$
$$+\dot{\psi}c_{4,1,z}+\dot{\theta}_{z,1}d_{4,1,z}$$
$$+\dot{\theta}_{3,1}e_{4,1,z}+\dot{P}_z]$$

$$\frac{\partial\omega^2_{x,0}}{\partial\dot{\theta}_{4,1}} = 0 \qquad\qquad \frac{\partial\omega^2_{y,0}}{\partial\dot{\theta}_{4,1}} = 0$$

$$\frac{\partial\omega^2_{z,0}}{\partial\dot{\theta}_{4,1}} = 0 \qquad\qquad \frac{\partial(\omega_{1,1,x})^=}{\partial\dot{\theta}_{4,1}} = 0$$

$$\frac{\partial(\omega_{1,1,y})^=}{\partial\dot{\theta}_{4,1}} = 0 \qquad\qquad \frac{\partial(\omega_{1,1,z})^=}{\partial\dot{\theta}_{4,1}} = 0$$

$$\frac{\partial(\omega_{z,1,x})^=}{\partial\dot{\theta}_{4,1}} = 0 \qquad\qquad \frac{\partial(\omega_{z,1,y})^=}{\partial\dot{\theta}_{4,1}} = 0$$

$$\frac{\partial(\omega_{z,1,z})^=}{\partial\dot{\theta}_{4,1}} = 0 \qquad\qquad \frac{\partial(\omega_{3,1,x})^=}{\partial\dot{\theta}_{4,1}} = 0$$

$$\frac{\partial(\omega_{3,1,y})^=}{\partial\dot{\theta}_{4,1}} = 0$$

$$\frac{\partial(\omega_{3,1,z})^=}{\partial\dot{\theta}_{4,1}} = 2[\dot{\theta}_{4,1}+\dot{\phi}S(\theta)C(\psi+\theta_{1,1}+\theta_{z,1})$$
$$-\dot{\theta}S(\psi+\theta_{1,1}+\theta_{z,1})+\dot{\theta}_{3,1}]$$

This completes the partials of the components of L with respect to the $\dot{q}_1$'s. The steps yet to be accomplished are, a) taking the time derivatives of the previous partials, b) taking the partial of L with respect to the

q_i's, and c) taking the partials of each of the constraint equations with respect to the q_i's and with respect to time.

With all the neccessary partial derivatives taken, Lagrange's equations can then be written in the form of Equation A1.10.

APPENDIX II

The following is a listing of the simulation program
and the input data files it uses.

```
C
C
C
C          THE MAIN ROUTINE IS USED TO SET INITIAL VALUES,
C          CONTROL THE TIMING LOOPS, AND CALL THE NECCESSARY
C          SUBROUTINES.
C
C
C
           CHARACTER*80 ID
           REAL INCR,IX0,IY0,IZ0,L(4,6),M(4,6),LM(4,6)
           REAL KM(4),KD,KP,KI,MB,IX(4,6),IY(4,6),IZ(4,6)
           REAL KPX(0:3),KPY(0:3),KPZ(0:3),KPHI(0:3)
           REAL KTHETB(0:3),KPSI(0:3)
           INTEGER LP,LQ,LR,AP,AQ,AR
           PARAMETER (V=17,IPOINT=3)
           DIMENSION THETA(4,6),DTHETA(4,6),RQ(12),DQ(6,6)
           DIMENSION QPX(V),QPY(V),QPZ(V),QPHI(V),QTHETB(V)
           DIMENSION QPSI(V),QEX(6,V),QEY(6,V),QEZ(6,V),T(4,6)
           DIMENSION R(4,6),DR(4,6),DDR(4,6),RJ(4),ERRINT(4,6)
           DIMENSION XW(6),YW(6),ZW(6),DRTHET(4,6)
           COMMON/B1/GR,MB
           COMMON/B2/THETA,L
           COMMON/B3/M,LM,IX,IY,IZ
           COMMON/B4/QPX,QPY,QPZ,QPHI,QTHETB,QPSI,QEX,QEY,QEZ
           COMMON/B6/N,K
           COMMON/B7/TIME,R,DR,DDR
           COMMON/B8/AP,AR,DTHETA,DQ
           COMMON/B9/K1,K2
           COMMON/B10/INCR
           COMMON/B11/IX0,IY0,IZ0
           COMMON/B12/PX,PY,PZ,PHI,THETAB,PSI
           COMMON/B13/LP,LQ,LR
           COMMON/B14/DRTHET
           COMMON/B15/KTHETA
           COMMON/B16/A
           COMMON/B17/LAMBDA
           COMMON/B18/RQ
           COMMON/B19/DPX,DPY,DPZ,DPHI,DTHETB,DPSI
           COMMON/B20/RTHETA
           COMMON/B21/T
           COMMON/B22/ATERM
           COMMON/B23/XW,YW,ZW
           COMMON/B24/ERRINT,LAMOLD
           COMMON/B25/KPX,KPY,KPZ,KPHI,KTHETB,KPSI
           COMMON/B26/RPX,RPY,RPZ,RPHI,RTHETB,RPSI
           COMMON/B27/DRPX,DRPY,DRPZ,DRPHI,DRTHEB,DRPSI
           COMMON/B28/DDRPX,DDRPY,DDRPZ,DDRPHI,DDRTHB,DDRPSI
           COMMON/B29/DQPX,DQPY,DQPZ,DQPHI,DQTHEB,DQPSI
C
```

```
C
C          THE PHYSICAL PARAMETERS OF THE SYSTEM ARE DEFINED HERE,
C
C
           GR=32.2
           PI=4.0*ATAN(1.0)
           MB=1.55
           IXO=0.250
           IYO=0.250
           IZO=0.436
           DO 10 J=1,6
           L(1,J)=0.75
           L(2,J)=0.25
           L(3,J)=1.5
           L(4,J)=2.0
           M(2,J)=0.0291
           M(3,J)=0.224
           M(4,J)=0.0839
           LM(2,J)=0.125
           LM(3,J)=0.75
           LM(4,J)=1.0
           IX(2,J)=0.000211
           IY(2,J)=0.000257
           IZ(2,J)=0.000257
           IX(3,J)=0.00104
           IY(3,J)=0.0425
           IZ(3,J)=0.0425
           IX(4,J)=0.000218
           IY(4,J)=0.0281
           IZ(4,J)=0.0281
10         CONTINUE
C
C
C          THE BODY POSITION AND ORIETATION ARE INITIALIZED
C          IN THIS SECTION.  PHI, THETAB, AND PSI ARE PURTERBED
C          SLIGHTLY TO OFFSET SINGULARITY PROBLEMS IN THE GAUSSX
C          SUBROUTINE.
C
C
           PX=0.0
           PY=0.0
           PZ=1.0
           PHI=1.0E-6
           THETAB=1.0E-6
           PSI=-1.0E-6
           DPX=0.0
           DPY=0.0
           DPZ=0.0
           DPHI=0.0
           DTHETB=0.0
```

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```
          DPSI=0.0
C
C
C              THE LEG JOINT ANGLES ARE INITIALIZED IN THIS SECTION.
C
C
          THETA(1,1)=0.0
          DO 100 J=2,6
          THETA(1,J)=THETA(1,J-1)+PI/3.0
100       CONTINUE
          DO 200 J=1,6
          THETA(2,J)=0.0
          THETA(3,J)=-0.601378
          THETA(4,J)=1.78067
200       CONTINUE
          DO 110 I=2,4
          DO 120 J=1,6
          DTHETA(I,J)=0.0
120       CONTINUE
110       CONTINUE
C
C
C              THE INITIAL SLOPES OF THE SPLINE FUNCTIONS
C              ARE DEFINED HERE.
C
C
          DO 105 I=1,4
          DO 106 J=1,6
          DRTHET(I,J)=0.0
106       CONTINUE
105       CONTINUE
C
          DQPX=0.0
          DQPY=0.0
          DQPZ=0.0
          DQPHI=0.0
          DQTHEB=0.0
          DQPSI=0.0
C
C
C              THE WORKING ARRAYS FOR THE RUNGE-KUTTA-GILL
C              SUBROUTINES ARE INITIALIZED HERE. RQ IS USED
C              IN SUBROUTINE RILEG AND DQ IS USED IN
C              SUBROUTINE RIARM
C
C
          DO 130 I=1,12
          RQ(I)=0.0
130       CONTINUE
C
```

```
          DO 140 I=1,6
          DO 150 J=1,6
          DQ(I,J)=0.0
150       CONTINUE
140       CONTINUE
C
C
C         THE STORAGE ARRAY FOR THE ERROR FUNCTION IS
C         INITIALIZED IN THIS SECTION.
C
C
          DO 160 I=1,4
          DO 170 J=1,6
          ERRINT(I,J)=0.0
170       CONTINUE
160       CONTINUE
C
C
C         THE GAINS AND INERTIA'S FOR CONTROL OF THE
C         NON SUPPORT LEGS ARE READ IN FROM THE FILE
C         'GAIN.DAT'.  THE VARIABLE 'PROPOR' IS A
C         PROPORTIONALITY CONSTANT USED IN ADJUSTING
C         T(3,J) FOR THE EFFECTS OF GRAVITY.
C
C
          OPEN(UNIT=10,FILE='GAIN.DAT',STATUS='OLD')
          READ(10,*)ID,KM(2)
          READ(10,*)ID,KM(3)
          READ(10,*)ID,KM(4)
          READ(10,*)ID,RJ(2)
          READ(10,*)ID,RJ(3)
          READ(10,*)ID,RJ(4)
          READ(10,*)ID,KD
          READ(10,*)ID,KP
          READ(10,*)ID,KI
          READ(10,*)ID,PROPOR
          CLOSE(10)
C
C
C         THE STARTING LOCATION OF THE ENDPOINTS OF EACH OF
C         LEGS IS DEFINED IN THIS SECTION.
C
C
          XW(1)=3.0
          YW(1)=0.0
          ZW(1)=0.0
          XW(2)=1.5
          YW(2)=2.598
          ZW(2)=0.0
          XW(3)=-1.5
```

