ON UNIMODULAR HYPERGRAPHS

by

GREGORY D. GOECKEL

B. S., Marymount College of Kansas, 1982

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1985

Approved by:

Richard [illegible]

Major Professor

TABLE OF CONTENTS

TABLES

ACKNOWLEDGEMENTS

I would sincerely like to thank my major professor Dick Greechie for his guidance and for the time he spent helping me with my thesis.

I would also like to thank Claude Berge for responding to a letter of mine concerning my thesis and his book.

CHAPTER 1

Throughout this thesis we will consider only finite graphs and hypergraphs.

Let X be a set of n distinct elements where n is a positive natural number. Let $x,y \in X$ ($x = y$ is allowed), then define $[x,y]$ to be a nonordered pair, i.e., $[x,y] = [y,x]$. Let $E \subseteq \{[x,y] : x,y \in X\}$ where $\cup\{\{x,y\} : [x,y] \in E\} = X$ then the pair $(X,E) = G$ is called a graph.

An element in X is called a vertex and an element $[x,y] \in E$ is called an edge connecting x and y, where x and y are endpoints of $[x,y]$.

A loop is an edge of the form $[x,x]$.

A chain of length q is a sequence $\mu = (x_1, \ldots ,x_{q+1})$ of vertices of G such that, for $i = 1, \ldots ,q$, $[x_i,x_{i+1}] \in E$. We say x_1 is the initial vertex and x_{q+1} is the terminal vertex. Let $\lambda(\mu) := q$.

A cycle is a chain μ such that $\lambda(\mu) > 1$, the initial and terminal vertices are the same and for x_i,x_{i+1},x_j and x_{j+1} in μ, we have that if $x_i = x_j$ then $x_{i+1} \neq x_{j+1}$.

An elementary cycle is a cycle in which the only vertex in the cycle that is repeated,is the initial and terminal vertex, which is repeated only then.

A connected graph is a graph such that for each $x,y \in X$, $x \neq y$, there exists a chain connecting x and y.

Let S be a subset of X, then S is called a stable set if no edge joins any two distinct vertices in S.

Let $G = (X,E)$ be a graph and let (S_1,S_2) be a partition of X where S_1 and S_2 are stable sets, then (S_1,S_2) is a bicoloring for G.

Lemma 1.1 Let G be a connected graph with a bicoloring, (S_1,S_2). Then for any $x \in S_1$, $y \in S_2$ and chain μ connecting x and y, we have $\lambda(\mu)$ is odd.

Proof Since G is connected, there is a chain μ connecting x and y. Since (S_1,S_2) is a bicoloring of X, μ is of the form $(x,y_1,x_1, \ldots ,y_p,x_p,y)$ for some p, where each x_i is in S_1 and each y_i is in S_2. Hence $\lambda(\mu) = 2p + 1$ which is odd.

Proposition 1.2 Let G be a graph with an odd cycle, then G has an elementary odd cycle.

Proof Assume G has an odd cycle, μ. If μ is elementary then we are finished. Assume μ is not elementary, then μ can be decomposed as (μ_1,μ_2) where either μ_1 or μ_2 is odd. Assume μ_1 is odd. If μ_1 is elementary then we are finished. If μ_1 is not elementary, then repeat the above process as many times as possible. Since μ had only finite length, there is only a finite number of times we can repeat the process. Let μ' be the final odd cycle. Then by the definition of elementary cycle, μ' is an odd elementary cycle.

Lemma 1.3 Let T be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$, then T is onto iff T^{-1} exists.

Proof For the proof of this lemma see [Curtis,Theorem 13.10].

Since any matrix M can be represented as a linear transformation T_M from $\mathbb{R}^n$ to $\mathbb{R}^n$ and any linear transformation can be represented as a matrix with respect to some basis, we have the following theorem.

Theorem 1.4 Let M be an integral $n \times n$ matrix, then $\det M = \pm 1$ iff for each integral $x \in \mathbb{R}^n$ there is an integral $x' \in \mathbb{R}^n$ such that $Mx' = x$.

Proof Let $\det M = \pm 1$ and M be integral, then M^{-1} is integral by the well known Cramer's Rule. Let $x \in \mathbb{R}^n$ be integral then $x' := M^{-1}x$ is integral such that $Mx' = M(M^{-1}x) = x$. Therefore for each integral $x \in \mathbb{R}^n$ there is an integral $x' \in \mathbb{R}^n$ such that $Mx' = x$.

Now let M be integral with the property that for each integral $x \in \mathbb{R}^n$ there is an integral $x' \in \mathbb{R}^n$ such that $Mx' = x$.

Define e_i to be the n-vector that has all zeros except for a 1 in the i^{th} position. Then for each $i \in \{1, \ldots, n\}$ there is an x_i such that $Mx_i = e_i$. Since $\{e_1, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$, M maps $\mathbb{R}^n$ onto $\mathbb{R}^n$. Hence by lemma 1.3 M^{-1} exists. Therefore by the hypothesis, $M^{-1}e_i$ is integral for each i. Note that $M^{-1}e_i$ is the i^{th} column of M^{-1}. Hence M^{-1} is integral.

Since $\det M^{-1} = (\det M)^{-1}$ and M^{-1} is integral, $\det M$ and $(\det M)^{-1}$ are integral. However, the only integers with this property are ± 1.

Let X be a set and $S \subseteq X$, then S^c is denoted to be the complement of S in X.

<u>Proposition 1.5</u> Let $P(J)$ be the power set of $J = \{1, \ldots ,n\}$, then $(P(J),+)$ is a group under "+" where "+" is defined, for $A,B \in P(J)$, by $A + B = (A \cap B^c) \cup (B \cap A^c) = (A \cup B) \cap (A \cap B)^c$.

<u>Proof</u> Clearly for all $A,B \in P(J)$, $(A + B) \in P(J)$, $A + \phi = \phi + A = A$ and $A + A = \phi$. Therefore $P(J)$ is closed under "+", there is an identity element and each element has an inverse. Let $A,B,C \in P(J)$, then

$$(A + B) + C =$$
$$[((A \cap B^c) \cup (A^c \cap B)) \cap C^c] \cup [((A \cup B) \cap (A \cap B)^c)^c \cap C] =$$
$$[(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)] \cup [((A \cap B) \cup (A \cup B)^c) \cap C] =$$
$$[(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)] \cup [(A \cap B \cap C) \cup ((A \cup B)^c \cap C)] =$$
$$[(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)] \cup [(A \cap B \cap C) \cup (C \cap A^c \cap B^c)] =$$
$$[(A \cap B^c \cap C^c) \cup (A \cap B \cap C)] \cup [(A^c \cap B \cap C^c) \cup (C \cap A^c \cap B^c)] =$$
$$[(A \cap (B \cup C)^c \cup (A \cap B \cap C)] \cup [A^c \cap ((B \cap C^c) \cup (B^c \cap C))] =$$
$$[A \cap ((B \cap C) \cup (B \cup C)^c)] \cup [A^c \cap ((B \cap C^c) \cup (B^c \cap C))] =$$
$$[A \cap ((B \cap C) \cap (B \cap C)^c)^c] \cup [A^c \cap ((B \cap C^c) \cup (B^c \cap C))] =$$
$$A + (B + C).$$

Hence we have associativity. Therefore $P(J)$ is a group under "+".

Let $Z_2 := \{0,1\}$ be the field of order 2. Let $A \in P(J)$ and $\alpha \in Z_2$, then define $\alpha A = A$ if $\alpha = 1$ and $\alpha A = \phi$ if $\alpha = 0$.

<u>Proposition 1.6</u> $P(J)$ is a vector space over Z_2 with the above definitions.

<u>Proof</u> Let $A,B,C \in P(J)$ and $\alpha,\beta \in Z_2$, then $(A + B) + C = A + (B + C)$, $A + B = B + A$, $A + \phi = A$, $A + A = \phi$, $\alpha(A + B) = \alpha A + \alpha B$,

$(\alpha + \beta)A = \alpha A + \beta A$, $(\alpha\beta)A = \alpha(\beta)A$ and $1 \cdot A = A$. Therefore $P(J)$ is a vector space over Z_2.

Let A be an $n \times m$ integral matrix with $J := \{1, \ldots, m\}$ indexing the columns and $I := \{1, \ldots, n\}$ indexing the rows. Let $K \in P(J)$, then define $\Delta K = \{i \in I : \Sigma_{j \in K} a_{ij}$ is even$\}$ and define $f_A(K) = (\Delta K)^c$.

<u>Proposition 1.7</u> Let A be an $n \times m$ integral matrix with J indexing the columns and I indexing the rows and let f_A be defined as above, then f_A is a linear function from $P(J)$ into $P(I)$.

<u>Proof</u> Let $S, T \in P(J)$. Then all that need be shown is $f_A(S + T) = f_A(S) + f_A(T)$.

$$f_A(S) + f_A(T) =$$
$$(f_A(S) \cup f_A(T)) \cap (f_A(S) \cap f_A(T))^c =$$
$$((\Delta S)^c \cup (\Delta T)^c) \cap ((\Delta S)^c \cap (\Delta T)^c)^c =$$
$$((\Delta S) \cap (\Delta T))^c \cap (((\Delta S) \cup (\Delta T))^c)^c =$$
$$(\Delta S \cup \Delta T) \cap (\Delta S \cap \Delta T)^c =$$
$$\Delta S + \Delta T.$$

Hence all that need be shown is $\Delta S + \Delta T = (\Delta(S + T))^c$, since $f_A(S + T) = (\Delta(S + T))^c$. Now since $\Delta S + \Delta T = (\Delta(S + T))^c$ iff $\Delta(S + T) \cap (\Delta S + \Delta T) = \phi$ and $\Delta(S + T) \cup (\Delta S + \Delta T) = I$, all that need be shown is $\Delta(S + T) \cap (\Delta S + \Delta T) = \phi$ and $\Delta(S + T) \cup (\Delta S + \Delta T) = I$.

