A PROBLEM IN DEPLETED FOURIER SERIES

by

EDISON GHEER

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INTRODUCTION

If a series \( \sum_{n=1}^{\infty} A_n \cos nx \) has all the terms present for \( n \) ranging from one to infinity, it is called a complete Fourier Cosine Series. If all terms divisible by \( p_1 \) are missing, it is called a depleted series; if all terms divisible by \( p_1 \) and \( p_2 \) are missing, it is called a doubly depleted series, etc. If \( p \) is the product of the \( k \) prime numbers \( p_1, p_2, \ldots, p_k \), we say the series is depleted by \( p \).

The problem under consideration is to determine the function that can be expressed by \( \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \) when this series has been depleted by \( p \).
CONVERGENCE OF \( \sum (\cos nx)/n^2 \)

A standard notation for an infinite series is
\[ u_1 + u_2 + u_3 + \ldots = \sum_{n=1}^{\infty} u_n = \sum u_n. \]
The nth partial sum of the series is
\[ S_n = u_1 + u_2 + u_3 + \ldots + u_n. \]
The sum of an infinite series is defined as the limit, as \( n \) increases indefinitely, of the sum of the first \( n \) terms:
\[ S = \lim_{n \to \infty} S_n; \]
provided the limit exists.

If \( \sum u_n \) has a sum \( S \), i.e. if \( S_n \) approaches a limit when \( n \) increases, the series is said to be convergent, or to converge to the value \( S \); if the limit does not exist, the series is divergent.

A series may diverge because \( S_n \) increases indefinitely as \( n \) increases; or it may diverge because \( S_n \) increases and decreases alternately, or oscillates, without approaching any limit. In the latter case the series is called oscillatory.

A necessary condition for convergence is that the general term approach zero as its limit;
\[ \lim_{n \to \infty} u_n = 0. \]
If in $\sum u_n$, $u_n$ is a function of $n$, we have the Integral Test for convergence or divergence of the series:

If the function $f(n)$ is defined not only for positive integral values, but for all positive values of $n$, and if $f(n)$ never increases with $n$, then the series $\sum u_n$ converges or diverges according as the integral $\int f(n)dn$ does or does not exist.

If $\sum u_n$ be a series of positive terms to be tested then by the Comparison Test:

(a) If a series $\sum a_n$ of positive terms, known to be convergent, can be found such that $u_n \leq a_n$, the series to be tested is convergent.

(b) If a series $\sum b_n$ of positive terms, known to be divergent, can be found such that $u_n \geq b_n$, the series to be tested is divergent.

The p-series, $\sum 1/n^p$, is convergent for $p > 1$ and divergent for $p \leq 1$. This can be shown by use of the Integral Test in the following way:

$$\frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} + \ldots = \sum \frac{1}{n^p}.$$  

The general term is $1/n^p$:

$$\int_{dn} \frac{n^p}{dn} = \int n^{-p}dn = n^{1-p}/(1-p)$$

Since this is a finite result for $p > 1$, the series is convergent. For $p \leq 1$, the integral fails to exist; therefore it is divergent.
To test the series

$$(\cos x)/1^2 + (\cos 2x)/2^2 + \ldots + (\cos nx)/n^2 + \ldots$$

compare with the series

$$1/1^2 + 1/2^2 + 1/3^2 + \ldots + 1/n^2 + \ldots$$

which is proved above to be convergent. Since

$$(\cos nx)/n^2 \leq 1/n^2$$

for all values of $n$, the series $\sum (\cos nx)/n^2$ converges.

If each term of the series is a function of a real variable $x$ for a closed interval $a \leq x \leq b$, we can write the series

$$u_1(x) + u_2(x) + u_3(x) + \ldots = \sum u_n(x);$$

and its nth partial sum is $S_n(x)$.

