A METHOD OF GENERATING n-BRANCH MILLMAN NETWORKS

by

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INTRODUCTION

The output voltage of a Millman network can be expressed as a combination of branch voltages and branch admittances of the network. Suppose it is desired that the output voltage follow a unit step function input to a specified degree. Branch admittances will be chosen to satisfy this condition provided branch voltages are suitable functions of time.

Solutions of the first-order vector differential equation $\dot{y} + My = x(t)$ will be used to generate branch admittances of Millman networks. The n x n matrix M will be a column-sum-one matrix with real, constant elements and with the vectors $x(t)$ and $y(0)$ chosen as column vectors of M or their linear combinations. Only bounded solutions will be considered, and positive-realness of admittances will be investigated to insure physical realizability of the impedors. By an impedor is meant that it is a generalized impedance element; i.e., an impedor may be a capacitor, inductor, or resistor, or any combination of them. Specific examples will be used in many instances to illustrate forms of solutions and other desirable characteristics.

Previous work in the areas, solution of vector differential equations, column-sum-one matrices, and positive-real functions will be discussed briefly.

Also, appendices will be devoted to column-sum-one matrices, a particular solution of the differential equation $\dot{y} + My = x$, bounded solution vectors by premultiplication, and selection of Millman network branch voltages.

PREVIOUS WORK

Our purpose is to give an account of previous work in areas encompassed by this thesis, which are solutions of vector differential equations, column-sum-one matrices, and positive-real functions.

Vector Differential Equations

Zadeh and Desoer (27) have used classical methods to solve first-order differential equations such as $\dot{x} = Ax + a(t)$, where A is an n x n matrix with real constant elements, to yield

$$x(t) = e^{A(t-t_0)}\ x(0) + \int_{t_0}^{t} e^{A(t-\tau)}\ a(\tau)d\tau \ .$$

Gantmacher (10), Ferrar (8), and Bellman (3) have also discussed classical methods of solution of $\dot{x} = Ax$ and similar equations. However, their interest was in the method, rather than the properties, of the solutions. Our interest lies in bounded solutions of the equation $\dot{y} + My = x$, and the above equation will obviously have unbounded solutions.

The classical solution will require calculations using matrix functions, so the use of Laplace transforms in solving vector differential equations is considered. For example, Hochstadt (12) has dealt with solutions of the equation $\dot{x} = Ax$, where A is again an n x n matrix with real, constant elements. Hochstadt used Laplace transforms to yield

$$sx(s) - x(0) = Ax(s)$$
$$x(s) = (sI - A)^{-1}\, x(0)$$

provided det $(sI - A) \neq 0$. The solutions of this equation will in general be unbounded, since for some $a_{ij} \neq 0$, det $(sI - A)$ will have right-half-plane zeros.

Column-sum-one Matrices

Column-sum-one matrices are of interest because the matrix M in the equation $\dot{y} + My = x$ is specified to be a column-sum-one matrix, and vectors $x(t)$ and $y(0)$ are derived from M. Considerable literature is available on column-sum-one matrices with respect to doubly stochastic (column and row-sum-one) and stochastic (row-sum-one) matrices.

Kemeny (13) discusses stochastic matrices and their properties in a chapter on finite Markov chains. Marcus (18, 19) discusses several theorems on doubly stochastic and nonnegative matrices. Bellman devotes two chapters of his book to Markov matrices and stochastic matrices. In fact, Bellman has solved the vector differential equation $\frac{dx}{dt} = Zx$ by transforming it into a difference equation. Here Z is a stochastic matrix.

Excellent bibliographies of literature on nonnegative and stochastic matrices are found in Wedderburn (25) and Gantmacher, Volume I.

Positive-real Functions

Positive-real functions are of importance since impedors of Millman networks are required to be physically realizable. As Brune (5) stated, "A necessary and sufficient condition for a real, rational function to be realizable as the driving point impedance of a one-port network is that it be a positive-real function."

Weinberg and Slepian (26) reviewed concepts, definitions, and theorems associated with positive-real functions and positive-real matrices. Impedors of Millman networks will be generated by vectors, rather than matrices, so parts of theorems on positive-real matrices given by Weinberg will not be applicable. For example, Theorem 4 of Weinberg stated that for each element $a_{ij}(s)$ of a positive-real matrix it was required that: (1) poles on the imaginary axis, Re $s = 0$, be simple, and (2) there must be no right-half-plane poles. These conditions should apply, whether positive-real vectors or matrices are considered. The remaining requirements of Theorem 4 were not applicable to vectors, so clearly these two conditions are only necessary ones for positive-real vectors. Further discussion of positive-real matrices may be found in Duffin (6) and Belevitch (2).

Lee (15) gave a summary of information on positive-real functions and discussed nonpositive-real functions. Arithmetic operations with positive-real and nonpositive-real functions were demonstrated; e.g.,

$$f(s) = \frac{1}{1+s} + \left[\frac{1}{1+s}\right]^n$$

is positive-real for $n \geq 1$ and $f(s)$ is the sum of a positive-real function and a partial-positive-real function. In what follows, our investigation of positive-real vectors will yield partial-positive-real and nonpositive-real functions.

PRELIMINARIES

Of particular interest will be the solutions of the first-order vector differential equation

$$\dot{y} + My = x(t) \qquad y(t)\Big|_{t=0} = y(0) \tag{1}$$

where $x(t)$, $y(t)$, and $y(0)$ are column n-vectors. M is an $n \times n$ column-sum-one matrix with real, constant elements.

The column n-vector u is defined to be $u = \left[1\ 1\ 1\ \ldots\ 1\right]'$ and is called the "sum vector". The prime, " ' ", indicates a transpose. With these identifications, $u'M = u'$.

Two lemmas may be stated without proof.

