CERTAIN STEADY STATE AND TRANSIENT TIME PHENOMENA

IN HEAT CONDUCTION

by

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INTRODUCTION

The purpose of this study was to determine the solution of a number of problems concerning the temperature within certain homogeneous, isotropic solids when subjected to given boundary conditions.

Ten problems were grouped with respect to the coordinate geometry used in each: First, those for which Cartesian coordinates were desirable; second, those for which cylindrical coordinates were desirable; and third, those for which spherical coordinates were desirable.

DERIVATION AND TRANSFORMATIONS

Derivation of the Fourier Heat Equation

By the second law of thermodynamics, heat flows from a hot body to a cold body. The flux of heat, or the rate at which heat is transferred across any surface S at any point P per unit area per unit time, is equal to

$$-KD_n u.$$

The minus sign indicates that heat flows from a point at higher temperature to a point at a lower temperature; K is the thermal conductivity of the substance, and D_n indicates differentiation along the outward drawn normal to the surface S.

If a homogeneous, isotropic solid is heated in any way and allowed to cool, the temperature at any point P (x, y, z) is a finite, continuous, and single valued function of x, y, z, and t. The first and second partial derivatives will also be finite, continuous, and single valued.

Consider a Cartesian element of volume centered about the point P (x, y, z) with edges parallel to the coordinate axes and with dimensions 2dx, 2dy, and 2dz. Let f represent the flux of heat, and its components be f_x, f_y, and f_z. Let A and B be the faces perpendicular to the X axis. The amount of heat which flows in through A at (x - dx) is equal to

$$4dydz\left[f_x - D_xf_xdx + \frac{1}{2!}D_x^2f_x(dx)^2 - \cdots\cdots\right].$$

That which flows out through face B at (x + dx) is equal to

$$4dydz\left[f_x + D_xf_xdx + \frac{1}{2!}D_x^2f_x(dx)^2 + \cdots\cdots\right]$$

Infinitesimals of order higher than two are neglected. The net gain in heat through these two faces is the difference

$$-8dxdydzD_xf_x.$$

By analogy, expressions for heat gained through the pairs of faces parallel to the Y and Z axes are

$$-8dxdydzD_yf_y$$

and

$$-8dxdydzD_zf_z,$$

respectively.

The element gains heat at the rate of

$$8dxdydzcpD_tu,$$

where c is the specific heat of the substance and p is the density.

These two expressions for the gain per unit volume in heat in the element of volume are equal, whence

$$-(D_x f_x + D_y f_y + D_z f_z) = c\rho D_t u.$$

Since

$$f_x = -KD_x u,$$

$$f_y = -KD_y u,$$

$$f_z = -KD_z u, \text{ then}$$

$$D_t u = a^2 \left[D_x^2 u + D_y^2 u + D_t^2 u \right], \qquad \text{I}$$

where $a^2 = \frac{K}{c\rho}$, the diffusivity.

This equation expresses mathematically in Cartesian coordinates the flow of heat by conduction in an isotropic, homogeneous solid, and is called Fourier's equation of heat conduction after Fourier (1768-1830), who was the first investigator in this field.

Transformation of Fourier Heat Equation into Cylindrical Coordinates

Due to the geometry of the boundaries assumed in some problems, it was deemed advantageous to carry out the analysis in coordinate systems other than Cartesian.

The transformation from Cartesian coordinates to cylindrical coordinates is

$$x = r \cos \phi \qquad r = \sqrt{x^2 + y^2}$$

$$y = r \sin \phi \qquad \phi = \text{arc tan } y/x$$

$$z = z, \qquad z = z.$$

By partial differentiation,

$$D_x u = D_r u D_x r + D_\phi u D_x \phi,$$

$$D_x^2 u = D_r u D_x^2 r + D_r^2 u D_x r^2 + D_\phi u D_x^2 \phi + D_\phi^2 u D_x \phi^2,$$

and by analogy,

$$D_y^2 u = D_r u D_y^2 r + D_r^2 u D_y r^2 + D_\phi u D_y^2 \phi + D_\phi^2 u D_y \phi^2.$$

Since the coordinate z in Cartesian coordinates carries directly over into cylindrical coordinates,

$$D_z^2 u = D_z^2 u.$$

By differentiation,

$$D_x r = \frac{x}{\sqrt{x^2 + y^2}} = \cos \phi,$$

$$D_y r = \frac{y}{\sqrt{x^2 + y^2}} = \sin \phi,$$

$$D_x \phi = \frac{-y}{x^2 + y^2} = -\frac{\sin \phi}{r},$$

$$D_y \phi = \frac{x}{x^2 + y^2} = \frac{\cos \phi}{r},$$

$$D_x^2 r = \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}} = \frac{\sin^2 \phi}{r},$$

$$D_y^2 r = \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}} = \frac{\cos^2 \phi}{r},$$

$$D_x^2 \phi = \frac{2xy}{(x^2 + y^2)^2} = \frac{2\cos \phi \sin \phi}{r^2},$$

$$D_y^2 \phi = \frac{-2xy}{(x^2 + y^2)^2} = \frac{-2\cos \phi \sin \phi}{r^2}.$$

Substitution of these values in Fourier's equation transforms the heat equation into cylindrical coordinates,

$$D_t u = a^2\left[D_r^2 u + \frac{1}{r}D_r u + \frac{1}{r^2}D_\phi^2 u + D_z^2 u\right]. \qquad \text{II}$$

