TOWARD A SOLUTION FOR THE OPTIMAL ALLOCATION OF INVESTMENT IN TRANSPORTATION NETWORK DEVELOPMENT

by 149

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SUMMARY

Realizing the effect of network improvement on the operation of a highway transportation system, this study develops a model which enables us to obtain an optimum investment policy for improving a transportation network.

Consider an area with the future traffic demand assumed to be known. The problem then is to build (or to improve) a transportation network to accommodate the demand with mimimum overall cost, assuming that the desired network configuration has already been determined. Travel time cost is assumed to be the only significant factor of operating cost. Therefore, the overall cost, or transportation cost, is assumed to be equal to the sum of investment cost and travel time cost.

A non-linear total travel time equation was developed to express the travel time as a function of the investment and of traffic volume. Three formulations associated with three different investment circumstances are presented. With the zonal interchanges considered as given information, the discrete maximum principle (1) was utilized to assign the trips and the investment to each link of the network based on the criterion of minimizing the overall cost. The maximum principle enables the location of the optimum without undue difficulties.

INTRODUCTION

The economic analysis of a transportation network provides valuable guidance in developing a comprehensive, long-range transportation plan which, as concluded by Zettel and Carll, is the basic objective of a transportation study (2). Being part of the public services and competing for the use of limited resources, the transportation system should be built and operated economically while at the same time it should meet the standards and goals of the community in order to promote growth and meet the needs of the economic activities. Specifically, the objectives of a transportation system have been summarized as: (3)

1. Provide a means for moving people and goods safely, freely and economically.
2. Provide a choice of mode.
3. Make the city a more attractive place to live.
4. Provide the means for fulfilling the needs and desires of the urban population within their ability to pay.

Theoretically, an optimal system which best fits the economic and social objectives would be the desired system and the evaluation of the system would be based on criteria which reflect these objectives. However, the evaluation would be very difficult if it were to be done quantitatively.

For example, The Chicago Area Transportation Study listed the following six criteria: (4)

1. Greater speed.
2. Increased safety.
3. Lower operating cost.
4. Economy in new construction.
5. Minimizing disruption.
6. Promoting better land development.

It is apparent that a single value system is not available for measuring all these criteria. Criteria three and four are easily expressed in dollars, and one and two can be converted into dollar cost. However, criteria five and six are not only intangible but also very difficult to measure quantitatively. For this reason, transportation system evaluations are generally restricted to economic analysis and leave the social objectives to be considered, somewhat subjectively, in the final selection of the optimal system. It is, although not perfect, the best approach available at present.

This study was an attempt to develop a mathematical model for the economic evaluation of a transportation system. Like many other studies, a single objective—to minimize the transportation cost—was adopted. Several surveys have shown that travel time is dominant as a factor in selecting a route (5,6). Therefore, to simplify the formulation, it was assumed that travel time was the only significant factor of operating cost thereby reducing the objective to minimizing the sum of the investment cost and travel time cost.

One of the major setbacks of linear programming type models is that unit travel time is assumed to be independent of traffic volume. To overcome the weakness, this study applied the Discrete Maximum Principle—a powerful optimum seeking method—which allows the use of a non-linear total travel time equation. It should be emphasized that although there is undoubtedly room for improvement in the particular functional relationship between travel time, assigned volume and investment cost, the primary purpose of this study was to demonstrate the usefulness and the ability of the discrete maximum principle in solving this type of non-linear optimization problem. Hopefully, by further research and modifications of the formulation, a more useful model may be evolved.

REVIEW OF LITERATURE

In the past few decades, several methods of economic analysis have been developed for use in transportation planning. Four principle methods are (a) annual cost method, (b) present worth method, (c) benefit-cost ratio method, and (d) rate of return method (7, 8, 9, 10). The relative advantages of each method are briefly described by Brennan and Rothrock (10).

Since World War II, the benefit-cost ratio method has been given a great deal of attention. In 1952, the American Association of State Highway Officials adopted this method and published an informational report on "Road User Benefit Analysis for Highway Improvement," the so-called, "Red Book" (8). Since that time, it has been accepted by many planning agencies. Grant, on the other hand, specially favors the use of the rate of return method. The application of this method and its merits have been discussed at length in his papers (9, 11). Many reports have compared the uses of the benefit-cost ratio method and the rate of return method in detail (11, 12). In general, when properly used, both methods will give the same results.

No matter which method was used, the analyses made in the past have restricted themselves to comparing alternatives for a single link or a single route of a network. The overall system effect of improvements was completely ignored or simply adjusted by using engineering judgment. Since the improvements will generally cause redistribution of traffic and since benefits on one route may cause a loss of benefits on other routes, this approach has sometimes resulted in an uneconomic or even retarded transportation plan.

Realizing this deficiency, some recent studies have compared alternatives through complete network analysis. In the Chicago Area Transportation Study, five alternative freeway systems were first developed. For each system, trip distribution and traffic assignment techniques were employed to obtain the traffic volume on each link of the system. Three methods, (a) benefit-cost ratio, (b) rate of return, and (c) annual cost, were then used to evaluate each system. They all resulted in the same answer (4, 13). This approach is in general quite satisfactory. However, because the number of alternatives to be compared was relatively small and the development of alternatives was largely based on engineering judgment, it is quite possible that the best system was not considered.

At the same time, more and more attention has been concentrated on the applications of optimization techniques to the transportation field. In 1958, Garrison and Marble (14) presented a linear programming formulation for the analysis of network improvement. Travel cost for each link was assumed to be constant and the investment was assumed to increase the capacity linearly. The objective was to miminize the sum of the investment and travel cost subject to constraints such as flow balance, budget, and capacity limits. The simplex algorithm was employed to seek the optimal solution. This paper leadsto increased interest in developing mathematical models in the following years.

Quandt (15), in 1960, presented a similar formulation. In his problem, a commodity was to be transported from n sources to m destinations, each node was connected with every other node by a direct link. Again, linear relations between shipment cost and volume, and between improvement cost and capacity were assumed. The problem was then formulated and solved by linear programming. Consideration was also given to problems with fixed budgets, indirect connection between nodes and multicommodity shipment.

Carter and Stowers (16), in 1963, again utilized linear programming to formulate a model for fund allocation for urban highway system capacity improvement. The basic formulation was the same as the previous papers except that each link was represented by two arcs, one with free flow capacity and normal operating cost, the other with higher operating cost (due to congestion) and capacity equal to the difference between possible and practical capacity. The ratio of the capacities of these two arcs was kept constant as the capacities were improved.

In 1964, Roberts and Funk (17) developed a linear programming model for the problem of adding links to a transportation system. The locations of possible additional links in the system were first decided. In seeking the optimum, the additional link was either completely built or not built at all. If the link were added, the cost was included in the objective function. If it were not added, the flow was blocked. In this formulation an integer programming technique was used. The paper also suggested a possible application of dynamic programming in treating the stage-wise construction problem. As a result, in 1966, Roberts, et. al. (18) combined the use of linear programming and dynamic programming techniques to solve a stage-wise link adding problem. The budget for each construction period was fixed. The method considered the budget at the Nth stage as the sum of the budgets from the first to the nth construction period and used the principle, "the links for stage N must be the subset of links for stage N + 1," to indicate the links which must be considered at stage N. At each stage, integer programming was used to select the links to be added. This method is considered useful for transportation system development in underdeveloped countries.

In the same year, Hay, Morlok and Charnes (19) presented a model for optimal planning of a two-mode urban transportation system. A two-mode system, private road and public transit, was to be built in an urban corridor. The road capital cost was linearly related to capacity and speed. Transit speed was fixed with the capacity linearly related to capital cost. The length of the transit route was also assumed fixed. The choice of mode was linearly related to the travel time ratio between road and transit. Again, the linear programming technique was used to formulate the problem and seek the optimum. The objective function was:

$$\min.\begin{Bmatrix}\text{annual road}\\ \text{capital cost}\end{Bmatrix}+\begin{Bmatrix}\text{annual vehicle}\\ \text{operating cost}\end{Bmatrix}+\begin{Bmatrix}\text{annual transit}\\ \text{capital cost}\end{Bmatrix}$$

$$+\begin{Bmatrix}\text{annual transit}\\ \text{operating cost}\end{Bmatrix}+\begin{Bmatrix}\text{Parking cost}\end{Bmatrix}$$

In this formulation, the travel time was excluded from operating cost and was treated as a constraint to reflect the minimum level of service desired and the maximum speed obtainable. For a true optimum, it was required to change the length of the transit route and run the program several times.

Distinct from the linear programming type models, Ridley (20) in 1965, developed a method for seeking the optimum investment policy to reduce the travel time in a transportation network. The unit travel time was assumed to be decreasing linearly with the investment. It was also assumed that the flow was far below the link capacity. The objective was to minimize the total travel time. Because the travel time was a function of both investment and traffic volume, the objective function was non-linear in nature. For some special cases, such as no budget limit, fixed traffic volume, fixed investment and single origin-destination, the formulation can be simplified into a linear programming model. For the general case which has budget and travel time constraints, the bounded subset method was utilized to search the optimum.

A common drawback of the above models is that unit travel time was assumed to be independent of traffic volume which is not at all close to reality. The linear assumption of travel time improvement in Hay's and Ridley's models is also questionable.

Although the above review shows some imperfection of the existing models, it is realized that the mathematical model is a powerful technique which we can apply in the field of transportation engineering in order to make a better analysis. Compared to the alternative comparison method, the mathematical model has merit in that: (19)

> "It selects that system which is optimal among all possible systems of a given type rather than merely examining a small number of alternatives."

There are, however, two difficulties with the mathematical models.

1. We are trying to express a highly complex real world phenomenon with a simple equation which is very difficult or may even be impossible.
2. The model sometimes becomes too complex to manipulate and too sophisticated to understand and

requires highly trained personnel to carry out the analysis.

DISCRETE MAXIMUM PRINCIPLE AND ITS RECENT APPLICATIONS IN TRANSPORTATION SYSTEMS ANALYSIS

In recent years, optimization has become more and more important in engineering systems analysis. Since World War II, many sophisticated techniques have been developed. Among them, Dynamic Programming and the Discrete Maximum Principle (1) were developed for the optimization of stagewise processes.

In 1964, Fan and Wang developed a discrete version of the maximum principle (1). Recently, it was demonstrated that this optimization technique was applicable to transportation systems analysis (22, 23, 24).

Consider an N stage process with state variables denoted by an s-dimensional vector, $X = (X_1, X_2, \ldots X_s)$, and decision variables denoted by a r-dimensional vector, $\theta = (\theta_1, \theta_2, \ldots \theta_r)$. The performance equations at the n-th stage are given as:

$$X_i^n = T_i^n \left(X_1^{n-1}, X_2^{n-1}, \ldots, X_s^{n-1};\ \theta_1^n, \theta_2^n, \ldots, \theta_r^n\right)$$

$$X_i^0 = \alpha_i$$

where, $i = 1, 2, \ldots, s$; $n = 1, 2, \ldots, N$ and α_i is constant.

A typical optimization problem associated with such a process is to find a set of θ^n, $n = 1, 2, \ldots, N$, subject to constraints:

$$\psi_i^n (\theta_1^n, \theta_2^n, \ldots, \theta_r^n) \leq 0 \qquad n = 1, 2, \ldots, N$$
$$i = 1, 2, \ldots, r$$

which optimizes the objcetive function,

$$S = \sum_{i=1}^{s} C_i \times X_i^N \qquad C_i = \text{constant}$$

The discrete maximum principle introduces an s-dimensional vector, Z^n, and a Hamiltonian function, H^n, which satisfy the following relations:

$$H^n = \sum_{i=1}^{s} Z_i^n \; T_i^n \; (X^{n-1}; \; \theta^n), \qquad N = 1, 2, \ldots, N$$

$$Z_i^{n-1} = \frac{H^n}{X_i^{n-1}} \qquad i = 1, 2, \ldots, s; \quad n = 1, 2, \ldots, N$$

and $Z_i^N = C_i$

The necessary condition for S to be a local extreme with respect to θ is

$$\frac{\partial H^n}{\partial \theta^n} = 0$$

when it is inside the boundary of the constraints, or

H^n = extreme

when it is on the boundary of the constraints.

If the objective function is a cumulated measure which can be expressed as

$$S = \sum_{n=1}^{N} \psi (X^{n-1}; \; \theta^n),$$

the algorithm can be extended by introducing an extra state variable X_{s+1}, defined as

$$X_{s+1}^0 = 0$$

$$X_{s+1}^n = X_{s+1}^{n-1} + \psi(X^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N$$

The new equations together with the original performance equations specify the process in s+1 variables where the objective function becomes

$$S = X_{s+1}^N = \sum_{n=1}^{N} \psi(X^{n-1}; \theta^n)$$

and the primal algorithm of the principle is restored.

In case that some of the state variables at the end stage, X_i^N, are fixed, the relation $Z_i^N = C_i$ no longer exists. The following equation

$$\frac{\partial H^N}{\partial \theta_K^N} = \sum_{j=1}^{s} Z_j \frac{X_i^N}{\theta_K^N} = 0$$

can be used for each fixed end stage variable. By solving these equations simultaneously, the Z_i^N values can be determined (21).

Recently, the discrete maximum principle has been successfully applied to the traffic assignment problem. In 1964, Yang and Snell (22) formulated the traffic assignment problem by considering each node of the network as a stage. The unit travel time on each link was assumed constant and a turning penalty was included in the travel time equation. The objective was to minimize the total travel time. Due to the linear characteristic of the objective function, the optimum searching procedure was reduced to a shortest path tree building routine which is similar to Moore's algorithm.

In 1966, Snell, Funk and Blackburn (23) developed a more complete model. In this model, travel time was non-linearly related to traffic volume. This characteristic is considered to be a step toward more realistic traffic assignment. In 1967, Funk and Snell (24) developed a procedure for an approximate multicopy traffic assignment problem. The results obtained from this procedure appear to be very close to the true optimum.

THE OBJECTIVE FUNCTION AND TRAVEL TIME EQUATION

As previously explained, the objective of this study was to minimize the sum of the investment cost and the travel time cost. The investment was an independent variable and it was assumed that it could be expressed in terms of dollars per mile. However unit travel time was, in general, dependent on traffic volume and roadway conditions. In other words, unit travel time was a function of both traffic volume and investment. The relationship among them was, in reality, very complex. In developing a mathematical model, it was generally necessary to make some assumptions and simplify the relationship in order to express the relationship by a relatively simple equation which was manageable and yet not too far from reality.

To express unit travel time as a function of traffic volume and investment, some basic characteristics were observed:

1. Unit travel time increased as the traffic volume increased.

2. Unit travel time decreased as the investment increased.

3. Unit travel time had a lower limit. (free flow travel time)

4. With constant travel time, service volume increased as the investment increased.

The typical relation between traffic volume and operating speed is shown in Fig. 1a. As the speed is inversely proportional to the travel time, this curve can be converted into a travel time-traffic volume relation curve as shown in Fig. 1b. The dotted part of the curve shows the relation under congested conditions. Therefore, under normal operating conditions, it is logical to assume that unit travel time (in hours per vehicle per mile) is linearly related to traffic volume and should have an equation of the following form:

$$t = K + K'V \tag{1}$$

where

t = unit travel time (hr/mi/veh)

K = free flow travel time (hr/mi/veh)

K' = slope of the curve in Fig. 1b ($hr^2/mi/veh^2$)

V = traffic volume per unit time (veh/hr)

Keeping basic characteristics in mind and further assuming that the free flow travel time is constant for each link and traffic volume served is proportional to investment for a constant travel time, an equation of the following form may be hypothesized:

$$t = K_1 + \frac{K_2}{\sigma} V \tag{2}$$

where

t = unit travel time (hr/mi/veh)

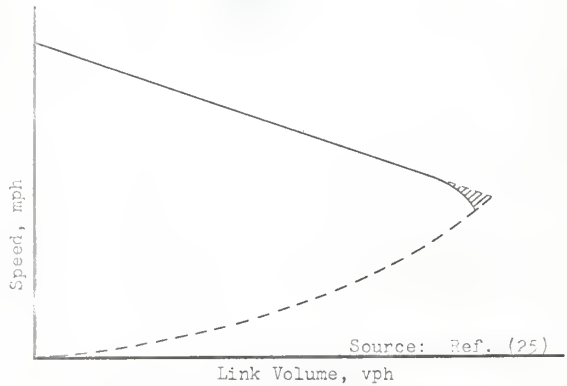


Fig. 1a Typical Speed-Volume Curve

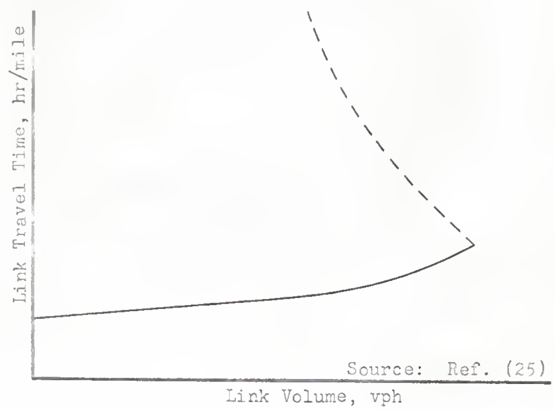


Fig. 1b Typical Travel Time-Volume Curve

K_1 = free flow travel time (hr/mi/veh). The magnitude depends on the maximum speed obtainable or regulated.

K_2 = coefficient of improvement (dollar-hr/mi^2/veh^2). Its magnitude depends on link location and reflects the difficulty of improvement.

θ = equivalent hourly investment per unit length (dollar/mi/hr).

V = traffic volume per unit time (veh/hr).

In the case where old facilities exist, the investment should be expressed as:

$$\theta = K_3 + \theta' \tag{3}$$

where, K_3, in dollars per mile per hour, represents the existing investment and θ', in dollars per mile per hour, is the additional investment.

The general form of the unit travel time equation then becomes

$$t = K_1 + \frac{K_2}{K_3 + \theta'} V \tag{4}$$

The characteristics of this equation are demonstrated in Figure 2a, 2b and 2c.

Let L be the length of the link and C_t the cost of time. The objective function then becomes

$$S = \theta' L + (K_1 V + \frac{K_2}{K_3 + \theta'} V^2) L C_t \tag{5}$$

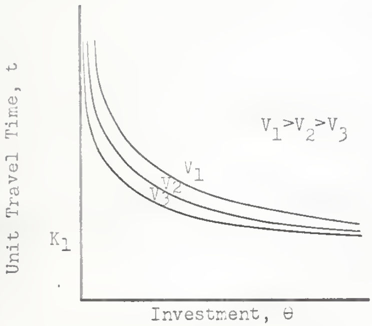


Fig. 2a Travel Time-Investment Curve With Fixed Volume

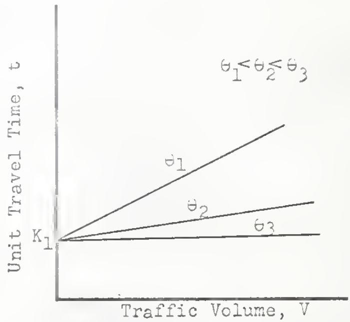


Fig. 2b Travel Time-Volume Curve With Fixed Investment

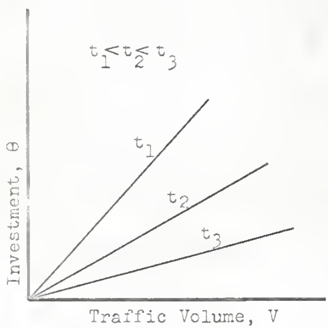


Fig. 2c Volume-Investment Curve With Fixed Travel Time

GENERAL FORMULATION OF THE PROBLEM

This section presents the general formulation of the optimal network improvement problem. Three investment conditions are considered which result in three different sets of equations and two slightly different methods of seeking the optimum.

The formulation is applied to rectangular networks only. However, for many non-rectangular networks, it is possible to modify them into rectangular forms by adding slack links. Two examples are shown in Fig. 3.