```
        YW(3)=2.598
        ZW(3)=0.0
        XW(4)=-3.0
        YW(4)=0.0
        ZW(4)=0.0
        XW(5)=-1.5
        YW(5)=-2.598
        ZW(5)=0.0
        XW(6)=1.5
        YW(6)=-2.598
        ZW(6)=0.0
C
C
C        THE DATA POINTS DEFINING THE PATH OF THE BODY AND
C        OF THE END POINTS OF THE LEGS ARE READ IN FROM THE
C        FILE 'INPUT.DAT'.  V IS THE NUMBER OF DATA POINTS
C        SUPPLIED FOR EACH OF THE BODY AND END POINT
C        VARIABLES.
C
C
        OPEN(UNIT=10,FILE='INPUT.DAT',STATUS='OLD')
        READ(10,*)ID
        READ(10,*)(QPX(I),I=1,V)
        READ(10,*)(QPY(I),I=1,V)
        READ(10,*)(QPZ(I),I=1,V)
        READ(10,*)(QPHI(I),I=1,V)
        READ(10,*)(QTHETB(I),I=1,V)
        READ(10,*)(QPSI(I),I=1,V)
        DO 210 J=1,6
        READ(10,*)ID
        READ(10,*)(QEX(J,I),I=1,V)
        READ(10,*)ID
        READ(10,*)(QEY(J,I),I=1,V)
        READ(10,*)ID
        READ(10,*)(QEZ(J,I),I=1,V)
210     CONTINUE
        CLOSE(10)
C
C
C        THE RESULTS OF THE SIMULATION ARE WRITTEN INTO
C        THE FILE 'OUTPUT.DAT'.
C
C
        OPEN(UNIT=10,FILE='OUTPUT.DAT',STATUS='NEW')
C
C
C        THE SUPPORT AND NON SUPPORT LEGS ARE DEFINED IN THIS
C        SECTION.  AP, AQ, AND AR, ARE THE NON SUPPORT LEGS.
C        LP, LQ, AND LR, ARE THE SUPPORT LEGS.
C
```

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```
C
        AP=1
        AQ=3
        AR=5
        LP=2
        LQ=4
        LR=6
C
C
C       THE VARIABLES DESCRIBING THE VARIOUS TIME PARAMETERS
C       ARE DEFINED IN THIS SECTION.  DELTA IS THE SPACING
C       OF THE INPUT DATA ALONG THE TIME AXES.  INCR IS THE
C       STEP SIZE FOR THE NUMERICAL INTEGRATION. DEL IS USED
C       IN THE CONTROL OF THE TIMING LOOPS AND THE TRANSFER
C       OF BODY WEIGHT FROM ONE SET OF LEGS TO THE OTHER.
C       TIME IS A COUNTER USED TO KEEP TRACK OF THE TIME
C       ALONG EACH SECTION OF THE SPLINE CURVES.  RTIME IS
C       A COUNTER USED TO KEEP TRACK OF THE TOTAL SIMULATION
C       TIME.
C
C
        DELTA=1.0
        INCR=0.1
        DEL=INCR/2.0
        TIME=0.0
        RTIME=0.0
C
C
C       THE SIMULATION STARTS HERE.
C
C
C       THE FIRST LOOP CONTROLS THE NUMBER OF SPLINE SECTIONS
C       THAT ARE CREATED.
C
C
        N=IPOINT-1
        DO 300 K=1,V-N
        TIME=0.0
C
C
C       SUBROUTINE PLOY2 CREATES THE NECCESSARY SPLINES AND
C       THE CORRESPONDING POLYNOMIAL COFFECIENTS.
C
C
        CALL POLY2
C
C
C       THIS LOOP CONTROLS THE INCREMENTING FOR THE
C       NUMERICAL INTEGRATION.
C
```

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```
C
          DO 400 TI=INCR,DELTA+0.5*INCR,INCR
          TIME=TI/DELTA
          RTIME=RTIME+INCR
C
C
C         SUBROUTINE SPEC2 CALCULATES THE VALUE OF THE INPUT
C         CURVES FOR THE CURRENT VALUE OF TIME.  SUBROUTINE
C         TORQUE CALCULATES THE CONTROL TORQUES FOR THE
C         SUPPORT LEGS.  SUBROUTINE LEG IS WHERE THE DYNAMIC
C         SIMULATION FOR THE BODY AND SUPPORT LEGS IS DONE.
C
C
          CALL SPEC2
          CALL TORQUE
          CALL LEG
C
C
C         THE JOINT TORQUES FOR THE NON SUPPORT LEGS ARE
C         SPECIFIED IN THIS SECTION. T(3,J) AND T(4,J)
C         BOTH CONTAIN CORRECTIONS TO OFFSET THE EFFECTS
C         OF GRAVITY.
C
C
          DO 600 J=AP,AR,2
          DO 650 I=2,4
          ERRINT(I,J)=ERRINT(I,J)+(R(I,J)-THETA(I,J))*INCR
          CALL CNTRL(RJ(I),KM(I),KD,KP,KI,DDR(I,J),DR(I,J),R(I,J),
     $              DTHETA(I,J),THETA(I,J),ERRINT(I,J),T(I,J))
650       CONTINUE
          CA=LM(3,J)*COS(THETA(3,J))
          CB=LM(4,J)*COS(THETA(4,J)+THETA(3,J))
          CC=2.0*CA+CB
          T(3,J)=T(3,J)-PROPOR*(M(3,J)*GR*CA+M(4,J)*GR*CC)
          T(4,J)=T(4,J)-M(4,J)*GR*CB
600       CONTINUE
C
C
C         SUBROUTINE ARM PERFORMS THE DYNAMICS FOR THE NON
C         SUPPORT LEGS.
C
C
          CALL ARM
C
C
C         THE OUTPUT FILE IS CREATED IN THIS SECTION. THE
C         WRITE COMMANDS AND FORMAT STATEMENT CAN BE CHANGED
C         TO FIT WHATEVER DATA NEEDS TO BE WRITTEN TO THE
C         OUTPUT FILE.
C
```

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```
C
5           FORMAT(A20,3E15.7)
            ID=' THETA(2,5)'
            WRITE(10,5)ID,R(2,5),THETA(2,5),RTIME
            ID=' THETA(3,5)'
            WRITE(10,5)ID,R(3,5),THETA(3,5),RTIME
            ID=' THETA(4,5)'
            WRITE(10,5)ID,R(4,5),THETA(4,5),RTIME
            ID=' PX'
            WRITE(10,5)ID,RPX,PX,RTIME
            ID=' PY'
            WRITE(10,5)ID,RPY,PY,RTIME
            ID=' PZ'
            WRITE(10,5)ID,RPZ,PZ,RTIME
            ID=' PHI'
            WRITE(10,5)ID,RPHI,PHI,RTIME
            ID=' THETAB'
            WRITE(10,5)ID,RTHETB,THETAB,RTIME
            ID=' PSI'
            WRITE(10,5)ID,RPSI,PSI,RTIME
C
C
C           THIS SECTION MONITORS THE RTIME VARIABLE SO THAT
C           THE TRANSFER OF BODY WEIGHT FROM ONE SET OF LEGS
C           TO THE OTHER CAN BE MADE AT THE DESIRED TIMES.
C           SUBROUTINE XYZ CALCULATES THE END POSITIONS OF
C           THE LEGS THAT ARE BECOMING THE SUPPORT LEGS.
C           THE INTEGRAL ERROR FUNCTION, ERRINT, IS RESET FOR
C           THE JOINT ANGLES.
C
C
            A=ABS(RTIME-2.0)
            B=ABS(RTIME-6.0)
            C=ABS(RTIME-10.0)
            D=ABS(RTIME-14.0)
            IF(A.LT.DEL.OR.B.LT.DEL.OR.C.LT.DEL.OR.D.LT.DEL)THEN
                    CALL XYZ
                    DUMBP=AP
                    DUMBQ=AQ
                    DUMBR=AR
                    AP=LP
                    AQ=LQ
                    AR=LR
                    LP=DUMBP
                    LQ=DUMBQ
                    LR=DUMBR
                    DO 800 I=2,4
                    DO 850 J=1,6
                    ERRINT(I,J)=0.0
850                 CONTINUE
```

