

Digitized by the Internet Archive  
in 2012 with funding from  
LYRASIS Members and Sloan Foundation

<http://archive.org/details/engineeringdesig00chow>

ENGINEERING DESIGN IN RELIABILITY CRITERION

by

DER-MEI CHOW

Diploma (Industrial Engineering)

Taipei Institute of Technology, Taiwan, 1971

---

A MASTER'S THESIS

submitted in partial fulfillment of the  
requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1978

Approved by:

  
Major Professor

Documents  
12  
2002  
74  
1113  
051  
02

TABLE OF CONTENTS

	PAGE
LIST OF FIGURES	iv
LIST OF TABLES	v
ACKNOWLEDGEMENTS	vi
CHAPTER 1 INTRODUCTION	1
CHAPTER 2 PROBABILISTIC ENGINEERING DESIGN	5
2.1 Distributions of stress and strength	5
2.2 General expression for reliability	14
2.3 Definition of safety factor and safety margin	19
2.4 Safety factor and reliability	20
2.5 Computation of moments of a function of random variables	22
CHAPTER 3 THE STRESS AND STRENGTH INTERFERENCE MODELS	26
3.1 Reliability computation for normally distributed stress and strength	26
3.2 Reliability computation for log normally distributed stress and strength	28
3.3 Reliability computation for exponentially distributed stress and strength	31
3.4 Reliability computation for gamma distributed stress and strength	32
3.5 Reliability computation for Weibull distributed stress and strength	35

3.6	Reliability computation for extreme value distributions	36
CHAPTER 4	TIME DEPENDENT STRESS-STRENGTH MODELS	41
4.1	Reliability computations for cyclic occurrences	45
4.2	Reliability in case of cumulative damage	53
CHAPTER 5	OPTIMIZATION IN ENGINEERING DESIGN	56
5.1	Statement of problem and model development	57
5.1.1	Statement of problem	57
5.1.2	Model development	60
5.2	Nonlinear goal programming for engineering design problem	69
5.2.1	Case 1	70
5.2.2	Case 2	74
5.2.3	Discussion	76
5.3	Sequential multiobjective problem solving technique (SEMOPS) for engineering design problem	76
5.3.1	The method	76
5.3.2	The design problem	81
5.3.3	Discussion	92
5.4	Discussion	92
CHAPTER 6	CONCLUSION AND DISCUSSION	94
REFERENCES		96
APPENDIXES		98

## LIST OF FIGURES

## Figure

1. Stress and strength factors and distributions	6
2. Normal distribution	9
3. Log normal distribution	10
4. Exponential distribution	12
5. Gamma distribution	13
6. Stress-strength interference	15
7. Enlarged portion of the interference area of figure 6	17
8. Circular cross-section shaft	58
9. A taxonomy of methods for multiple objective decision making	59

## LIST OF TABLES

## Table

1. Safety factors and reliability	21
2. Data for estimating cost function	65
3. Comparison of results for estimating the parameters of equations (5.7) and (5.8)	
4. Data for estimating density	68
5. Procedure for achieving an objective	71
6. Comparison of results of two methods	77
7. Results of the first cycle of SEMOPS	86
8. Results of the second cycle of SEMOPS	90
9. Results of the third cycle of SEMOPS	91

## ACKNOWLEDGEMENTS

The writer wishes to express his sincere gratitude to his major professor, Dr. C. L. Hwang, for his advice, encouragement, constructive criticism and personal interest taken in the preparation of this thesis. He also wishes to express appreciation to Dr. Frank A. Tillman and Dr. C. L. Huang for serving on his advisory committee. Finally, the writer would like to extend particular appreciation to his parents and wife Hsin-Yi for the endless love and moral support provided to make this thesis possible.



## CHAPTER 1 INTRODUCTION

It is a complex and uncertain world. Probability is used everywhere today. But, the word "probability" is often accepted rather skeptically by engineers; they think that they are employing an "exact science" despite much use of empiricism. We can say that engineering is a type of applied physics. Many physical phenomena can be described only by using the theory of probability. It is not surprising to discover that equipment failures which result from interactions of heat, electric and magnetic field, "static" loadings, and vibrations can best be described in probabilistic terms [16].

Some of the engineering design are still stuck on conventional design methodology by using "safety factors". It should be brought to the area of probabilistic design. Reliability is a terminology in the area of probabilistic design.

What is reliability? There are several definitions of reliability; one of the best is that of the National Aeronautics and Space Administration (NASA) [2] which defines "reliability" as "the probability of a device performing adequately for the period of time intended under the operating conditions encountered." Other definitions are: "reliability is a measure of the capacity of a piece of equipment to operate without failure when put into service" [16], and "The reliability of a system is the probability that, when operating under stated environmental conditions, the system will perform its intended function adequately for a specified interval of time" [10].

In this thesis, reliability is defined as: the probability of the strength of a component, which is a certain stress resisting capacity of the component, exceeding the stress induced by the operating conditions. If the stress exceeds this capacity, then failure results. It is unfortunate to use words such as "stress" or "strength". The imputed meaning is not necessarily restricted to mechanical loading in the minds of engineers. We can use it in a broader sense, applicable in a variety of situations well beyond the traditional mechanical or structural systems. "Stress" is used to denote any agency that tends to induce "failure", while "strength" denotes any agency that tends to resist "failure". "Failure" itself is taken to mean failure to function as intended; it is defined to have occurred when the actual stress exceeds the actual strength in the beginning [10].

Mathematical formulation of reliability is given by

$$\text{Reliability} = R = P(S > s) = P(S - s > 0) \quad (1.1)$$

where  $S$  is strength,  $s$ , stress, and  $P$ , probability function.

The detailed expression of reliability will be discussed in Chapter 2.

Literature survey: This thesis is mainly concerning about optimizing the following objectives; maximize reliability of stress-strength interference model, minimize total cost of a component, minimize weight of a component; subject to some absolute constraints. The problem will be solved by multiple objective decision making methods. The idea of this thesis is developed according to the following literature survey.

Since 1961, I. Bazovsky [2] has mentioned the stress-strength interference theory (S-S-I). In 1964, M. J. Bratt, G. Reethof, and

G. W. Weber [3] interpreted the stress-strength interference case and also developed some models for time varying stress-strength interference cases (S-S-T). In 1964, R. L. Disney, C. Lipson, and N. J. Sheth [4] developed the complete models of S-S-I of several different distributions. In 1972, Leonard Shaw, Martin Shooman, Robert Schutz [15] developed the several models for S-S-T models. In 1975, K. C. Kubar [9] studied the optimization technique of a component reliability problem with a cost constraint. The problem dealt mainly with one objective, subject to one or more constraints. But in the real world, the designer or decision maker would like to optimize several objectives at the same time rather than one objective.

Objective: The objective of this thesis is to solve the following multiple objectives problem. To find  $\underline{x}$  so as to

$$\text{Max } f_R(\underline{x})$$

$$\text{Min } f_C(\underline{x})$$

$$\text{Min } f_W(\underline{x})$$

$$\text{subject to } \underline{g}(\underline{x}) \leq 0$$

where  $f_R$ ,  $f_C$ , and  $f_W$  are the functions of reliability, cost, and weight respectively.  $\underline{x}$  denotes a decision variable vector  $(x_1, x_2, \dots, x_n)$ .

In solving this problem, it is necessary to replace the concept of optimum with that of best compromise. The problem will be solved by two methods of the Multiple Objective Decision Making-methods (MODM) [6], goal programming [7,9] and sequential multiobjective problem solving technique (SEMOPS) [14].

In order to develop the model as we mentioned above, Chapter 2 to Chapter 4 provide the background for getting the model. Chapter 2 discusses the distributions of stress or strength, the relationship between safety factor and reliability, and computation of moments of a function of random variables. Chapter 3 discusses several stress-strength interference models. Chapter 4 discusses several time dependent stress-strength interference models. In Chapter 5, the main part of this study, development of the problem, and how to use different proper methods to solve the problem are presented. Chapter 6 gives the conclusion and the extensions of the problem which remain unsolved.

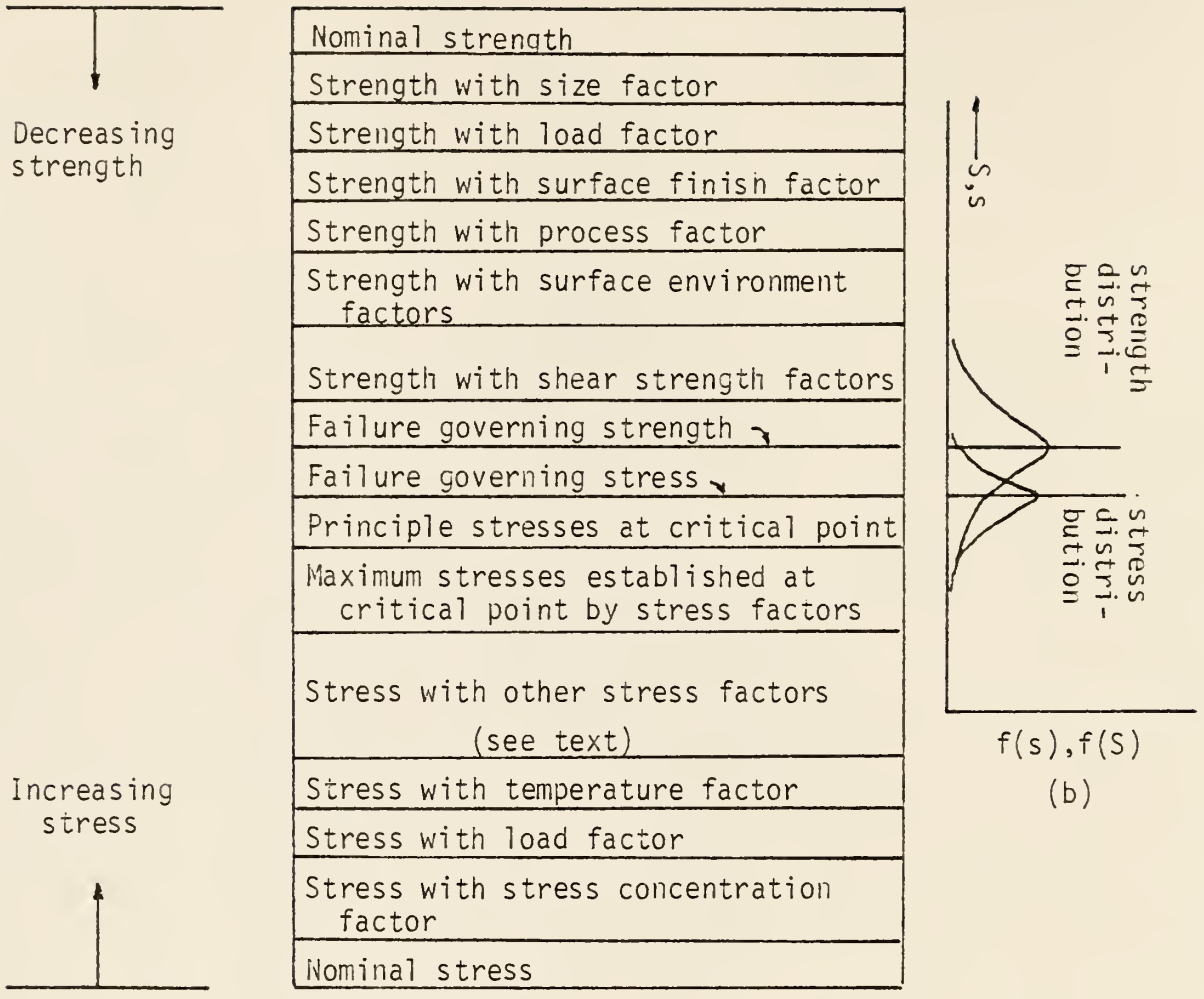
## CHAPTER 2. PROBABILISTIC ENGINEERING DESIGN

The conventional design approach, which is based on safety factors and safety margins, gives little indication of the probability of failure of the component. Some designers believed that a component failure could be completely eliminated by using a safety factor above a certain large magnitude. Actually, the failure probability may vary from a low to an extremely high value for the same safety factor. The safety factor can be used only when its value is based on considerable experience with parts similar to the one considered. Furthermore, the design variables and parameters are often random variables, which are ignored by the conventional design approach. Therefore, another design methodology which does consider the probability nature of the design is needed. Such a design methodology is called "probabilistic design." It identifies explicitly all the design variables and parameters which can determine both the stress and strength distributions (Fig. 1). When these two distributions are determined, the component reliability can be calculated.

### 2.1 Distributions of stress and strength

For the strength computations, considerations must be given to the properties of the material used. The factors which affect the strength may include, but are not necessarily limited to, the following: (Fig. 1) [12].

1. Size
2. Forming and manufacturing process
3. Surface finish



(a)

- (a) Stress increase and strength decrease resulting from the application of the respective stress and strength factors.
- (b) Stress and Strength distributions

Fig. 1 Stress and strength factors and distributions [11]

4. Loads
5. Heat treatment
6. Direct surface environment
7. Temperature
8. Time
9. Fretting corrosion
10. Surface treatment

The factors which affect the stress may include, but are not necessarily limited to, the following: (Fig. 1) [12]

1. Stress concentration factors
2. Load factors
3. Temperature stress factors
4. Forming or manufacturing stress factors
5. Surface treatment stress factors
6. Heat treatment stress factors
7. Assembly stress factors
8. Corrosion stress factors
9. Direct surface environment stress factors
10. Notch sensitivity factors.

Here, the determination of stress and strength should be limited to mechanical components where the fiber stress, in pounds per square inch, is the stress that governs failure. The theory that is developed is not limited in application to this case only.

The distributions of stress and strength generally can be classified into the following distributions:

(1) Normal distribution: (Fig. 2)

The probability density function of a normal distribution can be given by

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left(\frac{x-\bar{x}}{\sigma_x}\right)^2\right\} \quad -\infty < x < \infty \quad (2.1)$$

where  $\bar{x}$  = mean value of variable  $x$

$\sigma_x$  = standard deviation of variable  $x$

For example, the probability distribution of the length of human lives is found to be normal. Also, the probability distribution of ultimate tensile, yield and endurance strengths of steels are found to be normal. The probability distribution of the stress such as rocket motor thrust or the gas pressure in the cylinder heads of reciprocating engines are also found to be normal.

(2) Log-normal distribution: (Fig. 3)

The probability density function is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2} (\ln x - \mu)^2\right\}, \quad x > 0 \quad (2.2)$$

where  $\mu$  and  $\sigma$  are the mean and the standard deviation respectively of the variable  $\ln x$ , which is normally distributed. If we let  $y = \ln x$ , then

$dy = \left(\frac{1}{x}\right)dx$ . From equation (2.2), we have

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2} (y - \mu)^2\right\}, \quad -\infty < y < \infty \quad (2.3)$$

It is found that the strength properties of structural alloy materials often tend to follow a log normal distribution.



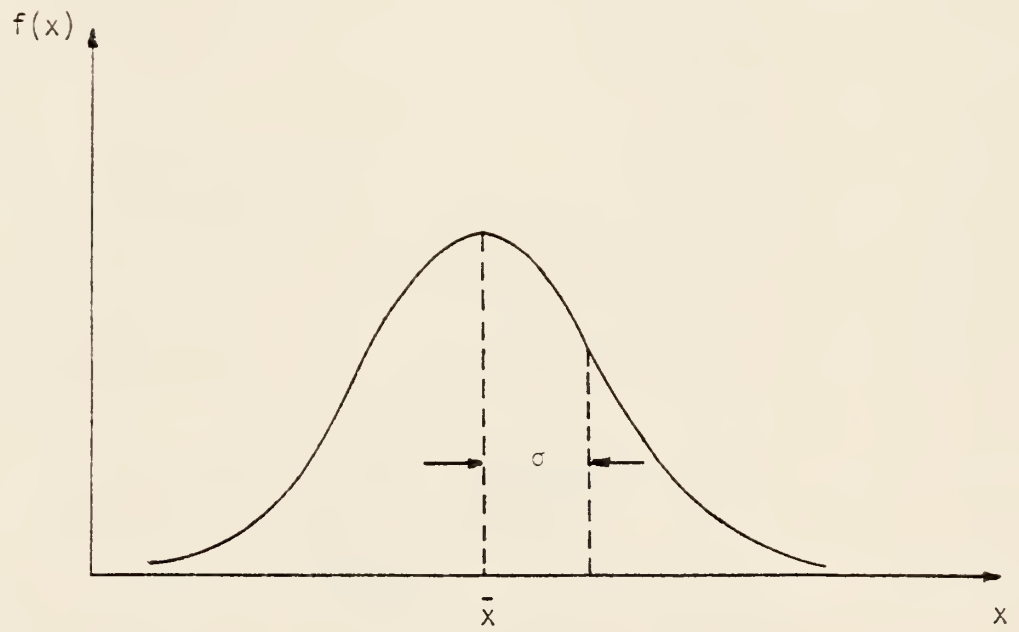


Figure 2 Normal distribution

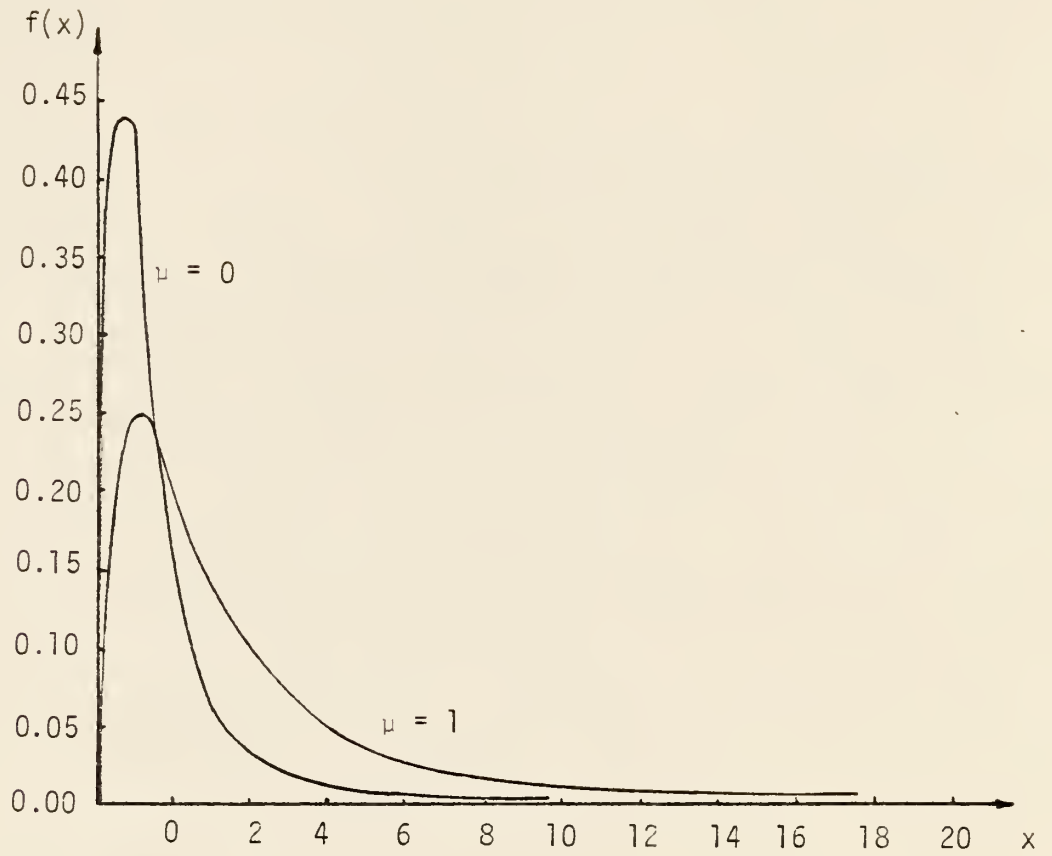


Figure 3. Log normal distribution when  $\sigma^2 = 1$

(3) Exponential distribution: (Fig. 4)