The series $\sum u_n(x)$ is uniformly convergent over the interval $(a,b)$ if there is a convergent series of positive constant terms, $\sum a_n$ say, such that $|u_n(x)| \leq a_n$ for all values of $n$ and $x$. Therefore the series

$$\sum (\cos nx)/n^2 = (\cos x)/1^2 + (\cos 2x)/2^2 + \ldots$$

is uniformly convergent over any interval.
The summation of the complete series
\[ \sum \frac{\cos nx}{n^2} = \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots \]
can be attained by expanding each term around \( x = \pi \) by means of Taylor's Series and summing the double series thus formed. Taylor's Series gives us:
\[ f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2! + \ldots, \]
where \( f(x) \) is expanded in powers of \( (x-a) \) in the vicinity of \( x = a \).

To find the sum of \( \sum \frac{\cos nx}{n^2} \) over the interval \( 0 \leq x \leq 2\pi \), we shall choose the mid-point \( x = \pi \) around which to expand the function.
\[
\begin{align*}
  f(x) &= \frac{\cos nx}{n^2} & f(\pi) &= (-1)/n^2 \\  f'(x) &= -(\sin nx)/n & f'(\pi) &= 0 \\  f''(x) &= -\cos nx & f''(\pi) &= (-1)^{n-1} \\  f'''(x) &= n \sin nx & f'''(\pi) &= 0 \\  f^\text{IV}(x) &= n^2 \cos nx & f^\text{IV}(\pi) &= (-1)^n/n^2 \\  f^\text{V}(x) &= -n^3 \sin nx & f^\text{V}(\pi) &= 0 \\
  & \vdots & & \vdots 
\end{align*}
\]
From the above we find
\[ \frac{\cos nx}{n^2} = (-1)^n/n^2 + (-1)^{n-1}(x-\pi)^2/2! + (-1)^n n^2 (x-\pi)^4/4! + \ldots; \]
hence

\[ \sum \left( \frac{\cos nx}{n^2} \right) = \sum \frac{(-1)^n}{n^2} + \frac{(x-\pi)^2}{2} \sum \frac{(-1)^{n-1}}{n^2} \]

\[ + \frac{(x-\pi)^4}{4} \sum \frac{(-1)^n}{n^2} + \frac{(x-\pi)^6}{6} \sum \frac{(-1)^{n-1}n^4}{n^2} + \ldots. \]

Therefore \( \sum \left( \frac{\cos nx}{n^2} \right) \) expands into an infinite set of infinite series commonly called a double series.

\[ \sum \left( \frac{\cos nx}{n^2} \right) = \left[ \sum \frac{-1}{1^2} + \sum \frac{1}{2^2} - \sum \frac{1}{3^2} + \sum \frac{1}{4^2} - \sum \frac{1}{5^2} + \ldots \right] \]

\[ + \frac{(x-\pi)^2}{2} \left[ \sum 1 - \sum 1 - \sum 1 - \sum 1 + \ldots \right] \]

\[ + \frac{(x-\pi)^4}{4} \left[ \sum -1 + \sum 4 - \sum 9 + \sum 16 - \sum 25 + \ldots \right] \]

\[ + \frac{(x-\pi)^6}{6} \left[ \sum 1 - \sum 16 + \sum 81 - \sum 256 + \ldots \right] \]

\[ \vdots \]

\[ + \frac{(-1)^3(x-\pi)^2(2s+1)}{(2s+2)!} \left[ \sum \frac{1-2s+3s^2}{s^2} - \ldots \right] \]

\[ \vdots \]

where \( s \) is equal to 0, 1, 2, 3, \ldots.

By the formula \( B_r = \frac{(2r)!}{(2^r-1)\pi^{2r}} \sum 1/n^{2r} \), we can find the \( \sum 1/n^2 \) by letting \( r = 1 \) and knowing that \( B_1 = 1/6 \), where \( B_1 \) is the first Bernoulli number.

\[ 1/6 = 1/\pi^2 \sum 1/n^2 \text{, or } \sum 1/n^2 = \pi^2/6. \]

Since the series

\[ -1 + \frac{1}{1^4} - \frac{1}{1^3} + \frac{1}{1^6} - \frac{1}{1^25} + \ldots = \sum \frac{-1}{n^2} + \sum \frac{2}{2n^2}, \]

then

\[ \sum \frac{(-1)}{n^2} = -\sum \frac{1}{n^2} + \frac{1}{2} \sum \frac{1}{n^2} = -\pi^2/6 + \pi^2/12 = -\pi^2/12. \]