130

```
800           CONTINUE
        ELSE
                  CONTINUE
        END IF
400     CONTINUE
300     CONTINUE
C
C
C       THIS SECTION IS HERE SO THAT THE PLOTTING PROGRAM
C       CAN DETECT THE END OF THE OUTPUT FILE.
C
C
        ID='END'
        DO 900 I=1,9
        WRITE(10,5)ID,1.0,2.0,3.0
900     CONTINUE
C
C
C
        CLOSE(10)
        STOP
        END
C*****************************************************************
C
C
C       SUBROUTINE TORQUE PRODUCES THE JOINT TORQUES FOR
C       THE SUPPORT LEGS.
C
C
        SUBROUTINE TORQUE
        DOUBLE PRECISION LAM(6),C(6,7)
        DIMENSION T(4,6),ERRINT(4,6),A(9,15),QT(6)
        REAL INCR,KM,KD,KP,KI,J,IXO,IYO,IZO
        REAL LAM1,LAM2,LAM3,LAM4,LAM5,LAM6,LAM7,LAM8,LAM9
        INTEGER P,Q,R
        CHARACTER*80 ID
        COMMON/B10/INCR
        COMMON/B11/IXO,IYO,IZO
        COMMON/B12/PX,PY,PZ,PHI,THETAB,PSI
        COMMON/B13/P,Q,R
        COMMON/B16/A
        COMMON/B19/DPX,DPY,DPZ,DPHI,DTHETB,DPSI
        COMMON/B21/T
        COMMON/B24/ERRINT,LAMOLD
        COMMON/B26/RPX,RPY,RPZ,RPHI,RTHETB,RPSI
        COMMON/B27/DRPX,DRPY,DRPZ,DRPHI,DRTHEB,DRPSI
        COMMON/B28/DDRPX,DDRPY,DDRPZ,DDRPHI,DDRTHB,DDRPSI
C
C
C       NEQ IS THE NUMBER OF VARIABLES SOLVED FOR IN THE
```

```
C          GAUSSX SUBROUTINE.  J IS THE MASS OF THE BODY.
C
C
           NEQ=6
           NEQ1=NEQ+1
           J=1.55
C
C
C          THE GAIN VALUES FOR THE BODY CONTROL SYSTEM ARE
C          READ IN FROM THE FILE 'BODY.DAT'.  ALSO READ IN
C          IS THE VARIABLE DEL WHICH IS USED IN THE GAUSSX
C          SUBROUTINE WHEN THE BODY ORIENTATION ANGLES GET
C          CLOSE TO ZERO.
C
C
           OPEN(UNIT=11,FILE='BODY.DAT',STATUS='OLD')
           READ(11,*)ID,KM
           READ(11,*)ID,KD
           READ(11,*)ID,KP
           READ(11,*)ID,KI
           READ(11,*)ID,DEL
           CLOSE(11)
C
C
C          SUBROUTINE ACOEF CALCULATES ALL OF THE PARTIAL
C          DERIVATIVES OF THE CONSTRAINT EQUATIONS.
C
C
           CALL ACOEF
C
C
C          THE GENERALIZED BODY CONTROL FORCES ARE DEVELOPED
C          IN THIS SECTION.  ERRINT IS THE INTEGRAL OF THE
C          ERROR FUNCTION.  T3 IS MODIFIED TO COUNTERACT THE
C          GRAVITATIONAL FORCE ON THE BODY.  NOTE:  THIS
C          CORRECTION FACTOR ON T3 IS ONLY VALID WHEN THE BODY
C          IS IN A NEAR LEVEL POSITION.
C
C
           ERRINT(1,1)=ERRINT(1,1)+(RPX-PX)*INCR
           CALL CNTRL(J,KM,KD,KP,KI,DDRPX,DRPX,RPX,DPX,PX,
          $          ERRINT(1,1),T1)
           ERRINT(1,2)=ERRINT(1,2)+(RPY-PY)*INCR
           CALL CNTRL(J,KM,KD,KP,KI,DDRPY,DRPY,RPY,DPY,PY,
          $          ERRINT(1,2),T2)
           ERRINT(1,3)=ERRINT(1,3)+(RPZ-PZ)*INCR
           CALL CNTRL(J,KM,KD,KP,KI,DDRPZ,DRPZ,RPZ,DPZ,PZ,
          $          ERRINT(1,3),T3)
           T3=T3+49.91
           ERRINT(1,4)=ERRINT(1,4)+(RPHI-PHI)*INCR
```

```
      J=IZO*COS(THETAB)+IYO*SIN(THETAB)
      KM=J/2.5
      CALL CNTRL(J,KM,KD,KP,KI,DDRPHI,DRPHI,RPHI,
     $           DPHI,PHI,ERRINT(1,4),T4)
      ERRINT(1,5)=ERRINT(1,5)+(RTHETB-THETAB)*INCR
      J=0.25
      KM=0.1
      CALL CNTRL(J,KM,KD,KP,KI,DDRTHB,DRTHEB,RTHETB,
     $           DTHETB,THETAB,ERRINT(1,5),T5)
      ERRINT(1,6)=ERRINT(1,6)+(RPSI-PSI)*INCR
      J=0.436
      KM=0.1744
      CALL CNTRL(J,KM,KD,KP,KI,DDRPSI,DRPSI,RPSI,
     $           DPSI,PSI,ERRINT(1,6),T6)
C
C
C     C(M,N) IS THE COEFFICIENT MATRIX USED BY THE GAUSSX
C     SUBROUTINE TO SOLVE FOR THE SET OF LAMBDA'S WHICH
C     RESULT FROM THE GENERALIZED BODY FORCES T1 THRU T6.
C     LAM(M) IS A WORKING SUBROUTINE USED TO TRANSFER THE
C     LAMBDA'S FROM THE GAUSSX SUBROUTINE TO THE TORQUE
C     SUBROUTINE. LAM1THRU LAM9 ARE THE RESULTING LAMBDA'S
C     USED TO CALCULATE THE SUPPORT LEG JOINT TORQUES.
C
C
      C(1,1)=DBLE(A(1,1)+A(2,1))
      C(1,2)=DBLE(A(1,1)+A(3,1))
      C(1,3)=DBLE(0.0)
      C(1,4)=DBLE(0.0)
      C(1,5)=DBLE(0.0)
      C(1,6)=DBLE(0.0)
      C(1,7)=DBLE(T1)
      C(2,1)=DBLE(0.0)
      C(2,2)=DBLE(0.0)
      C(2,3)=DBLE(A(4,2)+A(5,2))
      C(2,4)=DBLE(A(5,2)+A(6,2))
      C(2,5)=DBLE(0.0)
      C(2,6)=DBLE(0.0)
      C(2,7)=DBLE(T2)
      C(3,1)=DBLE(0.0)
      C(3,2)=DBLE(0.0)
      C(3,3)=DBLE(0.0)
      C(3,4)=DBLE(0.0)
      C(3,5)=DBLE(A(7,3)+A(9,3))
      C(3,6)=DBLE(A(8,3)+A(9,3))
      C(3,7)=DBLE(T3)
      C(4,1)=DBLE(A(1,4)+A(2,4))
      C(4,2)=DBLE(A(1,4)+A(3,4))
      C(4,3)=DBLE(A(4,4)+A(5,4))
      C(4,4)=DBLE(A(5,4)+A(6,4))
```

```
          C(4,5)=DBLE(0.0)
          C(4,6)=DBLE(0.0)
          C(4,7)=DBLE(T4)
          C(5,1)=DBLE(A(1,5)+A(2,5))
          C(5,2)=DBLE(A(1,5)+A(3,5))
          C(5,3)=DBLE(A(4,5)+A(5,5))
          C(5,4)=DBLE(A(5,5)+A(6,5))
          C(5,5)=DBLE(A(7,5)+A(9,5))
          C(5,6)=DBLE(A(8,5)+A(9,5))
          C(5,7)=DBLE(T5)
          C(6,1)=DBLE(A(1,6)+A(2,6))
          C(6,2)=DBLE(A(1,6)+A(3,6))
          C(6,3)=DBLE(A(4,6)+A(5,6))
          C(6,4)=DBLE(A(5,6)+A(6,6))
          C(6,5)=DBLE(A(7,6)+A(9,6))
          C(6,6)=DBLE(A(8,6)+A(9,6))
          C(6,7)=DBLE(T6)
          CALL GAUSSX(C,LAM,DEL,NEQ,NEQ1)
          LAM2=LAM(1)
          LAM3=LAM(2)
          LAM4=LAM(3)
          LAM6=LAM(4)
          LAM7=LAM(5)
          LAM8=LAM(6)
          LAM1=LAM2+LAM3
          LAM5=LAM4+LAM6
          LAM9=LAM7+LAM8
C
C
C         THE SUPPORT LEG JOINT TORQUES ARE COMPUTED IN THIS
C         SECTION.
C
C
          T(2,P)=-(A(1,7)*LAM1+A(4,7)*LAM4+A(7,7)*LAM7)
          T(3,P)=-(A(1,8)*LAM1+A(4,8)*LAM4+A(7,8)*LAM7)
          T(4,P)=-(A(1,9)*LAM1+A(4,9)*LAM4+A(7,9)*LAM7)
          T(2,Q)=-(A(2,10)*LAM2+A(5,10)*LAM5+A(8,10)*LAM8)
          T(3,Q)=-(A(2,11)*LAM2+A(5,11)*LAM5+A(8,11)*LAM8)
          T(4,Q)=-(A(2,12)*LAM2+A(5,12)*LAM5+A(8,12)*LAM8)
          T(2,R)=-(A(3,13)*LAM3+A(6,13)*LAM6+A(9,13)*LAM9)
          T(3,R)=-(A(3,14)*LAM3+A(6,14)*LAM6+A(9,14)*LAM9)
          T(4,R)=-(A(3,15)*LAM3+A(6,15)*LAM6+A(9,15)*LAM9)
          RETURN
          END
C*******************************************************************
C
C
C         SUBROUTINE GAUSSX IS A GAUSSIAN ELIMINATION
C         SUBROUTINE USED TO SOLVE THE COEFFICIENT MATRIX
C         C(M,N) IN THE TORQUE SUBROUTINE.
```

```
C
C
        SUBROUTINE GAUSSX(A,X,DEL,N,N1)
        DOUBLE PRECISION A(N,N1),X(N),BIG,DUMMY,S,AIJ
        DO 200 J=1,N
        J1=J+1
        IF(J1.GT.N) GOTO 980
        BIG=ABS(A(J,J))
        M=J
        DO 900 L=J1,N
        IF(ABS(A(L,J)).LE.BIG) GOTO 900
        M=L
        BIG=ABS(A(L,J))
900     CONTINUE
        DO 990 JJ=J,N1
        DUMMY=A(M,JJ)
        A(M,JJ)=A(J,JJ)
990     A(J,JJ)=DUMMY
980     CONTINUE
C
C       DEL IS THE SMALLEST VALUE THAT A(J,J) CAN TAKE ON.
C       THIS IS BECAUSE THE C(M,N) MATRIX BECOMES POORLY
C       DEFINED AT EXTREMELY SMALL BODY ORIENTATION ANGLES.
C
        IF(ABS(A(J,J)).LT.DBLE(DEL)) A(J,J)=DBLE(DEL)
C
C
        S=1.0/A(J,J)
        DO 201 K=J,N1
201     A(J,K)=A(J,K)*S
        DO 202 I=1,N
        IF(I-J) 203,202,203
203     AIJ=-A(I,J)
        DO 204 K=J,N1
204     A(I,K)=A(I,K)+AIJ*A(J,K)
202     CONTINUE
200     CONTINUE
        DO 300 I=1,N
300     X(I)=A(I,N1)
        RETURN
        END
C********************************************************************
C
C
C       SUBROUTINE CNTRL RETURNS THE CORRECTIVE FORCE WHEN
C       GAINS, INERTIA'S, AND INPUT SIGNAL'S SUPPLIED TO IT.
C       SUBROUTINE CNTRL IS USED BY THE NON SUPPORT LEGS AS
C       WELL AS THE BODY.
C
C
```

```
      SUBROUTINE CNTRL(J,KM,KD,KP,KI,DDR,DR,R,DC,C,ERRINT,T)
      REAL J,KM,KD,KP,KI
      T=J*DDR+KM*KD*(DR-DC)+KM*KP*(R-C)+KM*KI*ERRINT
      RETURN
      END
C*****************************************************************
C
C
C         SUBROUTINE POLY2 OVERSEES THE CREATION OF ALL THE
C         NECCESSARY SPLINE FUNCTIONS.  KPX, KPY, KPZ, KPHI,
C         KTHETB, KPSI, AND KTHETA STORE THE POLYNOMIAL
C         COEFFICIENTS OF THE SPLINES.  RTHETA STORES THE
C         DISCRETE JOINT ANGLES COMPUTED FROM THE INPUT DATA
C         OF THE BODY AND LEG END POSITIONS.
C
C
      SUBROUTINE POLY2
      PARAMETER (V=17,IPOINT=3)
      DIMENSION RTHETA(4,6,IPOINT),QPOLY(0:IPOINT-1)
      DIMENSION DRTHET(4,6)
      DIMENSION QPX(V),QPY(V),QPZ(V),QPHI(V),QTHETB(V)
      DIMENSION QPSI(V),QEX(6,V),QEY(6,V),QEZ(6,V)
      REAL KPX(0:3),KPY(0:3),KPZ(0:3),KPHI(0:3),KTHETB(0:3)
      REAL KPSI(0:3),KPOLY(0:3),KTHETA(4,6,0:3)
      COMMON/B4/QPX,QPY,QPZ,QPHI,QTHETB,QPSI,QEX,QEY,QEZ
      COMMON/B6/N,K
      COMMON/B9/K1,K2
      COMMON/B14/DRTHET
      COMMON/B15/KTHETA
      COMMON/B20/RTHETA
      COMMON/B25/KPX,KPY,KPZ,KPHI,KTHETB,KPSI
      COMMON/B29/DQPX,DQPY,DQPZ,DQPHI,DQTHEB,DQPSI
C
C
C         THE DISCRETE JOINT ANGLES ARE COMPUTED FROM THE
C         BODY AND LEG END POINT DATA IN THIS SECTION.
C
C
      IF(K.EQ.1)THEN
              DO 100 K1=1,IPOINT
              K2=K1
              CALL INVERS
100           CONTINUE
      ELSE
              DO 200 N1=1,N
              DO 300 J=1,6
              DO 400 I=2,4
              RTHETA(I,J,N1)=RTHETA(I,J,N1+1)
400           CONTINUE
300           CONTINUE
```

```
200             CONTINUE
                K1=K+N
                K2=IPOINT
                CALL INVERS
         END IF
C
C
C        THE INDIVIDUAL JOINT SPLINE FUNCTIONS ARE COMPUTED
C        IN THIS SECTION.  DRTHET IS THE SLOPE AT THE BEGGINING
C        OF THE CURRENT SPLINE INTERVAL.  QPOLY AND DQPOLY
C        ARE WORKING VARIABLES USED TO PASS PARAMETERS TO
C        THE SPLINE SUBROUTINE.
C
C
         DO 500 I=2,4
         DO 600 J=1,6
         DO 700 M=1,IPOINT
         QPOLY(M-1)=RTHETA(I,J,M)
700      CONTINUE
         DQPOLY=DRTHET(I,J)
         CALL SPLINE(QPOLY,KPOLY,DQPOLY)
         DRTHET(I,J)=DQPOLY
         DO 800 M=0,3
         KTHETA(I,J,M)=KPOLY(M)
800      CONTINUE
600      CONTINUE
500      CONTINUE
C
C
C        THE SPLINE FUNCTIONS FOR THE BODY POSITION AND
C        ORIENTATION ARE COMPUTED IN THIS SECTION. DQPX
C        THRU DQPSI ARE THE SLOPES AT THE BEGGINING OF
C        THE CURRENT SPLINE SECTION.
C
C
         CALL GSPLIN(QPX,KPX,DQPX)
         CALL GSPLIN(QPY,KPY,DQPY)
         CALL GSPLIN(QPZ,KPZ,DQPZ)
         CALL GSPLIN(QPHI,KPHI,DQPHI)
         CALL GSPLIN(QTHETB,KTHETB,DQTHEB)
         CALL GSPLIN(QPSI,KPSI,DQPSI)
         RETURN
         END
C*********************************************************
C
C
C        SUBROUTINE GSPLIN IS AN INTERMEDIATE SUBROUTINE
C        FOR THE COMPUTATION OF THE BODY SPLINES.  THE
C        BODY VARIABLES ARE PUT INTO WORKING VARIABLES
C        WHICH ARE THEN PASSED TO THE SPLINE SUBROUTINE.
```

```
C
C
      SUBROUTINE GSPLIN(QBODY,KBODY,DQBODY)
      PARAMETER(V=17,IPOINT=3)
      REAL KBODY(0:3)
      DIMENSION QBODY(V),QPOLY(0:IPOINT-1)
      COMMON/B6/N,K
      DO 100 M=0,N
      QPOLY(M)=QBODY(K+M)
100   CONTINUE
      CALL SPLINE(QPOLY,KBODY,DQBODY)
      RETURN
      END
C**************************************************************
C     .
C
C     THE SPLINE CURVES ARE COMPUTED IN SUBROUTINE SPLINE.
C     Q IS THE DATA SET USD TO CREATE THE SPLINE.  KP IS
C     THE POLYNOMIAL COEFFICIENTS THAT DESCRIBE THE SPLINE.
C     DQ IS THE SLOPE AT THE BEGGINING OF THE SPLINE CURVE
C     FOR THE CURRENT SECTION.
C
C
      SUBROUTINE SPLINE(Q,KP,DQ)
      PARAMETER (IPOINT=3)
      REAL KP(0:3)
      DIMENSION Q(0:IPOINT-1),P(IPOINT-1),R(IPOINT)
      DIMENSION S(IPOINT),T(IPOINT),A(-1:2)
C
C
C     IF LESS THAN 4 DATA POINTS ARE USED TO COMPUTE THE
C     SPLINE CURVES, THE RECURSIVE CONSTANTS ARE NOT NEEDED.
C
C
      N=IPOINT-1
      IF(N.LT.3) GOTO 200
C
C
C     THE RECURSIVE CONSTANTS USED TO COMPUTE A(1) ARE
C     CALCULATED IN THIS SECTION.
C
C
      P(1)=6.0
      P(2)=-24.0
      DO 300 I=3,N
      P(I)=-4.0*P(I-1)-P(I-2)
300   CONTINUE
      R(2)=-1.5
      R(3)=6.0
      DO 400 I=4,N+1
```

```
          R(I)=-4.0*R(I-1)-R(I-2)
400       CONTINUE
          S(2)=-0.5
          S(3)=2.0
          DO 500 I=4,N+1
          S(I)=-4.0*S(I-1)-S(I-2)
500       CONTINUE
          T(2)=-3.5
          T(3)=13.0
          DO 600 I=4,N+1
          T(I)=-4.0*T(I-1)-T(I-2)
600       CONTINUE
C
C
C         A(1) IS COMPUTED IN THIS SECTION WHEN 4 OR MORE
C         DATA POINTS ARE USED IN CREATING THE SPLINE.
C
C
          Z=0.0
          DO 700 I=1,N
          Z=Z+P(I)*Q(N+1-I)
700       CONTINUE
          Y=0.0
          DO 800 I=1,N-2
          Y=Y+P(I)*Q(N-1-I)
800       CONTINUE
          A(1)=(Z-Y+(R(N+1)-R(N-1))*Q(0)+(S(N+1)-S(N-1))*
     $    DQ)/(T(N-1)-T(N+1))
          GOTO 900
C
C
200       IF(N.EQ.1) GOTO 1000
C
C         THIS EQUATION IS USED WHEN 3 DATA POINTS ARE USED
C         TO CREATE THE SPLINES.
C
          A(1)=-1.0/6.0*DQ-0.5*Q(0)+2.0*Q(1)-0.5*Q(2)
          GOTO 900
C
C         THIS EQUATION IS USED WHEN 2 DATA POINTS ARE USED
C         TO CREATE THE SPLINES.
C
1000      A(1)=-1.0/3.0*DQ-Q(0)+2.0*Q(1)
C
C
C         A(0), A(-1), AND A(2) ARE CALCULATED IN THIS SECTION.
C
C
900       A(0)=1.5*Q(0)+0.5*DQ-0.5*A(1)
          A(-1)=A(1)-2.0*DQ
```

