OPTIMIZATION OF MANAGEMENT SYSTEMS BY THE FUNCTIONAL GRADIENT TECHNIQUE

by 45

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CHAPTER 1

INTRODUCTION

Multistage optimization problems of management systems arise in connection with processes developing in time in which one or more control variables must be controlled to achieve certain conditions. Out of all possible values for the control variables, the one which gives a certain maximum (or minimum) performance index while simultaneously keeping all the state and control variables within the specified constraints of the problem must be determined.

Various methods have been developed and applied to solve multistage and single stage problems. The Kuhn-Tucker method, quadratic programming and linear programming is in a sense an adjacent extreme point method which employs the simplex algorithm as the fundamental computational tool.

The calculus of variation is a classical tool for solving optimization problems. Until recently, however, computational difficulties limited this to solving simple problems only. Dynamic programming provides an entirely new concept of optimization and has been used quite extensively to solve management optimization problems. However, owing to the dimensionality difficulty and the limited fast memory capacity of present day computers, this technique cannot be applied to problems with a fairly large number of state variables. In view of the complexity of many industrial and management systems, this is a serious limitation.

The gradient technique or the method of steepest ascent is an elementary concept in solving optimization problems. It dates back to

Cauchey [4a] and, in a variational version, to Hardamard [5a]. It is based on the fact that if movement is made in the direction of the gradient of the objective function, movement is also made in the direction of the maximum rate of increase in the objective function. In recent years the computational appeal of the method has led to its adaptation in a variety of applications. The dynamic version of the technique, generally known as the functional or serial gradient technique, has been applied successfully to solve problems in aerospace, control and chemical engineering systems. [4-6, 8-13]

This report is a study of the way in which the gradient concept can be applied to the solution of optimization problems with constraints. The application of the concept can assume a variety of forms depending on the type of problem to be solved and on the manner it is modified to account for the constraints. This report is dealing with fixed and constraints for some state variables in general and with the inventory and production scheduling problem in particular.

CHAPTER 2

THE METHOD

1. A General Problem

An optimization problem in a fairly general form can be stated as follows: Determine $\theta(t)$ in the interval $t_0 \leq t \leq T$ so as to maximize

$$\phi = \phi(\underline{x}(T), T) \tag{1}$$

subject to the constraints

$$\psi = \psi(x(T), T) = 0 \tag{2}$$

$$\frac{dx}{dt} = f(\underline{x}(t), \underline{\theta}(t), t) \tag{3}$$

with t_0 and $\underline{x}(t_0)$ being given

where

$\underline{\theta}(t) = \begin{pmatrix} \theta_1(t) \\ \vdots \\ \theta_m(t) \end{pmatrix}$, an mxl matrix of control variables

$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, an nxl matrix of state variables which results from the choice of $\underline{\theta}(t)$ and the given values of $x(t_0)$

$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_p \end{pmatrix}$, an pxl matrix of terminal constraint functions each of which is a known function of $\underline{x}(T)$ and T.

If an integral is to be maximized, simply introduce an additional state variable x_{n+1} and an additional differential equation

$$\frac{dx_{n+1}}{dt} = q(\underline{x}(t), \underline{\theta}(t), t) \qquad (4)$$

where q is the integrand of the integral. Now $x_{n+1}(T)$ is maximized with the initial condition $x_{n+1}(t_0) = 0$.

2. Concepts

The gradient technique is basically a method in which the control variable is improved from a point far away from the optimum along the gradient direction. The gradient direction being referred to is the gradient that a particular control variable has with respect to the given objective function.

Figure 1 is a sketch of an optimum programming problem. The state variable x must satisfy the functional relationship

$$\frac{dx}{dt} = f(x,\theta)$$

where θ is the control variable. The functional relationship must be satisfied at all points between the two end points $t = t_0$ and $t = T$.

The problem of optimization basically arises when it is necessary to find the controls which, while satisfying the path, also optimizes the objective function ϕ. In addition to optimizing the process and satisfying the functional relationship, it might be required that the controls also meet certain additional final conditions on the time variable t or on the final values of the state variables.

To solve this problem by the gradient technique, a certain sequence

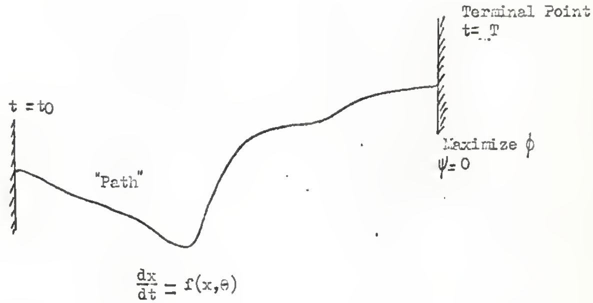

Fig.1 Symbolic Sketch Of Control Problems.

of values is assumed for the control variables. The gradient of the objective function with respect to the control is determined at all points along the path or trajectory. According to the gradient technique, it is best to improve the control variable along the gradient direction in order to reach the optimum as quickly as possible. The control variable is hence improved in the gradient direction by a certain predetermined step size at all points along the path. At each new point the gradient is redetermined and a step is again taken in this new direction. Proceeding in this manner, an optimum is reached. After determining the gradient direction and the step size, the new control variable can be computed from the relationship

$$\theta_{new}(t) = \theta_{old}(t) + K \left. \frac{\partial \phi}{\partial \theta} \right|_t$$

3. Functional Equations

Before beginning a numerical calculation based on the gradient method, two decisions must be made. First, a method of calculating the partial derivatives making up the gradient must be selected. Second, some scheme to determine the step size along the gradient must be devised. The direction of step along the gradient will be determined by whether a maximum or a minimum of the objective function is desired.

Two main methods are available to calculate partial derivatives. The simpler way is the independent perturbation of each element of the control variable. The partial derivatives can be estimated from these perturbations.

For example, consider the process of obtaining the gradient with respect to the control θ, the grid size being Δ. The first step is to assume a feasible value for θ_0 and to compute the value of the objective function S_0. The next step would be to purturb θ by a small amount $\Delta\theta$ and let the new value of the objective function be S_1. The gradient of the objective function with respect to the control may now be written as

$$\left.\frac{\partial S}{\partial \theta}\right|_t = \frac{S_1 - S_0}{\Delta\theta}$$

where $\left.\frac{\partial S}{\partial \theta}\right|_t$ is the partial derivative of the objective function with respect to the control variable at time t.

The above procedure has the advantages of simplicity and of high relative accuracy even in the presence of nonlinear effects. However, its major disadvantage is a rapid increase in computation time as the number of time increments involved increases. For example, increasing the number of time increments from 5 to 10 would increase the computer time required for the calculation of the gradients by almost 4 times. Thus, practical considerations limit the use of this technique to problems involving optimization over a relatively small number of control variables with a fairly small number of grid points.

The second method for calculating the partial derivatives which avoids the difficulty mentioned above is by setting up recurence relationships for the calculation of the derivatives.

Suppose there are n state variables and one control variable. Further assume that a feasible sequence of control variables $\theta(t)$, $t_0 \leq t \leq T$ can be obtained to start the calculations.

The performance equation as given in Eq. (3) may also be written as

$$x_i(t + \Delta t) = x_i(t) + f_i(x_1(t), x_2(t), x_n(t), \theta(t))\ \Delta t \tag{5}$$

where $x_i(t)$ and $i = 1, 2, \ldots n$ designate the value of the i^{th} element of the state vector at time t. Now, define

$S(x_1, x_2, \ldots, x_n) =$ The final value of the objective function is obtained by starting at the condition $x_1, x_2, \ldots, x_n$ and using the assumed control sequence $\theta(t)$, $t_0 \leqslant t \leqslant T$.

Using Eq. (5), then

$$S(x_1, x_2, \ldots, x_n) = S(x_1 + f_1\ \Delta t, x_2 + f_2 \Delta t, \ldots, x_n + f_n \Delta t). \tag{6}$$

It is necessary to determine

$$\left.\frac{\partial S(x_1, x_2, \ldots, x_n)}{\partial \theta}\right|_t .$$

This is the partial derivative of the final value of the objective function with respect to the control variable evaluated at time t. Using Eq. 6, then

$$\left.\frac{\partial S(x_1, x_2, \ldots, x_n)}{\partial \theta}\right|_t = \left.\frac{\partial S(x_1 + f_1 \Delta, \ldots, x_n + f_n \Delta)}{\partial \theta}\right|_t \tag{7}$$

Now, by the chain rule

$$\frac{\partial S(x_1 + f_1\Delta, \ldots, x_n + f_n\Delta)}{\partial\theta}\bigg|_t = \sum_{i=1}^{n}\left[\frac{\partial S}{\partial(x_i + f_i\Delta)}\bigg|_t \frac{\partial(x_i + f_i\Delta)}{\partial\theta}\bigg|_t\right] \tag{8}$$

Using Eq. (5),

$$\frac{\partial(x_i + f_i\Delta t)}{\partial\theta}\bigg|_t = \frac{\partial f_i}{\partial\theta}\bigg|_t \Delta \ . \tag{9}$$

Combining Eqs. (8) and (9), gives

$$\frac{\partial S}{\partial\theta}\bigg|_t = \sum_{i=1}^{n}\frac{\partial S}{\partial x_i}\bigg|_{t+\Delta}\frac{\partial f_i}{\partial\theta}\bigg|_t \Delta \ . \tag{10}$$

The left hand side of the above equation is the desired partial derivative of the objective function with respect to the control at a particular stage with a given Δt. The partial derivative $\frac{\partial f_i}{\partial\theta}\Big|_t$ may be calculated analytically. Thus one has a starting value and a recurence relationship for $\frac{\partial S}{\partial\theta}\Big|_t$. However, before making the calculation, a recurence relationship must also be obtained for $\frac{\partial S}{\partial x_i}\Big|_t$ $i = 1, 2, \ldots, n$. Differentiating Eq. 6 with respect to x_j gives

$$\frac{\partial S(x_1, x_2, \ldots, x_n)}{\partial x_j}\bigg|_t = \frac{\partial S(x_1 + f_1\Delta t, x_2 + f_2\Delta t, \ldots, x_n + f_n\Delta t)}{\partial x_j}\bigg|_t \tag{11}$$

or

$$\frac{\partial S}{\partial x_j}\bigg|_t = \sum_{i=1}^{n}\frac{\partial S}{\partial(x_i + f_i \ t)}\bigg|_t \frac{\partial(x_i + f_i\Delta t)}{\partial x_j}\bigg|_t \ . \tag{12}$$

But

$$\frac{\partial S}{\partial(x_i + f_i \Delta t)}\bigg|_t = \frac{\partial S}{\partial x_i}\bigg|_{t+\Delta t} \tag{13}$$

$$\frac{\partial(x_i + f_i \Delta t)}{\partial x_j}\bigg|_t = \frac{\partial f_i}{\partial x_j}\bigg|_t \Delta t \qquad i \neq j \tag{14}$$

and

$$\frac{\partial(x_i + f_i \Delta t)}{\partial x_j}\bigg|_t = 1 + \frac{\partial f_i}{\partial x_i}\bigg|_t \Delta t \qquad i = j \tag{15}$$

Combining Eqs. (11) through (15) results in

$$\frac{\partial S}{\partial x_j}\bigg|_t = \frac{\partial S}{\partial x_j}\bigg|_{t+\Delta t} + \sum_{i=1}^{n} \frac{\partial S}{\partial x_i}\bigg|_{t+\Delta t} \frac{\partial f_i}{\partial x_j}\bigg|_t \Delta t \; . \tag{16}$$

The recurence relationships obtained above are essentially equivalent to those developed by Bryson and Kelley [4, 6]. However, the developments due to Bryson and Kelley are considerably more complex, involving perturbations and adjoint equations.

The derivations outlined above were based on the situations where the control vector contained only one element. Problems with control vectors containing several elements do not cause any basic change in the development. However, there would be individual recurence relationships for the partial derivative of the objective function with respect to each control variable.

Now, a relationship must be developed to obtain an improved control variable based upon the gradients. As stated before, the new value of the control variable can be computed from the relationship

$$\theta_{i_{new}}(t) = \theta_{i_{old}}(t) + K \left.\frac{\partial S}{\partial \theta_i}\right|_t \tag{17}$$

Note that $\theta_i(t)$ is the value of the ith element of the control variable at time t. The scaler, K, may be thought of as the step size along the gradient. Thus the problem of obtaining a correction based on the gradients reduces to one of selecting a proper K. One feature of K is immediately obvious: its sign depends upon whether a maximum or a minimum is desired. Maximization problems require a positive K and minimization problems require a negative K.

A straight forward method of obtaining K would be to search over all reasonable values and select the one which gives the maximum improvement in the objective function. However, there is no way to specify the best range of K over which the search should be conducted. There is an additional difficulty of computation time. Since the objective function is defined at the final time, evaluation of a trial K requires a complete integration of the performance equations. If the process has a large number of state variables or a large number of time increments, this integration would require a great deal of computer time.

An alternative to the direct search for the determination of K is as follows. The expression $K \left.\frac{\partial S}{\partial \theta_i}\right|_t$ is basically the difference between the old and the new values of θ_i at time t. In incremental form this may be written as

$$\delta\theta_i(t) = K \frac{\partial S}{\partial \theta_i} \tag{18}$$

Suppose it is wished to estimate the total change in the objective function due to a series of changes in θ_i. One way to obtain this estimate for a process containing T time increments would be through the approximation

$$\Delta\phi \simeq \sum_{t=0}^{T} \left.\frac{\partial S}{\partial \theta_i}\right|_t \delta\theta_i(t) \ . \tag{19}$$

Combining Eqs. (18) and (19) gives

$$\Delta\phi = \sum_{t=0}^{T} \left.\frac{\partial S}{\partial \theta_i}\right|_t K \left.\frac{\partial S}{\partial \theta_i}\right|_t = K \sum_{t=0}^{T} \left(\left.\frac{\partial S}{\partial \theta_i}\right|_t\right)^2 \tag{20}$$

or

$$K = \frac{\Delta\phi}{\sum_{t=0}^{T} \left(\left.\frac{\partial S}{\partial \theta_i}\right|_t\right)^2} \tag{21}$$

Substituting the value of K from Eq. (21) into Eq. (17) yields

$$\theta_{i_{new}}(t) = \theta_{i_{old}}(t) + \frac{\Delta\phi \left.\frac{\partial S}{\partial \theta_i}\right|_t}{\sum_{t=0}^{T} \left(\left.\frac{\partial S}{\partial \theta_i}\right|_t\right)^2} \ . \tag{22}$$

In this equation, one must select a suitable value for $\Delta\phi$. In other words, it must be decided how much a change in the objective function is desired. If too small a value of $\Delta\phi$ is selected, many evaluations of the gradient will be required to obtain the optimum while too large a value of $\Delta\phi$ runs risk of obtaining no improvement at all. Notice that only a linear relationship is used in obtaining the gradients. A large $\Delta\phi$ can move the new controls outside the region where linearization is valid. Sometimes it may be possible to obtain a scheme for adjusting $\Delta\phi$ during the calculations to obtain a good balance between computation time and accuracy.

If there are no constraints, one would expect $\frac{\partial S}{\partial \theta_i}$ to approach zero as the maximum or minimum value of S is approached. Since the correction scheme requires division by $\sum_{t=0}^{T} \left(\frac{\partial S}{\partial \theta_i} \Big|_t \right)^2$, the computation will involve division by a very small number when the maximum or minimum is appraoched. The severity of this difficulty will be discussed in later chapters.

4. Functional Equations with Fixed End Conditions

Suppose that the following condition must also be satisfied at the end point:

$$z(x_1, x_2, x_n, t) = 0.$$

It is desired to compute the influence of the control variable θ on the final value z.

The arguments used to derive the relationships would be identical to those used in deriving the recurence relationships. The following recurence relationship for the additional final condition can be obtained

$$\frac{\partial z}{\partial \theta}\bigg|_t = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}\bigg|_{t+\Delta} \frac{\partial f_i}{\partial \theta}\bigg|_t \Delta \tag{23}$$

$$\frac{\partial z}{\partial x_j}\bigg|_t = \frac{\partial z}{\partial x_j}\bigg|_{t+\Delta} + \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}\bigg|_{t+\Delta} \frac{\partial f_i}{\partial x_j}\bigg|_t \Delta \tag{24}$$

with the final condition

$$\frac{\partial z}{\partial x_j}\bigg|_T = \frac{\partial z}{\partial x_j}\bigg|_T - \left(\frac{\partial \phi}{\partial t} \Big/ \frac{\partial \psi}{\partial t}\right)\bigg|_T \frac{\partial \psi}{\partial x_j}\bigg|_T . \tag{25}$$

Now let the improvement in the control variable take the form

$$\delta\theta_i(t) = K_1 \frac{\partial S}{\partial \theta_i}\bigg|_t + K_2 \frac{\partial z}{\partial \theta_i}\bigg|_t . \tag{26}$$

The constants K_1 and K_2 can be obtained by solving the following set of simultaneous equations:

$$\Delta\phi = K_1 \sum_{t=0}^{T} \left(\frac{\partial S}{\partial \theta}\bigg|_t\right)^2 + K_2 \sum_{t=0}^{T} \left(\frac{\partial S}{\partial \theta}\bigg|_t \frac{\partial z}{\partial \theta}\bigg|_t\right) \tag{27}$$

and

$$\Delta z = K_1 \sum_{t=0}^{T} \left(\frac{\partial S}{\partial \theta}\bigg|_t \frac{\partial z}{\partial \theta}\bigg|_t\right) + K_2 \sum_{t=0}^{T} \left(\frac{\partial z}{\partial \theta}\bigg|_t\right)^2 . \tag{28}$$

Due to the difficulties in finding the initial feasible trajectory, the term Δz, which represents the deviation from the desired final condition, can be thought of as a correction in the final value of the subsidiary condition.

It is to be noted that the expression

$$K_2 \frac{\partial z}{\partial \theta_i}\bigg|_t$$

in Eq. (26) is a penalty function imposed on the improvement of the control variable due to the auxiliary final condition which must be satisfied by the optimal sequence of the control vector. The penalty function, in general, reduces the rate of approach to the optimum.

Summarizing the above discussion, it is seen that if some end conditions at the final time have to be satisfied, Eq. (17) must be modified to

$$\theta_{i_{new}}(t) = \theta_{i_{old}}(t) + \left(K_1 \frac{\partial S}{\partial \theta_i}\bigg|_t + K_2 \frac{\partial z}{\partial \theta_i}\bigg|_t \right) \tag{29}$$

where the constants K_1 and K_2 and the expression for $\frac{\partial z}{\partial \theta_i}\Big|_t$ are computed from the relationships specified in Eqs. (23) through (28).