The probability density function is given by

$$f(x) = \lambda_x e^{-\lambda_x x}, \quad 0 \leq x < \infty \quad (2.4)$$

where the mean value of  $x$ ,  $\mu$ , and the standard deviation of  $x$ ,  $\sigma$ , was the following relations:

$$\mu = \frac{1}{\lambda_x}, \quad \sigma = \frac{1}{\lambda_x}$$

(4) Gamma distribution: (Fig. 5)

The probability density function is given by

$$f(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)}, \quad n > 0, \quad \lambda > 0, \quad 0 \leq x < \infty \quad (2.5)$$

where  $\lambda$  is called the scale parameter and  $n$  the shape parameter and  $\Gamma(n)$  is the gamma function.

(5) Weibull distribution

The probability density function for a Weibull distribution  $x$  is given by

$$f(x) = \left(\frac{\beta}{\theta}\right) \left(\frac{x-x_0}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{x-x_0}{\theta}\right)^\beta\right] \quad x_0 \leq x < \infty \quad (2.6)$$

where  $\beta$  is the slope parameter,  $\theta$  is the scale parameter, and  $x_0$  is the location parameter or lower bound of  $x$ .

The distribution becomes the exponential when  $\alpha = 1$ , a distribution skewed to the left when  $1 < \alpha < 3$ , the normal when  $\alpha = 3$ , approximately, and a distribution skewed to the right when  $\alpha > 3$ .

(6) The extreme value distributions:

In general, the extreme value distributions are applicable where the phenomena causing failure depend on the smallest or the largest value from a sequence of random variables.

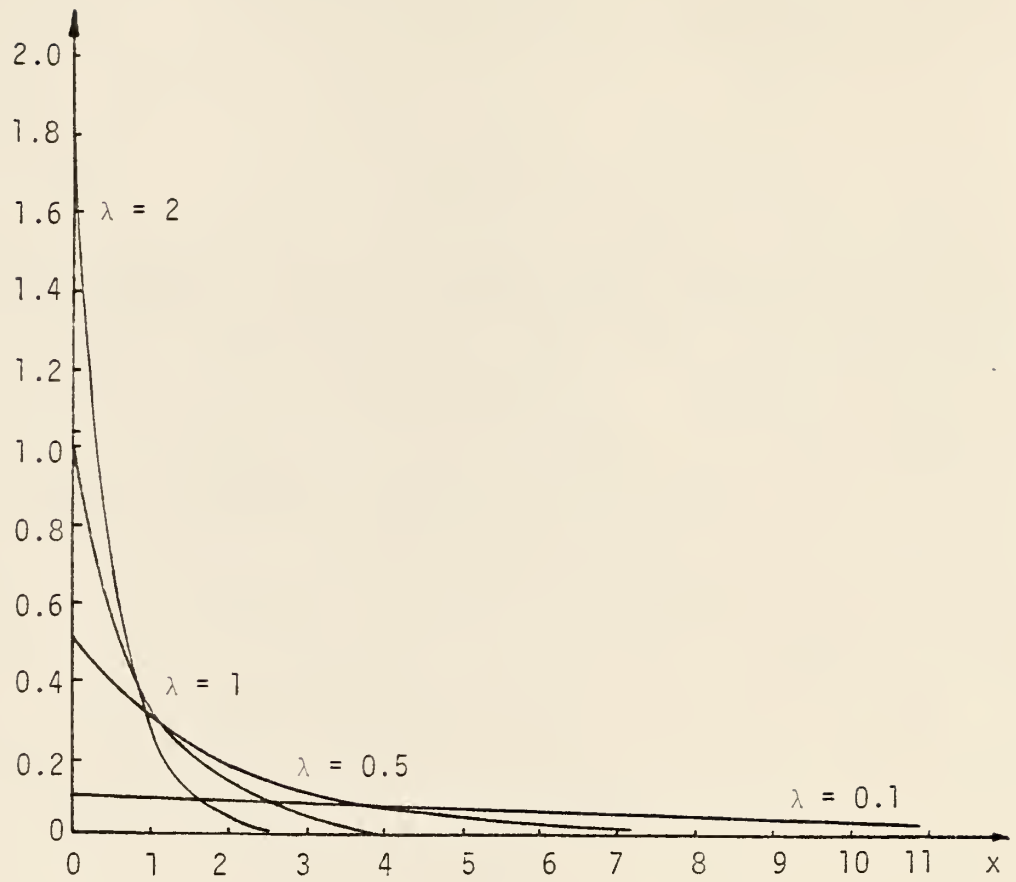


Figure 4. Exponential distribution

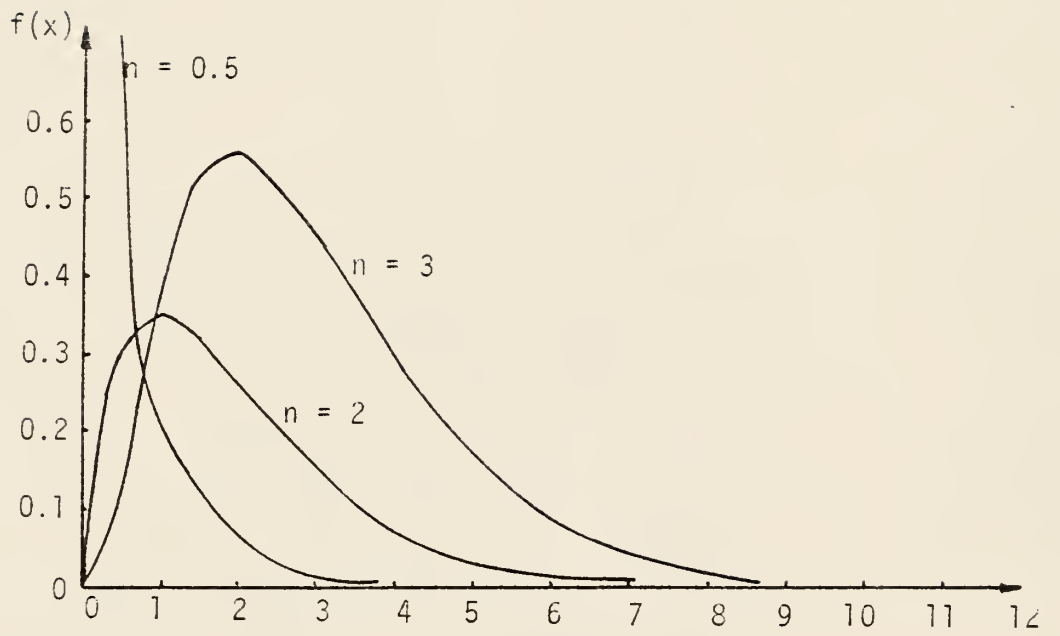


Figure 5 Gamma distribution

There are two type of distributions of extreme value, one is the smallest value distribution, another is the largest value distribution. If the cumulative function associated with  $f(x)$  is  $F(x) = \int_{-\infty}^x f(y)dy$ , then the distribution of the smallest value,  $y = \min(x_1, x_2, \dots, x_n)$ , in the sample size  $n$  drawn from the population is given by the probability density function

$$g_n(y) = nf(y)(1-F(y))^{n-1} \quad (2.7)$$

or by the cumulative distribution function

$$G_n(y) = \int_{-\infty}^{\infty} g_n(y)dy = 1 - (1 - F(y))^n \quad (2.8)$$

Also, the distribution of the largest value,  $z = \max(x_1, x_2, \dots, x_n)$ , in the sample size  $n$  drawn from the population is given by the probability density function

$$g_n(z) = nf(z)(F(z))^{n-1} \quad (2.9)$$

or by the cumulative distribution function

$$G_n(z) = \int_{-\infty}^{\infty} g_n(z)dz = (F(z))^n \quad (2.10)$$

The distribution of fatigue of metals or textiles, the breakdown of dielectrics and the corrosions of metals tend to follow extreme value distribution.

## 2.2 General expression for component reliability [10].

Let  $f_s(\cdot)$  and  $f_S(\cdot)$  be the density functions of the stress ( $s$ ) and the strength ( $S$ ) respectively. (see Fig. 6.) Then, by definition,

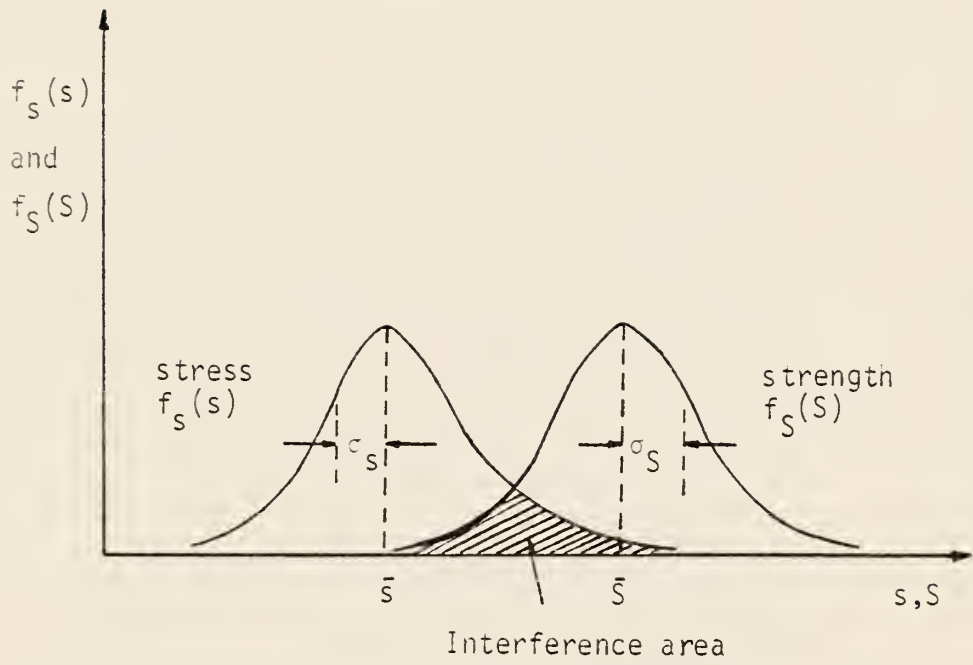


Figure 6 Stress-strength interference

$$\text{Reliability} = R = P(S > s) = P(S - s > 0) \quad (2.11)$$

The shaded portion in Fig. 6 is the interference area, which is the indication of the probability of failure. Let us magnify this interference area as shown in Fig. 7.

The probability of a stress value lying in a small interval  $ds$  is equal to the area of the element  $ds$  under the curve, that is

$$P\left(s_0 - \frac{ds}{2} \leq s \leq s_0 + \frac{ds}{2}\right) = f_s(s_0)ds \quad (2.12)$$

The probability that the strength  $S$  is greater than a certain stress  $s$  is

$$P(S > s_0) = \int_{s_0}^{\infty} f_S(S)dS \quad (2.13)$$

It is assumed that the stress and the strength random variables are independent and greater than or equal to zero. The probability for the stress value lying in the small interval  $ds$  and the strength  $S$  exceeding the stress given by this small interval  $ds$  is given by

$$f_s(s_0) \int_{s_0}^{\infty} f_S(S)dS \quad (2.14)$$

Because event A and event B are independent, then

$$P(A \cap B) = P(A) \cdot P(B) \quad (2.15)$$

Now, the reliability of the component is the probability that the strength  $S$  is greater than the stress  $s$  for all possible values of the stress  $s$  and is given by

$$R = \int_{-\infty}^{\infty} f_s(s) \left[ \int_s^{\infty} f_S(S)dS \right] ds \quad (2.16)$$



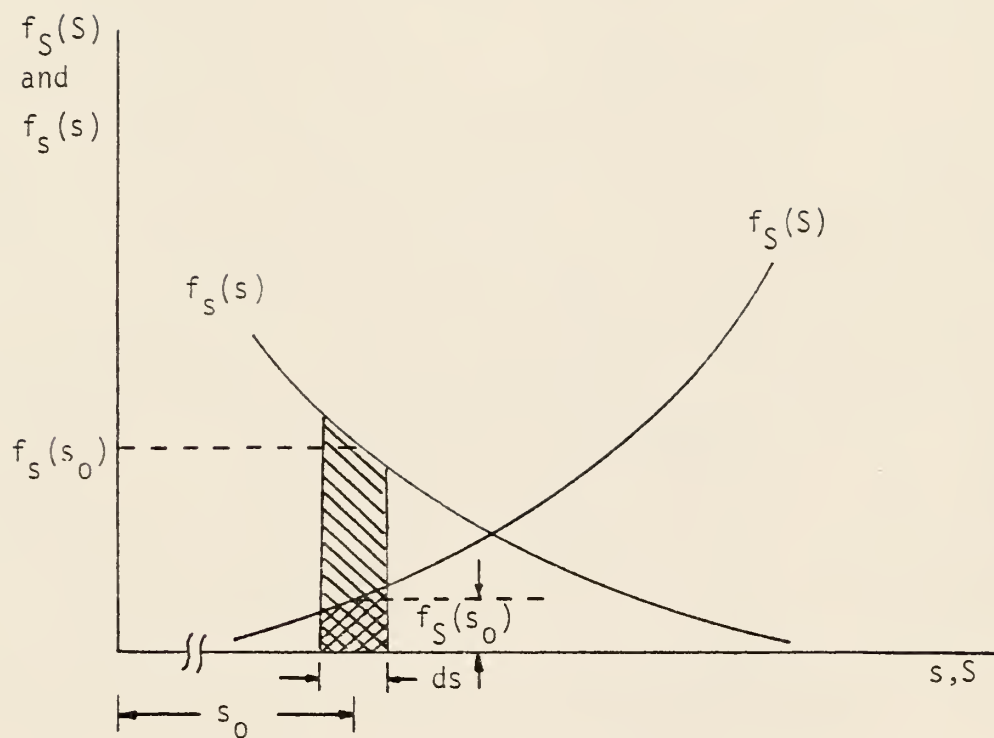


Figure 7. Enlarged portion of this interference area of Figure 6.

Similarly, the reliability of a component can be expressed by

$$R = \int_{-\infty}^{\infty} f_S(S) \left\{ \int_{-\infty}^S f_s(s) ds \right\} dS \quad (2.17)$$

The unreliability,  $\bar{R}$ , is defined as

$$\bar{R} = \text{probability of failure} = 1 - R = P(S \leq s) \quad (2.18)$$

Substituting for R from equation (2.16) yield

$$\begin{aligned} \bar{R} &= P(S \leq s) = 1 - \int_{-\infty}^{\infty} f_s(s) \left\{ \int_s^{\infty} f_S(S) dS \right\} ds \\ &= 1 - \int_{-\infty}^{\infty} f_s(s) [1 - F_S(s)] ds \\ &= \int_{-\infty}^{\infty} F_S(s) f_s(s) ds \end{aligned} \quad (2.19)$$

Similarly, using equation (2.17) we have

$$\begin{aligned} \bar{R} &= P(S \leq s) = 1 - \int_{-\infty}^{\infty} f_S(S) \left\{ \int_{-\infty}^S f_s(s) ds \right\} dS \\ &= 1 - \int_{-\infty}^{\infty} f_S(S) \cdot F_s(S) dS \\ &= \int_{-\infty}^{\infty} [1 - F_s(S)] f_S(S) dS \end{aligned} \quad (2.20)$$

Let  $y = S - s$ , then  $y$  is called the interference random variable. Now we can define the reliability as  $R = P(y > 0)$ . The density function,  $f_y(\cdot)$ , is then given by

$$f_y(y) = \int_s f_S(y+s)f_s(s)ds \quad (2.21)$$

Hence the probability of failure, which is  $y \leq 0$ , is given by

$$\bar{R} = \int_{-\infty}^0 f_y(y)dy = \int_{-\infty}^0 \int_{-y}^{\infty} f_S(y+s) \cdot f_s(s)dsdy \quad (2.22)$$

and the reliability by, which is  $y > 0$ , is given by

$$R = \int_0^{\infty} f_y(y)dy = \int_0^{\infty} \int_0^{\infty} f_S(y+s) \cdot f_s(s)dsdy \quad (2.23)$$

### 2.3 Definition of safety factor and safety margin [11]

A variety of definitions for safety factor has been proposed and used. These definitions are given as follows:

1. Ratio of ultimate strength in a component to the allowable or actual working stress. This can be written as

$$\text{Safety factor} = F = \frac{S_u}{S_w} \quad (2.24)$$

where  $S_u$  = ultimate strength, psi

$S_w$  = working stress, psi

2. Ratio of yeild strength in a component to the allowable or actual working stress.
3. Ratio of maximum safe load to normal service load.
4. Safety factor is the ratio of computed strength  $S$ , to the corresponding computed load  $\ell$ .
5. Ratio of damaging stress to maximum known working stress.
6. Ratio of mean strength to mean load.
7. Ratio of significant strength to significant stress.