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```
        A(2)=6.0*Q(1)-4.0*A(1)-A(0)
C
C
C       THE POLYNOMIAL COEFFICIENTS THAT DEFINE THE SPLINE
C       ARE CALCULATED IN THIS SECTION.
C
C

        KP(3)=(-A(-1)+3.0*A(0)-3.0*A(1)+A(2))/6.0
        KP(2)=(3.0*A(-1)-6.0*A(0)+3.0*A(1))/6.0
        KP(1)=(-3.0*A(-1)+3.0*A(1))/6.0
        KP(0)=(A(-1)+4.0*A(0)+A(1))/6.0
C
C
C       DQ IS SET EQUAL TO THE END SLOPE OF THIS SECTION.
C       WHEN THE SPLINES FOR THE NEXT SECTION ARE COMPUTED,
C       DQ WILL THEN BE USED AS THE BEGGINING SLOPE. THUS
C       ASSURING CONTINUITY OF THE FIRST DERIVATIVE.
C
C

        DQ=0.5*A(2)-0.5*A(0)
        RETURN
        END
C*****************************************************************
C
C
C       SUBROUTINE INVERS COMPUTES THE JOINT ANGLES THAT FIT
C       THE INPUT BODY AND LEG END POINT DATA.
C
C

        SUBROUTINE INVERS
        PARAMETER (V=17,IPOINT=3)
        DIMENSION QPX(V),QPY(V),QPZ(V),QPHI(V),QTHETB(V)
        DIMENSION QPSI(V),QEX(6,V),QEY(6,V)
        DIMENSION QEZ(6,V),RTHETA(4,6,IPOINT),THETA(4,6)
        REAL L(4,6)
        COMMON/B2/THETA,L
        COMMON/B4/QPX,QPY,QPZ,QPHI,QTHETB,QPSI,QEX,QEY,QEZ
        COMMON/B9/K1,K2
        COMMON/B20/RTHETA
C
C
C       SOME COMMON TERMS ARE COMPUTED IN THIS SECTION.
C
C

        CPHI=COS(QPHI(K1))
        SPHI=SIN(QPHI(K1))
        CTHETB=COS(QTHETB(K1))
        STHETB=SIN(QTHETB(K1))
        CPSI=COS(QPSI(K1))
        SPSI=SIN(QPSI(K1))
```

```
          TERM01=CPHI*CPSI-SPHI*CTHETB*SPSI
          TERM02=SPHI*CPSI+CPHI*CTHETB*SPSI
          TERM03=STHETB*SPSI
          TERM04=-CPHI*SPSI-SPHI*CTHETB*CPSI
          TERM05=-SPHI*SPSI+CPHI*CTHETB*CPSI
          TERM06=STHETB*CPSI
          TERM07=SPHI*STHETB
          TERM08=-CPHI*STHETB
C
C
C         THIS SECTION CALCULATES THE END POSITION OF THE
C         LEGS IN THE BODY (X0,Y0,Z0) COORDINATE SYSTEM.
C
C
          DO 100 J=1,6
          X=(QEX(J,K1)-QPX(K1))*TERM01+(QEY(J,K1)-QPY(K1))
     $     *TERM02+(QEZ(J,K1)-QPZ(K1))*TERM03
          Y=(QEX(J,K1)-QPX(K1))*TERM04+(QEY(J,K1)-QPY(K1))
     $     *TERM05+(QEZ(J,K1)-QPZ(K1))*TERM06
          Z=(QEX(J,K1)-QPX(K1))*TERM07+(QEY(J,K1)-QPY(K1))
     $     *TERM08+(QEZ(J,K1)-QPZ(K1))*CTHETB
C
C
C         THE JOINT ANGLES THAT FIT THE BODY AND LEG END
C         POSITION ARE COMPUTED HERE.
C
C
          C1=COS(THETA(1,J))
          S1=SIN(THETA(1,J))
          RTHETA(2,J,K2)=ATAN((C1*Y-S1*X)/(C1*X+S1*Y-L(1,J)))
          C2=COS(RTHETA(2,J,K2))
          C12=COS(THETA(1,J)+RTHETA(2,J,K2))
          S12=SIN(THETA(1,J)+RTHETA(2,J,K2))
          C4=((((X-L(1,J)*C1-L(2,J)*C12)/C12)**2+Z**2-L(4,J)**2
     $       -L(3,J)**2)/(2.0*L(3,J)*L(4,J)))
          S4=SQRT(1.0-C4**2)
          RTHETA(4,J,K2)=ATAN2(S4,C4)
          RTHETA(3,J,K2)=ATAN((L(4,J)*S4*(C12*X+S12*Y-L(1,J)*C2
     $                   -L(2,J))+Z*(L(4,J)*C4+L(3,J)))
     $                   /((L(4,J)*C4+L(3,J))*(L(1,J)
     $                   *C2+L(2,J)-C12*X-S12*Y)+Z*L(4,J)*S4))
100       CONTINUE
          RETURN
          END
C*************************************************************
C
C
C         SUBROUTINE SPEC2 COMPUTES THE VALUE, FIRST, AND
C         SECOND DERIVATIVE OF THE SPLINE CURVES THAT ARE
C         USED TO GUIDE THE WALKING ROBOT.
```

```
C
C
      SUBROUTINE SPEC2
      DIMENSION R(4,6),DR(4,6),DDR(4,6)
      REAL TIME,KTHETA(4,6,0:3),KPX(0:3),KPY(0:3)
      REAL KPZ(0:3),KPHI(0:3),KTHETB(0:3),KPSI(0:3)
      COMMON/B7/TIME,R,DR,DDR
      COMMON/B15/KTHETA
      COMMON/B25/KPX,KPY,KPZ,KPHI,KTHETB,KPSI
      COMMON/B26/RPX,RPY,RPZ,RPHI,RTHETB,RPSI
      COMMON/B27/DRPX,DRPY,DRPZ,DRPHI,DRTHEB,DRPSI
      COMMON/B28/DDRPX,DDRPY,DDRPZ,DDRPHI,DDRTHB,DDRPSI
      T2=TIME**2
      T3=TIME**3
C
C
C     THIS SECTION COMPUTES THE VALUES AND DERIVATIVES
C     FOR THE BODY.
C
C
      RPX=KPX(3)*T3+KPX(2)*T2+KPX(1)*TIME+KPX(0)
      DRPX=3.0*KPX(3)*T2+2.0*KPX(2)*TIME+KPX(1)
      DDRPX=6.0*KPX(3)*TIME+2.0*KPX(2)
      RPY=KPY(3)*T3+KPY(2)*T2+KPY(1)*TIME+KPY(0)
      DRPY=3.0*KPY(3)*T2+2.0*KPY(2)*TIME+KPY(1)
      DDRPY=6.0*KPY(3)*TIME+2.0*KPY(2)
      RPZ=KPZ(3)*T3+KPZ(2)*T2+KPZ(1)*TIME+KPZ(0)
      DRPZ=3.0*KPZ(3)*T2+2.0*KPZ(2)*TIME+KPZ(1)
      DDRPZ=6.0*KPZ(3)*TIME+2.0*KPZ(2)
      RPHI=KPHI(3)*T3+KPHI(2)*T2+KPHI(1)*TIME+KPHI(0)
      DRPHI=3.0*KPHI(3)*T2+2.0*KPHI(2)*TIME+KPHI(1)
      DDRPHI=6.0*KPHI(3)*TIME+2.0*KPHI(2)
      RTHETB=KTHETB(3)*T3+KTHETB(2)*T2+KTHETB(1)*TIME
     $       +KTHETB(0)
      DRTHEB=3.0*KTHETB(3)*T2+2.0*KTHETB(2)*TIME+KTHETB(1)
      DDRTHB=6.0*KTHETB(3)*TIME+2.0*KTHETB(2)
      RPSI=KPSI(3)*T3+KPSI(2)*T2+KPSI(1)*TIME+KPSI(0)
      DRPSI=3.0*KPSI(3)*T2+2.0*KPSI(2)*TIME+KPSI(1)
      DDRPSI=6.0*KPSI(3)*TIME+2.0*KPSI(2)
C
C
C     THE VALUES AND DERIVATIVES OF THE JOINT SPLINES
C     ARE COMPUTED IN THIS SECTION.  ALTHOUGH SPLINES
C     ARE CALCULATED FOR THE SUPPORT LEGS AS WELL AS
C     THE NON SUPPORT LEGS, ONLY THE NON SUPPORT LEGS
C     ARE GUIDED BY THESE SPLINES.
C
C
      DO 100 J=1,6
      DO 200 I=2,4
```