<u>Lemma 1</u>. If $u'M = u'$, then $u'M^n = u'$, $n = 0, 1, 2, \ldots$.

<u>Lemma 2</u>. If $u'M = u'$, then $u'e^{Mt} = u'e^t$.

The vectors $x(t)$ and $y(0)$ are required to be either column vectors of M or their linear combinations, which means that

$$u'x(t) = 1 \quad \forall t \text{ and } u'y(0) = 1 .$$

The following lemma results from the application of Lemmas 1 and 2 to the solution of (1):

<u>Lemma 3</u>. If M is such that $u'M = u'$, $u'y(0) = 1$, and

$u'x(t) = 1 \;\; \forall t,\ t \in (0, \infty)$, then the solution of (1) has the property that $u'y(t) = 1 \;\; \forall t$.

Solution vectors resulting from the use of column vectors of M for x(t) and y(0) are denoted y(i,j;t), where i and j indicate that:

$x(t)$ is the i^{th} column of M, $\quad i = 1, 2, \ldots, n$

and $\quad y(0)$ is the j^{th} column of M, $\quad j = 1, 2, \ldots, n$.

Other solution vectors will simply be denoted y(t) or y. The subscript k will indicate the k^{th} component of the vector; e.g., y_k.

The possible number of different y(i,j;t) depends upon the value of n and the rank of M. This number is less than or equal to n^2. When linear combinations of column vectors are allowed for vectors x(t) and y(0), an infinite number of solution vectors is possible.

SOLUTION OF THE DIFFERENTIAL EQUATION $\dot{y} + My = x$

The solution of the differential equation $\dot{y} + My = x(t)$ is a well known result, and may be written

$$y(t) = e^{-Mt}y(0) + \int_0^t e^{-M(t-\tau)}\, x(\tau)\,d\tau \ .$$

Laplace transforms will be used instead to solve equation (1) to avoid matrix functions and because it will be convenient to have solution vectors in the Laplace transform domain.

If $\overline{x}$ and $\overline{y}$ denote the Laplace transforms of x(t) and y(t) then our vector differential equation transforms into

$$s\bar{y} - y(0) + M\bar{y} = \bar{x}$$

$$(M + sI)\bar{y} = \bar{x} + y(0)$$

$$\bar{y} = (M + sI)^{-1} \left[\bar{x} + y(0)\right]$$

$$y(t) = \mathcal{L}^{-1} \left[(M + sI)^{-1} \left[\bar{x} + y(0)\right]\right] . \qquad (2)$$

Since $$(M + sI)^{-1} = \frac{\left[\text{adj } (M + sI)\right]'}{\det (M + sI)}$$

where "adj" denotes "adjoint" and "det" denotes "determinant", a necessary condition for solutions of (1) to exist is that

$$\det (M + sI) \neq 0 . \qquad (3)$$

The most interesting solution vectors to consider will be those which are bounded. Two definitions are stated:

Definition 1. A bounded solution vector is a vector such that each component of the vector is either constant or bounded for all time, $t \in (0, \infty)$.

Definition 2. An unbounded solution vector is a vector such that one or more components are unbounded functions of time.

PROPERTIES OF THE MATRIX M

Since it has been specified that x(t) and y(0) are column vectors of the matrix M, or linear combinations of these vectors, and that the value of det (M + sI) depends upon the elements of M, then the existence of bounded solution vectors depends entirely upon the form of the matrix M.

Some restrictions upon M have been given previously. M was to be an n x n matrix with real, constant elements and column sums equal to one. It is also important that M is such that det (M + sI) has no right-half-plane zeros, because of our interest in bounded solution vectors.

Let M be chosen a triangular matrix. This choice of form for M eliminates complex zeros of det (M + sI) from consideration, and terms of the form $e^{-kt} \cos k\omega t$ from solution vectors. Moreover, calculation complexity is avoided without obscuration of essential results.

If M is triangular, then negative diagonal elements are disallowed because of the resultant right-half-plane zeros in det (M + sI). Also, at most one zero diagonal element is allowed in M (which is equivalent to stating that the rank of M can only be n or n - 1) because terms of the form t^n, n = 1, 2, ..., would result in unbounded solution vectors.

The condition that M have no negative diagonal elements, with at most one zero diagonal element, is sufficient to guarantee bounded solution vectors, provided M is chosen a triangular form. Note that the condition of equation (3) is satisfied also.

The example in Appendix B for $M_n = \frac{1}{n} uu'$ illustrates the validity of the requirement of no negative, with at most one zero, diagonal element for M.

Det (M + sI) will then have the form

$$(s + a_{11})\ (s + a_{22})\ \ldots\ (s + a_{nn})$$

with at most one $a_{ii} = 0$. Since

$$\mathcal{L}^{-1}\left[\frac{n!}{s(s + 1)\ (s + 2)\ \ldots\ (s + n)}\right] = (1 - e^{-t})^n$$

then the form of det (M + sI) insures bounded solution vectors.

In fact, solution vectors for M under the preceding conditions will contain elements which are functions of exponential order, a set which will be denoted $\mathcal{E}$.

Further discussion of bounded solution vectors will be found in Appendix C.

The nonzero off-diagonal elements of M have been neglected up to now, but these elements will have no effect upon det (M + sI) because M is triangular. Nonzero off-diagonal elements do have some effect upon the form of elements of bounded solution vectors, however.

PROPERTIES OF SOLUTION VECTORS

Restrictions upon the elements of M which guaranteed bounded solution vectors were discussed in the preceding section. There are some important properties of this class of vectors which should be considered.