It may be seen that although definitions vary widely with appropriate interpretations of the terms used, the safety factor is the ratio of a particular strength value to a particular stress value. The central safety factor is the ratio of the central tendency measure of the strength distribution to the central tendency measure of the stress distribution.

The best concept such a safety factor can convey is how far the mean strength is removed from the mean stress. Sometimes designers believe that designing to a safety factor above some preconceived magnitude, usually above 2.5 would result in no component failure. On the contrary, with such, and even higher safety factors, the failure probability may vary from a very satisfactory low value to an intolerably high value. Furthermore, a safety factor of one, to most designers, implies that failure will occur 100% of the time because, presumably, there is no safety margin, whereas failure would actually occur only 50% of the time, if the stress and strength distributions are normal.

The basic safety margin definition used in most design books is

$$\text{Safety Margin} = M = F - 1 \quad (2.25)$$

This means that the safety margin is the amount by which the safety factor value exceeds unit. This definition again ignores the fact that stress and strength are distributed, as does the safety factor.

#### 2.4 Safety factor and reliability

From Table 1, we can see that the same safety factor, 2.5, can result the reliability from 0.987 to 1.0 and even the safety factor, 5.0, may only result in a reliability of 0.9738. It is not necessarily true that the higher value of safety factor can result in a higher reliability.

Table 1 Safety factors and reliability\* [11]

CASE No.	MEAN STRENGTH $\mu_S$	MEAN STRESS $\mu_s$	STRENGTH STANDARD DEVIATION $\sigma_S$	STRESS STANDARD DEVIATION $\sigma_s$	FACTOR OF SAFETY $\mu_S/\mu_s$	RELIABILITY R
1	50,000	20,000	2,000	2,500	2.5	1.0
2	50,000	20,000	8,000	3,000	2.5	0.9997
3	50,000	20,000	10,000	3,000	2.5	0.9979
4	50,000	20,000	8,000	7,500	2.5	0.9965
5	50,000	20,000	12,000	6,000	2.5	0.987
6	25,000	10,000	2,000	2,500	2.5	$0.9^{(6)}_4$
7	25,000	10,000	1,000	1,500	2.5	$0.9^{(16)}_6$
8	50,000	10,000	20,000	5,000	5.0	0.9738
9	50,000	40,000	2,000	2,500	1.25	0.99909
10	50,000	10,000	5,000	5,000	5.0	1.0

\* Assume that the stress and strength are normally distributed.

The relationship between safety factor  $F$  and reliability will be given as follows:

Letting safety factor  $F$  be a random variable, we define  $F$  as the ratio of the strength  $S$  to the stress  $s$ ; that is,  $F = \frac{S}{s}$ . Let  $\bar{S}$  and  $\bar{s}$  denote the mean strength and stress, respectively. Let  $F_c$  denote the central factor of safety, which is,  $F_c = \bar{S}/\bar{s}$ . Using the Bienayme-Chebyshev inequality and Taylor's series approximation, we can get the following relationship between central safety factor  $F_c$  and reliability  $R$  [10].

$$F_c \geq \frac{1}{(1 + V_s^2) - \sqrt{V_s^2 + V_s^2 \sqrt{\frac{R}{1-R}}}}$$

where

$$V_s = \frac{\sigma_s}{\bar{s}}, \quad V_S = \frac{\sigma_S}{\bar{S}} \quad (2.26)$$

## 2.5 Computation of moments of a function of random variables

Let us consider the case where  $x$  is a one-dimensional variable. The expression of  $Y = f(x)$  about the point  $x = \mu$ , which  $\mu$  is expected value of variable,  $x$ , by Taylor's series approximation up to the first three terms is

$$Y = f(x) = f(\mu) + (x-\mu)f'(\mu) + \frac{(x-\mu)^2}{2!} f''(\mu) + r \quad (2.27)$$

where  $r$  is the remainder. Taking the expectation of equation (2.27), we have

$$\begin{aligned}
E(Y) &\cong E[f(\mu)] + E\{xf'(\mu) - \mu f'(\mu)\} \\
&+ E\left\{\frac{1}{2} f''(\mu)(x-\mu)^2\right\} + E(r) \\
&\cong f(\mu) + \{ \mu f'(\mu) - \mu f'(\mu) \} + \frac{1}{2} f''(\mu)V(x) + E(r) \\
&\cong f(\mu) + \frac{1}{2} f''(\mu)V(x) \tag{2.28}
\end{aligned}$$

Equation (2.28) is an approximation for the expected value of  $Y$  because we have ignored the remained terms in the Taylor's series expansion. If the variance of  $x$  is small, then we may further ignore the second term in equation (2.28) to obtain

$$E(Y) = E[f(x)] \cong f(\mu) \tag{2.29}$$

The approximation value of  $V(Y)$ , is obtained by considering once again the Taylor's series expansion up to the first two terms:

$$Y = f(\mu) + (x-\mu)f'(\mu) + r \tag{2.30}$$

Taking the variance of equation (2.30) and ignoring the remainder term  $r$ , we have

$$\begin{aligned}
V(Y) &= V[f(\mu)] + V[(x-\mu)f'(\mu)] \\
&\cong [f'(\mu)]^2 V(x) \tag{2.31}
\end{aligned}$$

If we consider  $x$  is  $n$ -dimensional variables, that is

$$Y = f(x_1, x_2, \dots, x_n) = f(\underline{x}) \tag{2.32}$$

Let  $\mu = (\mu_1, \dots, \mu_n)$  and  $\sigma = (\sigma_1, \dots, \sigma_n)$  denote the vectors of the expected values and the standard deviations of  $x_1, x_2, \dots, x_n$

respectively. Then, by the Taylor's series expansion, we have

$$\begin{aligned}
 Y &= f(x_1, x_2, \dots, x_n) \\
 &= f(\mu_1, \mu_2, \dots, \mu_n) + \sum_{i=1}^n \frac{\partial f(\underline{x})}{\partial x_i} \Big|_{x=\mu} (x_i - \mu_i) \\
 &\quad + \frac{1}{2!} \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f(\underline{x})}{\partial x_i \partial x_j} \Big|_{x=\mu} (x_i - \mu_i)(x_j - \mu_j) + r \quad (2.33)
 \end{aligned}$$

Taking the expectation of equation (2.33), we have

$$\begin{aligned}
 E(Y) &= f(\mu_1, \dots, \mu_n) + \sum_{i=1}^n \frac{\partial f(\underline{x})}{\partial x_i} \Big|_{x=\mu} E(x_i - \mu_i) \\
 &\quad + \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f(\underline{x})}{\partial x_i \partial x_j} \Big|_{x=\mu} E[(x_i - \mu_i)(x_j - \mu_j)] + E(r) \quad (2.34)
 \end{aligned}$$

If  $x_1, x_2, \dots, x_n$  are independent variables, then, after deleting the zero terms, we get

$$\begin{aligned}
 E(Y) &= f(\mu_1, \dots, \mu_n) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 f(\underline{x})}{\partial x_i^2} \Big|_{x=\mu} V(x_i) + E(r) \\
 &\approx f(\mu_1, \dots, \mu_n) + \frac{1}{2} \sum_{i=1}^n \frac{\partial^2 f(\underline{x})}{\partial x_i^2} \Big|_{x=\mu} V(x_i) \quad (2.35)
 \end{aligned}$$



If we ignore the second term in equation (2.35) then,

$$E(Y) = f(\mu_1, \dots, \mu_n) \quad (2.36)$$

Now, considering only the first two terms in equation (2.33) and taking the variance, we have

$$\begin{aligned} V(Y) &= V[f(\underline{x}) \Big|_{x=\mu}] + V\left[\sum_{i=1}^n \frac{\partial f(\underline{x})}{\partial x_i} \Big|_{x=\mu} (x_i - \mu_i)\right] \\ &= \sum_{i=1}^n \left\{ \frac{\partial f(\underline{x})}{\partial x_i} \Big|_{x=\mu} \right\}^2 V(x_i) \end{aligned} \quad (2.37)$$

### CHAPTER 3 THE STRESS AND STRENGTH INTERFERENCE MODELS (S-S-I)

In order to compute the reliability we have to know the nature of the stress( $s$ ) and strength( $S$ ) random variables. As we mentioned before in Section 2-1, the distributions of stress or strength can be a normal, log-normal, exponential, gamma, Weibull, or extreme value distribution.

We can use equations (2.16) and (2.17) to compute the reliability.

They are

$$R = \int_{-\infty}^{\infty} f_S(s) \left\{ \int_s^{\infty} f_S(S) dS \right\} ds$$

or

$$R = \int_{-\infty}^{\infty} f_S(S) \left\{ \int_{-\infty}^S f_S(s) ds \right\} dS$$

where  $f_S(\cdot)$  and  $f_s(\cdot)$  are the probability density functions (pdf) of stress and strength, respectively. They can be one of distributions we mentioned above. Hence, the reliability of a component can be a different type of formulation. There are many combinations of different distributions. But, only some of them are derived, and the rest of them can be seen in references [4] and [10].

#### 3.1 Reliability computation for normally distributed strength and stress [10].

The pdf of a normally distributed stress( $s$ ) is given by

$$f_S(s) = \frac{1}{\sigma_S \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left[ \frac{s - \mu_S}{\sigma_S} \right]^2 \right\} \quad -\infty < s < \infty \quad (3.1)$$

and the pdf of a normally distributed strength ( $S$ ) is given by

$$f_S(S) = \frac{1}{\sigma_S \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left[ \frac{S - \mu_S}{\sigma_S} \right]^2 \right\} \quad -\infty < S < \infty \quad (3.2)$$

where

$\mu_S$  = mean value of strength

$\mu_s$  = mean value of stress

$\sigma_S$  = standard deviation of strength

$\sigma_s$  = standard deviation of stress

Let us define  $y = S - s$ . The random variable  $y$  is also normally distributed with mean  $\mu_y$ , that is

$$\mu_y = \mu_S - \mu_s \quad (3.3)$$

and a standard deviation  $\sigma_y$  is

$$\sigma_y = \sqrt{\sigma_S^2 + \sigma_s^2} \quad (3.4)$$

The reliability  $R$  can be expressed in terms of  $y$  which  $y > 0$ , by

$$R = P(y > 0) \\ = \int_0^{\infty} \frac{1}{\sigma_y \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{y - \mu_y}{\sigma_y} \right)^2 \right\} dy \quad (3.5)$$

If we let  $z = (y - \mu_y) / \sigma_y$ , then  $\sigma_y dz = dy$ . When  $y = 0$ , the lower limit of  $z$  is given by

$$z = \frac{0 - \mu_y}{\sigma_y} = - \frac{\mu_S - \mu_s}{\sqrt{\sigma_S^2 + \sigma_s^2}} \quad (3.6)$$

and when  $y \rightarrow +\infty$ , the upper limit of  $z \rightarrow +\infty$ . Therefore, equation (3.5) can be rewritten as

$$R = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\mu_S - \mu_s}{\sqrt{\sigma_S^2 + \sigma_s^2}}}^{\infty} e^{-\frac{z^2}{2}} dz \quad (3.7)$$

Obviously, the random variable  $z = (y - \mu_y) / \sigma_y$  is the standard normal variable. Hence, equation (3.7) can be changed to

$$R = 1 - \Phi \left( - \frac{\mu_S - \mu_s}{\sqrt{\sigma_S^2 + \sigma_s^2}} \right) \quad (3.8)$$

Reliability can be found by referring  $z = - \frac{\mu_S - \mu_s}{\sqrt{\sigma_S^2 + \sigma_s^2}}$  to the normal table.

### 3.2 Reliability computation for log-normally distributed strength and stress [10].

The log-normal density function is given by

$$f_p(p) = \frac{1}{p\sigma\sqrt{2\pi}} \exp \left[ - \frac{1}{2\sigma^2} (\ln p - \mu)^2 \right] \quad p > 0 \quad (3.9)$$

where  $p$  is the random variable. The parameters  $\mu$  and  $\sigma$  are the expected value and the standard deviation, respectively, of the variable  $\ln p$  which is normally distributed.

Let  $x = \ln p$ . Then  $dx = (1/p) dp$ . From equation (3.9), we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[ -\frac{1}{2\sigma^2} (x - \mu)^2 \right] \quad -\infty < x < \infty \quad (3.10)$$

If  $\overset{v}{p}$  denote the median of  $p$ , then we may write

$$0.5 = \int_0^{\overset{v}{p}} \frac{1}{p\sigma\sqrt{2\pi}} \exp \left[ -\frac{1}{2\sigma^2} (\ln p - \mu)^2 \right] dp \quad (3.11)$$

Using the transformation  $x = \ln p$ , we rewrite it as

$$0.5 = \int_{-\infty}^{\ln \overset{v}{p}} \frac{1}{\sigma\sqrt{2\pi}} \exp \left[ -\frac{1}{2\sigma^2} (x - \mu)^2 \right] dx \quad (3.12)$$

Yielding  $\mu = \ln \overset{v}{p}$  that is,  $\overset{v}{p} = e^\mu$ .

Returning to the original problem in which  $S$  and  $s$  are log normally distributed, we let  $y = S/s$ , which means  $\ln y = \ln S - \ln s$ .  $\ln y$  is normally distributed since both  $\ln S$  and  $\ln s$  are normally distributed.

The log normal density function is positively skewed and hence the median is a better and more convenient measure of the central tendency for the log normal distribution than the mean. The antilog of the mean of  $\ln S$  is the median of  $f_S(\cdot)$ , and that of  $\ln s$  the median of  $f_s(\cdot)$ ; that is,

$$\frac{V}{S} = e^{\mu_{\ln S}} \quad \text{or} \quad \mu_{\ln S} = \ln \frac{V}{S}$$

and

$$\frac{V}{s} = e^{\mu_{\ln s}} \quad \text{or} \quad \mu_{\ln s} = \ln \frac{V}{s}$$

where  $\frac{V}{S}$  and  $\frac{V}{s}$  are the medians of  $S$  and  $s$  respectively, By analogy, we add

$$\mu_{\ln y} = \ln \frac{V}{y} \quad (3.13)$$

since the variable  $y$  is also log normally distributed. But,

$$\mu_{\ln y} = \mu_{\ln S} - \mu_{\ln s} = \ln \frac{V}{S} - \ln \frac{V}{s} \quad (3.14)$$

Combining the two equations, (3.13) and (3.14), we get

$$\ln \frac{V}{y} = \ln \frac{V}{S} - \ln \frac{V}{s} = \ln \frac{s}{S} \quad (3.15)$$

and

$$\sigma_{\ln y} = \sqrt{\sigma_{\ln S}^2 + \sigma_{\ln s}^2} \quad (3.16)$$

From the definition of reliability, we have

$$R = P\left(\frac{S}{s} > 1\right) = P(y > 1) = \int_1^{\infty} f_y(y) dy$$

Let  $z = (\ln y - \mu_{\ln y})/\sigma_{\ln y}$ . Then  $z$  is the standard normal variable.