Let $i_o \in (\Delta S + \Delta T) = (\Delta S \cap (\Delta T)^c) \cup (\Delta T \cap (\Delta S)^c)$. Since $(\Delta S \cap (\Delta T)^c) \cap (\Delta T \cap (\Delta S)^c) = \phi$ we have either $i_o \in (\Delta S \cap (\Delta T)^c)$ or $i_o \in (\Delta T \cap (\Delta S)^c)$ but not both. Assume $i_o \in (\Delta S \cap (\Delta T)^c)$ then $\Sigma_{j \in S} a_{i_o j}$ is even and $\Sigma_{j \in T} a_{i_o j}$ is odd. Hence $(\Sigma_{j \in S} a_{i_o j} + \Sigma_{j \in T} a_{i_o j})$ is odd. Hence

$i_0 \varepsilon \Delta((S \cap T^c) \cup (T \cap S^c)) = \Delta(S + T)$. Similarly, if $i_0 \varepsilon (\Delta T \cap (\Delta S)^c)$, then $i_0 \notin \Delta(S + T)$. Therefore $(\Delta S + \Delta T) \cap \Delta(S + T) = \phi$.

Let $i_0 \varepsilon I$ to prove $\Delta(S + T) \cup (\Delta S + \Delta T) = I$.

Case 1 $\Sigma_{j\varepsilon S}\, a_{i_0 j}$ and $\Sigma_{j\varepsilon T}\, a_{i_0 j}$ are both even. Then $(\Sigma_{j\varepsilon(S\cap T^c)}\, a_{i_0 j} + \Sigma_{j\varepsilon(S^c\cap T)}\, a_{i_0 j})$ is even. Hence $i_0 \varepsilon \Delta(S + T)$.

Case 2 $\Sigma_{j\varepsilon S}\, a_{i_0 j}$ and $\Sigma_{j\varepsilon T}\, a_{i_0 j}$ are both odd. Then $(\Sigma_{j\varepsilon(S\cap T^c)}\, a_{i_0 j} + \Sigma_{j\varepsilon(S^c\cap T)}\, a_{i_0 j})$ is even. Hence $i_0 \varepsilon \Delta(S + T)$.

Case 3 $\Sigma_{j\varepsilon S}\, a_{i_0 j}$ is even and $\Sigma_{j\varepsilon T}\, a_{i_0 j}$ is odd. Then $i_0 \varepsilon (\Delta S \cap (\Delta T)^c)$. Hence $i_0 \varepsilon (\Delta S + \Delta T)$.

Case 4 $\Sigma_{j\varepsilon S}\, a_{i_0 j}$ is odd and $\Sigma_{j\varepsilon T}\, a_{i_0 j}$ is even. Then $i_0 \varepsilon (\Delta T \cap (\Delta S)^c)$. Hence $i_0 \varepsilon (\Delta S + \Delta T)$. Therefore $i_0 \varepsilon [(\Delta S + \Delta T) \cup \Delta(S + T)]$. Hence $(\Delta S + \Delta T) \cup \Delta(S + T) = I$. Therefore f_A is a linear function from $\mathcal{P}(J)$ into $\mathcal{P}(I)$.

Note that the need for the last three propositions is to help prove Theorem 3.15.

Lemma 1.8 Let A_n be an $n \times n$ matrix with $n \geq 3$ be that

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ & \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

then $\det(A_n)$ is 0 if n is even or 2 if n is odd.

Proof By expanding along the first column we see that

$$\det A_n = \det\begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} + (-1)^{n-1}\det\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix} =$$

$1 + (-1)^{n-1}$. Hence $\det A_n$ is 0 if n is even or 2 if n is odd.

CHAPTER 2

Let X be a finite set and let $E \subset P(X)$ where $\phi \notin E$ and $\cup E = X$, then we call the pair $(X,E) = H$ a hypergraph.

Each element in X is called a vertex and each $E \in E$ is called an edge.

A subhypergraph of $H = (X,E)$ generated by the set $A \subseteq X$ is defined to be the hypergraph $H_A = (A,E_A)$ where $E_A :=$ $\{E_A := (E \cap A) : E \in E$ and $(E \cap A) \neq \phi\}$.

In a hypergraph $H = (X,E)$ a chain of length q is defined to be a sequence $\eta = (x_1 E_1 x_2 E_2 \ldots x_q E_q x_{q+1})$ such that $x_1, \ldots ,x_q$ are distinct vertices of H, $E_1, \ldots ,E_q$ are distinct edges of H and for each $k \in \{1, \ldots ,q\}$, $x_k, x_{k+1} \in E_k$. Let $\lambda(\eta)$ denote the length of η.

A cycle of length q $(q > 1)$ is a chain of length q with $x_1 = x_{q+1}$.

A bicoloring of a hypergraph $H = (X,E)$ is a partition (S_1,S_2) of X such that if $E \in E$ with $|E| > 1$ then $E \not\subset S_1$ and $E \not\subset S_2$.

An equitable bicoloring of a hypergraph $H = (X,E)$ is a partition (S_1,S_2) of X such that, for each $E \in E$, $||E \cap S_1| - |E \cap S_2|| \leq 1$.

Proposition 2.1 An equitable bicoloring of a hypergraph $H = (X,E)$ is a bicoloring of H.

Proof Let (S_1,S_2) be an equitable bicoloring and $E \in E$ with $|E| > 1$, then since $||E \cap S_1| - |E \cap S_2|| \leq 1$ and $S_1 \cap S_2 = \phi$, $E \not\subset S_1$ and $E \not\subset S_2$. Therefore (S_1,S_2) is a bicoloring.

The following example shows that the converse is not true.

Example 2.2 Let $X := \{1,2,3,4\}$, $E := \{X\}$, $S_1 := \{1,2,3\}$ and $S_2 := \{4\}$, then (S_1,S_2) is a bicoloring but not an equitable bicoloring.

Proof Clearly (S_1,S_2) is a bicoloring and
$||X \cap S_1| - |X \cap S_2|| = 2$.

Let $G = (X,E)$ be a graph. Define $C_G = \{\mu : \mu$ is a cycle of $G\}$, $Ch_G = \{\mu : \mu$ is a chain of $G\}$. For each $\mu \in Ch_G$, let
$E_\mu := \{x : x$ is a vertex of $\mu\}$.

Proposition 2.3 Let $G = (X,E)$ be a graph with a bicoloring (S_1,S_2). Let $E = \{E_\mu : \mu \in C_G\}$, then (S_1,S_2) is a bicoloring of the hypergraph $H = (X,E)$.

Proof Let G be a graph with a bicoloring (S_1,S_2). Let be any element in C_G. Since (S_1,S_2) is a bicoloring for G, $E_\mu \not\subset S_1$ and $E_\mu \not\subset S_2$. Therefore (S_1,S_2) is a bicoloring for H.

A hypergraph $H = (X,E)$ is defined to be unimodular if for each $S \subseteq X$, where $|S| \geq 2$, the subhypergraph H_S has an equitable bicoloring.

Example 2.4 Let $X := \{1, \ldots ,n\}$, $n \geq 2$, and
$F := \{[i,i+1] : 1 \leq i \leq n\}$. Define $G = (X,F)$. Choose $C \in P(Ch_G)$ such that $\bigcup_{\mu\varepsilon C} E_\mu = X$, then for $E := \{E_\mu : \mu \in C\}$ we have
$H = (X,E)$ is a unimodular hypergraph.

Proof Let G be a graph defined as in the hypothesis. Choose $C \in P(Ch_G)$ such that $\bigcup_{\mu\varepsilon C} E_\mu = X$ (such a C exists since $\mu = (1, \ldots ,n)$ is a chain in G). Let $S \subseteq X$, where $|S| \geq 2$.

Define $I_j = \{i \in S : i \leqq j\}$, $S_1 = \{j : |I_j| \text{ is even}\}$ and $S_2 = \{j : |I_j| \text{ is odd}\}$. Then (S_1,S_2) is a partition of S. Let $E \in E_S$. If $|E|$ is even then by the definition of S_1 and S_2 and the structure of each chain in G, $|E \cap S_1| - |E \cap S_2| = 0$. Also by the same reasoning, if $|E|$ is odd, then $||E \cap S_1| - |E \cap S_2|| = 1$. Hence for each $E \in E_S$, $||E \cap S_1| - |E \cap S_2|| \leqq 1$. Therefore (S_1,S_2) is an equitable bicoloring for H_S. Therefore for each $S \subseteq X$, H_S has an equitable bicoloring. Therefore H is a unimodular hypergraph.

<u>Proposition 2.5</u> If H is a unimodular hypergraph, then each subhypergraph of H is also unimodular.

<u>Proof</u> Let H be a unimodular hypergraph and $S \subseteq X$, $|S| \geqq 2$, then for any $T \subseteq S$, $|T| \geqq 2$, we have $(H_S)_T = H_T$, which has an equitable bicoloring. Hence H_S is unimodular. Therefore each subhypergraph is unimodular.

<u>Theorem 2.6</u> Let $G = (X,E)$ be a graph with no odd cycles, then G has a bicoloring.

<u>Proof</u> <u>Case 1</u> G is connected.