Definitions

1. Objective Function - a function, which is to be minimized in this problem, representative of the total cost.
2. Zone Centroid - a point of trip origin or destination.
3. Node - a point where segments of the road system connect.
4. Link - a connection between two nodes representative of a segment of the road system.
5. Path - a series of connected links representative of a trip route.
6. Network - the combination of all links and nodes.

Notations

1. $X_j^{n,m}$ - state variables representing flows from node (n,m).

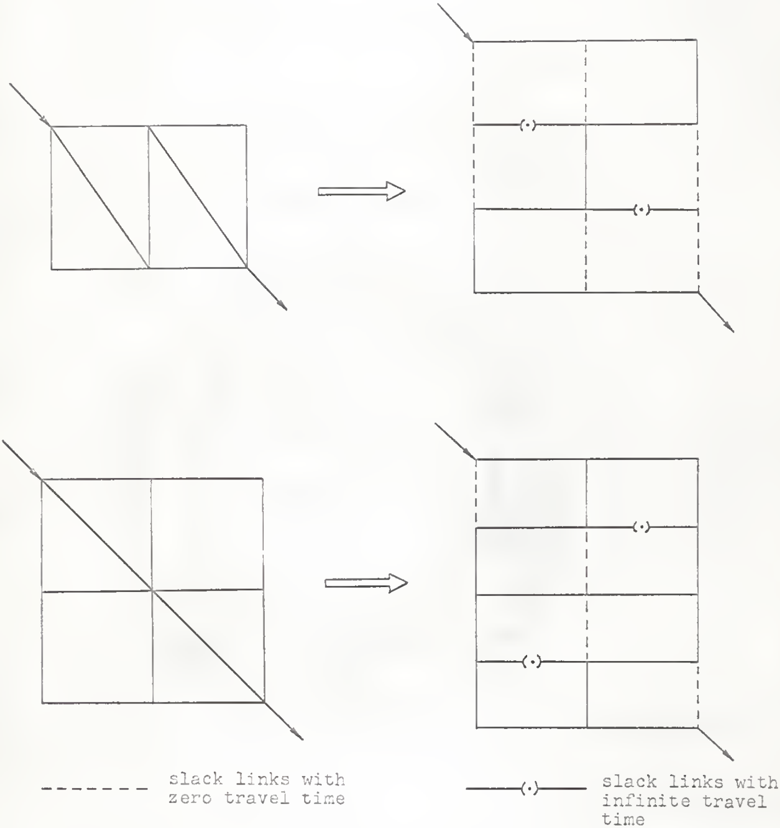


Fig. 3 Examples of Network Modification

2. $\theta_j^{n,m}$ - decision variables representing investments on links leaving node (n,m).

3. $K_{j1}^{n,m}$ - Free flow travel time on links leaving node (n,m).

4. $K_{j2}^{n,m}$ - coefficient of investment on links leaving node (n,m).

5. $K_{j3}^{n,m}$ - existing investment on links leaving node (n,m).

6. $L_j^{n,m}$ - link length on links leaving node (n,m).

7. $t_j^{n,m}$ - unit travel time on links leaving node (n,m).
   where, j = 1, for horizontal links.
   j = 2, for vertical links.

8. $X_3^{n,m}$ - state variable representing the total investment on horizontal links from node (1,1) through node (n,m).

9. $X_4^{n,m}$ - state variable representing the total investment on vertical links from node (1,1) through node (n,m).

10. $X_5^{n,m}$ - state variable representing the total travel time cost on horizontal links from node (1,1) through node (n,m).

11. $X_6^{n,m}$ - state variable representing the total travel time cost on vertical links from node (1,1) through node (n,m).

12. $X_7^{n,m}$ - state variable representing the total investment on both links from node (1,1) through node (n,m).

13. $\theta_5^{n,m}$ - decision variable representing the fraction of the vehicles departing node (n,m) on the horizontal link.

14. C_t - time cost.

15. $H^{n,m}$ - Hamiltonian function at node (n,m).

16. $V^{n,m}$ - input trips at node (n,m).

17. GI - total system budget.

18. $SI^{n,m}$ - section budget at node (n,m).

19. $Z_1^{n,m}, Z_2^{n,m}, \ldots, Z_7^{n,m}$ - adjoint variables associated with $X_1^{n,m}, X_2^{n,m}, \ldots, X_7^{n,m}$ respectively.

20. S - objective function.

Assumptions

1. No turn penalties.
2. Zone centroids coincide with the nodes.
3. Traffic directions are preassigned.
4. Traffic distribution is fixed.
5. Transportation network can be represented by a rectangularly arranged combination of links.
6. Travel time is the only factor that influences the traffic assignment.

7. Unit travel time on each link can be expressed as:

$$t_j^{n,m} = K_{j1}^{n,m} + \frac{K_{j2}^{n,m}}{\theta_j^{n,m} + K_{j3}^{n,m}} X_j^{n,m} \tag{6}$$

where

$j = 1$, for horizontal links

$j = 2$, for vertical links

Figure 4 shows a basic N x M rectangular network with node (N,M) as the destination and all other nodes as origins. With the input trips at each node obtained from a traffic distribution study, the problem is to find an investment policy under each investment condition such that the total cost is a minimum.

Investment With No Budget Constraint

In this case, the overall system budget was assumed unlimited. However, there are three special conditions which imply upper or lower limits of investment on each link.

The performance equations for a typical interior node as shown in Fig. 5 were developed as follows:

$$X_1^{n,m} = (X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m})\theta_3^{n,m} = AI^{n,m}\theta_3^{n,m} \tag{7}$$

$$X_2^{n,m} = (X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m})(1-\theta_3^{n,m}) = AI^{n,m}(1-\theta_3^{n,m}) \tag{8}$$

$$X_3^{n,m} = \theta_1^{n,m} L_1^{n,m} + X_3^{n,m-1} \quad , \quad \theta_1^{n,m} \geq 0 \tag{9}$$

$$X_4^{n,m} = \theta_2^{n,m} L_2^{n,m} + X_4^{n,m-1} \quad , \quad \theta_2^{n,m} \geq 0 \tag{10}$$

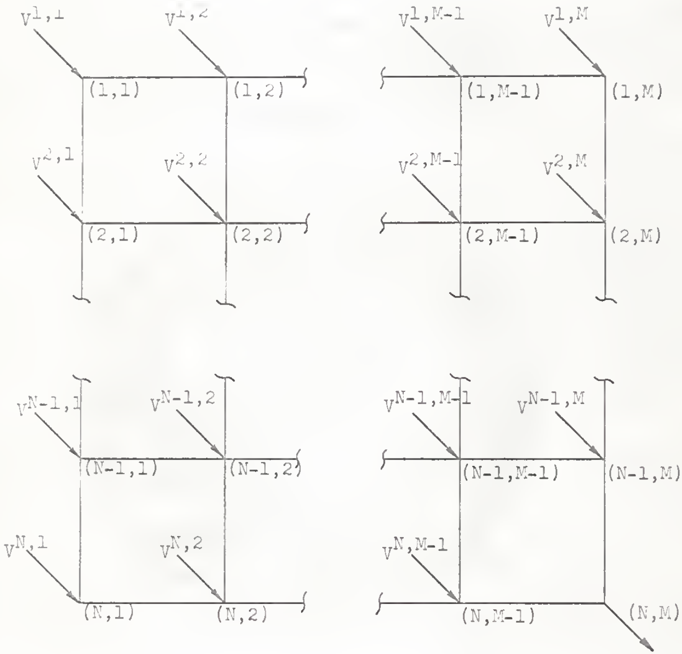


Fig. 4 Basic N x M Network

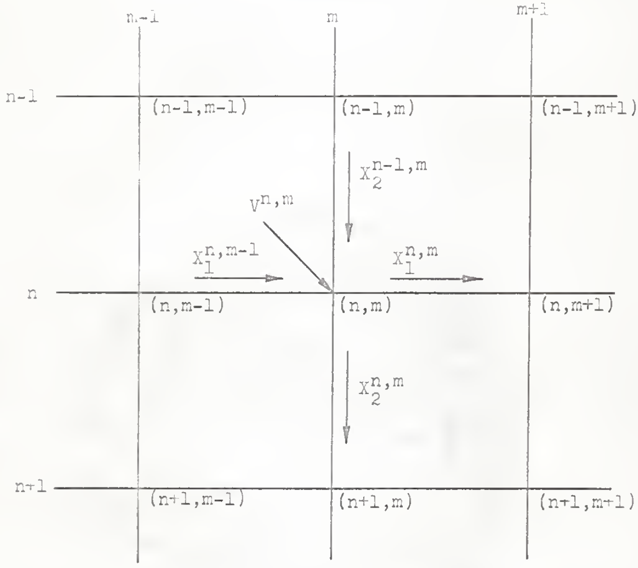


Fig. 5 Typical Interior Node of A Rectangular Network

$$X_5^{n,m} = K_{11}^{n,m}\ X_1^{n,m}\ L_1^{n,m}\ C_t + \frac{K_{12}^{n,m}\ L_1^{n,m}\ C_t}{\theta_1^{n,m} + K_{13}^{n,m}}\ (X_1^{n,m})^2 + X_5^{n,m-1}$$

$$= K_{11}^{n,m}\ AI^{n,m}\theta_3^{n,m}L_1^{n,m}C_t + \frac{K_{12}^{n,m}\ L_1^{n,m}\ C_t}{\theta_1^{n,m} + K_{13}^{n,m}}\ (AI^{n,m}\theta_3^{n,m})^2$$

$$+ X_5^{n,m-1} \tag{11}$$

$$X_6^{n,m} = K_{21}^{n,m}\ X_2^{n,m}\ L_2^{n,m}\ C_t + \frac{K_{22}^{n,m}\ L_2^{n,m}\ C_t}{\theta_2^{n,m} + K_{23}^{n,m}}\ (X_2^{n,m})^2 + X_6^{n,m-1}$$

$$= K_{21}^{n,m}\ AI^{n,m}(1-\theta_3^{n,m})L_2^{n,m}C_t + \frac{K_{22}^{n,m}\ L_2^{n,m}\ C_t}{\theta_2^{n,m} + K_{23}^{n,m}}\left[AI^{n,m}(1-\theta_3^{n,m})\right]^2$$

$$+ X_6^{n,m-1} \tag{12}$$

where $AI^{n,m} = X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m}$ (13)

$\theta_1^{n,m} \geq 0$

$\theta_2^{n,m} \geq 0$

and $0 \leq \theta_3^{n,m} \leq 1$

The Hamiltonian function at this node is defined as:

$$H^{n,m} = Z_1^{n,m}\ X_1^{n,m} + Z_2^{n,m}\ X_2^{n,m} + Z_3^{n,m}\ X_3^{n,m} + Z_4^{n,m}\ X_4^{n,m}$$

$$+ Z_5^{n,m}\ X_5^{n,m} + Z_6^{n,m}\ X_6^{n,m} \tag{14}$$

Substituting equations (7) to (12) into equation (14) and taking derivatives with respect to state variables, the adjoint variables are obtained as follows:

$$Z_1^{n,m-1} = \frac{\partial H^{n,m}}{\partial X_1^{n,m-1}} = Z_1^{n,m}\theta_3^{n,m} + Z_2^{n,m}(1-\theta_3^{n,m}) + Z_5^{n,m}K_{11}^{n,m}\theta_3^{n,m}L_1^{n,m}C_t$$

$$+ Z_6^{n,m}K_{21}^{n,m}(1-\theta_3^{n,m})\ L_2^{n,m}C_t$$

$$+ 2\ Z_5^{n,m}\ \frac{K_{12}^{n,m}\ L_1^{n,m}\ C_t}{\theta_1^{n,m} + K_{13}^{n,m}}\ AI^{n,m}(\theta_3^{n,m})^2$$

$$+2Z_6^{n,m}\ \frac{K_{22}^{n,m}\ L_2^{n,m}\ C_t}{\theta_2^{n,m} + K_{23}^{n,m}}\ AI^{n,m}(1-\theta_3^{n,m})^2 \qquad (15)$$

$$Z_2^{n-1,m} = \frac{\partial H^{n,m}}{\partial X_2^{n-1,m}} = Z_1^{n,m}\theta_3^{n,m} + Z_2^{n,m}(1-\theta_3^{n,m}) + Z_5^{n,m}K_{11}^{n,m}\theta_3^{n,m}L_1^{n,m}C_t$$

$$+ Z_6^{n,m}K_{21}^{n,m}(1-\theta_3^{n,m})\ L_2^{n,m}C_t$$

$$+ 2\ Z_5^{n,m}\ \frac{K_{12}^{n,m}\ L_1^{n,m}C_t}{\theta_1^{n,m} + K_{13}^{n,m}}\ AI^{n,m}(\theta_3^{n,m})^2$$

$$+ 2\ Z_6^{n,m}\ \frac{K_{22}^{n,m}\ L_2^{n,m}C_t}{\theta_2^{n,m} + K_{23}^{n,m}}\ AI^{n,m}(1-\theta_3^{n,m})^2 \qquad (16)$$

$$Z_3^{n,m-1} = \frac{\partial H^{n,m}}{\partial X_3^{n,m-1}} = Z_3^{n,m} \qquad (17)$$

$$Z_4^{n-1,m} = \frac{\partial H^{n,m}}{\partial X_4^{n-1,m}} = Z_4^{n,m} \tag{18}$$

$$Z_5^{n,m-1} = \frac{\partial H^{n,m}}{\partial X_5^{n,m-1}} = Z_5^{n,m} \tag{19}$$

$$Z_6^{n-1,m} = \frac{\partial H^{n,m}}{\partial X_6^{n-1,m}} = Z_6^{n,m} \tag{20}$$

The original conditions for the state variables are given as:

$$X_1^{o,o} = X_2^{o,o} = X_3^{o,o} = X_4^{o,o} = X_5^{o,o} = X_6^{o,o} = 0 \tag{21}$$

The objective function is

$$S = X_3^{N,M} + X_4^{N,M} + X_5^{N,M} + X_6^{N,M} \tag{22}$$

Therefore, by definition, the boundary conditions for the adjoint variables are:

$$Z_1^{N,M} = Z_2^{N,M} = 0 \tag{23}$$

$$Z_3^{N,M} = Z_4^{N,M} = Z_5^{N,M} = Z_6^{N,M} = 1 \tag{24}$$

Substituting equation (24) into equations (17) to (20), the following equation is derived.

$$Z_3^{n,m} = Z_4^{n,m} = Z_5^{n,m} = Z_6^{n,m} = 1 \qquad \text{for all } (n,m) \tag{25}$$

The Hamiltonian function then becomes:

$$H^{n,m} = Z_1^{n,m}X_1^{n,m} + Z_2^{n,m}X_2^{n,m} + X_3^{n,m} + X_4^{n,m} + X_5^{n,m} + X_6^{n,m} \tag{26}$$

In order to have S a minimum, the following conditions are necessary:

$$\frac{\partial H^{n,m}}{\partial \theta_1^{n,m}} = 0 \qquad \theta_1^{n,m} > 0$$

$$\frac{\partial H^{n,m}}{\partial \theta_2^{n,m}} = 0 \qquad \theta_2^{n,m} > 0$$

$$\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} = 0 \qquad 0 < \theta_3^{n,m} < 1$$

when $\left(\theta_1^{n,m}, \theta_2^{n,m}, \theta_3^{n,m}\right)$ is an interior point, or $H^{n,m}$ = minimum with respect to those $\theta_j^{n,m}$ which are at a boundary point of the constraints.

Substituting equations (7) to (12) into equation (26) and taking derivatives with respect to the various decision variables, the following equations are obtained:

$$\frac{\partial H^{n,m}}{\partial \theta_1^{n,m}} = L_1^{n,m} - \frac{K_{12}^{n,m}\ L_1^{n,m}\ C_t}{(\theta_1^{n,m} + K_{13}^{n,m})^2}\ (AI^{n,m}\theta_3^{n,m})^2 \tag{27}$$

$$\frac{\partial H^{n,m}}{\partial \theta_2^{n,m}} = L_2^{n,m} - \frac{K_{22}^{n,m}\; L_2^{n,m} C_t}{(\theta_2^{n,m} + K_{23}^{n,m})^2} \left[AI^{n,m}(1-\theta_3^{n,m})\right]^2 \tag{28}$$

$$\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} = (Z_1^{n,m} - Z_2^{n,m})AI^{n,m} + (K_{11}^{n,m}L_1^{n,m} - K_{21}^{n,m}L_2^{n,m})AI^{n,m}C_t$$

$$+ 2\frac{K_{12}^{n,m}\; L_1^{n,m}\; C_t}{\theta_1^{n,m} + K_{13}^{n,m}} (AI^{n,m})^2 \theta_3^{n,m}$$

$$- 2\frac{K_{22}^{n,m}\; L_2^{n,m}\; C_t}{\theta_2^{n,m} + K_{23}^{n,m}} (AI^{n,m})^2 (1-\theta_3^{n,m}) \tag{29}$$

Taking derivative of equation (29) with respect to $\theta_3^{n,m}$, the following equation is obtained:

$$\frac{\partial^2 H^{n,m}}{(\partial \theta_3^{n,m})^2} = 2\frac{K_{12}^{n,m}\; L_1^{n,m}\; C_t}{\theta_1^{n,m} + K_{13}^{n,m}} (AI^{n,m})^2 + 2\frac{K_{22}^{n,m}\; L_2^{n,m} C_t}{\theta_2^{n,m} + K_{23}^{n,m}} (AI^{n,m})^2 \tag{30}$$

Setting equations (27) and (28) equal to zero and applying the boundary conditions of the decision variables, the values of $\theta_1^{n,m}$ and $\theta_2^{n,m}$ can be obtained from the following equations:

$$\theta_1^{n,m} = \sqrt{K_{12}^{n,m} C_t}\; AI^{n,m}\theta_3^{n,m} - K_{13}^{n,m} \qquad \text{when } \theta_1^{n,m} > 0 \tag{31}$$

or

$$\theta_1^{n,m} = 0 \quad \text{when} \sqrt{K_{12}^{n,m}C_t}\, AI^{n,m}\theta_3^{n,m} - K_{13}^{n,m} \leq 0 \tag{32}$$

$$\theta_2^{n,m} = \sqrt{K_{22}^{n,m}C_t}\, AI^{n,m}(1-\theta_3^{n,m}) - K_{23}^{n,m} \quad \text{when } \theta_2^{n,m} > 0 \tag{33}$$

or

$$\theta_2^{n,m} = 0 \quad \text{when} \quad \sqrt{K_{22}^{n,m}C_t}\, AI^{n,m}(1-\theta_3^{n,m}) - K_{23}^{n,m} \leq 0 \tag{34}$$

When both $\theta_1^{n,m}$ and $\theta_2^{n,m}$ are greater than zero, equations (31) and (33) can be substituted into equation (29) to obtain the following equation

$$\begin{aligned} \frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} &= (Z_1^{n,m} - Z_2^{n,m})\, AI^{n,m} + (K_{11}^{n,m}L_1^{n,m} - K_{21}^{n,m}L_2^{n,m})AI^{n,m}C_t \\ &\quad + 2\sqrt{K_{12}^{n,m}C_t}\,L_1^{n,m}AI^{n,m} - 2\sqrt{K_{22}^{n,m}C_t}\,L_2^{n,m}AI^{n,m} \\ &= AI^{n,m}\Big[(Z_1^{n,m} - Z_2^{n,m}) + (K_{11}^{n,m}L_1^{n,m} - K_{21}^{n,m}L_2^{n,m})C_t \\ &\quad + 2\Big(\sqrt{K_{12}^{n,m}C_t}\; L_1^{n,m} - \sqrt{K_{22}^{n,m}C_t}\; L_2^{n,m}\Big)\Big] \end{aligned} \tag{35}$$

$\theta_3^{n,m}$ is eliminated by the substitution and the value of $\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}}$ becomes independent of $\theta_3^{n,m}$ as shown in equation (35). This implies that the value of $H^{n,m}$ is linearly related to $\theta_3^{n,m}$ and the extreme of $H^{n,m}$ with respect to $\theta_3^{n,m}$ occurs at a boundary. In this case, to obtain the minimum value of $H^{n,m}$, $\theta_3^{n,m} = 0$ when $\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} > 0$ or $\theta_3^{n,m} = 1$

when $\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} < 0$. If $\frac{H^{n,m}}{\theta_3^{n,m}} = 0$, $\theta_3^{n,m}$ can be any value between 0 and 1 because the value of $H^{n,m}$ is independent of $\theta_3^{n,m}$.