An illustration of the form of solution vectors is given by using the following matrix in equation (2):

$$M_1 = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\overline{y}(3,1) = \begin{bmatrix} \dfrac{1}{(3s + 1)} & \dfrac{3s}{(3s + 1)(3s + 2)} & \dfrac{s + 2}{s(3s + 2)} \end{bmatrix}' \qquad (4)$$

$$y(3,1;t) = \left[(\frac{1}{3} e^{-t/3}) \; (\frac{2}{3} e^{-2t/3} - e^{-t/3}) \; (1 - \frac{2}{3} e^{-2t/3}) \right]' .$$

M_1 is a nonnegative matrix with rank three so $0 \leq m_{ij} \leq 1$, where m_{ij} is an element of M_1, and i, j = 1, 2, ..., n. The solution set for nonnegative matrices will then have elements of the form:

$$k, \pm qe^{-pt}, k \pm qe^{-pt}, \text{etc.}$$

where $0 \leq k \leq 1$, $0 \leq p \leq 1$, and $0 \leq q \leq 1$. Obviously, for a given value of n, the number of terms occurring for each component of solution vectors will be less than or equal to n.

Determination of a Solution Basis[1]

Let M_t denote the class of n x n matrices which yields bounded solution vectors; i.e., M_t is a triangular matrix with real, constant elements, and has nonnegative diagonal elements (at most one zero element). The vectors x(t) and y(0) are column vectors of M_t or their linear combinations.

The solution set for M_t comprises a vector space $\mathcal{V}$ because the set is closed under addition and scalar multiplication. Consider the subset $\mathcal{Y}$ of the vector space $\mathcal{V}$ for which $u'M = u'$, which means that $u'x(t) = 1$ and $u'y(0) = 1$. We shall be interested in finding a set of basis (c.f. below) vectors for $\mathcal{Y}$. Vectors of the solution set $\mathcal{Y}$ are not closed under addition

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[1]This is not a basis in the ordinary sense, c. f. below.

and scalar multiplication, however, since column sums are not preserved under these operations. Thus the conventional definition of a basis [Fuller (9), Turnbull (24), Kemeny (13), and Ayres (1)] cannot be applied to $\mathcal{Y}$.

A set of basis vectors for the solution set $\mathcal{Y}$ can be found if the following definition is made:

Definition 3. A solution basis is a set of linearly independent vectors which span the solution set.

Elementary column transformations (E.C.T.) will be defined to include both scalar and functional multiplication of vectors, where the functions contain elements of the set $\mathcal{E}$, since elements of bounded solution vectors in $\mathcal{Y}$ are members of the set $\mathcal{E}$.

With these definitions in mind, Fuller's method of finding a set of basis vectors can be employed to find a solution basis for the solution set $\mathcal{Y}$. Validity of his method depends upon the fact that it can be shown that the column space of a matrix is unchanged under E.C.T. This implies that the same space is spanned by both the column vectors of a matrix and a simplest, or canonical form. Note that column sums are not necessarily preserved under E.C.T.

Fuller's process of reduction of a rectangular matrix to a canonical form using E.C.T. is performed by working down the rows, trying to obtain ones as diagonal elements with the rest of the elements to be zeros in columns with ones in them. When a zero vector appears in the final form, all those vectors of higher index will be zero also.

The nonzero column vectors of the canonical form are r in number and are linearly independent, whereas column vectors of the original matrix may be linearly dependent.

Using Fuller's method, a solution basis for the set of $y(i,j;t)$ (which result when $M = M_1$) in the solution set $\mathcal{Y}$ is obtained by first forming a 3 x 9 matrix with these solution vectors:

$$\begin{bmatrix} 1-\frac{2}{3}z & 0 & 0 & 0 & 0 & 1-z & 1-z & \frac{1}{3}z & \frac{1}{3}z \\ \frac{2}{3}z-\frac{1}{3}z^2 & 1-\frac{1}{3}z^2 & 0 & 1-z^2 & \frac{2}{3}z^2 & z-\frac{1}{3}z^2 & z-z^2 & 1-\frac{1}{3}z-\frac{1}{3}z^2 & \frac{2}{3}z^2-\frac{1}{3}z \\ \frac{1}{3}z^2 & \frac{1}{3}z^2 & 1 & z^2 & 1-\frac{2}{3}z^2 & \frac{1}{3}z^2 & z^2 & \frac{1}{3}z^2 & 1-\frac{2}{3}z^2 \end{bmatrix} \tag{5}$$

where $z = e^{-t/3}$, and reduction to canonical form yields:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{6}$$

The first three column vectors of (6) comprise a solution basis of the $y(i,j;t)$ (for $M = M_1$) in $\mathcal{Y}$. These vectors form a "natural" basis, since each is a column vector of the identity matrix I_3. Also, these basis vectors are linearly independent; in fact, they span the entire solution set $\mathcal{Y}$ for $n = 3$. This particular basis indicates that all solution vectors of dimension three in $\mathcal{Y}$ will form a solution basis of $\mathcal{Y}$.

Similarly, solution bases may be found for other matrices M which yield vectors of the solution set $\mathcal{Y}$. It is apparent that when M has dimension n, a solution basis will consist of vectors of I_n.

Note as $t \rightarrow \infty$, matrix (5) reduces to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{E.C.T.}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which yields the same basis vectors as obtained for $\mathcal{Y}$, $M = M_1$. This result should in general hold true for all M which yield solution vectors of the set $\mathcal{Y}$.

NETWORK APPLICATION OF SOLUTION VECTORS

Application of bounded solution vectors to generate "Millman" networks will now be considered.

First of all, a Millman network is simply a multiple Thevenin generator, and is so named because Millman investigated its output voltage [LePage and Seely (16)] . For instance, in Fig. 1 for n = 3:

$$\overline{E}_0 = \frac{\overline{E}_1\overline{Y}_1 + \overline{E}_2\overline{Y}_2 + \overline{E}_3\overline{Y}_3}{\overline{Y}_1 + \overline{Y}_2 + \overline{Y}_3}$$

where $\overline{E}_k$ and $\overline{Y}_k$ denote the voltage and admittance Laplace transforms, $(k = 1, 2, 3)$, and $\overline{E}_0$ is the Laplace transformed output voltage.