5. Computational Procedure

Knowing the recurence relationships, the step to step computational procedure for a problem without given end conditions are:

1. Assume a sequence of control variables.
2. Using the performance Eq. (5), determine the sequence of state variables.
3. Evaluate $\frac{\partial f_i}{\partial \theta}\Big|_t$ and $\frac{\partial f_i}{\partial x_j}\Big|_t$ for every t using the numerical values of the state vectors and controls.

4. From the final values of the state vector and the controls, evaluate $\frac{\partial S}{\partial x_i}$ at the end of the process.

5. Calculate $\left.\frac{\partial S}{\partial \theta_j}\right|_t$ by backward recursion of Eq. (16).

6. Calculate $\left.\frac{\partial S}{\partial \theta}\right|_t$ from Eq. (10).

7. Calculate the new control from Eq. (22).

8. Repeat Steps 2 through 7 until the gradient is so small that further improvement is not significant.

In case the problem involves satisfying some end conditions on the state variables, the only difference in the computational procedure would be in Step 7 where the improved control would now be computed from Eq. (26) instead of Eq. (22).

CHAPTER 3

APPLICATIONS

Three problems relating to the optimization of management systems have been solved by the gradient technique in this report. The first is a simple problem in the field of production control. Here both the initial and the final inventories are given. Thus this is an optimization problem with one fixed end condition. The sales in this case are assumed to be known. The second problem considers the diffusion model of advertising. The problem is to find the optimum advertising for the maximum profit with a given production rate. The model also controls the inventory level. The final problem considers both production and sales as variables with the operating temperature controlling the production rate and the diffusion model being used for the advertising.

3.1 A Production and Inventory Control Problem

3.1.1 The Model

Consider the solution of a simple problem in the field of production and inventory control using the gradient technique. Further consider the sales rate $Q(t)$ to be known with certainty and that the rate of change of the inventory level $I(t)$ is given by

$$\frac{dI(t)}{dt} = p(t) - Q(t) \tag{30}$$

where $p(t)$ is the production rate at time t. The problem is to minimize the cost function

$$C_T = \int_0^T \left[C_I(I(t))^2 + C_p \exp\,(p_m - p(t))^2\right]dt \tag{31}$$

where C_T is the total cost of inventory and production. C_I is the inventory carrying cost. C_p is the minimum production cost which occurs when the production rate equals p_m which can be considered as the capacity of the manufacturing plant. Since the plant is designed for capacity p_m, an increase in capacity may require additional equipment and manpower and thus can be very expensive. On the other hand, a decrease in capacity can be equally expensive because of maintenance of unused equipment and because of manpower which, due to contract agreements, cannot be easily reduced.

Sales, a known quantity, is given by the linear relation

$$Q(t) = a + b(t)\ . \tag{32}$$

The initial inventory is

$$I(0) = C \tag{33}$$

It is desired that the inventory level at final time t = 1 be I(T) = D.

Reformulating this problem according to the notations used previously results in $x_1(t) = I(t)$ and $\theta(t) = p(t)$. Equation (~~20~~ 30) now becomes

$$\frac{dx_1(t)}{dt} = \theta(t) - a - b(t) \tag{34}$$

$$x_1(0) = C \tag{35}$$

$$x_1(T) = D. \tag{36}$$

Let

$$x_2(t) = \int_0^t \, [C_I(x_1(t))^2 + C_p \exp \, (p_m - p(t))^2] \, dt. \tag{37}$$

Then

$$x_2(T) = C_T$$

and

$$\frac{dx_2}{dt} = C_I(x_1(t))^2 + C_p \exp \, (p_m - \overset{\delta(t)}{p(t)})^2 \tag{38}$$

with

$$x_2(0) = 0. \tag{39}$$

Equations (34) and (38) are the desired differential equations corresponding to Eq. (3) with n = 2. The initial conditions are given by Eqs. (35) and (39). The function to be minimized is

$$\phi = x_2 \; . \tag{40}$$

The terminal conditions are

$$\psi = t - T = 0 \tag{41}$$

and

$$z = x_1(T) - D = 0 \; . \tag{42}$$

3.1.2 The Recurence Equations

Considering t as the third state variable, the variational equations can be obtained easily. From Eq. (10),

$$\frac{\partial S}{\partial \theta}\bigg|_t = \frac{\partial S}{\partial x_1}\bigg|_{t+\Delta}\Delta - 2\,\frac{\partial S}{\partial x_2}\bigg|_{t+\Delta}\left[C_p(p_m - \theta)\,\exp\,(p_m - \theta)^2\right]\bigg|_t \tag{43}$$

Using Eq. (16) yields

$$\frac{\partial S}{\partial x_1}\bigg|_t = \frac{\partial S}{\partial x_1}\bigg|_{t+\Delta} + 2\,\frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} C_I\, x_1\bigg|_t\,\Delta \tag{44}$$

$$\frac{\partial S}{\partial x_2}\bigg|_t = \frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} . \tag{45}$$

The terminal conditions are obtained from Eq. (9) give

$$\frac{\partial S}{\partial x_1}\bigg|_T = 0 \tag{46}$$

$$\frac{\partial S}{\partial x_2}\bigg|_T = 1 . \tag{47}$$

An equation for $\frac{\partial S}{\partial t}$ can be obtained from Eq. (8). However, for the present problem this equation is not needed.

The variational equations for the second constraint

$$z = x_1(T) - D = 0$$

can be obtained in a similar manner. From Eq. (23),

$$\frac{\partial z}{\partial \theta}\bigg|_t = \frac{\partial z}{\partial x_1}\bigg|_{t+\Delta}\Delta - 2\,\frac{\partial z}{\partial x_2}\bigg|_{t+\Delta}\left\{C_p(p_m - \theta)\,\exp\,(p_m - \theta)^2\bigg|_t\right\}\Delta \tag{48}$$

From Eq. (24)

$$\left.\frac{\partial z}{\partial x_1}\right|_t = \left.\frac{\partial z}{\partial x_1}\right|_{t+\Delta} + 2\left.\frac{\partial z}{\partial x_2}\right|_{t+\Delta} C_I \left. x_1\right|_t \Delta \tag{49}$$

$$\left.\frac{\partial z}{\partial x_2}\right|_t = \left.\frac{\partial z}{\partial x_2}\right|_{t+\Delta} . \tag{50}$$

The terminal conditions can be obtained from Eq. (25).

$$\left.\frac{\partial z}{\partial x_1}\right|_T = 1 \tag{51}$$

$$\left.\frac{\partial z}{\partial x_2}\right|_T = 0. \tag{52}$$

From Eqs. (40) and (42), then

$$\left.\frac{\partial z}{\partial x_2}\right|_t = \left.\frac{\partial z}{\partial x_2}\right|_{t+\Delta} = 0 \tag{53}$$

Equations (39), (41) and (43) give

$$\left.\frac{\partial z}{\partial x_1}\right|_t = \left.\frac{\partial z}{\partial x_1}\right|_{t+\Delta} \Delta = \Delta \tag{54}$$

Substituting the values obtained in Eqs. (43) through (51) into Eqs. (27) and (25), the values of the constants K_1 and K_2 can be calculated. By substituting these values of K_1 and K_2 into Eq. (29) the improvement in Δz is obtained. The value of Δz is $(x_1(T) - D)$ and $\Delta\phi$ is the desired improvement in the objective function.

3.1.3 Numerical Results

The numerical values used are

$$a = 2 \qquad D = 9.25$$

$$b = 1 \qquad C_p = 0.001$$

$$C = 5 \qquad p_m = 5$$

$$C_I = 0.1 \qquad T = 1$$

$$\Delta = 0.01$$

This problem was solved on an IBM 360-50 computer. The convergence rate of the control variable, the production rate, is shown in Fig. (2). The convergence rates of the inventory level and the cost function are shown in Figs. (3) and (4), respectively. The Runge-Kutta integration formula was used to integrate Eqs. (39) and (38). The step size used was 0.01, which is the same as the Δ value used. A value of $\Delta\phi$ equal to $-$ 0.1 was used for the first 25 iterations and values of $\Delta\phi = -0.01$ and $\Delta\phi = -0.001$ were used for 26 to 72 and 73 to 126, respectively.

The last part of the production rate curve is very insensitive to the cost function x_2. Only five iterations were required to get a cost very near the optimum. However, the curve for the production rate at the fifth iteration is far from the optimum one. (See Fig. 2). The cost C_T at the fifth iteration was 5.25; the minimum cost was 5.17, a decrease of only 1.14%.

The convergence rate from the fifth iteration to the optimum is very slow. Approximately 90 iterations were required to improve the cost from 5.25 to 5.17. This difficulty comes from the fact that the gradient

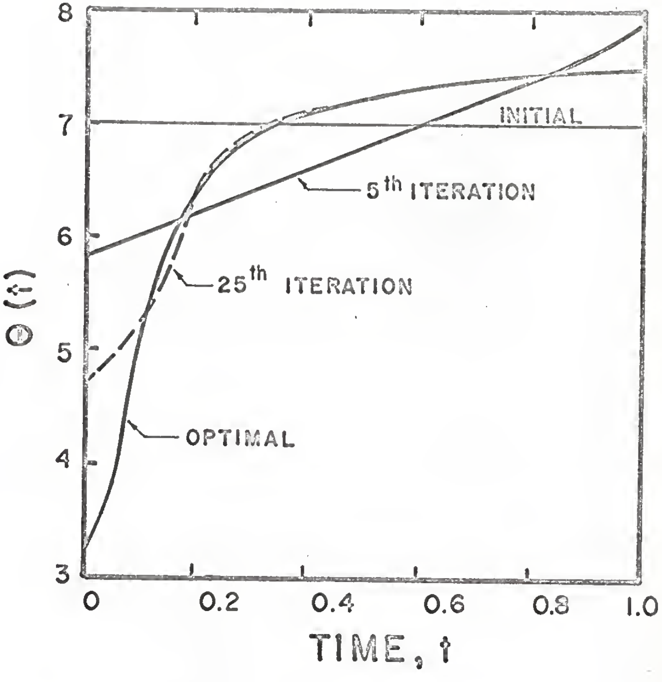

FIGURE 2. CONVERGENCE
RATE OF PRODUCTION

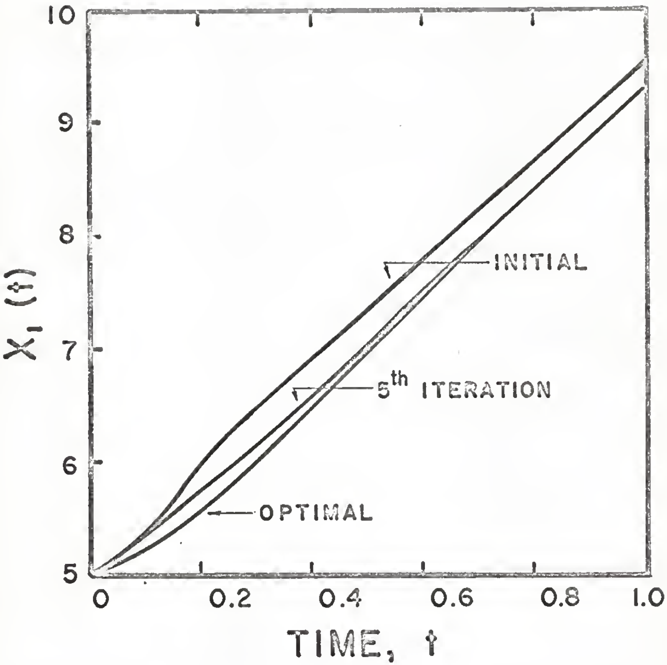

FIGURE 3. CONVERGENCE RATE OF INVENTORY

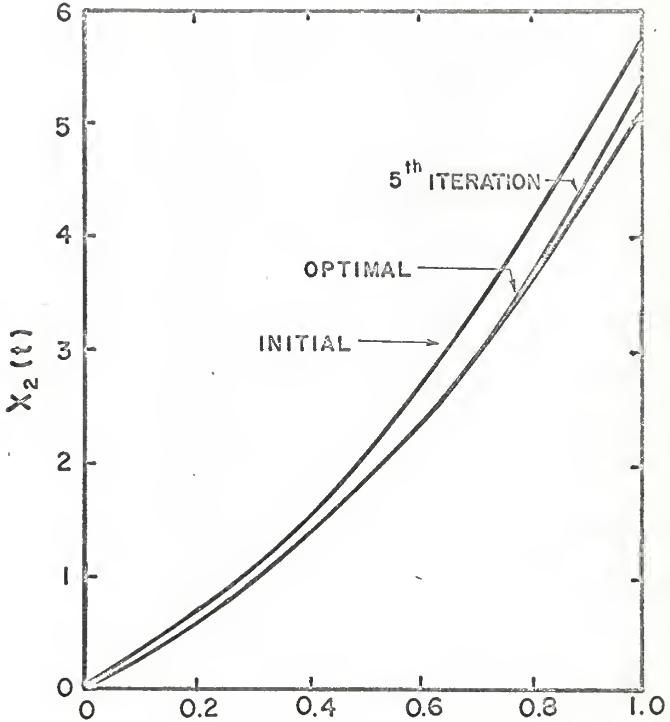

FIGURE 4. CONVERGENCE RATE OF COST.

is very small near the optimum. The end condition required of the inventory was satisfied at the second iteration, and the inventory converged to the required final value.

3.2 An inventory Control Problem with Advertising

3.2.1 The Model

Consider an inventory control model as shown in Fig. 5. With the production rate given, the problem is to balance the cost of advertising and the inventory level for maximum profit. The variables involved are production, inventory, sales and advertising. Production can be considered a function of time. Let production at time t be

$$P(t) = A + bt \tag{46}$$

where A and b are constants.

The rate of change of inventory I(t) is the difference between the production and sales at time t. If Q(t) represents the number of customers at time t and C_q represents the number of times bought by each customer, the rate of change of inventory level may be represented by

$$\frac{dI(t)}{dt} = p(t) - C_q Q(t) \tag{47}$$

To determine the rate of change of customers (informed persons) at time t, a diffusion model incorporating advertising will be used. Consider a group of people in which only a certain number possess a particular piece of information. Suppose that the total number of persons

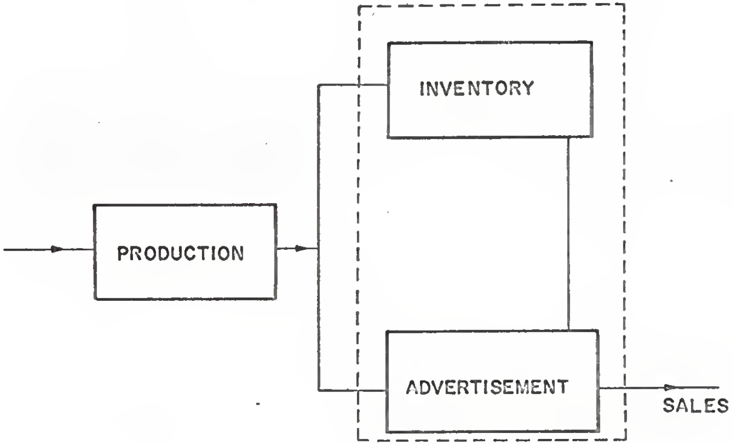

FIG: 5 BLOCK DIAGRAM OF AN INVENTORY CONTROL MODEL WITH ADVERTISEMENT in the group under consideration remains constant and that diffusion of information only occurs through personal contact. The number of contacts made by an average person in an arbitrary unit of time is given by a contact coefficient. In a contact, the contactee receives information if he does not have it; if he already has it; the contact is wasted so far as increasing the number of people who have the information is concerned.

Let

$Q(0)$ = the number of informed participants in the group at time 0.

N = the total number of participants in the group.

$Q(t)$ = the number of informed participants at time t.

$\frac{Q(t)}{N}$ = proportion of informed persons in the group at time t.

$1 - \frac{Q(t)}{N}$ = proportion of uninformed persons in the group at time t.

$C_q Q(t)dt$ = contacts made during a time interval dt.

The increase in the total number of informed persons during a short interval Δt is obtained by multiplying the number of contacts by the proportion of persons who do not possess the information, since only contacts with uninformed persons leads to an increase in the informed members.

$$dQ(t) = C_q Q(t)[1 - \frac{Q(t)}{N}]\Delta t \qquad (48)$$

The differential equations is

$$\frac{dQ(t)}{dt} = C_q Q(t)[1 - \frac{Q(t)}{N}] \tag{49}$$

Suppose that the information in the model given above is about the product of a firm and assume that the firm can influence the number of contacts by spending money for advertising. In particular, it can increase the number of contacts made by the informed persons (above the ones included in C) by an additional number per unit of time. Then

$aQ(t)dt$ = number of additional contacts made in the time interval dt.

Hence the differential cquation becomes

$$\frac{dQ(t)}{dt} = [C + a]\, Q(t)\, [1 - \frac{Q(t)}{N}]\ . \tag{50}$$

The net profit can be obtained from the equation:

Net Profit = Revenue - Cost of holding inventory - Cost of advertisement.

Since $C_q Q(t)$ units are sold at time t, the revenue is $FC_q Q(t)$. If I_m represents the optimal inventory level and C_I represents the inventory cost, the cost due to inventory is $C_I[I_m - I(t)]^2$. The cost of advertising is $C_A Q(t)\, A^2(t)$ and the total net profit over the duration of the process is

$$J = \int_0^{t_f} [FC_q Q(t) - C_I[I_m - I(t)]^2 - C_A Q(t) a^2(t)]dt. \tag{51}$$

3.2.2 <u>The Recurence Equations</u>

The state variables are I(t) and Q(t) and the control variable is the additional contact coefficient a(t). The objective is to maximize the total net profit J. The differential equations representing the process are Eqs. (46), (47) and (49) with the profit represented by J.