Let $x' \in X$, then define $S_1 = \{y \in X :$ there is a chain of odd length connecting x' and $y\}$ and $S_2 = \{y \in X :$ there is a chain of even length connecting x' and $y\} \cup \{x'\}$.

<u>Claim</u> (S_1,S_2) is a bicoloring of X.

<u>Proof of claim</u> Clearly if $x' \in S_1$, G would have an odd cycle.

If $x' \neq y \in (S_1 \cap S_2)$ then there exist chains μ_1 and μ_2 of minimal length such that μ_1 connects x' to y, μ_2 connects y to x', $\lambda(\mu_1)$ is odd and $\lambda(\mu_2)$ is even. Then $\mu = (\mu_1,\mu_2)$ is a cycle with $\lambda(\mu) = \lambda(\mu_1) + \lambda(\mu_2)$ which is odd. But G has no odd cycles, so $S_1 \cap S_2 = \phi$. Hence (S_1, S_2) is a partition of X.

Let $y_1, y_2 \in S_1$ (the argument for S_2 is similar) and assume $[y_1, y_2] \in E$. Then there exists chains μ_1 and μ_2 of minimal length such that μ_1 connects x' to y_1, μ_2 connects y_2 to x' and both $\lambda(\mu_1)$ and $\lambda(\mu_2)$ are odd. Then $\mu = (\mu_1, y_1, y_2, \mu_2)$ is a cycle such that $\lambda(\mu) = \lambda(\mu_1) + 1 + \lambda(\mu_2)$ which is odd. But G has no odd cycles, so $[y_1, y_2] \notin E$. Therefore (S_1, S_2) is a bicoloring of G.

Case 2 G is not connected.

Define $S_1 = U(S_{i1})$ and $S_2 = U(S_{i2})$ where (S_{i1}, S_{i2}) is a bicoloring for the i^{th} connected component of G which exist since each connected component is a connected graph with no odd cycles. Then (S_1, S_2) is a bicoloring for G.

The following theorem gives a sufficient condition for a hypergraph to be unimodular.

Theorem 2.7 If $H = (X, E)$ is a hypergraph without odd cycles, then H is unimodular.

Proof Let H have no odd cycles, then each subhypergraph of H has no odd cycles. For, if H_S has an odd cycle, $(a_1 E_1^* \dots a_{2n+1} E_{2n+1}^* a_1)$ where $\{a_1, \dots a_{2n+1}\} \subseteq S \subseteq X$ and for each i, $E_i^* = (S \cap E_i) \in E_S$ for some $E_i \in E$, then $(a_1 E_1 \dots a_{2n+1} E_{2n+1} a_1)$ would be an odd cycle of H. Hence, by the definition of a unimodular hypergraph, we need only show that a hypergraph H without odd cycles has an equitable bicoloring.

Let f be a 1-1 and onto map from $I := \{1, \ldots, |E|\}$ to E. For each $i \in I$, denote $E_i' = f(i)$. For each $i \in I$, let f_i be a 1-1 and onto map from $\{1, \ldots, r_i\}$ to E_i', where $|E_i'| = r_i$. If r_i is even define $F_i =$

$$\{[f_i(1),f_i(2)],[f_i(3),f_i(4)], \ldots ,[f_i(r_i-1),f_i(r_i)]\}.$$

If r_i is odd then define $F_i =$

$$\{[f_i(1),f_i(2)],[f_i(3),f_i(4)], \ldots ,[f_i(r_i-2),f_i(r_i-1)],[f_i(r_i),f_i(r_i)]\}.$$

Define $\hat{E} = \cup\{F_i : i \in I\}$, then consider the graph $G = (X,\hat{E})$.

Suppose that G contains an elementary odd cycle, $\mu = (a_1, \ldots ,a_{2p+1},a_1)$, for some positive integer p. We may assume that μ contains two disjoint edges from the same class, F_i. If not, then there would exist a cycle $(a_1E_1 \ldots a_{2p+1}E_{2p+1}a_1)$ which is odd and hence contradicting the fact that H has no odd cycles.

Without loss of generality, let these two edges be $[a_s,a_{s+1}]$ and $[a_t,a_{t+1}]$. Transform F_i by changing the above two edges into $[a_s,a_{t+1}]$ and $[a_t,a_{s+1}]$. Replace μ by either the sequence $\mu' = (a_1, \ldots ,a_s,a_{t+1}, \ldots ,a_{2p+1},a_1)$ or $\mu'' = (a_{s+1},a_{s+2}, \ldots ,a_t,a_{s+1})$ each of which are elementary cycles of odd length.

Repeat this process as many times as possible. Since μ had only finite length, this process will terminate. At the final step, the cycle will be elementary and odd. As stated above, this odd cycle will determine an odd cycle in H, which is a contradiction. Therefore G contains no elementary odd cycles. Proposition 1.2 implies that if G has odd cycles, then G has elementary odd cycles. Hence G contains no odd cycles.

Define S_1 and S_2 as in the proof of Theorem 2.6 then there is a bicoloring (S_1,S_2) of the vertices of G. Since any vertex in S_1 is adjacent only to itself and from the way in which $\hat{E}$ was constructed, we have $||E \cap S_1| - |E \cap S_2|| \leq 1$ for all $E \in \mathcal{E}$. Hence (S_1,S_2) is an equitable bicoloring of H.

The following example shows that a unimodular hypergraph can have odd cycles.

<u>Example 2.8</u> Define $X = \{1,2,3,4\}$, $E_1 = \{1,2\}$, $E_2 = \{2,3\}$, $E_3 = \{3,4\}$, $E_4 = \{4,1\}$, $E_5 = X$ and $\mathcal{E} = \{E_1,E_2,E_3,E_4,E_5\}$. then $H = (X,\mathcal{E})$ is unimodular and has odd cycles.

<u>Proof</u> Let $S \subset X$ with $|S| = 2$. Let (S_1,S_2) be a partition of S, then for all i, $||E_i \cap S_1| - |E_i \cap S_2|| \leq 1$.

Suppose $|S| = 3$, then partition S into (S_1,S_2) where $S_1 = \{1,3\}$ or $S_1 = \{2,4\}$. Clearly for $i \neq 5$, $||E_i \cap S_1| - |E_i \cap S_2|| = 0$. Now for $i = 5$ we have $|E_5 \cap S_1| - |E_5 \cap S_2| = 2 - 1 = 1$.

Finally, suppose $S = X$, then partition S into (S_1,S_2) where $S_1 := \{1,3\}$ and $S_2 :=\{2,4\}$. Then for each i, $|E_i \cap S_1| = |E_i \cap S_2|$. Hence for each $S \subseteq X$, where $|S| \geq 2$, H_S has an equitable bicoloring. Therefore H is unimodular with a cycle $(1E_1 2E_2 3E_5 1)$ which is odd.

CHAPTER 3

A <u>submatrix</u> of a matrix A is a marix formed by deleting rows and columns of A.

A <u>totally unimodular</u> matrix is a matrix such that every square submatrix has determinant $0,1$ or -1.

<u>Proposition 3.1</u> If A is totally unimodular, then each a_{ij} is either $0,1$ or -1.

<u>Proof</u> (a_{ij}) is a 1×1 square submatrix of A.

Let A_i and C_j denote the i^{th} row and j^{th} column, respectively, of the matrix A.

For the remainder of the thesis, the same letter will be used to denote a set of rows or columns and the matrix formed by these rows or columns. The context in which each will be used should cause no confusion.

Let $n \leq m$ and let S be an $n \times m$ integral matrix then for $\{d : d$ is a determinant for some $n \times n$ submatrix of $S\} := D$ define

$$\text{GCD } S = \begin{cases} 0 & \text{if } D = \{0\} \\ \text{greatest common divisor of all the elements in } D & \text{otherwise.} \end{cases}$$

<u>Lemma 3.2</u> Let m and n be relatively prime integers then there exists integers a and b such that $an + bm = 1$.

Proof The proof can be found in [Hernstein, Lemma 1.3.1].

Lemma 3.3 Let S be an $n \times m$ integral matrix then $S = UDV$ where U, D and V are integral, U is $n \times n$, D is $n \times m$, V is $m \times m$, $|\det U| = |\det V| = 1$ and D is diagonal.

Proof The proof will be by induction on $n + m$. Since the smallest $n + m$ can be is 2 we will start the induction at 2.

Let $n + m = 2$, then $n = 1$ and $m = 1$. Then $U := I_1$, $D := S$ and $V := I_1$ have the desired properties.

Assume that for each $(n + m) \in \{2, \ldots, p\}$ we have shown that there exists U, D and V with the desired properties.

Now assume $n + m = p + 1$.