Some divergent series are summable. By letting

\[ e^{-x}(u_0 + u_1x + u_2x^2/2! + u_3x^3/3! + \ldots) = e^{-x}u(x) \text{ and assum-} \]

\[ \text{See } \text{Bromwich, Theory of Infinite Series, Art. 93.} \]
ing that the coefficients $u_n$ are such that the series $u(x)$ converges for all values of $x$, we may give the following definition for a summable divergent series:

Provided that the integral $\int e^{-x}u(x)dx$ is convergent, we may agree to associate its value with the series $\sum u_n$, if this series is not convergent; this integral may then be called the "sum" of the series; and the series may be called summable. The sum may be denoted by the symbol $\mathcal{S}_u^\infty$.

This definition is due to Borel and is regarded as the fundamental definition.

Further, if $C$ is any factor independent of $n$,

$$\mathcal{S}_{Cu_n}^\infty = C \mathcal{S}_u^\infty.$$

Now, in the series $1 = 1 + 1 - 1 + 1 - \ldots$, $u(x) = 1 - x + x^2/2! - x^3/3! + \ldots = e^{-x}$, and so

$$\int e^{-x}u(x)dx = \int e^{-2x}dx = 1/2.$$

By letting

$$C = 1 + \cos \theta + \cos 2\theta + \cos 3\theta + \ldots$$

and

$$S = 0 + \sin \theta + \sin 2\theta + \sin 3\theta + \ldots,$$

we obtain $C + iS = 1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots$.

*See Bromwich, Theory of Infinite Series, Art. 102.*
from which we see that the associated function is
\[ u(x) = 1 + xe^{i\theta} + x^2e^{2i\theta}/2! + \ldots = e^{x^2i\theta}, \]
or
\[ u(x) = ex(\cos \theta + is\sin \theta). \]

Hence, provided that \( \theta \) is not zero or a multiple of \( 2\pi \), we find the sum
\[
\int_0^\infty e^{-xu(x)}dx = \int_0^\infty e^{-x(1-\cos \theta-is\sin \theta)}dx = \frac{1}{1-\cos \theta-is\sin \theta} = \frac{1}{2(1 + is\cot \theta/2)}.
\]
Therefore, the real part of
\[
\int_0^\infty e^{-xu(x)}dx = \frac{1}{2}e^{x^2i\theta}l^\frac{1}{2}(1^2 + l^2) = \frac{1}{2}e^{x^2i\theta}l^\frac{1}{2}.
\]
Hence we find
\[
\oint_{n=0}^\infty n^{2s}\cos n\theta = 0, \text{ and } \oint_{n=0}^\infty n^{2s-1}\sin n\theta = 0.
\]
Taking \( \theta = \pi \) in the first equation, we find the result:
\[
12s - 22s + 32s - 42s + \ldots = 0.
\]
Since the series \( \sum (\cos nx)/n^2 \) is uniformly convergent for all values of \( x \), we shall expect, as is the case, that each series in the double series is summable. Therefore
\[
\sum (\cos nx)/n^2 = -\pi^2/12 + (x-\pi)^2/(2\cdot 2) + (x-\pi)^4/4!(0) + \ldots
\]
where \( 0 \leq x \leq 2\pi \); or by taking any interval of length \( 2\pi \), we may write
\[
\sum (\cos nx)/n^2 = 1/12 \left\{ 3\left[x - (2k+1)\pi\right]^2 - \pi^2 \right\}
\]
where \( 2k\pi \leq x \leq 2(k+1)\pi \), and \( k = 0, 1, 2, 3, \ldots \).

See Bromwich, Theory of Infinite Series, Art. 109 and 110.
\( y_\phi = \sum (\cos n\phi x)/(n\phi)^2 \) is a series composed of all the terms of the complete series \( \sum (\cos nx)/n^2 \) divisible by \( \phi \).