To reformulate the problem in terms of the derivations, let $I(t)=x_1(t)$, $Q(t) = x_2(t)$ and $a(t) = \theta(t)$. In addition the following variable can be introduced:

$$x_3(t) = \int_0^t [FC_q Q(t) - C_I[I_m - I(t)]^2 - C_A\, Q(t)a^2(t)]dt \tag{52}$$

or

$$\frac{dx_3}{dt} = [FC_q Q(t) - C_I[I_m - I(t)]^2 - C_A\, Q(t)a^2(t)] \tag{53}$$

with

$$x_3(0) = 0$$

and

$$x_3(t_f) = J \ .$$

The objective now became the maximization of $\phi = x_3(t_f)$ with the terminal condition

$$\psi = t - T = 0 \ .$$

Using Eq. (12), these recursive relationships are obtained:

$$\left.\frac{\partial S}{\partial x_1}\right|_t = \left.\frac{\partial S}{\partial x_1}\right|_{t+\Delta} + \left(\left.\frac{\partial S}{\partial x_1}\right|_{t+\Delta} \left.\frac{\partial f_1}{\partial x_1}\right|_t + \left.\frac{\partial S}{\partial x_2}\right|_{t+\Delta} \left.\frac{\partial f_2}{\partial x_1}\right|_t + \left.\frac{\partial S}{\partial x_3}\right|_{t+\Delta} \left.\frac{\partial f_3}{\partial x_1}\right|_t\right) \Delta \tag{54}$$

or

$$\frac{\partial S}{\partial x_1}\bigg|_t = \frac{\partial S}{\partial x_1}\bigg|_{t+\Delta} + \left(\frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} (2C_I(I_m - x_1))\right) \Delta \tag{55}$$

$$\frac{\partial S}{\partial x_2}\bigg|_t = \frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} + \left(\frac{\partial S}{\partial x_1}\bigg|_{t+\Delta} \frac{\partial f_1}{\partial x_2}\bigg|_t + \frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} \frac{\partial f_2}{\partial x_2}\bigg|_t + \frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} \frac{\partial f_3}{\partial x_2}\bigg|_t\right) \Delta \tag{56}$$

or

$$\frac{\partial S}{\partial x_2}\bigg|_t = \frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} + \left(-\frac{\partial S}{\partial x_1}\bigg|_{t+\Delta} + \frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} (C+\theta)(1 - \frac{2x_2}{N}) + \frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} (F - C_A\theta^2)\right)\Delta \tag{57}$$

and

$$\frac{\partial S}{\partial x_3}\bigg|_t = \frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} + \left(\frac{\partial S}{\partial x_1}\bigg|_{t+\Delta} \frac{\partial f_1}{\partial x_3}\bigg|_t + \frac{\partial S}{\partial x_2}\bigg|_{\Delta} \frac{\partial f_2}{\partial x_3}\bigg|_t + \frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} \frac{\partial f_3}{\partial x_3}\bigg|_t\right) \Delta \tag{58}$$

or

$$\frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} = \frac{\partial S}{\partial x_3}\bigg|_t . \tag{59}$$

Terminal conditions obtained from Eq. (16) are

$$\frac{\partial S}{\partial x_1}\bigg|_T = 0 \tag{60}$$

$$\frac{\partial S}{\partial x_2}\bigg|_T = 0 \tag{61}$$

$$\left.\frac{\partial S}{\partial x_3}\right|_T = 1. \tag{62}$$

To find the recursive relationship for the gradient of the objective function with respect to the control variable, there is the general relationship

$$\left.\frac{\partial S}{\partial \theta}\right|_t = \sum_{i=1}^{n+1} \left.\frac{\partial S}{\partial x_i}\right|_{t+\Delta} \left.\frac{\partial f_i}{\partial \theta}\right|_t \Delta \tag{63}$$

$$= \left(\left.\frac{\partial S}{\partial x_2}\right|_{t+\Delta} (x_2 - \frac{x_2^2}{N}) - \frac{\partial S}{\partial x_3} (2\theta x_2 C_A)\right) \Delta \tag{64}$$

Equations (11) through (21) are the required recursive relationships for the solution of the above problem by the gradient technique.

3.2.3 Computational Procedure

The numerical values used were

$C_I = 0.15$	$A = 70$
$C_A = 1.5$	$b = 100$
$F = 10.0$	$C = 2.0$
$N = 150$	$C_q = 1.0$
$I_m = 50.0$	$\Delta t = .01, \quad T = 1$

The values of $\Delta\phi$ used were: $\Delta\phi = 40$ for first 17 iterations, and $\Delta\phi = .05$ for the remaining iterations. With the initial conditions as

$$x_1(0) = 20.0, \quad x_2(0) = 20.0 \quad \text{and} \quad x_3(0) = 0,$$

the step-by-step procedure followed for obtaining the solution was:

1. Integrate Eqs. (47), (50) and (51).
2. Obtain the end conditions for $\frac{\partial S}{\partial x_1}$, $\frac{\partial S}{\partial x_2}$, $\frac{\partial S}{\partial x_3}$, from Eqs. (60) through (62).
3. Calculate the values of $\frac{\partial S}{\partial x_1}$, $\frac{\partial S}{\partial x_2}$ and $\frac{\partial S}{\partial x_3}$ at the 101 grid points by means of backward recursion of Eqs. (55) through (59).
4. Calculate the gradient of the control variable $\frac{\partial S}{\partial \theta}$ at the 101 grid points by means of backward recursion of Eq. (64).
5. Calculate the improvement in the control variable θ by the relationship

$$\theta_{new} = \theta_{old} + \Delta\phi \frac{\partial S/\partial \theta}{\sum_{t=0}^{T} (\frac{\partial S}{\partial \theta})^2}$$

6. Repeat Steps (1) through (5) till no further improvement could be made.

3.2.4 Discussion of Results

Using an initial guess of 2.5 for the advertising, the control variable converged to the optimal in 80 iterations. The convergence rate of advertising is shown in Fig. 6. The profit function had a value of 25.0 with the assumed controls and converged to the optimal value of 580.0 in 80 iterations. It is seen that the rate of convergence during the initial stages of iteration was much faster than during the final stages. The reason for this is that the gradient is small and hence improvement is also small at the final stages of iteration. The trajectories of the

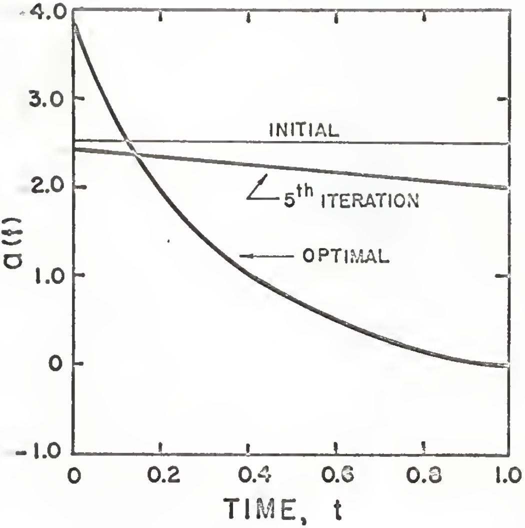

FIGURE 6. CONVERGENCE RATE OF a(t).

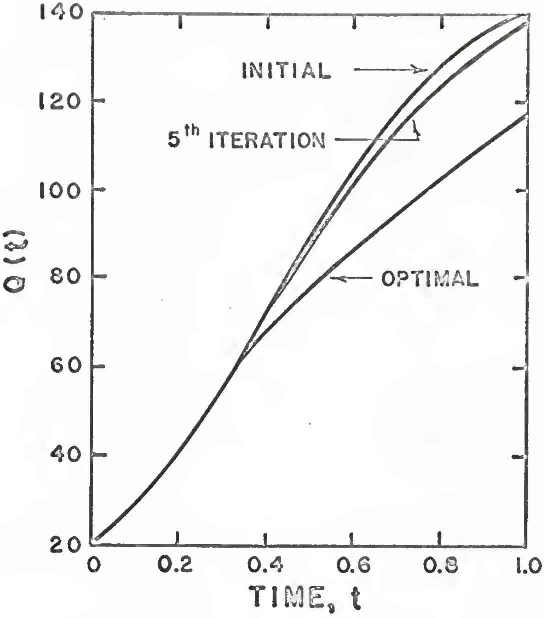

FIGURE 7. CONVERGENCE RATE OF Q(t).

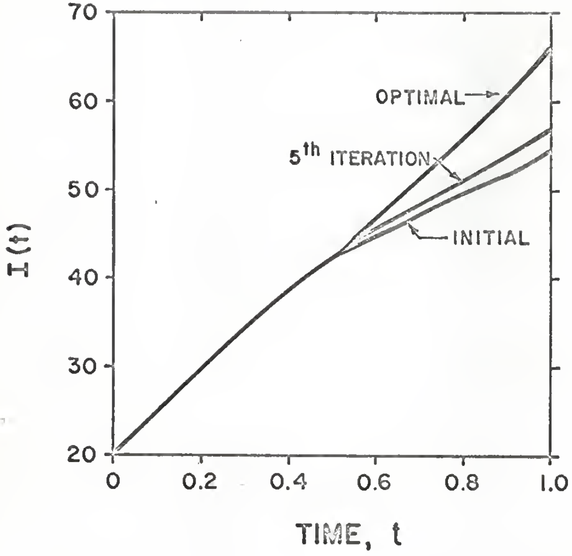

FIG: 8 CONVERGENCE RATE OF I(t)

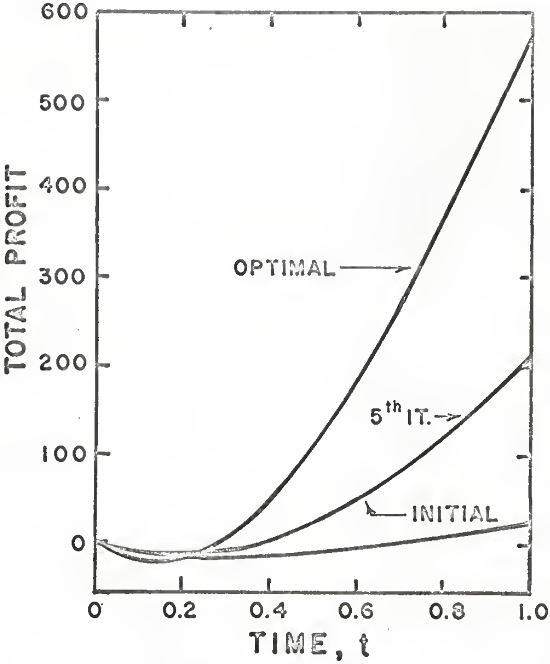

FIGURE 9. CONVERGENCE RATE OF PROFIT.

state and control variables are given in Figs. 6 through 9.

3.3.A Production, Inventory Control and Advertising Problem

3.3.1 The Model

Consider the manufacturing process shown in Fig. 10. The raw material is fed into two reactors in series and the Arrhenius reaction rate expression

$$k = G \exp\left(-\frac{E}{RT}\right)$$

will be used for the rate constants. The reactions are first order and can be expressed as

$$A \xrightarrow{k} B \xrightarrow{k} C$$

where k is the reaction rate constant, G is the frequency factor constant, E is the activation energy of the reaction, R is the gas constant and T is the temperature. Material B is the desired product for which inventory and advertising are assumed.

The transformation equations for the two reactions are

$$V_1 \frac{dx_1}{dt} = q(x_0 - x_1) - V_1 k_{a_1} x_1 \tag{65}$$

$$V_1 \frac{dy_1}{dt} = q(y_0 - y_1) - V_1 k_{b_1} y_1 + V_1 k_{a_1} V_1 \tag{66}$$

$$V_2 \frac{dx_2}{dt} = q(x_1 - x_2) - V_2 k_{a_2} x_2 \tag{67}$$

$$V_2 \frac{dy_2}{dt} = q(y_1 - y_2) - V_2 k_{b_2} y_2 + V_2 k_{a_2} x_2 \tag{68}$$

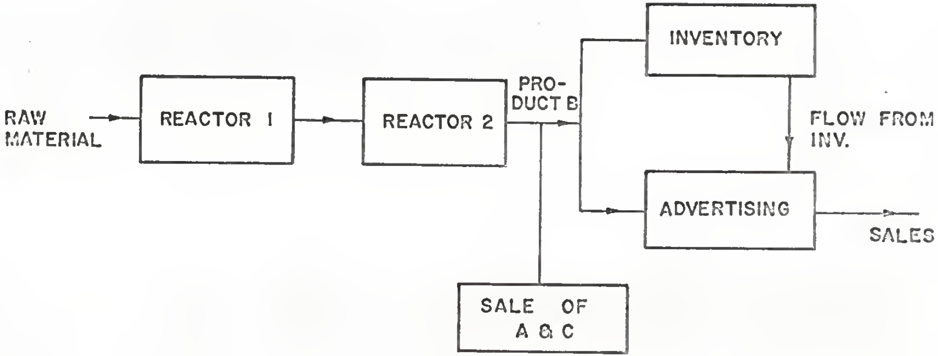

FIG: 10 BLOCK DIAGRAM OF THE MODEL

where V_1 and V_2 are the volumes of the first and second reactors, respectively, q is the flow rate, k_{a_i} and k_{b_i} represent the reaction rate for the first and second reactions, respectively, and x and y are the concentrations of A and B, respectively.

If C_q represents the number of items bought by each informed person, the change in inventory I(t) can be represented by the differential equation

$$\frac{dI(t)}{dt} = qy_2 - C_q\, K(t) \tag{69}$$

where K(t) is the number of informed persons at time t.

To determine the sales, the diffusion model of advertising discussed in Section 3.2 will be used and the differential equation is

$$\frac{dK(t)}{dt} = [C + a(t)]\, K(t)\, [1 - \frac{K(t)}{N}] \tag{70}$$

The profit equation can be written as

Net Profit = Income from sales of A + Income from sales of B
+ Income from sales of C - Cost of Inventory
- Cost of Advertising - Cost of Production

If I_m represents the optimal inventory level and T_{1m} the feed temperature, the profit equation can be written as

$$J = \int_0^{t_f} [C_1C_qK(t) + C_2qx_2 + C_3q(1-x_2-y_2) - C_I(I_m-I(t))^2 - C_A[a(t)K(t)]^2$$

$$- C_T[T_{1m}-T_1)^2 + (T_1-T_2)^2]]dt.$$

where C_1, C_2, C_3, C_I, C_A and C_T are the per unit costs of B, A, C, inventory, advertising, and production, respectively.

Introducing an additional state variable x_5 now gives

$$x_5(t) = \int_0^t [C_1 C_q K(t) + C_2 q x_2(t) + C_3 q(1-x_2-y_2) - C_I(I_m-x_3)^2 - C_a(\theta_3 x_4)^2$$

$$- C_T[(T_{1m}-T_1)^2 + (T_1-T_2)^2]]dt \qquad (71)$$

with $x_5(0) = 0$ and $x_5(t_f) = J$. Representing $K(t)$ by $x_4(t)$ and $I(t)$ by $x_3(t)$, a problem involving six state variables, x_1, y_1, x_2, y_2, x_3 and x_4 and three control variables, T_1, T_2 and $a(t)$, is involved. Let $V_1 = V_2 = V$; now the differential equations representing the process may be summarized as:

$$\frac{dx_1}{dt} = \frac{q(x_0 - x_1)}{V} - k_{a_1} x_1$$

$$\frac{dy_1}{dt} = \frac{q(y_0 - y_1)}{V} - k_{b_1} y_1 + k_{a_1} x_1$$

$$\frac{dx_2}{dt} = \frac{q(x_1 - x_2)}{V} - k_{a_2} x_2$$

$$\frac{dy_2}{dt} = \frac{q(y_1 - y_2)}{V} - k_{b_2} y_2 + k_{a_2} x_2$$

$$\frac{dx_3}{dt} = qy_2 - C_q x_4$$

$$\frac{dx_4}{dt} = (C + \theta_3)[x_4 - \frac{x_4^2}{N}]$$

$$\frac{dx_5}{dt} = C_1 C_q x_4 + C_2 q x_2 + C_3 q(1-x_2-y_2) - C_I(I_m-x_3)^2 - C_a(\theta_3 x_4)^2$$

$$- C_T[(T_{1m}-T_1)^2 + (T_1-T_2)^2]$$

with the given initial conditions $x_1(0) = x_1^0$, $y_1(0) = y_1^0$, $x_2(0) = x_2^0$, $y_2(0) = y_2^0$, $x_3(0) = x_3^0$, $x_4(0) = x_4^0$ and $x_5(0) = 0$.

The problem is to maximize $\phi = x_5(T)$ subject to the end condition constraint $\psi = t - T = 0$.

3.3.2 The Recurence Equations

For the end condition of the slope of the objective function with respect to the state variables, this equation exists:

$$\left.\frac{\partial S}{\partial x_j}\right|_T = \left.\frac{\partial \phi}{\partial x_j}\right|_T - \left.\left(\frac{\partial \phi}{\partial t} \Big/ \frac{\partial \psi}{\partial t}\right)\right|_T \left.\frac{\partial \psi}{\partial x_j}\right|_T \qquad j = 1, 2, \ldots, 7.$$

For the present case, then

$$\left.\frac{\partial S}{\partial x_1}\right|_T = 0 \tag{72}$$

$$\left.\frac{\partial S}{\partial y_1}\right|_T = 0 \tag{73}$$

$$\left.\frac{\partial S}{\partial x_2}\right|_T = 0 \tag{74}$$

$$\left.\frac{\partial S}{\partial y_1}\right|_T = 0 \tag{75}$$

$$\left.\frac{\partial S}{\partial x_3}\right|_T = 0 \tag{76}$$

$$\left.\frac{\partial S}{\partial x_4}\right|_T = 0 \tag{77}$$

and

$$\left.\frac{\partial S}{\partial x_5}\right|_T = 1. \tag{78}$$

The recursive relationship as derived in Chapter II are:

$$\left.\frac{\partial S}{\partial x_j}\right|_t = \left.\frac{\partial S}{\partial x_j}\right|_{t+\Delta} + \sum_{i=1}^{n+1} \left.\frac{\partial S}{\partial x_i}\right|_{t+\Delta} \left.\frac{\partial f_i}{\partial x_j}\right|_t \Delta$$

Using this equation for the state variables, then

$$\left.\frac{\partial S}{\partial x_1}\right|_t = \left.\frac{\partial S}{\partial x_1}\right|_{t+\Delta} + \left[\left.\frac{\partial S}{\partial x_1}\right|_{t+\Delta} \left.\left(-\frac{q}{V_1} - k_{a_1}\right)\right|_t \right.$$

$$\left. + \left.\frac{\partial S}{\partial y_1}\right|_{t+\Delta} \left.(k_{a_1})\right|_t + \left.\frac{\partial S}{\partial x_2}\right|_{t+\Delta} (q/V_2)\right] \Delta \tag{79}$$

$$\left.\frac{\partial S}{\partial y_1}\right|_t = \left.\frac{\partial S}{\partial y_1}\right|_{t+\Delta} + \left[\left.\frac{\partial S}{\partial y_1}\right|_{t+\Delta} \left.(-q/V_1 - k_{b_1})\right|_t \right.$$

$$\left. + \left.\frac{\partial S}{\partial y_2}\right|_{t+\Delta} (q/V_2)\right] \Delta \tag{80}$$

$$\frac{\partial S}{\partial x_2}\bigg|_t = \frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} + \left\{\frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} (- q/V_2 - k_{a_2})\bigg|_t\right.$$

$$\left. + \frac{\partial S}{\partial y_2}\bigg|_{t+\Delta} (k_{a_2})\bigg|_t + \frac{\partial S}{\partial x_5}\bigg|_{t+\Delta} (C_2 q - C_3 q)\right\} \Delta \qquad (81)$$

$$\frac{\partial S}{\partial y_2}\bigg|_t = \frac{\partial S}{\partial y_2}\bigg|_{t+\Delta} q + \frac{\partial S}{\partial x_5}\bigg|_{t+\Delta} (- q/V_2 - k_{b_2})\bigg|_t$$

$$+ \frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} q + \frac{\partial S}{\partial x_5}\bigg|_{t+\Delta} (- C_3 q)\Big\} \;\Delta \qquad (82)$$

$$\frac{\partial S}{\partial x_3}\bigg|_t = \frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} + \left\{\frac{\partial S}{\partial x_5}\bigg|_{t+\Delta} (2C_I(I_m - x_3))\bigg|_t\right\} \Delta \qquad (83)$$

$$\frac{\partial S}{\partial x_4}\bigg|_t = \frac{\partial S}{\partial x_4}\bigg|_{t+\Delta} + \left\{- \frac{\partial S}{\partial x_3}\bigg|_{t+\Delta} C_q + \frac{\partial S}{\partial x_4}\bigg|_{t+\Delta} (C + \theta_3)\right.$$

$$\left.(1 - \frac{2x_4}{N}) + \frac{\partial S}{\partial x_5}\bigg|_{t+\Delta} (C_1 C_q - 2C_a \theta_3^2 x_4)\right\} \Delta \qquad (84)$$

$$\frac{\partial S}{\partial x_5}\bigg|_t = \frac{\partial S}{\partial x_5}\bigg|_{t+\Delta} . \qquad (85)$$