When either $\theta_1^{n,m}$ or $\theta_2^{n,m}$ is equal to zero, or when both are equal to zero, equation (35) is no longer valid. Equation (29) is then set equal to zero and solved for the optimal value of $\theta_3^{n,m}$.

Special Case I: In an urban area, the available space for road construction is often limited. For example, a freeway with more than eight lanes would be very difficult to build near a CBD area. It is, therefore, necessary to set an upper limit on the size of the links. These limits can be expressed as limits on investment. Mathematically, they are expressed as:

$$K_{11}^{n,m} + \theta_1^{n,m} \leq \theta_{1\,max.}^{n,m}$$

$$K_{21}^{n,m} + \theta_2^{n,m} \leq \theta_{2\,max.}^{n,m}$$

Equations (31) and (33) are then replaced by:

$$\theta_1^{n,m} = \sqrt{K_{12}^{n,m} C_t}\; AI^{n,m} \quad \theta_3^{n,m} - K_{13}^{n,m} ,$$

$$0 < \theta_1^{n,m} \leq (\theta_{1\,max.}^{n,m} - K_{11}^{n,m}) \qquad (36)$$

$$\theta_2^{n,m} = \sqrt{K_{22}^{n,m} C_t \; AI^{n,m}(1-\theta_3^{n,m})} - K_{23}^{n,m} \, ,$$

$$0 < \theta_2^{n,m} \leq (\theta_{2\ \max.}^{n,m} - K_{21}^{n,m}) \tag{37}$$

The other equations remain unchanged.

Special Case II: In developing a urban transportation network, it is sometimes required to provide a minimum level of service for the entire area. For example, arterial streets would be distributed uniformly throughout the whole area. This criterion can be fulfilled by requiring a minimum amount of investment on each link. Mathematically, it can be expressed as:

$$K_{11}^{n,m} + \theta_1^{n,m} \geq \theta_{1\ \min.}^{n,m} \tag{38}$$

$$K_{21}^{n,m} + \theta_2^{n,m} \geq \theta_{2\ \min.}^{n,m} \tag{39}$$

Special Case III: When the conditions governing both special cases I and II exist, both upper and lower limits of investment should be applied to each link. Mathematically, they are expressed as:

$$\theta_{1\ \min.}^{n,m} \leq K_{11}^{n,m} + \theta_1^{n,m} \leq \theta_{1\ \max.}^{n,m} \tag{40}$$

$$\theta_{2\ \min.}^{n,m} \leq K_{21}^{n,m} + \theta_2^{n,m} \leq \theta_{2\ \max.}^{n,m} \tag{41}$$

The above formulation provides the equations for searching the optimum sequence of the decision variables, $\theta_1^{n,m}$, $\theta_2^{n,m}$ and $\theta_3^{n,m}$ and the associated values of the state variables.

The optimum seeking procedure developed for this problem is as follows:

1. Assume a set of decision variables, $\theta_3^{n,m}$.
2. Calculate $X_1^{n,m}$, $X_2^{n,m}$ and $AI^{n,m}$ by equations (7) (8) and (13) starting at n=m=1 and proceeding to n=N and m=M.
3. Calculate decision variables, $\theta_1^{n,m}$ and $\theta_2^{n,m}$, by equations (31) and (33) and check the boundary conditions for each special case.
4. Calculate the values of $X_i^{n,m}$, i = 3, 4, 5, 6, by equations (9) to (13) starting at n=m=1 and proceeding to n=N and m=M.
5. Calculate the adjoint vectors, $Z_i^{n,m}$, i = 1, 2, with the above $X_i^{n,m}$ values, by equations (15), (16), (23) and (25) starting at n=N, m=M and proceeding backward to n=m=1.
6. Using the above values of $X_i^{n,m}$ and $Z_i^{n,m}$, calculate $\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}}$ and $\frac{\partial^2 H^{n,m}}{(\partial \theta_3^{n,m})^2}$ by equations (29) and (30).
7. Adjust the values of $\theta_3^{n,m}$ by adding an amount equal to Δ, where

$$\Delta = - \frac{\dfrac{\partial H^{n,m}}{\partial \theta_3^{n,m}}}{\dfrac{\partial^2 H^{n,m}}{(\partial \theta_3^{n,m})^2}}$$

and check the boundary condition.

8. With the new values of $\theta_3^{n,m}$, return to step 2 and repeat the procedure until the value of the objective function, equation (22), is sufficiently close to the previous value to indicate adequate convergence.

Investment With Fixed Node Investment

In developing a large area trunk line system, the budget for each traffic section is sometimes predetermined. Suppose that we consider each node and its associated two links as a traffic section where a fixed budget is allocated, then a different formulation could be developed.

The budget condition in this case can be expressed as:

$$\theta_1^{n,m} + \theta_2^{n,m} = SI^{n,m} \tag{42}$$

where, $\theta_1^{n,m}$ and $\theta_2^{n,m}$ are total investments instead of investments per unit length as in the previous case. Let $\theta_1^{n,m}$ be the independent variable, then $\theta_2^{n,m}$ can be expressed as:

$$\theta_2^{n,m} = SI^{n,m} - \theta_1^{n,m} \tag{43}$$

Since the total investment is fixed, the individual investment costs are no longer included in the objective function and also will not be expressed as state variables. The performance equations for a typical interior node as shown in Fig. 5 can be written as follows:

$$X_1^{n,m} = (X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m})\theta_3^{n,m} = AI^{n,m}\theta_3^{n,m} \tag{44}$$

$$X_2^{n,m} = (X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m})(1-\theta_3^{n,m}) = AI^{n,m}(1-\theta_3^{n,m}) \tag{45}$$

$$X_5^{n,m} = K_{11}^{n,m}\ L_1^{n,m}C_t\ \ AI^{n,m}\ \theta_3^{n,m} + \frac{K_{12}^{n,m}\ L_1^{n,m}\ C_t}{\dfrac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}}\,(AI^{n,m}\theta_3^{n,m})^2 + X_5^{n,m-1} \tag{46}$$

$$X_6^{n,m} = K_{21}^{n,m}\ L_1^{n,m}C_t\ \ AI^{n,m}(1-\theta_3^{n,m}) + \frac{K_{22}^{n,m}\ L_2^{n,m}\ C_t}{\dfrac{SI^{n,m}-\theta_1^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}}\,[AI^{n,m}(1-\theta_3^{n,m})]^2 + X_6^{n,m-1} \tag{47}$$

where, $AI^{n,m} = X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m}$ (48)

$$0 \le \theta_3^{n,m} \le 1$$

and $0 \le \theta_1^{n,m} \le SI^{n,m}$

The Hamiltonian function in turn becomes

$$H^{n,m} = Z_1^{n,m} X_1^{n,m} + Z_2^{n,m} X_2^{n,m} + Z_5^{n,m} X_5^{n,m} + Z_6^{n,m} X_6^{n,m} \quad (49)$$

the objective function is

$$S = X_5^{N,M} + X_6^{N,M} \quad (50)$$

The values of the adjoint variables are as follows:

$$Z_1^{n,m-1} = Z_2^{n-1,m} = Z_1^{n,m}\theta_3^{n,m} + Z_2^{n,m}(1-\theta_3^{n,m}) + K_{11}^{n,m}L_1^{n,m}C_t(1-\theta_3^{n,m})$$

$$+ 2 \frac{K_{12}^{n,m} L_1^{n,m} C_t}{\frac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}} AI^{n,m}(\theta_3^{n,m})^2$$

$$+ 2 \frac{K_{22}^{n,m} L_2^{n,m} C_t}{\frac{SI^{n,m}-\theta_1^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}} AI^{n,m}(1-\theta_3^{n,m})^2 \quad (51)$$

$$Z_5^{n,m} = Z_6^{n,m} = 1 \qquad \text{for all } (n,m) \quad (52)$$

and

$$Z_1^{N,M} = Z_2^{N,M} = 0 \quad (53)$$

The initial values of state variables are

$$X_1^{o,o} = X_2^{o,o} = X_5^{o,o} = X_6^{o,o} = 0 \quad (54)$$

To find the minimum value of S, the following conditions are necessary

$$\frac{\partial H^{n,m}}{\partial \theta_1^{n,m}} = 0 \qquad 0 < \theta_1^{n,m} < SI^{n,m} \tag{55}$$

$$\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} = 0 \qquad 0 < \theta_3^{n,m} < 1 \tag{56}$$

when $\left(\theta_1^{n,m},\ \theta_3^{n,m}\right)$ is inside the boundary, or

$$H^{n,m} = \text{minimum} \tag{57}$$

when $\left(\theta_1^{n,m},\ \theta_3^{n,m}\right)$ is at a boundary point of the constraints.

Substituting equations (44) to (48) and (52) into equation (49) and then taking derivatives with respect to $\theta_1^{n,m}$ and $\theta_3^{n,m}$, the following equations are obtained:

$$\frac{\partial H^{n,m}}{\partial \theta_1^{n,m}} = - \frac{K_{12}^{n,m}\ C_t}{\left(\frac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}\right)^2} \left(AI^{n,m}\theta_3^{n,m}\right)^2$$

$$+ \frac{K_{22}^{n,m}\ C_t}{\left(\frac{SI^{n,m}-\theta_1^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}\right)^2} \left[AI^{n,m}(1-\theta_3^{n,m})\right]^2 \tag{58}$$

$$\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} = (Z_1^{n,m} - Z_2^{n,m})AI^{n,m} + (K_{11}^{n,m}\, L_1^{n,m} - K_{21}^{n,m}\, L_2^{n,m})C_t AI^{n,m}$$

$$+ 2\frac{K_{12}^{n,m}\, L_1^{n,m}\, C_t}{\dfrac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}}\,(AI^{n,m})^2\;\; \theta_3^{n,m}$$

$$- 2\frac{K_{22}^{n,m}\, L_2^{n,m}\, C_t}{\dfrac{SI^{n,m}-\theta_1^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}}\,(AI^{n,m})^{\,2}\;(1-\theta_3^{n,m}) \qquad (59)$$

Setting equation (58) equal to zero and solving for $\theta_1^{n,m}$, we obtain

$$\theta_1^{n,m} = \frac{\sqrt{K_{12}^{n,m}}(SI^{n,m} + K_{23}^{n,m} L_2^{n,m})\theta_3^{n,m} L_1^{n,m} \mp \sqrt{K_{22}^{n,m}}\, K_{13}^{n,m}(1-\theta_3^{n,m}) L_1^{n,m} L_2^{n,m}}{\sqrt{K_{22}^{n,m}}(1-\theta_3^{n,m}) L_2^{n,m} \mp \sqrt{K_{12}^{n,m}}\; \theta_3^{n,m}\; L_1^{n,m}}$$

$$, \ldots \quad (60)$$

Using equations (44) through (58), the optimum seeking procedure developed in the previous case was again applied to solve the problem.

Investment with Fixed System Budget:

It is not unusual for the total budget for a transportation system improvement to be predetermined. In this case, the total investment must be equal to the budget. This, then, becomes a fixed end point problem as described in reference (21).

The performance equations for a typical interial node as shown in Fig. 5 can be written as follows:

$$X_1^{n,m} = (X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m})\theta_3^{n,m} = AI^{n,m}\theta_3^{n,m} \tag{61}$$

$$X_2^{n,m} = (X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m})(1-\theta_3^{n,m}) = AI^{n,m}(1-\theta_3^{n,m}) \tag{62}$$

$$X_5^{n,m} = K_{11}^{n,m}L_1^{n,m}C_t AI^{n,m}\theta_3^{n,m} + \frac{K_{12}^{n,m}\ L_1^{n,m}C_t}{\dfrac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}}(AI^{n,m}\theta_3^{n,m})^2 + X_5^{n,m-1} \tag{63}$$

$$X_6^{n,m} = K_{21}^{n,m}L_2^{n,m}C_t AI^{n,m}(1-\theta_3^{n,m}) + \frac{K_{22}^{n,m}\ L_2^{n,m}\ C_t}{\dfrac{\theta_2^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}}\left[AI^{n,m}(1-\theta_3^{n,m})\right]^2 + X_6^{n,m-1} \tag{64}$$

$$X_7^{n,m} = \theta_1^{n,m} + \theta_2^{n,m} + X_7^{n,m-1} \tag{65}$$

where, $$AI^{n,m} = X_1^{n,m-1} + X_2^{n-1,m} + V^{n,m} \tag{66}$$

and $0 \leq \theta_3^{n,m} \leq 1$

Here, $\theta_1^{n,m}$ and $\theta_2^{n,m}$ are total investments on the horizontal and vertical links respectively at node (n,m).

Since total investment is a fixed amount, the objective function becomes:

$$S = X_5^{N,M} + X_6^{N,M} \tag{67}$$

The Hamiltonian function is

$$H^{n,m} = Z_1^{n,m} X_1^{n,m} + Z_2^{n,m} X_2^{n,m} + Z_5^{n,m} X_5^{n,m} + Z_6^{n,m} X_6^{n,m}$$

$$+ Z_7^{n,m} X_7^{n,m} \tag{68}$$

The boundary conditions are given as follows:

$$X_1^{o,o} = X_2^{o,o} = X_5^{o,o} = X_6^{o,o} = X_7^{o,o} = 0 \tag{69}$$

$$Z_1^{N,M} = Z_2^{N,M} = 0 \tag{70}$$

$$Z_5^{N,M} = Z_6^{N,M} = 1 \tag{71}$$

$$X_7^{N,M-1} = X_7^{N,M} = GI \tag{72}$$

Since $X_7^{N,M}$ is fixed, $Z_7^{N,M}$ becomes an unknown. However, with $X_7^{N,M-1}$ fixed, $Z_7^{N,M-1}$ can be obtained by solving the following equation:

$$\frac{\partial H^{N,M-1}}{\partial \theta_1^{N,M-1}} = - \frac{K_{12}^{N,M-1} \; C_t}{\left(\frac{\theta_1^{N,M-1}}{L_1^{N,M-1}} + K_{13}^{N,M-1}\right)^2} (AI^{N,M-1})^2 + Z_7^{N,M-1} = 0 \tag{73}$$

where, $\theta_1^{N,M-1} = GI - X_7^{N,M-2}$

$$\text{Therefore, } Z_7^{N,M-1} = \frac{K_{12}^{N,M-1} C_t (AI^{N,M-1})^2}{\left(\frac{GI-X_7^{N,M-2}}{L_1^{N,M-1}} + K_{13}^{N,M-1}\right)^2} \tag{74}$$

$$\text{Since } Z_7^{n,m-1} = \frac{\partial H^{n,m}}{\partial X_7^{n,m-1}} = Z_7^{n,m}$$

$$Z_7^{n,m} = Z_7^{N,M-1} \qquad \text{for all } (n,m) \tag{75}$$

The values of the adjoint variables are as follows:

$$Z_1^{n,m-1} = Z_2^{n-1,m} = Z_1^{n,m}\theta_3^{n,m} + Z_2^{n,m}(1-\theta_3^{n,m}) + K_{11}^{n,m}\theta_3^{n,m}L_1^{n,m}C_t$$

$$+ K_{21}^{n,m}(1-\theta_3^{n,m})L_2^{n,m}C_t + 2\,\frac{K_{12}^{n,m} \; L_1^{n,m} \; C_t}{\frac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}}\, AI^{n,m}(\theta_3^{n,m})^2$$

$$+ 2\,\frac{K_{22}^{n,m}\,L_2^{n,m}\,C_t}{\dfrac{\theta_2^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}}\,AI^{n,m}(1-\theta_3^{n,m})^2 \tag{76}$$

$$Z_5^{n,m} = Z_6^{n,m} = 1 \qquad \text{for all } (n,m) \tag{77}$$

$$Z_7^{n,m} = \frac{K_{12}^{N,M-1}C_t\,(AI^{N,M-1})^2}{\left(\dfrac{GI-X_7^{N,M-2}}{L_1^{N,M-1}} + K_{13}^{N,M-1}\right)^2} \quad \text{for all } (n,m) \tag{78}$$

The necessary conditions for S to be a local minimum is that:

$$\frac{\partial H^{n,m}}{\partial \theta_1^{n,m}} = 0 \qquad 0 < \theta_1^{n,m} < GI - X_5^{n,m-1} - X_6^{n,m-1} \tag{79}$$

$$\frac{\partial H^{n,m}}{\partial \theta_2^{n,m}} = 0 \qquad 0 < \theta_2^{n,m} < GI - X_5^{n,m-1} - X_6^{n,m-1} - \theta_1^{n,m} \tag{80}$$

$$\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} = 0 \qquad 0 < \theta_3^{n,m} < 1 \tag{81}$$

when $\left(\theta_1^{n,m},\ \theta_2^{n,m},\ \theta_3^{n,m}\right)$ is an interior point, or

$$H^{n,m} = \text{minimum} \tag{82}$$

when $\left(\theta_1^{n,m},\ \theta_2^{n,m},\ \theta_3^{n,m}\right)$ is at a boundary point of the constraints.

Substituting equations (61) to (66) and equation (77) into equation (68) and taking derivatives with respect to various decision variables, the following equations are obtained:

$$\frac{\partial H^{n,m}}{\partial \theta_1^{n,m}} = - \frac{K_{12}^{n,m}(AI^{n,m}\theta_3^{n,m})^2 C_t}{\left(\frac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}\right)^2} + Z_7^{N,M-1} \tag{83}$$

$$\frac{\partial H^{n,m}}{\partial \theta_2^{n,m}} = - \frac{K_{22}^{n,m}\ AI^{n,m}(1-\theta_3^{n,m})^2 C_t}{\left(\frac{\theta_2^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}\right)^2} + Z_7^{N,M-1} \tag{84}$$

$$\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}} = (Z_1^{n,m} - Z_2^{n,m})AI^{n,m} + (K_{11}^{n,m}L_1^{n,m} - K_{21}^{n,m}L_2^{n,m})\ C_t AI^{n,m}$$

$$+ 2\ \frac{K_{12}^{n,m}(AI^{n,m})^2 \theta_3^{n,m} L_1^{n,m} C_t}{\frac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}}$$

$$- 2\ \frac{K_{22}^{n,m}(AI^{n,m})^2 (1-\theta_3^{n,m})\ L_2^{n,m} C_t}{\frac{\theta_2^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}} \tag{85}$$

Taking derivative of equation (85) with respect to $\theta_3^{n,m}$, we obtain

$$\frac{\partial^2 Z^{n,m}}{(\partial\theta_3^{n,m})^2} = 2\,\frac{K_{12}^{n,m}(AI^{n,m})^2 L_1^{n,m} C_t}{\dfrac{\theta_1^{n,m}}{L_1^{n,m}} + K_{13}^{n,m}} + 2\,\frac{K_{22}^{n,m}(AI^{n,m})^2 L_2^{n,m} C_t}{\dfrac{\theta_2^{n,m}}{L_2^{n,m}} + K_{23}^{n,m}} \qquad (86)$$

Setting equations (83) and (84) equal to zero, we obtain the following:

$$\theta_1^{n,m} = \frac{K_{12}^{n,m}\,C_t}{Z_7^{N,M-1}}\,AI^{n,m}\,\theta_3^{n,m}\,L_1^{n,m} - K_{13}^{n,m}\,L_1^{n,m} \qquad (87)$$

$$\theta_2^{n,m} = \frac{K_{22}^{n,m}\,C_t}{Z_7^{N,M-1}}\,AI^{n,m}(1-\theta_3^{n,m})L_2^{n,m} - K_{23}^{n,m}\,L_2^{n,m} \qquad (88)$$

The optimum seeking procedure developed for this problem is as follows:

1. Assume a set of decision variables $\left(\theta_1^{n,m},\ \theta_2^{n,m},\ \theta_3^{n,m}\right)$.

2. Calculate values of $X_i^{n,m}$, $i = 1, 2, 5, 6, 7$ and $AI^{n,m}$ starting at n=m=1 and proceeding to n=N, m=M.

3. a.) For the first iteration, calculate $Z_7^{N,M-1}$ by equation (74) with the above $X_i^{n,m}$ and $AI^{n,m}$ values and go to step 4.

   b.) For the second and the following iterations, calculate $Z_7^{N,M-1}$ by equation (74) with the above $X_i^{n,m}$ and $AI^{n,m}$ values. This $Z_7^{N,M-1}$ value is then compared with the value obtained in the previous iteration. If the two values are sufficiently close, proceed to step 6. If they are not sufficiently close, proceed to step 4.