Transformation of Fourier Equation into Spherical Coordinates

The transformation is

$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$

$$y = r\sin\theta\sin\phi \qquad \theta = \text{arc tan}\,\frac{1}{z}\sqrt{x^2+y^2}$$

$$z = r\cos\theta, \qquad \phi = \text{arc tan}\,y/x.$$

We know that the following is true for any transformation:

$$D_x u = D_r u D_x r + D_\theta u D_x\theta + D_\phi u D_x\phi.$$

Applying this same formula to itself, we have

$$\begin{aligned} D_x^2 u = \Big[& D_r u D_x^2 r + D_x r D_r^2 u D_x r + D_{r\theta}^2 u D_x\theta + D_{r\phi}^2 u D_x\phi \\ & + D_\theta u D_x^2\theta + D_x\theta D_{r\theta}^2 u D_x r + D_\theta^2 u D_x\theta + D_{\theta\phi}^2 u D_x\phi \\ & + D_\phi u D_x^2\phi + D_x\phi D_{r\phi}^2 u D_x r + D_{\theta\phi}^2 u D_x\theta + D_\phi^2 u D_x\phi \Big]. \end{aligned}$$

By analogy for $D_y^2 u$ and $D_z^2 u$,

$$\begin{aligned} \frac{1}{a^2}D_t u = & \left[D_r u(D_x^2 r + D_y^2 r + D_z^2 r) + D_\theta u(D_x^2\theta + D_y^2\theta + D_z^2\theta) + D_\phi u(D_x^2\phi + D_y^2\phi + D_z^2\phi)\right] \\ & + \Big[D_r^2 u\left[(D_x r)^2 + (D_y r)^2 + (D_z r)^2\right] + D_\theta^2 u\left[(D_x\theta)^2 + (D_y\theta)^2 + (D_z\theta)^2\right] \\ & + D_\phi^2 u\left[(D_x\phi)^2 + (D_y\phi)^2 + (D_z\phi)^2\right]\Big] + \Big[2D_{r\theta}^2 u(D_x r D_x\theta + D_y r D_y\theta + D_z r D_z\theta) \\ & + 2D_{r\phi}^2 u(D_x r D_x\phi + D_y r D_y\phi + D_z r D_z\phi)\Big] + \left[2D_{\theta\phi}^2 u(D_x\theta D_x\phi + D_y\theta D_y\phi + D_z\theta D_z\phi)\right]. \end{aligned}$$

Upon differentiating the transformation the following relationships were found to be true:

$$(D_x r)^2 + (D_y r)^2 + (D_z r)^2 = 1,$$

$$D_x^2 r + D_y^2 r + D_z^2 r = \frac{2}{r},$$

$$(D_x \theta)^2 + (D_y \theta)^2 + (D_z \theta)^2 = \frac{1}{r^2},$$

$$D_x^2 \theta + D_y^2 \theta + D_z^2 \theta = \frac{\operatorname{Cot} \theta}{r^2},$$

$$(D_x \phi)^2 + (D_y \phi)^2 + (D_z \phi)^2 = \frac{1}{r^2 \sin^2 \theta},$$

$$D_x^2 \phi + D_y^2 \phi + D_z^2 \phi = 0,$$

$$D_x r D_x \theta + D_y r D_y \theta + D_z r D_z \theta = 0,$$

$$D_x r D_x \phi + D_y r D_y \phi + D_z r D_z \phi = 0,$$

$$D_x \theta D_x \phi + D_y \theta D_y \phi + D_z \theta D_z \phi = 0.$$

Substitution of these values in (A) yields

$$D_t u = \frac{a^2}{r^2}\left[D_r(r^2 D_r u) + \frac{1}{\sin \theta} D_\theta(\sin \theta \; D_\theta u) + \frac{1}{\sin^2 \theta} D_\phi^2 u\right]. \qquad \text{III}$$

SOLUTIONS OF PROBLEMS

Problem I

This problem found the temperature at any point within a rectangular bar of infinite extent after certain constant temperatures have been impressed upon the faces. The temperature was assumed to have reached steady state.

The boundary conditions for this solution, as given in the statement of the problem, are:

$$\begin{aligned} u &= T \text{ when } x = a, & &(a) & &(1)\\ u &= T \text{ when } y = b, & &(b)\\ u &= 0 \text{ when } x = 0, & &(c)\\ u &= 0 \text{ when } y = 0. & &(d) \end{aligned}$$

Since the temperature has reached the steady state, the change in temperature with respect to time will be equal to zero. The infinite bar has no end effects, so that the first and second derivatives of the temperature with respect to z will be equal to zero. Fourier's equation I then becomes

$$D_x^2u + D_y^2u = 0. \qquad (2)$$

To solve equation (2), assume $u = XY$ where

$$X = X(x),$$

$$Y = Y(y).$$

Making this substitution yields

$$YD_x^2X + XD_y^2Y = 0.$$

Divide by XY and transpose, whence

$$\frac{D_x^2X}{X} = -\frac{D_y^2Y}{Y}$$

Since the left member is a function of x alone and the right member is a function of y alone, each must be equal to some constant. Call this constant $-q^2$. Setting each side equal to this constant, two ordinary differential equations are found:

$$D_x^2X = -q^2X, \qquad (3)$$

$$D_y^2Y = q^2Y. \qquad (4)$$

These have the general solutions

$$X = A\sin qx + B\cos qx$$

and

$$Y = L\sinh qy + M\cosh qy,$$

whence

$$u = \left[A\sin qx + B\cos qx\right]\left[L\sinh qy + M\cosh qy\right]. \tag{5}$$