$\overline{E}_0$ can also be expressed as a combination of $\overline{\lambda}_k$ and $\overline{E}_k$,

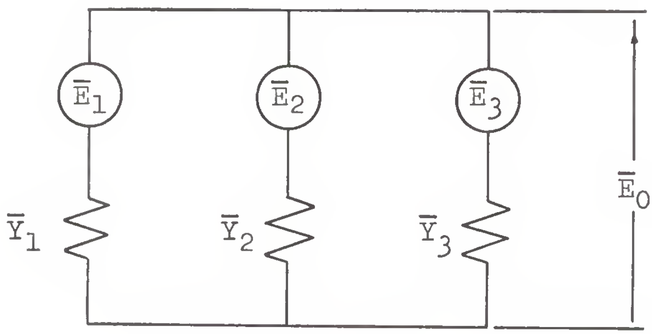

Fig. 1. Millman network, n = 3.

where $$\bar{\lambda}_k = \frac{\bar{Y}_k}{\sum_{k=1}^{n} \bar{Y}_k} \quad \text{and} \quad \sum_{k=1}^{n} \bar{\lambda}_k = 1 \quad ;$$

$$\bar{E}_0 = \bar{\lambda}_1\bar{E}_1 + \bar{\lambda}_2\bar{E}_2 + \dots + \bar{\lambda}_n\bar{E}_n \quad .$$

The question asked is, "How can $\bar{Y}_k$ and $\bar{E}_k$ be chosen such that $\bar{E}_0$ will follow a unit step function input; i.e.,

$$\bar{E}_0 = \frac{1}{s} \cdot \hat{1} \tag{7}$$

where $\hat{1}$ is an approximate identity [Rault (22)] of specified order?"

The branch admittances $\bar{Y}_k$ are chosen such that $\sum_{k=1}^{n} \bar{Y}_k = 1$, which means that $\bar{\lambda}_k = \bar{Y}_k$ for all k. Bounded solution vectors of the solution set $\mathcal{Y}$ may be used to generate branch admittances because their Laplace transforms $\bar{y}$ are such that $u's\bar{y} = 1$. Thus we can choose $s\bar{y}_k = \bar{Y}_k$.

Branch voltages $\bar{E}_k$ may be chosen as functions of Laplace transforms $\bar{e}_k$ such as 1, 1/s, and s to achieve the desired approximate identity in equation (7) for $\bar{E}_0$. An example to illustrate choosing $\bar{E}_k$ is found in Appendix D.

POSITIVE-REAL FUNCTIONS

Branch admittances $\overline{Y}_k$ of Millman networks that are generated by bounded solution vectors of the solution set $\mathcal{Y}$ will be required to be physically realizable as impedors, in which case the branch admittances must be positive-real functions (P.R.F.). [See Brune (5) and Hazony (11) for concise definitions.]

The following definitions will be useful:

Definition 4. A positive-real vector is a vector such that each component is a positive-real function.

Definition 5. A partial-positive-real function (P.P.R.F.) is a function which is positive-real in an infinite subdomain of the half-plane Re $s \geq 0$.

Definition 6. A nonpositive-real function (non-P.R.F.) is a function which is negative or equal to zero when Re $s \geq 0$; i.e., it is the negation of a positive-real function.

Transforms of bounded solution vectors, before or after multiplication by s to form $s\overline{y}$ (which are used to generate branch admittances of Millman networks), satisfy the two necessary conditions for positive-real vectors given by Weinberg; i.e., poles on the imaginary axis, Re $s = 0$, are simple, and there are no right-half-plane poles. Since the two conditions are necessary, but not sufficient, then there will be components of $s\overline{y}$ which are non-P.R.F. or P.P.R.F. These undesirable functions result from at least two conditions present in vectors $s\overline{y}$: multiple zeros on the imaginary axis Re $s = 0$, and right-half-plane zeros. These latter two conditions are seen to be the "duals" of Weinberg's necessary conditions for positive-real vectors to exist. Hence there will exist solution vectors of the solution set $\mathcal{Y}$ which, when $s\bar{y}$ is formed, will contain one or more components which are P.P.R.F. or non-P.R.F., with the rest being P.R.F. These P.P.R.F. or non-P.R.F. cannot be physically realized as impedors of a network. The question arises, "Can the P.P.R.F. or non-P.R.F. be combined by some arithmetic operation with a P.R.F. of the same vector to form a new P.R.F., and hence to get a physically realizable impedor?"

Arithmetic Operations

Lee (15) has shown that the sum of a P.R.F. and a P.P.R.F. may yield another P.R.F. The sum operation may be demonstrated for n = 3 using equation (4), for which it can be shown that

$$s\bar{y}_1(3,1) = \frac{s}{3s + 1} \text{ and } s\bar{y}_3(3,1) = \frac{s + 2}{3s + 2} \text{ are P.R.F., and}$$

$$s\bar{y}_2(3,1) = \frac{3s^2}{(3s + 1)(3s + 2)} \text{ is a P.P.R.F.}$$

The sum of a P.R.F. and a P.P.R.F. gives

$$sy_1(3,1) + sy_2(3,1) = \frac{6s^2 + 2s}{9s^2 + 9s + 2} \equiv A(s) \ .$$

The denominator of the sum A(s) is Hurwitz, and further calculations yield:

$$A(i\omega) = \frac{-6\omega^2 + 2i\omega}{(2 - 9\omega^2) + 9i\omega}$$

$$\text{Re } A(i\omega) = \frac{54\omega^4 + 6\omega^2}{(2 - 9\omega^2) + 81\omega^2} \geq 0 \text{ for all real } \omega.$$

Therefore the sum $\left[s\bar{y}_1(3,1) + s\bar{y}_2(3,1)\right]$ is a P.R.F. In like manner, the sum $\left[s\bar{y}_2(3,1) + s\bar{y}_3(3,1)\right]$ also yields a P.R.F. And in general, for the vectors being considered, the sum of non-P.R.F. or P.P.R.F. with P.R.F. will yield P.R.F.