4. Calculate new values of $\theta_1^{n,m}$ and $\theta_2^{n,m}$ using equations (87) and (88) and check the boundary conditions.

5. Return to step 2.

6. With the above $\theta_i^{n,m}$ and $X_i^{n,m}$ values, calculate $Z_1^{n,m}$ and $Z_2^{n,m}$ starting at n=N, m=M and proceeding backward to n=m=1 by the use of equations (70) through (76)

7. Using the above values of $X_i^{n,m}$ and $Z_i^{n,m}$, calculate $\frac{\partial H^{n,m}}{\partial \theta_3^{n,m}}$ and $\frac{\partial^2 H^{n,m}}{(\partial \theta_3^{n,m})^2}$ through the use of equations (85) and (86).

8. Adjust the values of $\theta_3^{n,m}$ by adding an amount equal to $\triangle$, where

$$\triangle = - \frac{\dfrac{\partial H^{n,m}}{\partial \theta_3^{n,m}}}{\dfrac{\partial^2 H^{n,m}}{(\partial \theta_3^{n,m})^2}}$$

and check the boundary conditions.

9. Return to step 2 and repeat the procedure until the value of the objective function [equation (67)] is sufficiently close to the previous value to indicate adequate convergence.

In the case where a minimum level of service is to be provided, the minimum investment can be treated as the existing facilities. The problem can then be solved by the general method without changing the algorithm. In other words, when the values of $K_{13}^{n,m}$ are less than the minimum required investment, set them equal to the minimum investment and deduct the difference from the total budget.

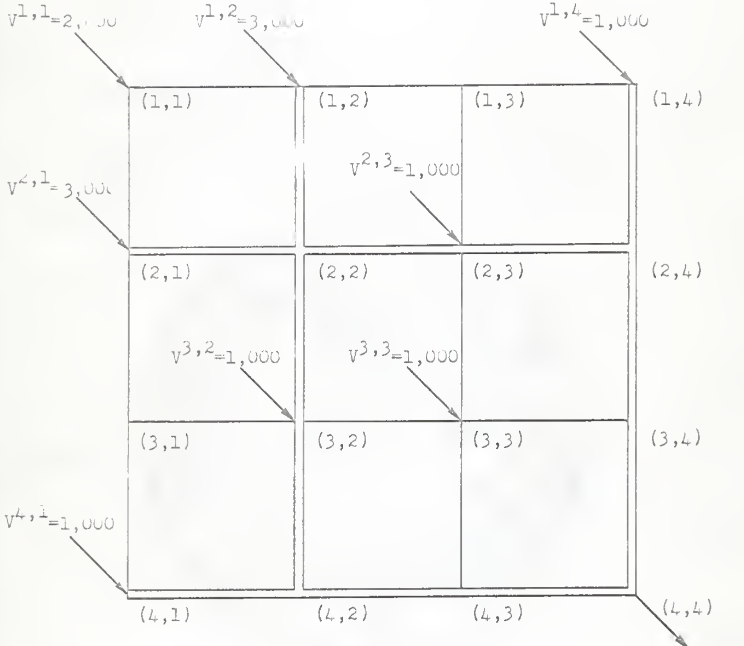
The above formulation provides solutions to a single-quadrant network, single-copy problem. To solve a multi-quadrant network, multi-copy problem, the procedures developed by Snell, et. al. (23, 24) can be employed. However, due to the capacity restriction of the available computer, this extension has not been accomplished.

EXAMPLES AND DISCUSSION

Several examples under different investment conditions are presented in this section to demonstrate the use of the model.

A hypothetical network was developed as shown in Fig. 6. Node (4,4) was assumed to be the centroid of the CBD. Peak hour trips which are produced in the other zones and destined to the CBD are also shown in the figure. All links have an equal length of one mile. The area was divided into two parts by a diagonal line which passes through nodes (1,4) and (4,1). The lower part which is adjacent to the CBD was assumed to be densely developed. The upper part was assumed to be less densely developed. Assuming the maximum speed in the densely developed area to be 60 mph and in the less densely developed area 70 mph, minimum travel times in these two areas become 0.0167 hour per mile and 0.0143 hour per mile respectively. Single line links represent existing local streets and double line links represent existing arterial streets.

Input data for the models are summarized in Table 1. Values of K_{i2} and K_{i3} are also indicated in Fig. 7 and Fig. 8 respectively. Since construction cost and right-of-way cost will not be the same in each area, two values of K_{i3} were assigned to the links even though these links represent the same type of facilities. For the same reason, in the link investment constraint model, links have different values for



input traffic in vph
all links have an equal length of 1 mile

Fig. 6 Hypothetical Network and Peak Hour Traffic Distribution

Nodes (r,m)	links (i)	$K_{i1}^{n,m}$	$K_{i2}^{n,m}$	$K_{i3}^{n,m}$	$\theta_{imax}^{n,m}$	$\theta_{imin}^{n,m}$	$V^{n,m}$	$SI^{n,m}$
1,1	1	0.0143	0.00003	8.0	80	10	2,000	40
	2	0.0143	0.00004	8.0	80	10		
1,2	1	0.0143	0.00006	8.0	80	10	3,000	40
	2	0.0143	0.00005	10.0	80	10		
1,3	1	0.0143	0.00008	8.0	80	15	0	60
	2	0.0143	0.00006	8.0	80	10		
1,4	1	0.0167	1.000	0.00001	100	15	1,000	40
	2	0.0167	0.00010	15.0	100	15		
2,1	1	0.0143	0.00005	10.0	80	10	3,000	40
	2	0.0143	0.00005	8.0	80	10		
2,2	1	0.0143	0.00006	10.0	80	10	0	60
	2	0.0143	0.00005	10.0	80	10		
2,3	1	0.0167	0.00010	15.0	100	15	1,000	80
	2	0.0167	0.00008	12.0	100	15		
2,4	1	0.0167	1,000	0.00001	100	15	0	50
	2	0.0167	0.00015	15:0	100	15		
3,1	1	0.0143	0.00006	8.0	80	10	0	60
	2	0.0143	0.00006	8.0	80	10		
3,2	1	0.0167	0.00008	12.0	100	15	1,000	80
	2	0.0167	0.00010	15.0	100	15		
3,3	1	0.0167	0.00015	12.0	100	15	1,000	100
	2	0.0167	0.00015	12.0	100	15		
3,4	1	0.0167	1.000	0.00001	100	15	0	60
	2	0.0167	0.00025	15.0	100	15		
4,1	1	0.0167	0.00008	15.0	100	15	1,000	40
	2	0.0167	1.000	0.00001	100	15		
4,2	1	0.0167	0.00015	15.0	100	15	0	50
	2	0.0167	1.000	0.00001	100	15		
4,3	1	0.0167	0.00020	15.0	100	15	0	60
	2	0.0167	1.00	0.00001	100	15		
4,4	1	0.0167	0.00001	0.00001	100	15	0	0
	2	0.0167	0.00001	0.00001	100	15		

i = 1 for horizontal links GI = \$300.00

i = 2 for vertical links C_t = \$1.55/hour

Table 1 Input Data of Example Problems

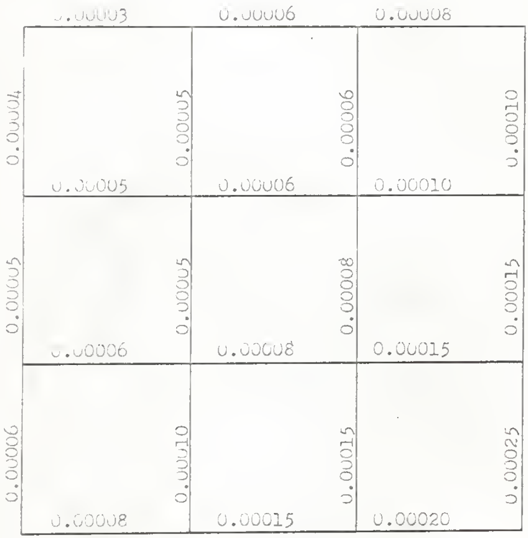


Fig. 7 K_{12} Values For The Example Problems

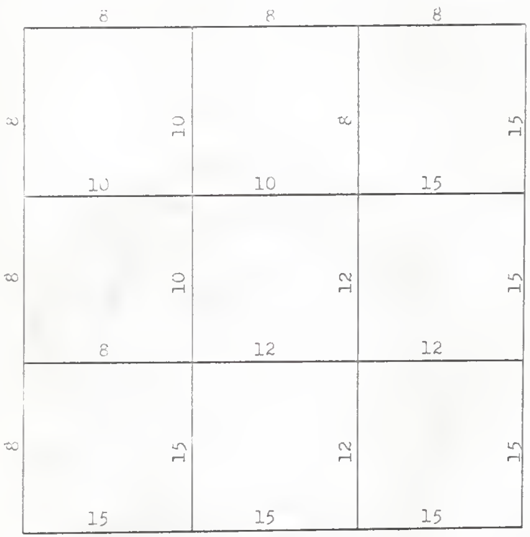


Fig. 8 K_{i3} Values For The Example Problems maximum and minimum investment levels. The derivation of these data is discussed in Appendix A. Time cost (C_t) is assumed to be \$1.55 per hour per vehicle as suggested by AASHO (8).

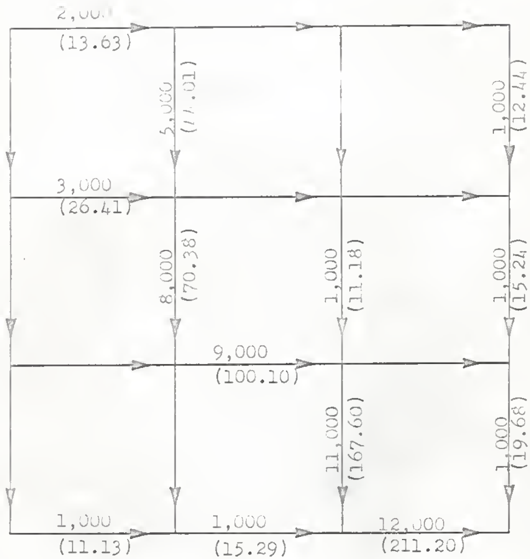
<u>Example 1</u>: Theoretical optimal system.

Suppose we are planning for a completely undeveloped area where no facilities exist and there is no budget limitation on link investment. A theoretical optimal system can then be developed to accommodate the predicted trip demand. Using the formulation of "Investment With No Budget Constraint" and letting $K_{i3}^{n,m} = 0$, for all (n,m), the resulting system is shown in Fig. 9. Notice that the system forms a shortest path tree in which only one route is built for each origin-destination pair and all trips are assigned to this route. This result coincides with the analysis discussed in page 33 which shows the linear characteristic of the problem under no limit condition.

<u>Example 2</u>: Optimal investment with upper and lower limits on link investment.

The hypothetical network shown in Fig. 6 is to be improved with the following conditions:

1. No system budget limit.
2. A minimum level of service (arterial street) is to be provided for the entire area.



Total Investment = \$ 718.63
Travel Time Cost = \$2,101.23
Total Cost = \$2,819.86

2,000 : Traffic volume
(13.63): Investment

Fig. 9 Optimal Investment and Traffic Assignment: Results of Example 1

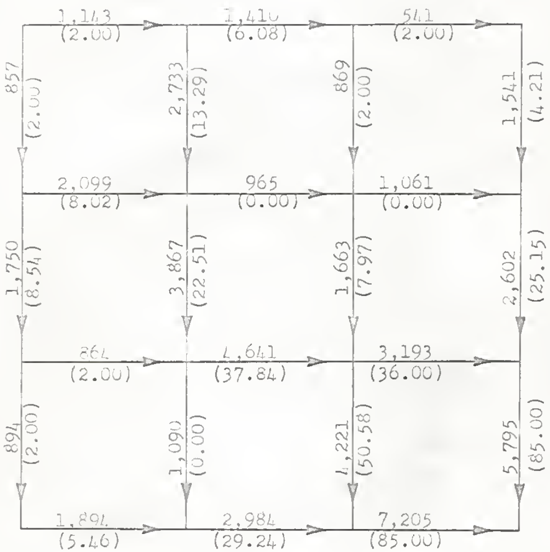
3. Roadway space obtainable is restricted.

The investment limits, $\theta_i$min. and $\theta_i$max., associated with conditions 2 and 3 are listed in Table 1. The formulation of this problem has been developed in the previous section under the category, "Investment with no budget constraint: special case III."

The results are shown in Fig. 10. Note that with the minimum level of service provided for the entire area, trips are assigned rather uniformly to take advantage of all facilities. When traffic is focused on the CBD, a space limitation is in effect which forces traffic to split and enter the CBD area from two directions. Considering existing facilities as part of the cost, total cost becomes $2,875.99 (2,603.99 + 272.00). Comparing this cost with the total cost in example 1 ($2,819.86), the difference is only about two percent. This indicates that providing a minimum level of service might be desirable in an urban area.

<u>Example 3</u>: Investment with fixed node budget.

This is the fixed node investment problem as formulated in a previous section. Investment for each node ($SI^{n,m}$) is listed in Table 1. Consider the area as completely undeveloped $K_{i3}^{n,m} = 0$), the resulting system is shown in Fig. 11. Regardless of the investment level, as compared with previous examples, travel time costs in this problem are greater than in the previous examples. This result demonstrates that



Total Investment = \$ 445.04
Travel Time Cost = \$2,158.95
Total Cost = \$2,603.99

1,143 : Traffic volume
(2.00) : Investment

Fig. 10 Optimal Investment and Traffic Assignment Results of Example 2

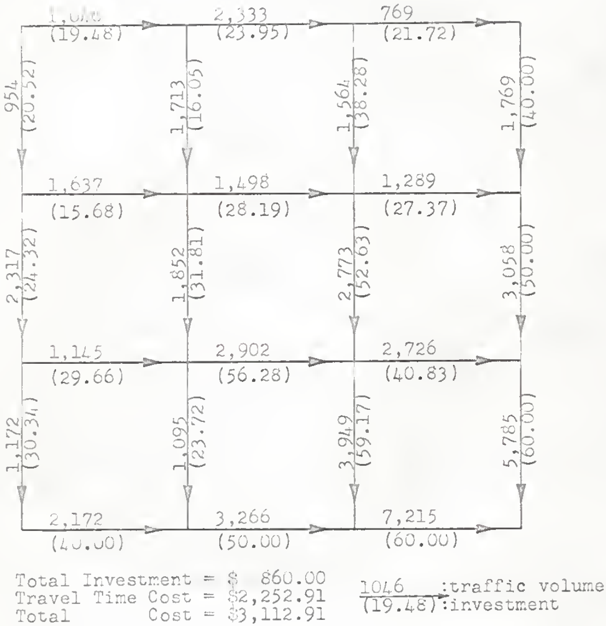


Fig. 11 Optimal Investment and Traffic Assignment: Results of Example 3

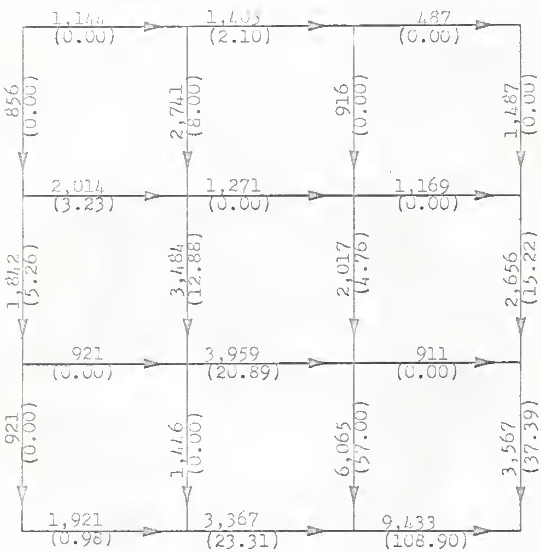
improper allocation of funds could be very costly. It also indicates the advantage of area-wise transportation system development which is carried out by a single authority.

Example 4: Investment with fixed system budget.

The hypothetical network as shown in Fig. 6 is to be improved with a total system budget of $300 (GI = 300, equivalent peak hour budget). The resulting traffic assignment and link investments are shown in Fig. 12. Most investment appears to be made along the shortest path trees as obtained from example 1 which are also the routes that will cost the least to improve. This is logical since the budget is substantially less than that required by a theoretical optimal system (compared with example 2). Comparing the costs with those obtained in example 2, it is evident that although investment cost decreases more than 30 percent, total cost increases only 1.4 percent. This again points out the advantage of area-wise transportation system development.

In order to verify the optimality of the results, two procedures were used.

1. Assuming a new set of decision variables ($\theta_1^{n,m}$, $\theta_2^{n,m}$, $\theta_3^{n,m}$) to start with, each problem was solved once more. The results were then compared with the previous ones. No significant differences between the two solutions of each problem were observed. Total costs are summarized in Table 2.



Total Investment = \$ 300.00
Travel Time Cost = \$2,339.38
Total Cost = \$2,639.38

1,144 : traffic volume
(0.00) : investment

Fig. 12 Optimal Investment and Traffic Assignment: Results of Example 4

Table 2 Number of Iterations, Approximate Computing Time Used And Total Costs for Example Problems

<table>
<tr><th>Example No.</th><th>Starting Point</th><th>Number of Iterations</th><th>Time Used (min.)</th><th>Total Cost ($)</th></tr>
<tr><td rowspan="2">1</td><td>$\theta_3^{n,m} = 0.7$</td><td>15</td><td>20</td><td>2,819.86</td></tr>
<tr><td>$\theta_3^{n,m} = 0.3$</td><td>25</td><td>30</td><td>2,819.87</td></tr>
<tr><td rowspan="2">2</td><td>$\theta_3^{n,m} = 0.7$</td><td>18</td><td>25</td><td>2,603.99</td></tr>
<tr><td>$\theta_3^{n,m} = 0.3$</td><td>18</td><td>25</td><td>2,604.39</td></tr>
<tr><td rowspan="2">3</td><td>$\theta_3^{n,m} = 0.3$</td><td>11</td><td>15</td><td>3,112.92</td></tr>
<tr><td>$\theta_3^{n,m} = 0.7$</td><td>12</td><td>16</td><td>3,112.95</td></tr>
<tr><td rowspan="2">4</td><td>$\theta_1^{n,m}=15$ $\theta_2^{n,m}=5$ $\theta_3^{n,m}= 0.3$</td><td>18</td><td>120</td><td>2,639.38</td></tr>
<tr><td>$\theta_1^{n,m}=10$ $\theta_2^{n,m}=10$ $\theta_3^{n,m}= 0.7$</td><td>20</td><td>160</td><td>2,639.38</td></tr>
</table>

final results, in the original computer output format, are presented in Appendix C. Although this is not a rigid proof of the optimality, it does indicate that the results are properly converged and, therefore, are likely the optimal solutions.