If each member is set equal to p^2 rather than $-q^2$ the general solutions are

$$X = N\sinh px + P\cosh px$$

and

$$Y = Q\sin py + R\cos py,$$

whence

$$u = \left[N\sinh px + P\cosh px\right]\left[Q\sin py + R\cos py\right]. \tag{6}$$

One such special form of (5) would be

$$u = A\sin qx \frac{\sinh qy}{\sinh qb}, \tag{7}$$

and a special form of (6) would be

$$u = B\sin py \frac{\sinh px}{\sinh pa}, \tag{8}$$

The sum of linear combinations of solutions (7) and (8) will also be a solution, since (2) is a linear partial differential equation, whence

$$u = \sum \left[A\sin qx \frac{\sinh qy}{\sinh qb} + B\sin py \frac{\sinh px}{\sinh pa}\right]. \tag{9}$$

For the conditions

$$x = 0$$

and

$$y = 0,$$

both terms of the solution vanish and boundary conditions (1,c) and (1,d) are satisfied. If the solution can be made to converge to the limit T for

$$x = a$$

and

$$y = b,$$

it will also satisfy (1,a) and (1,b), and will be the complete solution. If q and p are assigned the values $\frac{m\pi}{a}$, $\frac{n\pi}{b}$ respectively, solution (9) reduces to

$$u = \sum B_n \sin \frac{n\pi y}{b} \qquad (10)$$

when x = a, and to

$$u = \sum A_m \sin \frac{m\pi x}{a} \qquad (11)$$

when y = b. A_m and B_n are determined so that the series of sines (10) and (11) converge to the value T. The coefficients are determined by the usual Fourier series method.

$$A_m = \frac{2T}{a}\int_0^a \sin \frac{m\pi x}{a}\, dx = \frac{-2T}{m\pi}\left[1 - \cos m\pi\right]. \qquad (12)$$

A_m is therefore $\frac{-4T}{m\pi}$ for odd values of m and vanishes for even values of m. Likewise,

$$B_n = \frac{2T}{b}\int_0^b \sin \frac{n\pi y}{b}\, dy = \frac{-2T}{n\pi}\left[1 - \cos n\pi\right]. \qquad (13)$$

B_n is therefore $\frac{-4T}{n\pi}$ for odd values of n and vanishes for even values of n.

The final solution of this problem is

$$u = \frac{4T}{\pi}\left[\sum_{m=0}^{m=\infty}\left[\frac{1}{2m+1}\cdot\frac{\sinh\frac{(2m+1)\pi y}{a}}{\sinh\frac{(2m+1)\pi b}{a}}\sin\frac{(2m+1)\pi x}{a}\right] + \right.$$

$$\left.\sum_{n=0}^{n=\infty}\left[\frac{1}{2n+1}\cdot\frac{\sinh\frac{(2n+1)\pi x}{b}}{\sinh\frac{(2n+1)\pi a}{b}}\sin\frac{(2n+1)\pi y}{b}\right]\right]. \qquad (14)$$

This solution satisfies equation (2) and all the boundary conditions (1) and, therefore, yields valid results for the temperature u at any point within a rectangular bar of width a, thickness b, and infinite in length.

This solution reduces to

$$u = \frac{4T}{\pi}\sum_{k=1}^{k=\infty}\frac{1}{2k+1}\left[\frac{\sinh\,(2k+1)y}{\sinh\,(2k+1)\pi}\sin(2k+1)x + \right.$$

$$\left.\frac{\sinh\,(2k+1)x}{\sinh\,(2k+1)\pi}\sin\,(2k+1)y\right] \qquad (15)$$

for the special case when $a = b = \pi$.

Problem II

This problem found the temperature at any point within a right circular cylinder whose plane faces are kept at a constant temperature and whose convex surface is maintained at zero. The temperature was assumed to have reached steady state.

The boundary conditions for this solution, as given in the statement of the problem, are:

$$\begin{aligned} u &= T \text{ when } z = 0, & \text{(a)} \\ u &= T \text{ when } z = h, & \text{(b)} \\ u &= 0 \text{ when } r = a. & \text{(c)} \end{aligned} \qquad (16)$$

From the symmetry of the problem it is seen that u is independent of ϕ so that

$$D^2_{\phi}u = 0.$$

Since the temperature has reached steady state, u is also independent of time and

$$D_t u = 0.$$

The simplified equation II for this problem is

$$D^2_r u + \frac{1}{r}D_r u + D^2_z u = 0. \qquad (17)$$

Making the substitution

$$u = RZ,$$

where

$$R = R(r),$$

$$Z = Z(z),$$

equation (17) then becomes

$$\frac{D^2_r R}{R} + \frac{1}{r}\frac{D_r R}{R} = \frac{-D^2_z Z}{Z} = -k^2, \qquad (18)$$

by the same reasoning as was used in Problem I.

Equation (18) in turn may be broken up into the two ordinary differential equations;

$$D_r^2 R + \frac{1}{r} D_r R + k^2 R = 0, \tag{19}$$

$$D_z^2 Z - k^2 Z = 0. \tag{20}$$

Making the substitution,

$$x = kr,$$

transformed equation (19) into

$$D_x^2 R + \frac{1}{x} D_x R + R = 0. \tag{21}$$

This is Bessel's equation of order zero, which has the general solution

$$R = A\ J_o(kr) + B\ K_o(kr). \tag{22}$$

Equation (20) has the general solution

$$Z = P \sinh (kz) + Q \cosh (kz). \tag{23}$$

The complete general solution is a linear combination of products of (22) and (23),

$$u = \sum \left[A\ J_o(kr) + B\ K_o(kr)\right] \left[P \sinh (kz) + Q \cosh (kz)\right]. \tag{24}$$

From the nature of the problem, B in this solution must be equal to zero since K_o (kr) becomes infinite for $r = 0$.

