A NONLINEAR CONTROLLER FOR UNDERDAMPED SYSTEMS

by

JOSEPH C. WEBB

B. S., West Virginia University, 1959

A THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Chemical Engineering

KANSAS STATE UNIVERSITY
OF AGRICULTURE AND APPLIED SCIENCE

1962

TABLE OF CONTENTS

PURPOSE

The purpose in carrying out this work was to develop a nonlinear controller suitable for control of underdamped processes and to determine its stability by simulation on an analogue computer.

INTRODUCTION AND THEORETICAL BACKGROUND

Control Theory

In general, physical systems are passive in that they do not contain an active source of internal energy. Passive systems are inherently stable because there exists no reason for self-excitation due to an unbounded build-up or decay of energy within the system (1). A stable system does not necessarily imply that the value or quantity of the system output variable is desirable, or even tolerable, but only that the system output variable is a bounded function of a bounded input (2), (3). In most physical systems it is necessary not only to maintain the output variable as a bounded function of a bounded input but to maintain its value within narrow predetermined limits. In order to accomplish this the energy or material flow into an earlier stage of the system can be controlled or varied by information emerging from an advanced stage of the system. Systems in which the information from an advanced stage is returned or fed back into an earlier stage are called closed loop or feedback control systems (4). Feedback control systems may become unstable for certain system disturbances due to the propagation of these disturbances around the closed loop (5). Even at the risk of encountering instability the advantages of feedback control systems are so numerous that they are used in almost every physical process.

Feedback control systems usually consist of a sequential or parallel array of basic component interconnected to form one or more closed loops. The five basic components usually found in a feedback control system are (6): 1. the process, 2. the measuring means, 3. the error detecting mechanism, 4. the controller, 5. the final control element. From the standpoint of control, each of these components may be considered as information processing units; and the physical energy and material flows may be represented by functional analogue signals. As each signal passes through one of the information processing units, it may be transformed in magnitude, phase angle, or by some nonlinear relationship. The output signal from each element may be described by a differential equation which is a function of the input signal, time, and the element itself. By making a few simplifying assumptions concerning the nonlinearities, this equation can generally be solved for the LaPlace transform of the output signal divided by the LaPlace transform of the input signal. The ratio of these transforms, when it exists, is defined as the system transfer function (7). Hence, each system element may be represented by a box or block described by a transfer function. By connecting an array of these blocks by functional signal lines, a block diagram or signal flow diagram representing an actual physical system may be obtained.

Figure 1 is a simplified block diagram of a feedback control system with a single feedback control loop. The process output variable is measured and transformed by the measuring means into a signal representative of the condition it is desired to control. This signal is transmitted to the error detecting device and compared to the reference input or set point. If initially the difference or the error between the feedback variable and

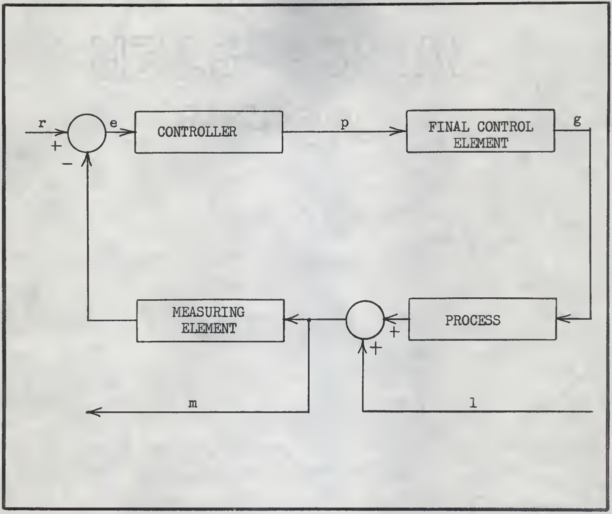


Figure 1. A simplified signal flow or block diagram of a feedback control system.

the set point is zero, there will not be a change in the signal transmitted to the controller; and, therefore, no control action will be taken. Now if the process is disturbed by a change in the process load variable, the difference between the feedback variable and the set point will no longer be equal to zero; and an error signal proportional to this difference will be transmitted to the controller. The controller will then transform the error signal into a control signal capable of actuating the final control element. The control signal may be either a linear or nonlinear function of the error. The final control element upon receiving the signal from the controller changes the energy or material flow into the process, altering the value of the process output variable. If all of the control functions and system parameters have been properly selected, the output variable will return back within acceptable limits in a reasonable length of time. If the system has been designed improperly, the resulting control action could initiate instability; or poor control could result. Since the process parameters are generally fixed, or at best adjustable within narrow limits, the controller functions or control modes along with their proper adjustment have received considerable attention.

Underdamped Systems

A system with a considerable amount of inertia in relation to the system restraining forces is described as underdamped (8), (9). When an underdamped system is disturbed, it tends to result in oscillations which die out gradually with time. If an underdamped system is included in a feedback control loop, these oscillations may be propagated around the loop in such a manner that each oscillation results in a new system disturbance.

Thus a closed loop system containing an underdamped element may be very sensitive to upsets. It is the function of the controller to generate an error compensating signal capable of damping out these oscillations as rapidly as possible with minimum area under the error-time curve (6).

If the controller gain is set at a very low value, the maximum magnitude of each oscillation will be greatly reduced each time around the loop; but the system will be sluggish, and the error may persist for a long period of time. As the controller gain is increased, the system response rate will increase; but the maximum magnitude of each oscillation around the loop will not be reduced as greatly as with the lower controller gain. As the controller gain is increased, a point will be reached where the maximum magnitude of each oscillation around the loop is reduced by about 75%. This system usually exhibits satisfactory control, and the area under the error-time curve is near a minimum (2). When the controller gain has been sufficiently increased, the system will begin to oscillate with constant magnitude and period. This condition is usually described as marginal stability, and the loop gain is at its ultimate or critical value (2). The frequency of the oscillation is called the ultimate frequency of the system (10). With only linear elements in the control loop the ultimate frequency is approximately equal to the undamped or the natural frequency of the system (11). This condition is usually considered undesirable from the control viewpoint because any further increase in gain will cause instability. When the system reaches the point of instability, the maximum magnitude of the oscillations will increase each time around the loop. Once instability has been initiated, the system output variable is completely independent of the system input variable (8). To restore stability the feedback loop must be disconnected or the loop gain sufficiently reduced.

Conventional Controllers and Controller Theory

When a difference or error between the system output variable and the reference input is detected, it is the function of the controller to generate a signal to reduce the error back within limits as rapidly as possible. The method the controller uses to generate this signal is called the control action or control mode. Modern industrial controllers usually consist of the sum of two or more control modes (12)

The most common linear process control mode is the proportional as described by the equation:

$$p(t) = Ae \tag{1}$$

where $p(t)$ is the controller output, A is a gain constant, and e is the system error. This equation should contain a constant term associated with the steady state or zero error output. Since the inclusion of this constant term does nothing to improve the clarity of the equation, it will be omitted in all controller equations in this thesis.

Equation (1) shows that the proportional control mode is simply the error multiplied by the constant A. Since the controller output is zero except when an error exists, the proportional control mode will allow a steady state error or offset to exist for sustained load changes. The popularity of the proportional mode is due to its inherent stabilizing effect resulting from the controller output's always being in phase with the error (13). Thus, the controller output is always in a direction tending to reduce the error, providing there are no time delays or phase lags within the control loop. However, in most physical systems time delays and phase lags exists; and the proportional control mode tends to increase the error for certain error signals. In order to offset this condition it is sometimes necessary, or desirable, to add phase angle shifting elements to the control loop.

If the error signal to the controller is regarded as a sine wave, there are two critical points per cycle where the controller output should be zero because there is a chance that both the error and its first time derivative will return to zero without any further control action. These critical points are approximately 135^{o} and 315^{o}, depending upon the characteristics of the system parameters (14). In industrial controllers the derivative mode is usually added to the proportional mode to reduce the error to zero at the critical points.

The equation describing the derivative mode is:

$$p(t) = AD(de/dt) \tag{2}$$

where D is a gain constant, or in transform notation:

$$\frac{P(s)}{E(s)} = ADs \tag{3}$$

where s is the LaPlacian dummy variable. The LaPlace transform of a variable will be represented by a capital letter with the dummy variable s in parentheses. Zero initial conditions will be assumed throughout this work.

Equations (2) and (3) show that the derivative control mode leads the error signal by 90^{o}. If the derivative control mode is combined with the proportional controller response the transfer function is given by the equation:

$$\frac{P(s)}{E(s)} = A + ADs \tag{4}$$

or $$\frac{P(s)}{E(s)} = A(1 + Ds) \tag{5}$$

If the error signal to the controller is:

$$e(t) = \sin \omega t \tag{6}$$

where ω is the angular frequency, then the controller output of Equation (5) is given by:

$$p(t) = \text{Im } A(1 + jD\omega)\exp j\omega t \tag{7}$$

where Im signifies that only the imaginary part of the final solution will be considered, and $j = (-1)^{\frac{1}{2}}$. Solving Equation (7) for its real part results in:

$$p(t) = A(1 + D^2\omega^2)^{\frac{1}{2}}\sin(\omega t + \arctan D\omega) \tag{8}$$

Equation (8) shows that the proportional plus derivative controller has a gain factor A which is modified by $(1 + D^2\omega^2)^{\frac{1}{2}}$. Since the angular frequency ω is included in the gain term of Equation (8), the derivative response adds a dynamic gain element to the control loop as opposed to the zero frequency gain supplied by the proportional mode alone (11). The phase angle associated with the proportional plus derivative mode is given by $\arctan(D\omega)$. In order to reduce the controller output to zero at the critical points the error frequency must be determined and the gain term D properly selected.

The integral mode is generally added to the controller for the purpose of reducing the steady state error or offset. The equation describing the integral mode is:

$$p(t) = AI\int e\,dt \tag{9}$$

where I is a gain constant. Equation (9) shows that the integral mode lags the error phase angle by 90^{o}. If the integral mode is combined with the proportional control mode the following transfer function is obtained:

$$\frac{P(s)}{E(s)} = A + AI/s \tag{10}$$

or $$\frac{P(s)}{E(s)} = A(1 + I/s) \tag{11}$$

If the error is again considered to be a sine wave, then Equation (11) becomes:

$$p(t) = \text{Im}\, A(1 + I/j\omega)\exp j\omega t \tag{12}$$

Solving Equation (12) results in the following:

$$p(t) = A\left[1 + (I/\omega)^2\right]^{\frac{1}{2}} \sin(\omega t - \arctan I/\omega) \tag{13}$$

The proportional gain constant A is now modified by $\left[1 + (I/\omega)^2\right]^{\frac{1}{2}}$, and becomes infinite when the error frequency approaches zero. Thus the integral control mode will not tolerate steady state errors since its zero frequency gain is infinite (5). The phase angle associated with the proportional plus integral controller lags the error phase angle by the term arctan (I/ω). Since it is undesirable to add any components to the control loop that will increase the phase angle lag, the integral mode should be employed only where offset cannot be tolerated (15).

For many physical processes the system parameters are related in such a manner that good control and stability are incompatible with only linear phase shifting elements in the control loop. For this and other reasons the introduction of nonlinear elements into the feedback control loop has been proposed (16), (17).

Nonlinear Control Systems

A feedback control system should be designed to hold the system error to as small a value as possible following a system load or setpoint change. If the absolute value of the error is large or departing from zero, a high loop gain or a small damping factor is desirable to increase the system response rate to controller corrections. Conversely, if the absolute value of the error is small or approaching zero, a low loop gain or a large damping factor is desirable to slow down the system response rate and prevent excessive oscillations. Since the system damping factor can alter the system response rate in essentially the same manner as the controller or the loop gain, the use of nonlinear damping which has an inverse relationship to the error or error rate has been proposed (17), (18), (19). Since the damping factor is an integral part of the system, this requires that the system transfer function be altered or varied during operation. This is usually feasible only in mechanical or electrical systems which fall under the classification of servomechanisms, and rarely, if ever, applicable to chemical processes or regulator control systems.