This series may be summed in the following way:

\[
\sum (\cos n\phi x)/(n\phi)^2 = 1/\phi^2 \sum (\cos nu)/n^2, \text{ where } u = \phi x.
\]

\[
\sum (\cos n\phi x)/(n\phi)^2 = 1/(12\phi^2) \left\{ 3 \left[ u - (2k+1)\pi \right]^2 - \pi^2 \right\}
\]

or

\[
\sum (\cos n\phi x)/(n\phi)^2 = 1/(12\phi^2) \left\{ 3 \left[ \phi x - (2k+1)\pi \right]^2 - \pi^2 \right\}
\]

where \( 2k\pi/\phi \leq x \leq 2(k+1)\pi/\phi \), and \( k = 1, 2, 3, \ldots \).
SUMMATION OF $\sum (\cos nx)/n^2$, DEPLETED BY SIX

By depleting a series by some number $p$, in this case 6, it is meant to omit all terms of the complete series divisible by any of the prime factors of $p$. In the series

$$y = \sum (\cos nx)/n^2, \text{ depleted by } 6 = (\cos x)/1^2 + (\cos 5x)/5^2 + (\cos 7x)/7^2 + (\cos 11x)/11^2 + (\cos 13x)/13^2 + \ldots,$$

we have

$$y = y_1 - y_2 - y_3 + y_6$$

where

$$y_1 = (\cos x)/1^2 + (\cos 2x)/2^2 + (\cos 3x)/3^2 + \ldots,$$

$$y_2 = (\cos 2x)/2^2 + (\cos 4x)/4^2 + (\cos 6x)/6^2 + \ldots = 1/2^2 \sum (\cos nu)/n^2, \text{ where } u = 2x,$$

$$y_3 = (\cos 3x)/3^2 + (\cos 6x)/6^2 + (\cos 9x)/9^2 + \ldots = 1/3^2 \sum (\cos nu)/n^2, \text{ where } u = 3x, \text{ and}$$

$$y_6 = (\cos 6x)/6^2 + (\cos 12x)/12^2 + (\cos 18x)/18^2 + \ldots = 1/6^2 \sum (\cos nu)/n^2, \text{ where } u = 6x.$$

Since

$$y' = \sum (\cos n\phi x)/(n\phi)^2 = 1/(12\phi^2) \left\{ 3 \left[ \phi x - (2k+1)\pi \right]^2 - \phi^2 \right\}$$

where $2k\pi/3 \leq x \leq 2(k+1)\pi/3$, the summation of the above series can be written.

$$y_2 = 1/(4\cdot12) \left\{ 3 \left[ 2x - (2k+1)\pi \right]^2 - \pi^2 \right\}, \text{ when } k\pi \leq x \leq (k+1)\pi.$$

$$y_3 = 1/(9\cdot12) \left\{ 3 \left[ 3x - (2k+1)\pi \right]^2 - \pi^2 \right\}, \text{ when } 2k\pi/3 \leq x \leq 2(k+1)\pi/3.$$
And 
\[ y_6 = \frac{1}{36.12} \left\{ 3 \left[ x - (2k+1)\pi \right]^2 - \pi^2 \right\}, \]
when \( k\pi/3 \leq x \leq (k+1)\pi/3. \)

Let \( v_6 \) be the integral part of \( v/6 \), and \( r_6 \) the remainder; then \( v = 6v_6 + r_6. \)

By making these substitutions for \( k \), we arrive at the summation of the depleted series.

\[ y = y_1 - y_2 - y_3 + y_6 \]
\[ = \frac{1}{12} \left\{ 3 \left[ x - (2v_6+1)\pi \right]^2 - \pi^2 \right\} \]
\[ - \frac{1}{48} \left\{ 3 \left[ 2x - (2v_5+1)\pi \right]^2 - \pi^2 \right\} \]
\[ - \frac{1}{108} \left\{ 3 \left[ 3x - (2v_4+1)\pi \right]^2 - \pi^2 \right\} \]
\[ + \frac{1}{432} \left\{ 3 \left[ 5x - (2v_2+1)\pi \right]^2 - \pi^2 \right\} \]
where \( v\pi/3 \leq x \leq (v+1)\pi/3. \)