P and Q were determined such that the second factor of (24) would be equal to unity for the values $z = 0$ and $z = h$. This form of (23) was found to be

$$\left[\frac{1 - \cosh kr}{\sinh kh} \sinh kz + \cosh kz\right]. \tag{25}$$

For these two conditions when $z = 0$ or $z = h$, the solution (24) becomes

$$u = \sum A\, J_o(kr). \tag{26}$$

From boundary conditions (16,a) and (16,c), the temperature for each of these two conditions must be T. The solution (26) was expanded into a series to converge to the value T as the number of terms was allowed to increase without bound. Condition (16,c) was satisfied if k was defined as follows:

Let $$k_m = \frac{p_m}{a},\ k_{m+1} = \frac{p_{m+1}}{a},\ \cdots\cdots,$$

where $$p_m,\ p_{m+1},\ p_{m+2},\ \cdots\cdots$$

are the mth, m+1st, m+2nd, ····· roots of the equation

$$J_o(p) = 0.$$

Then $$T = A_1 J_o(k_1 r) + A_2 J_o(k_2 r) + \cdots + A_m J_o(k_m r) + \cdots \tag{27}$$

satisfies condition (16,c), as each term vanishes when $r = a$. To determine the coefficients in the series (27), each side of the equation was multiplied by

$$r\, J_o(k_m r)\, dr,$$

and both sides integrated term by term, between the limits zero and a. The left member became

$$\int_0^a T\, r\, J_o(k_m r)\, dr. \tag{28}$$

Every term in the second member is equal to zero with the exception of

$$\int_{o}^{a} r \left[J_o(k_m r)\right]^2 dr.^{*}$$

Upon integration, this becomes

$$A_m \frac{a^2}{2} \left[\left[J_o(k_m a)\right]^2 + \left[J_1(k_m a)\right]^2 \right].^{**} \qquad (29)$$

But since $(k_m a)$ is a root of $J_o(x) = 0$, (29) reduces to

$$A_m \frac{a^2}{2} \left[J_1(k_m a)\right]^2. \qquad (30)$$

Upon integration, (28) becomes

$$\frac{aT}{k_m} J_1(k_m a).^{***} \qquad (31)$$

Equating (30) and (31) and solving for A_m gives

$$A_m = \frac{2T}{ak_m J_1(k_m a)}.$$

Combination of (25) and (27) gives the final form of the solution;

$$u = \sum_{m=1}^{m=\infty} \left[\frac{1-\cosh(k_m h)}{\sinh(k_m h)} \sinh(k_m z) + \cosh(k_m z) \right] \frac{2T}{ak_m J_1(k_m a)} J_o(k_m r). \quad (32)$$

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* Byerly Equation (2), Art. 125.

** Ibid Equation (10), Art. 122.

*** Ibid Equation (9), Art. 122.

Problem III

This problem found the temperature at any point within a hollow right circular cylinder of infinite length; the inner surface maintained at a constant temperature, the outer surface maintained at a different constant temperature. The temperature was assumed to have reached steady state.

The following boundary conditions must be satisfied:

$$u = T_a \text{ when } r = a, \qquad (a) \qquad (33)$$
$$u = T_b \text{ when } r = b. \qquad (b)$$

Since the cylinder is infinite, the temperature will not vary with the length coordinate, and the term of the Fourier heat equation II which involves the coordinate z will be zero. Also, since the temperatures are assumed to have reached the steady state the change in temperature with respect to time, $D_t u$, will be equal to zero and from the symmetry of the problem it is seen that u is independent of ϕ so that

$$D_\phi^2 u = 0.$$

The simplified equation for this problem is the ordinary differential equation

$$D_r^2 u + \frac{1}{r} D_r u = 0, \qquad (34)$$

in which u appears only in the derivatives.

The general solution of (34) is

$$u = C_1 \log r + C_2. \qquad (35)$$

From the boundary conditions (33,a) and (33,b), the values of the constants of integration were found to be:

$$C_1 = \frac{T_a - T_b}{\log \frac{a}{b}}$$

$$C_2 = T_a - \frac{T_a - T_b}{\log \frac{a}{b}} \log a.$$

Substituting these values in (35),

$$u = \frac{T_a - T_b}{\log \frac{a}{b}} \log r + \left[T - \frac{T_a - T_b}{\log \frac{a}{b}} \log a \right]. \qquad (36)$$

If the dimensions are special so that

$$a = e, \text{ and}$$
$$b = 1;$$

$$u = (T_a - T_b)\log r + T_b. \qquad (37)$$

Problem IV

This problem differs from Problem III in that the cylinder was of finite length, the plane surfaces at the ends maintained at zero. The other conditions remained identical. The temperature was assumed to have reached steady state.