Case I $m > 1$

Define S^* to be the first p columns of S, then by the induction hypothesis we can write $S^* = U^*D^*V^*$, where U^*, D^* and V^* have the desired properties corresponding to S^*. Let

$$V' := \begin{pmatrix} V^* & 0 \\ 0 & I_1 \end{pmatrix}$$

and $C'_m := (U^*)^{-1}C_m$ where C_m is the m^{th} column of S. Then C'_m is integral and $S = U^*(D^*, C'_m)V'$. Let $J' :=$ $\{j : d^*_{jj} \neq 0$ and $c'_{jm} \neq 0\}$. For each $j \in J'$, let $k_j := \gcd(d^*_{jj}, c'_{jm})$ then by Lemma 3.2 there exists integers a_j and b_j such that $a_j d^*_{jj} + b_j c'_{jm} = 1$. For each $j \in J'$ let

$$V_j := \begin{pmatrix} I_{j-1} & & 0 & 0 \\ 0 & a_j & 0 & -c'_{jm}/k_j \\ 0 & 0 & I_{m-j-1} & 0 \\ 0 & b_j & 0 & d^*_{jj}/k_j \end{pmatrix}$$

then V_j is integral and $\det V_j = \pm 1$. Let $I :=$ $\{i : c'_{im}$ is the only nonzero element in the i^{th} row$\}$ and for each $r \in \{1, \ldots, |I|\} := I'$, let $k'_r := \gcd(c'_{i_0 m}, \ldots, c'_{i_r m})$ where $i_0 < i_1 < \ldots < i_r$ and for each $i \in I \setminus \{i_0, \ldots, i_r\}$, $i_r < i$. By Lemma 3.2 we have for each $r \in I'$ there exists integers a'_r and b'_r such that $a'_r k'_{r-1}/k'_r + b'_r c'_{i_r m}/k'_r = 1$, where $k'_0 := c'_{i_0 m}$. For each $r \in I'$ let

$$U_r := \begin{pmatrix} I_{i_0 - 1} & 0 & 0 & 0 \\ 0 & a'_r & 0 & b'_r \\ 0 & 0 & I_{n-i_0-1} & 0 \\ 0 & -c'_{i_r m}/k'_r & 0 & k'_{r-1}/k'_r \end{pmatrix},$$

then U_r is integral and $\det U_r = \pm 1$.

If $i_0 \leqq \min\{m,n\}$ then let P be the permutation matrix permuting column i_0 with column m. Otherwise let P be the permutation matrix permuting row i_0 with row m.

For the following, the product over the empty set is the identity matrix. If $i_0 \leqq \min\{m,n\}$ then $U := U^*(\Pi_{r \in I'} U_r)^{-1}$, $V := (\Pi_{j \in J'} V_j)^{-1} P^t V'$ and $D := (\Pi_{r \in I'} U_r)((D^*, C'_m))(\Pi_{j \in J'} V_j)P$ have the desired properties. If $i_0 > \min\{m,n\}$ then

$U := U^*P^t(\Pi_{r\varepsilon I'}\ U_r)^{-1}$, $V := (\Pi_{j\varepsilon J'}\ V_j)^{-1}V'$ and

$D := P(\Pi_{r\varepsilon I'}\ U_r)((D^*,C'_m))(\Pi_{j\varepsilon J'}\ V_j)$ have the desired properties.

Case II $n > 1$

By Case I there exist U,D and V, with the desired properties, such that $S^t = UDV$. Hence $U' := V^t$, $D' := D^t$ and $V' := U^t$ have the desired properties for S. Hence by induction, S has the desired factorization.

<u>Lemma 3.4</u> Let S be an $n \times m$ integral matrix with $n \leqq m$ and let B be an $n \times n$ integral matrix such that $\det B = \pm 1$ then $\text{GCD } S = \text{GCD } BS$.

<u>Proof</u> Let $S' := BS$ and for $1 \leqq k \leqq n$ let S_{i_k} and S'_{i_k} be the i_k^{th} columns of S and S', respectively. Now since $S'_{i_k} = BS_{i_k}$, $(S'_{i_1}, \ldots ,S'_{i_n}) = B(S_{i_1}, \ldots ,S_{i_n})$ and $|\det (S'_{i_1}, \ldots ,S'_{i_n})| = |\det (S_{i_1}, \ldots ,S_{i_n})|$. Therefore by the definition of GCD, $\text{GCD } S = \text{GCD } BS$.

<u>Lemma 3.5</u> Let S be an integral $n \times m$ matrix with $\text{GCD } S = 1$ and let A be an integral $m \times m$ matrix with $\det A = \pm 1$ and SA diagonal, then $\text{GCD } SA = 1$.

<u>Proof</u> Let $\mathcal{M}_n := \{\{S_{i_1}, \ldots, S_{i_n}\} \subseteq S : S_{i_j}$ is the i_j^{th} column of S and $\{S_{i_1}, \ldots, S_{i_n}\}$ is a set of n distinct elements$\}$, then $\text{GCD}\, S$ = greatest common divisor of $\{\det M : M \in \mathcal{M}_n\}$.

Let $d_1, \ldots, d_n$ be the diagonal elements of SA, then since $\text{GCD}\, S \neq 0$ (which implies $\text{GCD}\, SA \neq 0$), we have that for each $i \in \{1, \ldots, n\}$, $d_i \neq 0$. Hence one sees that the first n rows of A^{-1} must be

$$\begin{pmatrix} a_{11}/d_1 & \cdots & a_{1m}/d_1 \\ & \vdots & \\ a_{n1}/d_n & \cdots & a_{nm}/d_n \end{pmatrix}.$$

Since $\det A = \pm 1$ we have by the well known Cramer's Rule that A^{-1} is integral. Hence for each i, d_i divides every element in the i^{th} row of S.

Let $M \in \mathcal{M}_n$ then $\det M =$

$$(\Pi d_i)\det \begin{pmatrix} m_{11}/d_1 & \cdots & m_{1n}/d_1 \\ & \vdots & \\ m_{n1}/d_n & \cdots & m_{nn}/d_n \end{pmatrix}.$$

Since fore each i, d_i divides every element in the i^{th} row of S,

$$\det \begin{pmatrix} m_{11}/d_1 & \cdots & m_{1n}/d_1 \\ & \vdots & \\ m_{n1}/d_n & \cdots & m_{nn}/d_n \end{pmatrix}$$

is integral. Therefore Πd_i divides $\det M$ for all $M \in \mathcal{M}_n$. Hence Πd_i divides $\text{GCD}\, S$. But $\text{GCD}\, S = 1$, so $\Pi d_i = \pm 1$. Since for each i, d_i is integral, we have $d_i = \pm 1$. Therefore $\text{GCD}\, SA = 1$.

<u>Corollary 3.6</u> Let S be an $n \times m$ integral matrix with $n \leq m$. Let UDV be a factorization of S by Lemma 3.3, and let $\text{GCD}\, S = 1$, then $\text{GCD}\, D = 1$.

Proof Since U^{-1} is integral with $\det U^{-1} = \pm 1$ we have by Lemma 3.4, $\mathrm{GCD}\ U^{-1}S = \mathrm{GCD}\ S$. Therefore, since V^{-1} is integral with $\det V^{-1} = \pm 1$, $U^{-1}SV^{-1} = D$ is diagonal and $\mathrm{GCD}\ U^{-1}S = \mathrm{GCD}\ S = 1$, we have by Lemma 3.5 $\mathrm{GCD}\ D = 1$.

Let $x,y \in \mathbb{R}^n$ then define $x \geqq y$ if for each $i \in \{1, \ldots ,n\}$, $x_i \geqq y_i$, where x_i and y_i are the i^{th} component of x and y respectively.

Let A be any $m \times n$ matrix and $b \in \mathbb{R}^m$, then define $P(b) := \{x \in \mathbb{R}^n : Ax \geqq b\}$. Notice that $P(b)$ is not necessarily bounded. For example, let A be the 1×1 matrix (1) then $P(2) = \{x \in \mathbb{R} : (1)x \geqq 2\}$ is clearly unbounded.

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$ and S a subset of the rows of A, then define $F_S = F_S(b) = \{x \in \mathbb{R}^n : Ax \geqq b$ and $A_i x = b_i$ if $A_i \in S\}$, $L_S(b) = \{x \in \mathbb{R}^n : A_i x = b_i$ for all $A_i \in S\}$ and G_S = the subspace of $\mathbb{R}^n$ spanned by the rows of S.

If F_S is not empty, then it is called a <u>face</u> of $P(b)$. If F_{S_1} and F_{S_2} are such that $F_{S_1} \subseteq F_{S_2}$, then F_{S_1} is called a <u>subface</u> of F_{S_2}.

<u>Lemma 3.7</u> Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$ and let $S,S^* \subseteq A$ be subsets of the rows of A such that $S \subseteq S^*$; moreover let F_{S^*} and F_S be faces, then:

i) $F_S \subseteq F_{S^*}$

ii) F_S is a minimal face iff $G_A = G_S$, i.e., iff S has the same rank as A.

<u>Proof of i)</u> Let $S \subseteq S^*$ and $x \in F_{S^*}$ then for all $A_i \in S$,

$A_i x = b_i$. Hence $x \in F_S$. Therefore $F_{S^*} \subseteq F_S$.

<u>Proof of ii)</u> Let S^* be all rows of A in which $G_S = G_{S^*}$, then A_i is a linear combination ofthe $A_i \in S$ iff $A_i \in S^*$. Therefore what must be proved is that $G_S = G_A$ iff $S^* = A$.

The first implication is proven by proving the contrapositive. If $S^* \neq A$, there is at least one row $A_k \in A \backslash S^*$. Then there is a vector y such that $A_i y = 0$ for all $A_i \in S$ and $A_k y < 0$.

Let $x' \in F_S$. As $A_k x' \geqq b_k$, there is a number $\lambda_k > 0$, for which $A_k(x' + \lambda_k y) = b_k$. For every $A_j \in A \backslash S^*$ such that $A_j y < 0$, the equation $A_j(x' + \lambda y) = b_j$ has a nonnegative solution. Let λ_j be that solution. Define $\lambda = \min \{\lambda_j\}$ and let j' be a value such that $\lambda = \lambda_{j'}$. Since λ_k exists, there is at least one λ_j, so λ exists. By the definition of λ, we have:

$$A(x' + \lambda y) \geqq b,$$

$$A_i(x' + \lambda y) = b_i \quad \text{for all} \quad A_i \in S \quad \text{and}$$

$$A_{j'}(x' + \lambda y) = b_{j'}.$$

Thus $F_{(S \cup A_{j'})}$ is not empty, and is therefore a proper subface of F_S, since $A_{j'} x' > b_{j'}$. Hence F_S is not minimal.