```
      R(I,J)=KTHETA(I,J,3)*T3+KTHETA(I,J,2)*T2+KTHETA(I,J,1)
     $        *TIME+KTHETA(I,J,0)
      DR(I,J)=3.0*KTHETA(I,J,3)*T2+2.0*KTHETA(I,J,2)*TIME+
     $        KTHETA(I,J,1)
      DDR(I,J)=6.0*KTHETA(I,J,3)*TIME+2.0*KTHETA(I,J,2)
200   CONTINUE
100   CONTINUE
      RETURN
      END
C*************************************************************
C
C
C     SUBROUTINE LEG CONTROLS THE DYNAMIC COMPUTATION
C     FOR THE BODY AND SUPPORT LEGS.
C
C
      SUBROUTINE LEG
      DIMENSION A(9,15),T(4,6),RQ(12),ATERM(27)
      DIMENSION Y(12),DY(12)
      REAL LAMBDA(9),INCR
      INTEGER P,Q,R
      COMMON/B10/INCR
      COMMON/B12/PX,PY,PZ,PHI,THETAB,PSI
      COMMON/B13/P,Q,R
      COMMON/B16/A
      COMMON/B17/LAMBDA
      COMMON/B18/RQ
      COMMON/B19/DPX,DPY,DPZ,DPHI,DTHETB,DPSI
      COMMON/B21/T
      COMMON/B22/ATERM
C
C
C     NEQ IS THE NUMBER OF FIRST ORDER EQUATIONS IN THE
C     RUNGE-KUTTA-GILL SUBROUTINE.
C
C
      NEQ=12
C
C
C     SUBROUTINE ACOEF CALCULATES THE PARTIAL DERIVATIVES
C     OF THE CONSTRAINT EQUATIONS.  SUBROUTINE ATERMS
C     CALCULATES SOME COMMON TERMS THAT ARE MADE UP OF THE
C     PARTIALS OF THE CONSTRAINT EQUATIONS.
C
C
      CALL ACOEF
      CALL ATERMS
C
C
C     THE GENERALIZED JOINT FORCES ARE SET EQUAL TO THE
```