The boundary conditions to be satisfied are:

$$\begin{array}{lll} u = 0 \text{ when } z = 0, & (a) & (38) \\ u = 0 \text{ when } z = b, & (b) & \\ u = T_c \text{ when } r = c, & (c) & \\ u = T_a \text{ when } r = a. & (d) & \end{array}$$

As in Problem II, equation (17) may be broken up into the two ordinary differential equations

$$D_r^2R + \frac{1}{r}D_r r - k^2R = 0, \tag{39}$$

$$D_z^2Z = -k^2Z. \tag{40}$$

These equations have the general solutions

$$R = P\,J_0(ikr) + Q\,K_0(ikr), \tag{41}$$

$$Z = A\sin(kz) + B\cos(kz), \tag{42}$$

whence

$$u = \left[A\sin(kz) + B\cos(kz)\right]\left[P\,J_0(ikr) + Q\,K_0(ikr)\right]. \tag{43}$$

Boundary condition (38,a) is satisfied if

$$B = 0.$$

Boundary condition (38,b) is satisfied if k is defined as follows:

$$k = \frac{m\pi}{b}.$$

The following relations must be true if conditions (38,c) and (38,d) are to be satisfied:

$$P_mJ_0\left(\frac{im\pi a}{b}\right) + Q_mK_0\left(\frac{im\pi a}{b}\right) = 1, \tag{44}$$

$$P_mJ_0\left(\frac{im\pi c}{b}\right) + Q_mK_0\left(\frac{im\pi c}{b}\right) = \frac{T_c}{T_a}, \tag{45}$$

whence

$$P_m = \frac{1}{T_a}\left[\frac{T_aK_0\left[\frac{(2m+1)\pi ci}{b}\right] - T_cK_0\left[\frac{(2m+1)\pi ai}{b}\right]}{J_0\left[\frac{(2m+1)\pi ai}{b}\right]K_0\left[\frac{(2m+1)\pi ci}{b}\right] - J_0\left[\frac{(2m+1)\pi ci}{b}\right]K_0\left[\frac{(2m+1)\pi ai}{b}\right]}\right] \tag{46}$$

and

$$Q_m = \frac{1}{T_a}\left[\frac{T_c J_o\left[\frac{(2m+1)\pi ai}{b}\right] - T_a J_o\left[\frac{(2m+1)\pi ci}{b}\right]}{J_o\left[\frac{(2m+1)\pi ai}{b}\right]K_o\left[\frac{(2m+1)\pi ci}{b}\right] - J_o\left[\frac{(2m+1)\pi ci}{b}\right]K_o\left[\frac{(2m+1)\pi ai}{b}\right]}\right]. \quad (47)$$

From (36,a), when $r = a$, (43) becomes

$$T_a = \sum_{m=0}^{m=\infty} A_m \sin\left(\frac{mz}{b}\right). \quad (48)$$

A_m is a Fourier coefficient over the interval $0 = x = b$,

$$A_m = \frac{2T_a}{b}\int_0^b \sin\left(\frac{mz}{b}\right) dz, \quad (49)$$

whence

$$\begin{aligned} A_m &= \frac{4T_a}{m} \qquad \text{when } m \text{ is odd,} \\ A_m &= 0 \qquad \text{when } m \text{ is even.} \end{aligned} \quad (50)$$

By substitution (43) becomes

$$u = \sum_{m=0}^{m=\infty}\left[\frac{4T_a}{m}\sin\left[\frac{(2m+1)\ z}{b}\right]\left[P_m J_o\left[\frac{(2m+1)\ ri}{b}\right] + Q_m K_o\left[\frac{(2m+1)\ ri}{b}\right]\right]\right], \quad (51)$$

where P_m and Q_m are defined in (46) and (47).

Problem V

This problem found the temperature at any point within a solid sphere subjected to the following conditions: One half the surface maintained at a constant temperature (T), the other half maintained at a constant temperature (-T), and the great circle dividing the two surfaces at temperature zero. The temperature was assumed to have reached steady state.

The following are the boundary conditions:

$$u = T \text{ when } 0 \leq \theta < \frac{\pi}{2} \text{ and } r = a, \quad \text{(a)} \qquad (52)$$

$$u = 0 \text{ when } \theta = \frac{\pi}{2} \text{ and } r = a, \quad \text{(b)}$$

$$u = -T \text{ when } \frac{\pi}{2} < \theta \leq \pi \text{ and } r = a. \quad \text{(c)}$$

Since the temperature is assumed to have reached steady state, the temperature u is independent of the time and, likewise, the change of temperature with respect to time, $D_t u$, must be zero in equation III.

Due to the circular symmetry of the problem, the temperature is seen to be independent of ϕ and, therefore,

$$D_\phi u = 0.$$

The simplified equation for this problem is then

$$D_r(r^2 D_r u) + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta u) = 0. \qquad (53)$$

Assume u of the form

$$u = r^n F(\theta); \qquad (54)$$

whence (53) becomes

$$m(m+1)r^m P(\theta) + \frac{r^m}{\sin\theta} D_\theta\left[\sin\theta \, D_\theta P(\theta)\right] = 0. \tag{55}$$

Divide by r^m

$$m(m+1)P(\theta) + \frac{1}{\sin\theta} D_\theta\left[(\sin\theta \, D_\theta P(\theta)\right] = 0. \tag{56}$$

This is Legendre's equation and has the general solution

$$P(\theta) = P_m(\cos\theta)^*,$$

a surface zonal harmonic of order m, if m is a positive integer. Substitution in (54) gives the general solution of (53)

$$u = A_m r^m P_m(\cos\theta). \tag{57}$$

To satisfy conditions (52,a) and (52,c), (57) must reduce to

$$T(u) = A_0 P_0(\cos\theta) + A_1 a P_1(\cos\theta) + \cdots A_m a^m P_m(\cos\theta) + \cdots, \tag{58}$$

where

$$T(u) = T \text{ when } 0 \leq \theta < \frac{\pi}{2},$$

$$T(u) = -T \text{ when } \frac{\pi}{2} < \theta \leq \pi.$$

The A_m in the solution (57) was obtained by multiplying both members of (58) by

$$P_m(\cos\theta)\, d(\cos\theta)$$

and integrating term by term between the limits zero and π.