After expanding and collecting, we get
\[ y = Ax + A_1\pi^2, \]
where
\[ A = -v_6 + v_5/2 + v_2/3 - v/6 - 1/6, \]
and
\[ A_1 = v_6(v_6+1) - v_3(v_3+1)/4 - v_2(v_2+1)/9 - v(v+1)/36 + 1/9 \]
when \( v\pi/3 \leq x \leq (v+1)\pi/3. \)

But \( v_6 = (v-r_6)/6, v_5 = (v-r_3)/3, \) and \( v_2 = (v-r_2)/2; \) therefore, by substitution, we get
\[ A = (r_6 - r_3 - r_2 - 1)/6, \]
and
\[ A_1 = -Av/3 + B \]
where
\[ B = \left[ r_6(r_6-6) - r_3(r_3-3) - r_2(r_2-2) + 4 \right]/36. \]

\( A \) has at most six different values, and \( B \) has at most six different values.
\[ y = A \frac{x}{\pi} - A \frac{x^2}{\pi^2} \] is a series of straight lines which can be represented by a Fourier series depleted by six.

By using different values of \( v \) we can find the remainders \( r_6, r_3, \) and \( r_2; \) and, therefore, values of \( A, B, \) and \( A_1, \) from which we can write the equations for the six regions in the interval from \( x = 0 \) to \( x = 2\pi. \)

<table>
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<tr>
<th>( v )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tr>
<td>( r_6 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>( r_3 )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( A )</td>
<td>-1/6</td>
<td>-1/3</td>
<td>-1/6</td>
<td>1/6</td>
<td>1/3</td>
<td>1/6</td>
</tr>
<tr>
<td>( B )</td>
<td>1/9</td>
<td>1/18</td>
<td>-1/18</td>
<td>-1/9</td>
<td>-1/18</td>
<td>1/18</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>1/9</td>
<td>1/6</td>
<td>1/18</td>
<td>-5/18</td>
<td>-1/2</td>
<td>-2/9</td>
</tr>
</tbody>
</table>

For \( v = 0, \) \[ y = (-\pi/6)x + \pi^2/9, \] \( 0 \leq x \leq \pi/3. \)

For \( v = 1, \) \[ y = (-\pi/3)x + \pi^2/6, \] \( \pi/3 \leq x \leq 2\pi/3. \)

For \( v = 2, \) \[ y = (-\pi/6)x + \pi^2/18, \] \( 2\pi/3 \leq x \leq \pi. \)

For \( v = 3, \) \[ y = (\pi/6)x - 5\pi^2/18, \] \( \pi \leq x \leq 4\pi/3. \)

For \( v = 4, \) \[ y = (\pi/3)x - \pi^2/2, \] \( 4\pi/3 \leq x \leq 5\pi/3. \)

For \( v = 5, \) \[ y = (\pi/6)x - 2\pi^2/9, \] \( 5\pi/3 \leq x \leq 2\pi. \)

From the inequality \( v\pi/3 \leq x \leq (v+1)\pi/3, \) we can determine \( v. \) For instance, let \( x = 200^\circ: \)

\[ v/3 < 200^\circ < (v+1)\pi/3 \]

\[ v < 600^\circ/180^\circ < v+1 \]
\[ v < 3 \quad \frac{1}{3} < v+1 \]

Therefore, \[ v = 3. \]

Since the cosine function has a period of \( 2\pi \), these equations will suffice for all values of \( x \) if properly selected.
SUMMATION OF $\sum (\cos nx)/n^2$, DEPLETED BY P

First we shall solve for the summation of the series when depleted by $p$ where $p$ is a prime number or is taken alone. With this condition in mind

$$y = \sum (\cos nx)/n^2, \text{ depleted by } p = y_1 - y_p.$$  

From equations derived in part three, it is obvious that

$$y_1 = \frac{1}{12} \left\{ 3 \left[ x - (2v+1)\pi \right]^2 - \pi^2 \right\},$$

and

$$y_p = \frac{1}{12p^2} \left\{ 3 \left[ px - (2v+1)\pi \right]^2 - \pi^2 \right\}.$$  