Nixon (3) has suggested that the product of the absolute value of the error and the error be added to the proportional mode response to increase the controller gain for large errors. This controller response is described by the equation:

$$p(t) = Ae + A_1|e|e \tag{14}$$

where A_1 is a gain constant. Since the sign of the nonlinear term is always the same as the sign of the proportional mode, the nonlinear term has the appearance of being in phase with the error but with a distorted wave form. For nonlinear systems the phase angle in not easily or accurately defined. The usual procedure is to excite the system with a sinusoidal forcing function and determine the Fourier series of the output wave form (19), (20). The phase angle associated with the system is given by the phase angle contributions of the first harmonics of the Fourier series. Applying this definition to the nonlinear term in Equation (14) results in the following wave form:

$$p(t) = A_1 \left|\sin\omega t\right| \sin\omega t \tag{15}$$

after substituting (-t) for (t) in Equation (15), the result is given by the following:

$$p(-t) = A_1 \left|\sin(-\omega t)\right| \sin(-\omega t) \tag{16}$$

or $$p(-t) = -A_1 \left|\sin\omega t\right| \sin\omega t \tag{17}$$

therefore:

$$p(t) = -p(-t) \tag{18}$$

and p(t) is an uneven function, and the Fourier series will contain only sine wave harmonics in its expansion (21). The Fourier series for Equation (14) is given by:

$$p(t) = A \sin\omega t + \sum_{n=1}^{n=\infty} a_n \sin n\omega t \tag{19}$$

where $n = 1, 2, 3, ---, n, --- \infty$, and a_n is the coefficient of the n th term of the series. An examination of Equation (19) shows that the first harmonics are:

$$p(\omega t) = A \sin\omega t + a_1 \sin\omega t \tag{20}$$

and the phase angle contribution is zero since the sum of two sine waves of the same frequency is another sine with that frequency. The wave form will be distorted by the harmonics so that the magnitude of the nonlinear term increases rapidly for large errors and contributes very little when the error is small. In order to be very effective on systems with phase angle lags or time delays, a phase angle lead component such as the derivative mode should be included in Equation (14).

A similar, but superior, controller mode is obtained by subtracting the third harmonic of the error from the first (22). If the error is regarded as a sine wave, then the controller output would be described by the equation:

$$p(t) = A_2 \sin\omega t - A_3 \sin 3\omega t \tag{21}$$

where A_2 and A_3 are gain constants. If A_2 and A_3 are chosen as follows:

$$A_2 = 3A_4/4 \tag{22a}$$

and $$A_3 = A_4/4 \tag{22b}$$

the controller output of Equation (21) is then:

$$p(t) = (3\,A_4/4)\,\sin\omega t \;-\; (A_4/4)\,\sin 3\omega t \tag{23}$$

or $$p(t) = A_4 \sin^3 \omega t \tag{24}$$

and $$p(t) = A_4 e^3 \tag{25}$$

which is a cubic response.

In consideration of the first harmonic of the controller response given in Equation (21), the cubic response does not supply a leading phase angle to the system. The cubic response does have the advantage of increasing the loop gain for large errors and decreasing the gain when the error is small. This can be seen by an examination of Equation (23) which shows that the sign of the third harmonic will be the same as the sign of the first harmonic when the error is between 60° to 120° and 240° to 300°. Therefore, the magnitude of the first harmonic will be increased by the value of the third harmonic term over these ranges where the error is largest. At all other points during one cycle the value of the third harmonic will be subtracted from the magnitude of the first harmonic, and the controller will have a low gain value for small errors.

In order to reduce the controller gain to zero at the critical points, the gain of the third harmonic term in Equation (21) could be increased above that required for a true cubic response. This would also decrease the response for small errors departing from zero and could cause the controller error correction to be in the wrong direction over the first portion of the error cycle. Since this could easily lead to instability, the derivative control mode could be added to the cubic response if a leading phase angle component is desired.

Bates (14) has proposed setting the loop gain at some suitable low value and increasing it to the limit whenever the error or the error rate reaches a maximum. Mathematically this could be accomplished by the addition of two nonlinear control modes to the controller. These are the error divided by the absolute value of its first time derivative and the first time derivative divided by the absolute value of the error or in equation form:

$$p(t) = A_5 \frac{e}{|de/dt|} + A_6 \frac{de/dt}{|e|} \qquad (26)$$

where A_5 and A_6 are gain constants. These terms would require low values for the gain coefficients since division by signals close to zero could result in extremely large gains which would initiate instability. In a practical system limiters would be required, or a somewhat different controller response could be selected.

Gibson (23) has suggested the use of a dual controller to provide high loop gain for large errors and a small loop gain for small errors. This system would require a relay switching device to select the controller which would be in operation. Each controller would be equipped with its own separate gain and phase angle shifting components. When the error is small the high gain controller would be switched out of the loop, and the system would be controlled by the low gain controller. When the system error exceeds some predetermined limit, the low gain controller would be switched out of the control loop; and the high gain controller would assume system control. This type controller could be very effective but would be complex and costly since two separate controllers along the necessary error sensing and switching mechanism would be required.

A somewhat more sophisticated extension of the dual mode controller is the programmed controller or the optimum switching relay servomechanism (17), (22), (23). This type controller is based on the theory that for every system error an optimum controller response can be determined. Theoretically this type controller could have any desired response for any linear or nonlinear function of the error. However, in practical systems the controller response is usually determined by some function of the error and its first time derivative (22).

An equation of the form:

$$y(t) \quad = \quad f(e,\ de/dt) \tag{27}$$

is determined by consideration of the phase plane plot of the error versus the error rate. Equation (27) when represented on the phase plane plot is called the optimum switching line (17). The controller response will then be determined by the value of y(t) in Equation (27). From a practical viewpoint the applications of the optimum switching relay servomechanism are somewhat limited due to the cost and complexity of the equipment required to optimize a given system.

Consideration of these ideas has led to the development and investigation of another nonlinear controller which is believed to have many desirable characteristics.

AN ERROR MAGNITUDE-RATE CONTROLLER

Controller Characteristics

The nonlinear control mode developed and analyzed for this thesis* is described by the equation:

$$p(t) = AB\,|e|\,de/dt \tag{28}$$

where A and B are gain constants. Since the controller output in Equation (28) is proportional to the product of the absolute value of the error magnitude and the error rate, it has been called the "magnitude-rate mode" in this thesis as an aid in its discussion.

Since the output wave form of the magnitude-rate mode is a nonlinear function of the error, its phase angle will be given by the phase angle contributions of the first harmonics of its Fourier expansion with a sinusoidal error forcing. Equation (28), with a sinusoidal error is given by the following:

$$p(t) = AB\omega\,|\sin\omega t|\cos\omega t \tag{29}$$

The Fourier expansion of Equation (29) has been derived in the Appendix and is described by the infinite series:

$$p(t) = -\frac{4AB\omega}{\pi}\sum_{n=1}^{n=\infty}\frac{1}{(2n - 3)(2n + 1)}\cos(2n - 1)\omega t \tag{30}$$

where $n = 1, 2, 3, ---, n, ---\infty$. Solving Equation (30) for n equals

---

* Selected sections of this thesis have been published by Dr. H. T. Bates, and the author. See Reference No. 14.

one results in the first harmonic of the series:

$$p(\omega t) = \frac{4AB\omega}{3\pi} \cos \omega t \tag{31}$$

which is the same as the derivative mode except for the form of the gain constant. In consideration of the first harmonic of the output wave form, the magnitude-rate mode has a 90° phase angle lead component.

Figure 2 shows the controller output wave forms of Equations (29) and (31) with the gain coefficients chosen to give both responses the same maximum magnitude. Since the sign of the magnitude-rate mode is determined by the sign of the first time derivative of the error, it always has the same sign as the derivative mode. However, the response of the magnitude-rate mode differs significantly from the response of the derivative mode. The points of greatest difference are: 0°, 180°, and 360°, over one error cycle. The difference at these points is due to the inclusion of $|\sin \omega t|$ in Equation (29) which forces the magnitude-rate mode to zero at these points while the derivative mode assumes its maximum value. Thus, the main purpose of the higher harmonics in Equation (30) is to reduce the first harmonic cosine wave to zero at these points. An expansion of Equation (30) for the first five terms of the Fourier series results in:

$$p(t) = \frac{4AB\omega}{3\pi} \cos \omega t - \frac{4AB\omega}{5\pi} \cos 3\omega t - \frac{4AB\omega}{21\pi} \cos 5\omega t - \frac{4AB\omega}{45\pi} \cos 7\omega t - \frac{4AB\omega}{77\pi} \cos 9\omega t \tag{32}$$

Equation (32) shows that the Fourier series representing the magnitude-rate mode consists only of odd harmonics with the sign of the higher harmonics

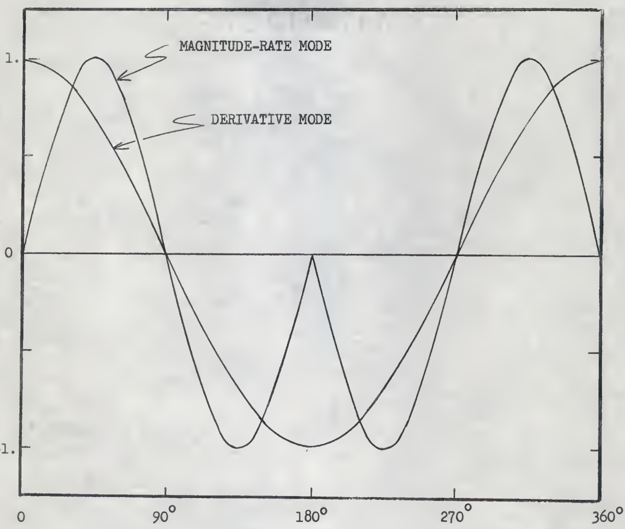


Figure 2. Comparison of the output wave forms for the magnitude-rate and the derivative control modes.

always being opposite that of the first harmonic at the points 0°, 180°, and 360°.

Considering only the first two harmonics of Equation (30) demonstrates the error possible in assuming that the phase angle of a nonlinear element is accurately described by the first harmonics of the Fourier series only. The difference between the coefficients of the first two terms of Equation (30) is:

$$\frac{4AB\omega}{3\pi} - \frac{4AB\omega}{5\pi} \tag{33}$$

or
$$\frac{4AB\omega}{\pi}\left[\frac{1}{3} - \frac{1}{5}\right] \tag{34}$$

Equation (34) shows that the coefficient of the first harmonic of the Fourier series is only 40% greater than the coefficient of the third harmonic. If only the magnitude of the coefficients is considered, the third harmonic has almost as much effect on the magnitude-rate mode as the first harmonic. However, it is known that most physical systems act as low pass filters (19). When this is the case, the higher harmonics are greatly attenuated and thus lose some of their effectiveness in defining system performance. Since the accuracy of assuming that the first harmonics determine the system response is generally within 10% error (20), the actual effect of the phase angle lead is probably between 75° and 85°.

Since the response of the magnitude-rate mode is zero anytime the error rate is zero, a control mode with a zero frequency gain must be included with it to prevent the system output variable from drifting or stabilizing at a new value. Either the proportional or the integral mode is suitable for this purpose although each has its disadvantages. If the integral mode is used in the loop, an additional phase angle lag is introduced for consideration; but there will not be a steady-state error. If the proportional mode is used, the system will tolerate steady-state errors; but no additional phase angles are included in the loop. Since steady-state errors have less effect on system stability than phase lags, the proportional mode was chosen to supply a zero frequency gain (16). In addition, the proportional response can be used as a "standard" for comparison to help clarify the effect of the magnitude-rate response.