The second implication is also proven by proving the contrapositive.

Let F_S not be minimal, then it has some proper subface $F_{(S \cup A_j)}$. Hence there exists $x_1, x_2 \in F_S$ such that $A_j x_1 = b_j$ and $A_j x_2 > b_j$. Therefore $A_j x$ varies as x ranges over F_S. But for $A_i \in S$, $A_i x = b_i$ is constant as x varies over F_S. Hence A_j cannot be a linear conbination of the $A_i \in S$, so $A_j \in A \backslash S^*$. Hence $S^* \neq A$. Therefore $S^* = A$ iff $G_S = G_A$.

If b is an m-tuple and S is a set of r linearly independent rows of A, then let b_S be the "subvector" consisting of the r components of b which correspond to the rows of S. Let $\tilde{b}$ always represent an r-tuple. The components of $\tilde{b}$ and b_S will be indexed by the indices used for the rows of S, not by the integers 1 to r.

<u>Lemma 3.8</u> Suppose S is a set of r linearly independent rows of A, where $r = \text{rank } A$, then for any $\tilde{b}$ there is a b such that:

E1 $b_S = \tilde{b}$ and,

E2 F_S is a minimal face of $P(b)$.

<u>Proof</u> As S is a set of linearly independent rows, the equation $Sx = \tilde{b}$ has at least one solution, call it y. Define b as follows:

$$b = \begin{cases} \tilde{b}_i & \text{if } A_i \in S \\ [A_i y] & \text{if } A_i \notin S. \end{cases}$$

Hence $b_S = \tilde{b}$ and b is integral. Also $y \in F_S$, so $F_S \neq \phi$. By Lemma 3.7, F_S is a minimal face, so E2 is satisfied.

<u>Lemma 3.9</u> Suppose S^* is a set of rows of a matrix A that has the same rank as A, and $S \subseteq S^*$ is a set of $r := \text{rank } A$ linearly independent rows of A. Then for any b such that $F_S{*}$ is a face, $F_S{*} = L_S$.

<u>Proof</u> Let y be a fixed element in $F_S{*}$ and let x be any element of L_S. Since for all $A_i \in S^*$ we have $A_i y = b_i$, $F_S{*} \subseteq L_S$.

Since S had rank r, any row $A_k \in A$ can be expressed as a linear combination of rows $A_i \in S$, i.e., $A_k = \Sigma\alpha_{k_i} A_i$

Since $A_i x = A_i y = b$ for all $A_i \in S$, $A_k x = \Sigma\alpha_{k_i} A_i x = \Sigma\alpha_{k_i} A_i y = A_k y$. Hence $L_S \subseteq F_S{*}$. Therefore $F_S{*} = L_S$.

Lemma 3.10 Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$, then any minimal face of $P(b)$ can be expressed in the form F_S where S is a set of $r := \text{rank } A$ linearly independent rows of A.

Proof Assume the face is F_{S^*}. By Lemma 3.7, S^* must have rank r. Let S be a set of r linearly independent rows of S^*. By Lemma 3.8 $F_{S^*} = L_S = F_S$.

Lemma 3.11 Let A be an $m \times n$ integral matrix. Let S be a set of linearly independent rows of A and let S have the same rank as A, then the following are equivalent:

i) $L_S(\tilde{b})$ contains an integral point for every intergral $\tilde{b}$

ii) $\text{GCD } S = 1$

Proof By Lemma 3.3 there exists integral matrices U,D and V such that $S = UDV$, $|\det U| = |\det V| = 1$ and D is diagonal. By Lemma 3.6 $\text{GCD } S = \text{GCD } D$. Clearly $\text{GCD } D = |d_{11} \cdot \ldots \cdot d_{rr}|$ where $r := \text{rank } A$. Hence ii) is equivalent to the condition that for $i \in \{1, \ldots r\}$, $d_{ii} = \pm 1$.

Suppose that $d_{jj} = k > 1$ for some j. Without loss of generality we may assume that $j = 1$. Let $\tilde{e}$ be the r-tuple $(1,0, \ldots ,0)^t$, let $\tilde{b} := U\tilde{e}$ and let $x \in L_S(\tilde{b})$. Then $Sx = UDVx = \tilde{b} = U\tilde{e}$. Hence $DVx = \tilde{e}$. Clearly the first component of $y = Vx$ is $1/k$. Hence y is not integral. Therefore x cannot be integral. Therefore $F_S(\tilde{b})$ contains no integral point. Hence if $\text{GCD } S \neq 1$ then i) cannot hold.

Assume $d_{ii} = \pm 1$ for $i \in \{1, \ldots ,r\}$. Let $x \in L_S(\tilde{b})$ and set $Vx = (y_1, \ldots ,y_r,y_{r+1}, \ldots ,y_n)$. Then $U^{-1}\tilde{b} = DVx =$

$(d_{11}y_1, \ldots, d_{rr}y_r)$, and so $y_1, \ldots, y_r$ are integral. Let $y = (y_1, \ldots, y_r, 0, \ldots, 0)$, then $V^{-1}y$ is integral. Since $Dy = DVx$, $S(V^{-1}y) = UDV(V^{-1}y) = UDy = UDVx = \tilde{b}$. Thus $V^{-1}y \in L_S(\tilde{b})$. Therefore ii) implies i).

<u>Theorem 3.12</u> If A is totally unimodular, b is integral and $P(b)$ is nonempty, then $P(b)$ contains an integral point.

<u>Proof</u> Let $F_S{}^*$ be some minimal face of $P(b)$. By Lemma 3.10 this face can be expressed as F_S where S consists of $r := \text{rank } A$ linearly independent rows of A. By Lemma 3.9, $F_S(b) = L_S(b_S)$. Since A is totally unimodular and S is of full rank we have $\text{GCD } S = 1$. By Lemma 3.11, $L_S(\tilde{b})$ must contain an integral point. Therefore $P(b)$ contains an integral point.

Let A be an $m \times n$ matrix, $b, b' \in \mathbb{R}^m$ and $c, c' \in \mathbb{R}^n$ where $b \leqq b'$ and $c \leqq c'$ then define $P(b,b';c,c') = \{x \in \mathbb{R}^n : b \leqq Ax \leqq b' \text{ and } c \leqq x \leqq c'\}$.

<u>Theorem 3.13</u> A is totally unimodular iff for every integral b, b', c and c' such that $P(b,b';c,c')$ is nonempty, then $P(b,b';c,c')$ contains an integral point.

<u>Proof</u> Assume A is totally unimodular and $P(b,b';c,c')$ is nonempty. Consider $A_* := \begin{pmatrix} A \\ -A \\ I \\ -I \end{pmatrix}$. Clearly A_* is totally unimodular.

For $x \in P(b,b';c,c')$, $b \leqq Ax \leqq b'$ and $c \leqq x \leqq c'$ iff $Ax \leqq b$, $-Ax \leqq -b'$, $x \leqq c$ and $-x \leqq -c'$. Hence $P(b,b';c,c') =$

$P_*((b_1, \ldots, b_m, -b'_1, \ldots, -b'_m, c_1, \ldots, c_n, -c'_1, \ldots, -c'_n)^t)$.
Since $P_*(\cdot)$ is nonempty, we have by Theorem 3.12 $P(b,b';c,c')$ contains an integral point.

Suppose A is integral but not totally unimodular. Since A is not totally unimodular there is a $p \times p$ submatrix A^* of A such that $\det A^* \neq 0$ or 1. Hence by Theorem 1.4, there is a non-integral $x \in \mathbb{R}^p$ such that A^*x is integral. Let I,K be the sets of indices that indexes the rows and columns, respectively, of A^*. Define

$$x' = \begin{cases} x'_i = x_i & i \in K \\ x'_i = 0 & i \notin K, \end{cases} \quad b = \begin{cases} b_i = A_i x' & i \in I \\ b_i = [A_i x'] & i \notin I, \end{cases} \quad b' = \begin{cases} b'_i = b_i & i \in I \\ b'_i = b_i + 1 & i \notin I, \end{cases}$$

$$c = \begin{cases} c_i = [x_i] & i \in K \\ c_i = 0 & i \notin K \end{cases} \quad \text{and} \quad c' = \begin{cases} c'_i = c_i + 1 & i \in K \\ c'_i = 0 & i \notin K. \end{cases}$$

Then b,b',c and c' are integral and $P(b,b';c,c')$ is well defined and nonempty.

Define $A' = \{A_i \in A : i \in I\}$. Since $\det A^* \neq 0$ and $A^*x = b_{A'} = A'x'$ and by the definition of c and c', we have $\{x'\} = P(b,b';c,c')$. Hence $P(b,b';c,c')$ contains no integral points. Therefore there are integral b,b',c and c' such that $P(b,b';c,c')$ is nonempty and does not contain any integral points.