Next, consider the subtraction operation. Define Z_1 = P.R.F. and Z_2 = P.P.R.F. Now for $Z_1 = s\bar{y}_1(3,1)$ and $Z_2 = s\bar{y}_2(3,1)$, the subtraction $(Z_1 - Z_2)$ is performed.

$$Z_{12} = Z_1 - Z_2 = \frac{2s}{9s^2 + 9s + 2}$$

The denominator of Z_{12} is again Hurwitz, and further

$$Z_{12}(i\omega) = \frac{2i\omega}{(2 - 9\omega^2) + 9i\omega}$$

$$\text{Re } Z_{12}(i\omega) = \frac{18\omega^2}{(2 - 9\omega^2) + 81\omega^2} \geq 0 \text{ for all real } \omega,$$

so Z_{12} is a P.R.F. Likewise, for $Z_1 = s\bar{y}_3(3,1)$, Z_{12} is a P.R.F. Note $Z_{21} = -Z_{12}$ cannot be a P.R.F. In general, then, the subtraction Z_{12} yields P.R.F., whether Z_2 be non-P.R.F. or P.P.R.F.

Multiplication of P.R.F. with non-P.R.F. or P.P.R.F. will not in general yield P.R.F. Also, since 1/P.R.F. = P.R.F., the quotients Z_1/Z_2 or Z_2/Z_1 will not yield P.R.F.

In conclusion, there are at least two operations, addition and subtraction Z_{12}, which can be used to combine P.P.R.F. or non-P.R.F. with P.R.F. to yield new P.R.F. So when P.P.R.F. or non-P.R.F. occur in vectors $s\bar{y}$, they may be readily eliminated by one of these operations. Non-unique elementary matrices may be used to perform these operations; e.g., addition can be performed symbolically to yield

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \text{P.R.F.} \\ \text{P.R.F.} \\ \text{P.P.R.F.} \end{bmatrix} = \begin{bmatrix} \text{P.R.F.} \\ \text{P.R.F.} \end{bmatrix} .$$

In terms of Millman networks, this means that some of the networks generated by solution vectors will have fewer branches, but each branch admittance will be physically realizable. Clearly, these results may be extended for any value of n.

The addition and subtraction operations could be eliminated completely if a matrix M could be found (for a particular value of n, of course) such that resulting solution vectors yielded only positive-real components. Such matrices can be shown to exist.

The general case $n = 3$ is considered. First, $s\bar{y}(3,1)$ is studied, because of the $s\bar{y}(i,j)$, it singularly contained a partial-positive-real component. The matrix employed here is

$$M_2 = \begin{bmatrix} a & 0 & 0 \\ c & b & 0 \\ d & e & 1 \end{bmatrix}$$

where $a + c + d = 1$, $b + e = 1$, and $a, b > 0$ so that rank

$M_2 = n = 3.$

Using equation (2) gives

$$\bar{y}(3,1) = \begin{bmatrix} \dfrac{a}{s + a} \\ \dfrac{cs}{(s + a)(s + b)} \\ \dfrac{ds^2 + s + a}{s(s + a)(s + 1)} - \dfrac{ces}{(s + a)(s + b)(s + 1)} \end{bmatrix} .$$

In determining the conditions such that $s\bar{y}(3,1)$ will be positive-real, it is noted first that the denominator of each component is Hurwitz. Further calculations are as follows:

$s\bar{y}_1(3,1)$ is positive-real since $a > 0$

$$s\bar{y}_2(3,1) = \frac{cs^2}{s^2 + (a + b)s + ab} \equiv G_2(s)$$

$$G_2(i\omega) = \frac{-c\omega^2}{(ab - \omega^2) + i(a + b)\omega}$$

$$\text{Re } G_2(i\omega) = \frac{c\omega^4 - abc\omega^2}{(ab - \omega^2)^2 + (a + b)^2\omega^2} .$$

Thus $s\bar{y}_2(3,1)$ can be positive-real only for $c = 0$, since rank $M_2 = 3$. Using $c = 0$, it can be shown that $s\bar{y}_3(3,1)$ is positive-real when $0 \leq d \leq \frac{1}{a}$. Summarizing, the conditions such that $s\bar{y}(3,1)$ has positive-real components are:

$$a = 1 - d \tag{8}$$

$$b = 1 - e \tag{9}$$

$$c = 0 \tag{10}$$

and
$$0 \le d \le \frac{1}{a} \quad . \tag{11}$$

Conditions (8) to (10) can be employed further. The following calculations include an allowance for all possible vectors x(t) and y(0); i.e., x(t) and y(0) are column vectors of M_2 or linear combinations of them. Let

$$x(t) = \left[f \ g \ h\right]' , \quad y(0) = \left[m \ n \ p\right]'$$

where
$$f + g + h = 1 \quad \text{and} \quad m + n + p = 1 \ . \tag{12}$$

Using (2) and substituting in c = 0, d = 1 - a, and e = 1 - b, gives

$$\bar{y} = \bar{y}_g \equiv \begin{bmatrix} \dfrac{ms+f}{s(s+a)} \\ \dfrac{ns^2 + (g+an)s + ag}{s(s+a)(s+b)} \\ \dfrac{ps^2 + \left[h + b(p+n) - n\right] s + b(g+h) - g}{s(s+b)(s+1)} + \dfrac{(ms+f)(a-1)}{s(s+a)(s+1)} \end{bmatrix} .$$