If the proportional mode is combined with the magnitude-rate mode, the total controller response will be:

$$p(t) = Ae + AB|e|\,de/dt \tag{35}$$

or, after factoring out the gain constant A:

$$p(t) = A(e + B|e|\,de/dt) \tag{36}$$

which shows the reason for inclusion of two gain factors in Equation (28). As an aid in visualizing the controller response of the proportional plus magnitude-rate mode, it is convenient to let the error be represented by a sine wave; then Equation (36) becomes:

$$p(t) = A(\sin\omega t + B\omega\,|\sin\omega t|\cos\omega t) \tag{37}$$

Figure 3 shows the controller output wave form of Equation (37) for $A = 1$; $B\omega = 0, 1, 2$ and 5. When $B\omega$ equals zero, the controller response reduces to the proportional mode described by Equation (1). When $B\omega$ is not equal to zero, the proportional response is distorted from a true sine wave by the value of the magnitude-rate mode. The degree of distortion depends upon the magnitude of the error frequency ω and the gain factor B.

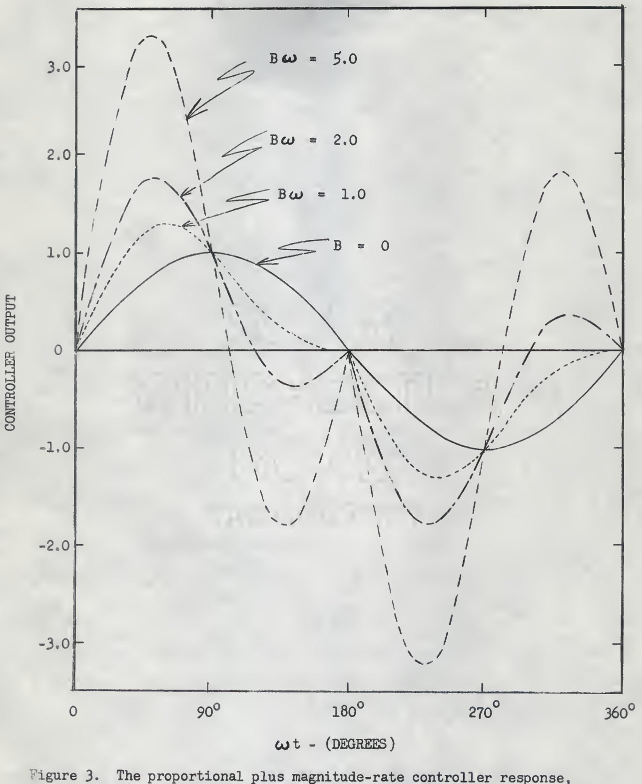


Figure 3. The proportional plus magnitude-rate controller response, $A(e + B|e|de/dt)$, for a sinusoidal error, with $A = 1.0$ for several values of $B\omega$.

When the error is departing from zero, the sign of magnitude-rate mode is the same as the sign of the proportional mode; and the controller gain increases rapidly in an effort to restore the system error to zero. As the error continues to depart from zero, but with a decreasing error rate, the value of the magnitude-rate mode begins to have less effect on the proportional mode; and the controller response begins to decrease. When the error rate is zero, the output of the magnitude-rate mode is zero; and the controller response is reduced to that of the proportional mode alone, further exemplifying that the magnitude-rate mode connot be used without a zero frequency gain element in the loop. When the error turns and begins to approach zero, the sign of the magnitude-rate mode will be opposite that of the proportional mode; and the controller gain will be decreased. By an appropriate choice of the gain constant B for a constant operating frequency the controller output may be made to approach zero very slowly as the error approaches zero ($B\omega = 1$), or it may be made to reverse sign to correct for the error overshoot before it occurs ($B\omega \geqq 2$). For any choice of $B\omega$ the controller output will always be reduced to zero when the error is zero. As the error crosses zero and continues to increase in magnitude, the controller output increases rapidly, and the cycle, as previously described, is repeated.

It should be pointed out that the gain of the magnitude-rate mode, like the derivative mode, is modified by the error rate or error frequency. At very high frequencies the gain of the magnitude-rate mode could become very large in comparison to the proportional response, and the loop gain could be sufficiently increased to initiate instability. When $B\omega \geqq 10$ the effect of the proportional mode is negligible, and the Equation (37) is approximately:

$$p(t) = AB\omega\,|\sin\omega t|\cos\omega t \tag{38}$$

with a maximum error of about 10%. Equation (38) can be written as (24):

$$p(t) = \begin{cases} +\dfrac{AB\omega}{2}\sin 2\omega t, & \text{when} \quad 0 \leq t \leq \dfrac{\pi}{\omega} \\[2ex] -\dfrac{AB\omega}{2}\sin 2\omega t, & \text{when} \quad \dfrac{\pi}{\omega} \leq t \leq \dfrac{2\pi}{\omega} \end{cases} \tag{39}$$

and the magnitude-rate mode has the appearance of being a second harmonic sine wave in direct contradiction of the Fourier series of Equation (30) which contains only odd cosine harmonics. The disappearance of the second harmonic sine waves in the Fourier series is apparently due to the sign reversal on each half cycle exhibited by Equation (39). An explanation of the way in which the tendency toward instability at high frequencies is somewhat avoided can be seen by an examination of Equations (32) or (39). Attenuation of high frequency signals, and in particular the higher harmonics, results in reducing the effect of the magnitude-rate mode as described by Equation (32) since it is essentially made up of higher harmonics. In addition to this, the sign reversal once each half cycle in Equation (39) results in a smoothing effect as it passes through a low pass filter system. On systems capable of responding to high frequency signals a low gain constant B would be desirable to reduce the controller gain in these frequency ranges.

The significant features of the magnitude-rate mode may be summarized as follows:

1. The controller gain will be increased for all errors departing from

zero.

2. The controller gain will be decreased for all errors approaching zero.

3. The controller output will be zero anytime the error or the error rate is zero.

4. The magnitude-rate mode has a leading phase angle of approximately 80°.

5. The tendency toward instability at high frequencies is reduced by the attenuation of the higher harmonics representing the magnitude-rate mode.

6. Since the magnitude-rate mode does not have a zero frequency gain, it must be used in combination with some other control mode such as the proportional or integral.

Experimental Procedure

Due to the nonlinearity of the magnitude-rate mode, an analytical determination of its effect in a feedback control system would have been exceedingly difficult to obtain. Since an analogue computer capable of simulating the nonlinearity was available, analogue simulation of feedback control loop containing the magnitude-rate mode was performed as part of the experimental work for this thesis.

The analogue computer used was the Kansas State Engineering Experimental Station Analogue Computer (KEESAC). High gain DC amplifiers were used to perform the linear arithmetic operations. Ten turn Helipot precision potentiometers with calibrated dials were used to adjust the control functions and the closed loop gain. Diodes were used for voltage rectification in order to obtain the absolute value of the error signal. Servo-multipliers were used for voltage multiplication. A two channel Brush recorder was used to record two variables simultaneously with respect to time. A Hewlett-Packard function generator was used to generate the sinusoidal forcing functions for the open loop response.

Figure 4 shows the analogue computer signal flow diagram used for both the open loop and the closed loop response. All resistor values are in megohms and capacitor values are in microfarads.

Amplifier 1 was used to obtain the system error by comparison of the feedback variable and the reference input. Switch S_2 was used to disconnect the feedback variable for the open loop response. Switch S_1, a micro-switch, was used as a gate for the step function inputs for the closed loop response and for the sinusoidal forcing functions for the open loop frequency response. The voltage for the step function was supplied directly from the output of an amplifier to prevent potentiometer loading effects.

Amplifier 3 was used in conjunction with the two diodes for full-wave voltage rectification in order to obtain the absolute value of the error. Figure 5(b) shows a recording of a sinusoidal error and its absolute value. The diodes used did not have an exact cut-off value; so the corners are somewhat rounded at the points where the error signal changes sign. The exact mathematical significance of the ten megohm resistor was not determined, but system noise was greatly reduced by its inclusion in the circuit. The resistor did not appear to have an effect on the rounding of the corners.

Amplifier 4 was used to obtain an approximation of the first time derivative of the error. Since noise is generally an inherent feature of an exact differentiation circuit, a 0.01 microfarad capacitor was employed in the feedback path (25). The transfer function of Amplifier 4 is then:

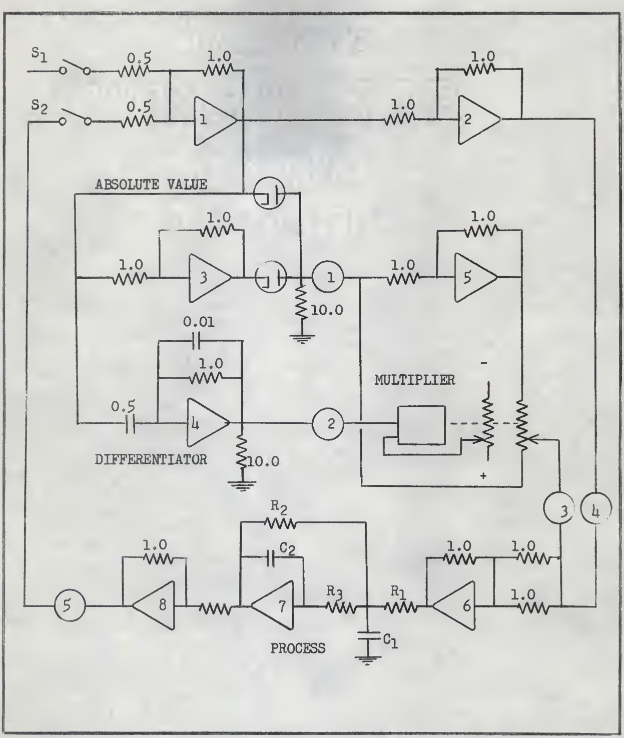


Figure 4. Analogue computer diagram for the proportional plus magnitude-rate controller and a second order process. All resistors are given in megohms and capacitors in microfarads.

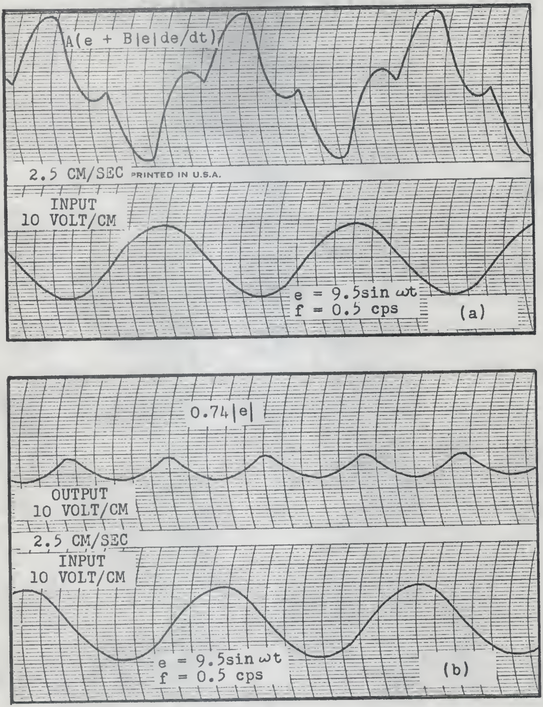


Figure 5. (a) Wave form of the proportional plus magnitude-rate circuit for a sinusoidal forcing function.
(b) Wave form of the absolute value of the error circuit for a sinusoidal error.

$$F_o(s) = -\frac{0.5s}{1 + 0.01s} \tag{40}$$

or for a sinusoidal input signal:

$$f_o(t) = \text{Im}\left[-\frac{0.5j\omega}{1 + 0.01j\omega}\right] \exp j\omega t \tag{41}$$

then:

$$f_o(t) = -\left(\frac{0.5\,\omega}{\left[1 + (0.01\omega)^2\right]^{\frac{1}{2}}}\right) \cos(\omega t - \arctan 0.01\omega) \tag{42}$$

or $$f_o(t) \cong -0.5\omega\cos\omega t \tag{43}$$

therefore:

$$de/dt = -2f_o(t) \tag{44}$$

with negligible error. Figure 6 shows the first time derivative of a sinusoidal signal using the approximation circuit described in Equation (40) at an angular frequency of π radians per second. The error present is not detectable on this recording.