```
C          SUPPORT LEG JOINT TORQUES IN THIS SECTION.
C
C
           T7=T(2,P)
           T8=T(3,P)
           T9=T(4,P)
           T10=T(2,Q)
           T11=T(3,Q)
           T12=T(4,Q)
           T13=T(2,R)
           T14=T(3,R)
           T15=T(4,R)
C
C
C          THE LAGRANGIAN MULTIPLIERS ARE COMPUTED IN THIS
C          SECTION.
C
C
           DEN1=A(1,7)*ATERM(1)-A(4,7)*ATERM(2)+A(7,7)*ATERM(3)
           LAMBDA(1)=(-T7*ATERM(1)+T8*ATERM(4)-T9*ATERM(5))/DEN1
           LAMBDA(4)=(T7*ATERM(2)-T8*ATERM(6)+T9*ATERM(7))/DEN1
           LAMBDA(7)=(-T7*ATERM(3)+T8*ATERM(8)-T9*ATERM(9))/DEN1
           DEN2=A(2,10)*ATERM(10)-A(5,10)*ATERM(11)+A(8,10)
          $      *ATERM(12)
           LAMBDA(2)=(-T10*ATERM(10)+T11*ATERM(13)-T12*ATERM(14))
          $          /DEN2
           LAMBDA(5)=(T10*ATERM(11)-T11*ATERM(15)+T12*ATERM(16))
          $          /DEN2
           LAMBDA(8)=(-T10*ATERM(12)+T11*ATERM(17)-T12*ATERM(18))
          $          /DEN2
           DEN3=A(3,13)*ATERM(19)-A(6,13)*ATERM(20)+A(9,13)
          $      *ATERM(21)
           LAMBDA(3)=(-T13*ATERM(19)+T14*ATERM(22)-T15*ATERM(23))
          $          /DEN3
           LAMBDA(6)=(T13*ATERM(20)-T14*ATERM(24)+T15*ATERM(25))
          $          /DEN3
           LAMBDA(9)=(-T13*ATERM(21)+T14*ATERM(26)-T15*ATERM(27))
          $          /DEN3
C
C
C          THE STATE VARIABLES ARE DEFINED HERE.
C
C
           Y(1)=PX
           Y(2)=PY
           Y(3)=PZ
           Y(4)=PHI
           Y(5)=THETAB
           Y(6)=PSI
           Y(7)=DPX
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                    Y(8)=DPY
                    Y(9)=DPZ
                    Y(10)=DPHI
                    Y(11)=DTHETB
                    Y(12)=DPSI
                    DY(1)=DPX
                    DY(2)=DPY
                    DY(3)=DPZ
                    DY(4)=DPHI
                    DY(5)=DTHETB
                    DY(6)=DPSI
                    DY(7)=DDPX
                    DY(8)=DDPY
                    DY(9)=DDPZ
                    DY(10)=DDPHI
                    DY(11)=DDTHEB
                    DY(12)=DDPSI
C
C
C                   SUBROUTINE RILEG IS THE RUNGE-KUTTA-GILL NUMERICAL
C                   INTEGRATION ROUTINE USED TO CALCULATE THE POSITION,
C                   VELOCITY, AND ACCELERATION OF THE BODY.
C
C
                    CALL RILEG(NEQ,INCR,Y,DY,RQ)
C
C
C                   THE STATE VARIABLES ARE PASSED BACK TO THERE PHYSICAL
C                   VARIABLES IN THIS SECTION.
C
C
                    PX=Y(1)
                    PY=Y(2)
                    PZ=Y(3)
                    PHI=Y(4)
                    THETAB=Y(5)
                    PSI=Y(6)
                    DPX=Y(7)
                    DPY=Y(8)
                    DPZ=Y(9)
                    DPHI=Y(10)
                    DTHETB=Y(11)
                    DPSI=Y(12)
                    DDPX=DY(7)
                    DDPY=DY(8)
                    DDPZ=DY(9)
                    DDPHI=DY(10)
                    DDTHEB=DY(11)
                    DDPSI=DY(12)
C
```

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C
C          THE DERIVATIVES OF THE CONSTRAINT EQUATIONS ARE
C          CALCULATED AT THE NEW BODY POSITION.  THE COMMON
C          TERMS CONSISTING OF THESE DERIVATIVES ARE ALSO
C          COMPUTED AT THE NEW BODY POSITION.  SUBROUTINE
C          LEGVEL CALCULATES THE VELOCITY OF THE SUPPORT LEG
C          JOINT ANGLES.  SUBROUTINE LEGPOS CALCULATES THE
C          JOINT ANGLES OF THE SUPPORT LEGS BASED ON THE
C          POSITION OF THE BODY AND THE ENDS OF THE SUPPORT
C          LEGS.
C
C
           CALL ACOEF
           CALL ATERMS
           CALL LEGVEL
           CALL LEGPOS
           RETURN
           END
C***************************************************************
C
C
C          SUBROUTINE ACOEF CALCULATES THE PARTIAL DERIVATIVES
C          OF THE CONSTRAINT EQUATIONS.
C
C
           SUBROUTINE ACOEF
           DIMENSION A(9,15),THETA(4,6),TERM(17)
           REAL L(4,6)
           INTEGER P,Q,R
           COMMON/B2/THETA,L
           COMMON/B12/PX,PY,PZ,PHI,THETAB,PSI
           COMMON/B13/P,Q,R
           COMMON/B16/A
C
C
C          ALL OF THE COEFFICIENTS ARE INITIALIZED TO ZERO.
C
C
           DO 100 KC=1,9
           DO 200 KD=1,15
           A(KC,KD)=0.0
200        CONTINUE
100        CONTINUE
C
C
C          SUBROUTINE TERMS COMPUTES THE COMMON TERMS USED
C          IN DEFINING THE A'S (PARTIAL DERIVATIVES OF THE
C          CONSTRAINT EQUATIONS).
C
C
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      CALL TERMS(P,TERM)
C
C
C     VALUES ARE ONLY ASSIGNED TO THE COEFFICIENTS WITH
C     NON ZERO VALUES.
C
C
      A(1,1)=1.0
      A(1,4)=TERM(1)
      A(1,5)=TERM(2)
      A(1,6)=TERM(3)
      A(1,7)=TERM(4)
      A(1,8)=TERM(5)
      A(1,9)=TERM(6)
      A(4,2)=1.0
      A(4,4)=TERM(7)
      A(4,5)=TERM(8)
      A(4,6)=TERM(9)
      A(4,7)=TERM(10)
      A(4,8)=TERM(11)
      A(4,9)=TERM(12)
      A(7,3)=1.0
      A(7,5)=TERM(13)
      A(7,6)=TERM(14)
      A(7,7)=TERM(15)
      A(7,8)=TERM(16)
      A(7,9)=TERM(17)
      CALL TERMS(Q,TERM)
      A(2,1)=1.0
      A(2,4)=TERM(1)
      A(2,5)=TERM(2)
      A(2,6)=TERM(3)
      A(2,10)=TERM(4)
      A(2,11)=TERM(5)
      A(2,12)=TERM(6)
      A(5,2)=1.0
      A(5,4)=TERM(7)
      A(5,5)=TERM(8)
      A(5,6)=TERM(9)
      A(5,10)=TERM(10)
      A(5,11)=TERM(11)
      A(5,12)=TERM(12)
      A(8,3)=1.0
      A(8,5)=TERM(13)
      A(8,6)=TERM(14)
      A(8,10)=TERM(15)
      A(8,11)=TERM(16)
      A(8,12)=TERM(17)
      CALL TERMS(R,TERM)
      A(3,1)=1.0
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        A(3,4)=TERM(1)
        A(3,5)=TERM(2)
        A(3,6)=TERM(3)
        A(3,13)=TERM(4)
        A(3,14)=TERM(5)
        A(3,15)=TERM(6)
        A(6,2)=1.0
        A(6,4)=TERM(7)
        A(6,5)=TERM(8)
        A(6,6)=TERM(9)
        A(6,13)=TERM(10)
        A(6,14)=TERM(11)
        A(6,15)=TERM(12)
        A(9,3)=1.0
        A(9,5)=TERM(13)
        A(9,6)=TERM(14)
        A(9,13)=TERM(15)
        A(9,14)=TERM(16)
        A(9,15)=TERM(17)
        RETURN
        END
C**********************************************************
C
C
C       SUBROUTINE TERMS COMPUTES THE NECCESSARY TERMS FOR
C       CALCULATING THE PARTIAL DERIVATIVES OF THE
C       CONSTRAINT EQUATIONS.
C
C
        SUBROUTINE TERMS(I,TERM)
        DIMENSION TERM(17),THETA(4,6)
        REAL L(4,6)
        COMMON/32/THETA,L
        COMMON/B12/PX,PY,PZ,PHI,THETAB,PSI
C
C
C       SOME COMMON TERMS ARE COMPUTED HERE.
C
C
        TERM01=COS(PSI+THETA(1,I)+THETA(2,I))
        TERM02=SIN(PSI+THETA(1,I)+THETA(2,I))
        TERM03=COS(PSI+THETA(1,I))
        TERM04=SIN(PSI+THETA(1,I))
        TERM05=COS(THETA(3,I)+THETA(4,I))
        TERM06=SIN(THETA(3,I)+THETA(4,I))
        TERM07=TERM05*L(4,I)+COS(THETA(3,I))*L(3,I)+L(2,I)
        TERM08=-TERM06*L(4,I)-SIN(THETA(3,I))*L(3,I)
        TERM09=COS(PHI)
        TERM10=SIN(PHI)
        TERM11=COS(THETAB)
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        TERM12=SIN(THETAB)
        TERM13=COS(PSI)
        TERM14=SIN(PSI)
C
C
C
C       THE TERMS USED IN SUBROUTINE ACOEF ARE CALCULATED
C       HERE.
C
C
        TERM(1)=-TERM10*TERM01*TERM07-TERM10*TERM03*L(1,I)
     $          -TERM09*TERM11*TERM02*TERM07-TERM09*TERM11
     $          *TERM04*L(1,I)-TERM09*TERM12*TERM06*L(1,I)
     $          -TERM09*TERM12*L(3,J)
        TERM(2)=TERM10*TERM12*TERM02*TERM07+TERM10*TERM12
     $          *TERM04*L(1,I)-TERM10*TERM11*TERM06*L(4,I)
     $          -TERM10*TERM11*SIN(THETA(3,I))*L(3,I)
        TERM(3)=-TERM09*TERM02*TERM07-TERM09*TERM04*L(1,I)
     $          -TERM10*TERM11*TERM01*TERM07-TERM10*TERM11
     $          *TERM03*L(1,I)
        TERM(4)=-TERM09*TERM02*TERM07-TERM10*TERM11*TERM01
     $          *TERM07
        TERM(5)=TERM09*TERM01*TERM08-TERM10*TERM11*TERM02
     $          *TERM08-TERM10*TERM12*TERM05*L(4,I)-TERM10
     $          *TERM12*COS(THETA(3,I))*L(3,I)
        TERM(6)=-TERM09*TERM01*TERM06*L(4,I)+TERM10*TERM11
     $          *TERM02*TERM06*L(4,I)-TERM10*TERM12*TERM05
     $          *L(4,I)
        TERM(7)=TERM09*TERM01*TERM07+TERM09*TERM03*L(1,I)
     $          -TERM10*TERM11*TERM02*TERM08-TERM10*TERM11
     $          *TERM04*L(1,I)-TERM10*TERM12*TERM06*L(4,I)
     $          -TERM10*TERM12*SIN(THETA(3,I))*L(3,I)
        TERM(8)=-TERM09*TERM12*TERM02*TERM07-TERM09*TERM12
     $          *TERM04*L(1,I)+TERM09*TERM11*TERM06*L(4,I)
     $          +TERM09*TERM11*SIN(THETA(3,I))*L(3,I)
        TERM(9)=-TERM10*TERM02*TERM07-TERM10*TERM04*L(1,I)
     $          +TERM09*TERM11*TERM01*TERM07+TERM09*TERM11
     $          *TERM03*L(1,I)
        TERM(10)=-TERM10*TERM02*TERM07+TERM09*TERM11*TERM01
     $          *TERM07
        TERM(11)=TERM10*TERM01*TERM08+TERM09*TERM11*TERM02
     $          *TERM08+TERM09*TERM12*TERM05*L(4,I)+TERM09
     $          *TERM12*COS(THETA(3,I))*L(3,I)
        TERM(12)=-TERM10*TERM01*TERM06*L(4,I)-TERM09*TERM11
     $          *TERM02*TERM06*L(4,I)+TERM09*TERM12*TERM05
     $          *L(4,I)
        TERM(13)=TERM11*TERM02*TERM07+TERM11*TERM04*L(1,I)
     $          +TERM12*TERM06*L(4,I)+TERM12*SIN(THETA(3,I))
     $          *L(3,I)
        TERM(14)=TERM12*TERM01*TERM07+TERM12*TERM03*L(1,I)
        TERM(15)=TERM12*TERM01*TERM07
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      TERM(16)=TERM12*TERM02*TERM08-TERM11*TERM05*L(4,I)
     $          -TERM11*COS(THETA(3,I))*L(3,I)
      TERM(17)=-TERM12*TERM02*TERM06*L(4,I)-TERM11*TERM05
     $          *L(4,I)
      RETURN
      END
C**************************************************************
C
C
C      SUBROUTINE ATERMS IS USED TO CALCULATE SOME COMMON
C      TERMS INVOLVING THE CONSTRAINT EQUATION DERIVATIVES.
C
C
      SUBROUTINE ATERMS
      DIMENSION A(9,15),ATERM(27)
      COMMON/B16/A
      COMMON/B22/ATERM
C
C
C      THE COMMON TERMS ARE CALCULATED HERE.
C
C
      ATERM(1)=A(4,8)*A(7,9)-A(4,9)*A(7,8)
      ATERM(2)=A(1,8)*A(7,9)-A(1,9)*A(7,8)
      ATERM(3)=A(1,8)*A(4,9)-A(1,9)*A(4,8)
      ATERM(4)=A(4,7)*A(7,9)-A(4,9)*A(7,7)
      ATERM(5)=A(4,7)*A(7,8)-A(4,8)*A(7,7)
      ATERM(6)=A(1,7)*A(7,9)-A(1,9)*A(7,7)
      ATERM(7)=A(1,7)*A(7,8)-A(1,8)*A(7,7)
      ATERM(8)=A(1,7)*A(4,9)-A(1,9)*A(4,7)
      ATERM(9)=A(1,7)*A(4,8)-A(1,8)*A(4,7)
      ATERM(10)=A(5,11)*A(8,12)-A(5,12)*A(8,11)
      ATERM(11)=A(2,11)*A(8,12)-A(2,12)*A(8,11)
      ATERM(12)=A(2,11)*A(5,12)-A(2,12)*A(5,11)
      ATERM(13)=A(5,10)*A(8,12)-A(5,12)*A(8,10)
      ATERM(14)=A(5,10)*A(8,11)-A(5,11)*A(8,10)
      ATERM(15)=A(2,10)*A(8,12)-A(2,12)*A(8,10)
      ATERM(16)=A(2,10)*A(8,11)-A(2,11)*A(8,10)
      ATERM(17)=A(2,10)*A(5,12)-A(2,12)*A(5,10)
      ATERM(18)=A(2,10)*A(5,11)-A(2,11)*A(5,10)
      ATERM(19)=A(6,14)*A(9,15)-A(6,15)*A(9,14)
      ATERM(20)=A(3,14)*A(9,15)-A(3,15)*A(9,14)
      ATERM(21)=A(3,14)*A(6,15)-A(3,15)*A(6,14)
      ATERM(22)=A(6,13)*A(9,15)-A(6,15)*A(9,13)
      ATERM(23)=A(6,13)*A(9,14)-A(6,14)*A(9,13)
      ATERM(24)=A(3,13)*A(9,15)-A(3,15)*A(9,13)
      ATERM(25)=A(3,13)*A(9,14)-A(3,14)*A(9,13)
      ATERM(26)=A(3,13)*A(6,15)-A(3,15)*A(6,13)
      ATERM(27)=A(3,13)*A(6,14)-A(3,14)*A(6,13)
      RETURN
```

```
          END
C***************************************************************
C
C
C         SUBROUTINE RILEG IS THE RUNGE-KUTTA-GILL NUMERICAL
C         INTEGRATION SUBROUTINE USED TO COMPUTE THE BODY
C         POSITION AND FIRST AND SECOND DERIVATIVES.
C
C
          SUBROUTINE RILEG(NEQ,H,Y,DY,Q)
          DIMENSION A(2)
          DIMENSION Y(NEQ),DY(NEQ),Q(NEQ)
          A(1)=0.2928932188134524
          A(2)=1.707106781186547
          H2=0.5*H
C
C
C         SUBROUTINE LEGEQ IS WHERE THE STATE EQUATIONS ARE.
C
C
          CALL LEGEQ(NEQ,Y,DY)
          DO 13 I=1,NEQ
          B=H2*DY(I)-Q(I)
          Y(I)=Y(I)+B
13        Q(I)=Q(I)+3.0*B-H2*DY(I)
          DO 20 IJ=1,2
          CALL LEGEQ(NEQ,Y,DY)
          DO 20 I=1,NEQ
          B=A(IJ)*(H*DY(I)-Q(I))
          Y(I)=Y(I)+B
20        Q(I)=Q(I)+3.0*B-A(IJ)*H*DY(I)
          CALL LEGEQ(NEQ,Y,DY)
          DO 26 I=1,NEQ
          B=0.166666666666666*(H*DY(I)-2.0*Q(I))
          Y(I)=Y(I)+B
26        Q(I)=Q(I)+3.0*B-H2*DY(I)
          RETURN
          END
C***************************************************************
C
C
C         SUBROUTINE LEGEQ SUPPLIES THE STATE EQUATIONS TO
C         THE RILEG SUBROUTINE.
C
C
          SUBROUTINE LEGEQ(NEQ,Y,DY)
          DIMENSION Y(NEQ),DY(NEQ),A(9,15)
          REAL LAMBDA(9),MB,IXO,IYO,IZO
          REAL M11,M22,M33,M44,M45,M46,M54,M55,M64,M66
          COMMON/B1/GR,MB
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      COMMON/B11/IX0,IY0,IZ0
      COMMON/B16/A
      COMMON/B17/LAMBDA
C
C
C     SOME COMMON TERMS ARE CALCULATED IN THIS SECTION.
C
C
      CPHI=COS(Y(4))
      SPHI=SIN(Y(4))
      CTHETB=COS(Y(5))
      STHETB=SIN(Y(5))
      CPSI=COS(Y(6))
      SPSI=SIN(Y(6))
      M11=MB
      M22=MB
      M33=MB
      M44=STHETB**2*(SPSI**2*IX0+CPSI**2*IY0)+CTHETB**2*IZ0
      M45=STHETB*CPSI*SPSI*(IX0-IY0)
      M46=CTHETB*IZ0
      M54=M45
      M55=CPSI**2*IX0+SPSI**2*IY0
      M64=M46
      M66=IZ0
      Q1=0.0
      Q2=0.0
      Q3=0.0
      Q4=0.0
      Q5=0.0
      Q6=0.0
      DO 100 I=1,9
      Q1=Q1-LAMBDA(I)*A(I,1)
      Q2=Q2-LAMBDA(I)*A(I,2)
      Q3=Q3-LAMBDA(I)*A(I,3)
      Q4=Q4-LAMBDA(I)*A(I,4)
      Q5=Q5-LAMBDA(I)*A(I,5)
      Q6=Q6-LAMBDA(I)*A(I,6)
100   CONTINUE
      Q3=Q3+MB*GR
      G1=0.0
      G2=0.0
      G3=0.0
      G4=Y(10)*Y(11)*(2.0*CTHETB*STHETB*(SPSI**2*IX0
     $    +CPSI**2*IY0-IZ0))
     $    +Y(10)*Y(12)*(2.0*STHETB**2*CPSI*SPSI*(IX0-IY0))
     $    +Y(11)**2*(CTHETB*CPSI*SPSI*(IX0-IY0))
     $    +Y(11)*Y(12)*(STHETB*(CPSI**2-SPSI**2)
     $    *(IX0-IY0)-STHETB*IZ0)
      G5=-Y(10)**2*(CTHETB*STHETB*(SPSI**2*IX0-CPSI**2*IY0
     $    -IZ0))
```