2. One decision variable was selected arbitrarily and its value was changed from one percent to ten percent (somewhat arbitrarily but related to its original value). Keeping the values of other decision variables unchanged, the total cost was calculated. This cost was then compared with the one previously obtained. All perturbations resulted in a higher cost which indicates that the results obtained from this model are at least very close to a local minimum. The costs resulting from several perturbations for each problem are shown in Appendix C.

Due to storage limitations of the available computer (IBM 1620), no attempt was made to apply this technique to a more complex network. The above examples are restricted to one quadrant, single copy problems. Therefore only a limited comparison of the results to the real world is possible at this stage of development.

Computer programs, one for each budget condition, written for use on the IBM 1620 computer are presented in Appendix B. The number of iterations and approximate computing time used for each example problem are summarized in Table 2.

CONCLUSIONS

A new technique for the analysis of transportation system investment problems has been presented in this study. Considering each node of a rectangular network as a stage, the discrete maximum principle was utilized to formulate a transportation system model. Investment models under different investment conditions were investigated and search procedures were developed to obtain the optimal investment policy and to assign trips to the network. This provides a broad application of the model to solve various problems which have specific investment restrictions.

As opposed to linear programming models, this model is capable of solving transportation system investment problems with travel time being non-linearly related to traffic volume and investment cost. The formulations presented in this study are applicable to one quadrant network, single copy problems. With minor modifications, the technique would be equally applicable to a more complex network once a larger computer is available.

The optimum seeking procedures appear quite efficient and yield reasonable results as shown by the example problems. Although direct comparison of the results with the real world is not feasible with the limitation of computer capacity, the model does represent a significant step toward more realistic analysis of transportation systems.

Although the travel time function derived in this study could be further improved and the objective function might include additional variables, this study has demonstrated the usefulness and the ability of the discrete maximum principle in solving this type of non-linear optimization problems. It also indicates that the discrete maximum principle could be a powerful tool in transportation system analysis.

RECOMMENDATIONS FOR FURTHER RESEARCH

The relationships between travel time, traffic volume and investment cost are complex. Although the non-linear travel time equation developed in this study is considered to be more realistic than a linear approximation, there is undoubtedly room for further improvement. Two improvements which might be considered are: (a) the desirability of capacity restriction on links, (b) the relationships of free flow travel time to investment.

Previously, it was mentioned that an approximate procedure has been developed to solve multicopy problems. However, this procedure will become invalid when the system budget is restricted, which is not uncommon in the real world. Therefore, to develop a useful model, an improved technique is required. One possible approach is to consider each copy as a large stage and each node becomes a small stage inside the larger stage. The objective function of each copy becomes the Hamiltonian of the larger stage. By this concept, the discrete maximum principle could be utilized to develop a complete model for multicopy transportation problems.

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APPENDIX A

VALUES OF CONSTANTS IN UNIT TRAVEL TIME EQUATION

APPENDIX A

VALUES OF CONSTANTS IN UNIT TRAVEL TIME EQUATION

Unit travel time has been expressed as:

$$t = K_1 + \frac{K_2}{\sigma + K_3}$$

t = unit travel time (hr/mi/veh)

K_1 = free flow travel time (hr/mi/veh). The magnitude depends on the maximum speed obtainable or regulated.

K_2 = coefficient of improvement (dollar-hr/mi^2/veh^2). Its magnitude depends on link location and reflects the difficulty of improvement.

K_3 = existing investment (dollar/mi/hr)

σ = equivalent hourly investment per unit length (dollar/mi/hr)

V = traffic volume per unit time (veh/hr)

In this section, a set of K values are derived from real world data reported by other researchers. The purpose of this section is two fold:

1. To justify the fitness of the equation.
2. To obtain a set of K values for the example problems.

Values of K_1

The K_1 value is equal to the reciprocal of the maximum speed obtainable or regulated in each area. Several common values are shown in Table A-1.

Table A-1

Maximum Speed (mph)	K_1 Values (hr/mile)
70	0.0143
60	0.0167
50	0.0200

For the example problems, maximum speeds were assumed to be 70 mph in less densely developed areas and 60 mph in densely developed areas. The K_1 values are therefore, 0.0143 hours per mile and 0.0167 hours per mile respectively.

Values of K_2

1. Near CBD Area:

The average cost of an 8 lane freeway near the CBD, as estimated by Aitken (19), is \$15,500,000.00 per mile. Assuming 30 year life and 6% interest, annual cost is equal to \$1,130,000.00 per mile. If we further assume peak hour traffic is 10% of daily traffic, the equivalent peak hour cost becomes:

$$\$1,130,000 \times \frac{1}{300} \times \frac{1}{10} = \$314 \text{ per mile per hour}$$

This freeway can handle 1100 vph per lane at unit travel time of 0.02 hr/mile. Assuming K_1 = 0.0143 hr/mi/veh (70 mph speed), K_2 is derived as follows:

$$0.0143 + \frac{K_2}{314} (1100 \times 8) = 0.020$$

or

$$K_2 = 0.00207 \text{ dollar-hr/mi}^2\text{/veh}^2 \qquad \text{(A-2)}$$

Using Haikalis' data and adjusting for the downtown area (19), an arterial street with 2,000 vph volume at unit travel time of 0.0333 hour per mile costs \$3,400,000 per mile or \$250,000 per mile annually. Equivalent peak hour cost becomes:

$$\$250{,}000 \times \frac{1}{360} \times \frac{1}{10} = \$69.5 \text{ per mile per hour.}$$

Assuming $K_1 = 0.025$ hr/mi/veh (40 mph speed), K_2 is derived as follows:

$$0.025 + \frac{K_2}{69.5}\ 2{,}000 = 0.0333,$$

$$K_2 = 0.000288 \text{ dollar-hr/mi}^2\text{/veh}^2 \qquad \text{(A-3)}$$

2. Average Urban Area:

The overall average cost for an 8 lane urban freeway is \$5,000,000 per mile as estimated by Moskowitz (26). Assuming 30 year life and 6% interest, equivalent peak hour cost becomes:

$$\$5{,}000{,}000 \times 0.0726 \times \frac{1}{360} \times \frac{1}{10} = \$101 \text{ per mile per hour.}$$

Using Fig. 3.[illegible] in the "Highway Capacity Manual" (25), a typical freeway with 70 mph average highway speed can handle 1800 vph per lane at a speed of 45 mph. The K_2 value is derived as follows:

$$K_1 = 0.0143 \text{ hr/mi/veh}$$

$$0.0143 + \frac{K_2}{101} (1800 \text{ X } 8) = 0.0222$$

$$K_2 = 0.0000553 \text{ dollar-hr/mi}^2\text{/veh}^2$$

Summarized from "Automobile Transportation Systems: Cost Characteristics" (27), Table A-2 shows relationships among volume, average speed and cost for three types of urban roads. Using these values and the assumed maximum speeds and average lanes, Table A-3 is obtained. The K_2 values are, then, derived as follows:

local street: $0.0286 + \frac{K_2}{11.1} \text{ X } 1000 = 0.05$

$$K_2 = 0.000227 \text{ dollar-hr/mi}^2\text{/veh}^2 \qquad \text{(A-5)}$$

arterial street: $0.025 + \frac{K_2}{19.2} \text{ X } 2800 = 0.308$

$$K_2 = 0.0000398 \text{ dollar-hr/mi}^2\text{/veh}^2 \qquad \text{(A-6)}$$

freeway: $0.0143 + \frac{K_2}{161} \text{ } 10800 = 0.0182$

$$K_2 = 0.0000582 \text{ dollar-hr/mi}^2\text{/veh}^2 \qquad \text{(A-7)}$$

Table A-2 Cost Characteristics of Urban Highways

	Local Street	Arterial	Freeway
Practical Capacity (vph/lane)	500	700	1800
Average Speed (mph)	20	25-40	45-65
ROW Cost ($/mile)	250,000	450,000	4-8 million
Construction Cost ($/mile)	300,000	500,000	4-6 million
Total Cost ($/mile)	550,000	950,000	8-14 million

source = Ref. (28)

Table A-3 Cost and Travel Time of Urban Highways

	Local Street	Arterial	Freeway
No. of lanes	2	4	6
Total Volume (vph)	1,000	2,800	10,800
Average Speed (mph)	20	32.5	55
Total Cost ($/mile)	550,000	950,000	8-14 million
Equivalent Peak -hour cost ($/mile)	11.1	19.2	161-282
Assumed maximum speed (mph)	35	40	70
Minimum unit travel time (hr/mile)	0.0286	0.025	0.0143
Average Travel Time (hr/mile)	0.05	0.0308	0.0182

source = Ref. (28)

$$0.0143 + \frac{K_2}{202} \; 10800 = 0.0182$$

$$K_2 = 0.000102 \text{ dollar-hr/mi}^2/\text{veh}^2 \qquad \text{(A-8)}$$

3. Rural Area:

Cost data for rural highways is not generally available. However, the cost of a rural freeway may be assumed as equal to the lowest cost of a freeway in an urban area.

On this basis an 8 lane freeway will cost about $3,000,000 per mile (27). Using Fig. 3.38 in the "Highway Capacity Manual" (25), a typical freeway with 70 mph average highway speed can handle 1800 vph per lane at 45 mph speed. Equivalent peak hour cost becomes:

$$\$3{,}000{,}000 \text{ X } 0.0726 \text{ X } \frac{1}{360} \text{ X } \frac{1}{10} = \$60.5 \text{ per mile per hour}$$

The K_2 value is derived as follows:

$$0.0143 + \frac{K_2}{60.5} (1800 \text{ X } 8) = 0.0222$$

$$K_2 = 0.00003322 \text{ dollar-hr/mi}^2/\text{veh}^2 \qquad \text{(A-9)}$$

Excluding equation A-5, K_2 values are summarized in Table A-4.

Table A-4

Type of Area	Range of K_2 Value
CBD	0.000207-0.000288
Average Urban Area	0.0000398-0.000102
Rural	0.0000332

[illegible] K_2 [illegible] in an avera[illegible] urban area [illegible] caused [illegible] land [illegible] way costs as shown in Table [illegible]. In general, K_2 [illegible] airly consistent in each area. This [illegible] a fairly good correlation between the equation and the real world situation.

Values of K_3

The K_3 value represents the existing facilities in terms of cost per mile per hour. Equivalent peak hour cost, for each type of road, derived in the previous sections indicates the average values of K_3.

The K values used in the example problems are summarized in Table 1.