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* Byerly, Chapter I, Art. 9.

The left member became

$$T\int_{0}^{\frac{\pi}{2}} P_m(\cos\theta)\,d(\cos\theta) - T\int_{\frac{\pi}{2}}^{\pi} P_m(\cos\theta)\,d(\cos\theta), \quad (59)$$

while the right member was term by term identically equal to zero with the exception of the term

$$A_m a^m \int_{0}^{\pi} \Big|P_m(\cos\theta)\Big|^2 d(\cos\theta).^{*} \qquad (60)$$

Equation (59) vanished for m an even integer with the exception of m equal to zero. For m an odd integer, (59) integrated into

$$2T\,(-1)^{\frac{m+1}{2}} \frac{1}{m(m+1)} \frac{3\cdot5\cdot7\ \cdots\cdots m}{2\cdot4\cdot6\ \cdots(m-1)}.^{**} \qquad (61)$$

Equation (60) integrated into

$$\frac{A_m a^m}{2m+1}.^{***} \qquad (62)$$

Equating (61) and (62), and solving for A_m gave

$$A_m = \frac{2\cdot T\cdot(-1)^{\frac{m+1}{2}}\ 3\cdot5\cdot7\ \cdots\cdots m}{a^m\, m(m+1)\ 2\cdot4\cdot6\cdots\cdots(m-1)}, \qquad (63)$$

remembering that m must be an odd integer.

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* Byerly, Art. (91), Ex. 2.

** Ibid, Art. (91), Ex. 1.

*** Ibid, Art. (91), Ex. 3.

Substitution of this value for A_m and the quantity

$$2n+1$$

for m in (57) and (63) gave the solution

$$u = \sum_{n=0}^{n=\infty} \frac{T\,(-1)^{n+1}\, 3\cdot 5\cdot 7 \cdots (2n-1)}{2a^{2n+1}\,(n+2)\, n!}\, r^{2n+1} P_{2n+1}(\cos\theta). \qquad (64)$$

Problem VI

This problem found the temperature at any point within a solid hemisphere; the convex surface maintained at a constant temperature, the plane surface maintained at zero. The temperature was assumed to have reached steady state.

Since the temperature is assumed to have reached steady state the temperature u is independent of the time and likewise the change of temperature with respect to time, $D_t u$, must be zero in the equation.

Also, due to the symmetry of the problem the temperature is independent of ϕ and, therefore,

$$D_\phi u = 0.$$

The simplified equation is then

$$D_r(r^2 D_r u) + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta u) = 0. \qquad (65)$$

Particular solutions of this equation must be found which satisfy the following boundary conditions,

$$u = o \text{ when } \theta = \frac{\pi}{2}. \qquad (a) \qquad\qquad (65)$$

$$u = T \text{ when } r = a. \qquad (b)$$

In problem V, equation (52) was found to have the particular solution

$$u = A_m r^m P_m(\cos \theta). \qquad (57)$$

Condition (62,a) is satisfied by a solution of type (63) since

$$P_{2m+1}(\cos \theta) = 0 \text{ when } \theta = \frac{\pi}{2}$$

for all even values of m. Condition (65,b) is satisfied by expanding (57) into a series which converges to the limit T when the number of terms is allowed to increase without bound. Such a series was exhibited in problem V, equation (64). The solution of this problem then is the same as that found for problem V:

$$u = \sum_{n=o}^{n=\infty} \frac{T\,(-1)^{n+1}\; 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2a^{2n+1}\;(n+2)\; n!}\, r^{2n+1} P_{2n+1}(\cos \theta). \qquad (64)$$

Problem VII

This problem found the transient temperature at any point within a solid hemisphere originally at zero with the boundary conditions impressed; the convex surface maintained at a constant temperature, the plane surface maintained at zero.

The following are the boundary conditions impressed:

$$u = 0 \text{ when } t = 0, \qquad (a) \qquad (66)$$

$$u = 0 \text{ when } \theta = \frac{\pi}{2}, \qquad (b)$$

$$u = T_c \text{ when } r = c. \qquad (c)$$

Equation III applies to this problem but due to the symmetry of the problem

$$D_\phi u = 0.$$

The simplified equation is

$$D_t u = \frac{a^2}{r^2}\left[D_r(r^2 D_r u) + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta u)\right]. \qquad (67)$$

Assume u of the form

$$u = T_c - V. \qquad (68)$$

The variable portion of the solution must be subject to the following conditions in order that the final solution shall satisfy the original boundary conditions of the problem:

$$V = T_c \text{ when } t = 0, \qquad (a) \qquad (69)$$

$$V = 0 \text{ when } r = c, \qquad (b)$$

$$V = T_c \text{ when } \theta = \frac{\pi}{2}, \qquad (c)$$

$$V = 0 \text{ when } t = \infty. \qquad (d)$$

Assume

$$V = R \cdot Q \cdot T \qquad (70)$$

where

$$R = R(r),$$

$$Q = Q(\theta),$$

$$T = T(t).$$

Equation (67) may be broken up into three ordinary differential equations:

$$D_t T = -q^2 a^2 T, \tag{71}$$

$$D_r^2 R + \frac{2}{r} D_r R + \left[q^2 - \frac{m(m+1)}{r^2} \right] R = 0, \tag{72}$$

and

$$\frac{1}{\sin\theta} D_\theta (\sin\theta \; D_\theta Q) - m(m+1)Q = 0. \tag{73}$$

Equation (71) has the particular solution

$$T = e^{-q^2 a^2 t}. \tag{74}$$

Making the substitutions

$$x = qr, \text{ and } Z = R(qr)^{\frac{1}{2}},$$

equation (72) became

$$D_x^2 Z + \frac{1}{x} D_x Z + \left[1 - \frac{(m+\frac{1}{2})^2}{x^2} \right] Z = 0. \tag{75}$$

This is Bessell's equation of order $(m+\frac{1}{2})$ and the general solution is

$$Z = AJ_{m+\frac{1}{2}}(x) + BK_{m+\frac{1}{2}}(x). \tag{76}$$

The constant B must be identically zero since $K_{m+\frac{1}{2}}(x)$ becomes infinite for $x = 0$, which is inconsistent with physical experience.