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```fortran
      $       +Y(10)*Y(12)*(STHETB*(CPSI**2-SPSI**2)*(IXO-IYO)
      $       +STHETB*IZO)
      $       -Y(11)*Y(12)*(2.0*CPSI*SPSI*(IXO-IYO))
          G6=-Y(10)**2*(STHETB**2*CPSI*SPSI*(IXO-IYO))
      $       -Y(10)*Y(11)*(STHETB*(CPSI**2-SPSI**2)*(IXO-IYO)
      $       +STHETB*IZO)
      $       -Y(11)**2*(CPSI*SPSI*(IXO-IYO))
          DET=M44*M55*M66-M45**2*M66-M46**2*M55
C
C
C         THE STATE EQUATIONS ARE DEFINED IN THIS SECTION.
C
C
          DY(1)=Y(7)
          DY(2)=Y(8)
          DY(3)=Y(9)
          DY(4)=Y(10)
          DY(5)=Y(11)
          DY(6)=Y(12)
          DY(7)=(-Q1-G1)/M11
          DY(8)=(-Q2-G2)/M11
          DY(9)=(-Q3-G3)/M11
C
C
C         THIS SECTION WAS ADDED BECAUSE THE MASS MATRIX
C         BECOMES SINGULAR WHEN THE BODY ORIENTATION ANGLES
C         TEND TOWARDS ZERO.
C
C
          IF(DET.LE.(1.0E-10))THEN
                  DY(10)=(-G4-Q4)/(2.0*IZO)
                  DY(11)=(-G5-Q5)/IXO
                  DY(12)=(-G6-Q6)/(2.0*IZO)
          ELSE
          DY(10)=(M55*M66*(-Q4-G4)-M45*M66*(-Q5-G5)
      $         -M46*M55*(-Q6-G6))/DET
          DY(11)=(-M45*M66*(-Q4-G4)+(M44*M66-M46**2)*(-Q5-G5)
      $         +M45*M46*(-Q6-G6))/DET
          DY(12)=(-M46*M55*(-Q4-G4)+M45*M46*(-Q5-G5)
      $         +(M44*M55-M45**2)*(-Q6-G6))/DET
          END IF
          RETURN
          END
C******************************************************************
C
C
C         SUBROUTINE LEGVEL COMPUTES THE JOINT VELOCITIES
C         OF THE SUPPORT LEGS.
C
C
```

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```
      SUBROUTINE LEGVEL
      DIMENSION A(9,15),ATERM(27),DTHETA(4,6),DQ(6,6)
      INTEGER P,Q,R,AP,AR
      REAL L(4,6)
      COMMON/B8/AP,AR,DTHETA,DQ
      COMMON/B13/P,Q,R
      COMMON/B16/A
      COMMON/B19/DPX,DPY,DPZ,DPHI,DTHETB,DPSI
      COMMON/B22/ATERM
C
C
C     SOME COMMON TERMS ARE CALCULATED HERE.
C
C
      C1=-(A(1,1)*DPX+A(1,2)*DPY+A(1,3)*DPZ+A(1,4)*DPHI
     $    +A(1,5)*DTHETB+A(1,6)*DPSI)
      C2=-(A(2,1)*DPX+A(2,2)*DPY+A(2,3)*DPZ+A(2,4)*DPHI
     $    +A(2,5)*DTHETB+A(2,6)*DPSI)
      C3=-(A(3,1)*DPX+A(3,2)*DPY+A(3,3)*DPZ+A(3,4)*DPHI
     $    +A(3,5)*DTHETB+A(3,6)*DPSI)
      C4=-(A(4,1)*DPX+A(4,2)*DPY+A(4,3)*DPZ+A(4,4)*DPHI
     $    +A(4,5)*DTHETB+A(4,6)*DPSI)
      C5=-(A(5,1)*DPX+A(5,2)*DPY+A(5,3)*DPZ+A(5,4)*DPHI
     $    +A(5,5)*DTHETB+A(5,6)*DPSI)
      C6=-(A(6,1)*DPX+A(6,2)*DPY+A(6,3)*DPZ+A(6,4)*DPHI
     $    +A(6,5)*DTHETB+A(6,6)*DPSI)
      C7=-(A(7,1)*DPX+A(7,2)*DPY+A(7,3)*DPZ+A(7,4)*DPHI
     $    +A(7,5)*DTHETB+A(7,6)*DPSI)
      C8=-(A(8,1)*DPX+A(8,2)*DPY+A(8,3)*DPZ+A(8,4)*DPHI
     $    +A(8,5)*DTHETB+A(8,6)*DPSI)
      C9=-(A(9,1)*DPX+A(9,2)*DPY+A(9,3)*DPZ+A(9,4)*DPHI
     $    +A(9,5)*DTHETB+A(9,6)*DPSI)
C
C
C
C     THE ANGULAR VELOCITIES OF THE SUPPORT LEG JOINTS
C     ARE CALCULATED IN THIS SECTION.
C
C
      DEN1=A(1,7)*ATERM(1)-A(4,7)*ATERM(2)+A(7,7)*ATERM(3)
      DTHETA(2,P)=(C1*ATERM(1)-C4*ATERM(2)+C7*ATERM(3))
     $       /DEN1
      DTHETA(3,P)=(-C1*ATERM(4)+C4*ATERM(6)-C7*ATERM(8))
     $       /DEN1
      DTHETA(4,P)=(C1*ATERM(5)-C4*ATERM(7)+C7*ATERM(9))
     $       /DEN1
      DEN2=A(2,10)*ATERM(10)-A(5,10)*ATERM(11)+A(8,10)
     $       *ATERM(12)
      DTHETA(2,Q)=(C2*ATERM(10)-C5*ATERM(11)+C8*ATERM(12))
     $       /DEN2
      DTHETA(3,Q)=(-C2*ATERM(13)+C5*ATERM(15)-C8*ATERM(17))
```

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```
      $               /DEN2
        DTHETA(4,Q)=(C2*ATERM(14)-C5*ATERM(16)+C8*ATERM(13))
      $               /DEN2
        DEN3=A(3,13)*ATERM(19)-A(6,13)*ATERM(20)+A(9,13)
      $        *ATERM(21)
        DTHETA(2,R)=(C3*ATERM(19)-C6*ATERM(20)+C9*ATERM(21))
      $               /DEN3
        DTHETA(3,R)=(-C3*ATERM(22)+C6*ATERM(24)-C9*ATERM(26))
      $               /DEN3
        DTHETA(4,R)=(C3*ATERM(23)-C6*ATERM(25)+C9*ATERM(27))
      $               /DEN3
        RETURN
        END
C*****************************************************************
C
C
C       THE SUPPORT LEG JOINT ANGLES ARE COMPUTED IN
C       THE SUBROUTINE LEGPOS.
C
C
        SUBROUTINE LEGPOS
        DIMENSION THETA(4,6),XW(6),YW(6),ZW(6)
        INTEGER P,Q,R
        REAL L(4,6)
        COMMON/B2/THETA,L
        COMMON/B12/PX,PY,PZ,PHI,THETAB,PSI
        COMMON/B13/P,Q,R
        COMMON/B23/XW,YW,ZW
        COMMON/B6/N,K
C
C
C       SOME COMMON TERMS ARE COMPUTED HERE.
C
C
        CPHI=COS(PHI)
        SPHI=SIN(PHI)
        CTHETB=COS(THETAB)
        STHETB=SIN(THETAB)
        CPSI=COS(PSI)
        SPSI=SIN(PSI)
C
C
C       THE END POSITIONS OF THE SUPPORT LEGS IN THE
C       BODY (X0,Y0,Z0) COORDINATE SYSTEM ARE CALCULATED
C       IN THIS SECTION.
C
C
        DO 100 J=P,R,2
        X=(CPHI*CPSI-SPHI*CTHETB*SPSI)*(XW(J)-PX)+
      $   (SPHI*CPSI+CPHI*CTHETB*SPSI)*(YW(J)-PY)+
```

```
      $     (STHETB*SPSI)*(ZW(J)-PZ)
            Y=(-CPHI*SPSI-SPHI*CTHETB*CPSI)*(XW(J)-PX)+
      $     (-SPHI*SPSI+CPHI*CTHETB*CPSI)*(YW(J)-PY)+
      $     (STHETB*CPSI)*(ZW(J)-PZ)
            Z=(SPHI*STHETB)*(XW(J)-PX)+
      $     (-CPHI*STHETB)*(YW(J)-PY)+
      $     CTHETB*(ZW(J)-PZ)
C
C
C
C           THE SUPPORT LEGS JOINT ANGLES ARE CALCULATED HERE.
C
C
            C1=COS(THETA(1,J))
            S1=SIN(THETA(1,J))
            THETA(2,J)=ATAN((C1*Y-S1*X)/(C1*X+S1*Y-L(1,J)))
            C12=COS(THETA(1,J)+THETA(2,J))
            S12=SIN(THETA(1,J)+THETA(2,J))
            TERM01=(X-L(1,J)*C1-L(2,J)*C12)/C12
            C4=(TERM01**2+Z**2-L(4,J)**2-L(3,J)**2)
      $     /(2.0*L(3,J)*L(4,J))
            S4=SQRT(1.0-C4**2)
            THETA(4,J)=ATAN2(S4,C4)
            C2=COS(THETA(2,J))
            TERM02=L(4,J)*S4*(C12*X+S12*Y-L(1,J)*C2-L(2,J))
      $     +Z*(L(4,J)*C4+L(3,J))
            TERM03=-(L(4,J)*C4+L(3,J))*(C12*X+S12*Y-L(1,J)*C2
      $     -L(2,J))+Z*S4*L(4,J)
            THETA(3,J)=ATAN(TERM02/TERM03)
100         CONTINUE
            RETURN
            END
C*****************************************************************
C
C
C           SUBROUTINE ARM CONTROLS THE DYNAMIC COMPUTATION
C           FOR THE NON SUPPORT LEGS.
C
C
            SUBROUTINE ARM
            DIMENSION THETA(1,6),DTHETA(4,6),DDTHET(4,6),DQ(4,6)
            DIMENSION Y(6),DY(6)
            INTEGER AP,AR
            REAL INCR,L(4,6)
            COMMON/B2/THETA,L
            COMMON/B8/AP,AR,DTHETA,DQ
            COMMON/B10/INCR
C
C
C           NEQ IS THE NUMBER OF FIRST ORDER EQUATIONS USED
C           IN THE RUNGE-KUTTA-GILL NUMERICAL INTEGRATION
```