APPENDIX B

COMPUTER PROGRAMS

```
C     [illegible]
      DIMENSION HV(4,4),VV(4,4),[illegible](4,4),[illegible](4,4),[illegible](4,4),
     1 AHH(3),[illegible]1(4,4),V(4,4),Z[illegible](4,4),Z[illegible](4,4),AL1(4,4),AL2(4,4),
     2CH1(4,4),CH2(4,4),CH3(4,4),CV1(4,4),CV2(4,4),CV3(4,4)
     3 ,SH1(4,4),SH2(4,4),SV1(4,4),SV2(4,4)
    1 FORMAT(4I4)
    2 FORMAT(7F1[illegible].4)
    3 FORMAT(15H ENTER [illegible] DATA)
    4 FORMAT(2I4,7F1[illegible].[illegible])
    5 FORMAT(14H [illegible] UN OUT)
    6 FORMAT(1[illegible]H [illegible] OF PROGRAM)
    7 FORMAT(I3,4F15.4)
    8 FORMAT(3F2[illegible].6)
    9 FORMAT(74H  ROW COL      HV              VV             HIHV           VIHV
     1        ZH            ZV)
   10 FORMAT(3X,2I3,3X,4E15.[illegible])
   11 FORMAT(7F11.4)
   12 FORMAT(6F12.5)
  [illegible] READ 1, N, M, IC1
      READ [illegible], DELTA, AK[illegible], T, AK,A[illegible]
      READ 2, DASM1, DASM2, DASM3
      PUNCH 4, N,M,DELTA,AK
      COSTP=[illegible].[illegible]
      KEYSO=
      DO 1 2 I=1,N
      DO 1 2 J=1,M
      READ 2, V(I,J), AL1(I,J),AL2(I,J)
      PUNCH 12, V(I,J),AL1(I,J),AL2(I,J)
      IF(I-N) 2 1,2 2,1112
  [illegible]1 IF(J-M) 3 1,3 2,1112
  [illegible]1 D3(I,J)=DASM3
      GO TO 1 2
  [illegible] D3(I,J)= .
      GO TO 1 2
  [illegible] D3(I,J)=1.
  1 2 CONTINUE
      DO 1 3 I=1,N
      DO 1 3 J=1,M
      READ 2, CH1(I,J),CH2(I,J),CH3(I,J),SH1(I,J),SH2(I,J)
      READ 2, CV1(I,J),CV2(I,J),CV3(I,J),SV1(I,J), SV2(I,J)
      PUNCH 11, CH1(I,J),CH2(I,J),CH3(I,J), SH1(I,J),SH2(I,J)
  1 3 PUNCH 11, CV1(I,J),CV2(I,J),CV3(I,J), SV1(I,J), SV2(I,J)
      ITER=
1     IF(ITER-IC1) 1[illegible],1111,1112
1[illegible] ITER=ITER+1
      IF(ITER-IC1)[illegible],[illegible],[illegible]
  [illegible] KEYSO=1
  [illegible] DO 211 I=1,N
      DO 211 J=1,M
      IF(I-1) 11[illegible],[illegible],[illegible]
  21[illegible] IF(J-1) 1112,[illegible],[illegible]
```

```
 4[illegible] [illegible]
 4[illegible] HV(I,J)=[illegible](I,J)[illegible](I,J)
      [illegible](I,J)=[illegible](I,J)[illegible].-[illegible]3(I,J)[illegible]
      GO TO [illegible]
 41[illegible] [illegible](I,J)=[illegible](I,J-1)+V(I,[illegible])
      GO TO [illegible]
 [illegible]1[illegible] IF(J-1[illegible])[illegible],[illegible]1,[illegible]
 [illegible]1[illegible] [illegible]I(I,J)=[illegible](I-1,J)+V(I,J)
      GO TO [illegible]
  12 [illegible]I(I,[illegible](I,J-1)+VV(I-1,J)+V(I,J)
      GO TO [illegible]
 [illegible] CONTINUE
      DO 4 [illegible],
      DO 4[illegible] [illegible]1,[illegible]
      IF(HV(I,J))1112,1[illegible],2[illegible]
 1[illegible]5 [illegible]1(I,J)=[illegible].
      GO TO 115
 [illegible] D1(I,J)=HV(I,J)*(CH2(I,J)*T)**[illegible].[illegible]-CH3(I,J)
      IF(D1(I,J)-SH1(I,J))1[illegible]6,1[illegible],1[illegible]7
 1[illegible]6 [illegible]1(I,J)=SH1(I,J)
      GO TO 115
 1[illegible]7 IF(D1(I,J)-SH2(I,J))1[illegible],1[illegible]8,1[illegible]
 1[illegible] D1(I,J)=SH2(I,J)
 115 IF(VV(I,J))1112,[illegible],17[illegible]
 17[illegible] [illegible]2(I,J)=VV(I,J)*(CV2(I,J)*T)**[illegible].5-CV[illegible](I,J)
      IF([illegible]2(I,J)-SV1(I,J))2[illegible]6,2[illegible]6,2[illegible]7
 [illegible]6 [illegible]2(I,J)=SV1(I,J)
      GO TO 4
 [illegible]7 IF([illegible]2(I,J)-SV2(I,J))4[illegible],2[illegible]8,2[illegible]8
 [illegible] [illegible]2(I,J)=SV2(I,J)
      GO TO 4
 [illegible] [illegible]2(I,J)=[illegible].
   4 CONTINUE
      COSTH=[illegible].
      COSTV=[illegible].
      COSTT=[illegible].
      DO 221 I=1,[illegible]
      DO 221 J=1,[illegible]
      COSTH=COSTH+[illegible]1(I,J)*AL1(I,J)
      COSTV=COSTV+[illegible]2(I,J)*AL2(I,J)
 [illegible]21 COSTT=COSTT+(CH1(I,J)*HV(I,J)+C[illegible]2(I,J)[illegible](I,J)[illegible]*2/([illegible](I,J)+
     1 CH3(I,J)))*T*AL1(I,J)+(CV1(I,J)*VV(I,J)+CV2(I,J)*VV(I,J)**2/
     2 ([illegible]2(I,J)+CV3(I,J)))*T*AL2(I,J)
      COSTI=COSTH+COSTV
      COST=COSTI+COSTT
      PUNCH 7, ITER,T,[illegible]
      PUNCH [illegible], COST,COSTI,COSTT
      IF(SENSE [illegible]ITCH 2)12[illegible],24[illegible]
 12[illegible] TYPE [illegible],COST
 [illegible] DELT[illegible]=[illegible]SF([illegible](COST-COSTP)/COST)
      IF(DELT[illegible]-DELT[illegible])11[illegible],11[illegible],11[illegible]2
 11[illegible]1 IF(KEYS[illegible]-1)11[illegible],11[illegible],111
```

```
1   ? [illegible]=1
      GO TO [illegible]
1112 [illegible]EYSC=[illegible]
11 4  DO 231 I=1,[illegible]
      DO 231 J=1,[illegible]
      LN=N+1-I
      LM=M+1-J
      [illegible]P=LN+1
      [illegible]P=LM+1
      IF(LN-[illegible])611,[illegible]
 612  IF(LM-[illegible])611,61[illegible]
 61[illegible] D3(LN,L[illegible])=1.
      ZH(LN,L[illegible])=[illegible].
      ZV(LN,L[illegible])=[illegible].
      GO TO 2[illegible]1
 611  IF(J-1)1111,[illegible]
 712  ZH(LN,L[illegible])=ZH([illegible])+[illegible])*(1.-[illegible])+
     1 (CH1(L[illegible],[illegible]P)*D3(L[illegible])+2.*CH2(L[illegible])*HV(LN,[illegible])*D3(L[illegible])/
     2 (D1(LN,[illegible]P)+CH3(LN,[illegible])))*T*AL1(LN,[illegible]P)+(CV1(LN,[illegible])*(1.-D3(LN,[illegible])+
     3 2.*CV2(LN,[illegible]P)*VV(LN,[illegible]P)*(1.-D3(LN,[illegible]P))/(D2(LN,[illegible]P)+CV3(LN,[illegible]))*
     4 T*AL2(LN,[illegible]P)
      IF(I-1)1111,811,711
 [illegible] ZV(LN,L[illegible])=ZH(LN,L[illegible])
      GO TO 231
 711  ZV(LN,LM)=ZH(NP,L[illegible])*D3(NP,LM)+ZV(NP,LM)*(1.-D3(N[illegible],L[illegible]))+
     1 (CH1(NP,LM)*D3(NP,L[illegible])+2.*CH2(N[illegible],L[illegible])*HV(NP,LM)*D3([illegible]P,L[illegible])/
     2 (D1(NP,LM)+CH3(N[illegible])))*T*AL1(NP,L[illegible])+(CV1([illegible],L[illegible])*(1.-D3(NP,L[illegible])+
     3 2.*CV2(NP,LM)*VV(NP,LM)*(1.-D3(NP,L[illegible]))/(D2(NP,LM)+CV3(NP,LM))*T*
     4 T*AL2(NP,LM)
      IF(J-1)1111,815,2[illegible]1
 81[illegible] ZH(LN,L[illegible])=ZV(LN,L[illegible])
 [illegible] CONTINUE
      IF(SENSE SWITCH 3)1502,15[illegible]
1[illegible] IF(KEYS[illegible]-1)15[illegible]4,1[illegible]2,1112
15[illegible] [illegible]NCH [illegible]
      DO 3[illegible] I=1,N
      DO [illegible] J=1,M
      PUNCH [illegible] I,J,HV(I,J),VV(I,J),[illegible]1(I,J),[illegible]2(I,J),[illegible]3(I,J),
     1 ZH(I,J),ZV(I,J)
 [illegible] CONTINUE
1[illegible] IF(SENSE SWITCH 1)15[illegible]1,15[illegible]7
15[illegible]1 READ 2,[illegible]KK
1[illegible]7 DO 241 I=1,N
      DO 241 J=1,[illegible]
      IF(I-N)11[illegible],116,111[illegible]
 11[illegible] IF(J-M)117,11[illegible],111[illegible]
 116  [illegible]3(I,J)=1.
      GO TO 241
 118  D3(I,J)=[illegible].
      GO TO 241
 117  IF(AI(I,J))111[illegible],121,1[illegible]
 121  AI(I,J)=1.
```

```
11[illegible] [illegible](I,J)-[illegible](1.[illegible](I,[illegible])+[illegible]1[illegible],[illegible](I.J)+2.[illegible](I,J)*[illegible](I,J)
    1 *AI(I,J)**2/([illegible]1(I.[illegible])+[illegible](I.[illegible])*T*[illegible](I,J)-(CV1(I,J)*AI(I,J)+2.
    2 *(V2(I.J)*(1.-[illegible]2(I.J))*AI(I,J)**2/([illegible](I,J)+CV3(I,J)))*T*AL2(I,J)
      [illegible]=(2.[illegible]H2(I,J)*AI(I,J)**2/([illegible]1(I,J)+[illegible]3(I,J)))*T*AL1(I,J)+
    1 (2.*CV2(I,J)*AI(I,J)**2/([illegible](I,J)+[illegible](I,J)))*T*AL2(I,J)
      IF(ABSF([illegible])-A[illegible])217,217,[illegible]1
 [illegible]1 [illegible]3(I.J)=[illegible](I,J)-A[illegible]/[illegible]
      IF(D3(I,J)-1.)257,262,[illegible]
 [illegible] D3(I,J)=1.
      GO TO 217
 [illegible]7 IF(D3(I.J))2[illegible],[illegible],21[illegible]
 [illegible] D3(I.J)=[illegible].
 [illegible]7 GO TO 241
 [illegible]1 CONTINUE
      COSTP=COST
      GO TO 1[illegible]
 1111 PUNCH 6
      TYPE 3
      PAUSE
      GO TO 1
 1112 TYPE 5
      STOP
      END
```

```
[illegible] IX[illegible]
      [illegible](4,4),[illegible](4,4),[illegible](4,4),[illegible](4,4),[illegible](4,4),[illegible](4,4),
     1 AH[illegible](3), [illegible](4,4),V(4,4),Z[illegible](4,4),ZV(4,4),AL1(4,4),[illegible](4,4),
     [illegible]CH1(4,4),[illegible](4,4),C[illegible](4,4),C[illegible]1(4,4),[illegible](4,4),CV3(4,4)
    1 FORMAT(4I4)
    2 FORMAT(7F1[illegible].4)
    3 FORMAT(15H [illegible]TH[illegible]ATA)
    4 FORMAT(2I4,7F1[illegible].4)
    5 FORMAT(14H ERR[illegible]OUT)
    6 FORMAT(15H E[illegible] OF [illegible]A[illegible])
    7 FORMAT(I3,4F15.[illegible])
    8 FORMAT(3E2[illegible].6)
    9 FORMAT(74H [illegible]   HV        VV        HINV       VINV       [illegible]
     1         ZH         [illegible])
   10 FORMAT(3X,2I3,3X,4[illegible]1[illegible].6)
   11 FORMAT(7F11.4)
   12 FORMAT(6F12.5)
   1[illegible] READ 1, [illegible], [illegible], IC1
      READ 2, [illegible]ELTA, AKK, T, AK,AFK,[illegible]
      READ 2, [illegible]1, DASM2, [illegible]3
      PUNCH 4, [illegible]ELTA,DAS[illegible],52
      COSTP=[illegible]
      KEYSO=
      DO 1[illegible]2 I=1,N
      DO 1[illegible]2 J=1,[illegible]
      READ 2, V(I,J), AL1(I,J),AL2(I,J),[illegible]1(I,J)
      PUNCH 12, V(I,J),AL1(I,J),AL2(I,J),[illegible]1(I,J)
      IF(I-N) 2[illegible]1,2[illegible]2,1112
  [illegible] IF(J-[illegible]) 3[illegible]1,3[illegible]2,1112
  [illegible]1 D3(I,J)=DAS[illegible]
      GO TO 1[illegible]2
  [illegible] D3(I,J)=[illegible].
      GO TO 1[illegible]2
  [illegible] [illegible]3(I,J)=1.
  [illegible] CONTINUE
      DO 1[illegible]3 I=1,N
      DO 1[illegible]3 J=1,[illegible]
      READ 2, CH1(I,J),CH2(I,J),CH3(I,J)
      READ 2, CV1(I,J),CV2(I,J),CV3(I,J)
      PUNCH 11, CH1(I,J),CH2(I,J),CH3(I,J)
  1[illegible] PUNCH 11, CV1(I,J),CV2(I,J),CV3(I,J)
      [illegible]=5
      ITER=
 1[illegible] IF(ITER-IC1) 1[illegible]1,1111,111[illegible]
 1[illegible] ITER=ITER+1
      IF(ITER-IC1)[illegible]21,3[illegible],[illegible]11[illegible]
  [illegible]22 KEYSO=1
  [illegible]21 DO 211 I=1,N
      DO 211 J=1,[illegible]
      IF(I-1)111[illegible],[illegible]11,[illegible]1[illegible]
  [illegible]11 IF(J-1) 111[illegible],411,41[illegible]
```

```
411 AI(I,J)=V(I,J)
47  HQ(I,J)=AI(I,J)*[illegible]
    VV(I,J)=AI(I,J)*(1.-D3(I,J))
    GO TO [illegible]
412 AI(I,J)=HV(I,J-1)+V(I,J)
    GO TO 43
 12 IF(J-1) 1112,511,512
511 AI(I,J)=VV(I-1,J)+V(I,J)
    GO TO 43
512 AI(I,J)=HV(I,J-1)+VV(I-1,J)+V(I,J)
    GO TO 43
211 CONTINUE
    DO 4  I=1,N
    DO 4  J=1,M
    IF(I-N)31,32,1112
 32 D1(I,J)=S1(I,J)
    D2(I,J)=0.
    GO TO 4
 31 IF(J-M)33,34,1112
 34 D2(I,J)=S1(I,J)
    D1(I,J)=0.
    GO TO 4
 33 D1(I,J)=(CH2(I,J)**0.5*AL1(I,J)*D3(I,J)*(S1(I,J)+CV3(I,J)*AL2(I,J
   1 )-CV2(I,J)**0.5*AL2(I,J)*(1.-D3(I,J))*CH3(I,J)*AL1(I,J))/(
   2 CV2(I,J)**0.5*AL2(I,J)*(1.-D3(I,J))+CH2(I,J)**0.5*AL1(I,J)*
   3 D3(I,J))
    IF(D1(I,J)-S1(I,J))107,106,106
106 D1(I,J)=S1(I,J)
    GO TO 115
107 IF(D1(I,J)-S2)108,108,115
108 D1(I,J)=S2
115 D2(I,J)=S1(I,J)-D1(I,J)
  4 CONTINUE
    COSTH=0.
    COSTV=0.
    COSTT=0.
    DO 221 I=1,N
    DO 221 J=1,M
    COSTH=COSTH+D1(I,J)
    COSTV=COSTV+D2(I,J)
221 COSTT=COSTT+(CH1(I,J)*HV(I,J)*AL1(I,J)+CH2(I,J)*(AL1(I,J)*HV(I,J))
   1 **2/(D1(I,J)+CH3(I,J)*AL1(I,J)))*T+(CV1(I,J)*VV(I,J)*AL2(I,J)+CV2
   2 (I,J)*(AL2(I,J)*VV(I,J))**2/(S1(I,J)-D1(I,J)+CV3(I,J)*AL2(I,J)))
   3 *T
    COSTI=COSTH+COSTV
    COST=COSTI+COSTT
    PUNCH 7, ITER,T,AKK
    PUNCH 8, COST,COSTI,COSTT
    IF(SENSE SWITCH 2)125,243
125 TYPE 8,COST
243 DELTB=ABSF(ABSF(COST-COSTP)/COST)
```

```
      IF(NFEL-NC)1003,1101,1102
 1101 IF(KEY-1)1103,1111,1103
 1103 KEYSC=1
      GO TO 1104
 1102 KEYSC=
 1104 DO 231 I=1,N
      DO 231 J=1,M
      LN=N+1-I
      LM=M+1-J
      NP=LN+1
      MP=LM+1
      IF(LN-N)611,612,1111
  612 IF(LM-M)611,613,1111
  613 D3(LN,LM)=1.
      ZH(LN,LM)= .
      ZV(LN,LM)= .
      GO TO 231
  611 IF(J-1)1111,712,711
  712 ZH(LN,LM)=ZH(LN,MP)*D3(LN,MP)+ZV(LN,MP)*(1.-D3(LN,MP))+(CH1(LN,MP)
     1*AL1(LN,MP)*D3(LN,MP)+2.*CH2(LN,MP)*AL1(LN,MP)**2*HH(LN,MP)*D3(LN,
     2MP)/(D1(LN,MP)+CH3(LN,MP)*AL1(LN,MP))+CV1(LN,MP)*AL2(LN,MP)*(1.-
     2D3(LN,MP))+2.*CV2(LN,MP)*AL2(LN,MP)**2*VV(LN,MP)*(1.-D3(LN,MP))/(
     3 S1(LN,MP)-D1(LN,MP)+CV3(LN,MP)*AL2(LN,MP)))*T
      IF(I-1)1111,811,711
  811 ZV(LN,LM)=ZH(LN,LM)
      GO TO 231
  711 ZV(LN,LM)=ZH(NP,LM)*D3(NP,LM)+ZV(NP,LM)*(1.-D3(NP,LM))+(CH1(NP,LM)
     1*AL1(NP,LM)*D3(NP,LM)+2.*CH2(NP,LM)*AL1(NP,LM)**2*HH(NP,LM)*D3(NP,
     2LM)/(D1(NP,LM)+CH3(NP,LM)*AL1(NP,LM))+CV1(NP,LM)*AL2(NP,LM)*(1.-
     3D3(NP,LM))+2.*CV2(NP,LM)*AL2(NP,LM)**2*VV(NP,LM)*(1.-D3(NP,LM))/(
     4 S1(NP,LM)-D1(NP,LM)+CV3(NP,LM)*AL2(NP,LM)))*T
      IF(J-1)1111,815,231
  815 ZH(LN,LM)=ZV(LN,LM)
  231 CONTINUE
      IF(ITER-NC)22,23,1112
   22 IF(KEYSC-1)25,26,1112
   23 NC=NC+5
   26 PUNCH 9
      DO 3   I=1,N
      DO 3   J=1,M
      PUNCH 4, I,J,HV(I,J),VV(I,J),D1(I,J),D2(I,J),D3(I,J),
     1 ZH(I,J),ZV(I,J)
    3 CONTINUE
   25 IF(SENSE SWITCH 1)151,157
  151 READ 2, AKK
  157 DO 241 I=1,N
      DO 241 J=1,M
      IF(I-N)119,116,1112
  119 IF(J-M)117,116,1112
  116 D3(I,J)=1.
      GO TO 241
```

```
1[illegible] [illegible](I,[illegible]=[illegible].
      [illegible] TO 2[illegible]1
117 IF(AI(I,J))[illegible]
121 AI(I.J)=[illegible].
122 [illegible]H=ZH(I.J)-ZV(I.J)+([illegible]AL1(I,J)[illegible]*T+(2.*
    1 CH2(I.J)*AL1(I.J)**2*[illegible]V(I.J)/([illegible]1(I.[illegible])+CH3(I.J)[illegible])-2.*
    2 CV2(I.J)*AL2(I.J)**2*VV(I.J)/([illegible]1(I,J)-[illegible]1(I,J)+[illegible](I.J)*AL2(I.[illegible]
    3 *T
      [illegible]H=(2.*(CH2(I.J)*AL1(I.J)**2*AI(I.J)/([illegible]1(I,J)+[illegible](I,J)*[illegible]L1(I.J))+
    1 2.*CV2(I,J)*AL2(I.J)**2*AI(I.J)/([illegible]1(I,J)-[illegible]1(I,J)+CV3(I.J)*
    2 AL2(I,J)))*T
      IF(ABSF(DH)-[illegible]K)217,217,[illegible]1
[illegible] DD3=AKK*DH/DDH
      [illegible]=ABSF([illegible]3)
      IF(ADD[illegible]-S3)271.271.2[illegible]
[illegible]71 [illegible]3(I,J)=D3(I,J)-[illegible]
      GO TO 2[illegible]2
[illegible] IF(DD3)27[illegible],27[illegible].274
[illegible]73 D3(I.J)=D3(I.J)+S3
      GO TO 252
[illegible]74 D3(I,J)=D3(I,J)-S3
[illegible] IF(D3(I.J)-1.)25[illegible].26[illegible],26[illegible]
[illegible] D3(I,J)=1.
      GO TO 217
[illegible] IF(D3(I,J))26[illegible],26[illegible],217
[illegible] [illegible]3(I,J)=[illegible].
217 GO TO 241
[illegible]41 CONTINUE
      COSTP=COST
      GO TO 1[illegible]
1111 PUNCH 6
      TYPE 3
      PAUSE
      GO TO 1
1112 TYPE [illegible]
      STOP
      END
```

```
      OPTIMAL T[illegible] [illegible] X[illegible] Y[illegible]T
      DIMENSION HV(4,4),VV(4,4),[illegible](4,4),[illegible](4,4),[illegible](4,4),X[illegible](4,4),
     1          [illegible]I(4,4),V(4,4),[illegible](4,4),ZV(4,4),AL1(4,4),AL2(4,4)
     2 ,CH1(4,4),CH2(4,4),CH3(4,4),CV1(4,4),CV2(4,4),CV3(4,4)
    1 FORMAT(4I4)
      FORMAT(7F10.4)
      FORMAT(15H ENTER [illegible] DATA)
    6 FORMAT(2I4,7E1 .4)
      FORMAT(14H E[illegible] [illegible]T)
      FORMAT(15H END OF PROGRAM)
    7 FORMAT(I3,4F15.4)
      FORMAT([illegible]F[illegible] .6)
      FORMAT(74H [illegible] HV          VV          HINV        VINV       [illegible]
    1         ZH              ZV)
   11 FORMAT(7F11.4)
      READ 1, N, M, IC, NCI
      READ 2, DELTA, T, GI, AKK1, AK[illegible] ,DZ
      READ 2, DASM1, DASM2, DASM3
      READ 2, S1,S2,S3,S4
      PUNCH 6, N,M,DELTA,GI,S1,S2,S4,DASM1,DASM[illegible]
      COSTP=0.
      KEYSC=
      DO 3  I=1,N
      DO 3  J=1,M
      READ 2,  V(I,J),AL1(I,J),AL2(I,J)
      PUNCH 7, V(I,J),AL1(I,J),AL2(I,J)
      D1(I,J)=DASM1
      D2(I,J)=DASM2
   33 D3(I,J)=DASM3
    3 CONTINUE
      DO 40 I=1,N
      DO 40 J=1,M
      READ 2,  CH1(I,J),CH2(I,J),CH3(I,J),CV1(I,J),CV2(I,J),CV3(I,J)
    4 PUNCH 11,CH1(I,J),CH2(I,J),CH3(I,J),CV1(I,J),CV2(I,J),CV3(I,J)
      ITER=0
      NC=NC1
 1[illegible]  IF(ITER-IC)41,1111,1112
   41 ITER=ITER+1
      ZSP=0.
      IF(ITER-IC)42,43,1112
   43 KEYSC=1
   42 DO 50 I=1,N
      DO 50 J=1,M
      IF(I-N)21,22,1112
   22 IF(J-M)23,24,1112
   24 D3(I,J)=1.
      D1(I,J)= .
      D2(I,J)=0.
      GO TO 59
   23 D3(I,J)=1.
      D2(I,J)= .
```

```
      GO TO [illegible]
   [illegible] IF(J-[illegible],[illegible],1111
   [illegible] [illegible]2(I,J)= .
      [illegible]1(I,J)= .
    5 IF(I-1)111[illegible],44,4[illegible]
   44 IF(J-1)111[illegible],45,47
   45 [illegible]I(I,J)=V(I,J)
   46 [illegible]V(I,J)=[illegible](I,J)*[illegible](I,J)
      VV(I,J)=[illegible](I,J)*(1.-[illegible](I,J))
      GO TO 5[illegible]
   47 [illegible]I(I,J)=HV(I,J-1)+V(I,J)
      GO TO 46
   4[illegible] IF(J-1)1112,[illegible]1,[illegible]2
   [illegible]1 [illegible]I(I,J)=VV(I-1,J)+V(I,J)
      GO TO 46
   [illegible] [illegible]I(I,J)=HV(I,J-1)+VV(I-1,J)[illegible](I,J)
      GO TO 46
   [illegible] CONTINUE
      AKK2=A[illegible]
      [illegible]2=[illegible]2
   [illegible] DO 6 I=1,[illegible]
      DO 6[illegible] J=1,[illegible]
      IF(I-[illegible])[illegible]1,[illegible],1112
   [illegible] IF(J-M+1)2[illegible]1,2[illegible]3,2[illegible]4
   [illegible] [illegible]1(I,J)=GI-X5(I,J-1)
      GO TO 62
   [illegible] [illegible](I,J)=X5(I,J-1)
      GO TO [illegible]
   [illegible] IF(I-1)1112,61,6[illegible]
   6[illegible] IF(J-1)1112,6[illegible],64
   [illegible] X5(I,J)=[illegible]1(I,J)+[illegible]2(I,J)
      GO TO 66
   [illegible] IF(J-1)[illegible],65,[illegible]4
   [illegible] X5(I,J)=X5(I-1,[illegible])+[illegible]1(I,J)+[illegible](I,J)
      GO TO 66
   [illegible]4 [illegible](I,J)=X5(I,J-1)+[illegible]1(I,J)+[illegible](I,J)
   66 IF(X5([illegible],J)-[illegible]I)6[illegible],6[illegible],7[illegible]
   7[illegible] [illegible]D=[illegible]1(I,J)+[illegible]2(I,J)
      [illegible]1(I,J)=[illegible]1(I,J)-(X5(I,J)-GI)[illegible]1(I,J)/[illegible]
      [illegible]2(I,J)=[illegible]2(I,J)-(X5(I,J)-[illegible]I)[illegible]2(I,J)/[illegible]
      X5(I,J)=GI
    [illegible] CONTINUE
      COSTH= .
      COSTV= .
      COSTT= .
      DO 7 I=1,[illegible]
      DO 7 J=1,[illegible]
      COSTH=COSTH+[illegible]1(I,J)
      COSTV=COSTV+[illegible]2(I,J)
    7 [illegible]TT=COSTT+(([illegible]2(I,J)*[illegible](I,J)**2/([illegible](I,J)/[illegible]1(I,J)+[illegible](I,J)))*
     1 [illegible]1(I,J)*T+((V2(I,J)*V[illegible](I,J)**2/([illegible](I,J)/[illegible](I,J)+[illegible](I,J)))*
```

```
   1PE2(I,J)*1+(...(I,J)...V)...(I,J)+...(I,J)...(I,J)...L2(I,J)))
      ...TI=...TH+...TV
      ...T=...TT+...TI
      ...  7, IT...,T,A...1,
      ...  ...,  ...T,...TI,...
      IF(...  ...ITCH 2) 67,68
   67 TYPE ..., ...T
   68 ...ELT...=...SF(A...F(...T-C...T...)...)
      Z5=CH2(M,...-1)*HV(M,M-1)**...(...(M,M-2))/...(M,M-1)+...(M,M-...)
     1 ***T
      DZ=ABSF((Z5-Z5P)/Z5)
      IF(DZ-...DZ)211,211,212
  211 Z5P=Z5
      DO 1 2 I=1,...
      DO 1 2 J=1,...
      ...H2=Z5      -CH2(I,J)*HV(I,J)**2*T/(...1(I,J)/AL1(I,J)+...3(I,J))**
      ...H2=2.*CH2(I,J)*HV(I,J)**2*T/AL1(I,J)/(...1(I,J)/AL1(I,J)+
     1 ...3(I,J))**3
      ...=Z5      -CV2(I,J)*VV(I,J)**2*T/(...2(I,J)/AL2(I,J)+CV3(I,J))**2
      ...=2.*CV2(I,J)*VV(I,J)**2*T/AL2(I,J)/(D2(I,J)/AL2(I,J)+
     1 CV3(I,J))**3
      ...=AK...*...H2/(D...H2+...         ...1)
      ...3=AK2*...3/(D...3+...           ...1)
  132 IF(AD2-S2)131,133,133
  133 AD2=S2
      GO TO 135
  131 IF(AD2+S2)134,135,135
  134 AD2=-S2
  135 D1(I,J)=D1(I,J)-AD2
      IF(D1(I,J))123,123,124
  123 D1(I,J)=0.
  124 IF(AD3-S2)136,137,137
  137 AD3=S2
      GO TO 139
  136 IF(AD3+S2)138,139,139
  138 AD3=-S2
  139 D2(I,J)=D2(I,J)-AD3
      IF(D2(I,J))125,125,126
  125 D2(I,J)=0.
  126 IF(D1(I,J)-S3)141,142,142
  141 DT1=((CH2(I,J)*T/Z5        )**  .5*...(I,J)-CH3(I,J))*AL1(I,J)
      IF(D1(I,J)-DT1)143,142,142
  143 D1(I,J)=DT1
  142 IF(D1(I,J)-S4)147,147,148
  148 D1(I,J)=S4
  147 IF(D2(I,J)-S3)144,145,145
  144 DT2=(((CV2(I,J)*T/Z...        )**  .5*...(I,J)-CV3(I,J))*AL2(I,J)
      IF(D2(I,J)-DT2)146,145,145
  146 D2(I,J)=DT2
  145 IF(D2(I,J)-S4)14...,1...,1...
  1.. D2(I,J)=S4
```

```
1   GO TO 1
1   CONTINUE
    IF(SENSE SWITCH 1)1 4,1
1 4 READ 2, KEZ,
1 5 GO TO 1
211 DO 8  I=1,N
    DO 8  J=1,M
    LN=N+1-I
    LM=M+1-J
    NP=LN+1
    MP=LM+1
    IF(LN-N)81,  ,1112
    IF(LM-M)81,  ,1112
    D3(LN,LM)=1.
    ZH(LN,LM)= .
    ZV(LN,LM)= .
    GO TO
 81 IF(J-1)1112,84,8
 85 ZH(LN,LM)=ZH(LN,  )*D3(L ,  )+(CH1(LN,  )*T (  ,  )+2.*CH (LN,
   1 HV(LN, P)*D3(LN, P)/(D1(LN, P)*AL1(L , P)+CH3(L , P)))*AL1(L ,
   2 *T+ZV(LN,MP)*(1.-D3(LN, P))+(CV1(LN, P)*(1.-D3(LN, P))+2.*
   3 CV2(LN,MP)*VV(LN, P)*(1.-D3(LN, P))/(D2(LN, P)/AL2(L , P)+
   4 CV3(LN,MP)))*AL2(LN,MP)*T
    IF(I-1)1112,85,84
 86 ZV(LN,LM)=ZH(LN,LM)
    GO TO
 84 ZV(LN,L )=ZH(NP,LM)*D3(NP,LM)+(CH1(NP,LM)*D3(NP,L )+2.*CH2(NP,L
   1 HV(NP,L  )*D3(NP,L )/(D1(NP,LM)*AL1(NP,LM)+CH3(NP,LM)))*AL1(NP,L
   2 *T+ZV(NP,LM)*(1.-D3(NP,LM))+(CV1(NP,L )*(1.- 3(NP,L ))+2.*
   3 CV2(NP,L )*VV(NP,L )*(1.-D3(NP,LM))/( 2(N ,L )/AL2(  ,L )+
   4 CV3(NP,LM)))*AL2(NP,LM)*T
    IF(J-1)1112,87,8
 87 ZH(LN,LM)=ZV(LN,LM)
 8  CONTINUE
    IF(ITER-NC)6 ,1 2,1112
 6  IF(DELT -DELTA)18 ,18 ,
    KEYSO=
    IF(SENSE SWITCH 4)183,98
182 NC=NC+NC1
183 PUNCH
    DO 9  I=1,N
    DO 9  J=1,M
    PUNCH 4, I,J,HV(I,J),VV(I,J), 1(I,J), 2(I,J), 3(I,J),Z (I,J),
   1 ZV(I,J)
 9  CONTINUE
    IF(KEYSO-1)75,97,1112
 75 IF(SENSE SWITCH 5)97,96
 97 DO 120 I=1,N
    DO 12  J=1,
12  PUNCH 11, X5(I,J),Z5
 96 IF(DELT -DELTA)71,71,72
```

```
 [illegible] IF(KEY [illegible]-1)[illegible]
 [illegible] KEYSO=1
     GO TO [illegible]
  71 KEYSO=
 [illegible] IF(SENSE SWITCH [illegible])[illegible]4,[illegible]
 [illegible] READ 2, AKK1, [illegible]
 [illegible] DO 1 1 I=1,[illegible]
     DO 1 1 J=1,[illegible]
 1 7 IF(AI(I,J))1112,[illegible]
 111 AII=1.
     GO TO 112
 113 AII=AI(I,J)
 112 DH1=ZH(I,J)-ZV(I,J)+([illegible])*[illegible]L[illegible](I,J)-(V1(I,J)*[illegible]L[illegible]*T+2.[illegible]
   1 CH2(I,J)*AII*D3(I,J)/[illegible](I,J)/[illegible](I,J)+CH3(I,J))*[illegible](I,J)*T-2.[illegible]
   2 CV2(I,J)*AII*(1.-[illegible]3(I,J))/([illegible]2(I,J)/AL2(I,J)+CV3(I,J))*AL2(I,J)*
     [illegible]DH1=2.*CH2(I,J)*AII [illegible] *AL1(I,J)/([illegible]1(I,J)/AL1(I,J)+CH3(I,J))*T+
   1 2.*CV2(I,J)*AII [illegible] *AL2(I,J)/([illegible]2(I,J)/AL2(I,J)+CV3(I,J))*T
     AD1=AKK1*DH1/(D2H1+ . [illegible] 1)
     IF(ABSF(AD1)-S1)121,121,122
 122 IF(AD1)162,162,172
 162 AD1= .-S1
     GO TO 121
 172 AD1=S1
 121 D3(I,J)=D3(I,J)-AD1
     IF(D3(I,J)-1.)151,152,152
 152 D3(I,J)=1.
     GO TO 1 1
 151 IF(D3(I,J))153,153,1 1
 153 D3(I,J)= .
 1 1 CONTINUE
     COSTP=COST
     GO TO 1 [illegible]
1111 PUNCH 6
     TYPE 3
     PAUSE
     GO TO 1 [illegible]
1112 TYPE 5
     STOP
     END
```