Since the derivative of a step function is infinite, limiters probably should have been used on the output of the derivative circuit. This was not done because the output of the derivative circuit was fed directly into the servo-multiplier. Since the frequency response characteristics of the servo-multiplier were so poor at high frequencies, it was felt that this sudden high voltage input would have very little effect of the servo-multiplier output (8).

Amplifier 5 was used in conjunction with the servo-multiplier to obtain the product of the absolute value of the error and the error rate. The constant voltage for the reference potentiometer was approximately ± 25

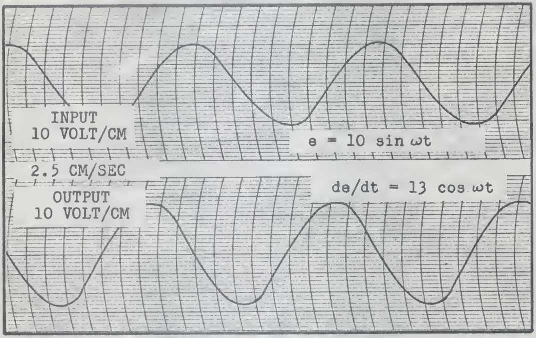


Figure 6. Output wave form of the derivative approximation circuit for a sinusoidal input.

volts. Figure 7 (a) and (b) are the outputs of the servo-multiplier for a sinusoidal error. At 0.5 cycles per second the servo-multiplier followed very closely. At 2.0 cycles per second the output wave was completely distorted from the second harmonic sine waves given by Equation (39). In general, the servo-multipliers followed rather closely up to about 1.5 cycles per second. At other times the servo-multiplier exhibited instability or failed to follow at lower frequencies for no apparent reason. It was usually necessary to switch between the two operating servo-multipliers every few minutes to allow the servo-multiplier that had been in use to settle down and quit "hunting". The cause or the solution to this problem was not determined.

Amplifier 6 was used to obtain the sum of the proportional plus magnitude-rate mode. Potentiometers 3 and 4 were used to adjust the relative magnitude of the two controller modes. A gain factor of ten was used in Amplifier 6 because it was found that the noise from the servo-multiplier was reduced by having Potentiometer 3 at a low value (0.2 to 0.5). Figure 5(a) is the sum of the proportional plus magnitude-rate mode for a sinusoidal error.

Amplifier 7 was used to simulate the second order processes with a damping factor of approximately 0.63. The transfer function of Amplifier 7 is derived in the Appendix, and is given by:

$$\frac{V_o(s)}{V_i(s)} = -\frac{R_2}{R_1}\left[\frac{1}{R_2C_2R_3C_1s^2 + C_2(R_2 + R_3 + R_2R_3/R_1)s + 1}\right] \qquad (45)$$

The second order process definition used in this work is given by the transfer function:

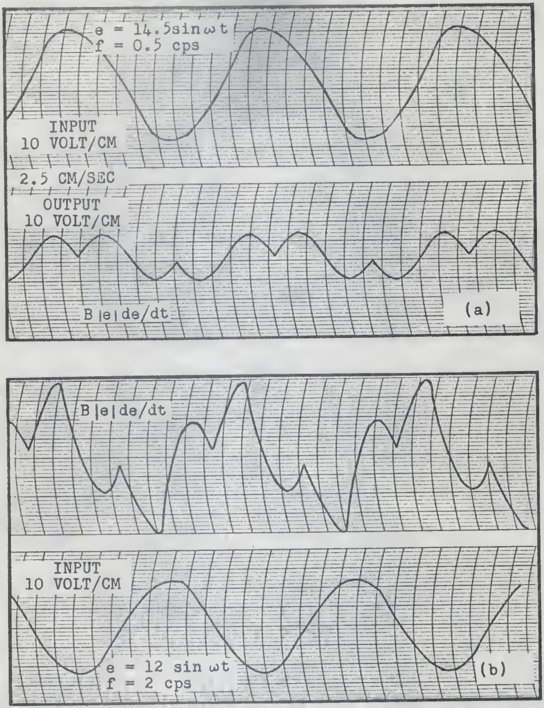


Figure 7. Output wave form of the servo-multiplier generating the product of the absolute value of the error and the error rate.
(a) At low frequencies the servo followed closely.
(b) Servo failed to follow at high frequencies.

$$\frac{V_o(s)}{V_i(s)} = \frac{K}{T^2s^2 + 2\zeta Ts + 1} \tag{46}$$

where K is the gain factor, T is the time constant, and ζ is the damping factor. Comparing Equations (45) and (46) results in the following system parameters:

$$K = -\frac{R_2}{R_1} \tag{47}$$

$$T = (R_2C_2R_3C_1)^{\frac{1}{2}} \tag{48}$$

$$\zeta = \frac{(R_2 + R_3 + R_2R_3/R_1)C_2}{2(R_2C_2R_3C_1)^{\frac{1}{2}}} \tag{49}$$

and $$\omega_n = \frac{1}{(R_2C_2R_3C_1)^{\frac{1}{2}}} \tag{50}$$

where ωn is the natural frequency of the process. Table 1 in the Appendix lists all of the resistor and capacitor values and the calculated process parameters. Table 2 below lists only the process parameters for the two second order processes simulated.

Table 2. Second order process parameters.

Parameters	Run Number 1	Run Number 2	Units
K	-0.5	-2.5	volts/volt
T	2.0	0.316	seconds
ζ	0.625	0.633	-
ω_n	0.500	3.16	radians/sec
f_n	0.0796	0.503	cycles/sec

The resistor and capacitor values in Equation (45) were chosen to give a damping factor of approximately the same value for the two runs. This was done in order to allow a comparison of the two systems based on their normalized angular frequency ω/ω_n. To have made the damping factors exactly the same for the range of natural frequencies desired would have required "odd" size capacitors and resistors.

Amplifier 8 was used for sign inversion for the closed loop response. Potentiometer 5 was used for final closed loop gain adjustment.

Closed Loop Response

If a step function change is made in the setpoint of the feedback control loop, the zero frequency or the steady state error E(0) is given by the following expression (6):

$$E(0) = \frac{C}{1 + G(0)} \tag{50}$$

where C is the amplitude of the step function, and G(0) is the total zero frequency gain of the control loop. In order to have an adjustable zero frequency gain, the setting of Potentiometer 5 in Figure 4 was chosen as the system loop gain parameter. Then, the zero frequency gain of the loop is given by the equation:

$$G(0) = K(0)\ P_5 \tag{51}$$

where K(0) is the maximum zero frequency gain of the loop, and P_5 is the dial setting of Potentiometer 5. Since the magnitude-rate mode does not have a zero frequency gain component, the total steady state gain of the loop is determined by the product of the proportional gain constant and the zero frequency gain of the other elements in the control loop. For convenience, the proportional gain constant A was chosen to represent the total zero frequency gain of the loop, or:

$$A = K(0)\, P_5 \tag{52}$$

Combining Equations (50), (51), and (52) and solving for A results in:

$$A = \frac{C - E(0)}{E(0)} \tag{53}$$

By exciting the system in Figure 4 with a step function input and allowing the error to reach its steady state value, the gain constant A was determined by Equation (53). The maximum steady state gain K(0) was determined by Equation (52).

The gain constant B was determined by opening the loop and generating a sinusoidal error, or:

$$e(t) = E_m \sin \omega t \tag{54}$$

where E_m is the maximum amplitude of the sine wave. The output of the magnitude-rate mode is given by:

$$p(t) = B\omega |E_m \sin \omega t| E_m \cos \omega t \tag{55}$$

Equation (55) can be written as:

$$p(t) = \frac{B\omega E_m^2}{2} \sin 2\omega t, \quad \text{when} \quad 0 \leq t \leq \frac{\pi}{\omega} \tag{56}$$

If the maximum amplitude P_m of Equation (56) is known, then the gain constant B can be calculated by:

$$B = \frac{2P_m}{E_m^2 \omega} \tag{57}$$

From Equations (52) and (57) the proportional response is given by:

$$p(t) = K(0)\ P_5 e \tag{58}$$

and the proportional plus magnitude-rate response is:

$$p(t) = K(0)\ P_5(e + B|e|de/dt) \tag{59}$$

The closed loop response of Run 1 was obtained for both the proportional plus magnitude-rate controller and the proportional controller alone. The step function change in the system setpoint and the system error were recorded simultaneously on the two channel Brush recorder.

The total zero frequency loop gain was found to be $25P_5$ to give a proportional response of:

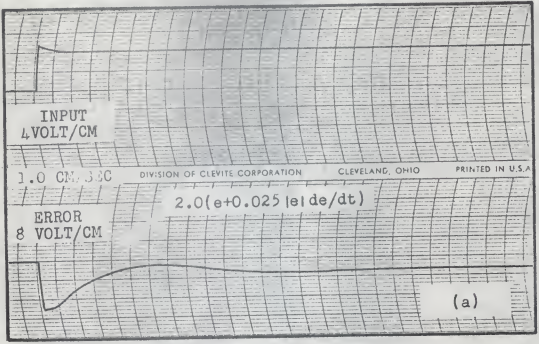
$$p(t) = 25P_5 e \tag{60}$$

and the gain constant B was set to give a proportional plus magnitude-rate response of:

$$p(t) = 25P_5\,(e + 0.025|e|de/dt) \tag{61}$$

Figures 8 through 14 are comparisons of the closed loop response of Run 1 with a step function input of 4.0 volts. It should be noted that the step function is doubled since Amplifier 1 has a gain of 2.0. The curves labeled (a) are for the proportional plus magnitude-rate controller system. The curves labeled (b) are for the system with only the proportional mode controller.

Figure 8 is a comparison of the two systems at the very low gain of



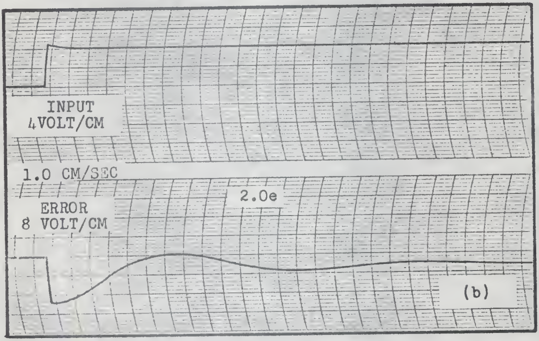


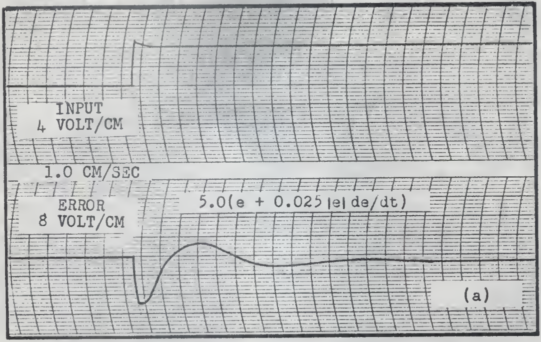
Figure 8. Closed loop response of the second order process to a step function. Run 1. Potentiometer 5 = 0.08.
(a) With the proportional plus magnitude-rate mode controller.
(b) With the proportional mode only.