Then

$$y = \frac{1}{12} \left\{ 3 \left[ x - (2v+1)\pi \right]^2 - \pi^2 \right\} - \frac{1}{12p^2} \left\{ 3 \left[ px - (2v+1)\pi \right]^2 - \pi^2 \right\}$$

$$= \left[ \frac{(2v+1)}{2p} - \frac{(2v+1)}{2} \right] \pi x - \left[ \frac{(2v+1)^2}{4} - \frac{1}{12p^2} \right] \pi^2$$

$$= A \pi x + A_1 \pi^2$$

where

$$A = \frac{(2v+1)}{2p} - v_p - \frac{1}{2},$$

and

$$A_1 = \frac{(2v+1)^2}{4p^2} - \frac{(2v+1)^2}{4p^2} - \frac{1}{12p^2}.$$  

Since $v_p = \left( v - r_p \right)/p$, where $v_p$ is the integral part of $v/p$ and $r_p$ the remainder, by substituting this value for $v_p$ we get

$$y = A \pi x + A_1 \pi^2$$

when

$$A = \frac{r_p}{p} - \frac{1}{2},$$

and

$$A = -2A_v/p + B$$

where

$$B = \frac{r_p(r_p-p)/p^2 + (p^2-1)/6p^2}{p},$$

when

$$2v\pi/p \leq x \leq 2(v+1)\pi/p.$$
Now let us consider the case \( y = \sum (\cos nx)n^2 \), depleted by \( p \), where \( p_1 \) and \( p_2 \) are the prime factors of \( p \), and \( p \equiv p_1p_2 \).

Here \( y = y_1 - y_{p_1} - y_{p_2} + y_p \)

\[
= \frac{1}{12} \left\{ \frac{3}{2} \left[ x - (2v+1)\pi \right]^2 - \pi^2 \right\} - \frac{1}{12p_1^2} \left\{ \frac{3}{2} \left[ p_1x - (2v+1)\pi \right]^2 - \pi^2 \right\} \\
+ \frac{1}{12p_2^2} \left\{ \frac{3}{2} \left[ p_2x - (2v+1)\pi \right]^2 - \pi^2 \right\}
\]

when \( 2v\pi/p \leq x \leq 2(v+1)\pi/p \).

By simplifying and substituting \((v-r_p)/p\) for \( v_p \), \((v-r_{p_1})/p_1\) for \( v_{p_1} \), and \((v-r_{p_2})/p_2\) for \( v_{p_2} \), we get

\[ y = A_{11}x + A_{11}^2 \]

where

\[ A = (r_p - r_{p_1} - r_{p_2})/p + (p_1 + p_2 - p - 1)/2p, \]

and

\[ A_{11} = -2AV/p + B \]

when

\[ B = \frac{1}{p^2} \left[ r_p(r_p - p) - r_{p_1}(r_{p_1} - p_1) - r_{p_2}(r_{p_2} - p_2) \right] \]

\[ + \frac{1}{6p^2} \left[ p^2 - p_1^2 - p_2^2 + 1 \right] \]

in the interval \( 2v\pi/p \leq x \leq 2(v+1)\pi/p \).

We shall take one more case before generalizing in the number of prime factors of \( p \). That is \( y = \sum (\cos nx)/n^2 \) where \( p_1, p_2, \) and \( p_3 \) are the three prime factors of \( p \), and \( p = p_1p_2p_3 \). With an increasing number of prime factors of \( p \), the problem rapidly increases in complexity.

\[ y = y_1 - y_{p_1} - y_{p_2} - y_{p_3} + y_{p_1p_2} + y_{p_1p_3} + y_{p_2p_3} - y_p \]
\[ y = \frac{1}{12} \left( 3 \left[ x - (2v_p + 1) \pi \right]^2 - \pi^2 \right) - \frac{1}{12p_1^2} \left( 3 \left[ p_1x - (2v_{p2p3} + 1) \pi \right]^2 - \pi^2 \right) \\
- \frac{1}{12p_2^2} \left( 3 \left[ p_2x - (2v_{p1p5} + 1) \pi \right]^2 - \pi^2 \right) + \frac{1}{12p_3} \left( 3 \left[ p_3x - (2v_{p2} + 1) \pi \right]^2 - \pi^2 \right) + \frac{1}{12p_2^2p_3} \left( 3 \left[ p_2p_3x - (2v_{p1} + 1) \pi \right]^2 - \pi^2 \right) - \frac{1}{12p_2^2} \left( 3 \left[ px - (2v + 1) \pi \right]^2 - \pi^2 \right) \]