Lemma 3.14 Let A be an $m \times n$ totally unimodular matrix then for all $x \in \{0,1,-1\}^n$ there exists a vector $y \in \{0,1,-1\}^n$ such that $y \equiv x \pmod 2$ and for each $A_i \in A$,

$$A_i y = \begin{cases} A_i y = 0 & \text{if } A_i x \equiv 0 \pmod 2 \\ A_i y = \pm 1 & \text{otherwise} \end{cases}$$

Proof Let $x \in \{0,1,-1\}^n$ and $Ax = a$. Define d^v for $v = 1,2$

to be

$$d_i^v = \begin{cases} 0 & \text{if } x_i \equiv 0(\text{mod } 2) \\ 1/2(x_i - 1) & \text{if } x_i \equiv 1(\text{mod } 2) \text{ and } v = 1 \\ 1/2(x_i + 1) & \text{if } x_i \equiv 1(\text{mod } 2) \text{ and } v = 2. \end{cases}$$

Define b^v for $v = 1,2$ to be

$$b_i^v = \begin{cases} 1/2(a_i) & \text{if } a_i \equiv 0(\text{mod } 2) \\ 1/2(a_i - 1) & \text{if } a_i \equiv 1(\text{mod } 2) \text{ and } v = 1 \\ 1/2(a_i + 1) & \text{if } a_i \equiv 1(\text{mod } 2) \text{ and } v = 2. \end{cases}$$

Note that $b^1 \leqq 1/2(a) \leqq b^2$ and $d^1 \leqq 1/2(x) \leqq d^2$. Hence $1/2(x) \varepsilon P(b^1,b^2;d^1,d^2)$. Therefore by Theorem 3.13 there exists $x' \varepsilon \{0,1,-1\}^n \cap P(b^1,b^2;d^1,d^2)$. Set $y = x - 2x'$, then $y \equiv x(\text{mod } 2)$ and $A_iy = A_i(x - 2x') = a_i - 2(1/2(a_i)) = 0$ if $a_i \equiv 0(\text{mod } 2)$, $A_iy = a_i - 2(1/2(a_i - 1)) = 1$ if $a_i = b_i^1$ and $A_iy = a_i - 2(1/2(a_i + 1)) = -1$ if $a_i = b_i^2$.

<u>Theorem 3.15</u> Let A be a $0,1$ $m \times n$ matrix with I indexing the rows and K indexing the columns, then A is totally unimodular iff for each $J \subseteq K$ there exists a partition (J_1,J_2) of J such that for each $i \varepsilon I$, $|\Sigma_{j\varepsilon J_1} a_{ij} - \Sigma_{j\varepsilon J_2} a_{ij}| \leqq 1$.

<u>Proof</u> Assume A is totally unimodular, then let A^* be any submatrix of A, then clearly A^* is totally unimodular. So all that is needed to be shown is that for A, there are $K_1,K_2 \subset K$ such that for each $i \varepsilon I$, $|\Sigma_{j\varepsilon J_1} a_{ij} - \Sigma_{j\varepsilon J_2} a_{ij}| \leqq 1$.

The proof will be by induction on $p = |K|$.

Let $p = 2$ then clearly there exists $x \varepsilon \{1,-1\}^p$ such that $Ax \varepsilon \{0,1,-1\}^m$.

Suppose that for any $m \times p$ totally unimodular matrix, where

$2 \leqq p \leqq n$, there exists $x \varepsilon \{1,-1\}^p$ such that $Ax \varepsilon \{0,1,-1\}^m$.

Assume that A is an $m \times n$ totally unimodular matrix. Then for $(C_1, \ldots, C_{n-1})$ there exists an $x \varepsilon \{1,-1\}^{n-1}$ such that $(C_1, \ldots, C_{n-1})x \varepsilon \{0,1,-1\}^m$. Define

$$x' = \begin{cases} x'_i = x_i & \text{for } i = 1, \ldots, n-1 \\ x'_n = 1 & \text{otherwise} \end{cases} .$$

If $Ax \varepsilon \{0,1,-1\}^m$ then we are finished. Suppose not, then by Lemma 3.14 there exists a $y \varepsilon \{1,-1\}^n$ such that $Ay \varepsilon \{0,1,-1\}^m$. Therefore by induction any $m \times p$ totally unimodular matrix A is such that there exists $x \varepsilon \{1,-1\}^p$ such that $Ax \varepsilon \{0,1,-1\}^m$.

Let A be an $m \times n$ totally unimodular matrix and $x \varepsilon \{1,-1\}^n$ such that $Ax \varepsilon \{0,1,-1\}^m$. Define $J_1 = \{j : x_j = 1\}$ and $J_2 = \{j : x_j = -1\}$. If either J_1 or J_2 is empty then clearly there is another choice for x. Assume neither J_1 nor J_2 are empty, then (J_1,J_2) is a partition of J and for each $i \varepsilon I$

$$|\Sigma_{j \varepsilon J_1} a_{ij} - \Sigma_{j \varepsilon J_2} a_{ij}| \leqq 1.$$

Assume that for each $J \subseteq K$ there are $J_1, J_2 \subset J$ such that (J_1,J_2) is a partition of J and for each $i \varepsilon I$

$$|\Sigma_{j \varepsilon J_1} a_{ij} - \Sigma_{j \varepsilon J_2} a_{ij}| \leqq 1.$$

Suppose A contains a square submatrix, A^*, such that the determinant of any square submatrix of A^*, but not equal to A^*, is $0,1$ or -1 and $\det A^* \neq 0,1$ or -1. Let I_o index the rows of A^* and K_o index the columns of A^*.

Let $f_{A^*} : P(K_o) \to P(I_o)$ as in Chapter 1.

Suppose f_{A^*} is not 1-1, then there exists $J \subseteq K_o$ such that $f_{A^*}(J) = \phi$. Hence $\Delta J = I_o$. Since there exists $J_1, J_2 \subset J$ such that

J_1, J_2 partition J and for each $i \in I_o$, $|\Sigma_{j\epsilon J_1} a_{ij} - \Sigma_{j\epsilon J_2} a_{ij}| \leq 1$ and $\Delta J = I_o$, we have for each $i \in I_o$, $\Sigma_{j\epsilon J_1} a_{ij} - \Sigma_{j\epsilon J_2} a_{ij} = 0$.

Define

$$x = \begin{cases} x_j = 1 & j \in J_1 \\ x_j = -1 & j \in J_2 \\ x_j = 0 & \text{otherwise,} \end{cases}$$

then $A^* x = 0$. But since $x \neq 0$, this contradicts the fact that $\det A^* \neq 0$.

Suppose $f_A{*}$ is 1-1, then for some $i_o \in I_o$, there exists a unique $J \subseteq K_o$ such that $f_A{*}(J) = \{i_o\}$. Since there exists a partition (J_1, J_2) of J such that for each $i \in I_o$, $|\Sigma_{j\epsilon J_1} a_{ij} - \Sigma_{j\epsilon J_2} a_{ij}| \leq 1$, we have for each $i \in I_o$, $i \neq i_o$, $\Sigma_{j\epsilon J_1} a_{ij} - \Sigma_{j\epsilon J_2} a_{ij} = 0$. Without loss of generality we may assume $\Sigma_{j\epsilon J_1} a_{i_oj} - \Sigma_{j\epsilon J_2} a_{i_oj} = 1$. Let $k := |K_o|$ and define $(C_1, \ldots, C_k) = A^*$. Without loss of generality we may assume $1 \in J$ and $i_o = 1$. Then for $C_1^* := \Sigma_{j\epsilon J_1} C_j - \Sigma_{j\epsilon J_2} C_j$, $c_{11}^* = 1$ and $c_{i1}^* = 0$ otherwise. Hence $\det A^* = \det (C_1, \ldots, C_k) = \det (C_1^*, \ldots, C_k) = \det A^{**}$, where A^{**} is formed by deleting the first row and column of A^*. But since A^{**} is a square submatrix of A^* and is not equal to A^*, $\det A^{**} = 0, 1$ or -1. But this contradicts the fact that $\det A^* \neq 0, 1$, or -1.

Hence $f_A{*}$ is both 1-1 and not 1-1, which is absurd. Therefore every submatrix of A has determinant $0, 1$ or -1. Therefore A is totally unimodular.

Note that the totally unimodular matrix A in Theorem 3.14 cannot be a $0, 1, -1$ matrix, since

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

is totally unimodular and $|a_{12} - a_{11}| = |a_{11} - a_{12}| = 2$. The reader should notice that Berge incorrectly states this theorem in his book for this reason.

Corollary 3.16 The incidence matrix of a unimodular hypergraph is totally unimodular.

Proof Let $H = (X,E)$ be a unimodular hypergraph then for each $S \subseteq X$, $|S| \geqq 2$, there exists $S_1, S_2 \subset S$ such that S_1, S_2 partition S and for each $E_i \varepsilon E$, $||E_i \cap S_1| - |E_i \cap S_2|| \leqq 1$. Let A be the incidence matrix of H. Let I index the rows of A and K index the columns of A. Then there exists $J \subseteq K$ such that each $j \varepsilon J$ corresponds to an $x \varepsilon S$. Hence there exists $J_1, J_2 \subset J$ such that J_1, J_2 partition J and for each $i \varepsilon I$ we have $|E_i \cap S_1| = \Sigma_{j\varepsilon J_1} a_{ij}$ and $|E_i \cap S_2| = \Sigma_{j\varepsilon J_2} a_{ij}$. Hence for each $i \varepsilon I$, there is an $E_i \varepsilon E$ such that $1 \geq ||E_i \cap S_1| - |E_i \cap S_2|| = |\Sigma_{j\varepsilon J_1} a_{ij} - \Sigma_{j\varepsilon J_2} a_{ij}|$. Since S was any subset of X, we have for every $J \subseteq K$ there exists $J_1, J_2 \subset J$ such that J_1, J_2 partitions J and for each $i \varepsilon I$, $|\Sigma_{j\varepsilon J_1} a_{ij} - \Sigma_{j\varepsilon J_2} a_{ij}| \leqq 1$. Therefore, by Theorem 3.15, A is totally unimodular.

Define $1_n, 0_n \varepsilon \mathbb{R}^n$ to be such that every component of 1_n is 1 and every component of 0_n is 0.