A = 2.00. The system response is essentially the same for both controllers except that the proportional plus magnitude-rate mode reached its steady state error value slightly sooner than the proportional mode alone. Since the error frequency was low, the magnitude-rate mode contributed very little to the total controller response.

When the loop gain had been increased to A = 5.0, the error rate had been increased enough to make the presence of the magnitude-rate mode more evident. From Figure 9, it is seen that the maximum magnitude of the first error overshoot is reduced by about 25% by the action of the magnitude-rate mode. The proportional mode system required over two cycles. The system response rate was increased by about 35% by the inclusion of the magnitude-rate mode.

Figure 10 is the response of the two systems when the loop gain had been increased to A = 10.0. Both systems exhibited their approximate optimum response. The proportional plus magnitude-rate mode was completely lined out after two oscillations while the proportional mode required in excess of three oscillations. The height of the first error overshoot was approximately the same for both systems, but the response rate of the proportional plus magnitude-rate mode was increased by approximately 60%.

In Figure 11 the proportional plus magnitude-rate mode was almost to the point of instability. The magnitude of each oscillation was decreased very little each time around the loop. The system containing the proportional mode was not unstable, but several cycles were required for the error to line out at its zero frequency value. The frequency of oscillation of the nonlinear system was 0.66 cycles per second as compared to an undamped natural frequency of 0.0796 cycles per second for the second order process.



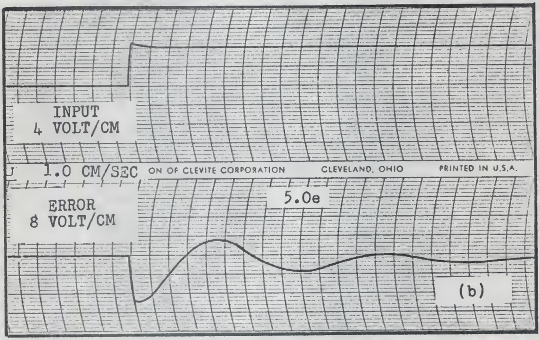
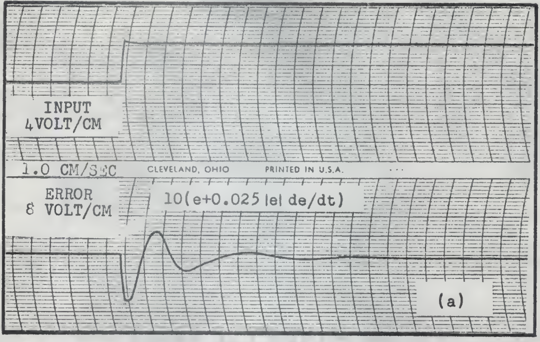


Figure 9. Closed loop response of the second order process to a step function. Run 1. Potentiometer 5 = 0.2.
(a) With the proportional plus magnitude-rate mode controller.
(b) With the proportional mode only.



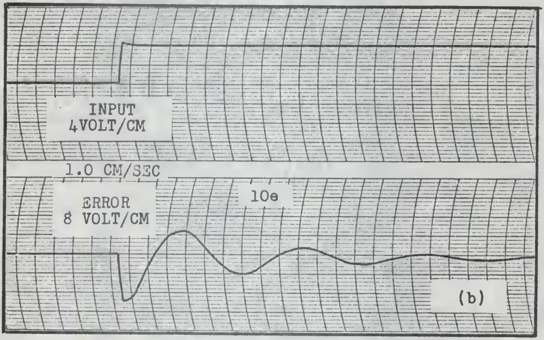
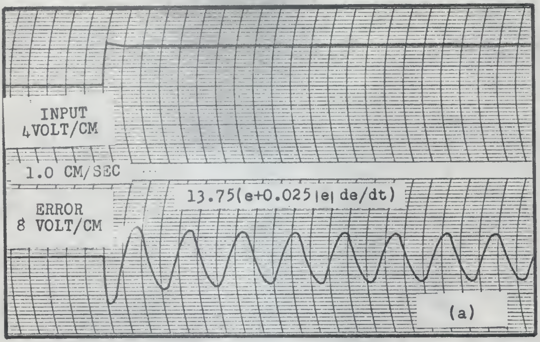


Figure 10. Closed loop response of the second order process to a step function. Run 1. Potentiometer $5 = 0.4$.
(a) With the proportional plus magnitude-rate mode controller.
(b) With the proportional mode only.



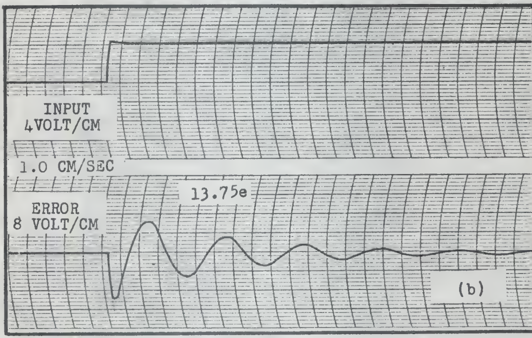


Figure 11. Closed loop response of the second order process to a step function. Run 1. Potentiometer 5 = 0.55.
(a) With the proportional plus magnitude-rate mode controller.
(b) With the proportional mode only.

When the loop gain had been decreased to A = 12.5, the system containing the proportional plus magnitude-rate mode was stable; but four cycles were required for the error to reach its zero frequency value. It is interesting to note that the error was essentially reduced to its steady state value during the last oscillation only. This could indicate saturation or limiting of some component within the loop. This was not verified since none of the intermediate voltages in the loop were monitored.

Figures 13 and 14 in the Appendix are additional recordings of the closed loop response of the two control systems.

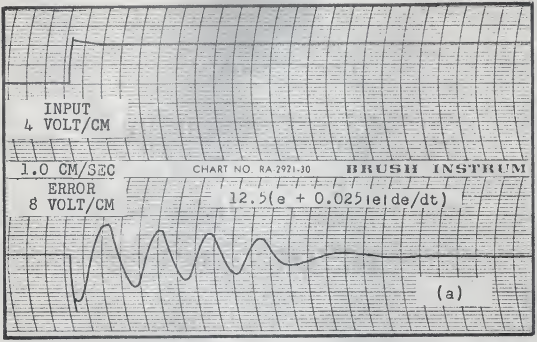
The closed loop response of Run 2 was obtained for the proportional plus magnitude-rate mode only. The zero frequency gain was found to be $2.0P_5$. In order to make the effect of the magnitude-rate mode more pronounced, the gain constant B was increased by approximately 28% over Run 1; or in terms of the controller response:

$$p(t) = 2.0P_5(e = 0.032|e|de/dt) \tag{61}$$

The response of Equation (61) to a sinusoidal error is shown in Figure 15 at 0.5 cycles per second.

Figure 16(a) is the closed loop response of Run 2 with A = 0.40. The effect of the magnitude-rate mode predominates even at this relatively low loop gain. Since the sign of the magnitude-rate mode is opposite the sign of the proportional mode as the error approaches zero, the error rate is reduced to zero before the error reaches its steady state value. After the error rate levels off, the magnitude-rate response is reduced to zero; and the proportional response slowly forces the error to its steady state value.

When the gain is increased to A = 0.7 in Figure 16(b), the error rate



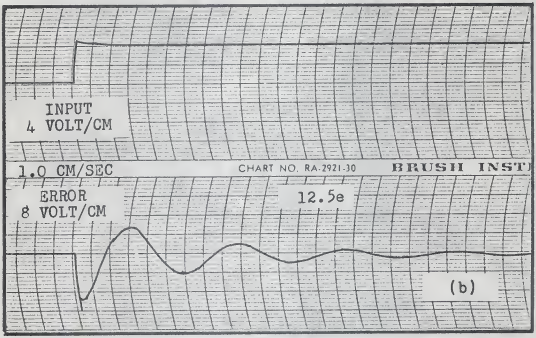


Figure 12. Closed loop response of the second order process to a step function. Run 1. Potentiometer 5 = 0.5.
(a) With the proportional plus magnitude-rate mode controller.
(b) With the proportional mode only.

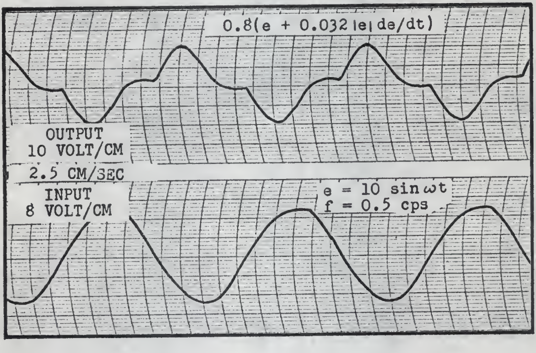
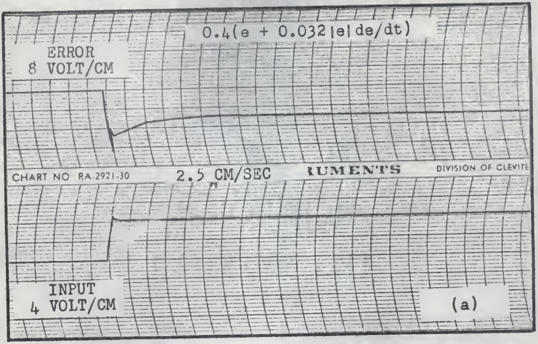


Figure 15. Wave form of the proportional plus magnitude-rate mode controller response for Run 2.



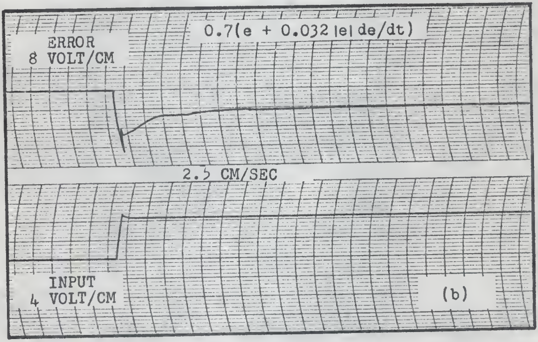


Figure 16. Closed loop response of the second order process with the proportional plus magnitude-rate mode. Run 2.
(a) Potentiometer 5 = 0.20.
(b) Potentiometer 5 = 0.35.

is reduced to zero for a considerable length of time and then approaches its steady state value very slowly.

When A is increased to 0.9, the approximate optimum system response is obtained for the 4.0 volt step input. The error rate levels off once and then approaches its steady state value rapidly. There is no error overshoot.

When the loop gain is increased to A = 1.2 in Figure 17(a), the system is unstable. The magnitude of the oscillations is increased each time around the loop. The frequency of oscillation is 1.38 cycles per second as compared to the undamped natural frequency of 0.503 cycles per second for the second order process.

Figure 18 in the Appendix shows two additional closed loop responses for Run 2.

Open Loop Response

The open loop responses of the second order processes (Runs 1 and 2) were obtained with the proportional plus magnitude-rate controller. Sinusoidal forcing functions of various frequencies between 0.03 and 1.2 cycles per second were used. The sinusoidal input and the second order process output wave were recorded simultaneously on the dual channel Brush recorder. The zero frequency output magnitude was assumed to be given by the maximum output wave amplitude at 0.03 cycles per second. The zero frequency magnitude was required to calculate the magnitude ratio for the frequency response curves. The phase angle lag was determined at corresponding points where the process output wave and the sinusoidal input amplitudes crossed zero on the recordings. No attempt was made to adjust the output wave phase angle or the magnitude for nonlinear distortion.