The recurence relation for the control variables are:

$$\frac{\partial S}{\partial \theta_j}\bigg|_t = \sum_{i=1}^{n+1} \frac{\partial S}{\partial x_i}\bigg|_{t+\Delta} \frac{\partial f_i}{\partial \theta_j}\bigg|_t \;\Delta$$

Applying this equation to Eqs. (1) through (7) gives

$$\frac{\partial S}{\partial T_1}\bigg|_t = \left[\frac{\partial S}{\partial x_1}\bigg|_{t+\Delta} \left(- x_1 \frac{k_{a_1}}{T_1}\right)\bigg|_t + \frac{\partial S}{\partial y_1}\bigg|_{t+\Delta}\right.$$

$$\left(x_1 \frac{\partial k_{a_1}}{\partial T_1} - y_1 \frac{\partial k_{b_1}}{\partial T_1}\right)\bigg|_t + \frac{\partial S}{\partial x_5}\bigg|_{t+\Delta}$$

$$\left.\left(- C_T\left(- 2\ (T_{1m} - T_1)\big|_t + 2(T_1 - T_2)\big|_t\right)\right)\right] \Delta \tag{86}$$

$$\frac{\partial S}{\partial T_2}\bigg|_t = \left[\frac{\partial S}{\partial x_2}\bigg|_{t+\Delta} \left(- x_2 \frac{\partial k_{a_2}}{\partial T_2}\right)\bigg|_t + \frac{\partial S}{\partial y_2}\bigg|_{t+\Delta} \left(x_2 \frac{\partial k_{a_2}}{\partial T_2}\right.\right.$$

$$\left.\left. - y_2 \frac{\partial k_{b_2}}{\partial T_2}\right)\bigg|_t + \frac{\partial S}{\partial x_5}\bigg|_{t+\Delta} \left(- C_T\left(- 2(T_1 - T_2)\big|_t\right)\right)\right] \Delta \tag{87}$$

and

$$\frac{\partial S}{\partial \theta_3}\bigg|_t = \left[\frac{\partial S}{\partial x_4}\bigg|_{t+\Delta} \left(x_4 - \frac{x_4^2}{N}\right) + \frac{\partial S}{\partial x_5}\bigg|_{t+\Delta} \left(-2\ C_{a\ 3} x_4^2\right)\big|_t\right] \Delta\ . \tag{88}$$

The improvement in the control is given by

$$\theta_{j_{new}}(t) = \theta_{j_{old}}(t) + \frac{\Delta\phi_j \dfrac{\partial S}{\partial \theta_j}\bigg|_t}{\sum\limits_{t=0}^{T} \dfrac{\partial S}{\partial \theta_j}\bigg|_t} \qquad j = 1, 2, 3 \tag{89}$$

where $\Delta\phi_j$ is the desired improvement in the objective function due to the jth control.

Equations (65) through (89) are the desired equations for the solution of the problem by the gradient technique. It should be noted that since the decision vector is multidimenstional, individual improvements $\Delta\phi_j$ are suggested for each control.

3.3. Numerical Results

Based on the model stated above, the problem was solved using different parameters and different starting values. The parameters and starting values used are summarized in Table 1. The initial conditions for the seven state variables used are shown in Table 2. The results are discussed in the following sections for each of the problems shown in Table 1.

Problem 1a: It can be seen from Fig. 11 that the optimal temperature profile for T_1 had a value of about 362°K at time t_0 and 339°K at t_f. The concentration of A in Fig. 12 fell to a value of .458 at time 0.65 and raised to 0.484 at t_f. The concentration of B in Fig. 13 arises to 0.465 at t=0.65 and falls to 0.454 at t_f. The profit as can be seen from Fig. 20, with the initial controls was $72.04 and at the 20th iteration it was to $97.05, an increase of 34.5%. However, after the 20th iteration it took another 200 iterations for the profit to reach its optimal value of $107.04. This slow increase of only 1.0% in 200 iterations was due to the slow convergence rate near the optimal. At the 20th iteration the sum of $(\frac{\partial S}{\partial T_1})$ was 0.4×10^{-3}. This sum was 0.2×10^{-5} for the optimal profile. Since the value of the gradient is very small at this point, any further improvement was not significant. For the control T_2, the sum of $(\frac{\partial S}{\partial T_2})^2$ at the initial iteration was 0.3×10^{-4}. At the 20th it was 0.6×10^{-3} and at the optimal it was 0.3×10^{-4}. It can be seen in this case that the gradient at the

Table 1. Parameters and Initial Approximations

INITIAL GUESS	PARAMETERS	Prob. 1a	Prob. 1b	Prob. 2
	R	2.0	2.0	2.0
	q	60.0	60.0	60.0
	V_1	12.0	12.0	12.0
	G_a	$.535 \times 10^{11}$	$.535 \times 10^{11}$	$.535 \times 10^{11}$
	E_a	18000.0	18000.0	18000.0
	V_2	12.0	12.0	12.0
	G_b	$.461 \times 10^{18}$	$.461 \times 10^{18}$	$.461 \times 10^{18}$
	E_b	30000.0	30000.0	30000.0
	C_q	1.0	1.0	1.0
	C	1.0	1.0	1.0
	N	100.0	100.0	100.0
	C_1	5.0	5.0	5.0
	C_2	0.0	0.0	0.0
	C_3	0.0	0.0	0.0
	C_I	1.0	1.0	1.0
	C_A	0.0002	0.0002	0.01
	C_T	0.005	0.005	0.005
	x_0	0.53	0.53	0.53
	y_0	0.43	0.43	0.43
	T_{1m}	$340.0^{\circ}K$	$340.0^{\circ}K$	$340.0^{\circ}K$
T_1		$330.0^{\circ}K$	$345.0^{\circ}K$	$345.0^{\circ}K$
T_2		$330.0^{\circ}K$	$345.0^{\circ}K$	$345.0^{\circ}K$
a		8.0	10.0	3.0

Table 2. Initial Conditions for the State Variables

Variable	Prob. 1a	Prob. 1b	Prob. 2
$x_1(0)$	0.53	0.53	0.53
$y_1(0)$	0.43	0.43	0.43
$x_2(0)$	0.53	0.53	0.53
$y_2(0)$	0.43	0.43	0.43
$x_3(0)$	8.00	8.00	8.00
$x_4(0)$	0.10	0.10	1.00
$x_5(0)$	0.00	0.00	0.00

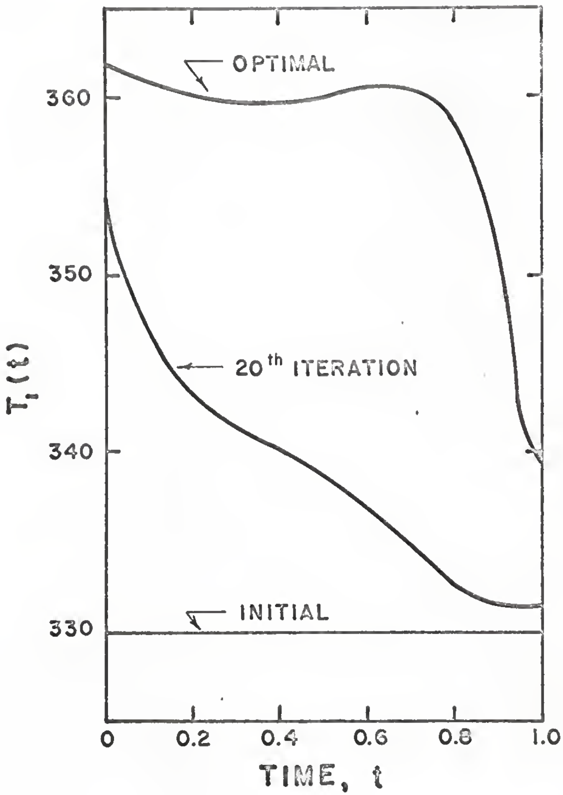

FIGURE 11. CONVERGENCE RATE IN PROBLEM 1a.

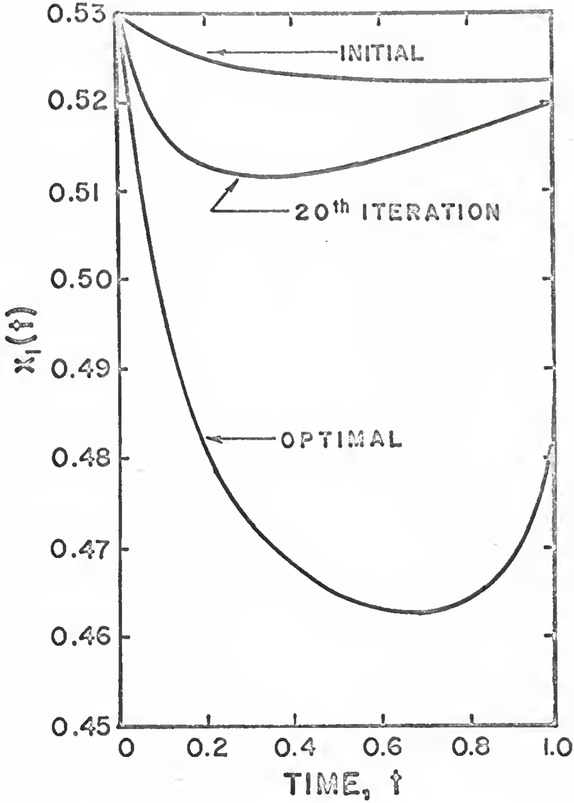

FIGURE 12. CONVERGENCE RATE OF x_1 IN PROBLEM 1a.

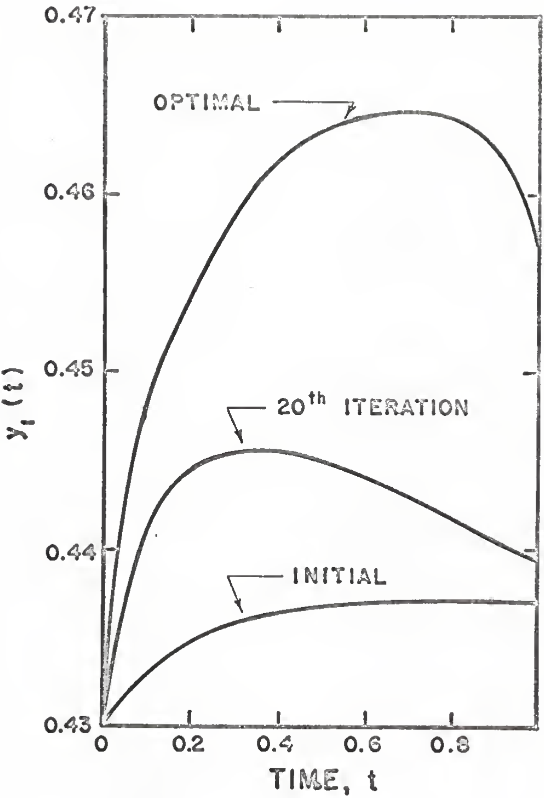

FIGURE 13. CONVERGENCE RATE OF y_1 IN PROBLEM 1a.

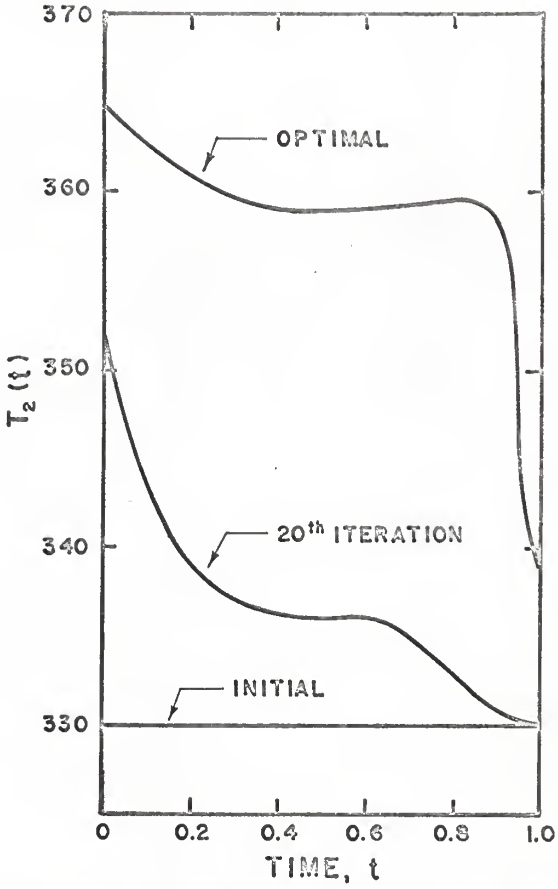

FIGURE 14. CONVERGENCE RATE TF T_2 IN PROBLEM 1a.

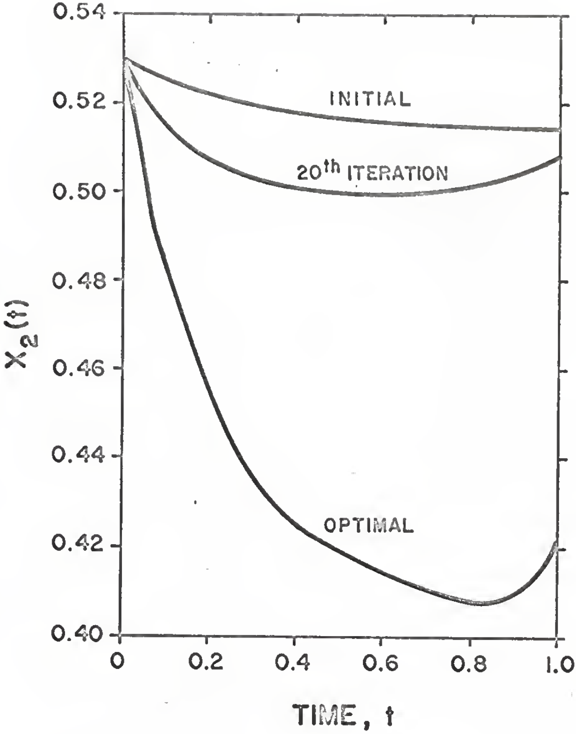

FIG: 15 CONVERGENCE RATE OF X_2 IN PROBLEM Ia.

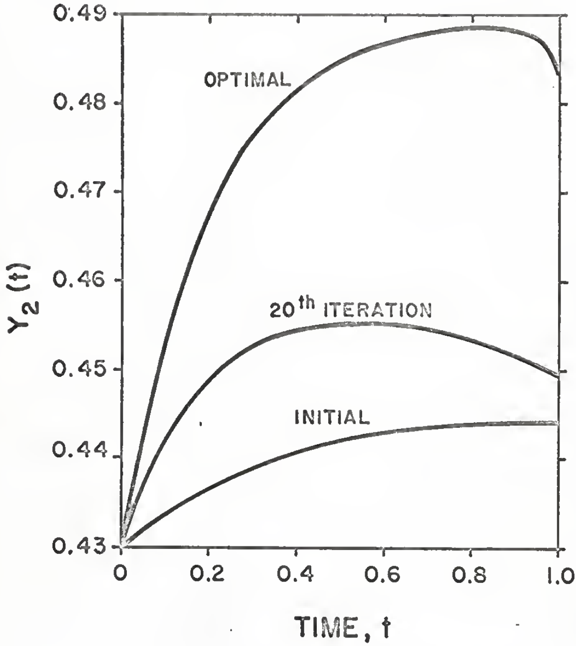

FIG: 16 CONVERGENCE RATE OF Y_2 IN PROBLEM 1a.

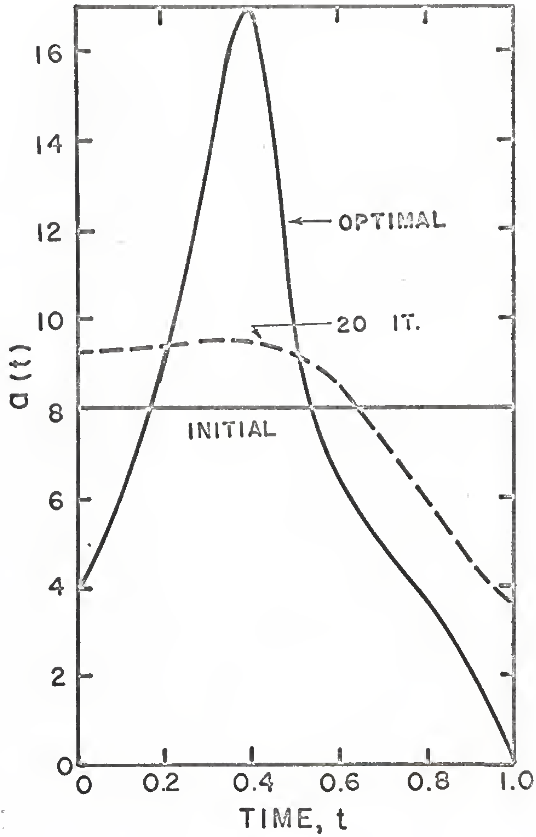

FIGURE 17. CONVERGENCE RATE OF a IN PROBLEM 1a.