APPENDIX C

RESULTS

EXAMPLE 1 , SOLUTION 1

13 1.38[illegible] 6.[illegible]

2[illegible].[illegible]06[illegible]E+02 71.[illegible]E+[illegible] 21.[illegible]1251[illegible]E+02

ROW	COL	HV	[illegible]	HINV	VINV	D.VAR	ZH	[illegible]
1	1	.1999E+ 4	.18[illegible]E- 2	.1363[illegible]+ 2	.1517E- 4	.9999E-[illegible]	.2451E-[illegible]	.2451[illegible]-
1	2	.9171E- 3	.4999E+ 4	.7845E-05	.44[illegible]1E+ 2	.1834E-06	.2070E-0[illegible]	.2[illegible]54E-
1	3	.[illegible]E-99	.9171E- 3	[illegible]	.7845E- 5	.0000E-99	.1724E-0[illegible]	.1656[illegible]-
1	4	.[illegible]E-99	.1[illegible]E+ 4	[illegible]	.1244E+ 2	.[illegible]E-99	.1216E-[illegible]	.1216E-
2	1	.3[illegible]E+ 4	.[illegible]E- 3	.2641E+ 2	.767[illegible]E- 5	.9999E-00	.2054E-0[illegible]	.2[illegible]70E-
2	2	.4792E+01	.7995E+ 4	.4621E-[illegible]1	.7038E+ 2	.599[illegible]E-03	.1656E-[illegible]	.1656E-
2	3	.[illegible]E-99	.10[illegible]4E+ 4	.[illegible]E-99	.1118E+02	.[illegible]E-99	.1216E-[illegible]	.1174E-
2	4	.[illegible]E-99	.10[illegible]E+ 4	.000[illegible]E-99	.1524E+02	.0000E-99	.6525E-01	.6525E- 1
3	1	.[illegible]E-99	.9000E-03	.[illegible]E-99	.7679E-05	.0000E-99	.1656E-00	.1656[illegible]-
3	2	.8992E+04	.3136E+01	.1001E+03	.5904E-01	.9996E-00	.1174E-00	.1174E-
3	3	.1947E-01	.1099E+ 5	.2959E-03	.1676E+03	.1770E-05	.6525E-01	.61[illegible]- 1
3	4	.[illegible]E-99	.10[illegible]E+ 4	.[illegible]E-99	.1968E+ 2	.0000E-99	.[illegible]00E-99	.[illegible]
4	1	.1[illegible]E+ 4	.[illegible]E-99	.1113E+ 2	.[illegible]E-99	.1[illegible]E+ 1	.1174E-[illegible]	.1174E-
4	2	.1[illegible]3E+ 4	.[illegible]E-99	.1529E+ 2	.[illegible]E-99	.1[illegible]E+01	.61[illegible]9E-01	.6109[illegible]- 1
4	3	.1199E+ 5	.[illegible]E-99	.2112E+ 3	.[illegible]E-99	.1[illegible]E+01	.[illegible]E-99	.[illegible]
4	4	.130[illegible]E+ 5	.[illegible]E-99	.1618E+ 3	.[illegible]E-99	.1[illegible]0E+01	.[illegible]000E-99	.[illegible]

EXAMPLE 1 , SOLUTION 2

25 1.55[illegible] 5.[illegible]0

28.19872[illegible]E+02 71.[illegible]63529E+01 21.[illegible]12358E+02

ROW	COL	HV	VV	HINV	VINV	D.VAR	ZH	Z[illegible]
1	1	.1999E+ 4	.34[illegible]E- 2	.1363E+ 2	.2577E- 4	.9999E-[illegible]	.2451E-[illegible]	.2451E-
1	2	.1837E- 4	.4999E+ 4	.[illegible]E-99	.44[illegible]1E+ 2	.3674E-08	.1912E-0[illegible]	.2[illegible]54E-
1	3	.6122E- 8	.1836E- 4	.[illegible]E-99	.[illegible]E-99	.3332E-03	.1724E-00	.[illegible]-
1	4	.[illegible]E-99	.1[illegible]E+ 4	.[illegible]E-99	.1244E+ 2	.0000E-99	.1216E-0[illegible]	.1216E-
2	1	.3[illegible]0E+ 4	.3[illegible]E- 3	.2641E+02	.1893E-05	.999[illegible]-00	.2054E-00	.2[illegible]63[illegible]-
2	2	.8844E+ 1	.7991E+ 4	.[illegible]529E-01	.7[illegible]34E+ 2	.1165E-02	.1656E-00	.1656E-
2	3	.2962E- 3	.1[illegible]8E+ 4	.7613E-05	.112[illegible]E+02	.[illegible]676E-06	.1216E-00	.1174E-
2	4	.[illegible]E-99	.10[illegible]E+ 4	.[illegible]E-99	.1524E+ 2	.[illegible]E-99	.6525E-01	.6525E-
3	1	.1365E-04	.2863E- 3	.[illegible]E-99	.1761E- 5	.455[illegible]E-01	.1656E-00	.1656E-
3	2	.8967E+ 4	.4009E+ 1	.1[illegible]+ 3	.4991E-01	.9995E-00	.1174E-00	.1174E-
3	3	.9771E-02	.1097E+ 5	.147[illegible]E-0[illegible]	.1676E+03	.8886E-06	.6525E-01	.6109E- 1
3	4	.[illegible]E-99	.1[illegible]E+ 4	.[illegible]E-99	.1968E+ 2	.[illegible]E-99	.[illegible]E-99	.[illegible]
4	1	.1[illegible]E+ 4	.[illegible]E-99	.1113E+ 2	.[illegible]E-99	.1000E+01	.1174E-00	.1174E- 0
4	2	.1[illegible]4E+ 4	.[illegible]E-99	.153[illegible]E+ 2	.[illegible]E-99	.1[illegible]E+01	.6109E-01	.61[illegible]- 1
4	3	.1199E+ 5	.[illegible]E-99	.2112E+ 3	.[illegible]E-99	.1[illegible]E+ 1	.[illegible]E-99	.[illegible]
4	4	.1299E+ 5	.[illegible]E-99	.1618E+ 3	.[illegible]E-99	.1[illegible]E+01	.[illegible]E-99	.[illegible]

EXAMPLE 2 , SOLUTION [illegible]

18 [illegible] [illegible]

26.[illegible]E+[illegible] 44.[illegible]E+01 21.[illegible]E+02

ROW	COL	HV	VV	HINV	VINV	D.VAR	ZH	ZV
1	1	.1143E+ 4	[illegible]	[illegible]E+01	[illegible]E+ 1	.5716E-[illegible]	.2548E-[illegible]	.2[illegible]4[illegible]E-
1	2	.1410E+04	.27[illegible]E+ 4	[illegible]E+01	.14[illegible]E+ 2	.3404E-00	.2134E-0[illegible]	.215[illegible]E-
1	3	.54[illegible]9E+03	.8697E+03	[illegible]E+01	.2[illegible]E+ 1	.3834E-00	.1779E-0[illegible]	.1751E-
1	4	.0000E-99	.1540E+04	[illegible]E-99	.4105[illegible]E+01	[illegible]E-99	.1271E-00	.1271[illegible]-
2	1	.2099E+04	.1757E+ 4	[illegible]48[illegible]E+01	.6946E+01	.5443E-00	.2150E-00	.2136[illegible]-
2	2	.9653E+03	.3866E+ 4	[illegible]E-99	.24[illegible]E+02	.1997E-[illegible]	.1751E-[illegible]	.175[illegible]-
2	3	.1061E+04	.1773E+ 4	[illegible]E-99	.7752E+01	.3743E-0[illegible]	.1271E-0[illegible]	.127[illegible]E-
2	4	[illegible]E-99	.26[illegible]2E+ 4	[illegible]E-99	.2468E+ 2	[illegible]E-99	.7[illegible]8[illegible]E- 1	.7[illegible]E- [illegible]
3	1	.8641E+03	.8931E+03	.2[illegible]E+01	.2[illegible]E+01	.4917E-00	.1752E-00	.175[illegible]E-
3	2	.4640E+04	.109[illegible]E+ 4	.5967[illegible]E+02	[illegible]E-99	.8096E-00	.1270E-00	.1269[illegible]-
3	3	.3193E+04	.4220E+ 4	.566[illegible]E+02	.5235E+02	.4307E-00	.7080E-01	.70[illegible]E- [illegible]
3	4	[illegible]E-99	.5795E+ 4	[illegible]E-99	.65[illegible]E+ 2	[illegible]E-99	[illegible]E-99	[illegible]E-99
4	1	.1893E+04	.0000E-99	[illegible]E+01	[illegible]E-99	.1000E+01	.1269E-00	.1269E-
4	2	.2963E+ 4	[illegible]E-99	.3[illegible]491E+ 2	[illegible]E-99	.1[illegible]E+01	.7[illegible]55E- 1	.7[illegible]
4	3	.72[illegible]4E+ 4	[illegible]E-99	[illegible]E+ 2	[illegible]E-99	.1[illegible]E+01	[illegible]E-99	[illegible]
4	4	.13[illegible]E+ 5	[illegible]E-99	.1[illegible]E+ 3	[illegible]E-99	.1[illegible]E+01	[illegible]E-99	[illegible]

EXAMPLE 2 , SOLUTION 2

19 1.5500 1.[illegible]

26.[illegible]43[illegible]57E+02 44.[illegible]46[illegible]3E+01 21.6[illegible]9271E+02

ROW	COL	HV	VV	HINV	VINV	D.VAR	ZH	ZV
1	1	.1143E+ 4	.8566[illegible]+ 3	.2[illegible]E+01	.2[illegible]E+ 1	.5716E-[illegible]	.2552E-[illegible]	.25[illegible]E-
1	2	.1435E+ 4	.27[illegible]7[illegible]+ 4	.5847E+01	.1383E+02	.3465E-00	.2138E-00	.2155 -
1	3	.5425E+03	.[illegible]33[illegible]+ 3	.2[illegible]E+ 1	.2[illegible]E+ 1	.3778E-	.1782E-[illegible]	.17[illegible] -
1	4	[illegible]E-99	.1542E+ 4	[illegible]E-99	.42[illegible]5E+01	[illegible]E-99	.1274E-[illegible]	.1274-
2	1	.2[illegible]74E+ 4	.1782E+ 4	[illegible]26[illegible]E+ 1	[illegible]E+ 1	.5376E-0[illegible]	.2155E-0[illegible]	.2140E-
2	2	[illegible]842E+03	.3797E+04	[illegible]E-99	.2342E+ 2	.2[illegible]58E-0[illegible]	.1751E-00	.1757[illegible] -
2	3	.1[illegible]45E+04	.183[illegible]E+ 4	[illegible]E-99	.64[illegible]4E+ 1	.3632E-00	.1274E-00	.127[illegible] -
2	4	[illegible]E-99	.2567E+ 4	[illegible]E-99	.2445E+ 2	[illegible]E-99	.7105E- 1	.71[illegible]5E-[illegible]
3	1	.8655E+ 3	.9168E+03	[illegible]E+01	.2[illegible]E+ 1	.4856E-00	.1757E-0[illegible]	.174[illegible]E-
3	2	.4393E+04	.126[illegible]E+ 4	[illegible]E+ 2	.61[illegible]4E-	.7757E-0[illegible]	.127[illegible]E-0[illegible]	.1267[illegible]-
3	3	.324[illegible]E+ 4	.3964E+ 4	[illegible]741[illegible]+ 2	.4075E+ 2	.4485E-[illegible]	.71[illegible]5E-01	.7[illegible]- [illegible]
3	4	[illegible]E-99	.5623E+ 4	[illegible]E-99	.65[illegible]E+ 2	[illegible]E-99	[illegible]E-99	[illegible]
4	1	.1916E+ 4	[illegible]E-99	.6345[illegible]+ 1	[illegible]E-99	.1000E+01	.1267E-00	.1267E-[illegible]
4	2	.3186E+04	[illegible]E-99	.335[illegible]+ 2	[illegible]E-99	.1[illegible]E+01	.7034E-01	.7[illegible]34[illegible]- 1
4	3	.7171E+ 4	[illegible]E-99	.25[illegible]E+ 2	[illegible]E-99	.1[illegible]E+ 1	[illegible]E-99	[illegible]
4	4	.13[illegible]+ 5	[illegible]-99	.1[illegible]E+03	[illegible]E-99	.1[illegible]E+ 1	[illegible]E-99	[illegible]

EXAMPLE [illegible] , [illegible]

12 1.[illegible] .[illegible]

[illegible] [illegible] [illegible]