To find the new limits of integration, when  $y = 1$ ,

$$z = \frac{\ln 1 - \mu_{\ln y}}{\sigma_{\ln y}} = - \frac{\ln \frac{V}{S} - \ln \frac{V}{s}}{\sqrt{\sigma_{\ln S}^2 + \sigma_{\ln s}^2}} \quad (3.17)$$

and when  $y \rightarrow +\infty$ ,  $z \rightarrow +\infty$ . Then, the reliability can be rewritten as

$$R = \frac{1}{\sqrt{2\pi}} \int_{\frac{-\ln S - \ln s}{\sqrt{\sigma_{\ln S}^2 + \sigma_{\ln s}^2}}}^{\infty} e^{-z^2/2} dz \quad (3.18)$$

### 3.3 Reliability computation for exponentially distributed strength and stress [10]

The pdf of an exponentially distributed strength  $S$  is given by

$$f_S(S) = \lambda_S e^{-\lambda_S S} \quad 0 \leq S < \infty \quad (3.19)$$

and that for stress  $s$  is

$$f_s(s) = \lambda_s e^{-\lambda_s s} \quad 0 \leq s \leq \infty \quad (3.20)$$

Using equation (2.16), we have

$$\begin{aligned} R &= \int_0^{\infty} f_s(s) \left[ \int_s^{\infty} f_S(S) dS \right] ds \\ &= \int_0^{\infty} \lambda_s e^{-\lambda_s s} \left[ \int_s^{\infty} \lambda_S e^{-\lambda_S S} dS \right] ds \\ &= \int_0^{\infty} \lambda_s e^{-\lambda_s s} [e^{-\lambda_S s}] ds \\ &= \int_0^{\infty} \lambda_s e^{-(\lambda_s + \lambda_S)s} ds \end{aligned}$$

$$= \frac{\lambda_S}{\lambda_S + \lambda_s} \quad (3.21)$$

If the mean value of strength is denoted by  $\mu_S$ , then

$$\begin{aligned} \mu_S = E(S) &= \int_0^{\infty} f_S(S) \cdot S dS \\ &= \int_0^{\infty} \lambda_S e^{-\lambda_S S} \cdot S dS \\ &= \frac{1}{\lambda_S} \end{aligned} \quad (3.22)$$

and the mean value of stress by  $\mu_s$  is

$$\mu_s = E(s) = \frac{1}{\lambda_s} \quad (3.23)$$

then

$$R = \frac{\mu_S}{\mu_S + \mu_s} \quad (3.24)$$

### 3.4 Reliability computation for gamma distributed strength and stress

[10]

The gamma density function of a random variable  $x$  is given by

$$f(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} \quad n > 0, \lambda > 0, 0 \leq x < \infty \quad (3.25)$$

where  $\lambda$  is called the scale parameter and  $n$  the shape parameter and  $\Gamma(n)$  is the gamma function. The pdf for strength is



$$f_S(S) = \frac{\lambda^m}{\Gamma(m)} S^{m-1} e^{-\lambda S} \quad \lambda > 0, \quad m > 0, \quad 0 \leq S < \infty \quad (3.26)$$

and the pdf for stress is

$$f_s(s) = \frac{u^n}{\Gamma(n)} s^{n-1} e^{-us} \quad u > 0, \quad n > 0, \quad 0 \leq s < \infty \quad (3.27)$$

Using equation (2.21), we have, as before, for  $y = S-s$

$$\begin{aligned} R &= \int_0^\infty f_S(y+s) f_s(s) ds \\ &= \frac{\lambda^m u^n}{\Gamma(m)\Gamma(n)} \int_0^\infty \int_0^\infty (y+s)^{m-1} e^{-\lambda(y+s)} s^{n-1} e^{-us} ds dy \end{aligned} \quad (3.28)$$

Let  $v = s/y$ ,  $r = u/\lambda$ , and  $dv = (1/y)ds$ , then

$$R = \frac{\lambda^m u^n}{\Gamma(m)\Gamma(n)} \int_0^\infty y^{m+n-1} e^{-(1+(1+r)v)y\lambda} dy \int_0^\infty (1+v)^{m-1} v^{n-1} dv \quad (3.29)$$

also let

$$\int_0^\infty y^{m+n-1} e^{-[(1+(1+r)v)\lambda]y} dy = \frac{\Gamma(m+n)}{(1+(1+r)v)^{m+n} \lambda^{m+n}}$$

then we have

$$R = \frac{r^n \Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^\infty \frac{(1+v)^{m-1} v^{n-1}}{(1+(1+r)v)^{m+n}} dv \quad (3.30)$$

where  $r = u/\lambda$ . If let  $w = rv/(1+(1+r)v)$ , we have

$$dw = \left\{ \frac{r}{(1+(1+r)v)} - \frac{(1+r)vr}{(1+(1+r)v)^2} \right\} dv = \frac{vr}{(1+(1+r)v)^2} dv$$

$$R = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^{r/(1+r)} (1-w)^{m-1} w^{n-1} dw$$

Four special cases are briefly discussed here.

1. If  $\lambda = \mu = 1$ , then

$$R = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^{\infty} \frac{(1+v)^{m-1} v^{n-1}}{(1+2v)^{m+n}} dv = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^{1/2} (1-w)^{m-1} w^{n-1} dw \quad (3.31)$$

2. If  $m = n = 1$ , then  $S$  and  $s$  are exponentially distributed with

$$R = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \int_0^{r/1+r} dw = \frac{r}{1+r} = \frac{u}{u+\lambda} \quad (3.32)$$

Equation (3.32) is the same as equation (3.21).

3. If  $m \neq 1$ , and  $n = 1$ , then the strength has a gamma distribution and the stress has an exponential distribution. In this case, we have

$$k = \frac{\Gamma(m+1)}{\Gamma(m)\Gamma(1)} \int_0^{r/1+r} (1-w)^{m-1} dw = 1 - \left(\frac{1}{1+r}\right)^m = 1 - \left(\frac{\lambda}{\mu+\lambda}\right)^m \quad (3.33)$$

4. If  $m=1$ , and  $n \neq 1$ , then the strength has an exponential distribution and the stress has a gamma distribution. In this case, we have

$$R = \frac{\Gamma(n+1)}{\Gamma(1)\Gamma(n)} \int_0^{r/(1+r)} w^{n-1} dw = \frac{n\Gamma(n)}{\Gamma(n)} \left(\frac{r}{1+r}\right)^n \frac{1}{n} = \left(\frac{r}{1+r}\right)^n \quad (3.34)$$

3-5 Reliability computation for Weibull distributed strength and stress. [10]

The pdf of Weibull distribution has been mentioned in Section 2-1. The pdf for the strength and the stress are given by

$$f_S(S) = \frac{\beta_S}{\theta_S} \left(\frac{S-S_0}{\theta_S}\right)^{\beta_S-1} \exp\left[-\left(\frac{S-S_0}{\theta_S}\right)^{\beta_S}\right], \quad S_0 \leq S < \infty \quad (3.35)$$

and

$$f_s(s) = \frac{\beta_s}{\theta_s} \left(\frac{s-s_0}{\theta_s}\right)^{\beta_s-1} \exp\left[-\left(\frac{s-s_0}{\theta_s}\right)^{\beta_s}\right], \quad s_0 \leq s < \infty \quad (3.36)$$

respectively. The probability of failure given in equation (2.20) will be

$$\begin{aligned} \bar{R} &= P(S \leq s) = \int_{-\infty}^{\infty} [1 - F_S(S)] f_S(S) dS \\ &= \int_{S_0}^{\infty} \exp\left[-\left(\frac{S-s_0}{\theta_S}\right)^{\beta_S}\right] \frac{\beta_S}{\theta_S} \left(\frac{S-s_0}{\theta_S}\right)^{\beta_S-1} \exp\left[-\left(\frac{S-s_0}{\theta_S}\right)^{\beta_S}\right] dS \end{aligned} \quad (3.37)$$

where

$$F_S(S) = 1 - \exp\left[-\left(\frac{S-s_0}{\theta_S}\right)^{\beta_S}\right]$$

Let  $q = \left(\frac{S-s_0}{\theta_S}\right)^{\beta_S}$

then

$$dq = \frac{\beta_S}{\theta_S} \left( \frac{S-S_0}{\theta_S} \right)^{\beta_S-1} dS \quad (3.38)$$

and

$$S = q^{1/\beta_S} \theta_S + S_0 \quad (3.39)$$

then we have

$$\bar{R} = P(S \leq s) = \int_0^\infty e^{-q} \exp\left[-\left(\frac{\theta_S}{\theta_s} q^{1/\beta_S} + \frac{S_0-s_0}{\theta_s}\right)^{\beta_S}\right] dq \quad (3.40)$$

$$R = P(S \geq s) = \int_{-\infty}^0 e^{-q} \exp\left[-\left(\frac{\theta_S}{\theta_s} q^{1/\beta_S} + \frac{S_0-s_0}{\theta_s}\right)^{\beta_S}\right] dq$$

### 3-6 Reliability computation for extreme value distributions

As we mentioned in chapter 2. The pdf of the smallest value distribution is given by

$$g_n(x) = nf(x)[1-F(x)]^{n-1} \quad (3.41)$$

and the cumulative distribution function is given by

$$G_n(x) = 1 - [1 - F(x)]^n$$

where

$$F(x) = \int_{-\infty}^x f(x)dx$$

There are three asymptotic distributions for the smallest value.

They have the following form:

Type I

$$G_n(x) = 1 - \exp[-\exp(\frac{x-x_0}{\theta})] \quad -\infty < x < +\infty, \quad \theta > 0 \quad (3.43)$$

Type II

$$G_n(x) = 1 - \exp[-(-\frac{x-x_0}{\theta})^{-\beta}] \quad -\infty < x \leq x_0, \quad \theta > 0, \beta > 0 \quad (3.44)$$

Type III

$$G_n(x) = 1 - \exp[-(\frac{x-x_0}{\theta})^\beta] \quad x_0 \leq x < \infty, \quad \theta > 0, \beta > 0 \quad (3.45)$$

Each type of these distributions arises when certain conditions are met. Type I arises when the underlying density function tends to zero exponentially as  $x \rightarrow +\infty$ . On the other hand, if the range of the density function is unbounded from below and if for some  $\theta > 0, \beta > 0$  we have

$$\lim_{x \rightarrow +\infty} (-x)^\theta F_x(x) = \beta$$

the limiting distribution of the smallest value is of Type II. Type III distribution arises when the following conditions are met:

- 1). The range for the underlying density function is bounded from below (i.e.,  $x \geq x_0$ ).
- 2).  $F_x(x)$  behaves like  $(\theta-x_0)^\beta$ , for some  $\theta > 0, \beta > 0$  and as  $x \rightarrow \infty$ .

The pdf of the largest value is given by

$$h_n(x) = nf(x)[F(x)]^{n-1} \quad (3.46)$$

and the cumulative distribution function is given by

$$H_n(x) = [F(x)]^n \quad (3.47)$$

There are also three asymptotic distributions for the largest value.

Type I

$$H_n(x) = \exp[-\exp[-(\frac{x-x_0}{\theta})]] \quad -\infty < x < \infty, \theta > 0 \quad (3.48)$$

Type II

$$H_n(x) = \exp[-(\frac{x-x_0}{\theta})^{-\beta}] \quad x \geq x_0, \theta > 0, \beta > 0 \quad (3.49)$$

Type III

$$H_n(x) = \exp[-(-\frac{x-x_0}{\theta})^\beta] \quad x \leq x_0, \theta > 0, \beta > 0 \quad (3.50)$$

Let us consider the case where the strength has the Weibull distribution and the stress has the Type II largest extreme value distribution, we have

$$f_S(s) = \frac{\beta_S}{\theta_S} (\frac{s-s_0}{\theta_S})^{\beta_S-1} \exp[-(\frac{s-s_0}{\theta_S})^{\beta_S}] \quad s_0 \leq s < \infty \quad (3.51)$$

and

$$F_S(s) = \exp[-(\frac{s-s_0}{\theta_S})^{\beta_S}] \quad s_0 \leq s < \infty, \theta_S > 0, \beta_S > 0 \quad (3.52)$$

Substituting these two equations into equation (2.17) gives

$$R = \int_{S_0}^{\infty} \exp\left[-\left(\frac{S-s_0}{\theta_S}\right)^{\beta_S}\right] \frac{\beta_S}{\theta_S} \left(\frac{S-s_0}{\theta_S}\right)^{\beta_S-1} \exp\left[-\left(\frac{S-s_0}{\theta_S}\right)^{\beta_S}\right] dS \quad (3.53)$$

Let

$$q = \left(\frac{S-s_0}{\theta_S}\right)^{\beta_S}$$

then

$$dq = \frac{\beta_S}{\theta_S} \left(\frac{S-s_0}{\theta_S}\right)^{\beta_S-1} dS$$

and

$$q^{1/\beta_S} \cdot \theta_S + S_0 = S$$

Hence

$$R = \int_0^{\infty} e^{-q} \exp\left[-\left[\frac{\theta_S}{\theta_S} q^{1/\beta_S} + \frac{S_0-s_0}{\theta_S}\right]^{\beta_S}\right] dq \quad (3.54)$$

Similarly, the reliability in other case can be derived which either strength or stress has extreme value distribution. For example:

- 1). Weibull distributed strength and Type III smallest extreme value distributed stress:

$$R = \int_0^{\infty} e^{-q} \exp\left[-\left(\frac{\delta_S - \theta_S \ln q - S_0}{\theta_S}\right)^{\beta_S}\right] dq \quad (3.55)$$

where

$$q = \exp\left[-\left(\frac{S-\delta_S}{\theta_S}\right)\right]$$

- 2). Type I smallest extreme value strength and Weibull distributed stress:

$$R = \int_0^{\infty} e^{-q} \exp\left[-\exp\left[\frac{\theta_S}{\theta_S} \left(q^{1/\beta_S} - \frac{\delta_S - s_0}{\theta_S}\right)\right]\right] dq \quad (3.56)$$

where

$$q = \left(\frac{s-s_0}{\theta_S}\right)^{\beta_S}$$

- 3). Both stress and strength tend to be Type I largest extreme value distribution:

$$R = \int_0^{\infty} e^{-q} \exp\left[-\exp\left[\frac{\theta_S}{\theta_S} \left(\ln q - \frac{\delta_S - \delta_S}{\theta_S}\right)\right]\right] dq \quad (3.57)$$

where

$$q = \exp\left[-\left(\frac{s-\delta_S}{\theta_S}\right)\right]$$

There still have many combinations of different strength and stress distributions, which we did not derive here. But, the derivation is quiet similar to what we derived here. The equations we derived before almost include the integration. Those integrations cannot integrate analytically. Hence, a numerical integration method is used, which is shown in Appendix A.



## CHAPTER 4. TIME DEPENDENT STRESS-STRENGTH MODELS (S-S-T)

In Chapter 3, the stress-strength interference (S-S-I) models have been presented. However, the important time-factor is not considered. As time proceeds, everything changes. A physiological or psychological function changes as a human ages. A radioactive element reduces its weight as time passes. Concerning society, population changes when time passes. Similarly, also affects the stress or strength distribution. The time factor can be classified into the following two categories: [1.5]

### 1). Cyclic Occurrences:

The random variable, such as stress or strength, changes according to cyclic pattern. These variations may be due to natural seasonal changes or day/night temperature cycles or man made on-off, up-down, cycles. The cyclic changes have a fixed period and a nonconstant time between cycles. The reliability is a function of the number of cycles,  $n$ , rather than time. Examples are the operating cycles of a relay or the number of take offs and landings of an aircraft.

### 2). Random Occurrences

In this category, the times between variable occurrences are random rather than known. If it is assumed that stresses occur randomly and that each stress occurrence is independent of each other, then we have a Poisson probability law for stress occurrence. Random occurrences include not only the Poisson probability law but other occurrence laws as well.

The stress and the strength may vary during a long operating interval, in relation to the passage of time or to the number and/or severity of previous stresses. We will use the following classification for such time variations.

A). Aging - Aging describes changes with time in the parameters of the model. Most commonly this is modeled as a shift of the mean and/or variance, e.g., a linear decrease in strength. A simple example is the corrosion in a liquid cooling system.

B). Cyclic Damage - An item may experience a change in its strength as the device undergoes repeated operating cycles. Thus, the strength density is a function of the number of cycles  $n$ . An example of this phenomena is the shortened life of a light bulb which is subject to many on-off cycles rather than allowed to burn continuously.

C). Cumulative Damage - A device is said to suffer cumulative damage when its decrease in strength is determined by the magnitude and the number of previous stresses. An example would be air leakage from a space craft due to meteorite collisions puncturing the skin. Larger meteorites making larger holes creating more air leakage may be assumed.

These two terms cyclic occurrences and cyclic damage, are often confused as are random occurrence and aging. The similarity between cyclic occurrences and cyclic damage is in considering cycles  $n$  rather than time period. The main difference between them is that the cyclic damage is for strength only, whereas cyclic occurrences is for both strength and stress. The similarity and difference between random occurrences and aging are the same as between cyclic occurrences and cyclic damage. The only difference between former and latter is one for time period and another for cycles  $n$ .

The uncertainty about the stress and the strength variables may be classified in the following three categories: 1) deterministic stress or strength, 2) random-fixed stress or strength, 3) random-independent stress or strength.

1). Deterministic stress or strength

The variable is either constant or varies in some known predictable manner. If both stress and strength are known, the failure is deterministic rather than probabilistic. The device succeeds if the strength is greater than the stress, and fails if the strength is not greater than the stress. Of course known or deterministic implies that the manufacturing process is well controlled so the parameters are predictable or a simple nondestructive test is available to determine the parameters. This control rarely exists in the real world, but, in some cases, acceptable approximation may be used.

2). Random-fixed stress or strength

We are interested in the behavior of the variable with respect to time or cycles. Strength is a random variable at any particular instant of time. It is assumed that enough data has been recorded in the past to determine a probability density function for strength or stress. It is also assumed that any test to determine the variable precisely is too costly or destructive. The word "fixed" in this classification refers to the behavior of the random variable with respect to time and/or cycles. This means that the random variable changes in time in a known manner.

Let  $f_{S_0}(S_0)$  be the pdf of the strength random variable  $S_0$  at the initial time. Then the strength  $S(t)$  at any instant of time  $t$  is given by

$$S(t) = S_0 \phi(t) \quad (4.1)$$

Where  $\phi(t)$  is a known or given function that describes exactly how strength decreases with time. For example, let

$$\phi(t) = (1 - 0.0001 t) \quad (4.2)$$

which means that with the passage of each unit of time, strength decreases by 0.01% of the initial strength  $S_0$ . Similarly, let  $S_k$  denote the strength for the  $k$ th application of load, and let us assume that  $S_k$  is a function of  $S_0$  and the occurrence number  $k$ . Thus, we have

$$S_k = S_0 k^{-\alpha} \quad (4.3)$$

or

$$S_k = S_0 + bk \quad (4.4)$$

In the above equations, the strength decreases with the number of load occurrences for  $\alpha > 0$  and  $b < 0$ , respectively. If increase in strength occurs because of work hardening, we have  $\alpha < 0$  and  $b > 0$ .

### 3). Random-independent stress or strength

In this category, not only is a variable value sufficiently unknown so that it is well described as a random variable, but successive variable with respect to time and/or cycles are so unrelated as to be statistically independent. Observation of one variable value gives no information about the size of the subsequent value. Successive stresses are generally independent. Strength will vary randomly and will be independently from cycle to cycle only if it is being affected by other

environmental factors, such as temperature and vibrations, which are independent of the process.