Corollary 3.17 Let H be unimodular and A its incidence matrix, then the extreme points of $P(1_m, 1_m; 0_n, 1_n)$ are integral.

Proof This follows directly from Theorem 3.13.

Proposition 3.18 Let $H = (X,E)$ be a hypergraph and A its incidence matrix, then every point in $P(1_m,1_m;0_n,1_n)$ is a convex combination of the extreme points of $P(1_m,1_m;0_n,1_n)$.

Proof Let $\{y_1, \ldots ,y_k\}$ be the extreme points of $P(\cdot)$ and $x \in P(\cdot)$. Clearly $P(\cdot)$ is contained in the linear span of its extreme points. Hence $x = \alpha_1 y_1 + \ldots + \alpha_k y_k$. Since $\{x,y_1, \ldots ,y_k\} \subseteq P(\cdot)$, $\Sigma_1^k \alpha_i = 1$. Therefore every point in $P(\cdot)$ is a convex combination of the extreme points of $P(\cdot)$.

Using Corollary 3.16 we have the following theory.

Let $H = (X,E)$ be a hypergraph and $\mu := (y_1 E_1 y_2 \ldots y_k E_k y_1)$ be a cycle of H, then μ is said to be a T-cycle of H if for $Y := \{y_1, \ldots ,y_k\}$, $|Y \cap E_i| = 2$ for $i = 1, \ldots ,k$.

Theorem 3.19 Let $H = (X,E)$ be a unimodular hypergraph, then H has no odd T-cycles.

Proof Suppose H has an odd T-cycle, $(y_1 E_1 y_2 \ldots y_k E_k y_1)$. Let A be the incidence matrix for H and A' be the incidence matrix of $H' = (Y,\{E_1 \cap Y, \ldots ,E_k \cap Y\})$. Clearly A' is a submatrix of A and by Lemma 1.8 $|\det A'| = 2$. But this contradicts the fact that A is totally unimodular.

CHAPTER 4

Let ω be a function mapping X into $[0,1]$ and $H = (X,E)$ be a hypergraph, then ω is called a state on H if for all $E \varepsilon E$, $\Sigma_{x \varepsilon E}\, \omega(x) = 1$. Define $X_\omega = \{x \varepsilon X : \omega(x) > 0\}$, then X_ω is called the support of ω. Two states ω_1 and ω_2 are said to be equal if for all $x \varepsilon X$, $\omega_1(x) = \omega_2(x)$. Define $\Omega(H) = \{\omega : \omega$ is a state on $H\}$.

Proposition 4.1 Let $H = (X,E)$ be a hypergraph and A be its incidence matrix, then $\Omega(H)$ is isomorphic to $P(1_m,1_m;0_n,1_n)$.

Proof Define $\Phi : \Omega(H) \to P(\cdot)$ by $\Phi(\omega) = (\omega(x_1), \ldots ,\omega(x_n))^t$. Let $\omega_1,\omega_2 \varepsilon \Omega(H)$ where $\omega_1 \neq \omega_2$ then by the definition of two states not being equal we have $\Phi(\omega_1) = (\omega_1(x_1), \ldots ,\omega_1(x_n))^t \neq (\omega_2(x_1), \ldots ,\omega_2(x_n))^t = \Phi(\omega_2)$. Hence Φ is 1-1. Let $y \varepsilon P(\cdot)$ and define $\omega_y(x_i) = y_i$. By the definition of y, ω_y is a state on H. Hence Φ is onto.

Let $a,b \varepsilon \mathbb{R}$ be such that $a > 0$, $b > 0$ and $a + b = 1$. Define $\oplus_{a,b}$ to be such that for all $\omega_1,\omega_2 \varepsilon \Omega(H)$, $\omega_1 \oplus_{a,b} \omega_2 = a\omega_1 + b\omega_2$. Hence $\Phi(\omega_1 \oplus_{a,b} \omega_2) = \Phi(a\omega_1 + b\omega_2) = ((a\omega_1 + b\omega_2)(x_1), \ldots ,(a\omega_1 + b\omega_2)(x_n))^t = a(\omega_1(x_1), \ldots ,\omega_1(x_n))^t + b(\omega_2(x_1), \ldots ,\omega_2(x_n))^t = a\Phi(\omega_1) + b\Phi(\omega_2) = \Phi(\omega_1) \oplus_{a,b} \Phi(\omega_2)$. Hence Φ is an isomorphism. Therefore $\Omega(H)$ is isomorphic to $P(\cdot)$.

Let $H = (X,E)$ be a hypergraph, then a set T is called a transversal of H if for all $E \varepsilon E$, $T \cap E \neq \phi$.

A transversal T of H is said to be thin if for all $E \varepsilon E$, $|T \cap E| = 1$. A state, Ψ, on H is said to be <u>dispersion free</u> if for all $x \varepsilon X$, $\Psi(x) \varepsilon \{0,1\}$.

<u>Lemma 4.2</u> Let $H = (X,E)$ be a hypergraph, then each dispersion free state corresponds to some thin transversal and each thin transversal corresponds to some dispersion free state.

<u>Proof</u> Let H have a thin transversal T, then define

$$\Psi(x) = \begin{cases} 1 & \text{if } x \varepsilon T \\ 0 & \text{if } x \not\varepsilon T. \end{cases}$$

Since T is thin, we have that for all $E \varepsilon E$, $\Sigma_{x \varepsilon E} \Psi(x) = 1$.

Suppose there exists a dispersion free state, Ψ, then define $T = \{x \varepsilon X : \Psi(x) = 1\}$. Since Ψ is dispersion free, we have that for all $E \varepsilon E$, $|T \cap E| = 1$.

<u>Lemma 4.3</u> Let $H = (X,E)$ be unimodular and A be the incidence matrix, then every extreme point in $P(1_m, 1_m; 0_n, 1_n)$ corresponds to a dispersion free state of H.

<u>Proof</u> This follows from Proposition 4.1, Corollary 3.17 and Lemma 4.2.

<u>Corollary 4.4</u> Let $H = (X,E)$ be unimodular and $\Omega(H) \neq \phi$, then for $\omega \varepsilon \Omega(H)$, $\omega = \Sigma_1^k p_i \Psi_i$ where $\Sigma_1^k p_i = 1$, for each i, $p_i > 0$ and Ψ_i is dispersion free.

<u>Proof</u> This follows from Proposition 3.18 and Lemma 4.3.

Let A be an $n \times n$ matrix, then A is called bistochastic if for each i we have that $\sum_{j=1}^{n} a_{ij} = 1$ and for each j we have that $\sum_{i=1}^{n} a_{ij} = 1$ and for each i and j, $a_{ij} \geqq 0$.

Corollary 4.5 Let A be a bistochastic matrix, then $A = \Sigma_1^k p_i P_i$ where, for each i, $0 < p_i \leqq 1$ and P_i is a permutation matrix.

Proof Let $X = \{x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn}\}$ and $E = \{E_i, F_i : E_i = \{x_{i1}, \ldots, x_{in}\}, F_i = \{x_{1i}, \ldots, x_{ni}\}\}$, then $H = (X, E)$ is a hypergraph. Hence for $i \neq j$, $E_i \cap E_j = \phi$ and $F_i \cap F_j = \phi$. Also for each i and j, $E_i \cap F_j = \{x_{ij}\}$. Therefore H contains no odd cycles. By Theorem 2.7, H is unimodular.

Let A be bistochastic, then the function $\omega(x)$, where $\omega(x_{ij}) := a_{ij}$, is a state on H. By Corollary 4.4 $\omega(x) = \Sigma_1^k p_i \Psi_i(x)$ where for each i, we have $0 < p_i \leqq 1$ and Ψ_i is a dispersion free state and $\Sigma_1^k p_i = 1$. Note that for Ψ_q to be a state, we have that if $1 \leqq i_o \leqq n$ is fixed, then for some $1 \leqq j_o \leqq n$, $\Psi_q(x_{i_o j_o}) = 1$ and for $j \neq j_o$, $\Psi_q(x_{i_o j}) = 0$. Similarly if $1 \leqq j^* \leqq n$ is fixed for some $1 \leqq i^* \leqq n$, $\Psi_q(x_{i^* j^*}) = 1$ and for $i \neq i^*$, $\Psi_q(x_{ij^*}) = 0$. Therefore the matrix formed by $((\Psi_q(x_{ij}))_{ij})$ is a permutation matrix.

Clearly the matrix formed by $((\omega(x_{ij}))_{ij})$ is A. By the above

$$A = ((\omega(x_{ij}))_{ij}) = ((\sum_{q=1}^{k} p_q \Psi_q(x_{ij}))_{ij}) = \sum_{q=1}^{k} p_q((\Psi_q(x_{ij}))_{ij}) =$$

$\sum_{q=1}^{k} p_q P_q$ where for each q, P_q is the permutation matrix given by $((\Psi_q(x_{ij}))_{ij})$.

Note that this corollary is a result of P. Hall quoted in G. Berkhoff's paper.

Let $H = (X, E)$ be a hypergraph and $x, y \in X$ with $x \neq y$, then define $x \perp y$ if for some $E \in E$, $\{x,y\} \subseteq E$. Let $\Omega(H) \neq \phi$ and $S \subseteq \Omega(H)$ then S is said to be full if, for any $x, y \in X$ where $x \neq y$, $x \perp y$ iff $\omega(x) + \omega(y) > 1$ for some $\omega \in S$. Let ∇ denote the set of all dispersion free states of $\Omega(H)$.

Example 4.6 Let $H = (X, E)$ be the hypergraph represented by Figure 1 where each line is an edge of H. Then $\Omega(H)$ is full and ∇ is not full.