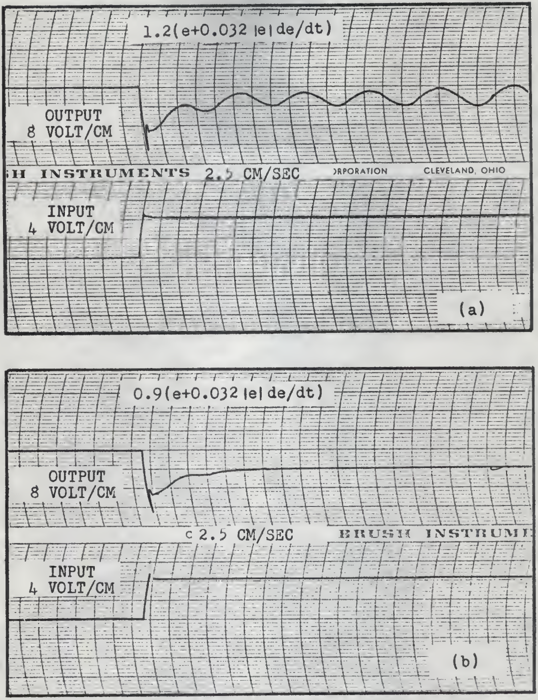


Figure 17. Closed loop response of the second order process with the proportional plus magnitude-rate mode. Run 2.
(a) Potentiometer $5 = 0.60$.
(b) Potentiometer $5 = 0.45$.

The proportional plus magnitude-rate mode controllers were adjusted to the same values as used in the corresponding closed loop response:

For Run 1:

$$p(t) = A(e + 0.025\,|e|\,de/dt) \tag{62}$$

For Run 2:

$$p(t) = A(e + 0.032\,|e|\,de/dt) \tag{63}$$

Figure 19 shows the frequency response curves for both Runs 1 and 2. The normalized angular frequency ω/ω_n was used on the horizontal axis for convenience in plotting the curves. The dashed lines on Figure 19 are the calculated response curves for the second order process with a sinusoidal forcing function.

The frequency response curves were very odd, as shown by the additional resonant "bump". Immediately after the second resonant bump the magnitude ratio and the phase angle dropped off rapidly, approaching that of the second order process with the sinusoidal input. For Run 1 the approximate point of the rapid decay was at 0.5 cycles per second or at the normalized frequency of 1.52. For Run 2 the frequency was 0.76 cycles per second or at the normalized frequency of 1.52.

The proportional plus magnitude-rate mode can be approximated by the equation:

$$p(t) = A \sin\omega t + \frac{4AB\omega}{3\pi} \cos\omega t \tag{64}$$

where the higher harmonics in the Fourier expansion of the magnitude-rate mode have been omitted due to their attentuation through the linear second

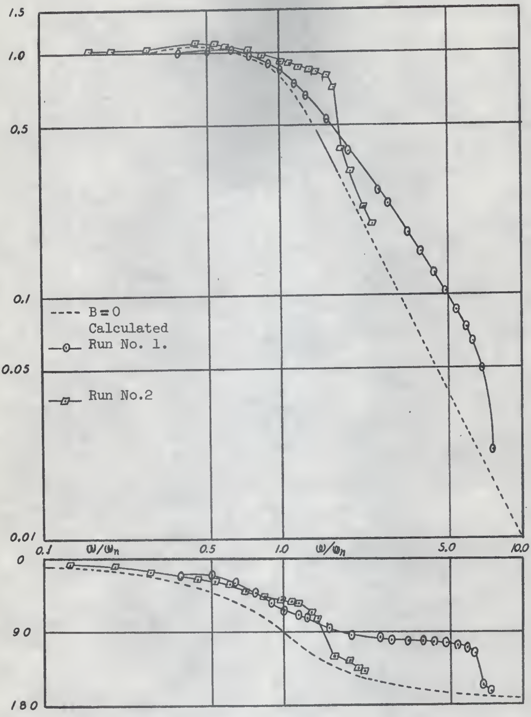


Figure 19. Open loop frequency response of the second order process with the forcing function $A(\sin\omega t + B\omega|\sin\omega t|\cos\omega t)$.

order process. The open loop frequency response of $A\sin\omega t$ is shown by the dashed lines on Figure 19. The first harmonic cosine wave has a leading phase angle of 90°, and the gain is increased by the inclusion of the angular frequency ω in the numerator. Therefore, at the higher frequencies the cosine wave would have the effect of shifting the dashed line representing the phase angle upward toward the -90° phase angle line. The increase in gain of the cosine wave would also have the effect of increasing the magnitude ratio at higher frequencies. Since the inclusion of the cosine wave in Equation (64) can only result in a positive increase in the phase angle and an increase in the magnitude the rapid decay exibited by the system must be due to a component failure within the loop. Since the servo-multipliers were usually accurate up to frequencies approaching 1.5 cycles per second, some other element within the loop must have failed. The most logical points are the capacitors used on the second order process or a faulty amplifier. The same capacitors were not used in both runs, but they were of the same type. The same amplifiers were used in both runs.

Figure 20 is the output wave form of Run 2 with a sinusoidal forcing function. At low frequencies the output wave was distorted only slightly from a true sine wave. When the frequency had been increased to 0.18 cycles per second, the output wave form was highly distorted by the magnitude-rate mode. When a frequency of 0.46 cycles per second had been reached, the effect of the higher harmonics were greatly reduced; and the wave form again appeared to be a sine wave, demonstrating the accuracy of Equation (64) in representing the proportional plus magnitude-rate mode at high frequencies.

If the error to the controller is a sinusoidal wave, then the pro-

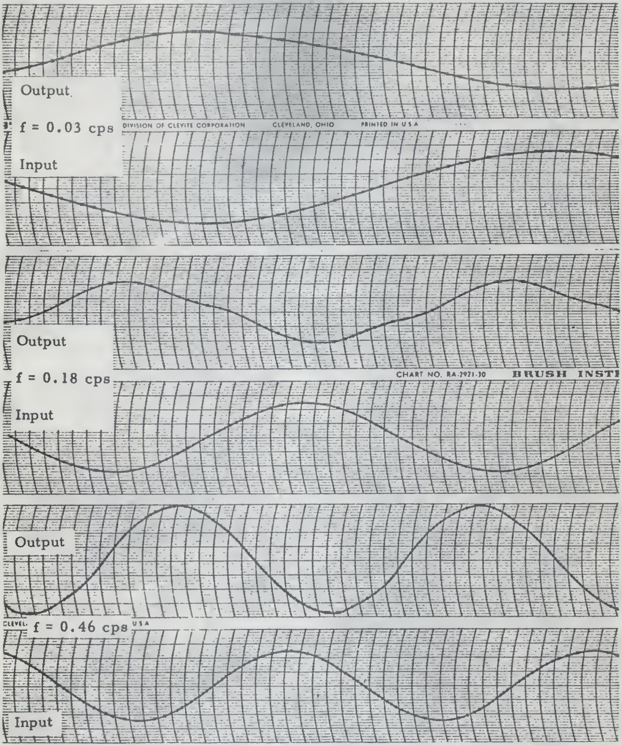


Figure 20. Open loop response of the second order process, with the proportional plus magnitude-rate controller. Run 2.

portional plus magnitude-rate response is:

$$p(t) = A(\sin \omega t + B\omega |\sin\omega t| \cos\omega t) \tag{65}$$

The LaPlace transform of Equation (65) has been derived in the Appendix and is given by:

$$P(s) = A\left[\frac{\omega}{s^2 + \omega^2} + \frac{B\omega^2}{s^2 + 4\omega^2} \tanh \frac{\pi s}{2\omega}\right] \tag{66}$$

If the process is described by the transfer function:

$$\frac{M(s)}{P(s)} = \frac{1}{T^2 s^2 + 2\xi Ts + 1} \tag{67}$$

then the process output is:

$$M(s) = \frac{P(s)}{T^2 s^2 + 2\xi Ts + 1} \tag{68}$$

$$M(s) = \frac{A\omega}{(s^2 + \omega^2)(T^2 s^2 + 2\xi Ts + 1)} + \frac{AB\omega^2}{(s^2 + 4\omega^2)(T^2 s^2 + 2\xi Ts + 1)} \tanh \frac{\pi s}{2\omega} \tag{69}$$

The inverse LaPlace transform of Equation (69) has previously been obtained (14) but will not be repeated here due to its complex nature and the more complete experimental results included in this thesis.

CONCLUSIONS

The nonlinear controller response:

$$p(t) = AB|e|de/dt \tag{70}$$

has been shown to have several desirable characteristics. The most significant of these are:

1) It provides an increase in the loop gain for all errors departing from zero.
2) The loop gain is decreased for all errors approaching zero.
3) The loop phase angle can be increased up to approximately 80°.
4) The controller response will be zero anytime the error or the error rate is zero.

The closed loop response of an underdamped process was obtained for both the proportional plus magnitude-rate controller and the proportional controller. The system with the proportional plus magnitude-rate mode improved the system response to step function setpoint changes. The system response rate was increased up to 60%. The number of oscillations required for the error to line out were reduced by approximately 50%. By increasing the gain cf the magnitude-rate mode the error could be made to approach its steady state value rapidly without any error overshoot or oscillations.

While this work substantiates the usefulness of the magnitude-rate mode for control of underdamped processes, additional studies should be made. A method of determining the optimum setting of the gain constant B should be determined. Its effect when used on third or higher order systems should be studied.

NOMENCLATURE

A	Proportional mode gain constant
A_i	Controller gain constants, $i = 1, 2, 3, \ldots\ldots$
a_o	Constant term in Fourier series
a_n	Harmonic sine wave constants in Fourier series
B	Magnitude-rate mode gain constant
b_n	Harmonic cosine wave constants in Fourier series
C	Step function amplitude
C_i	Capacitors in second order process circuit
E_m	Maximum amplitude of the sinusoidal error signal
$E(s)$	LaPlace transform of the error signal
$e(t)$	Error signal as a function of time
f_n	Undamped natural frequency of the second order process
G	Amplifier gain
$G(0)$	Steady state loop gain
$G(s)$	LaPlace transform of the manipulated variable
$g(t)$	Manipulated variable as a function of time
I	Integral mode gain constant
Im	Coefficient of the imaginary term
J	Electrical junction
j	Imaginary unit
$K(0)$	Maximum value of the loop gain
k	Length of the period for a periodic function
l	Load variable
$M(s)$	LaPlace transform of the process output signal
$m(t)$	Process output signal as a function of time

n	Integers, $n = 1, 2, 3, \ldots\ldots n, \ldots.$
P_5	Dial setting of Potentiometer 5
P_m	Maximum amplitude of the controller output signal
$P(s)$	LaPlace transform of the controller output signal
$p(t)$	Controller output signal as a function of time
R_1	Resistor in the second order process
r	Reference input signal or the system set point
s	LaPlacian dummy variable
T	Process time constant, seconds
t	time
V_a	Voltage between Junction a and ground
V_g	Voltage between Junction g and ground
V_i	System input voltage
V_o	System output voltage
$y(t)$	Optimum switching line for a programmed controller
α	Angular frequency, radians per second
β	Angular frequency, radians per second
ξ	Second order process damping factor
ω	Angular frequency, radians per second
ω_n	Undamped natural frequency, radians per second

LITERATURE CITED

(1) Zoss, L.M., H.S. Wilson.
Basic Control Considerations, ISA Journal., 9(1), p. 26 (1962)

(2) Murphy, G.J.
Basic Automatic Control Theory, D. Van Nostrand Co., Inc., Princeton, N. J., p. 13, 197 (1957).

(3) Nixon, F.E.
Principles of Automatic Control, Prentice-Hall, Inc., Englewood Cliffs, N. J., p. 98, 339 (1956).

(4) Seely, S.
Electron-Tube Circuits, McGraw-Hill Book Co., New York, N. Y., p. 147 (1958).