#### 4.1 Reliability computations for cyclic occurrences

The reliability of stress-strength interference cases without considering the time factor are presented earlier in this paper. The individual expressions for reliability with respect to time are developed here. We obtain these expressions by combining the three levels of uncertainty classifications for both stress and strength and two types of cycles occurrences. There should have 18 (= 3x3x2) expressions plus aging, cyclic damage and cumulative damage. Totally, 21 expressions should be developed. But not all of them are going to be developed here. Because aging and cyclic damage are special cases in the cyclic occurrences and random occurrences respectively, the reliability of random occurrences will not be developed here either. Only nine cases of the cyclic occurrences and cumulative damage are developed.

In the cyclic occurrences, we like to use  $R_n$ , the reliability after  $n$  cycles (the probability of not having a failure on any one of the  $n$  cycles), rather than  $R(t)$ , the reliability at time  $t$ , the argument  $t$  being continuous, because the former can be converted to the latter very simply when cycle times are deterministically known. For example

$$R(t) = R_n \quad t_n < t \leq t_{n+1} \quad n = 1, 2, \dots \quad (4.5)$$

where  $t_i$  is the instant in time at which the  $i$ th cycle occurs.

Case 1. deterministic stress and deterministic strength [10]

Let  $s_i$  and  $S_i$ ,  $i = 1, 2, \dots, n$  denote the stress and strength,

respectively, on the  $i$ th cycle. Then

$$R_n = P[E_1, E_2, \dots, E_n] \quad (4.6)$$

where  $R_n$  = reliability after  $n$  cycles of occurrences

$E_i$  = event of no failure occurs on the  $i$ th cycle

Hence, we have

$$R_n = \begin{cases} 0 & \text{if } s_i > S_i & \text{for some } i \quad 1 \leq i \leq n \\ 1 & \text{if } s_i \leq S_i & \text{for all } i \quad 1 \leq i \leq n \end{cases}$$

Case 2. deterministic stress and random-fixed strength [10]

Let  $s_0$ , a constant, denote the stress and  $S_i$  denote the strength on the  $i$ th cycle. The,

$$S_i = S_0 - a_i \quad i = 1, 2, \dots$$

where  $a_i \geq 0$  are known constants. Further, the  $a_i$ 's are assumed to be nondecreasing in time. The pdf of  $S_0$ ,  $f_{S_0}(S_0)$ , is assumed to be known. Then

$$\begin{aligned} P[E_n] &= P(s_n \leq S_n) \\ &= P(s_0 \leq S_0 - a_n) \\ &= P(S_0 \geq s_0 + a_n) \\ &= \int_{s_0 + a_n}^{\infty} f_{S_0}(S_0) dS_0 \end{aligned}$$

But

$$\begin{aligned}
 R_n &= P[E_1, E_2, \dots, E_n] \\
 &= P[E_1 | E_2, E_3, \dots, E_n] \times P[E_2, E_3, \dots, E_n] \\
 &= P[E_1 | E_2, E_3, \dots, E_n] \times P[E_2 | E_3, E_4, \dots, E_n] \\
 &\quad \times P[E_3, E_4, \dots, E_n] \\
 &= P[E_1 | E_2, E_3, \dots, E_n] \times P[E_2 | E_3, E_4, \dots, E_n] \\
 &\quad \times \dots \times P[E_{n-1} | E_n] \times P[E_n] \quad (4.7)
 \end{aligned}$$

All but the last term ( $P[E_n]$ ) in the R.H.S. of equation(4.7) are 1's because of the restrictions on the  $a_i$ 's which cause the strength  $S_i$  to decrease in time. Hence,

$$R_n = P[E_n] = \int_{s_0 + a_n}^{\infty} f_{S_0}(S_0) dS_0. \quad (4.8)$$

Let  $R_{n,n-1}$  be the conditional reliability for the  $n$ th cycle given that the device has survived the previous  $(n-1)$  cycles. Then

$$R_{n,n-1} = P[E_n | E_1, E_2, \dots, E_{n-1}] = \frac{P[E_1, \dots, E_n]}{P[E_1, \dots, E_{n-1}]} = \frac{R_n}{R_{n-1}} \quad (4.9)$$

Case 3. deterministic stress and random-independent strength [10]

Let a constant  $s_0$  denote the stress. Let  $f_{S_i}(S)$  be the pdf of the random variable strength  $S_i$  on the cycle  $i$ . Since successive values of

$S_i$  are independent, we get

$$R_n = P[E_1, E_2, \dots, E_n] = P[E_1] \times P[E_2] \times \dots \times P[E_n] \quad (4.10)$$

where

$$P[E_i] = P(s_0 \leq S_i) = \int_{s_0}^{\infty} f_{S_i}(S) dS \quad i = 1, 2, \dots, n$$

In particular, if the pdf remains unchanged over time, that is, if

$$f_{S_1}(S) = f_{S_2}(S) = \dots = f_{S_n}(S) = f_S(S) \quad (4.11)$$

then

$$R_n = (P(E_i))^n = \left[ \int_{s_0}^{\infty} f_S(S) dS \right]^n$$

Case 4. random-fixed stress and deterministic strength [10]

This case is similar to case 2 but the roles of stress and strength are reversed. Let  $s_i = s_0 + b_i$   $i = 1, 2, \dots$  denote the stress in cycle  $i$ , where  $b_i$ 's are known nonnegative constants, nondecreasing in time. Let a constant  $S_0$  denote the strength, and the pdf of  $s_0$ ,  $f_{s_0}(R_0)$ , is known. Then

$$\begin{aligned} R_n &= P[E_n] \\ &= P[s_n \leq S_0] \\ &= P(s_0 + b_n \leq S_0) \\ &= P(s_0 \leq S_0 - b_n) \end{aligned}$$



$$= \int_0^{S_0 - b_n} f_{s_0}(s_0) ds_0 \quad (4.12)$$

Case 5. random fixed stress and random-fixed strength [10]

Let the stress be given by

$$s_i = s_0 + b_i$$

and the strength

$$S_i = S_0 - a_i \quad i = 1, 2, \dots$$

where  $a_i$ 's and  $b_i$ 's have the same restrictions as in cases 2 and 4, respectively. The pdf's  $f_{s_0}(s_0)$  and  $f_{S_0}(S_0)$  are assumed to be known. It is required that the stress is nondecreasing and the strength is non-increasing. Hence, arguments identical to those in case 2 would yield

$$\begin{aligned} R_n &= P[E_n] \\ &= P(s_n \leq S_n) \\ &= P(s_0 + b_n \leq S_0 - a_n) \\ &= P(s_0 \leq S_0 - a_n - b_n) \\ &= \int_0^{\infty} f_{S_0}(S_0) \int_0^{S_0 - a_n - b_n} f_{s_0}(s_0) ds_0 dS_0 \end{aligned} \quad (4.13)$$

In special case, when  $a_i = b_i = 0$ , the equation (4.13) can be recognized as the standard expression for SSI model.

Case 6. random-fixed stress and random-independent strength [10]

Let  $f_s(s)$  denote the pdf of a random variable  $s$ . The successive random strengths  $S_1, S_2, \dots, S_n$  are independent, and identically distributed with pdf,  $f_S(S)$ , by assumption

Thus

$$R_n = P[E_1, E_2, \dots, E_n]$$

The  $E_i$  event means that  $E_i \quad (S_i > s)$

Hence

$$\begin{aligned} R_n &= P[(S_1 > s) \cap (S_2 > s) \cap (S_3 > s) \cap \dots \cap (S_n > s)] \\ &= P[\min(S_1, S_2, \dots, S_n) > s] \end{aligned} \quad (4.14)$$

The distribution function of the random variable

$$S_{\min} = \min(S_1, S_2, \dots, S_n) \quad (4.15)$$

is the smallest value distribution which has been mentioned in Section 2.1.

$$G_n(S) = 1 - [1 - F_S(S)]^n$$

Now, equation (4.18) can be rewritten as

$$R_n = P[S_{\min} > s] \quad (4.16)$$

Hence, using equation (2.21), we have

$$R_n = \int_0^{\infty} f_s(s) [1 - F_S(S)]^n ds \quad (4.17)$$

Case 7. random-independent stress and deterministic strength [10]

This case is an exact reversal of case 3. By exchange between stress  $s$  and strength  $S$ , we have

$$R_n = P(E_1) \times P(E_2) \times \dots \times P(E_n) \quad (4.18)$$

where

$$P(E_i) = P(s_i \leq S_0)$$

$$= \int_0^{S_0} f_{s_i}(s) ds$$

where  $S_0$  is the known strength and  $f_{s_i}(s)$  represents the pdf of the random variable stress  $s_i$  on the  $i$ th cycle. In particular, if  $f_{s_1}(s) = f_{s_2}(s) = \dots = f_{s_n}(s) = f_s(s)$ ,

then

$$R_n = (P(E_1))^n = \left\{ \int_0^{S_0} f_s(s) ds \right\}^n \quad (4.19)$$

Case 8. random-independent stress and random-fixed strength [10]

This case is similar to case 6 and we have

$$\begin{aligned} R_n &= P[(s_1 < S) \cap (s_2 < S) \cap \dots \cap (s_n < S)] \\ &= P[\max(s_1, \dots, s_n) < S] \end{aligned} \quad (4.20)$$

Let  $s_{\max} = \max(s_1, s_2, \dots, s_n)$ . Then the largest value distribution of  $s_{\max}$  is given by

$$F_n(s) = [F_s(s)]^n$$

where

$$F_s(s) = \int_0^s f_s(s) ds$$

and hence

$$R_n = \int_0^{\infty} f_S(S) [F_s(s)]^n dS \quad (4.21)$$

Case 9. random-independent stress and strength [10]

Let  $f_{s_i}(s)$  and  $f_{S_i}(S)$  denote the pdf's of stress  $s_i$  and strength  $S_i$ , respectively, in cycle  $i = 1, 2, \dots$ . Since  $s_i$ 's and  $S_i$ 's are independent, then

$$\begin{aligned} R_n &= P[E_n, E_{n-1}, \dots, E_1] & (4.22) \\ &= P(E_n) \times P(E_{n-1}) \times \dots \times P(E_1) \\ &= \prod_{i=1}^n P(E_i) \end{aligned}$$

$$\text{where } P(E_i) = P(s_i \leq S_i) \quad (4.23)$$

$$= \int_0^{\infty} f_{s_i}(s) \int_s^{\infty} f_{S_i}(S) dS ds$$

In particular, if  $f_s(\cdot)$  and  $f_S(\cdot)$  do not change with time, then

$$R_n = \prod_{i=1}^n P(E_i) = (P(E_1))^n \quad (4.24)$$

where  $P(E_1)$  is the reliability of the stress-strength interference models which were discussed in Chapter 3.

#### 4.2 Reliability in case of cumulative damage

As we mentioned before, cumulative damage of a device is when its decrease in strength is determined by the size as well as the number of previous stresses. In the cumulative damage models, if it is assumed that the cumulative damage weakens the strength of the part at each occurrence by an amount proportional to the applied stress, a cumulative damage law based on the sum of the applied stresses is given by

$$S_n = S_i - \sum_{i=1}^n c_i S_i \quad (4.25)$$

where  $c_i$ 's are proportionality constants.

Let us formulate the situation where the strength changes with time or the number of load occurrences. Let  $S_i$  denote the strength after  $i$  load occurrences, and  $S_i$  be modeled as a deterministic function of the initial (random) strength  $S_0$  and  $i$ . Thus, we have

$$S_i = h(S_0, i)$$

One example of the above function may be

$$S_i = S_0 \phi(i) \quad (4.26)$$

where  $\phi(i)$  is a monotonically decreasing function of  $i$ ,  $i = 1, 2, \dots$ . Let us assume that the stress  $s_0, s_1, s_2, \dots, s_n$  are independent with distribution function  $F_{s_0}, F_{s_1}, F_{s_2}, \dots, F_{s_n}$ , respectively. Then

$$R_n = P[(s_0 < S_0) \cap (s_1 < S_1) \cap (s_2 < S_2) \cap \dots \cap (s_n < S_n)] \quad (4.27)$$

Since the strength  $S_i$  is a known function of the initial strength  $S_0$ , equation (4.27) may be rewritten as

$$R_n = P[(s_0 < S_0) \cap (s_1 < h(S_0, 1)) \cap \dots \cap (s_n < h(S_0, n))] \quad (4.28)$$

If the initial strength,  $S_0$ , is between  $S$  and  $S+dS$ , the probability of success (defined by event  $E$ ) as given by equation (4.29) may be expressed by the following, because of the independence of the stresses:.

$$P[E|S < S_0 < S + dS] = P(s_0 < S) P(s_1 < h(S, 1)) \dots P(s_n < h(S, n)) \quad (4.29)$$

Hence, the reliability  $R_n$  is

$$R_n = \int_0^{\infty} F_{s_0}(S) F_{s_1}(h(S, 1)) \dots F_{s_n}(h(S, n)) f_S(S) dS \quad (4.30)$$

where  $f_S(S)$  is the pdf of the initial strength. If  $S_i = S_0 \phi(i)$  and all the stresses are independent and identically distributed, we have

$$R_n = \int_0^{\infty} f_S(S) \left\{ \prod_{i=0}^n F_S(S\phi(i)) \right\} ds \quad (4.31)$$

$R_n$  given by equation (4.31) may be approximated as follows. Let  $\bar{F}_S = 1 - F_S$  and  $S_0 = h(S_0, 0)$ . Then

$$R_n = \int_0^{\infty} f_S(S) \left[ \prod_{i=0}^n (1 - \bar{F}_S(h(S, i))) \right] ds \quad (4.32)$$

Now, the following inequality exists,

$$\prod_{i=0}^n (1 - \bar{F}_{S_i}(h(S,i))) > 1 - \sum_{i=0}^n \bar{F}_{S_i}(h(S,i)) \quad (4.33)$$

and thus the lower bound on  $R_n$  is given by

$$R_n > 1 - \sum_{i=1}^n \int_0^{\infty} f_S(S) \bar{F}_{S_i}(h(S,i)) dS \quad (4.34)$$

If  $\sum_{i=0}^n \bar{F}_{S_i}(h(S,i)) \leq 1$ , then, the lower bound given by equation (4.34)

is fairly close to  $R_n$ . Now, the probability of failure on the  $i$ th stress application is

$$P_f(i) = \int_0^{\infty} f_S(S) \bar{F}_{S_i}(h(S,i)) dS \quad (4.35)$$

Hence,  $R_n$  can be expressed as

$$R_n > 1 - \sum_{i=0}^n P_f(i) \approx \exp \left[ - \sum_{i=0}^n P_f(i) \right] \quad (4.36)$$

The relations given by equation (4.36) are close approximations to  $R_n$

when

$$\sum_{i=0}^n P_f(i) \ll 1 .$$

## CHAPTER 5 OPTIMIZATION IN ENGINEERING DESIGN

In the previous chapters, the emphasis of the study is on using reliability as a criterion of the engineering design. However, in real life, not only reliability, but also other criteria are to be considered in the design work. Generally, the following three criteria are most likely to be considered:

- (1) the reliability
- (2) the cost
- (3) the weight

Traditionally, only one of these criteria has been considered. The optimal solution for one criterion is surely not an optimal solution for others. In most of the cases, the solution is not even satisfactory, if other criteria were also considered. However, the decisionmaker (DM) always wants to attain more than one objective or goal in selecting the course of action. One of the major reasons for the scarcity of multiple objective formulation and consideration in literature is that until recently, almost all the solution strategies developed involved a single objective function.

The growing tendency to incorporate more and diverse criteria in a system design and the resulting difficulty in consolidating such criteria have provided an impetus to the development of multiple criteria methodology in various forms.

Recently, multiple objective analysis has been applied to a wide variety of problems [6]. However, no paper has been found applying multiple objective decision making methods (MODM) in the field of reliability analysis and design policy making.



In this chapter, two methods of MODM [6] are to be used to determine the optimal policy of a particular design problem:

- (1) Nonlinear goal programming [7]
- (2) Sequential multiobjective problem solving technique [SEMOPS][14].

Section 5.1 states the problem and to develop the models for three criteria. Section 5.2 applies the nonlinear goal programming to the problem. Section 5.3 applies the SEMOPS method to the problem. Section 5.4 discusses of results.

## 5.1 Statement of problem and model development

A particular design problem is presented here. The model for three criteria is developed.

### 5.1.1 Statement of problem

A critical tensile element (Fig. 8) is to be designed. The load  $P$  acting on the element is a random variable, which is known and tends to be a normal distribution with mean,  $\bar{P}$ , and standard deviation,  $\sigma_P$ . The tensile element has a circular cross-section. The radius,  $r$ , is a random variable because of the manufacturing tolerance. The ultimate tensile strength of the material used for the element is a random variable because the properties of the material vary. Normally distributed radius,  $r$ , and ultimate tensile strength,  $S$ , are assumed. [10]

The problem is to determine the following variables:

- (1) the mean ultimate tensile strength, which can be used to decide the material
- (2) the standard deviation of ultimate tensile strength

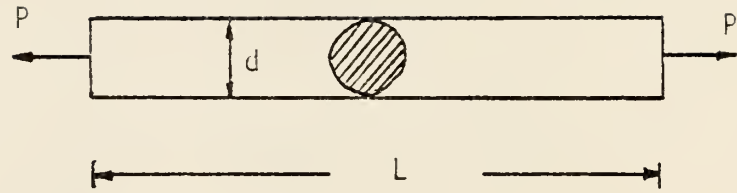


Figure 8 Circular cross-section shaft.