From (76),

$$R = A_k (qr)^{-\frac{1}{2}} J_{m+\frac{1}{2}}(qr). \tag{77}$$

Equation (73) is Legendre's equation and has the general solution

$$Q = B_m P_m(\cos\theta), \tag{78}$$

whence

$$V = \frac{e^{-q^2 a^2 t} B_m P_m(\cos\theta) \; A_k J_{m+\frac{1}{2}} (q_k r)}{(qr)^{\frac{1}{2}}}. \tag{79}$$

In order to satisfy (69,c) n must be limited to the even values and B_n must have the value

$$B_{2n} = \frac{1}{P_{2n}(0)},$$

in order that (78) is equal to unity and it remains only to expand (77) into a series having as a limit T_0 as the number of terms is increased without bound. Such a series is

$$T_0 = \frac{A_0}{(q_0 r)^{\frac{1}{2}}} J_{2n+\frac{1}{2}}(q_0 r) + \frac{A_1}{(q_1 r)^{\frac{1}{2}}} J_{2n+\frac{1}{2}}(q_1 r) +$$

$$\ldots\ldots \frac{A_k}{(q_k r)^{\frac{1}{2}}} J_{2n+\frac{1}{2}}(q_k r) + \ldots. \qquad (80)$$

To determine the A_k's, multiply both members of (80) by

$$(q_k r)^{\frac{3}{2}} J_{2n+\frac{1}{2}}(q_k r)dr \qquad (81)$$

and integrate term by term between the limits zero and a. All terms in the right member vanish with the exception of the kth, whence*

$$T_0 \int_0^a (q_k r)^{\frac{3}{2}} J_{2n+\frac{1}{2}}(q_k r)dr = A_k \int_0^a q_k r \left[J_{2n+\frac{1}{2}}(q_k r)\right]^2 dr. \qquad (82,a)$$

Integrating and solving for A_k gives

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* Byerly, page 233, Ex. 1.

$$A_k = \frac{2T_c (q_k)^{\frac{3}{2}} \int_0^c r^{\frac{3}{2}} J_{2n+\frac{1}{2}}(q_k r)}{c^2 \left[J_{2n+\frac{3}{2}}(q_k c) \right]^2} . * \qquad (82,b)$$

The final solution of the problem is

$$u = T_c - \sum_{n=0}^{n=\infty} \frac{P_{2n}(\cos\theta)}{P_{2n}(0)} \sum_{k=0}^{k=\infty} e^{-q_k^2 a^2 t} A_k (q_k r)^{-\frac{1}{2}} J_{2n+\frac{1}{2}}(q_k r) \quad (83)$$

where A_k is defined in (82).

Problem VIII

This problem found the temperature at any point within the solid portion of a hollow sphere; the inner surface maintained at zero, the outer surface maintained at a constant temperature. The temperature was assumed to have reached steady state.

The boundary conditions were:

$$u = 0 \text{ when } r = c, \qquad (a) \qquad\qquad (84)$$

$$u = T \text{ when } r = d. \qquad (b)$$

It will be seen from the symmetry of the problem that the temperature will be independent of both θ and ϕ and, therefore,

$$D_\theta u = 0 \text{ and } D_\phi u = 0.$$

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*Byerly, page 233, Ex. 2.

Since the temperature was assumed to have reached steady state, the temperature will also be independent of the time t, and

$$D_t u = 0.$$

The simplified Fourier equation for this problem is then

$$D_r(r^2 D_r u) = 0, \tag{85}$$

or

$$2rD_r u + r^2 D_r^2 u = 0, \tag{86}$$

whose general solution is

$$u = \frac{-C_1}{r} + C_2. \tag{87}$$

In order to satisfy condition (84,a) the following relation between the constants of integration must be true:

$$\frac{C_1}{c} = C_2. \tag{88}$$

In order to satisfy (84,b)

$$\frac{C_1}{d} + T = C_2; \tag{89}$$

whence,

$$C_1 = \frac{c\ d\ T}{(d-c)}, \tag{90}$$

$$C_2 = \frac{dT}{(d-c)}. \tag{91}$$

The final solution is

$$u = -\frac{c\ d\ T}{r(d-c)} + \frac{dT}{(d-c)}. \tag{92}$$

Problem IX

This problem found the temperature at any point within a hollow hemisphere; the outer surface maintained at a constant temperature, the plane surface and the inner surface maintained at zero. The temperature was assumed to have reached steady state.

The boundary conditions are:

$$u = 0 \text{ when } r = a, \qquad (a) \qquad\qquad (93)$$

$$u = 0 \text{ when } \theta = \frac{\pi}{2}, \qquad (b)$$

$$u = T \text{ when } r = A. \qquad (c)$$

Since the temperature is assumed to have reached steady state,

$$D_t u = 0.$$

Also, due to the symmetry of the problem,

$$D_\phi u = 0.$$

The simplified Fourier equation,

$$D_r(r^2 D_r u) + \frac{1}{\sin\theta} D_\theta(\sin\theta \, D_\theta u) = 0, \qquad (53)$$

has been shown in Problem VI to have the particular solution

$$u = A_m r^m P_m(\cos\theta). \qquad (94)$$

It may, in like manner, be readily shown to have the particular solution

$$u = B_m r^{-(m+1)} P_m(\cos\theta).^* \qquad (95)$$

* Byerly, page 17.