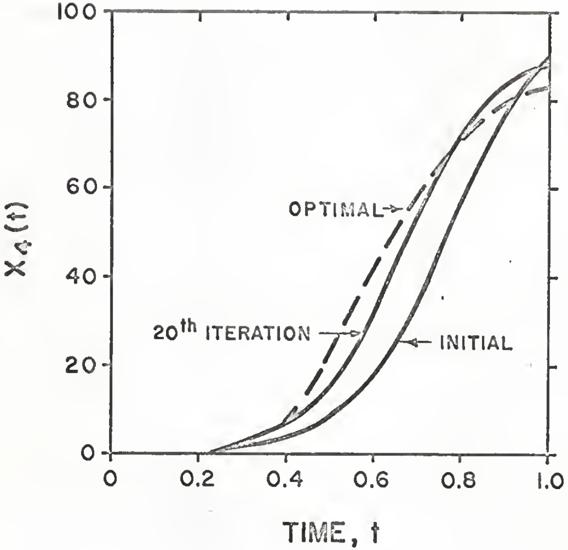

FIG: 18 CONVERGENCE RATE OF X_4 IN PROBLEM Ia.

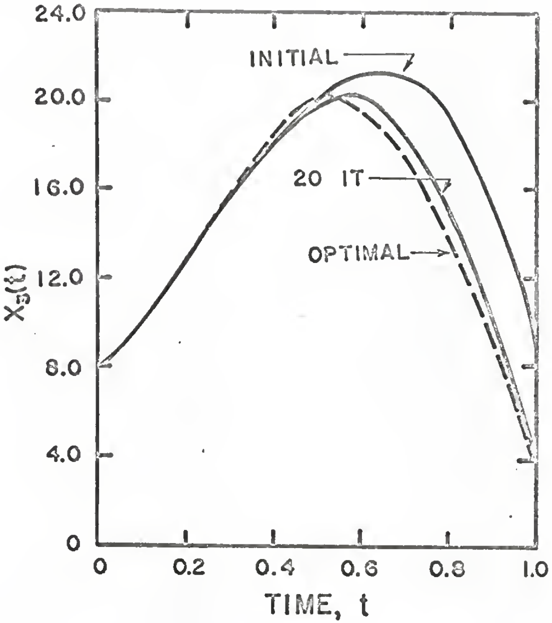

FIGURE 19. CONVERGENCE RATE OF X_3 IN PROBLEM 1a

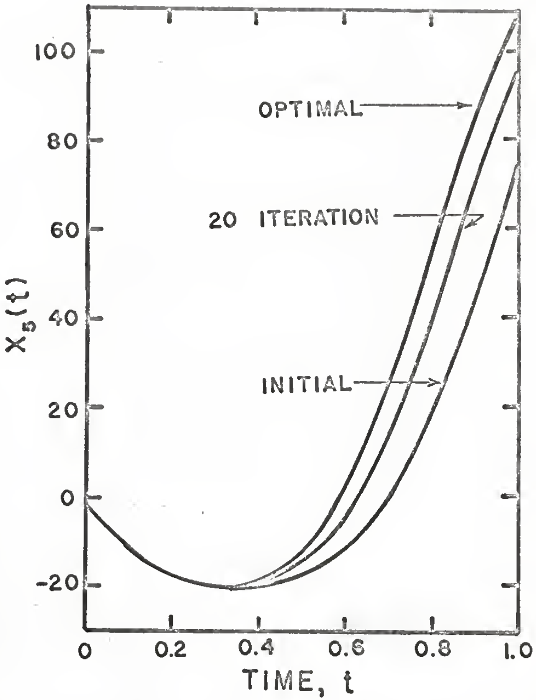

FIGURE 20. CONVERGENCE RATE OF PROFIT IN PROBLEM 1a.

20th iteration was steeper than the initial gradient and the gradient at the optimal was not very different from the gradient at the initial guessed control. Since the problem was to optimize the process with respect to the three controls, it was not necessary to have the same gradient for all the controls. The problem was to adjust the step size for each control so that the optimum could be obtained with the minimum amount of computation.

In the case of the advertising, (please refer to Fig. 17) the gradient at the initial iteration was 3.0. This indicates that the initial guess was fairly far from the optimum and is also evident from the plot in Fig. 17. At the 20th iteration the gradient was 0.2; at the optimal it was 0.3×10^{-1}. From Fig. 18 it is seen that the major change in the number of informed persons occurred during the initial stages of the process. This fact is also evident from Fig. 17 where the additional contact coefficient rises to 16.95 at $t = 0.4$ and then falls to a value of 0 at the final time. It should be noted that a non-negativity constraint was used for the additional contact coefficient. The plots of temperature and concentrations of products A and B versus time in the second reactor can be seen from Fig. 14 through 16.

<u>Problem 1b</u>: In this problem, the same parameters were used as in problem 1a except that different starting values were used for the controls.

From Fig. 30 it is seen that the profit at the initial iteration was \$55.90 and at the 20th iteration it increased to \$80.50, an increase of 45%. However, after the 20th iteration it required 180 further iterations to improve the profit to \$107.16, which is the same result as in problem 1a. The percentage improvement was only 26% in 180 iterations.

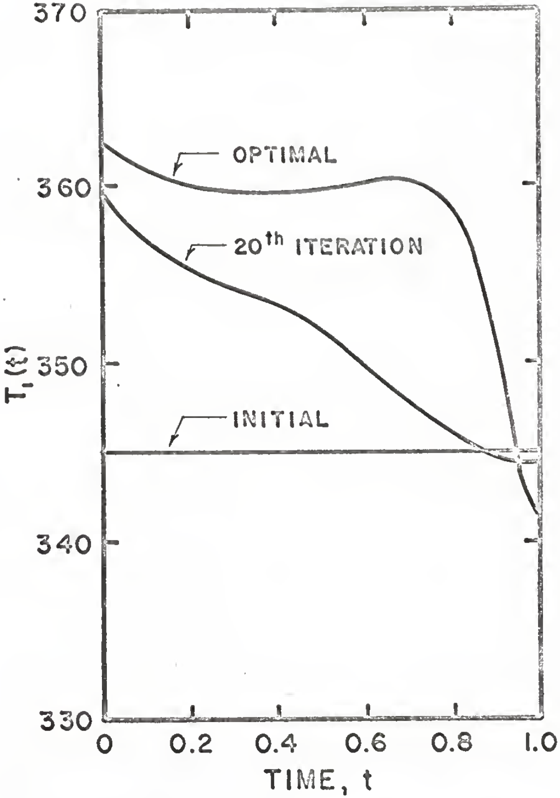

FIGURE 21. CONVERGENCE RATE OF T_1 IN PROBLEM 1b.

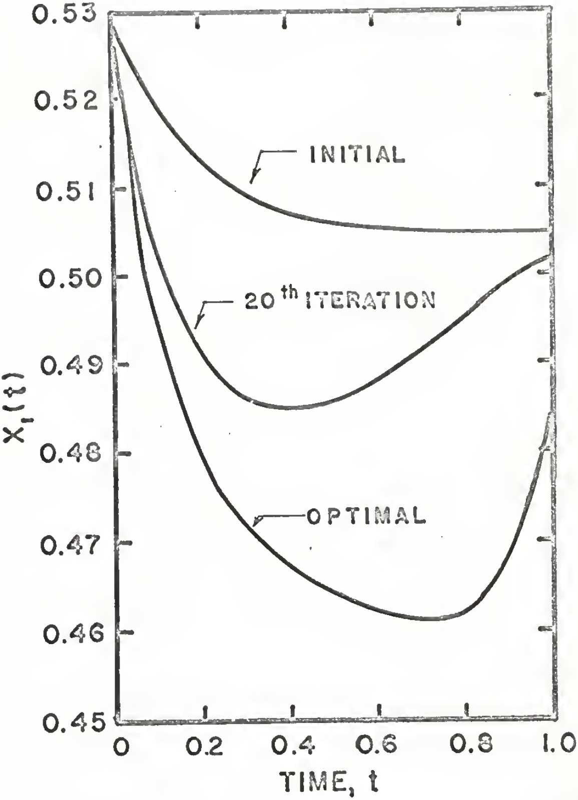

FIGURE 22. CONVERGENCE RATE OF X_1 IN PROBLEM 1b.

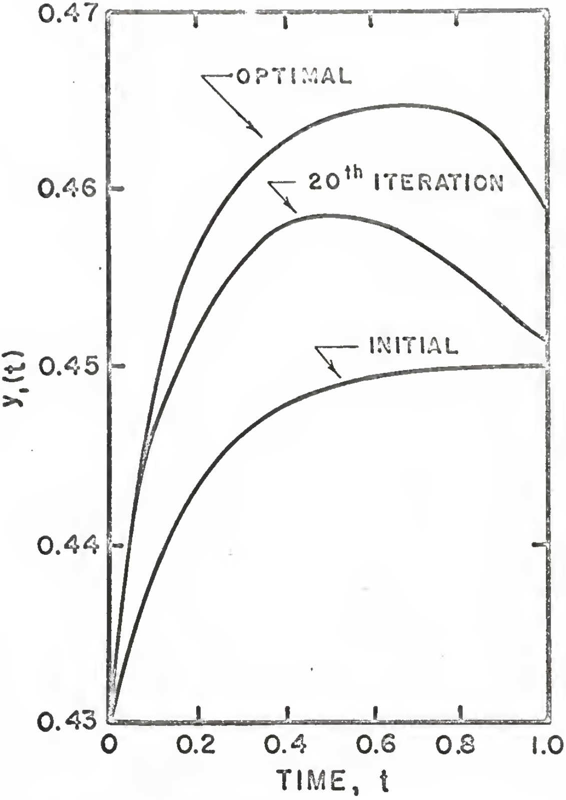

FIGURE 23. CONVERGENCE RATE OF y_1 IN PROBLEM 1b.

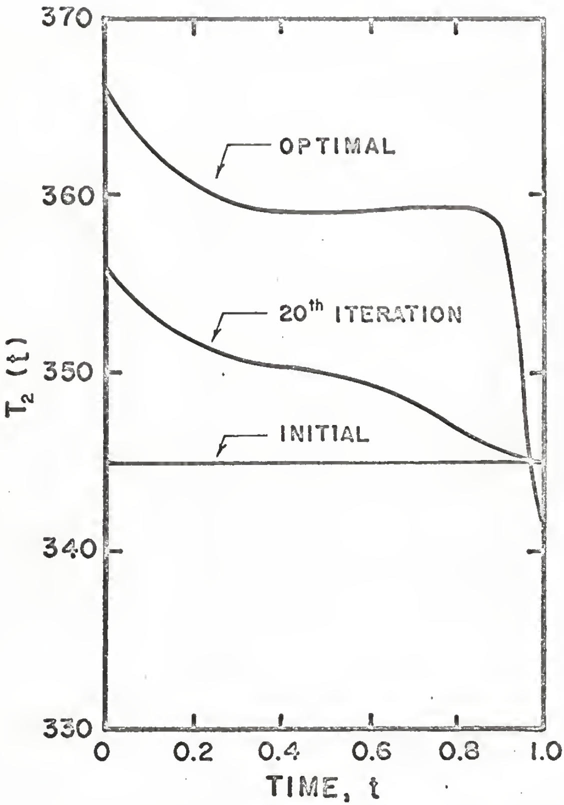

FIGURE 24. CONVERGENCE RATE OF T_2 IN PROBLEM 1b.

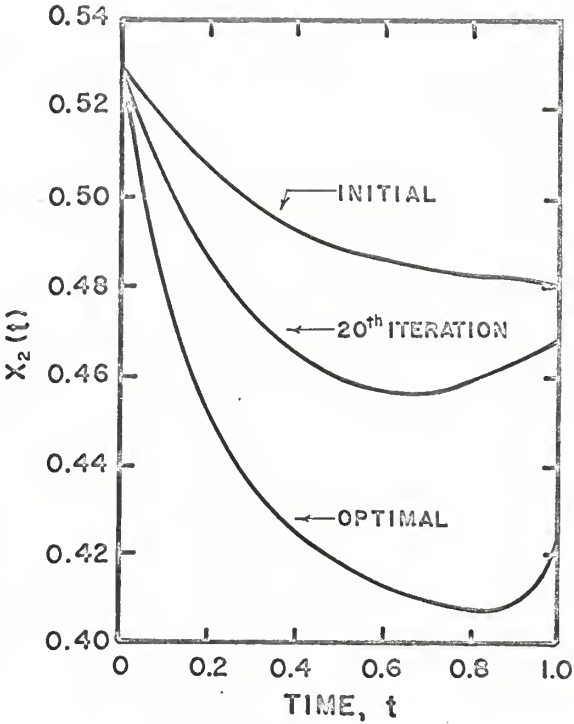

FIGURE 25. CONVERGENCE RATE OF X_2 IN PROBLEM 1b.

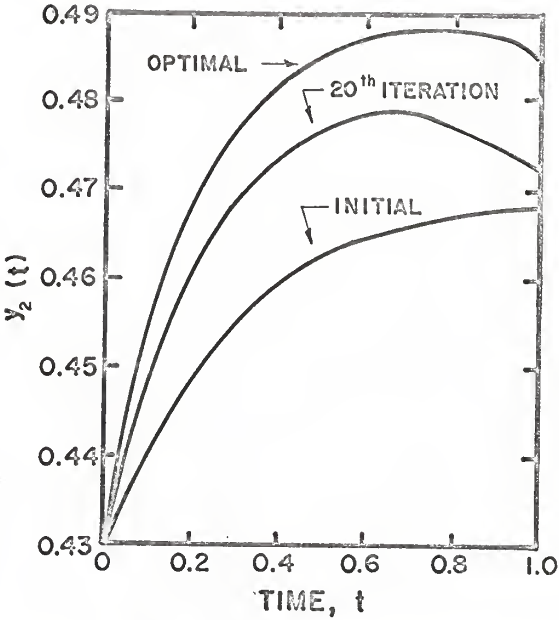

FIGURE 26. CONVERGENCE RATE OF y_2 IN PROBLEM 1b.

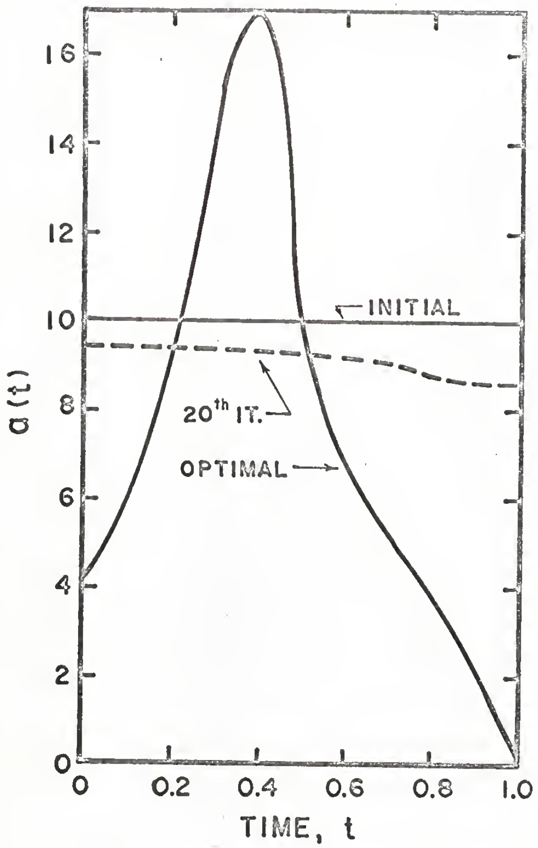

FIGURE 27. CONVERGENCE RATE OF a IN PROBLEM 1b.

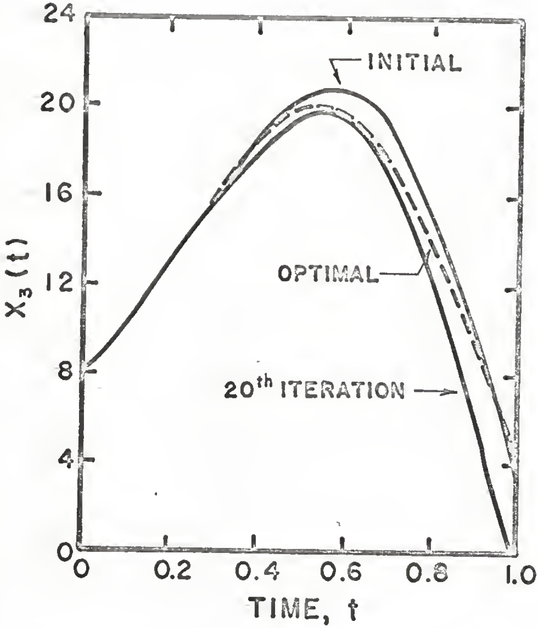

FIGURE 28. CONVERGENCE RATE OF X_3 IN PROBLEM 1b.

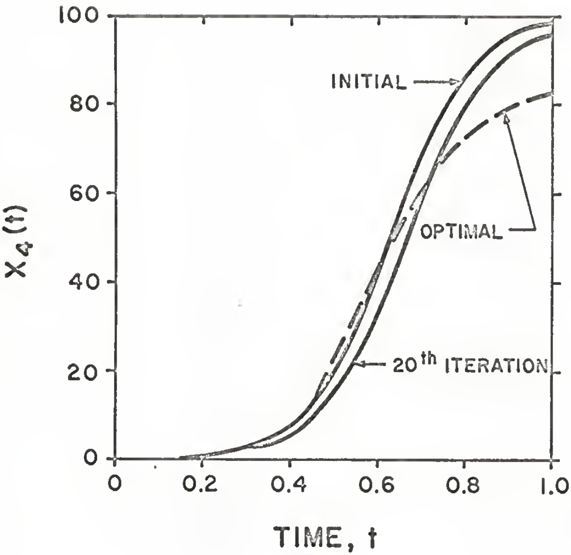

FIG: 29 CONVERGENCE RATE OF Q(t) IN PROBLEM Ib.

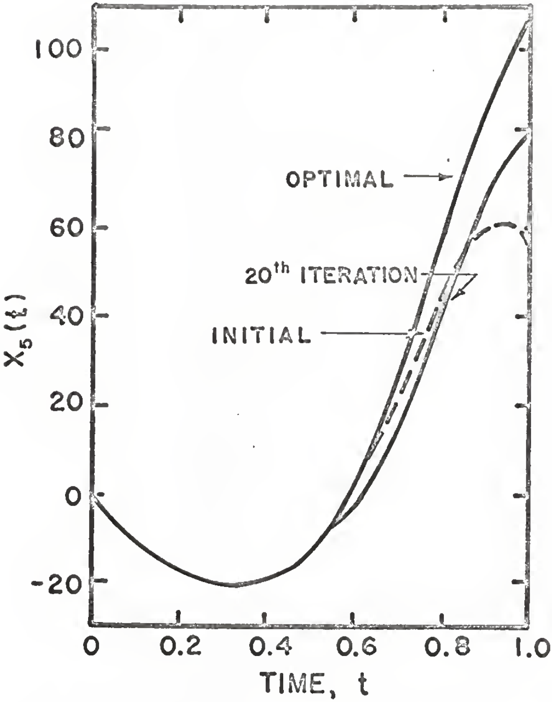

FIGURE 30. CONVERGENCE RATE OF PROFIT IN PROBLEM 1b.