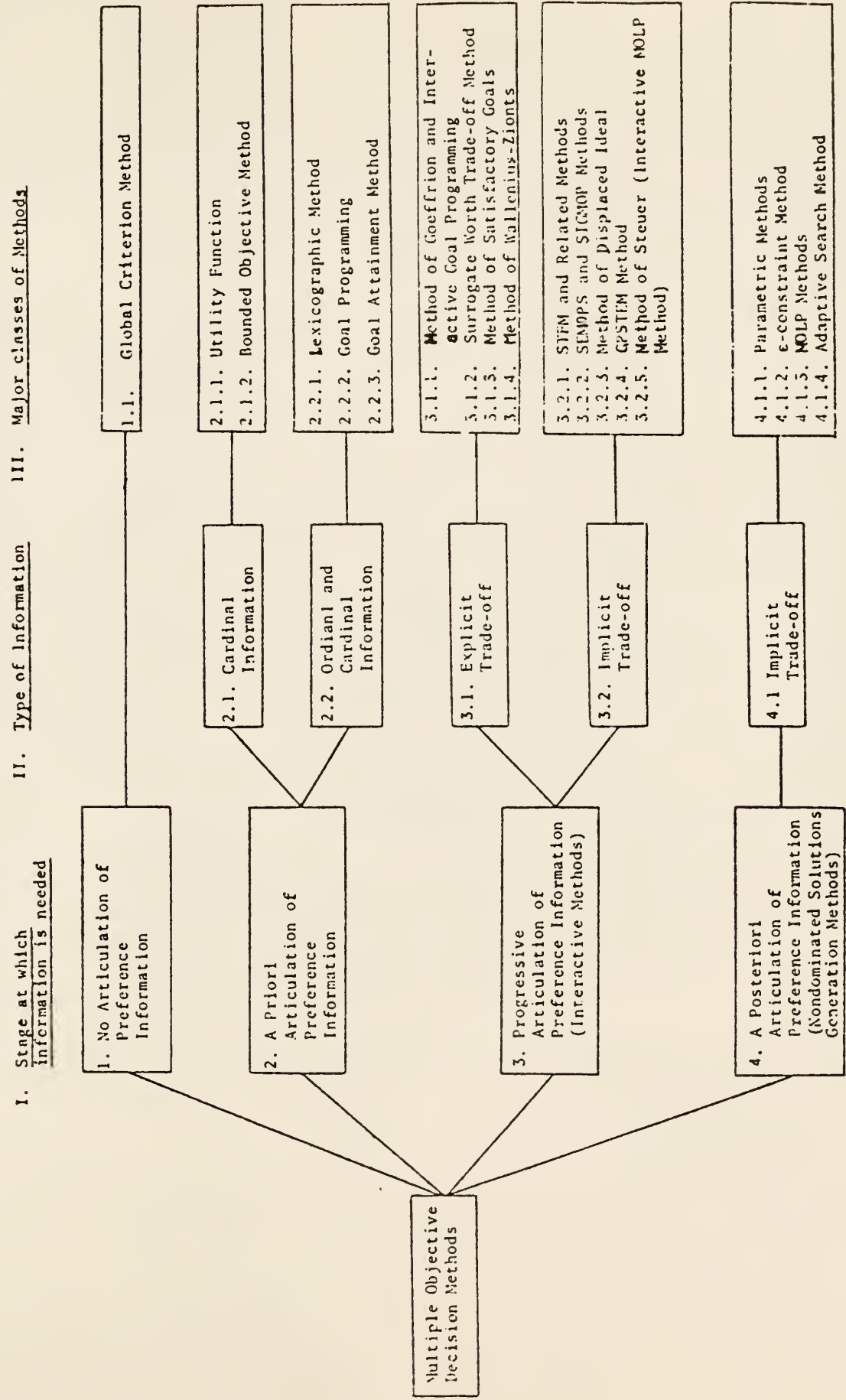


Fig. 9 A taxonomy of methods for multiple objective decision making.

- (3) the radius of the element
- (4) the manufacturing tolerance fraction

The problem has the following three criteria which are to be considered in most of engineering design:

- (1) the element reliability
- (2) the total cost of element
- (3) the total weight of element

In determining the optimal design policy, Kapur [10] selected reliability as the criterion and used the Lagrangian multiplier method.

In this chapter, two methods for multiple objective decision making are used for obtaining the optimal design policy based on three criteria.

#### 5.1.2 Model development

The problem is to determine (1) mean tensile strength (2) standard deviation of tensile strength (3) radius of element (4) the manufacturing tolerance fraction based on the following criteria:

- (1) the element reliability
- (2) the total cost of component
- (3) the total weight of component

#### Notations:

- $\bar{A}$  = the mean value of cross-section,  $\text{in}^2$
- $b_i$  = the constants of the  $i$ th parameter
- $C$  = the total cost, dollars
- $D$  = the density,  $\text{lb/in}^3$
- $L$  = the length of the tensile element, in

- $\bar{P}$  = the mean value of load, lb  
 $R$  = the reliability  
 $\bar{r}$  = the mean value of radius, in  
 $\bar{S}$  = the mean value of ultimate tensile strength, psi  
 $\bar{s}$  = the mean value of tensile stress, psi  
 $W$  = the total weight, lb  
 $Z$  = the standard normal variable  
 $\alpha$  = the manufacturing tolerance fraction  
 $\sigma_A$  = the standard deviation of cross-section, in<sup>2</sup>  
 $\sigma_P$  = the standard deviation of load, lb  
 $\sigma_r$  = the standard deviation of radius, in  
 $\sigma_S$  = the standard deviation of ultimate tensile strength, psi  
 $\sigma_s$  = the standard deviation of tensile stress, psi

The three criteria which are mentioned above are evaluated as follows:

(1) The element reliability

The strength and stress are assumed to be normally distributed.

From equation (3.8), we can derive

$$Z = \frac{\mu_S - \mu_s}{\sqrt{\sigma_S^2 + \sigma_s^2}} = \frac{\bar{S} - \bar{s}}{\sqrt{\sigma_S^2 + \sigma_s^2}} \quad (5.1)$$

Because Z value is closely related to the reliability in equation (3.8), equation (5.1) is used to represent the reliability criterion in this problem in order to avoid integration during the computation process.

The tensile stress,  $s$ , is given by  $s = P/A$  where  $A = \pi r^2$ . Using the Taylor's series approximation it can be derived that  $\bar{A} = \pi \bar{r}^2$  and  $\sigma_A = 2\pi \bar{r} \sigma_r$ . If the tolerance on the radius of the circular cross-section is a fraction  $\alpha$  of  $\bar{r}$ , then  $3\sigma_r = \alpha \bar{r}$ ; that is,  $\sigma_r = (\alpha/3)\bar{r}$ . Now, using equations (2.36) and (2.37), we have

$$\bar{s} = \bar{P}/\bar{A} = \bar{P}/\pi \bar{r}^2 \quad (5.2)$$

and

$$\sigma_s^2 = \sigma_p^2 \left(\frac{1}{\bar{A}}\right)^2 + \sigma_A^2 \left(\frac{\bar{P}}{\bar{A}^2}\right)^2 \quad (5.3)$$

Substituting the value of  $\bar{A} = \pi \bar{r}^2$  and  $\sigma_A^2 = 4\pi^2 \bar{r}^4 \alpha^2/9$  into equation (5.3) yields

$$\sigma_s^2 = \frac{\sigma_p^2 + (4/9)\alpha^2 \bar{P}^2}{\pi^2 \bar{r}^4} \quad (5.4)$$

Substitution of equations (5.2) and (5.4) into equation (5.1), we have

$$Z = \frac{\bar{S} - \bar{P}/\pi \bar{r}^2}{\sqrt{\sigma_s^2 + \left\{ \frac{\sigma_p^2 + (4/9)\alpha^2 \bar{P}^2}{\pi^2 \bar{r}^4} \right\}}} \quad (5.5)$$

This relates the several variables to the  $Z$  values directly and to the expected reliability indirectly.

(2) The total cost for making or purchasing the element

The cost of the element depend on the values of  $\bar{S}$ ,  $\sigma_s$ ,  $\bar{r}$ , and  $\alpha$ .

The cost function was proposed by Kapur [10]

$$C = b_1 \bar{S}^{b_2} + b_3 \sigma_S^{b_4} + b_5 \bar{S}^{b_6} + b_7 \sigma_S^{b_8} \quad (5.6)$$

The parameters of each variable were estimated by using regression analysis and by assuming other variables are fixed. The variables,  $s$  and  $\sigma_S$ , in equation (5.6) are slightly different from the variables,  $\bar{r}$  and  $\alpha$ , of this problem, but they are still correlated. Hence, equation (5.6) can be rewritten as

$$C = b_1 \bar{S}^{b_2} + b_3 (\sigma_S / \bar{S})^{b_4} + b_5 \bar{r}^{b_6} + b_7 \alpha^{b_8} \quad (5.7)$$

The second terms in equation (5.6) and equation (5.7) are different, because the same value of  $\sigma_S$  will not cost the same for different values of  $\bar{S}$ . For example, the cost for  $\sigma_S = 4,000$  is different when  $\bar{S} = 20,000$  from  $\bar{S} = 14,000$ . Hence, the ratio of  $\sigma_S$  to  $\bar{S}$  is to be preferred in the second term.

Another cost function is proposed as follows:

$$C = b_1 \bar{S}^{b_2} \cdot b_3 (\sigma_S / \bar{S})^{b_4} \cdot b_5 \bar{r}^{b_6} \cdot b_7 \alpha^{b_8} \quad (5.8)$$

The difference between equations (5.7) and (5.8) is the assumption. Equation (5.7) assumes that each variable is not interrelated with others and equation (5.8) assumes that they are interrelated. For example, with a higher value of  $\bar{S}$ , usually a harder material, not only does it cost more for the material, but also it costs more to produce accurately.

A set of data is needed to estimate the parameters of cost function by using a nonlinear models program (NONLIN)(Appendix B). Those data,

which include four variables and related cost, can be collected by asking the cost analysis department in a factory. The set of empirical data, which is shown in Table 2, was collected from a mechanic of Dept. of Physics, K.S.U and a machine shop in Manhattan area. Using NONLIN computer package, the estimated parameters and the sum of squares and iterations for equations (5.7) and (5.8) are shown in Table 3. From Table 3, equation (5.8) is selected as the cost function here according to less iterations and smaller sum of square, that is

$$C = 0.5726(\bar{S}/1000.0)^{1.04} \cdot 0.5726(\sigma_S/\bar{S})^{-0.4116} \cdot 0.5726(\bar{r})^{0.801} \cdot 0.5726 \cdot (10000 \cdot \alpha)^{-0.407}$$

where the constants in the parentheses of the first term and fourth term are used to prevent the overflow and underflow during the computing processing.

(3) The total weight of the element

The weight function is given by

$$W = \text{Density} \cdot \text{Volume} \quad (5.10)$$

The volume is equal to  $\pi \bar{r}^2 L$ , where  $\pi$  and  $L$  are known, and  $\bar{r}$  is variable. Hence, equation (5.10) can be rewritten as

$$W = \text{Density} \cdot \pi \cdot \bar{r}^2 \cdot L \quad (5.11)$$

The density in equation (5.11) need to be estimated. The density is mostly related to mean tensile strength  $\bar{S}$ .



Table 2 Data for estimating cost function

sample no	variables	S ksi	$\sigma_s$ ksi	r in	$\alpha$ x0.001	cost dollars
1		85.0	5.10	1.0	4.0	20.0
2		100.0	10.80	1.0	4.0	40.0
3		85.0	5.10	2.0	4.0	30.0
4		180.0	10.80	3.0	4.0	106.0
5		40.0	2.40	1.0	4.0	15.0
6		40.0	2.40	3.0	4.0	30.0
7		85.0	5.10	3.0	4.0	50.0
8		180.0	10.80	3.0	1.0	145.0
9		180.0	10.80	3.0	2.0	180.0
10		180.0	10.80	10.0	4.0	270.0
11		65.0	3.90	1.0	7.0	18.0
12		65.0	3.90	2.0	4.0	25.0
13		65.0	3.90	2.0	2.0	29.0
14		85.0	2.55	1.0	4.0	27.0
15		85.0	2.55	2.0	4.0	35.0
16		200.0	5.4	1.0	4.0	60.0
17		180.0	1.8	1.0	4.0	90.0
18		180.0	14.4	1.0	4.0	30.0
19		180.0	18.0	1.0	4.0	25.0
20		85.0	5.1	1.0	6.0	16.0
21		85.0	5.1	3.0	2.0	68.0

Table 3 Comparison of results for estimating the parameter of equations (5.7) and (5.8)

Equation	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$	Iterations	Sum of square
(5.7)	2.280	0.7769	-41.40	-0.2013	5.286	1.324	4.829	1.826	46	1123.95
(5.8)	0.5726	1.04	0.5726	0.4116	0.5726	0.801	0.5726	0.407	7	706.64

The density can be written as

$$D = b_1 \bar{S}^{b_2} \quad (5.12)$$

A set of data, in Table 4, which include  $\bar{S}$  and  $D$ , is collected from ASTM standard [1]. Using the NONLIN computer package, the parameters can be estimated, that are,

$$D = 0.0746(\bar{S}/1000.0)^{0.3} \quad (5.13)$$

where the constant, 1000.0, is used to prevent overflow during the computing processing.

Then, equation (5.11) can be rewritten as

$$W = 0.0746 (\bar{S}/1000.0)^{0.3} \pi r^2 \cdot L. \quad (5.14)$$

### Constraints

(1) Limits of mean strength

$$b_{u1} \leq \bar{S} \leq b_{L1}$$

(2) Limits of standard deviation of strength

$$b_{u2} \leq \sigma_S \leq b_{L2}$$

(3) Limits of mean radius

$$b_{u3} \leq \bar{r} \leq b_{L3}$$

(4) Limits of tolerance fraction

Table 4 Data for estimating the parameters of density

	S (ksi)	Density (lb/cu in)
1	65.0	0.270
2	85.0	0.285
3	82.0	0.283
4	120.0	0.310
5	95.0	0.290
6	180.0	0.335
7	43.0	0.100
8	55.0	0.163
9	165.0	0.323
10	100.0	0.317
11	70.0	0.277

$$b_{u4} \leq \alpha \leq b_{L4}$$

(5) Limits of ratio of standard deviation to mean strength.

$$b_{u5} \leq \sigma_S/\bar{S} \leq b_{L5}$$

These constraints are obtained according to the ranges of each variables in the data set in Table 2 and 4.

#### Numerical example:

Let  $x_1, x_2, x_3, x_4$  represent the variables  $\bar{S}, \sigma_S, \bar{r}, \alpha$  respectively and  $f_1(\underline{x}), f_2(\underline{x}), f_3(\underline{x})$  represent  $Z, C, W$  respectively.

The following numerical values are assumed

$$\bar{P} = 4000 \text{ lb,}$$

$$\sigma_p = 100 \text{ lb}$$

$$L = 6.0 \text{ in,}$$

$$b_{u1} = 200,000 \text{ psi,}$$

$$b_{L1} = 40,000 \text{ psi}$$

$$b_{u2} = 18,000 \text{ psi,}$$

$$b_{L2} = 2,000 \text{ psi}$$

$$b_{u3} = 10.00 \text{ in,}$$

$$b_{L3} = 0.07 \text{ in}$$

$$b_{u4} = 0.0070,$$

$$b_{L4} = 0.0009$$

$$b_{u5} = 0.10,$$

$$b_{L5} = 0.01$$

## 5.2 Nonlinear goal programming for engineering design problem [9]

For the nonlinear goal programming the decision-maker (DM) has to setup the goal and rank of importance for each objective [9].

For each equality or inequality, the negative deviational variable  $d^-$  and positive deviational variable  $d^+$  are introduced, which  $d^-, d^+ \geq 0$  and  $d^- \cdot d^+ = 0$ .

$$\text{If } f_j(\underline{x}) \begin{matrix} \geq \\ = \\ \leq \end{matrix} b_j,$$

then

$$f_j(\underline{x}) + d_j^- - d_j^+ = b_j$$

The three possibilities ( $\begin{matrix} \geq \\ = \\ \leq \end{matrix}$ ) may be achieved by minimizing a linear function of the deviational variables as shown in Table 5.

### 5.2.1 Case 1:

Suppose the DM states the following objectives in order of priority:

$$\text{Max } f_1(\underline{x})$$

$$\text{min } f_2(\underline{x})$$

$$\text{min } f_3(\underline{x})$$

and the goals are setup by the DM:

$$f_1(\underline{x}) \geq 4.0 \quad \text{that is } R \geq 0.99996833$$

$$f_2(\underline{x}) \leq 0.140$$

$$f_3(\underline{x}) \leq 0.00040$$

Hence, the design problem can be formulated as follows:

Table 5 Procedure for achieving an objective [9]

Objective	Procedure
(a) Equal or exceed $b_i$	minimize $d_i^-$
(b) Equal or be less than $b_i$	minimize $d_i^+$
(c) Equal $b_i$	minimize $d_i^- + d_i^+$

$b_i$  is the right hand side of inequality  $i$  or equality  $i$

Find  $\underline{x} = (x_1, x_2, x_3, x_4)$  so as to minimize

$$a = [(d_1^- + d_2^- + d_3^- + d_4^- + d_5^- + d_6^- + d_7^- + d_8^- + d_9^- + d_{10}^-), (d_{11}^-), (d_{12}^+), (d_{13}^+)]$$

subject to

$$G_1: x_1 - 40,000.0 + d_1^- - d_1^+ = 0$$

$$G_2: 200,000.0 - x_1 + d_2^- - d_2^+ = 0 \quad (5.15)$$

$$G_3: x_2 - 2,000.0 + d_3^- - d_3^+ = 0$$

$$G_4: 18,000.0 - x_2 + d_4^- - d_4^+ = 0$$

$$G_5: x_3 - 0.07 + d_5^- - d_5^+ = 0$$

$$G_6: 10.0 - x_3 + d_6^- - d_6^+ = 0$$

$$G_7: x_4 - 0.0009 + d_7^- - d_7^+ = 0$$

$$G_8: 0.007 - x_4 + d_8^- - d_8^+ = 0$$

$$G_9: 0.1 - x_2/x_1 + d_9^- - d_9^+ = 0$$

$$G_{10}: x_2/x_1 - 0.01 + d_{10}^- - d_{10}^+ = 0$$

$$G_{11}: \frac{x_1 - 1273.23/x_3^2}{\sqrt{x_2^2 + \frac{1013.2 + 720434.1 x_4^2}{x_3^4}}} + d_{11}^- - d_{11}^+ = 4.0$$

$$G_{12}: 0.5726 (x_1/1000.0)^{1.04} \cdot 0.5726 (x_2/x_1)^{-0.4116} \cdot 0.5726 (x_3)^{0.801} \cdot$$

$$0.5726 (1000.0 \cdot x_4)^{-0.407} + d_{12}^- - d_{12}^+ = 0.14$$



$$G_{13}: 0.0746(x_1/1000.0)^{0.3} \cdot 18.85 x_3^2 + d_{13}^- - d_{13}^+ = 0.0004$$

The first ten goals are the absolute constraints in the original problem and the last three goals are the three objectives.