Note that the denominator polynomials of each component are Hurwitz. Remaining conditions for $s\bar{y}_g$ to have all components positive-real are as follows:

$$s\bar{y}_{g1} = \frac{ms + f}{s + a} \equiv H_1(s)$$

$$H_1(i\omega) = \frac{f + mi\omega}{a + i\omega}$$

$$\text{Re } H_1(i\omega) = \frac{af + m\omega^2}{a^2 + \omega^2} \quad .$$

Component $s\bar{y}_{g1}$ will be positive-real only when

$$m \geq 0 \quad \text{and} \quad f \geq 0 \; . \tag{13}$$

The second component of $s\bar{y}_g$ is

$$s\bar{y}_{g2} = \frac{ns^2 + (g + an)s + ag}{s^2 + (a + b)s + ab} \equiv H_2(s)$$

$$H_2(i\omega) = \frac{(ag - n\omega^2) + i\omega(g + an)}{(ab - \omega^2) + i\omega(a + b)}$$

$$\text{Re } H_2(i\omega) = \frac{n\omega^4 + (a^2n + bg)\omega^2 + a^2bg}{(ab - \omega^2)^2 + \omega^2(a + b)^2} \; .$$

The conditions for $s\bar{y}_{g2}$ to be positive-real are:

$$n \geq 0 \tag{14}$$

$$a^2bg \geq 0, \text{ which implies } g \geq 0 \tag{15}$$

and

$$a^2n + bg \geq 0 \; . \tag{16}$$

The third component may be written as the sum of two fractions:

$$s\bar{y}_{g3} = \frac{(ms+f)(a-1)}{(s+a)(s+1)} + \frac{ps^2 + \left[h + b(p+n) - n\right]s + b(p+h) - g}{(s+b)(s+1)}$$

$$\equiv H_{31}(s) + H_{32}(s) \; .$$

The sum of two P.R.F. is again a P.R.F. [Lee (15)], so that $s\bar{y}_{g3}$ may be broken into two parts, and conditions may be developed for each part to be positive-real. For the first part of $s\bar{y}_{g3}$,

$$H_{31}(i\omega) = \frac{(a - 1)f + im(a - 1)\omega}{(a - \omega^2) + i\omega(a - 1)}$$

$$\text{Re } H_{31}(i\omega) = \frac{(a - 1)\left[\left[m(a - 1) - f\right]\omega^2 + af\right]}{(a - \omega^2)^2 + \omega^2(a - 1)^2} .$$

The conditions will then be

$$m(a - 1) - f \leq 0 \quad \text{since } a - 1 \leq 0$$

but $\quad (a - 1)af \leq 0$ implies $f \geq 0$.

Therefore $H_{31}(s)$ will be positive-real when

$$m(1 - a) \geq f \geq 0 . \tag{17}$$

For the second part of $s\bar{y}_{g3}$, one obtains

$$\overline{H}_{32}(i\omega) = \frac{b(p + h) - g - p\omega^2 + i\omega[h + b(p + n) - n]}{(b - \omega^2) + i\omega(b + 1)}$$

$$\text{Re } H_{32}(i\omega) = \frac{p\omega^4 + [g - bp + b^2(p+n) + h - n]\omega^2 + b^2(p+h) - bg}{(b - \omega^2)^2 + \omega^2(b + 1)^2} .$$

The resulting conditions are:

$$p \geq 0 \tag{18}$$

$$b(p + h) \geq g \tag{19}$$

and

$$b^2p + b^2n + h + g \geq n + bp . \tag{20}$$

Since it is possible that $[f\ g\ h]' = [0\ \ 0\ \ 1]'$, then apparently

$$h \geq 0 \tag{21}$$

which satisfies (12), (14), and (20).

Compiling conditions (8) through (21) results in the facts that M_2 must be a nonnegative matrix, and that the vectors $x(t)$

and y(0) must be nonnegative in order that all components of $s\bar{y}$ are positive-real. Therefore the range of values for elements of M, x(t), and y(0) must be between 0 and 1, since $u'x(t) = 1$ and $u'y(0) = 1$.

An example of a matrix which satisfies these criteria is:

$$M_3 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 1/2 & 1/2 & 1 \end{bmatrix} .$$

The matrix M_n, as given in Appendix B, is an example for which each component of $s\bar{y}$ is a P.R.F.

SUMMARY

A method of generating impedors of Millman networks using solutions of the vector differential equation $\dot{y} + My = x$ was presented in this paper. The impedors were required to be physically realizable, and an investigation of solution vectors was made to insure this requirement was fulfilled.

It was stated that M was a column-sum-one matrix with real, constant elements and that vectors x(t) and y(0) would be column vectors of M or their linear combinations. This implied that solutions of the differential equation are column-sum-one vectors for all time.

Bounded solution vectors were considered. The matrix M was chosen to be triangular. To guarantee bounded solution vectors, it was necessary to require that there be no negative diagonal elements with at most one zero diagonal element. The latter condition was a sufficient one. Elements of bounded solution vectors were found to be functions of exponential order and of time. A solution basis for the solution set $\mathcal{Y}$ was determined to be all solution vectors of that size in $\mathcal{Y}$.

Branch voltages were chosen as functions of Laplace transforms 1, s, 1/s as demonstrated in Appendix D, so that the Millman network output voltage would follow a unit step function input to a desired degree.

Branch admittances $\overline{Y}_k$ were chosen so that the sum of all the $\overline{Y}_k$ was one. With this fact, bounded solution vectors were used to generate branch admittances because their Laplace transforms were such that $u's\overline{y} = 1$. To physically realize branch admittances, it was necessary that vectors $s\overline{y}$ have all their components positive-real.