(5) Caldwell, W.I., G.A. Coon., and L.M. Zoss.
Frequency Response for Process Control, McGraw-Hill Book Co., Inc., New York, N. Y., p. 27-29, 177 (1959).

(6) Ceaglske, N.H.
Automatic Process Control for Chemical Engineers, John Wiley and Sons, Inc., New York, N. Y., p. 5, 185-187 (1956).

(7) Eckman, D.P.
Automatic Process Control, John Wiley and Sons, Inc., New York, N. Y., p. 300 (1958).

(8) Lauer, H., R.N. Lesnick., and L.E. Matson.
Servomechanism Fundamentals, McGraw-Hill Book Co., Inc., New York, N. Y., p. 91-92, 252, 420 (1960).

(9) Leedham, C.D.
Putting Control Theory into Practice, ISA Journal., 8(6), p. 53 (1961).

(10) Tucker, G.K., and D.M. Wills.
A Simplified Technique of Control System Engineering, Minneapolis-Honeywell Regulator Co., Philadelphia, Pa., p. 212-224 (1960).

(11) Powell, B.E.
How to Design Control Loop from Frequency Response Data, ISA Journal, 5(4), p. 34 (1958).

(12) Wilson, H.S., and L.M. Zoss.
Combination of Control Modes, ISA Journal, 9(6), p. 35 (1962).

(13) Zoss, L.M., and H.S. Wilson.
On-Off and Proportional Control Modes, ISA Journal, 9(4), p. 49 (1962).

(14) Bates, H.T., and J.C. Webb.
Nonlinear Controller for Underdamped Systems, Chemical Engineering Progress, 57(9), p. 49 (1961).

(15) Wilson, H.S., and L.M. Zoss.
Reset and Rate Control Modes, ISA Journal, 9(5), p. 46 (1962).

(16) Meyer, C.F.
A Graphical Analysis Technique for a Contactor Servo, Sperry Engineering Review, 19(1), p. 10 (1959).

(17) Stout, T.M.
Nonlinearity in Control Systems, Pt. III, Control Engineering, 3(4), p. 77-85 (1956).

(18) Truxal, J.C.
Automatic Feedback Control System Synthesis, McGraw-Hill Book Co., Inc., New York, N. Y., p. 653 (1955).

(19) Kochenburger, R.J.
Nonlinearity in Control Systems, Pt. II, Control Engineering, 3(3), p. 82-91 (1956).

(20) Stout, T.M.
Basic Analysis Methods for Nonlinear Systems, Instruments and Automation, 30(2), p. 263 (1957).

(21) Kerchner, R.M., and G.F. Corcoran.
Alternating-Current Circuits, John Wiley and Sons, Inc., New York, N. Y., p. 161-188 (1958).

(22) Cosgriff, R.L.
Nonlinear Control Systems, McGraw-Hill Book Co., Inc., New York, N. Y., p. 160-171 (1958).

(23) Gibson, J.E.
Nonlinear System Design, Pt. III, Control Engineering, 6(2), p. 82-85 (1959).

(24) Perry, J.H.
Chemical Engineers Handbook, McGraw-Hill Book Co., Inc., New York, N. Y., p. 72 (1950).

(25) Korn, G.A., and T.M. Korn.
Electronic Analogue Computers, McGraw-Hill Book Co., Inc., New York, N. Y., p. 19 (1952).

(26) Churchill, R.V.
Introduction to Complex Variables and Applications, McGraw-Hill Book Co., Inc., New York, N. Y., p. 40 (1948).

(27) Hudson, R.G.
The Engineers Manual, John Wiley and Sons, Inc., New York, N. Y., p. 46 (1950).

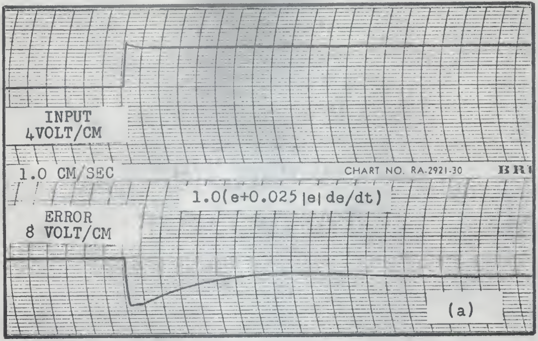
(28) Churchill, R.V.
Operational Mathematics, McGraw-Hill Book Co., Inc., New York, N. Y., p. 54-56 (1958).

ACKNOWLEDGMENT

The author is very grateful to Professor Herbert T. Bates for the introduction to control theory. My sincere appreciation for his interest and consultation while supervising this project.

Also, my sincere appreciation and apologies to my wife, Mary Lou, and daughters, Deborah Kay and Mary Jo, for their inconveniences while this work was being completed.

APPENDIX I



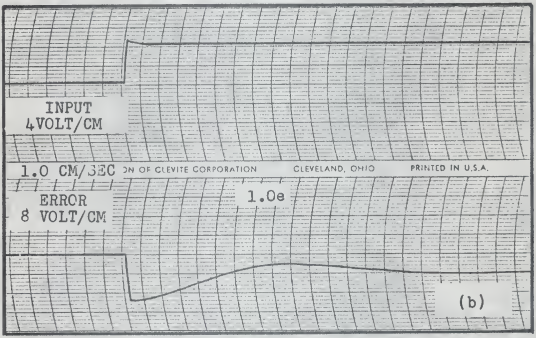


Figure 13. Closed loop response of the second order process to a step function. Run 1. Potentiometer 5 = 0.04.
(a) With the proportional plus magnitude-rate mode controller.
(b) With the proportional mode only.

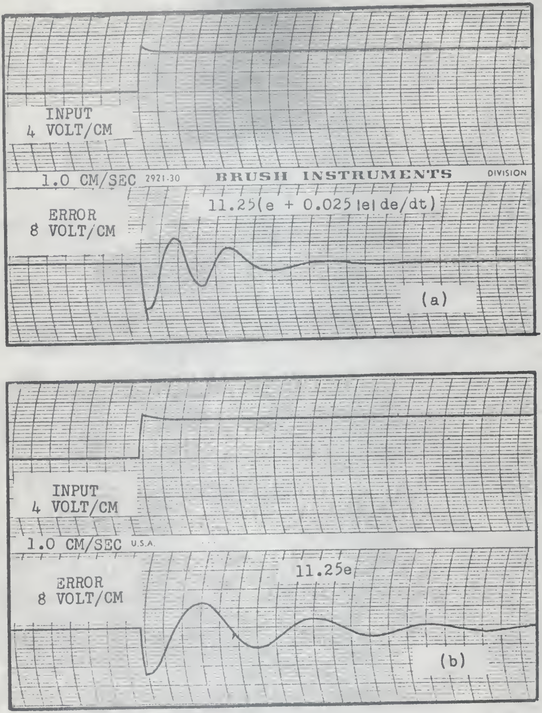


Figure 14. Closed loop response of the second order process to a step function. Run 1. Potentiometer 5 = 0.45.
(a) With the proportional plus magnitude-rate mode controller.
(b) With the proportional mode only.

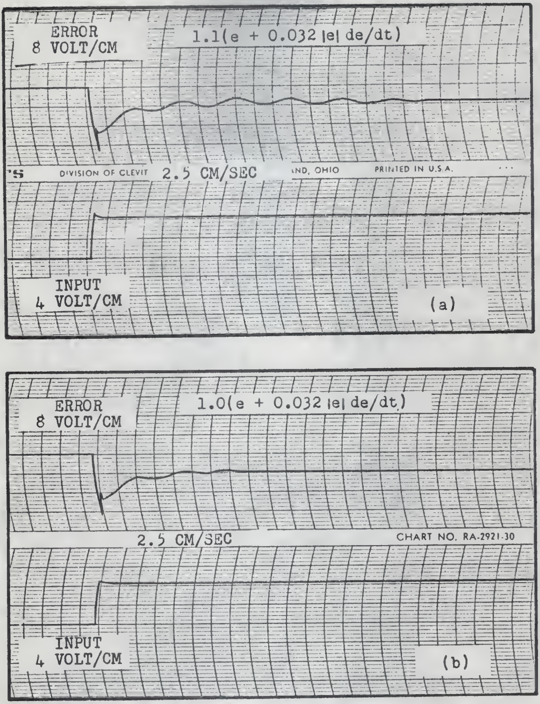


Figure 18. Closed loop response of the second order process with the proportional plus magnitude-rate mode. Run 2.
(a) Potentiometer 5 = 0.55.
(b) Potentiometer 5 = 0.50.

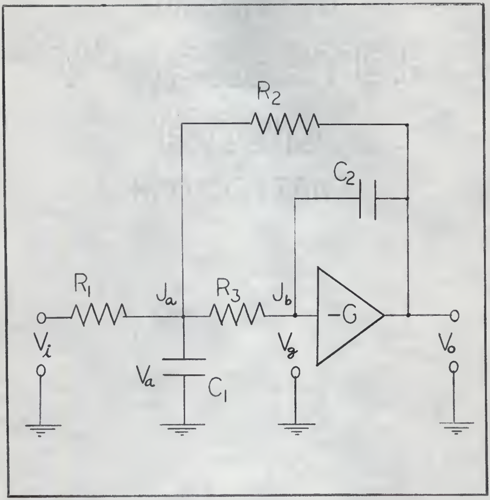


Figure 21. Second order process with a damping factor less than one.

APPENDIX II

DERIVATION OF THE FOURIER SERIES OF THE MAGNITUDE-RATE CONTROL MODE

The equation for the magnitude-rate mode is given by:

$$g(t) = AB|e|de/dt \tag{71}$$

If the error signal to the controller is given by:

$$e(t) = \sin\omega t \tag{72}$$

then $$g(t) = AB\omega|\sin\omega t|\cos\omega t \tag{73}$$

or as previously shown:

$$g(t) = \begin{cases} +\dfrac{AB\omega}{2}\sin 2\omega t, & \text{when } 0 \leqq t \leqq \dfrac{\pi}{\omega} \\[2ex] -\dfrac{AB\omega}{2}\sin 2\omega t, & \text{when } \dfrac{\pi}{\omega} \leqq t \leqq \dfrac{2\pi}{\omega} \end{cases} \tag{74}$$

Since g(t), as represented by Equation (74), is periodic and single-valued with a finite number of discontinuities in one cycle, it may be represented by a Fourier series (21):

$$g(t) = a_o + \sum_{n=1}^{n=\infty} a_n \sin n\omega t + \sum_{n=1}^{n=\infty} b_n \cos n\omega t \tag{75}$$

where $n = 1, 2, 3, ---, n, ---\infty$.

Now let:

$$g(\omega t + \pi) = AB\omega|\sin(\omega t + \pi)|\cos(\omega t + \pi) \tag{76}$$

Equation (76) reduces to:

$$g(\omega t + \pi) = -AB\omega|\sin\omega t|\cos\omega t \tag{77}$$

Then from Equations (73) and (77), there results:

$$g(\omega t) = -g(\omega t + \pi) \tag{78}$$

and $g(\omega t)$ is said to contain half wave symmetry and the Fourier series will contain no even harmonics and a_o will be equal to zero (21).

Now let:

$$g(-\omega t) = AB\omega \left| \sin(-\omega t) \right| \cos(-\omega t) \tag{79}$$

Equation (79) can be reduced to:

$$g(-\omega t) = AB\omega \left| \sin\omega t \right| \cos\omega t \tag{80}$$

Therefore:

$$g(\omega t) = g(-\omega t) \tag{81}$$

and $g(\omega t)$ is defined as an even function of ωt and will contain no sine wave harmonics in its Fourier series (21).