ROW	COL	HV	VV	HINV	VINV	.VAR	ZH	[illegible]
1	1	.1[illegible]46E+ 4	[illegible]	.1[illegible]4[illegible]	[illegible]	.[illegible]251E-0[illegible]	[illegible]	[illegible]
1	2	.[illegible]533E+04	.1[illegible]+ 4	[illegible]	[illegible]	.[illegible]766E-0[illegible]	[illegible]	[illegible]
1	3	.7669E+03	.1[illegible]E+04	.[illegible]E+02	.[illegible]E+02	.3295E-0[illegible]	.1[illegible]45E-00	.19[illegible] -
1	4	.[illegible]E-99	.17[illegible]+ 4	[illegible]	.4[illegible]E+02	.0000E-9[illegible]	.1549E-00	.154[illegible] -
2	1	.1656E+04	.2[illegible]+ 4	[illegible]+02	.2451E+02	.4140E-0[illegible]	.2277E-0[illegible]	.22[illegible] -
2	2	.1496E+04	.18[illegible] +04	.2[illegible]E+02	.[illegible]E+ 2	.4473E-0[illegible]	.1960E-0[illegible]	.19[illegible] -
2	3	.1269E+04	.277[illegible]E+ 4	.2737E+02	.1262E+02	.3174E-00	.1549E-00	.1574[illegible]
2	4	.[illegible]E-99	.30[illegible]E+ 4	[illegible]E-9[illegible]	.[illegible]E+02	.0000E-99	.1006[illegible]-0[illegible]	.1[illegible]
3	1	.1145E+ 4	.117[illegible]E+04	.2966E+ 2	.[illegible]3E+02	.4944E-0[illegible]	.1963E-0[illegible]	.1[illegible]6[illegible]
3	2	.2902E+ 4	.1094E+ 4	.[illegible]62[illegible]E+ 2	.2[illegible]71E+ 2	.7262E-0[illegible]	.1574E-0[illegible]	.15[illegible] -
3	3	.2726E+04	.[illegible]4[illegible]E+04	.4[illegible]83E+0[illegible]	.[illegible]16E+ 2	.4083E-0[illegible]	.1006E-0[illegible]	.1[illegible]4[illegible] -
3	4	.[illegible]E-99	.578[illegible]E+ 4	[illegible]E-9[illegible]	.6[illegible]E+02	.0000E-99	.[illegible]E-99	.[illegible] -
4	1	.2171E+ 4	.[illegible]E-99	.4[illegible]E+ 2	.[illegible]E-99	.1000E+01	.1566E-[illegible]	.15[illegible]6[illegible] -
4	2	.3205E+04	.[illegible]E-99	.5[illegible]E+02	.[illegible]E-99	.1000E+01	.1[illegible]04E-0[illegible]	.1[illegible]4[illegible] -
4	3	.7214E+04	.[illegible]E-99	.[illegible]E+ 2	.[illegible]E-99	.1[illegible]E+01	.[illegible]E-99	.[illegible]E-
4	4	.1300E+ 5	.[illegible]E-99	.[illegible]E-99	.[illegible]E-99	.1000E+01	.[illegible]E-99	.[illegible] -

EXAMPLE 2 . SOLUTION 2

12 1.[illegible]5 .[illegible]

[illegible]1.12[illegible]1[illegible]E+ 2 [illegible]5.[illegible]7[illegible]+01 22.[illegible]16E+02

ROW	COL	HV	VV	HINV	VINV	[illegible].VAR	ZH	[illegible]
1	1	.[illegible]E+ 3	.10[illegible]4E+04	.[illegible]47[illegible]E+02	.21[illegible]2E+02	.4[illegible]77E-00	.2660E-[illegible]	.26[illegible] -
1	2	.2201E+04	.1714E+04	.[illegible]372[illegible]E+02	.1027E+02	.5709E-0[illegible]	.2257E-0[illegible]	.227[illegible] -
1	3	.7697E+03	.1511E+04	.2221E+02	.[illegible]E+02	.[illegible]575E-0[illegible]	.1[illegible]52E-0[illegible]	.1[illegible] -
1	4	.[illegible]E-99	.176[illegible]E+ 4	[illegible]	.4[illegible]E+02	.0000E-9[illegible]	.1556E-00	.1[illegible] -
2	1	.1635E+04	.2[illegible]E+04	.1540E+02	.[illegible]453E+02	.4[illegible]8[illegible]E-0[illegible]	.2277E-0[illegible]	.22[illegible] -
2	2	.1533E+04	.1816E+04	.[illegible]882E+02	.[illegible]117E+02	.4577E-0[illegible]	.1960E-0[illegible]	.1[illegible] -
2	3	.1344E+04	.27[illegible]E+ 4	.2861[illegible]+02	.[illegible]1[illegible]E+02	.3324E-0[illegible]	.1556E-0[illegible]	.1571[illegible] -
2	4	.[illegible]E-99	.3114E+ 4	[illegible]E-9[illegible]	.[illegible]E+ 2	.[illegible]E-99	.1[illegible]07E-0[illegible]	.1[illegible]
3	1	.1225E+ 4	.114[illegible]E+ 4	.[illegible]4E+ 2	.[illegible]E+02	.[illegible]174[illegible]-0[illegible]	.1[illegible]62E-0[illegible]	.1[illegible]57[illegible] -
3	2	.2915E+04	.1127E+ 4	[illegible]	.[illegible]E+02	.7[illegible]11E-0[illegible]	.1571E-0[illegible]	.15[illegible] -
3	3	.2684E+04	.393[illegible]E+ 4	.4[illegible]E+[illegible]	.[illegible]41E+ 2	.4[illegible]58E-0[illegible]	.1007E-0[illegible]	.1[illegible] -
3	4	.[illegible]E-99	.579[illegible]E+04	[illegible]	.[illegible]E+ 2	.[illegible]E-99	.[illegible]E-99	.[illegible] -
4	1	.[illegible]143E+04	.[illegible]E-99	.4[illegible]E+ 2	.[illegible]E-99	.1[illegible]E+01	.1565E-0[illegible]	.15[illegible] -
4	2	.527[illegible]E+04	.[illegible]E-99	.[illegible]E+02	.[illegible]E-99	.1[illegible]E+ 1	.1012E-0[illegible]	.1[illegible] -
4	3	.72[illegible]E+04	.[illegible]E-99	.[illegible]E+02	.[illegible]E-99	.1[illegible]E+ 1	.[illegible]E-99	.[illegible] -
4	4	.15[illegible]E+05	.[illegible]E-99	.[illegible]E-99	.[illegible]E-99	.1[illegible]E+01	.[illegible]E-99	.[illegible] -

[illegible] PLE 4 , SOLUTION 1

1 [illegible] 1.5[illegible] 1. .[illegible]

2[illegible].[illegible]E+ [illegible] [illegible]E+ 1 [illegible].[illegible]E+[illegible]

ROW	COL	HV	VV	HINV	VINV	C.VAR	Z1	Z2
1	1	.1144E+ 4	.[illegible]+ [illegible]	.[illegible]	.[illegible]	.[illegible]721E-[illegible]	.[illegible]	.[illegible]
1	2	.14[illegible]3E+ 4	.274[illegible]+ 4	.[illegible]+ 1	.[illegible]4E+ 1	.[illegible]386E-[illegible]	.[illegible]	.[illegible]41[illegible]
1	3	.466[illegible]E+ 3	.[illegible]16[illegible]+ 3	.[illegible]	.[illegible]E-99	.[illegible]469[illegible]	.[illegible]	.1[illegible]
1	4	.[illegible]E-99	.146[illegible]+ 4	.[illegible]	.[illegible]E-99	.[illegible]E-[illegible]	.145[illegible]	.14[illegible]
2	1	.2[illegible]13E+04	.1[illegible]4[illegible]E+ 4	.[illegible]2[illegible]+01	.[illegible]E+01	.[illegible]222E-0[illegible]	.2413E-[illegible]	.[illegible]
2	2	.127[illegible]E+04	.346[illegible]E+ 4	.[illegible]E-99	.12[illegible]E+ 2	.2072E-[illegible]	.1955E-0[illegible]	.1[illegible]
2	3	.1169E+04	.2017[illegible]+ 4	.[illegible]E-99	.47[illegible]E+01	.[illegible]67[illegible]E-0[illegible]	.1453E-0[illegible]	.1[illegible]
2	4	.[illegible]E-99	.265[illegible]E+ 4	.[illegible]	.1[illegible]2E+ 2	.[illegible]E-99	.7864E-01	.7[illegible]4[illegible]
3	1	.[illegible]2[illegible]E+ 3	.[illegible]212E+ 3	.[illegible]E-9[illegible]	.[illegible]E-99	.4999E-0[illegible]	.1955E-[illegible]	.1[illegible]
3	2	.39[illegible]E+ 4	.1446E+ 4	.[illegible]8[illegible]+02	.[illegible]E-99	.7323E-0[illegible]	.1398E-[illegible]	.1[illegible]
3	3	.[illegible]1[illegible]5[illegible]+ 3	.6[illegible]65[illegible]+ 4	.[illegible]E-9[illegible]	.[illegible]7[illegible]E+02	.1305E-[illegible]	.7864E-01	.73[illegible]
3	4	.[illegible]E-99	.3567E+ 4	.[illegible]E-9[illegible]	.[illegible]E+ 2	.[illegible]00E-99	.[illegible]E-99	.[illegible]
4	1	.1921E+04	.[illegible]E-99	.[illegible]788E-0[illegible]	.[illegible]E-99	.1000E+01	.1398E-0[illegible]	.139[illegible]
4	2	.[illegible]367E+ 4	.[illegible]E-99	.[illegible]331E+ 2	.[illegible]E-99	.1[illegible]00E+01	.7307E-01	.73[illegible]7[illegible]
4	3	.9432E+04	.[illegible]E-99	.1[illegible]89E+0[illegible]	.[illegible]E-99	.1[illegible]0[illegible]E+01	.[illegible]E-99	.[illegible]
4	4	.130[illegible]E+05	.[illegible]E-99	.[illegible]E-99	.[illegible]E-99	.1[illegible]+01	.[illegible]00E-99	.[illegible]

[illegible]EXAMPLE 4 , SOLUTION 2

2 [illegible] 1.[illegible] 1.[illegible] .2[illegible]

26.[illegible]2E+ 2 [illegible].[illegible]1E+ 1 2[illegible].[illegible]38[illegible]E+0[illegible]

ROW	COL	HV	VV	HINV	VINV	C.VAR	Z1	Z2
1	1	.11[illegible]E+ 4	.[illegible]61[illegible]+ 3	.[illegible]	.[illegible]E-9[illegible]	.56[illegible]	.[illegible]7[illegible]	.[illegible]
1	2	.14[illegible]E+04	.[illegible]7[illegible]E+04	.[illegible]+01	.7[illegible]4E+01	.[illegible]396[illegible]	.[illegible]	.241[illegible]
1	3	.4603E+03	.[illegible]251E+ 3	.[illegible]E-9[illegible]	.[illegible]E-9[illegible]	.3417E-[illegible]	.2[illegible]18E-[illegible]	.1[illegible]4[illegible]
1	4	.[illegible]E-99	.146[illegible]+ 4	.[illegible]E-99	.[illegible]E-99	.[illegible]E-99	.1453E-[illegible]	.14[illegible]
2	1	.[illegible]E+ 4	.1[illegible]5[illegible]E+ 4	.[illegible]74[illegible]+01	.[illegible]E+01	.[illegible]191E-0[illegible]	.2413E-0[illegible]	.2[illegible]
2	2	.127[illegible]E+04	.3461E+04	.[illegible]E-99	.1274E+02	.[illegible]692E-0[illegible]	.[illegible]954E-0[illegible]	.1[illegible]
2	3	.1166E+04	.2[illegible]34E+ 4	.[illegible]E-9[illegible]	.[illegible]5E+01	.[illegible]044E-0[illegible]	.1453E-0[illegible]	.1[illegible]
2	4	.[illegible]E-99	.2647E+ 4	.[illegible]E-9[illegible]	.1[illegible]12E+02	.[illegible]E-99	.7863E-01	.7[illegible]
3	1	.[illegible]246E+0[illegible]	.[illegible]322[illegible]+ [illegible]	.[illegible]E-9[illegible]	.[illegible]E-99	.4979E-[illegible]	.1955E-[illegible]	.1[illegible]
3	2	.3936[illegible]E+04	.145[illegible]+ 4	.[illegible]71E+ 2	.[illegible]E-[illegible]	.73[illegible]E-[illegible]	.1398E-[illegible]	.1[illegible]
3	3	.91[illegible]E+ 3	.6[illegible]5[illegible]E+ 4	.[illegible]E-[illegible]	.[illegible]E+02	.1306E-[illegible]	.7863E-01	.73[illegible]
3	4	.[illegible]E-99	.3557[illegible]+04	.[illegible]E-9[illegible]	.[illegible]7E+02	.[illegible]E-99	.[illegible]E-99	.[illegible]
4	1	.1932E+04	.[illegible]-9[illegible]	.[illegible]84[illegible]+ 1	.[illegible]E-9[illegible]	.1[illegible]E+01	.1[illegible]8[illegible]E-[illegible]	.13[illegible]
4	2	.[illegible]382E+ 4	.[illegible]E-99	.[illegible]49[illegible]+ [illegible]	.[illegible]E-99	.1[illegible]E+ 1	.73[illegible]6E-0[illegible]	.730[illegible]
4	3	.[illegible]442E+ 4	.[illegible]E-99	.[illegible]9[illegible]E+ [illegible]	.[illegible]E-99	.1[illegible]E+ 1	.[illegible]00E-99	.[illegible]
4	4	.130[illegible]E+ 5	.[illegible]E-99	.[illegible]E-[illegible]	.[illegible]E-99	.1[illegible]0E+01	.[illegible]E-99	.[illegible]

RESULTS OF PERTURBATION , PROBLEM 1

Decision Variables being Perturbed	Original Values	Values after Perturbation	Resulting Total Costs
$\theta_3^{1,1}$	1.0000	.9500	622818.51
$\theta_2^{1,2}$	44.0100	40.0000	2820.25
$\theta_1^{2,3}$	.0000	.1000	2819.95
$\theta_2^{2,4}$	15.2400	16.0000	2819.88
$\theta_3^{3,2}$	1.0000	.9500	31390309.00
$\theta_2^{3,4}$	19.6800	18.0000	2820.01
$\theta_1^{4,1}$	11.1300	12.0000	2819.91
$\theta_1^{4,3}$	211.2000	215.0000	2819.91

ORIGINAL TOTAL COST = $2,819.86

RESULTS OF PERTURBATION , PROBLEM 2

Decision Variables Being Perturbed	Original Values	Values after Perturbation	Resulting Total Costs
$\theta_2^{1,1}$	2.0000	2.2000	2604.10
$\theta_3^{1,3}$	.3834	.4000	2604.01
$\theta_2^{2,2}$	24.0300	23.0000	2604.03
$\theta_3^{2,3}$	.3743	.3600	2604.05
$\theta_1^{3,1}$	2.0000	2.2000	2604.06
$\theta_3^{3,3}$	.4307	.4200	2604.07
$\theta_1^{4,1}$	6.0800	6.5000	2604.00
$\theta_1^{4,3}$	85.0000	84.0000	2604.62

ORIGINAL TOTAL COST = $2,603.99

RESULTS OF PERTURBATION , PROBLEM 3

Decision Variables Being Perturbed	Original Values	Values after Perturbation	Resulting Total Costs
$\theta_3^{1,1}$	.5231	.5000	3112.95
$\theta_1^{1,3}$	21.7300	22.5000	3112.91
$\theta_2^{2,2}$	31.8000	31.0000	3112.92
$\theta_3^{2,3}$	.3174	.3100	3112.95
$\theta_1^{3,1}$	29.6600	31.0000	3112.92
$\theta_3^{3,3}$	.4083	.4000	3112.96

ORIGINAL TOTAL COST = $3,112.90

RESULTS OF PERTURBATION , PROBLEM 4

Decision Variables Being Perturbed	Original Values	Values after Perturbation	Resulting Total Costs
$\theta_3^{1,1}$	.5721	.5500	
$\theta_3^{1,1}$	.5721	.5500	2639.49
$\theta_2^{1,2}$	8.0040	.8800	
$\theta_2^{2,1}$	5.2560	4.4560	2654.22
$\theta_3^{2,2}$	.2672	.2800	
$\theta_2^{2,3}$	.2672	.2800	2639.52
$\theta_2^{2,4}$	4.7630	5.0000	
$\theta_3^{3,1}$	15.2200	14.9800	2639.46
$\theta_3^{3,1}$	.4999	.4600	
$\theta_1^{4,1}$	.4999	.4600	2639.65
$\theta_1^{4,2}$	.9788	1.0000	
$\theta_1^{4,2}$	23.3100	23.2900	2639.44
$\theta_1^{4,3}$	23.3100	23.0000	
	108.9000	109.2100	2639.45

ORIGINAL TOTAL COST = $2,639.38

LIST OF SYMBOLS

Discrete Maximum Principle

X	state variable.
θ	decision variable.
T	transformation operator.
n	the n-th stage.
N	the N-th stage or the total number of stages.
s	total number of state variables in each stage.
r	total number of decision variables in each stage.
H	the Hamiltonian.
z	adjoint variable.
c	constant in objective function.
S	objective function.

General Formulation of the Problem

$X_j^{n,m}$	state variables representing flows from node (n,m).
$\theta_j^{n,m}$	decision variables representing investments on links leaving node (n,m).
$K_{j1}^{n,m}$	free flow travel time on links leaving node (n,m).
$K_{j2}^{n,m}$	coefficient of investment on links leaving node (n,m).
$K_{j3}^{n,m}$	existing investment on links leaving node (n,m).
$L_j^{n,m}$	link length on links leaving node (n,m).

$t_j^{n,m}$	unit travel time on links leaving node (n,m). where, j = 1, for horizontal link. j = 2, for vertical link.
$X_3^{n,m}$	state variable representing the total investment on horizontal links from node (1,1) through node (n,m).
$X_4^{n,m}$	state variable representing the total investment on vertical links from node (1,1) through (n,m).
$X_5^{n,m}$	state variable representing the total travel time cost on horizontal links from node (1,1) through node (n,m).
$X_6^{n,m}$	state variable representing the total travel time cost on vertical links from node (1,1) through (n,m).
$X_7^{n,m}$	state variable representing the total investment on both links from node (1,1) through node (n,m).
$\theta_3^{n,m}$	decision variable representing the fraction of the vehicles departing node (n,m) on the horizontal link.
C_t	time cost.
$H^{n,m}$	Hamiltoniam function at node (n,m).
$V^{n,m}$	input trips at node (n,m).
GI	total system budget.
$SI^{n,m}$	section budget at node (n,m).
$Z_1^{n,m}, Z_2^{n,m}, \ldots, Z_7^{n,m}$	adjoint variables associated with $X_1^{n,m}, X_2^{n,m}, \ldots, X_7^{n,m}$ respectively.
S	objective function.

TOWARD A SOLUTION FOR THE OPTIMAL ALLOCATION
OF INVESTMENT IN TRANSPORTATION NETWORK
DEVELOPMENT

by

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AN ABSTRACT OF A MASTER'S THESIS

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ABSTRACT

Following an introduction to the purpose of the economic analysis of transportation networks, the objective and criterion problems are briefly discussed. A single objective is selected for this study. A review of the literature shows the historical development of the methods of economic analysis. Major drawbacks of the traditional methods developed in the past are described. The merits and demerits of mathematical models are presented.

The discrete maximum principle is introduced with a brief review of its recent applications to transportation systems analysis. Starting with a description of the relationships between travel time, traffic volume and investment cost, a non-linear total travel time equation is developed which expresses travel time as a function of traffic volume and investment.

The purpose of this research was to formulate an optimal network improvement model (in equation forms as described) by the discrete maximum principle. Utilizing these equations, optimum seeking procedures were then developed. Three investment conditions were considered which resulted in three different sets of equations and two slightly different ways of seeking the optimum. Three special cases which implied limits on link investment were also described. This formulation provided a broad application of the technique to

problems with various constraints and assumed conditions.

Finally, four examples were presented to demonstrate the usefulness of the technique in different investment conditions. Derivation of the data used in example problems and the computer programs developed to solve the problem were presented in Appendix B.