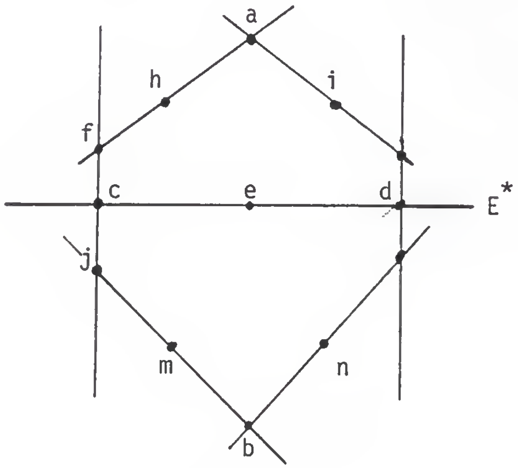

Figure 1

Proof Suppose Ψ is a dispersion free state on H where $\Psi(a) = \Psi(b) = 1$. Then $\Psi(c) = \Psi(d) = 1$. However $\Sigma_{x \in E^*} \Psi(x) = 2$, which is a contradiction. Hence ∇ is not full.

Using Chart 1 below by picking any 2 vertices, x and y, such that $x \not\perp y$, you will find a row which represents a state on H such that $\omega(x) + \omega(y) > 1$. Hence $\Omega(H)$ is full.

<u>Proposition 4.7</u> Let $H = (X, E)$ be a hypergraph with $S \subseteq \Omega(H)$ full, then $\Omega(H)$ is full.

<u>Proof</u> Let $x, y \in X$ where $x \neq y$ and $x \perp y$, then since S is full there exists some $\omega \in S \subseteq \Omega(H)$ such that $\omega(x) + \omega(y) > 1$.

<u>Theorem 4.8</u> Let $H = (X, E)$ be unimodular then ∇ is full iff $x, y \in X$, where $x \not\perp y$ and $x \neq y$, implies there is some thin transversal containing x and y.

<u>Proof</u> This is immediate from Lemma 4.2.

<u>Example 4.9</u> Let H be the hypergraph represented by Figure 2, where each line in Figure 2 is an an edge of H. Then ∇ is full and H is not unimodular.

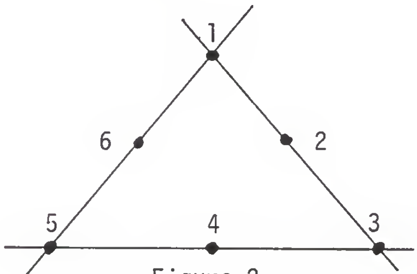

Figure 2

<u>Proof</u> Use Table 2 as in Example 4.6 to see that ∇ is full. Consider the incidence matrix of the subhypergraph $H_{\{1,3,5\}}$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

By Lemma 1.8 this matrix has determinant 2. Hence H is not unimodular.

Hence unimodularity does not imply a full set of dispersion free states. Note that the quest for finding what conditions need to be on a hypergraph in order that it have a full set of dispersion free states still continues.

Proposition 4.10 Let H be a hypergraph with ∇ full, then there is a state $\omega \in \Omega(H)$ such that $\omega(x) > 0$ for all $x \in X$.

Proof Let $\nabla = \{\Psi_1, \ldots, \Psi_k\}$ then clearly $\omega = (1/k)\Sigma_1^k \Psi_i$ is a state on H and for all $x \in X$, $\omega(x) > 0$.

The next example shows the converse is not true.

Example 4.11 Let H be the hypergraph represented by Figure 3, where each line is an edge of H, then ∇ is not full and there exists a state $\omega \in \Omega(H)$ such that $\omega(x) > 0$ for all $x \in X$.

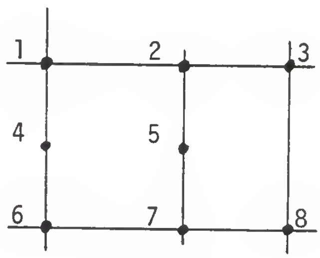

Figure 3

Proof Suppose Ψ is a state such that $\Psi(6) = \Psi(2) = 1$, then $\Psi(3) + \Psi(8) = 0$, which contradicts the fact that Ψ is a state. Hence ∇ is not full. Define ω by

$\omega(4) = \omega(5) = \omega(3) = \omega(8) = 1/2$ and $\omega(1) = \omega(2) = \omega(6) = \omega(7) = 1/4$, then ω is a state on H with the desired properties.

<u>Theorem 4.12</u> Let $H = (X, \mathcal{E})$ be a hypergraph such that for each $E \in \mathcal{E}$ there exists an $x^* \in E$ such that for all $E^* \in \mathcal{E}\ \{E\}$, $x^* \notin E^*$, then ∇ is full.

<u>Proof</u> Let $x, y \in X$, $x \neq y$ and $x \not\perp y$. Let Ψ be a 0,1 function where $\Psi(x) = \Psi(y) = 1$. For each $E \in \mathcal{E}$ such that $x \notin E$ or $y \notin E$ choose an $x^* \in E$ and let $\Psi(x^*) := 1$. Then clearly Ψ is a dispersion free state. Therefore ∇ is full.

	a	i	g	d	k	n	b	m	j	c	f	h	e
Ψ_1	1	0	0	0	1	0	0	1	0	1	0	0	0
Ψ_2	1	0	0	1	0	1	0	0	1	0	0	0	0
Ψ_3	1	0	0	0	1	0	0	0	1	0	0	0	1
Ψ_4	0	0	1	0	0	0	1	0	0	1	0	1	0
Ψ_5	0	1	0	1	0	0	1	0	0	0	1	0	0
Ψ_6	0	0	1	0	0	0	1	0	0	0	1	0	1
Ψ_7	0	1	0	1	0	1	0	0	1	0	0	0	1
Ψ_8	0	1	0	0	1	0	0	1	0	1	0	1	0
Ψ_9	0	1	0	1	0	1	0	1	0	0	1	0	0
Ψ_{10}	0	1	0	0	1	0	0	1	0	0	1	0	1
Ψ_{11}	0	0	1	0	0	1	0	1	0	1	0	1	0
Ψ_{12}	0	0	1	0	0	1	0	0	1	0	0	1	1
ω_1	3/4	0	1/4	1/4	1/2	0	1/2	0	1/2	1/4	1/4	0	1/2

Table 1

	1	2	3	4	5	6
Ψ_1	0	1	0	0	1	0
Ψ_2	0	0	1	0	0	1
Ψ_3	1	0	0	1	0	0

Table 2

BIBLIOGRAPHY

Berge, C., Graphs and Hypergraphs. North-Holland Publishing Co., Amsterdam, 1973, p. 463 - 472.

Greechie, R. J. and Miller, F. R., "On Structures Related to States on an Emperical Logic I. Weights on Finite Spaces." K.S.U. Technical Report No. 14, April 1970.

Greechie, R. J. and Miller, F. R., "On Structures Related to States on an Emperical Logic II. Weights and Duality on Finite Spaces." K.S.U. Technical Report No. 26, August, 1972.

Greechie, R. J., "Orthomodular Lattices Admitting No States." Journal of Combinatorial Theory, Vol. 10,No. 2, March 1971, p. 119 - 132.

Ghouila-Houri, A., "Charactérisation des Matrices Totalement Unimodulaires." C. R. Acad. Sc. Paris, 254, 1962, p. 1192 - 1194.

Gudder, S. P., Kläy, M. P. and Rüttiman, G. T., "States on Hypergraphs." 1980 Mathematics Subject Classification (AMS): Primary: 05C65; Secondary: 81B10.

Hoffman, A. J. and Kruskal, J. B., "Integral Boundary Points of Convex Polyhedra." Ann. of Math. Studies, 38, Princeton, 1956, p. 223 - 332.

Tamir, A., "On Totally Unimodular Matrices." Networks,6, p. 373 - 382.

Birkhoff, G., "Tres observationes sobre el linear Algebra," Rev. Univ. Nac. Tucuman, A 5, 1946, p. 147 - 148.

Curtis, C., Linear Algebra an Intrductory Approach, Allyn and Bacon, Inc., 1977.

Hernstein, I., Topics in Algebra, John Wiley & Sons, 1975.

ON UNIMODULAR HYPERGRAPHS

by

GREGORY D. GOECKEL

B. S., Marymount College of Kansas, 1982

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1985

ABSTRACT

Chapter 1 covers the essential theory of graphs and linear algebra that is necessary for the theory of unimodular hypergraphs.

Chapter 2 introduces the hypergraph and unimodular hypergraph. Two of the most important results of this chapter are:

i) each subhypergraph of a unimodular hypergraph is unimodular;

ii) if a hypergraph has no odd cycles, then it is unimodular.

Chapter 3 contains the three main theorems of this thesis. The first one is a theorem by Hoffman and Kruskal that characterizes an $m \times n$ totally unimodular matrix by a polyhedron in $\mathbb{R}^n$. The second is a theorem by Ghouila-Houri that characterizes a 0,1 totally unimodular matrix by being able to partition the columns into two sets in such a way that for any row, if you sum over one set of columns and subtract that from the sum over the other set of columns then that difference is less than or equal to 1 in absolute value. The last theorem is by Berge and is a corollary to the first two theorems that allows us to tap into the theory of totally unimodular matrices and use it to better characterize unimodular hypergraphs.

Chapter 4 deals with the applications to the theory developed in Chapter 3. The main theorem in this chapter says that any state on a unimoldular hypergraph can be writen as a convex combination of dispersion free states. A corollary to this theorem is that any bistochastic matrix can be written as a convex combination of permutation matrices.