The concentration of B in the second reactor as can be seen from Fig. 26 is seen to converge quite rapidly to a point close to the optimal. However, the rate of convergence from this point to the optimal is very slow.

For the temperature in the first reactor, the value of the sum of $(\frac{\partial S}{\partial T_1})^2$ at the initial iteration was 0.9×10^{-3}; this value was 0.2×10^{-5} at the optimum. A value of $\Delta\phi$ equal to 0.1 was used for the first fifty iterations and values of $\Delta\phi = 0.01$ and $\Delta\phi = 0.001$ were used for 51 to 99 and 100 to 150, respectively. Thereafter the value of $\Delta\phi$ was successively reduced and the optimal was reached with a $\Delta\phi$ value of 0.001 for T_1. As has been stated previously, the initial guess for advertising was far from the optimal. This necessitated the use of $\Delta\phi$ = 1.0 for the first 70 iterations; thereafter $\Delta\phi = 0.5$ was used for iterations 71 - 100 and $\Delta\phi = 0.01$ was used for iterations 101 through 150. The value of $\Delta\phi$ for the temperature in the second reactor was 0.1 for the first fifty iterations and an optimal was reached with $\Delta\phi = 0.001$.

The factor that governs the choice of a proper value for $\Delta\phi$ is the relative position of the current control with respect to the optimal control. In nearly all practical situations, the proper choice of $\Delta\phi$ must be obtained by a trial and error procedure. However, if a certain extimate for the location of the optimal control exists, the required computation to obtain the optimal could be greatly reduced. The temperature and concentration profiles in the two reactors are shown in Figures 21 through 29.

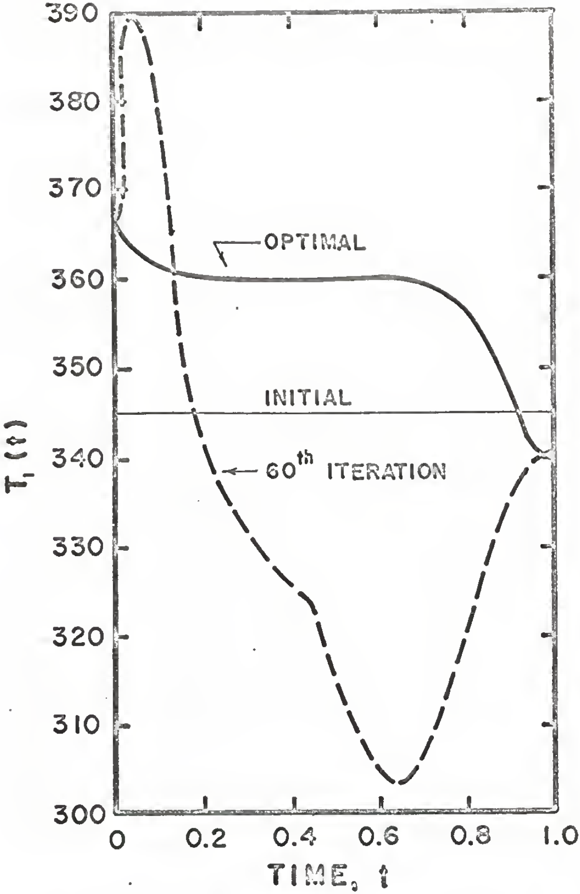

FIGURE 31. CONVERGENCE RATE OF T_1 IN PROBLEM 2.

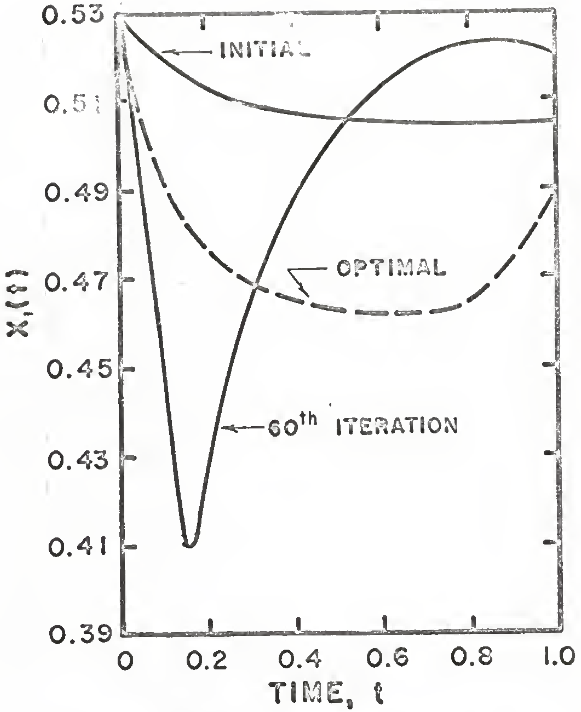

FIGURE 32 CONVERGENCE RATE OF X_1 IN PROBLEM 2.

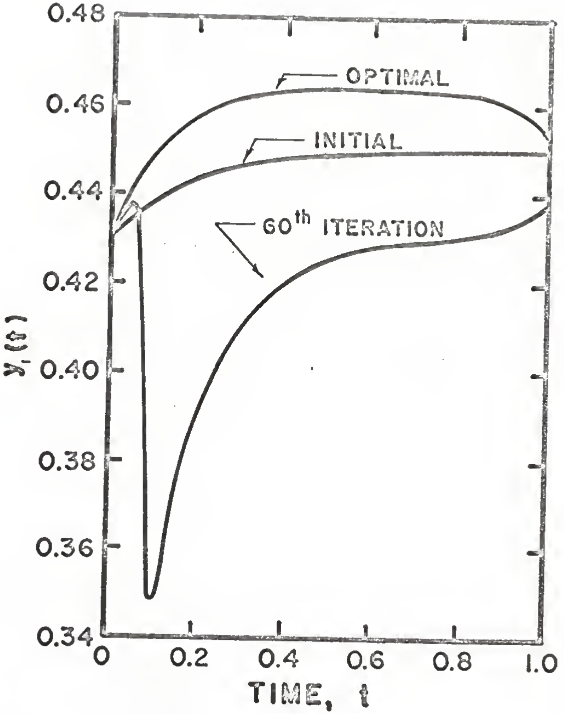

FIGURE 33. CONVERGENCE RATE OF y_1 IN PROBLEM 2.

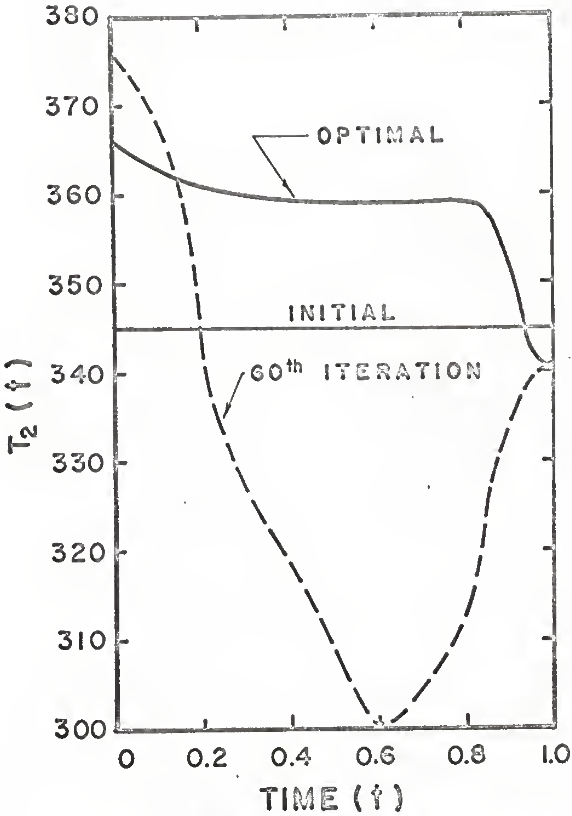

FIGURE 34. CONVERGENCE RATE OF T_2 IN PROBLEM 2.

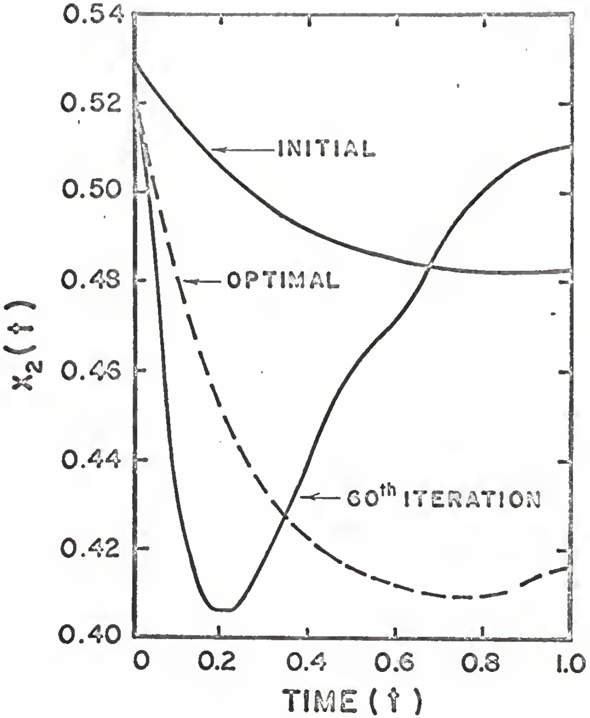

FIGURE 35. CONVERGENCE RATE OF x_2 IN PROBLEM 2.

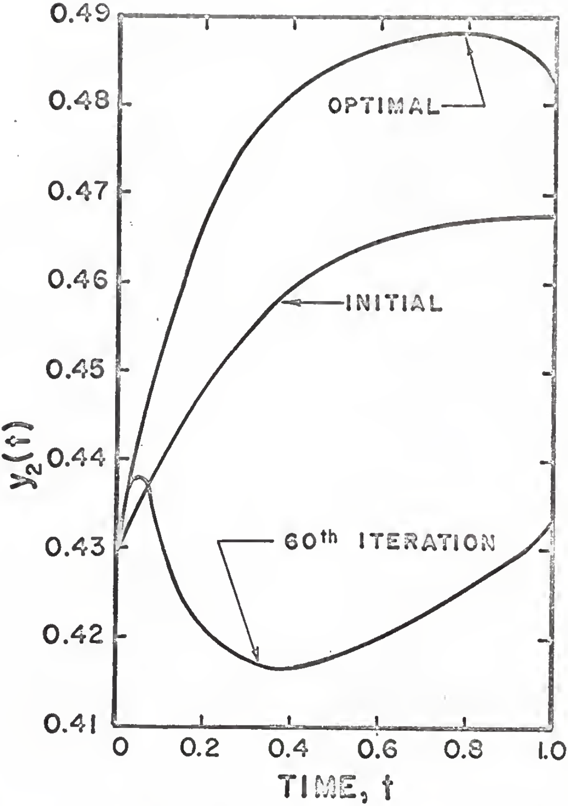

FIGURE 36. CONVERGENCE RATE OF y_2 IN PROBLEM 2.

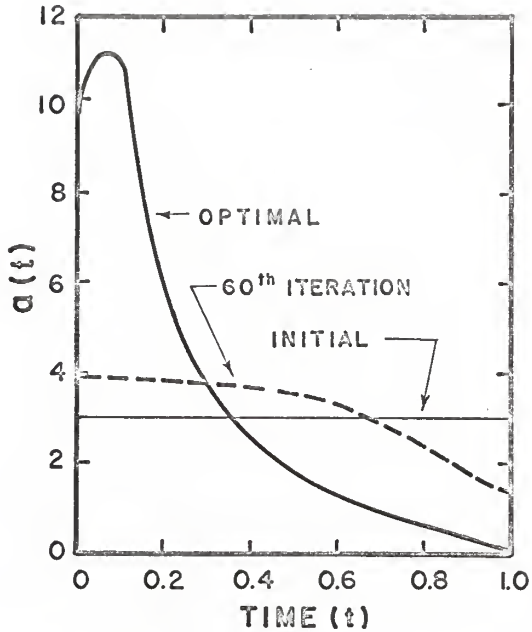

FIGURE 37. CONVERGENCE RATE OF a FOR PROBLEM 2.

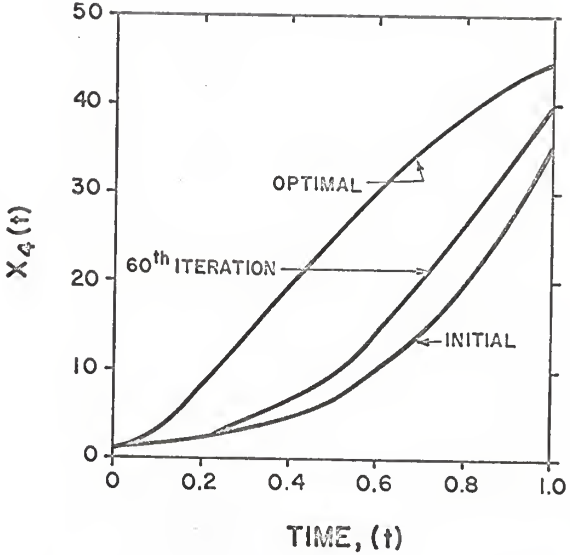

FIG: 38 CONVERGENCE RATE OF Q(t) IN PROBLEM 2.

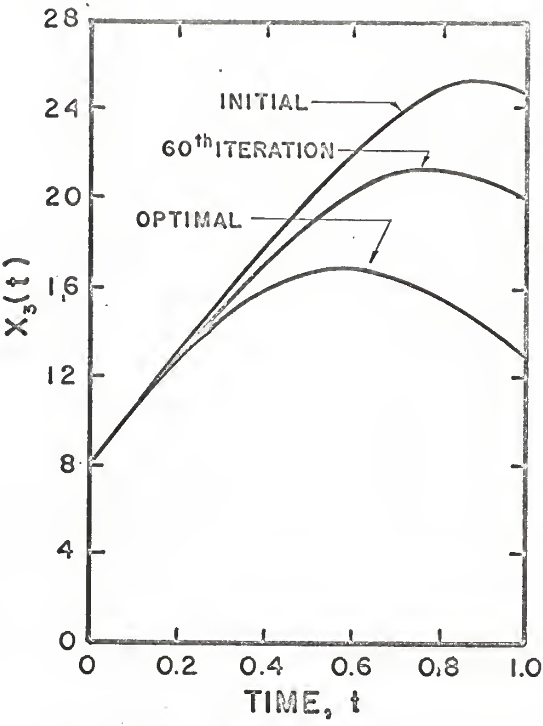

FIGURE 39. CONVERGENCE RATE OF INVENTORY IN PROBLEM 2.

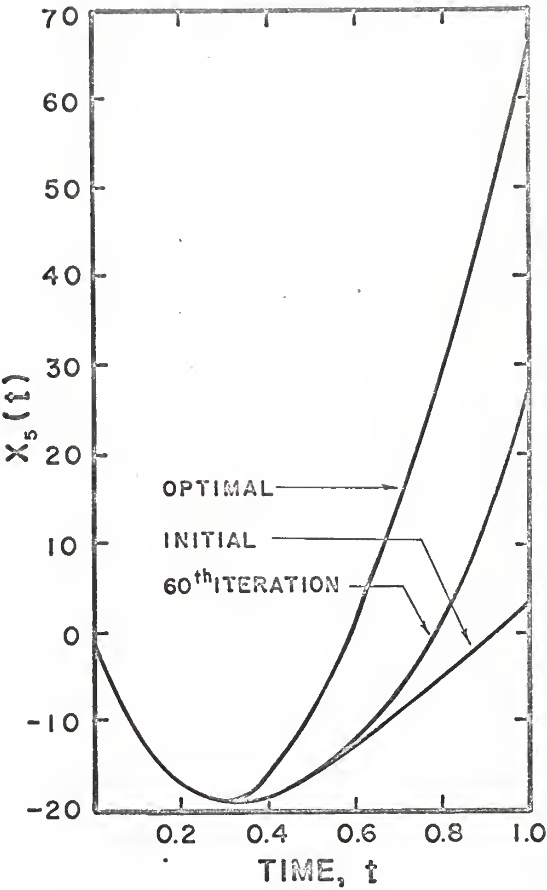

FIGURE 40. CONVERGENCE RATE OF PROFIT IN PROBLEM 2.

Problem 2

The model used for this problem was the same as that used for problems 1a and 1b except for different values of the parameters and initial guesses for controls.

The profit with the initially guessed control was \$4.10 and the optimal profit was \$66.05. Again it can be seen that the convergence rate is very slow when the current result is near the optimum. The profit at the 60th iteration was \$27.50. Another 200 iterations were required to reach the optimal. The temperature, concentration, inventory, sales and profit profiles are plotted in Figures 31 through 40.

CHAPTER 4

CONCLUSION AND DISCUSSION

Although the three problems solved in Section 3.3 have different parameters and different starting values, they have certain characteristics in common. To study the common characteristics, consider the detailed behavior of the system at any particular iteration.

At time t_0, the number of informed persons in the group is low so the number of items sold is low. There is a large number of uninformed persons and hence advertising is high. Since the system is already in production, there is a tendency for stock to go into inventory. As time goes on, the sales or the number of informed persons increases and the additional contact coefficient also increases because it is profitable to do so. The production rate of B rises because it has to meet the sales requirement as well as to maintain the required inventory level.

There comes a stage when the sales is much more profitable than slight variations in the inventory level and the inventory begins to fall. In this case this comes at about $t = 0.55$. However, when the inventory has become sufficiently low, the inventory cost in the total profit equation becomes important. It should be noted that a time delay exists between the fall in the inventory level and the fall in the production rate. This time delay is quite natural in any practical situation and is very well presented in the model stated above.

The technique illustrated above could easily be extended to optimize stagewise processes such as a transportation problem. It could also be extended to optimize multiproduct multifacility scheduling problems with complex interconnections.

The choice of a proper value for $\Delta\phi$ is of great importance if this technique is to be applied to actual optimization situations. It has been found from experience that the value of $\Delta\phi$ should be reduced when the gradient direction for the control changes sign together with a major drop in the profit function. If some effective logic is developed to automatically adjust $\Delta\phi$ once the gradient direction changes sign, the required computation time can be greatly reduced. The author used a logic in which the value of $\Delta\phi$ was reduced by half once the gradients at all the computed grid points changed sign. This method failed because after the reduction the value of $\Delta\phi$ was either still too large, causing a further reduction in its value, or was too small, causing the convergence rate to be extremely slow.

In the case of the production and inventory model with fixed end conditions, it was found that by using values for the control variable far from the optimal, the cost C_T went above reasonable limits. An emperical rule was adopted of asking for only part improvement in the inventory at each iteration. Encouraging results were obtained.

As the problem of slow convergence near the optimal still persists, it is suggested that the first variation of the gradient technique presented above should be used to get good starting values for the controls and other iterative optimization techniques such as the second variation or the quasilinearization technique be used to reach the optimum.