Transforms of bounded solution vectors, before or after multiplication by s to form $s\overline{y}$, satisfied the two necessary conditions for positive-real vectors given by Weinberg. Since these conditions were not sufficient ones, the existence of partial-positive-real or nonpositive-real functions in components of vectors $s\overline{y}$ was explained.

Since P.P.R.F. or non-P.R.F. cannot be realized physically as impedors, two alternatives were considered (specifically for the case $n = 3$). First the P.P.R.F. or non-P.R.F. could be added with P.R.F. of that vector, or the P.P.R.F. or non-P.R.F. could be subtracted from P.R.F. In each case, a new P.R.F. resulted, which could be physically realized.

The second alternative was to find conditions for the matrix M such that only positive-real vectors $s\bar{y}$ resulted. Such a matrix was found for rank of M equal to 3, and it had non-negative elements, as did the vectors x(t) and y(0).

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APPENDICES

APPENDIX A

Column-sum-one Matrices

Our purpose here will be to consider methods of generating column-sum-one matrices, and some of their properties.

Any n x n matrix N with real elements may be converted to a column-sum-one matrix by summing the elements of each column, dividing each element by that column sum, and forming new columns for each of the n columns. For example, if

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}, \qquad M_1 = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} .$$

Some other examples are $M = I_n, \frac{1}{n} uu'$ for $n = 1, 2, \ldots$.

Definition A-1. $e_i = \left[0\ 0\ ..\ 0\ 1\ 0\ \ldots\ 0\right]'$, where element 1 occurs i places from the left end of the vector.

An interesting property of M_1 is that

$$M_1{}^k = \frac{1}{3^k}\begin{bmatrix} 1 & 0 & 0 \\ 2^k - 1 & 2^k & 0 \\ 3^k - 2^k & 3^k - 2^k & 3^k \end{bmatrix}, \text{ where } k > 0$$

and

$$\lim_{k \to \infty} M_1{}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = e_3 u' .$$

This may be stated as a general result.

Lemma A-1. If M is a triangular n x n matrix with non-negative real elements such that $u'M = u'$, then

$$\lim_{k \to \infty} M^k = e_i u' .$$

Proof. Theorem 1, pages 392-3 of Kemeny (13) is used when M and $e_i u'$ are transposed to have row sums equal to one.

Diagonal Composition

More elaborate methods of generating column-sum-one matrices employ diagonal summing and "diagonal composition" of matrices.

Given two n x n matrices R and S with real coefficients such that $u'R = 0' = u'S$, form the diagonal sum $R \oplus S$:

$$R \oplus S = \begin{bmatrix} R & 0 \\ 0 & S \end{bmatrix} .$$

$R \oplus S$ can be further non-uniquely partitioned into

$$R \oplus S = \begin{bmatrix} r_{11} & r_{12} & 0 & 0 \\ r_{21} & r_{22} & 0 & 0 \\ 0 & 0 & s_{11} & s_{12} \\ 0 & 0 & s_{21} & s_{22} \end{bmatrix} .$$

Definition A-2. "Diagonal composition" of R and S is the result of diagonal squeezing of $R \oplus S$ to obtain

$$R \boxplus S = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} + s_{11} & s_{12} \\ 0 & s_{21} & s_{22} \end{bmatrix}$$

provided the addition of r_{22} and s_{11} is conformable.

<u>Lemma</u> <u>A-2</u>. If $u'R = 0' = u'S$, then $R \boxplus S$ is a column-sum-zero matrix of order one lower than $R \oplus S$.

If A and B are column-sum-one matrices, then non-unique column-sum-one matrices can be constructed from

$$I_m \pm \left[(I_n - A) \boxplus (I_n - B)\right] .$$

Of course, other column-sum-one matrices can be used in place of identity matrices.

APPENDIX B

A Particular Solution of the Differential Equation $\dot{y} + My = x$

An interesting solution of equation (1) is considered here. The matrix M used in the equation is

$$M = M_n = \frac{1}{n} uu' \qquad n = 1, 2, \ldots ,$$

and

$$M_{n+1} = \begin{bmatrix} \dfrac{uu'}{n+1} & \dfrac{u}{n+1} \\ \dfrac{u'}{n+1} & \dfrac{1}{n+1} \end{bmatrix} .$$

Since M_n has the form of a dyadic $\xi\eta'$, its rank will be one for all n, n = 1, 2, ... M_n is also an example of a doubly-stochastic matrix.

It is convenient to use M_{n+1} in calculation of the solution from equation (1).

The inverse $(M_n + sI_n)^{-1}$ is obtained by right-augmentation of $(M_{n+1} + sI_{n+1})$ with the identity matrix I_{n+1} and use of elementary row transformations until the identity matrix appears on the left side of the augmented form. The matrix on the right is then the inverse, which is shown in the following:

$$\frac{1}{s(n+1)(s+1)}\left[\begin{array}{cc|cc} I_n & 0 & (n+1)(s+1)I_n - uu' & -u \\ 0' & 1 & -u' & n + s(n+1) \end{array}\right].$$

Then the inverse is

$$(M_n + sI_n)^{-1} = \frac{1}{ns(s+1)}\begin{bmatrix} n(s+1)I_{n-1} - vv' & -v \\ -v' & ns + n - 1 \end{bmatrix}$$

where $v = [1\ 1\ .....\ 1]'$ has dimension $n - 1$. The method used by Edelblute (7) will yield the same result for the inverse.

When $x(t)$ and $y(0)$ are column vectors of M_n, or linear combinations of them, then $x(t) \equiv y(0)$. Applying equation (2) yields:

$$\bar{y} = \frac{1}{n^2s^2}\begin{bmatrix} n(s + 1)I_{n-1} - vv' & -v \\ -v' & ns + n - 1 \end{bmatrix}\begin{bmatrix} v \\ 1 \end{bmatrix}$$

$$\bar{y} = \frac{u}{ns}.$$

Then $y(t) = \frac{1}{n}u$, and $x(t) \equiv y(0) \equiv y(t)$.