A sum of linear combinations of these two solutions will also be a solution since (83) is a linear partial differential equation. Such a combination is

$$u = B_{2m+1} \frac{\left[\dfrac{a^{2m+2}}{r^{2m+2}} - \dfrac{r^{2m+1}}{a^{2m+1}}\right]}{\left[\dfrac{a^{2m+2}}{A^{2m+2}} - \dfrac{A^{2m+1}}{a^{2m+1}}\right]} P_{2m+1}(\cos \theta), \qquad (96)$$

which satisfied conditions (93), provided

$$T = B_{2m+1} P_{2m+1}(\cos \theta). \qquad (97)$$

To determine the B's, multiply both sides by

$$P_{2m+1}(\cos \theta)\, d(\cos \theta)$$

and integrate between zero and $\frac{\pi}{2}$. Since surface zonal harmonics are orthogonal functions in the interval $0 \leq \theta \leq \frac{\pi}{2}$,

$$\int_{\frac{\pi}{2}}^{0} P_m(\cos \theta)\, P_n(\cos \theta)\, d(\cos \theta) = 0,^{*}$$

m and n both odd or both even,

$$\int_{\frac{\pi}{2}}^{0} \left[P_m(\cos \theta)\right]^2 d(\cos \theta) = \frac{1}{2m+1}.^{**}$$

$$\int_{\frac{\pi}{2}}^{0} P_m(\cos \theta)\, d(\cos \theta) = \frac{1}{m(m+1)} \frac{3 \cdot 5 \cdot 7 \cdots\cdots m}{2 \cdot 4 \cdot 6 \cdots\cdots (m-1)}.^{***}$$

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* Byerly, page 172, Ex. 2.
** Ibid, page 72, Ex. 3.
*** Ibid, page 72, Ex. 1.

After integrating, (97) becomes

$$T\left[\frac{1}{(2m+2)}\cdot\frac{3\cdot5\cdot7\cdots\cdots(2m-1)}{2\cdot4\cdot6\cdots\cdots(2m)}\right]=B_{2m+1}\,\frac{1}{4m+3}. \qquad (98)$$

Solving for B_{2m+1} and substituting in (97) gives the final solution

$$u=\sum_{m=o}^{m=\infty}\left[\frac{T(4m+3)\;3\cdot5\cdot7\cdots(2m-1)}{4(m+1)\;(m!)}\;\frac{\left[\frac{a^{2m+2}}{r^{2m+2}}-\frac{r^{2m+1}}{a^{2m+1}}\right]}{\left[\frac{a^{2m+2}}{A^{2m+2}}-\frac{A^{2m+1}}{a^{2m+1}}\right]}\;P_{2m+1}(\cos\theta)\right]. \qquad (99)$$

Problem X

This problem found the transient temperature at any point within the solid of the preceding problem, the initial temperature being zero throughout and the same surface temperatures applied.

The boundary conditions are:

$u = 0$ when $t = 0$, (a) (100)

$u = 0$ when $r = b$, (b)

$u = 0$ when $\theta = \frac{\pi}{2}$ (c)

$u = T_c$ when $r = c$. (d)

Due to the symmetry of the problem, the temperature is independent of the angle ϕ and, therefore,

$$D_\phi u = 0.$$

Equation III with this modification is

$$D_t u = \frac{a^2}{r^2}\left[D_r(r^2 D_r u) + \frac{1}{\sin\theta}\,D_\theta(\sin\theta\, D_\theta u)\right]. \qquad (67)$$

This equation was shown in Problem VII to have the solution

$$u = T - V, \qquad (101)$$

where V is of the form

$$V = \frac{B_n P_n(\cos\theta)\, e^{-q^2 a^2 t} A_k J_{n+\frac{1}{2}}(q_k r)}{(q_k r)^{\frac{1}{2}}}. \qquad (79)$$

The conditions V must satisfy, in order that (101) shall satisfy conditions (100), are:

$$V = T_0 \text{ when } t = 0, \qquad (a) \qquad (102)$$

$$V = T_s \text{ when } r = b, \qquad (b)$$

$$V = T_s \text{ when } \theta = \frac{\pi}{2} \qquad (c)$$

$$V = 0 \text{ when } r = o. \qquad (d)$$

In Problem VII, it was found that the following solution of (67) satisfied conditions (100,a), (100,c), and (100,d). Upon examination it will be seen that it also satisfies (100,b) and is, therefore, the solution of this problem.

$$u = T - \sum_{n=o}^{n=\infty} \frac{P_{2n}(\cos\theta)}{P_{2n}(0)} \sum_{k=o}^{k=\infty} e^{-q_k^2 a^2 t} A_k(q_k r) \quad J_{2n+\frac{1}{2}}(q_k r) \qquad (83)$$

where A_k is defined in (82).

CONCLUSION

The numerical value of the temperature in any one of the above problems may be computed by the use of the general solutions given for each. Furthermore, it may be established that these solutions are the equivalent of any solution which might be obtained by another method and, therefore, are the unique solutions for the temperature in each particular case.

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LITERATURE CITED

Byerly, William E.

Fourier's Series and Spherical Harmonics. Boston. Ginn and Co. 287 p. 1893.

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