This problem is solved by using the nonlinear goal programming computer package of Department of Industrial Engineering, K.S.U. [7].

The initial starting point:  $\underline{x}^0 = (42200, 3752.5, 0.20, 0.004)$ .

The step size:  $\Delta \underline{x}^0 = (50.0, 10.0, 0.10, 0.001)$

After 32 iterations, the final solution is

$$\underline{x}^* = (40650, 4065, 0.23, 0.007)$$

$$\underline{a}^* = (0.0, 0.0, 1.681, 0.22446)$$

The results from the achievement function  $\underline{a}^*$  mean that the first priority, which consists of ten absolute constraints, has been satisfied and the second priority, which is  $Z \geq 4.0$ , has been satisfied too. The third and fourth priorities have not been satisfied, which have 1.681 and 0.22446 far from their goals respectively. The final values for three criteria can be obtained by substituting  $\underline{x}^*$  into three criterion functions, we have

$$Z \text{ value} = 4.0 \text{ that is, Reliability} = 0.999,968,33$$

$$\text{Total cost} = 1.821 \text{ dollars}$$

$$\text{Total weight} = 0.22486 \text{ lb}$$

The optimal design policy is that the material for the element should have the ultimate tensile strength,  $\bar{S} = 40650$  psi, and the standard deviation,  $\sigma_S = 4065$  psi, the radius of the element is 0.23 in with manufacturing

tolerance fraction 0.7%. The results give the reliability of 0.999,968,33, the cost of 1.821 dollars, and the weight of 0.22486 lb.

According to ASTM standard [1], the ultimate tensile strength of Aluminum Alloy 5052 is similar to the final solution of  $\bar{S} = 40650$  ps:

### 5.2.2 Case 2:

If the DM feels that the cost of one component, 1.821 dollars, which is from the result of Case 1, cannot compete with the same products from other companies, then the DM would like to reduce the goal for cost criterion to 1.8 dollars and put in a priority over other two criteria. The problem becomes:

$$\text{Min } f_2(\underline{x})$$

$$\text{Max } f_1(\underline{x})$$

$$\text{Min } f_3(\underline{x})$$

and the goals are set up by the DM:

$$f_2(\underline{x}) \leq 1.8$$

$$f_1(\underline{x}) \geq 4.0$$

$$f_3(\underline{x}) \leq 0.2$$

Hence, the problem can be formulated as the same as (5.15)

The differences are that the orders of the second and third priorities of

$$\underline{a} = [(d_1^- + d_2^- + d_3^- + d_4^- + d_5^- + d_6^- + d_7^- + d_8^- + d_9^- + d_{10}^-), (d_{12}^+), (d_{11}^-), (d_{13}^+)]$$

are exchanged, in addition to the goals on  $f_2$  and  $f_3$ , that is,

$$G_{12}: 0.5726 \cdot (x_1/1000.0)^{1.04} \cdot 0.5726 \cdot (x_2/x_1)^{-0.4116} \cdot 0.5726(x_3)^{0.801} \cdot$$

$$0.5726 \cdot (1000.0 \cdot x_4)^{-0.407} + d_{12}^- - d_{12}^+ = 1.8$$

$$G_{13}: 0.0746 \cdot (x_1/1000.0)^{0.3} \cdot 18.85 \cdot x_3^2 + d_{13}^- - d_{13}^+ = 0.2$$

The nonlinear goal programming problem was solved by:

The initial starting point:  $\underline{x}^0 = (42200, 3752.5, 0.20, 0.004)$

The step size:  $\underline{\Delta x}^0 = (50.0, 10.0, 0.10, 0.001)$

After 48 iterations, the final solution is

$$\underline{x}^* = (42625.5, 3785.0, 0.2, 0.007)$$

$$\underline{a}^* = (0.0, 0.0, 1.211, 0.0)$$

The results from the achievement function  $\underline{a}^*$  mean that the goals of the first, second and fourth priorities have been satisfied. The third priority, which is  $Z \geq 4.0$ , has not been satisfied, which has 1.211 far from the goal. The final values for three criterion functions can be obtained by substituting  $\underline{x}^*$  into those criterion functions, we have

Z value = 2.789, that is, Reliability = 0.99730

Total cost = 1.8 dollars

Total weight = 0.17 lb

The optimal design policy is that the material for the element should have the ultimate tensile strength,  $\bar{S} = 42625.5$  psi, the standard deviation,  $\sigma_S = 3785$  psi, and the radius of element is 0.2 in with manufacturing tolerance fraction 0.7%. The element reliability is equal to 0.99730, the cost is 1.8 dollars, and the weight is 0.17 lb.

According to ASTM standard [1], the ultimate tensile strength of Aluminum 5086 is similar to the optimal solution of  $\bar{S} = 42625.5$  psi.

### 5.2.3 Discussion:

The comparison of the results of Case 1 and Case 2 is given in Table 6. The results are different between two cases, because the ranking of importance and goal of each criterion which have been selected by the DM are different. It can be concluded that the nonlinear goal programming problem is very sensitive to the ranking of importance and goal of each criterion function. From Table 6, the values of  $X_3$ , that is the radius of element, is reduced, the value of three criterion functions are also reduced. It can also be concluded that the radius is the most sensitive variable in this two cases.

## 5.3 Sequential Multiobjective Problem Solving technique (SEMOPS) for engineering design problem

### 5.3.1 The method

A Sequential Multiobjective Problem Solving technique (SEMOPS), proposed by Monarchi, Kisiel, and Duckstein [14], allows the DM to trade off one objective vs another in an interactive manner. Semops cyclically

Table 6 Comparison of results of two methods

Method		Criteria		Decision variables				
		Z (R)	C	W	X <sub>1</sub>	X <sub>2</sub>	X <sub>3</sub>	X <sub>4</sub>
Nonlinear Goal Programming	Case 1	4.0 (0.999968)	1.821	0.225	40650	4065.0	0.23	0.007
	Case 2	2.789 (0.99730)	1.800	0.170	42625	3785.0	0.20	0.007
SEMOPS		2.515 (0.99379)	2.098	0.150	45772	3102.5	0.184	0.0064

uses a surrogate objective function based on goals and the DM's aspirations about achieving the objectives. The goal levels are conditions imposed on the DM by external forces and the aspiration levels are the attainment levels of the objectives that the DM personally desires to achieve. (Goals do not change, but aspiration levels change as each iterative cycle goes.)

Let  $\underline{A} = (A_1, \dots, A_k)$  be the DM's aspirations levels, and  $\underline{f}(\underline{x}) = (f_1(\underline{x}), \dots, f_k(\underline{x}))$  be the multiple objective functions. Five types of objectives and the corresponding dimensionless indicator of attainment,  $\underline{d}$ , are:

(1) at most:

$$f_i(\underline{x}) \leq A_i; \quad d_i = f_i(\underline{x}) / A_i$$

(2) at least:

$$f_i(\underline{x}) \geq A_i; \quad d_i = A_i / f_i(\underline{x})$$

(3) equals:

$$f_i(\underline{x}) = A_i; \quad d_i = \frac{1}{2} \left\{ \frac{A_i}{f_i(\underline{x})} + \frac{f_i(\underline{x})}{A_i} \right\}$$

(4) within an interval:

$$A_{iL} \leq f_i(\underline{x}) \leq A_{iU}; \quad d_i = \left( \frac{A_{iU}}{A_{iL} + A_{iU}} \right) \left( \frac{A_{iL}}{f_i(\underline{x})} + \frac{f_i(\underline{x})}{A_{iU}} \right)$$

(5) outside an interval:

$$f_i(\underline{x}) \leq A_{iL} \text{ or } f_i(\underline{x}) \geq A_{iU}; \quad d_i = \left( \frac{A_{iL} + A_{iU}}{A_{iU}} \right) \left( \frac{1}{\frac{A_{iL}}{f_i(\underline{x})} + \frac{f_i(\underline{x})}{A_{iU}}} \right)$$

Types (1), (2) and (4) are the most common. In each instance, values of  $d_i \leq 1$  imply that the objective is satisfied. Except for the first type (at most), the  $d_i$  are all nonlinear functions of an objective function that may itself be nonlinear.

The algorithm generates information under the guidance of the DM so that he/she can make a decision. Information concerning the interrelationships between objectives is in terms of how achievement or non-achievement of one objective affects the aspiration levels of other objectives.

The cyclical optimization of a surrogate objective function  $s$ , (5.16), is the mechanism by which information is generated for the DM. The word surrogate is used in recognition of the fact that the true preference function of the individual is unknown. Let  $T'$  be the subset of the set of  $T$  objectives as those objectives making up  $s$  at a given iteration of the decision-making process. Thus,

$$s \equiv \sum_{t \in T'} d_t \quad (5.16)$$

is defined as the surrogate objective function. The value of each  $d_t$  in  $s$  reflects whether the  $t$ -th objective has been satisfied; unsatisfied objectives have values  $> 1$ .

Operationally, Semops is a three-step algorithm involving setup, iteration, and termination. Setup involves transforming the original problem into a principal surrogate objective function problem and a set of auxiliary problems involving surrogate objective functions. The iteration step is the interactive segment of the algorithm and involves a cycling

between an optimization phase (by analyst) and an evaluation phase (by the DM) until a satisfaction is reached, which terminates the algorithm.

The first iteration,  $i=1$ , solves the principal problem, and a set of  $T$  auxiliary problems formed as follows, where the aspiration level of each objective is given as the goal of each objective, i.e.,  $A_i = b_i$ ,  $i = 1, 2, \dots, T$ .

The principal problem:

$$\left. \begin{array}{l} \min \quad s_1 = \sum_{t=1}^T d_t \\ \text{s.t.} \quad \underline{x} \in X \end{array} \right\} \quad (5.17)$$

The set of auxiliary problems,  $\ell = 1, 2, \dots, T$ :

$$\left. \begin{array}{l} \min \quad s_{1\ell} = \sum_{\substack{t=1 \\ t \neq \ell}}^T d_t \\ \text{s.t.} \quad \underline{x} \in X \\ \quad \quad f_{\ell}(\underline{x}) \geq A_{\ell} \end{array} \right\} \quad (5.18)$$

Solving (5.17) and (5.18) forms the optimization phase. The resulting policy vector and objectives for the principal problem and the set of auxiliary problems are presented to, and are used in the evaluation phase by the DM. The impact of an action on the attainment of the other objective is assessed, and a new aspiration level for an objective is set.

In general, the  $i$ th iteration solves the following principal problem, and a set of auxiliary problems.



The principal problem:

$$\left. \begin{aligned}
 \min \quad s_j &= \sum_{t \in T'} d_t \\
 \text{s.t.} \quad \underline{x} &\in X \\
 f_j(\underline{x}) &\geq A_j, \quad j \in (T-T')
 \end{aligned} \right\} \quad (5.19)$$

The set of auxiliary problems,  $\ell \in T'$  (the number of  $T' = T - i + 1$ ):

$$\left. \begin{aligned}
 \min \quad s_{i\ell} &= \sum_{\substack{t \in T' \\ t \neq \ell}} d_t \\
 \text{s.t.} \quad \underline{x} &\in X \\
 f_j(\underline{x}) &\geq A_j \quad \text{for } \forall j, \quad j \in (T-T') \\
 f_\ell(\underline{x}) &\geq A_\ell, \quad \text{for one } \ell, \ell \in T'
 \end{aligned} \right\} \quad (5.20)$$

The optimization phase solves (5.19) and (5.20). The resulting solutions are used in the evaluation phase and a guidance is given by the DM for the next iteration cycle.

### 5.3.2 The design problem:

This study of finding the optimal design policy involves three goals and four decision variables. Their various formulations are expressed below:

Goals and criterion functions

$$f_1(\underline{x}) = \frac{x_1 - 1273.23/x_3^2}{\sqrt{x_2^2 + \frac{1013.2 + 720434.1 \cdot x_4^2}{x_3^4}}} \geq A_1$$

$$f_2(\underline{x}) = 0.5726(x_1/1000.0)^{1.04} \cdot 0.5726(x_2/x_1)^{-0.4116} \cdot 0.5726(x_3)^{0.801} \cdot 0.5726(1000.0 - x_4)^{-0.407} \leq A_2$$

$$f_3(\underline{x}) = 0.0746(x_1/1000.0)^{0.3} \cdot 18.85x_3^2 \leq A_3$$

Goal levels (initial aspiration levels)

$$GL_1 = 4.0, \quad GL_2 = 1.7 \text{ dollars}, \quad GL_3 = 0.2 \text{ lb}$$

where  $GL_1 = 4.0$  is equivalent to, Reliability = 0.99996833.

Relevant range of  $f_j(\underline{x})$

$$\Gamma(f_1(\underline{x})) = [-40.0, 10.0], \quad \Gamma(f_2(\underline{x})) = [0.14, 10.0]$$

$$\Gamma(f_3(\underline{x})) = [0.0005, 10.0]$$

Constraint

$$\left. \begin{aligned}
 200,000 &\leq x_1 \leq 40,000 \\
 18,000 &\leq x_2 \leq 2,000 \\
 10.0 &\leq x_3 \leq 0.07 \\
 0.007 &\leq x_4 \leq 0.0009 \\
 0.1 &\leq x_2/x_1 \leq 0.01
 \end{aligned} \right\} \underline{g}(\underline{x}) \leq 0$$

Set up:

In order to avoid the negative value of  $d_i$ , the criterion functions and the aspiration levels are needed to be normalized as follow:

$$Y_i = \frac{f_i(\underline{x}) - f_i(\underline{x})_L}{f_i(\underline{x})_U - f_i(\underline{x})_L}$$

$$AN_i = \frac{A_i - f_i(\underline{x})_L}{f_i(\underline{x})_U - f_i(\underline{x})_L}$$

where  $f_i(\underline{x})_L$  and  $f_i(\underline{x})_U$  are the lower and upper relevant range of  $f_i(\underline{x})$ .

Hence, the three criterion function are normalized:

$$Y_1 = \frac{f_1(\underline{x}) + 40.0}{50.0}, \quad Y_2 = \frac{f_2(\underline{x}) - 0.14}{9.86}$$

$$Y_3 = \frac{f_3(\underline{x}) - 0.0005}{9.9995}$$

The initial aspiration levels are assumed to be equal to the goal levels,

$A_i = GL_i$ ,  $i = 1, 2, 3$  and so the values of  $AN_i$  are:

$$AN_1 = \frac{44.0}{50.0} = 0.88, \quad AN_2 = \frac{1.56}{9.86} = 0.1582$$

$$AN_3 = \frac{0.1995}{9.9995} = 0.01995$$

The three indicators of attainment are:

$$d_1 = AN_1/Y_1, \quad d_2 = Y_2/AN_2, \quad d_3 = Y_3/AN_3$$

The first cycle:

The principal problem to be solved on the first cycle is

$$\min \quad s_1 = d_1 + d_2 + d_3$$

$$\text{s.t.} \quad \underline{g(x)} \leq 0$$

Also as part of the first cycle, we construct three auxiliary problems which attempt to satisfy each of the goals in turn. If  $f_1(\underline{x}) \geq 4.0$  is entered as a constraint, then  $d_1$  is deleted from the surrogate objective function giving

$$\left. \begin{array}{l} \min \quad S_{1.1} = d_2 + d_3 \\ \text{s.t.} \quad \underline{g(x)} \leq 0 \\ \quad \quad f_1(\underline{x}) \geq 4.0 \end{array} \right\}$$

Similarly, the second, and third auxiliary problems are:

$$\left. \begin{array}{l} \min \quad S_{1.2} = d_1 + d_3 \\ \text{s.t} \quad \underline{g}(\underline{x}) \leq 0 \\ \quad \quad f_2(\underline{x}) \leq 1.7 \end{array} \right\}$$

$$\left. \begin{array}{l} \min \quad S_{1.3} = d_1 + d_2 \\ \text{s.t} \quad \underline{g}(\underline{x}) \leq 0 \\ \quad \quad f_3(\underline{x}) \leq 0.2 \end{array} \right\}$$

Each problem is solved by using Sequential Unconstrained Minimization Technique (SUMT) computer package of Hwang et al. [8]. The results for the principal problem and three auxiliary problems are shown in Table 7. From Table 7, none of the results is satisfied. Hence, the procedure goes into the evaluation phase. On examination of those numbers, it is apparent that the change of goal 2, which is cost, is relatively independent of attainment or nonattainment of the other goals. Also, the DM feels that he/she would like to raise the goal 2 level (cost) in order to get higher value of goal 1 level. So, it seems reasonable to choose an aspiration level for goal 2 and enter it as a constraint.