APPENDIX C

Bounded Solution Vectors by Premultiplication

Previously, the equation $\dot{y} + My = x$ was constructed from the triangular matrix M and vectors x(t) and y(0) were column vectors of M or their linear combinations. Bounded solution vectors were guaranteed when diagonal elements were nonnegative, and the rank of M was n or (n - 1).

Suppose M has nonnegative diagonal elements. Consider a procedure which transforms unbounded solution vectors resulting when the rank of M is less than or equal to (n - 2) into bounded solution vectors.

Unbounded terms resulting when the rank of M is less than or equal to (n - 2) are of the form t^m, m = 1, 2, ... Consider premultiplying solution vectors y(t) containing these unbounded terms by e^{-t}. An equivalent operation would be to replace s by s + 1 in Laplace transformed vectors $\overline{y}$. Then unbounded terms of the form t^m will be "neutralized". Observe, however, that no longer will u'y(t) = 1, but u'y(t) = e^{-t} when the "premultiplication" operation is applied to unbounded solution vectors.

An example illustrates results of the premultiplication operation. Consider M_4, with rank (n - 2).

$$M_4 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 1 & 1 \end{bmatrix} , \quad x(t) = y(0) = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}' .$$

Then $\bar{y} = \begin{bmatrix} 0 \\ \frac{2}{3s} + \frac{2}{3s^2} \\ \frac{1}{3s} - \frac{2}{3s^2} \end{bmatrix}$ and $y(t) = \begin{bmatrix} 0 \\ \frac{2}{3} + \frac{2}{3} t \\ \frac{1}{3} - \frac{2}{3} t \end{bmatrix}$.

If now the premultiplication operation is performed on $\bar{y}$, s is replaced by s + 1 to obtain:

$$\bar{y} = \begin{bmatrix} 0 \\ \frac{2}{s(s+1)} + \frac{2}{3(s+1)^2} \\ \frac{1}{3(s+1)} - \frac{2}{3(s+1)^2} \end{bmatrix} \text{ and } y(t) = \begin{bmatrix} 0 \\ \frac{2}{3} e^{-t} + \frac{2}{3} te^{-t} \\ \frac{1}{3} e^{-t} - \frac{2}{3} te^{-t} \end{bmatrix} .$$

Note how the terms $\frac{2}{3}$ t, $-\frac{2}{3}$ t are "neutralized" to form bounded terms: $\frac{2}{3} te^{-t}$, $-\frac{2}{3} te^{-t}$.

APPENDIX D

Selection of Millman Network Branch Voltages

Branch voltages of a Millman network of three branches will be chosen to satisfy equation (7); i.e., the output voltage $\bar{E}_0 = \frac{1}{s} \cdot \hat{1}$, where in this case $\hat{1}$ is an approximate identity of zeroth order.

Let branch admittances be chosen from the Laplace transform of the solution vector y(1,1;t); i.e., $s\bar{y}_k(1,1) = \bar{Y}_k$, $k = 1, 2, 3$, and

$$s\bar{y}(1,1) = \left[\frac{s+1}{3s+1} \quad \frac{3s(s+1)}{(3s+1)(3s+2)} \quad \frac{s}{3s+2} \right]' .$$

Branch voltages are chosen using the $\bar{e}_k$: 1, s, 1/s in the formula $\bar{E}_k = \frac{1}{s} \cdot \frac{\bar{e}_k}{\sum_{k=1}^{3} \bar{e}_k}$, which yields:

$$\bar{E}_1 = \frac{1}{s(1 + s + s^2)}, \quad \bar{E}_2 = \frac{1}{1 + s + s^2}, \quad \text{and } \bar{E}_3 = \frac{s}{1 + s + s^2} .$$

Then the output voltage is:

$$\begin{aligned} \bar{E}_0 &= \bar{E}_1\bar{Y}_1 + \bar{E}_2\bar{Y}_2 + \bar{E}_3\bar{Y}_3 \\ &= \frac{1}{s} \cdot \frac{2 + 5s + 6s^2 + 4s^3 + 3s^4}{2 + 11s + 20s^2 + 18s^3 + 9s^4} . \end{aligned}$$

Then $\bar{E}_0$ follows a unit step function input to a zeroth order approximate identity.

A METHOD OF GENERATING n-BRANCH MILLMAN NETWORKS

by

RICHARD D. TEICHGRAEBER

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A method for generating impedors of Millman networks, using solutions of the vector differential equation $\dot{y} + My = x$ is presented in this thesis. The impedors are required to be physically realizable, and hence positive-real functions. A study of solution vectors is made to insure that this requirement is fulfilled.

Previous work on solution of first-order vector differential equations is reviewed. Literature on column-sum-one matrices as it occurs in studies of stochastic and doubly-stochastic matrices is also reviewed. A theorem on positive-real matrices given by Weinberg and Slepian is also presented.

Only bounded solution vectors are considered. The matrix M is chosen triangular, and nonnegative diagonal elements of M (at most one zero element) guarantee bounded solution vectors.

Millman networks are required to have an output voltage that follows a unit step function input, which is satisfied by choosing suitable functions of time for branch voltages and by using bounded solution vectors to generate branch admittances, and hence branch impedors.

Weinberg's theorem, when applied to vectors indicates that not all solution vectors generating branch admittances are positive-real. Either use of arithmetic operations to combine non-positive-real or partial-positive-real functions to obtain positive-real components, or determination of conditions on the matrix M for solution vectors to possess positive-real components is necessary to guarantee physical realizability of impedors.