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APPENDIX A Numerical Solution of Differential Equations

As seen in the previous sections, the gradient method requires the solution of sets of first order differential equations. It would be appropriate at this stage to discuss some of the numerical techniques for the solution of such differential equations.

Ordinary differential equations are generally classified according to the degree of difficulty in obtaining numeric solutions. They could be classified as

1. Initial value problems
   a) Linear differential equations
   b) Nonlinear differential equations
2. Boundary value problems
   a) Linear differential equations
   b) Nonlinear differential equations

As long as the differential equations are initial value problems, numerical techniques can solve both linear and nonlinear problems with equal ease. However, for boundary value problems, the degree of difficulty in solving a nonlinear cquations is far greater than that for solving a linear equation.

The three main methods generally used for the numerical solution of differential equations can be classified as

1. The single step technique
2. The multiple step technique

Under the single step technique, the Euler and Range-Kutta integration methods would be discussed while the Milnes method would be discussed in the multiple step technique.

Euler Method [8]

Suppose that one wishes to solve the set of ordinary first order differential equations

$$\frac{dx_i}{dt} = g_i(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_s); \quad i = 1, 2, \ldots, n$$
$$j = 1, 2, \ldots, s$$

where x_i is the state variable and y_j is the control variable.

The Euler method is basically an approximation of the above differential equation in the form

$$\frac{\Delta x_i}{\Delta t} \simeq \frac{dx_i}{dt} = g_i\ (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_s)\ .$$

This approximation leads rather naturally to the equation

$$x_i^{k+1} = x_i^k + \Delta t \cdot g_i(x_1^k, x_2^k, \ldots, x_n^k, y_1^k, y_2^k, \ldots, y_s^k)$$

where x_i^{k+1} indicates the value of x_i at $(k+1)\Delta t$, x_i^k is the value of x_i at $k\Delta t$ and $g_i(x_1^k, x_2^k, \ldots, x_n^k, y_1^k, y_2^k, \ldots, y_s^k)$ indicates the value of g_i evaluated at time $k\Delta t$. One can easily see how the above equation can be used in conjunction with the values of x_i at zero time to obtain values of x_i^k for any specified k.

Since the approximation on which the Euler method is based becomes exact only when $\Delta t \rightarrow 0$ and finite Δt's are required for numerical calculations, one cannot, in general, expect the Euler method to be very accurate. Various modifications of the basic Euler method are available which reduce the accuracy problem.

In any computation involving the Euler method, one must select the time interval Δt very carefully. Large Δt values lead to gross errors while small Δt values cause excessively long computation times. The suitability of the Euler method for use with a specific problem depends upon there being a Δt which is an adequate compromise between the two effects.

Runge-Kutta Method

Like the Euler method, the Runge-Kutta method is designed for use with sets of first order ordinary differential equations.

The basic method is to write the Taylor series expansion for small purturbation of the variables about the initial conditions. The Taylor series is terminated after a suitable number of terms and a series of algebraic and operator manupulations is performed which leads to lumping the various derivatives into terms which may be evaluated from formulas. The most popular version of the Runge-Kutta is the system which results from retention of terms of up to fourth order in the original Taylor series.

The solution for a system of n equations of the form

$$\frac{dx_i}{dt} = g_i(x_1, x_2, \ldots, x_n); \quad i = 1, 2, \ldots, n$$

can be lead through application of the formulas

$$x_i^{k+1} = x_i^k + (R_{i,1} + 2\,R_{i,2} + 2R_{i,3} + R_{i,4})$$

where

$$R_{i,1} = \Delta t \cdot g_i(x_1^k, x_2^k, \ldots, x_n^k)$$

$$R_{i,2} = \Delta t \cdot g_i(x_1^k + \frac{1}{2} R_{1,1};\ x_2^k + \frac{1}{2} R_{2,1};\ \dots\ x_n^k + \frac{1}{2} R_{n,1})$$

$$R_{i,3} = \Delta t \cdot g_i(x_1^k + \frac{1}{2} R_{1,2};\ x_2^k + \frac{1}{2} R_{2,2};\ \dots\ x_n^k + \frac{1}{2} R_{n,2})$$

and

$$R_{i,4} = \Delta t \cdot g_i(x_1^k + R_{1,3};\ x_2^k + R_{2,3};\ \dots\ x_n^k + R_{n,3})\ .$$

The index i goes from 1 to n. As usual, x_i^k indicates the value of x_i at a time of $k\Delta t$. $R_{i,j}$ indicates the j^{th} Runge-Kutta coefficient for equation i.

The use of these equations to numerically solve the system of equations is straight-forward. The procedure is:

1. From a knowledge of x_1^k, x_2^k, ..., x_n^k, calculate $R_{1,1}$; $R_{2,1}$; ..., $R_{n,1}$.
2. From x_1^k, x_2^k, ..., x_n^k and $R_{1,1}$; $R_{2,1}$; ..., $R_{n,1}$ calculate $R_{1,2}$; $R_{2,2}$; ..., $R_{n,2}$.
3. From x_1^k, x_2^k, ..., x_n^k and $R_{1,2}$; $R_{2,2}$; ..., $R_{n,2}$, calculate $R_{1,3}$; $R_{2,3}$; ..., $R_{n,3}$.
4. From x_1^k, x_2^k, ..., x_n^k and $R_{1,3}$; $R_{2,3}$; ..., $R_{n,3}$, calculate $R_{1,4}$; $R_{2,4}$; ..., $R_{n,4}$.
5. Calculate x_i^{k+1} from x_i^k, $R_{i,1}$; $R_{i,2}$; $R_{i,3}$ and $R_{i,4}$.
6. Repeat steps 1 → 5 until the final value of A is reached.

The problem of accuracy is not of great importance in the Runge-Kutta integration scheme as the truncation error is of the fifth order. However, the stability problem sometimes arises as the process might not converge.

Multiple Step Method [8]

For the solution of the differential equation of the type

$$\frac{dx}{dt} = f(x,t),$$

the formulas for the recurrence relationship could be represented by

$$x(t_{k+1}) - x(t_{k-r})$$

$$= h(\ t,\ f(x(t_k),\ t_k),$$

$$f(x(t_{k-1}),\ x(t_{k-1}),\ \ldots\ f(x(t_{k-n}),\ t_{k-n}))$$

where r and n are positive integers.

To evaluate $x(t_{k+1})$, the values of $x(t_k)$, $x(t_{k-1})$, ..., $x(t_{k-n})$ and $x(t_{k-r})$ must be known. Hence it is seen that it is not possible to calculate $x(t_{k+1})$ directly from the initial value x^0. Also, to start the calculation the points $x(t_{k-1})$, $x(t_{k-2})$, ... must be known through another integration method.

To increase the accuracy, two integration formulas are generally used in the multiple step method. The first formula, known as the open end integration formula, is used to predict the approximate value of $x(t_{k+1})$. Then the second formula, or closed end formula, is used to generate a more accurate $x(t_{k+1})$. This latter formula may be iterated to obtain as accurate an answer as desired.

These two formulas form a predictor correcter scheme which is a powerful numerical tool. Milnes method is probably the best known multiple step integration formula. The predictor for this method is

$$x(t_{k+1}) = x(t_{k-3}) + \frac{4}{3}\,\Delta t\,[2f(x(t_k), t_k) - f(x(t_{k-1}), t_{k-1}) + 2f(x(t_{k-2}), t_{k-2}))]$$

and the correcter is

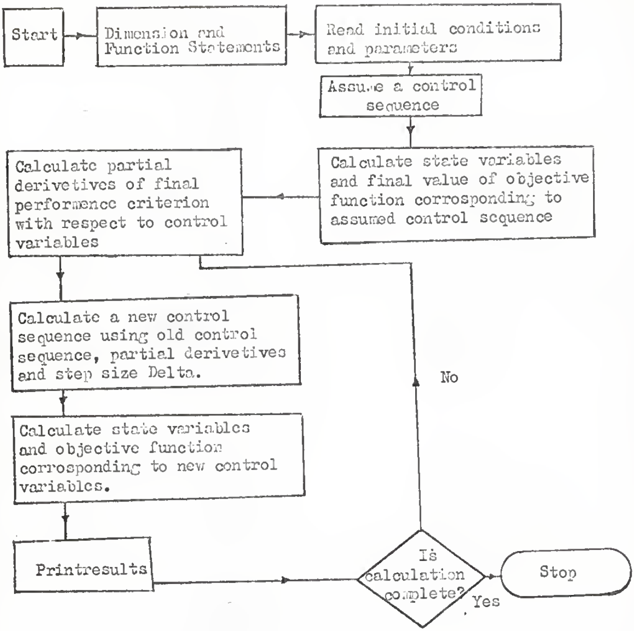
$$x(t_{k+1}) = x(t_{k-1}) + \frac{1}{3}\,\Delta t\,[f(x(t_{k+1}), t_{k+1}) + 4f(x(t_k), t_k) + f(x(t_{k-1}), t_{k-1})] \;.$$

To begin the integration, the starting values at the three grid points t_k, t_{k-1} and t_{k-2} can be obtained by a single step integration formula or by using Taylor series.

The single step methods have a number of advantages in terms of the use of digital computers. First in using the multiple step methods the starting values must be calculated by some other methods; no such predictions or corrections are necessary for the single step methods. Second, during the integration process several different values of the integration step Δt may be necessary in solving the same equations. It is not easy to reduce the integration step Δt for the multiple step methods as the integration proceeds. Some kind of interpolation formula must be used to reduce this step size.

Because of its high relative accuracy and ease of computation, the Runge-Kutta method is used for the numerical solution of the differential equations encountered in this report.