```
C         SUBROUTINE RIARM.
C
C
          NEQ=6
C
C
C         THE STATE VARIABLES FOR THE NON SUPPORT LEGS ARE
C         DEFINED IN THIS SECTION.
C
C
          DO 100 J=AP,AR,2
          Y(1)=THETA(2,J)
          Y(2)=THETA(3,J)
          Y(3)=THETA(4,J)
          Y(4)=DTHETA(2,J)
          Y(5)=DTHETA(3,J)
          Y(6)=DTHETA(4,J)
          DY(1)=DTHETA(2,J)
          DY(2)=DTHETA(3,J)
          DY(3)=DTHETA(4,J)
          DY(4)=DDTHET(2,J)
          DY(5)=DDTHET(3,J)
          DY(6)=DDTHET(4,J)
C
C
C         SUBROUTINE RIARM IS THE NUMERICAL INTEGRATION
C         SUBROUTINE USED TO COMPUTE POSITION, VELOCITY,
C         AND ACCELERATION OF THE NON SUPPORT LEG JOINTS.
C
C
          CALL RIARM(J,NEQ,INCR,Y,DY,DQ)
          THETA(2,J)=Y(1)
          THETA(3,J)=Y(2)
          THETA(4,J)=Y(3)
          DTHETA(2,J)=Y(4)
          DTHETA(3,J)=Y(5)
          DTHETA(4,J)=Y(6)
          DDTHET(2,J)=DY(4)
          DDTHET(3,J)=DY(5)
          DDTHET(4,J)=DY(6)
100       CONTINUE
          RETURN
          END
C*****************************************************************
C
C
C         SUBROUTINE RIARM IS A RUNGE-KUTTA-GILL NUMERICAL
C         INTEGRATION ROUTINE USED FOR CALCULATING THE
C         DYNAMICS FOR THE NON SUPPORT LEGS.
C
```

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```
C
        SUBROUTINE RIARM(J,NEQ,H,Y,DY,Q)
        DIMENSION A(2)
        DIMENSION Y(NEQ),DY(NEQ),Q(6,6)
        A(1)=.2928932188134524
        A(2)=1.707106781186547
        H2=0.5*H
C
C
C       SUBROUTINE ARMEQ SUPPLIES THE EQUATIONS OF STATE
C       FOR THE NON SUPPORT LEGS.
C
C
        CALL ARMEQ(J,NEQ,Y,DY)
        DO 13 I=1,NEQ
        B=H2*DY(I)-Q(J,I)
        Y(I)=Y(I)+B
13      Q(J,I)=Q(J,I)+3.0*B-H2*DY(I)
        DO 20 IJ=1,2
        CALL ARMEQ(J,NEQ,Y,DY)
        DO 20 I=1,NEQ
        B=A(IJ)*(H*DY(I)-Q(J,I))
        Y(I)=Y(I)+B
20      Q(J,I)=Q(J,I)+3.0*B-A(IJ)*H*DY(I)
        CALL ARMEQ(J,NEQ,Y,DY)
        DO 26 I=1,NEQ
        B=0.1666666666666666*(H*DY(I)-2.0*Q(J,I))
        Y(I)=Y(I)+B
26      Q(J,I)=Q(J,I)+3.0*B-H2*DY(I)
        RETURN
        END
C****************************************************************
C
C
C       SUBROUITNE ARMEQ SUPPLIES THE STATE EQUATIONS USED
C       BY SUBROUTINE RIARM.
C
C
        SUBROUTINE ARMEQ(J,NEQ,Y,DY)
        DIMENSION T(4,6),THETA(4,6),Y(NEQ),DY(NEQ),DDTHET(4,6)
        REAL M(4,6),LM(4,6),IX(4,6),IY(4,6),IZ(4,6),L(4,6),MB
        REAL M11,M22,M23,M33
        COMMON/B3/M,LM,IX,IY,IZ
        COMMON/B2/THETA,L
        COMMON/B1/GR,MB
        COMMON/B21/T
C
C
C       SOME COMMON TERMS ARE CALCULATED HERE.
C
```

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C
      C3=COS(Y(2))
      S3=SIN(Y(2))
      C4=COS(Y(3))
      S4=SIN(Y(3))
      C34=COS(Y(2)+Y(3))
      S34=SIN(Y(2)+Y(3))
      M11=M(2,J)*LM(2,J)**2+M(3,J)*(C3*LM(3,J)+L(2,J))**2
     $    +M(4,J)*(C34*LM(4,J)+C3*L(3,J)+L(2,J))**2+IZ(2,J)
     $    +IX(3,J)*S3**2+IY(3,J)*C3**2+IX(4,J)*S34**2
     $    +IY(4,J)*C34**2
      M22=M(3,J)*LM(3,J)**2+M(4,J)*(LM(4,J)**2
     $    +2.0*C4*L(3,J)*LM(4,J)+L(3,J)**2)+IZ(3,J)+IZ(4,J)
      M23=M(4,J)*(LM(4,J)**2+C4*L(3,J)*LM(4,J))+IZ(4,J)
      M33=M(4,J)*LM(4,J)**2+IZ(4,J)
      G1=Y(4)*Y(5)*(-2.0*M(3,J)*S3*LM(3,J)*(C3*LM(3,J)
     $    +L(2,J))-2.0*M(4,J)*(C34*S34*LM(4,J)**2+C3*S3
     $    *L(3,J)**2+SIN(2.0*Y(2)+Y(3))*L(3,J)*LM(4,J)+S34
     $    *L(2,J)*LM(4,J)+2.0*S3*L(2,J)*L(3,J))
     $    +2.0*(IX(3,J)-IY(3,J))*C3*S3+2.0*(IX(4,J)-IY(4,J))
     $    *C34*S34)+
     $    Y(4)*Y(6)*(-2.0*M(4,J)*S34*LM(4,J)*(C34*LM(4,J)
     $    +C3*L(3,J)+L(2,J))+2.0*(IX(4,J)-IY(4,J))*C34*S34)
      G2=Y(5)*Y(6)*(-2.0*M(4,J)*S4*LM(3,J)*LM(4,J))-Y(6)**2
     $    *(M(4,J)*S4*L(3,J)*LM(4,J))
      G3=Y(5)*Y(6)*(-2.0*M(4,J)*S4*L(3,J)*LM(4,J))
      Q1=0.0
      Q2=Y(4)**2*(M(3,J)*S3*LM(3,J)*(C3*LM(3,J)+L(2,J))
     $    +M(4,J)*(C34*S34*LM(4,J)**2+SIN(2.0*Y(2)
     $    +Y(3))*L(3,J)*LM(4,J)+S34*L(2,J)*LM(4,J)+
     $    C3*S3*L(3,J)**2+S3*L(2,J)*L(3,J))-(IX(3,J)
     $    -IY(3,J))*C3*S3-(IX(4,J)-IY(4,J))*C34*S34)-
     $    M(3,J)*GR*C3*LM(3,J)-M(4,J)*GR*(C34*LM(4,J)
     $    +C3*L(3,J))
      Q3=Y(4)**2*S34*(M(4,J)*LM(4,J)*(C34*LM(4,J)+C3*L(3,J)
     $    +L(2,J))-(IX(4,J)-IY(4,J))*C34)+
     $    Y(5)**3*(-M(4,J)*S4*L(3,J)*LM(4,J))-
     $    M(4,J)*GR*C34*LM(4,J)
C
C
C
      THE STATE EQUATIONS ARE DEFINED HERE.
C
C
      DY(1)=Y(4)
      DY(2)=Y(5)
      DY(3)=Y(6)
      DY(4)=(T(2,J)-Q1-G1)/M11
      DY(5)=(M33*(T(3,J)-Q2-G2)-M23*(T(4,J)-Q3-G3))
     $     /(M33*M22-M23**2)
      DY(6)=(M22*(T(4,J)-Q3-G3)-M23*(T(3,J)-Q2-G2))
```

```
      $           /(M33*M22-M23**2)
          RETURN
          END
C************************************************************************
C
C
C         SUBROUTINE XYZ CALCULATES THE END POSITIONS, IN THE
C         WORLD (XW,YW,ZW) COORDINATE SYSTEM,  OF THE LEGS
C         THAT ARE MAKING THE TRANSITION FROM NON SUPPORT TO
C         THE SUPPORT PHASE.
C
C
          SUBROUTINE XYZ
          INTEGER AP,AR
          REAL L(4,6)
          DIMENSION THETA(4,6),DTHETA(4,6),DQ(6,6),XW(6),YW(6)
          DIMENSION ZW(6)
          COMMON/B2/THETA,L
          COMMON/B8/AP,AR,DTHETA,DQ
          COMMON/B12/PX,PY,PZ,PHI,THETAB,PSI
          COMMON/B23/XW,YW,ZW
C
C
C         SOME COMMON TERMS ARE CALCULATED HERE.
C
C
          CPHI=COS(PHI)
          SPHI=SIN(PHI)
          CTHETB=COS(THETAB)
          STHETB=SIN(THETAB)
          CPSI=COS(PSI)
          SPSI=SIN(PSI)
          TERM01=CPHI*CPSI-SPHI*CTHETB*SPSI
          TERM02=-CPHI*SPSI-SPHI*CTHETB*CPSI
          TERM03=SPHI*STHETB
          TERM04=SPHI*CPSI+CPHI*CTHETB*SPSI
          TERM05=-SPHI*SPSI+CPHI*CTHETB*CPSI
          TERM06=-CPHI*STHETB
          TERM07=STHETB*SPSI
          TERM08=STHETB*CPSI
          DO 100 J=AP,AR,2
          C1=COS(THETA(1,J))
          S1=SIN(THETA(1,J))
          C3=COS(THETA(3,J))
          S3=SIN(THETA(3,J))
          C12=COS(THETA(1,J)+THETA(2,J))
          S12=SIN(THETA(1,J)+THETA(2,J))
          C34=COS(THETA(3,J)+THETA(4,J))
          S34=SIN(THETA(3,J)+THETA(4,J))
C
```

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```
C
C          THIS SECTION CALCULATES THE LEG END POSITIONS IN
C          THE BODY (X0,Y0,Z0) COORDINATE SYSTEM.
C
C
           X=C12*(L(4,J)*C34+L(3,J)*C3+L(2,J))+L(1,J)*C1
           Y=S12*(L(4,J)*C34+L(3,J)*C3+L(2,J))+L(1,J)*S1
           Z=-L(4,J)*S34-L(3,J)*S3
C
C
C          THE LEG END POSITIONS IN THE WORLD (XW,YW,ZW)
C          COORDINATE SYSTEM ARE COMPUTED HERE.
C
C
           XW(J)=TERM01*X+TERM02*Y+TERM03*Z+PX
           YW(J)=TERM04*X+TERM05*Y+TERM06*Z+PY
           ZW(J)=TERM07*X+TERM08*Y+CTHETB*Z+PZ
100        CONTINUE
           RETURN
           END
C************************************************************
```

161

INPUT FILE 'GAIN.DAT'

'KM(2)=',0.204
'KM(3)=',0.167
'KM(4)=',0.0448
'RJ(2)=',0.511
'RJ(3)=',0.417
'RJ(4)=',0.112
'KD=',30.0
'KP=',100.0
'KI=',160.0
'PROPOR=',0.7

INPUT FILE 'BODY.DAT'

'KM=',0.62
'KD=',17.5
'KP=',30.0
'KI=',25.0
'DEL=',1.0E-6

The input data file 'INPUT.DAT' contains the information in

Table 3.1.

VITA

Mark E. Baldridge

Candidate for the Degree of

Master of Science

Thesis: DEVELOPMENT OF A SIMULATION MODEL
        FOR THE STUDY OF ADVANCED CONTROL
        CONCEPTS FOR ARTICULATED MECHANISMS

Major Field:
      Mechanical Engineering

Biographical:

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            Science degree from Kansas State University, with
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            start in March, 1986 in the Government Systems
            Facility.

DEVELOPMENT OF A SIMULATION MODEL
FOR THE STUDY OF ADVANCED CONTROL
CONCEPTS FOR ARTICULATED MECHANISMS

by

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B.S., Kansas State University, 1984

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ABSTRACT

This paper investigates the use of polynomial spline
functions to guide a walking robot. The use of spline
functions allows for small amounts of input data to be
converted into smooth, continuous, paths to guide the
walking robot.

The kinematic and dynamic equations are developed for
a six-legged walking robot. The mathematical model is of a
robot with a flat cylindrical body with six three degree of
freedom legs spaced uniformly around the body.

A recursive formula is developed for computing the
spline curve, thus eleminating the need for performing
matrix inversions. Also developed is a method for
constructing a spline curve thru a large number of data
points using only a few data points at a time.

The use of polynomial spline functions as inputs to
the control system allows the use of feedforward control.
By using a feedforward loop, the control system has the
ability to anticipate the input signal. A method is
developed that allows the supporting legs to act in unison
to drive the body along its correct path by using the body
position and orientation as inputs to the control system.

The equations which describe the dynamics of the
walking robot, along with the recursive spline mathematics,

was implemented in a computer simulation. Performace of the
simulation are presented, as are recomendations for further
study.