Since \( v = (v - r_p)/p \), \( v_{p1} = (v - r_{p1})/p_1 \), \( v_{p2} = (v - r_{p2})/p_2 \), \( v_{p1p2} = (v - r_{p1p2})/p_1p_2 \), \( v_{p1p3} = (v - r_{p1p3})/p_1p_3 \), and \( v_{p2p3} = (v - r_{p2p3})/p_2p_3 \), we may substitute these values and simplify to attain

\[ y = A\pi^2 + A_1\pi^2 \]
where \( A = 1/p(r_p - r_{p1p2} - r_{p1p3} - r_{p2p3} - r_{p1} - r_{p2} - r_{p3}) + 1/2p(p_{p1p2} + p_{p1p3} + p_{p2p3} - p_1 - p_2 - p_3 - p + 1) \),
and \( A_1 = -2Av/p + B \)
where \( B = 1/p^2 \left[ r_p(r_p - p) - r_{p1p2} - r_{p1p2} - r_{p1p2} \right] \\
r_{p1p3} - r_{p2p3} \]
\[ + r_{p1} (r_p - p_{p1}) + r_{p2} (r_p - p_{p2}) + r_{p3} (r_p - p_{p3}) \]
\[ + 1/6p^2 \left[ p_{p1p2}^2 - p_{p1p3}^2 - p_{p2p3}^2 - p_1^2 - p_2^2 + p_3^2 \right] \]
when \( 2v\pi/p \leq x \leq 2(v + 1)\pi/p \).
With the results of these three particular cases of depleted series we can write from mathematical induction a general solution for the summation of the series
\[ \sum (\cos nx)/n^2 \] depleted by any number \( p \) which was a finite number \( k \) of prime factors \( p_1, p_2, p_3, \ldots, p^1, \ldots, p^k \), and \( p = p_1 p_2 p_3 \ldots p_k \).

\[ \sum (\cos nx)/n^2, \text{ depleted by } p = A/B + A_1 \pi^2 \text{ where } \]

\[
A = \frac{1}{p} \left[ r_{q_k} - \sum r_{q_{k-1}} + \sum r_{q_{k-2}} - \ldots + (-1)^{k-1} \sum r_{q_1} \right] \\
+ \frac{1}{2p} \left[ \sum q_{k-1} - \sum q_{k-2} + \ldots + (-1)^k \sum q_1 - p + (-1)^{k-1} \right],
\]

and \( A_1 = -2Av/p + B \) where

\[
B = \frac{1}{p^2} \left[ r_{q_k} (r_{q_k} - q_k) - \sum r_{q_k-1} (r_{q_{k-1}} - q_{k-1}) + \sum r_{q_{k-2}} (r_{q_{k-2}} - q_{k-2}) - \ldots + (-1)^{k-1} \sum r_{q_1} (r_{q_1} - q_1) \right] + \frac{1}{6p^2} \left[ q_{k}^2 - \sum q_{k-1}^2 \right. \\
\left. + \sum q_{k-2}^2 - \ldots + (-1)^{k-1} \sum q_1^2 + (-1)^{k} \right],
\]

when \( 2\pi n/p \leq x \leq 2(\pi+1)\pi/p \). Here \( q_m \) is the products of the numbers in the combinations of the \( k \) prime factors of \( p \) taken \( m \) of them at a time, and in \( \sum r_{q_m} (r_{q_m} - q_m) \) the same value of \( q_m \) is used in all three positions simultaneously.
CONCLUSION

\[ f(x) = A\pi x + A_1\pi^2 \] , a series of straight lines, is a function which can be expressed by \( \sum (\cos nx)/n^2 \) when depleted by \( p \).

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