APPENDIX B Flow Charts



APPENDIX C

COMPUTER PROGRAMS

```
) FORMAT(8F8.3,I1,I3)
. FORMAT(F8.6)
) FORMAT(1H ,10HITERATION ,I3)
) FORMAT(1H ,3E16.8)
) FORMAT(1H ,51H           SUM    SUM OF DTHETA/DZSUM OF DTH/DZ+DS/DZ)
 DIMENSIONT(200),P1(200),P2(200),P3(200),P4(200),Q1(200),Q2(200),
1Q3(200),Q4(200),S(200),X1(200),DX(200),DX2(200),X2(200),SX1(200),
2ST(200),E(200),Z(200),PSZ(200)
 DO 10 I=1,101
 H=I-1
 X=H*D
) T(I)=X*4.75+3.
) SUM=0
 SUMT=0
 SUMP=0
 DO 4 I=1,101
 S(I)=I-1
 X1(1)=5.
 X2(1)=0.
 P1(I)=(T(I)-A-B*D*S(I))*D
 P2(I)=D*(T(I)-A-B*(D*S(I)+D/2.))
 P3(I)=P2(I)
 P4(I)=D*(T(I)-A-B*(D*S(I)+D))
 DX(I)=(1./6.)*(P1(I)+2.*P2(I)+2.*P3(I)+P4(I))
 X1(I+1)=X1(I)+DX(I)
 Q1(I)=(CI*((X1(I)**2)+CP*EXP((PA-T(I))**2))*D
 Q2(I)=(CI*(X1(I)+.5*PI(I))**2+CP*EXP((PA-T(I))**2))*D
 Q3(I)=(CI*(X1(I)+.5*P2(I))**2+CP*EXP((PA-T(I))**2))*D
 Q4(I)=(CI*(X1(I)+P3(I))**2+CP*EXP((PA-T(I))**2))*D
 DX2(I)=(1./6.)*(Q1(I)+2.*Q2(I)+2.*Q3
 DX2(I)=(1./6.)*(Q1(I)+2.*Q2(I)+2.*Q3(I)+Q4(I))
 X2(I+1)=X2(I)+DX2(I)
 E(I)=S(I)*D
 CONTINUE
 FORMAT(1H ,'ITER',3X,'TIME',2X,'INVENTORY',4X,'COST',10X,'THETA',
110X,'DS/DX1',10X,'DS/DZ',10X,'DTH/DZ')
 PRINT 300
 DO 5 I=1,100
 SX1(101)=0.
 N=101-I
 SX1(N)=SX1(N+1)+2.*D*CI*X1(N)
 ST(N)=D*(SX1(N+1)-2.*CP*(PA-T(N))*EXP((PA-T(N))**2))
 SUM=SUM+ST(N)**2
 ST(101)=ST(100)
 SUM=SUM+ST(101)**2
 DO 15 I=1,101
 TZ(I)=D
 SUMT=SUMT+TZ(I)**2
 DO 6 I=1,101
 PRINT 200,I,E(I),X1(I),X2(I),T(I),SX1(I),ST(I),TZ(I)
 PRINT 700
 PRINT 500,SUM,SUMT,SUMP
 K=K+1
 IF(K-L)70,50,50
 DTH=9.25-X1(101)
 A2=(DEL*SUMP-DTH*SUM)/(SUMP**2-SUMT*SUM)
 A1=(DEL-A2*SUMP)/SUM
 DO 7 I=1,101
 T(I)=T(I)+(A1*ST(I)+A2*TZ(I))
 PRINT 400,K
```

```
DIMENSIONP(102),X2(102),X1(102),X3(102),A(102),E(102)
DIMENSIONSX1(102),SX2(120),SX3(120),ST(102)
FORMAT(1H ,5H TIME,6X,9HINVENTORY,10X,5HSALES,9X,6HPROFIT,
.2X,13HADVERTISEMENT,5X,10HPRODUCTION,7X,8HGRADIENT)
FORMAT(1H ,F5.2,6E15.4)
FORMAT(1H ,9HITERATION,I6)
FORMAT(E15.9)
FORMAT(8F8.3,2I4)
FORMAT(2F5.2)
FORMAT(1H ,8F8.3,2I4)
FORMAT(1H ,2F5.2)
FORMAT(3F10.2)
FORMAT(1H ,3F10.2)
FORMAT(F20.8)
READ 18,X1(1),X2(1),X3(1)
READ 14,AB,B,C,PO,PI,F,CI,CA,K,L
READ15,D,DEL
PRINT19,X1(1),X2(1),X3(1)
PRINT16,AB,B,C,PO,PI,F,CI,CA,K,L
PRINT17,D,DEL
DO 9 I=1,101
A(I)=2.5
SUM=0
DO 1 I=1,101
T=I-1
T=T*D
P(I)=AB+B*T
A1=(P(I)-X2(I))*D
B1=((C+A(I))*(X2(I)-(X2(I)**2)/PO))*D
A2=(P(I)-X2(I)-B1/2.)*D
B2=((C+A(I))*(X2(I)+B1/2.-((X2(I)+B1/2.)**2)/PO))*D
B3=((C+A(I))*(X2(I)+B2/2.-((X2(I)+B2/2.)**2)/PO))*D
B4=((C+A(I))*(X2(I)+B3-((X2(I)+B3)**2)/PO))*D
A3=(P(I)-X2(I)-B2/2.)*D
A4=(P(I)-X2(I)-B3)*D
C1=(F*X2(I)-((PI-X1(I))**2)*CI-CA*(A(I)**2)*X2(I))*D
C2=(F*(X2(I)+B1/2.)-((PI-(X1(I)+A1/2.))**2)*CI-CA*(A(I)**2
1)*(X2(I)+B1/2.))*D
C3=(F*(X2(I)+B2/2.)-((PI-(X1(I)+A2/2.))**2)*CI-CA*(A(I)**2
1)*(X2(I)+B2/2.))*D
C4=(F*(X2(I)+B3)-((PI-(X1(I)+A3))**2)*CI-CA*(A(I)**2)*(X2(I
1)+B3))*D
X1(I+1)=X1(I)+(A1+2.*A2+2.*A3+A4)*1./6.
X2(I+1)=X2(I)+(B1+2.*B2+2.*B3+B4)*1./6.
X3(I+1)=X3(I)+(C1+2.*C2+2.*C3+C4)*1./6.
E(I)=T
DO 2 I=1,100
SX1(101)=0
SX2(101)=0
SX3(101)=1
N=101-I
SX1(N)=SX1(N+1)+(SX3(N+1)*2.*CI*(PI-X1(N)))*D
SX2(N)=SX2(N+1)+(-SX1(N+1)+SX2(N+1)*(C+A(N))*(1.-2.*X2(N)/PO)
1-SX3(N+1)*(CA*(A(N)**2)-F))*D
SX3(N)=SX3(N+1)
ST(N)=(SX2(N+1)*(X2(N)-(X2(N)**2)/PO)-SX3(N+1)*2.*A(N)*X2(N)*CA)
1*D
SUM=SUM+ST(N)**2
ST(101)=ST(100)
SUM=SUM+ST(101)**2
```

```
  PRINT10
  DO 3 I=1,101,5
3 PRINT11 ,E(I),X1(I),X2(I),X3(I),A(I),P(I),ST(I)
  K=K+1
  IF(K.GE.18)DEL=.05
  IF(K-L)4,5,5
, DO 6 I=1,101
) A(I)=A(I)+DEL*ST(I)/SUM
  PRINT12,K
  GO TO 7
; DO 8 I=1,101
} WRITE(2,13)A(I)
  STOP
  END
.Y
        20.          0.
   100.      2.        150.       50.     10.       .15      1.5      0   120
  40.
```

```
 DIMENSIONT1(102),T2(102),T3(102),E(102),S(102),SX1(102),SY1(102)
 DIMENSION X1(102),Y1(102),X2(102),Y2(102),X3(102),X4(102),X5(102)
 DIMENSIONSX2(102),SY2(102),SX3(102),SX4(102),SX5(102),ST1(102)
 DIMENSIONST2(102),ST3(102),ZX1(102),ZX2(102),ZX3(102),ZX4(102)
 DIMENSIONZX5(102),ZY1(102),ZY2(102),ZT1(102),ZT2(102),ZT3(102)
 DIMENSIONPSZ1(102),PSZ2(102),PSZ3(102)
 DIMENSION AA1(102),AB1(102),AA2(102),AB2(102),OA1T1(102),
1OB1T1(102),OA2T2(102),OB2T2(102)
 DIMENSION OLT1(102),OLT2(102),OLT3(102)
 FORMAT(1H ,6HDELT1=,F9.6,6HDELT2=,F9.6,6HDELT3=,F9.6)
 FORMAT(1H ,2F8.2)
 FORMAT(1H ,7F8.2,2I5)
 FORMAT(1H ,8F10.4)
 FORMAT(1H ,7F10.4)
 FORMAT(1H ,2E20.3,2F10.0)
 FORMAT(1H ,2F10.3)
 FORMAT(3E11.4)
 FORMAT(1H ,3F8.2)
 FORMAT(1H ,F4.2,6E15.4)
 FORMAT(1H ,F4.2,3E15.4,E18.7,3E15.4)
 FORMAT(1H ,4HDEL=,F5.2)
 FORMAT(1H ,4HTIME,7X,8HCONC. 1A,7X,8HCONC.1 B,9X,6HTEMP 1,7X,8HCON
1C. 2A,   7X,8HCONC.2 B,9X,6HTEMP 2)
 FORMAT(1H ,4HTIME,6X,9HINVENTORY,10X,5HSALES,7X,8HADV.COST,12X,6HP
1ROFIT,10X,5HGR.T1,10X,5HGR.T2,10X,5HGR.T3)
 FORMAT(1H ,9E12.6)
 FORMAT(1H ,8X,4HSUM1,8X,4HSUM2,8X,4HSUM3,7X,5HSUMZ1,7X,5HSUMZ2,7X,
15HSUMZ3,7X,5HPSUM1,7X,5HPSUM2,7X,5HPSUM3)
 FORMAT(1H ,9HITERATION,I4)
 FORMAT(3E20.8)
 FORMAT(2F4.2)
 READ700,X0,Y0
 FORMAT(7F8.2,3I2)
 READ600,X1(1),Y1(1),X2(1),Y2(1),X3(1),X4(1),X5(1),K,L,M
 FORMAT(8F10.4)
 FORMAT(7F10.4)
 READ800,CQ,C,P,AM,T1M,Z1,Z2,Z3
 READ850,CI,CA,R,Q,V1,V2,CT
 FORMAT(2E10.3,2F7.0)
 READ900,GA,GB,EA,EB
 FORMAT(5F6.3)
 FORMAT(2F10.2)
 READ 150,D,DELT1,DELT2,DELT3,OLCOS
 DO 11 I=1,101
 T1(I)=340
 T2(I)=340
 T3(I)=2.
 OLT1(I)=T1(I)
 OLT2(I)=T2(I)
 OLT3(I)=T3(I)
 PRINT101,X0,Y0
 PRINT102,X1(1),Y1(1),X2(1),Y2(1),X3(1),X4(1),X5(1),K,L
 PRINT103,CQ,C,P,AM,T1M,Z1,Z2,Z3
 PRINT104,CI,CA,R,Q,V1,V2,CT
 PRINT105,GA,GB,EA,EB
 PRINT106,D,DELT1
 DO 15 I=1,101
 PRINT108,T1(I),T2(I),T3(I)
 A=10.
 SUM1=0
```

```
 SUM2=0
 SUM3=0
 SUMZ1=D
 SUMZ2=D
 SUMZ3=0
 PSUM1=0
 PSUM2=0
 PSUM3=D
 SUMT1=D
 SUMT2=0
 SUMT3=0
 DD1I=1,101
 S(I)=I-1
 AA1(I)=GA*EXP(-EA/(R*T1(I)))
 AB1(I)=GB*EXP(-EB/(R*T1(I)))
 AA2(I)=GA*EXP(-EA/(R*T2(I)))
 AB2(I)=GB*EXP(-EB/(R*T2(I)))
 A1=(Q*(X0-X1(I))/V1-AA1(I)*X1(I))*D
 A2=(Q*(XD-X1(I)-A1/2.)/V1-AA1(I)*(X1(I)+A1/2.))*D
 A3=(Q*(X0-X1(I)-A2/2.)/V1-AA1(I)*(X1(I)+A2/2.))*D
 A4=(Q*(X0-X1(I)-A3)/V1-AA1(I)*(X1(I)+A3))*D
 B1=(Q*(Y0-Y1(I))/V1-AB1(I)*Y1(I)+AA1(I)*X1(I))*D
 B2=(Q*(Y0-Y1(I)-B1/2.)/V1-AB1(I)*(Y1(I)+B1/2.)+AA1(I)*(X1(I)+A1
1/2.))*D
 B3=(Q*(Y0-Y1(1)-B2/2.)/V1-AB1(I)*(Y1(I)+B2/2.)+AA1(I)*(X1(I)+
1A2/2.))*D
 B4=(Q*(Y0-Y1(I)-B3)/V1-AB1(I)*(Y1(I)+B3)+AA1(I)*(X1(I)+A3))*D
 X1(I+1)=X1(I)+(1./6.)*(A1+2.*A2+2.*A3+A4)
 Y1(I+1)=Y1(I)+(1./6.)*(B1+2.*B2+2.*B3+B4)
 C1=(Q*(X1(I)-X2(I))/V2-AA2(I)*X2(I))*D
 C2=(Q*(X1(I)-X2(I)+A1/2.-C1/2.)/V2-AA2(I)*(X2(I)+C1/2.))*D
 C3=(Q*(X1(I)-X2(I)+A2/2.-C2/2.)/V2-AA2(I)*(X2(I)+C2/2.))*D
 C4=(Q*(X1(I)-X2(I)+A3-C3)/V2-AA2(I)*(X2(I)+C3))*D
 D1=(Q*(Y1(I)-Y2(I))/V2-AB2(I)*Y2(I)+AA2(I)*X2(I))*D
 D2=(Q*(Y1(I)-Y2(I)+B1/2.-D1/2.)/V2-AB2(I)*(Y2(I)+D1/2.)+AA2(I)*
1(X2(I)+C1/2.))*D
 D3=(Q*(Y1(I)-Y2(I)+B2/2.-D2/2.)/V2-AB2(I)*(Y2(I)+D2/2.)+AA2(I)*
1(X2(1)+C2/2.))*D
 D4=(Q*(Y1(I)-Y2(I)+B3-D3)/V2-AB2(I)*(Y2(I)+D3)+AA2(I)*(X2(I)+C3
1))*D
 X2(I+1)=X2(I)+(1./6.)*(C1+2.*C2+2.*C3+C4)
 Y2(I+1)=Y2(I)+(1./6.)*(D1+2.*D2+2.*D3+D4)
 F1=((C+T3(I))*X4(I)*(1.-X4(I)/P))*D
 F2=((C+T3(I))*(X4(I)+F1/2.)*(1.-(X4(I)+F1/2.)/P))*D
 F3=((C+T3(I))*(X4(I)+F2/2.)*(1.-(X4(I)+F2/2.)/P))*D
 F4=((C+T3(I))*(X4(I)+F3)*(1.-(X4(I)+F3)/P))*D
 X4(I+1)=X4(I)+(1./6.)*(F1+2.*F2+2.*F3+F4)
 E1=(Q*Y2(I)-CQ*X4(I))*D
 E2=(Q*(Y2(I)+D1/2.)-CQ*(X4(I)+F1/2.))*D
 E3=(Q*(Y2(I)+D2/2.)-CQ*(X4(I)+F2/2.))*D
 E4=(Q*(Y2(I)+D3)-CQ*(X4(I)+F3))*D
 X3(I+1)=X3(I)+(1./6.)*(E1+2.*E2+2.*E3+E4)
 G1=(Z1*CQ*X4(I)+Z2*Q*X2(I)+Z3*Q*(1.-X2(I)-Y2(I))-CI*(AM-X3(I))**2
1-CT*((T1M-T1(I))**2+(T1(I)-T2(I))**2)-CA*((T3(I)*X4(I))**2.))*D
 G2=(Z1*CQ*(X4(I)+F1/2.)+Z2*Q*(X2(I)+C1/2.)+Z3*Q*(1.-X2(I)-C1/2.-Y2
1(I)-D1/2.)-CI*(AM-X3(I)-E1/2.)**2          -CT*((T1M-T1(I))**2+
2(T1(I)-T2(I))**2)-CA*((T3(I)*(X4(I)+F1/2.))**2.))*D
 G3=(Z1*CQ*(X4(I)+F2/2.)+Z2*Q*(X2(I)+C2/2.)+Z3*Q*(1.-X2(I)-C2/2.-Y2
1(I)-D2/2.)-CI*(AM-X3(I)-E2/2.)**2          -CT*((T1M-T1(I))**2+
2(T1(I)-T2(I))**2)-CA*((T3(I)*(X4(I)+F2/2.))**2.))*D
```

```
 G4=(Z1*CQ*(X4(I)+F3)+Z2*Q*(X2(I)+C3)+Z3*Q*(1.-X2(I)-C3-Y2(I)-D3)
1-CI*(AM-X3(I)-E3)**2              -CT*((T1M-T1(I))**2+(T1(I)-T2(I))*
2*2)-CA*((T3(I)*(X4(I)+F3))**2.))*D
 X5(I+1)=X5(I)+(1./6.)*(G1+2.*G2+2.*G3+G4)
 E(I)=S(I)*0
 DO 2 I=1,100
 SX1(101)=0
 SY1(101)=0
 SX2(101)=0
 SY2(101)=0
 SX3(101)=0
 SX4(101)=0
 SX5(101)=1
 N=101-I
 SX1(N)=SX1(N+1)+(SX1(N+1)*(-Q/V1-AA1(N))+SY1(N+1)*AA1(N)+SX2(N+1)
1*Q/V2)*D
 SY1(N)=SY1(N+1)+(SY1(N+1)*(-Q/V1-AB1(N))+SY2(N+1)*Q/V2) *D
 SX2(N)=SX2(N+1)+(SX2(N+1)*(-Q/V2-AA2(N))+SY2(N+1)*AA2(N)+SX5(N+
11)*(Z2*Q-Z3*Q))*D
 SY2(N)=SY2(N+1)+(SY2(N+1)*(-Q/V2-AB2(N))+SX3(N+1)*Q+SX5(N+1)*(-Z
13*Q))*D
 SX3(N)=SX3(N+1)+SX5(N+1)*2.*C1*(AM-X3(N))*D
 SX4(N)=SX4(N+1)+(-SX3(N+1)*CQ+SX4(N+1)*(C+T3(N))*(1.-2.*X4(N)/P)
1-SX5(N+1)*(2.*CA*(T3(N)**2)*X4(N)           -CQ*Z1))*D
 SX5(N)=SX5(N+1)
 DA1T1(N)=(GA*EA/(R*(T1(N)**2)))*EXP(-EA/(R*T1(N)))
 DB1T1(N)=(GB*EB/(R*(T1(N)**2)))*EXP(-EB/(R*T1(N)))
 DA2T2(N)=(GA*EA/(R*(T2(N)**2)))*EXP(-EA/(R*T2(N)))
 DB2T2(N)=(GB*EB/(R*(T2(N)**2)))*EXP(-EB/(R*T2(N)))
 ST1(N)=(SX1(N+1)*(-X1(N)*DA1T1(N))+SY1(N+1)*(+X1(N)*DA1T1(N)-Y1
1(N)*DB1T1(N))+SX5(N+1)*(-CT*(-2.*(T1M-T1(N))+2.*(T1(N)-T2(N)))))*D
 ST2(N)=(-SX2(N+1)*X2(N)*DA2T2(N)+SY2(N+1)*(X2(N)*DA2T2(N)-Y2(N)
1*DB2T2(N))+SX5(N+1)*(CT*(2.*(T1(N)-T2(N)))))*D
 ST3(N)=(SX4(N+1)*X4(N)*(1.-X4(N)/P)- SX5(N+1)*2.*CA*(X4(N)**2)*T3(
1N))*D
 SUMT1=SUMT1+ST1(N)
 SUMT2=SUMT2+ST2(N)
 SUMT3=SUMT3+ST3(N)
 SUM1= SUM1+ST1(N)**2
 SUM2= SUM2+ST2(N)**2
 SUM3= SUM3+ST3(N)**2
 ST1(101)=ST1(100)
 ST2(101)=ST2(100)
 ST3(101)=ST3(100)
 SUM1=SUM1+ST1(101)**2
 SUM2=SUM2+ST2(101)**2
 SUM3=SUM3+ST3(101)**2
 SUMT1=SUMT1+ST1(101)
 SUMT2=SUMT2+ST2(101)
 SUMT3=SUMT3+ST3(101)
 IF(X5(101).GE.OLCOS)GO TO 113
 DO 37 I=1,101
 T1(I)=OLT1(I)
 T2(I)=OLT2(I)
 T3(I)=OLT3(I)
 IF(SUMT1.GT.0.)GO TO 30
 IF(OLST1.GT.0.)DELT1=DELT1/2.
 GO TO 31
 IF(OLST1.LT.0.)DELT1=DELT1/2.
 IF(SUMT2.GT.0.)GO TO 32
```

```
 IF(OLST2.GT.D.)CELT2=DELT2/2.
 GO TO 33
 IF(OLST2.LT.O.)CELT2=DELT2/2.
 IF(SUMT3.GT.D.)GO TO 34
 IF(DLST3.GT.O.)DELT3=DELT3/2.
 GO TO 35
 IF(OLST3.LT.O.)DELT3=DELT3/2.
 PRINT36,DELT1,CELT2,DELT3
 GO TO 2D
 OLST1=SUMT1
 OLST2=SUMT2
 OLST3=SUMT3
 DO 3 I=1,1DO
 ZX1(1D1)=0
 ZY1(101)=0
 ZX2(1D1)=0
 ZY2(1D1)=0
 ZX3(1D1)=1
 ZX4(1D1)=D
 ZX5(1D1)=D
 N=1D1-I
 ZX1(N)=ZX1(N+1)+(ZX1(N+1)*(-Q/V1-AA1(N))+ZY1(N+1)*AA1(N)+ZX2(N+1)
1*Q/V2)*D
 ZY1(N)=ZY1(N+1)+(ZY1(N+1)*(-Q/V1-AB1(N))+ZY2(N+1)*Q/V2) *D
 ZX2(N)=ZX2(N+1)+(ZX2(N+1)*(-Q/V2-AA2(N))+ZY2(N+1)*AA2(N)+ZX5(N+
11)*(Z2*Q-Z3*Q))*D
 ZY2(N)=ZY2(N+1)+(ZY2(N+1)*(-Q/V2-AB2(N))+ZX3(N+1)*Q+ZX5(N+1)*(-Z
13*Q))*D
 ZX3(N)=ZX3(N+1)+ZX5(N+1)*2.*CI*(AM-X3(N))*D
 ZX4(N)=ZX4(N+1)+(-ZX3(N+1)*CQ+ZX4(N+1)*(C+T3(N))*(1.-2.*X4(N)/P)
1-ZX5(N+1)*(2.*CA*(T3(N)**2)*X4(N)          -CQ*Z1))*D
 ZX5(N)=ZX5(N+1)
 ZT1(N)=(ZX1(N+1)*(-X1(N)*DA1T1(N))+ZY1(N+1)*(+X1(N)*DA1T1(N)-Y1
1(N)*DB1T1(N))+ZX5(N+1)*(-CT*(-2.*(T1M-T1(N))+2.*(T1(N)-T2(N)))))*D
 ZT2(N)=(-ZX2(N+1)*X2(N)*DA2T2(N)+ZY2(N+1)*(X2(N)*CA2T2(N)-Y2(N)
1*DB2T2(N))+ZX5(N+1)*(CT*(2.*(T1(N)-T2(N)))))*D
 ZT3(N)=(ZX4(N+1)*X4(N)*(1.-X4(N)/P)- ZX5(N+1)*2.*CA*(X4(N)**2)*T3(
1N))*D
 SUMZ1=SUMZ1+ZT1(N)**2
 SUMZ2=SUMZ2+ZT2(N)**2
 SUMZ3=SUMZ3+ZT3(N)**2
 ZT1(1D1)=ZT1(1CO)
 ZT2(1D1)=ZT2(100)
 ZT3(101)=ZT3(1CO)
 SUMZ1=SUMZ1+ZT1(1D1)**2
 SUMZ2=SUMZ2+ZT2(1D1)**2
 SUMZ3=SUMZ3+ZT3(101)**2
 DO 4 I=1,101
 PSZ1(I)=ST1(I)*ZT1(I)
 PSZ2(I)=ST2(I)*ZT2(I)
 PSZ3(I)=ST3(I)*ZT3(I)
 PSUM1=PSUM1+PSZ1(I)
 PSUM2=PSUM2+PSZ2(I)
 PSUM3=PSUM3+PSZ3(I)
 IF(M)21,21,22
 PRINT1DO
 DO 5 I=1,101
 PRINT2DD,E(I),X1(I),Y1(I),T1(I),X2(I),Y2(I),T2(I)
 PRINT 125
 DC 51 I=1,1D1
```

```
PRINT 250,E(I),X3(I),X4(I),T3(I),X5(I),ST1(I),ST2(I),ST3(I)
GO TO 111
PRINT 100
DO 56 I=1,1C1,5
PRINT200,E(I),X1(I),Y1(I),T1(I),X2(I),Y2(I),T2(I)
PRINT 125
DO 57 I=1,101,5
PRINT 250,E(I),X3(I),X4(I),T3(I),X5(I),ST1(I),ST2(I),ST3(I)
PRINT 300
PRINT 400,SUM1,SUM2,SUM3,SUMZ1,SUMZ2,SUMZ3,PSUM1,PSUM2,PSUM3
K=K+1
IF(K-L)6,10,10
DELZ=10.-X3(101)
TSUM1=SUM1+SUM2+SUM3
TPSUM=PSUM1+PSUM2+PSUM3
TSUMZ=SUMZ1+SUMZ2+SUMZ3
A1=DFL/TSUM1
OLCOS=X5(101)
DO 7 I=1,101
OLT1(I)=T1(I)
OLT2(I)=T2(I)
OLT3(I)=T3(I)
T1(I)=T1(I)+DELT1*  ST1(I)/SUM1
T2(I)=T2(I)+DELT2*  ST2(I)/SUM2
T3(I)=T3(I)+DELT3*ST3(I)/SUM3
IF(T3(I).LT.0.)GO TO 71
GO TO 7
T3(I)=0
CONTINUE
PRINT 5C0,K
GO TO 20
DO 61 I=1,101
WRITE(2,251) T1(I),T2(I),T3(I)
STOP
END
Y
.05
5   0.05    0.95     0.05   1.00    0.10     0.00   050 5
1.        1.       100.       10.     300.       1.         .50         0
01        0.1       2.0      100.0    20.0       20.00   0.0005
 E 11 .461 E 18 18000. 30000.
  1.    1.    .1     -9
```

OPTIMIZATION OF MANAGEMENT SYSTEMS
BY THE FUNCTIONAL GRADIENT TECHNIQUE

by

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ABSTRACT

The objective of this report is to apply the functional gradient technique to optimize management systems. The basic methodology in the functional gradient technique is to obtain a set of functional equations which gives the gradient of the objective function with respect to the control. The objective function is then improved in this gradient direction. If the state variables have to satisfy certain final conditions, a penalty function is introduced in the equations.

A set of problems in the field of production, inventory control and advertising have been solved with the purpose of making a critical study of the technique. The first problem solved considers sales to be fixed and production controlled in order to minimize cost and maintain a certain desired inventory level at the final time. In Section 3.2 the diffusion model is used to determine sales with the production considered as a constant. The control here is advertising and the objective is to maximize profit. In Section 3.3 a set of three problems, each with six state variables and three control variables, is solved. The results obtained are satisfactory but the inherent difficulty of slow convergence rate near the optimal still persists.