Using Table 7, the DM can assess the impact of such an action on the attainment of the other goals. This assessment is made as follows. The results of the principal problem and the second auxiliary problem are

Table 7 Results of the first cycle where  $A^0 = (4.0, 1.7, 0.2)$

Kind of problem	$s_1 = \sum_{t \in T} d_t$	$x^*$	$f(\underline{x})$		
			Z [R]	C	W
Principle	$s_1 = d_1 + d_2 + d_3 = 2.531$	(45756, 3712.3, 0.1282, 0.006995)	-7.545, [0]	1.408,	0.0727
Auxiliary problem 1	$s_{11} = d_2 + d_3 = 2.256$	(45867, 2921.1, 0.1960, 0.005998)	4.180, [0.99998]	2.333,	0.1701
Auxiliary problem 2	$s_{12} = d_1 + d_3 = 1.683$	(45682, 3452.8, 0.1499, 0.006355)	-3.794, [0.0003]	1.670,	0.0939
Auxiliary problem 3	$s_{13} = d_1 + d_2 = 2.158$	(45808, 3408.3, 0.1532, 0.006986)	-2.298, [0.02898]	1.686,	0.1037

Goal	Principal Problem	Auxiliary Problem
1	-7.545	-3.794
2	1.408	1.670
3	0.0727	0.0929

where goal 1 is pure number, goal 3 is the total weight in lbs.

The new aspiration level for goal 2 on the next cycle is  $A_2^1$ . Let  $q$  be the proportion  $(A_2^1 - 1.670)/(1.408 - 1.670)$ . Then the approximate effects of adding these constraints are

Goal	Approximate Effects
1	$q \cdot (-7.545 - (-3.794)) + (-3.794)$
3	$q \cdot (0.0727 - 0.0929) + (0.0929)$

The DM can try out several values of  $A_2^1$  until he believes that he can accept the estimated effect on the other goals. This procedure estimates the results for the principal problem of the next cycle. It assumes that  $df_1(\underline{x})/df_2(\underline{x})$  and  $df_3(\underline{x})/df_2(\underline{x})$  are constants, that means  $f_1(\underline{x})$  and  $f_2(\underline{x})$  or  $f_3(\underline{x})$  and  $f_2(\underline{x})$  are linearly related. The DM sets a new aspiration level of 2.1 for goal 2 and estimates the results as

Goal	Results
1	2.361
3	0.1261

where goal 1 is pure number, goal 3 is the total weight in lbs. The DM is satisfied with these new aspiration levels and to 2.1 for goal 2 is not greatly different from original goal of 1.7

### The second cycle

The new aspiration level  $A^1 = (4.0, 2.1, 0.2)$  and goal 2 is entered into constraint. Proceeding to the second cycle, only two auxiliary problems are necessary solve because one goal has been added to the constraint set for this principal program.

The principal problem:

$$\begin{aligned} \min \quad & s_2 = d_1 + d_3 \\ \text{s.t.} \quad & \underline{g}(\underline{x}) \leq 0 \\ & f_2(\underline{x}) \leq 2.1 \end{aligned}$$

The auxiliary problem 1:

$$\begin{aligned} \min \quad & s_{2.1} = d_3 \\ \text{s.t.} \quad & \underline{g}(\underline{x}) \leq 0 \\ & f_1(\underline{x}) \geq 4.0 \\ & f_2(\underline{x}) \leq 2.1 \end{aligned}$$

The auxiliary problem 2:

$$\begin{aligned} \min \quad & s_{2.3} = d_1 \\ \text{s.t.} \quad & f_2(\underline{x}) \leq 2.1 \\ & f_3(\underline{x}) \leq 0.2 \\ & \underline{g}(\underline{x}) \leq 0 \end{aligned}$$



Each problem is solved by SUMT computer package of Hwang et al [8]. The results for the principal problem and two auxiliary problems are shown in Table 8. From Table 8, the auxiliary problem 1 does not have a solution, because the feasible region is empty. The rest of results show that none of results is to be satisfied, but goals 2 and 3 have been satisfied. So, the DM decides to decrease the aspiration level of goal 1; that means the DM reduces the Z values, reducing the element reliability. Finally, the DM decides to set a new aspiration level of 2.5 for goal 1 and add it to be constraint. The DM argues that a new aspiration 2.5 for goal 1, which corresponds to the reliability of 0.99379, can be accepted.

The third cycle

The new aspiration level  $A^2 = (2.5, 2.1, 0.2)$  and both goals 1 and 2 are entered as constraints. The principal problem is:

$$\begin{aligned} \min \quad & s_3 = d_3 \\ \text{s.t} \quad & \underline{g}(\underline{x}) \leq 0 \\ & f_1(\underline{x}) \geq 2.5 \\ & f_2(\underline{x}) \leq 2.1 \end{aligned}$$

and there is no auxiliary problem. The problem is solved by using SUMT(LAI) computer package [8]. The results are tabulated in Table 9. From the Table, the DM decides that he is content with the results and terminates the procedure.

Table 8 Results of the second cycle where  $A^1 = (4.0, 2.1, 0.2)$

Kind of problem	$s_1 = \sum_{t \in T} t \epsilon_7$	$x^*$	$f(x)$		
			$z$ [R]	C	W
Principle	$s_2 = d_1 + d_3 = 1.709$	(45688.7, 3285.3, 0.1467, 0.00461)	-3.732, [0.00038]	1.952,	0.0953
Auxiliary problem 1	$s_{21} = d_3$	no feasible solution			
Auxiliary problem 2	$s_{23} = d_1 = 1.034$	(45766.7, 3097.5, 0.1841, 0.00642)	2.525, [0.99421]	2.098,	0.1500

Table 9 Results of the third cycle where  $A^2 = (2.5, 2.1, 0.2)$

Principle  $s_3 = d_3 = 0.749$  (4.5771.7, 3102.5, 0.1840, 0.00644) 2.515, 2.098, 0.1499  
 [0.99379]

### 5.3.3 Discussion

The results in Table 9, which are satisfied by the DM, are

$$x_1 = 45772, \quad x_2 = 3102.5, \quad x_3 = 0.184, \quad x_4 = 0.0064$$

and  $f_1(\underline{x}) = 2.515$      $f_2(\underline{x}) = 2.09$ ,     $f_3(\underline{x}) = 0.15$

The results mean that the material should have the mean ultimate tensile strength,  $\bar{S} = 45772$  psi, and standard deviation,  $\sigma_S = 3102.5$  psi, and the radius of the element is 0.184 in with manufacturing tolerance fraction 0.64%. The reliability is equal to 0.99379, the cost is 2.09 dollars, and the weight is 0.15 lb. Alluminum alloy 5456 is selected for the material.

### 5.4 Discussion

The two methods of MODM are used to solve the design problem in this chapter. Two cases of nonlinear goal programming are studied in the Section 5.2. The results of two methods are tabulated in Table 6. The different results between the two cases in the nonlinear goal programming are discussed in Section 5.2.3. It is noted that the results are influenced by the DM's ranking and goal of each criterion. The solution is very sensitive to the goal vector set for the criterion and ordinal ranking given by the DM.

The optimization technique used in SEMOPS does not solve problems; it generates information so that the DM can look at the inconsistent constraint set and selected an acceptable alternative. The key concept of SEMOPS is its interactive nature. This prevents the problem of specifying the DM's preference structure by allowing him to keep within himself the

transformation that he makes to convert numbers into value judgements. The DM can develop a ranking of goals as he receives information concerning the feasible alternatives. He may also revise his preferences during the course of the iterations. In the design problem, it shows that SEMOPS prompts the modification of conflicting aspiration levels so that an acceptable solution can be determined. SEMOPS accomplishes this by revealing to the DM the extent to which his aspirations will have to be modified to achieve a feasible alternative.

Two methods have been discussed above. The preference of the methods is still based on the assumption and concept of each method. If the DM has some preference and goal for each criterion, then the goal programming is selected to use. Otherwise, the SEMPOS is preferred.

## CHAPTER 6 CONCLUSION AND DISCUSSION

This study discusses the relationship between reliability and safety. It also discusses several models for computing the component reliability. The main point of this study is to develop the design model which is based on three criteria:

- (1) the component reliability
- (2) the total cost for making or purchasing the component
- (3) the total weight of the component

To determine the optimal design policy for finding (1) the mean ultimate tensile strength (2) the standard deviation of strength (3) radius of the component (4) the manufacturing tolerance fraction, which satisfy the three criteria, two methods of multiple objective decision making methods are used. They are nonlinear goal programming, and the Sequential Multiobjective Problem Solving technique (SEMOPS). The advantages of goal programming are that the DM does not need to give the numerical weights for the objectives, he only needs to give an ordinal ranking of them. The disadvantages is that goal programming is very sensitive to the goal vector set for the objectives and ordinal ranking given by the DM. Goal programming has been widely used in many MODM problems. Sequential Multiobjective Problem Solving techniques (SEMOPS) belongs to an interactive method. The advantages of this method are:

- (1) it can be used to solve nonlinear problems, and (2) the DM can reevaluate his desirable achievement levels for the objectives.

The disadvantages are: (1) since the optimization phase uses an algorithm based on linear programming; the amount of linearization work

is quite large; (2) it is handicapped by the requirement of differentiability of objective functions and constraint functions. The disadvantages may be eliminated if a proper nonlinear programming method is used such as SUMT method.

The model presented in this article only restricted to three objectives; reliability function, cost function, and weight function. But in the real world's design, the designer needs to consider more criteria. Hence, more objectives will be involved such as the reliability for shear stress and strength interference model or for compressive stress and strength interference model. How to estimate the cost according to these parameters will need further studies.

In the model, one of the objectives concerns reliability. We use  $Z$ , the standard normal variable, to represent the reliability because the integration in the reliability model only can be solved by numerical integration, which needs too much computing time.

In this thesis the reliability model presented belongs to an SSI model. The SST models are not considered, due to the following reasons: a) Some of SST models rarely happen, but they are almost the same as SSI models, if we can solve SSI models, there is no problem to do SST models. b) although some of the SST models are more realistic than some SSI models, their solutions are too costly in computer time, and frequently inaccurate.

## REFERENCES

1. American Society of Testing and Materials "1977 Annual Book of ASTM Standards".
2. Bazovsky, I., Reliability Theory and Practice, Englewood Cliffs, N.J., Prentice-Hall, 1961.
3. Bratt, M. J., Reethoff, G., and Weber, G. W., "A Method for Time Varying and Interfering Stress/Strength Probability Density Distributions with Consideration for Failure Incidence and Property Degradation", Proceedings Third Annual Aerospace Reliability and Maintainability Conference, 1964, pp. 566-575.
4. Disney, R. L., Sheth, N. J., and Lipson, C., "The Determination of the Probability of Failure by Stress/Strength Interference Theory", Proceedings of Annual Symposium on Reliability, 1968, pp. 417-422.
5. Epstein, B., "Elements of the Theory of Extreme Value", Technometrics, Vol. 2, No. 1, February 1960.
6. Hwang, C. L., and Masud, A., "Multiple Objective Decision Making - Methods and Application: A State-of-the Art Survey", presented at the joint meeting of TIMS/ORSA, New York, May 1-3, 1978.
7. Hwang, C. L., Paidy, S., Masud, A., and Raju, K. M., "Nonlinear Goal Programming Method", to be published.
8. Hwang, C. L., Tillman, F. A., Lai, K. C., and Fan, L. T., "Optimization of system reliability by the sequential unconstrained minimization techniques," IEEE Trans. Reliability, Vol R-24, pp 133-135 (1975).



9. Ignizio, J. P., Goal Programming and Extensions, Lexington, Mass. 1976.
10. Kapur, K. C., and Lamberson, L. R., "Reliability in Engineering Design", Wiley, New York, 1977.
11. Kececioglu, D., and Cormier, D., "Designing a Specified Reliability Directly into a Component," Proceedings Third Annual Aerospace Reliability and Maintainability Conference, 1964, pp 546-565.
12. Kececioglu, D., and Haugen, E. B., "A Unified Look at Design Safety Factors, Safety Margins and Measures of Reliability", Annals of Assurance Sciences - Seventh Reliability and Maintainability Conference, 1968, pp. 520-530.
13. Mischke, C., "A Method of Relating Factor of Safety and Reliability," Journal of Engineering for Industry, Transactions of ASME, August 1970, pp. 537-542.
14. Monarchi, D. E., C. C. Kisiel, and L. Duckstein, "Interactive Multi-objective Programming in Water Resources" A Case Study", Water Resources Research, Vol. 9, No. 4, pp. 837-850, 1973.
15. Shaw, L., Shooman, M., and Schatz, R., "Time Dependent Stress/Strength Methods for Non-electrical and Electrical Systems", Annals of Reliability and Maintainability Symposium, 1972, pp. 186-197.
16. Smith, C. O., Introduction to Reliability in Design, McGraw-Hill, 1976.

## APPENDIX A

The Numerical Integration Program

```

COMPILER OPTIONS - NAME= MAIN,OPT=02,LINECNT=60,SIZE=0000K,
SOURCE,EBCDIC,NOLIST,NCDECK,LOAD,MAP,NOEDIT,IO,NOXREF
C *****
C *
C * NUMERICAL INTEGRATION *
C *
C *****
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION X(50)
READ(5,1)(X(I),I=1,4)
1 FORMAT(4F10.2)
N1=100
Z=-1.*(X(1)-1273.23/(X(3)**2))/DSQRT((X(2)**2)+((1013.2+720434.1*(
IX(4)**2))/(X(3)**4)))
IF(Z)40,50,60
40 RX1=DEXP(-(Z**2)/2)
RX0=1.0
H1=(0.-Z)/N1
M=N1/2
SUM=RX0+RX1
DO15 J=1,M
VVRR=0.0+(2*J-1.0)*H1
RX=DEXP(-(VVRR**2)/2)
SUM=SUM+4*RX
15 CONTINUE
IF(M-1)26,26,13
12 M1=M-1
DO25 J=1,M1
VVRR=0.0+2*J*M1
RX=DEXP(-(VVRR**2)/2)
SUM=SUM+2*RX
25 CONTINUE
26 TN=M1/3.*SUM/2.5066
TN=TN+0.5
GO TO 100
60 RX1=DEXP(-(Z**2)/2)
RX0=1.0
H1=(Z-0.)/N1
M=N1/2
SUM=RX0+RX1
DO85 J=1,M
VVRR=0.0+(2*J-1.0)*H1
RX=DEXP(-(VVRR**2)/2)
SUM=SUM+4*RX
85 CONTINUE
IF(M-1)36,36,115
115 M1=M-1
DO65 J=1,M1
VVRR=0.0+2*J*M1
RX=DEXP(-(VVRR**2)/2)
SUM=SUM+2*RX
65 CONTINUE
36 TN=M1/3.*SUM/2.5066
TN=0.5-TN
GO TO 100
50 TN=0.5
GO TO 100
100 WRITE(6,2)TN
2 FORMAT('1',F10.5)
STOP
END

```

## APPENDIX B

Nonlinear Models Program (NONLIN)

DEPARTMENT OF STATISTICS AND THE STATISTICAL LABORATORY - NON-LINEAR PROGRAM

TITLE, COST FUNCTION

JCB .. XPRS0525

DATA, 5.

LABEL(1)=STRENGTH, STDS, RADIUS, ALPHAT, COST

PARMLAB(1)=81,82,83,84,85,86,87,88

METHOD, MARQUARDT

ITERATIONS, 60

PARAMETERS, 8

INIT(1)=0.7,0.7,0.7,0.7,0.7,0.7,0.7,0.7

OUTPUT, STEPS

MODEL, 5=F(1,2,3,4)

END OF PARAMETERS

FCRMT(F10.1,F10.2,F10.1,F10.1,F10.1)

```

SUBROUTINE TRANS(X)
DIMENSION X(1)
RETURN
END

```

```

SUBROUTINE FUNG(B,F,X)
REAL*8 F,B(1)
REAL*4 X(100)
COMMON NRAW,NTRANS,ICROP,AMCC
C
F=(B(1)*X(1)**B(2))*(B(3)/(X(2)/X(1))**B(4))*(B(5)*X(3)**B(6))*(B(
17)/X(4)**B(8))
C
PLACE FUNCTIONS HERE
C
RETURN
END

```

```

SUBROUTINE LGRANG(B,G)
REAL*8 G(1),B(1)
COMMON NRAW,NTRANS,ICROP,AMCC
COMMON/CCNST/NITER,NPARM,ACCNST,NVAR,N,NCES
C
PLACE CONSTRAINTS HERE
C
RETURN
END

```

ENGINEERING DESIGN IN RELIABILITY CRITERION

by

DER-MEI CHOW

Diploma (Industrial Engineering)

Taipei Institute of Technology, Taiwan, 1971

---

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the  
requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1978

The conventional design approach based on a "safety factor" gives little indication of failure probability of a component. In reality, the failure probability may vary from a low to an intolerably high value for the same safety factor. The stress-strength interference theory used in this thesis is a better method to calculate the reliability of a component. Time dependent stress-strength interference theory is also discussed.

This thesis studies an optimization problem for an engineering design using the stress-strength interference theory. The problem consists of three objectives, which are: (1) maximization of the component reliability, (2) minimization of the total cost, and (3) minimization of the weight. Two methods of multiple objective decision making are used. They are nonlinear goal programming and the Sequential Multiobjective Problem Solving technique (SEMOPS). Goal programming requires a priori articulation of preference information; that is, the decision-maker needs to give goals and an ordinal ranking of objectives, while SEMOPS is an interactive method.

A design of a tensile element of a critical component of a system is presented as an illustrated example. The coefficients of the two objective functions (cost and weight) in this example are estimated from the real data by the regression technique. The optimization problem is formed to have three objectives mentioned before, and four decision variables, mean tensile strength, standard deviation of tensile strength, radius of the element, and manufacturing tolerance fraction. The results obtained by using the nonlinear goal programming approach



indicate that they are very sensitive to the ordinal ranking of the importance of the three objectives in the problem. However, the results from SEMOSP are different from nonlinear goal programming, because they indicate that they come from the trade-off procedure among three objectives in an interactive manner.