Equation (75) can now be reduced to the following:

$$g(t) = \sum_{n=1}^{n=\infty} b_{2n-1} \cos(2n-1)\omega t \tag{82}$$

where b_{2n-1} has been substituded for b_n so that n will assume consecutive values, and:

$$b_{2n-1} = \frac{1}{k/2} \int_0^k g(t) \cos(2n-1)\omega t \tag{83}$$

where k is the period. Since g(t) has half wave symmetry, b_{2n-1} may be evaluated over one half the period and the result multiplied by 2, or:

$$b_{2n-1} = \frac{2}{k/2}\int_0^{k/2} g(t)\cos(2n-1)\omega t \tag{84}$$

Substituting Equation (74) into Equation (84):

$$b_{2n-1} = \frac{4}{2\pi/\omega}\int_0^{\pi/\omega} \frac{AB\omega}{2}\sin 2\omega t\,\cos(2n-1)\omega t \tag{85}$$

The integral in Equation (85) is given by (27):

$$\int \sin\alpha t\,\cos\beta t = \frac{\cos(\alpha-\beta)t}{2(\alpha-\beta)} - \frac{\cos(\alpha+\beta)t}{2(\alpha+\beta)} \tag{86}$$

After substituting Equation (86) into Equation (85) and supplying the limits the following expression results:

$$b_{2n-1} = \frac{AB\omega^2}{\pi}\left\{\frac{\cos(3-2n)\omega t}{2(3-2n)} - \frac{\cos(1+2n)\omega t}{2(1+2n)}\right\}_0^{\pi/\omega}$$

After substituting in the limits and evaluating, Equation (87) reduces to the following expression:

$$b_{2n-1} = \frac{AB\omega^2}{\pi\omega}\left[\frac{1}{2(3 - 2n)} + \frac{1}{2(3 - 2n)}\right.$$

$$\left.\frac{1}{2(1 + 2n)} + \frac{1}{2(1 + 2n)}\right] \tag{88}$$

which reduces to:

$$b_{2n-1} = \frac{AB\omega}{\pi}\left[\frac{1}{3 - 2n} + \frac{1}{1 + 2n}\right] \tag{89}$$

Equation (89) reduces to:

$$b_{2n-1} = -\frac{4AB\omega}{\pi}\left[\frac{1}{(2n - 3)(2n + 1)}\right] \tag{90}$$

Substituting Equation (90) into Equation (82) results is the equation for the Fourier series of the magnitude-rate control mode:

$$g(t) = -\frac{4AB\omega}{\pi}\sum_{n=1}^{n=\infty}\frac{1}{(2n - 3)(2n + 1)}\cos(2n - 1)\omega t \tag{91}$$

DERIVATION OF THE LAPLACE TRANSFORM FOR THE PROPORTIONAL PLUS MAGNITUDE-RATE CONTROLLER RESPONSE

The equation describing the proportional plus magnitude-rate controller is given by:

$$g(t) = A(e + B|e|de/dt) \tag{92}$$

Let the error signal be given by:

$$e(t) = \sin\omega t \tag{93}$$

Substituting Equation (93) into Equation (92) and expanding result in the following equation:

$$g(t) = A \sin\omega t + AB\omega|\sin\omega t|\cos\omega t \tag{94}$$

Equation (94) can be written as:

$$g(t) = g_1(t) + g_2(t) \tag{95}$$

where:

$$g_1(t) = A \sin\omega t \tag{96}$$

and

$$g_2(t) = AB\omega|\sin\omega t|\cos\omega t \tag{97}$$

Equation (97) can be written as:

$$g_2(t) = \begin{cases} +\dfrac{AB\omega}{2}\sin 2\omega t, & \text{when} \quad 0 \leqq t \leqq \dfrac{\pi}{\omega} \\[2ex] -\dfrac{AB\omega}{2}\sin 2\omega t, & \text{when} \quad \dfrac{\pi}{\omega} \leqq t \leqq \dfrac{2\pi}{\omega} \end{cases} \tag{98}$$

Since $g_2(t)$ is periodic and has a finite number of discontinuities in one period, its LaPlace transform is given by the expression (28):

$$G_2(s) = \frac{1}{1 - \exp(-ks)} \int_0^k \exp(-st)\, g_2(t)dt \tag{99}$$

where k is the period of the function $g_2(t)$.

Substituting Equation (98) into Equation (99) results in the following integrals:

$$G_2(s) = \frac{AB\omega}{2\left[1 + \exp(-2\pi/\omega)\right]} \left\{ \int_0^{\pi/\omega} \exp(-st)\sin 2\omega t dt - \int_{\pi/\omega}^{2\pi/\omega} \exp(-st)\sin 2\omega t dt \right\} \tag{100}$$

The integrals in Equation (100) may be evaluated by the expression (27)

$$\int \exp(-st)\sin 2\omega t dt = \frac{\exp(-st)(-s \sin 2\omega t - 2\omega \cos 2\omega t}{s^2 + 4\omega^2} \tag{101}$$

Performing the integrations in Equation (101) and substituting in the limits results in the following equation for the LaPlace transform of $g_2(t)$:

$$G_2(s) = \frac{2AB\omega^2}{2\left[1 - \exp(-2\pi s/\omega)\right]\left[s^2 + 4\omega^2\right]} \Big[- \exp(-\pi s/\omega)$$

$$+ 1 + \exp(-2\pi s/\omega) - \exp(-\pi s/\omega) \Big] \tag{102}$$

Equation (102) can be rearranged to give:

$$G_2(s) = \frac{2AB\omega^2\left[1 + 2\exp(-\pi s/\omega) + \exp(-2\pi s/\omega)\right]}{2\left[s^2 + 4\omega^2\right]\left[1 - \exp(-2\pi s/\omega)\right]} \tag{103}$$

which on further simplification yields:

$$G_2(s) = \left[\frac{AB\omega^2}{s^2 + 4\omega^2}\right]\left[\frac{1 - \exp(-\pi s/\omega)}{1 + \exp(-\pi s/\omega)}\right] \tag{104}$$

since (28):

$$\tanh \frac{\pi s}{2\omega} = \left[\frac{1 - \exp(-\pi s/\omega)}{1 + \exp(-\pi s/\omega)}\right] \tag{105}$$

Substituting Equation (105) into Equation (104) results in the following expression:

$$G_2(s) = \frac{AB\omega^2}{s^2 + 4\omega^2} \tanh \frac{\pi s}{2\omega} \tag{106}$$

The LaPlace transform of $g_1(t)$ is given by

$$G_1(s) = \frac{A\omega}{s^2 + \omega^2} \tag{107}$$

Since:

$$G(s) = G_1(s) + G_2(s) \tag{108}$$

the Laplace transform of the proportional plus magnitude-rate mode is given by the sum of Equation (106) and Equation (107), or:

$$G(s) = \frac{A\omega}{s^2 + \omega^2} + \frac{AB\omega^2}{s^2 + 4\omega^2}\tanh\frac{\pi s}{2\omega} \tag{109}$$

which after factoring out the common constant A results in the final form:

$$G(s) = A\left[\frac{\omega}{s^2 + \omega^2} + \frac{B\omega^2}{s^2 + 4\omega^2}\tanh\frac{\pi s}{2\omega}\right] \tag{110}$$

DERIVATION OF THE SECOND ORDER PROCESS TRANSFER FUNCTION

From Figure 21 if the gain G is large:

$$V_g = 0 \tag{111}$$

The current summation to junction J_b is:

$$\frac{V_a}{R_3} + V_o s C_2 = 0 \tag{112}$$

Solving Equation (112) for V_a:

$$V_a = - V_o s C_2 R_3 \tag{113}$$

The current summation to junction J_a is:

$$\frac{V_i - V_a}{R_1} - V_a s C_1 + \frac{V_o - V_a}{R_2} - \frac{V_a}{R_3} = 0 \tag{114}$$

Which after elimination of V_a results in:

$$\frac{V_o}{V_i} = - \frac{R_2}{R_1} \left[\frac{1}{C_1 C_2 R_2 R_3 s^2 + C_2(R_2 + R_3 + R_2 R_3/R_1)s + 1} \right] \tag{115}$$

APPENDIX III

Table 1. The second order process circuit constants and parameters.

Constants		Run Number 1	2
R_1	megohms	2.00	0.20
R_2	megohms	1.00	0.50
R_3	megohms	1.00	1.00
C_1	microfarads	4.00	2.00
C_2	microfarads	1.00	0.10
K	-	-0.50	-2.50
T	seconds	2.00	0.316
ω_n	radians/sec	0.500	3.160
f_n	cycles/sec	0.0796	0.503
ζ	-	0.625	0.633

Table 3. Experimental open loop frequency response data for the second order process of Run 1 with the proportional plus magnitude-rate controller forcing.

f	ω/ω_n	Magnitude Ratio	Phase Angle Degrees
0.03	0.377	1.000	-22.3
0.04	0.502	1.020	-20.7
0.05	0.628	1.030	-29.2
0.06	0.754	0.965	-44.0
0.07	0.879	0.901	-55.4
0.08	1.005	0.855	-62.5
0.09	1.130	0.737	-68.2
0.10	1.26	0.661	-74.8
0.12	1.51	0.526	-85.0
0.15	1.89	0.396	-91.1
0.20	2.51	0.275	-94.7
0.22	2.77	0.240	-98.1
0.26	3.27	0.184	-99.9
0.30	3.77	0.151	-102.0
0.34	4.27	0.123	-102.8
0.38	4.77	0.102	-105.1
0.42	5.27	0.088	-106.7
0.46	5.78	0.074	-109.0
0.50	6.78	0.065	-116.3
0.55	6.91	0.050	-155.2
0.60	7.54	0.023	-164.0

Table 4. Experimental open loop frequency response data for the second order process of Run 2 with the proportional plus magnitude-rate controller forcing.

f	ω/ω_n	Magnitude Ratio	Phase Angle Degrees
0.03	0.06	1.00	-4.9
0.08	0.13	1.02	-7.9
0.10	0.20	1.02	-11.3
0.15	0.28	1.02	-18.1
0.22	0.44	1.10	-26.1
0.26	0.52	1.06	-28.9
0.30	0.60	1.04	-32.0
0.38	0.72	1.00	-40.0
0.42	0.84	0.981	-47.5
0.50	1.00	0.910	-49.6
0.54	1.08	0.904	-53.8
0.58	1.16	0.862	-58.9
0.66	1.32	0.848	-68.0
0.70	1.40	0.825	-74.6
0.76	1.52	0.801	-86.0
0.82	1.64	0.704	-122.6
0.90	1.80	0.387	-127.1
0.98	1.96	0.312	-132.0
1.10	2.20	0.233	-135.2
1.20	2.40	0.198	-137.0

A NONLINEAR CONTROLLER FOR UNDERDAMPED SYSTEMS

by

JOSEPH C. WEBB

B. S., West Virginia University, 1959

AN ABSTRACT OF A THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Chemical Engineering

KANSAS STATE UNIVERSITY
OF AGRICULTURE AND APPLIED SCIENCE

1962

ABSTRACT

The effect of the proportional plus magnitude-rate controller response $A(e + B|e|de/dt)$ when included in a closed loop system containing an underdamped process has been determined on an analogue computer. In addition, the closed loop response of the underdamped process with the proportional mode only was obtained for comparison purposes.

The closed loop response curves have shown that the proportional plus magnitude-rate controller can increase the system response rate up to 60% over that of the proportional mode when used alone. The time required for the oscillations to die out was decreased by approximately 50% due to the action of the nonlinear term. The height of the first overshoot was decreased up to 25%. By the proper choice of the gain constant B the error could be made to approach its steady state value rapidly without any oscillations or error overshoot.

The magnitude-rate control mode has been shown to have a high gain for all errors departing from zero and a low gain for all errors approaching zero. The magnitude-rate mode has a leading phase angle of approximately 80°.