A TRIANGLE AND ITS CIRCLES

by

NELLE MAY COOK

A.B., Hiram College, 1913; B.S., Phillips University, 1923

A THESIS

submitted in partial fulfillment of the requirements

for the degree of

MASTER OF SCIENCE

KANSAS STATE AGRICULTURAL COLLEGE

1929

TABLE OF CONTENTS

INTRODUCTION

This dissertation is an attempt to present a consistent and systematic account of some of the various theories of the triangle and its circles from the standpoint of modern geometry and to solve a number of original exercises in connection with these theorems.

In the science of geometry the triangle and circle have always played an important role. However, a wonderful advance has been made in this subject during the latter half of the Nineteenth Century. That new theorems should have been found in recent time is remarkable when we consider that these figures were subjected to close examination by the Greeks and the many geometers who followed them. Recently developed properties have been added by Steiner, Feuerberg, Lemoine, Neuberg, Crelle, Jacobi, Brocard and a countless number of others.

Crelle (1780-1855) a German mathematician who first discovered the Brocard points said, "It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in properties."(1)

This treatise is intended to give interesting facts concerning the different circles associated with a triangle. A complete account would be impossible but some of the recently developed properties of the triangle and circle will be presented.

One of the outstanding features of modern geometry as contrasted with ancient or Euclidean is that on the whole it is descriptive rather than metrical. In elementary geometry the triangle is considered as a portion of a plane bounded by three straight lines. In modern geometry the triangle is considered as a system of three non-concurrent infinite lines in which the three intersections are taken into account or as a system of three points which are not collinear in which the three straight lines of connection are taken into account.

Notation

In general the following notation will be used for the triangle A B C.

A,B,C,--------------vertices of triangle.
A',B',C',-----------mid-points of sides.
D,E,F,--------------feet of the altitudes.
G-----------------Centroid.
H-----------------orthocentre
O-----------------circumcentre
O'----------------incentre
I_1, I_2, I_3-------------excentres
X,Y,Z--------------points of contact of the in-circle.
X',Y',Z'-----------points of contact of the ex-circles on BC,CA,AB respectively.

r----------------radius of inscribed circle.

R----------- -----radius of circumscribed circle.

r_1, r_2, r_3----------ex-radii.

a,b,c,-----------lengths of sides.

s----------------semi perimeter.

N----------------Nagel point.

K----------------symmedian point.

$\frac{1}{2}R$---------------radius of nine-point circle.

BRIEF REVIEW OF CIRCLES RELATED TO A TRIANGLE IN ELEMENTARY GEOMETRY AND TRIGONOMETRY

In elementary geometry are found many facts concerning the inscribed, circumscribed and escribed circles. To find the radii of these circles in terms of the sides of the triangle furnishes an interesting study.

For $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

$$r_1 = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}$$

$$r_2 = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-b}$$

$$r_3 = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-c}$$

Then many theorems are based on the relationship between these circles and the parts of the triangles concerned. A useful theorem in plane geometry states that in any triangle the product of any two sides is equal to the diameter of the circumscribed circle multiplied by the perpendicular drawn to the third side from the vertex of the opposite angle. (ab = a Rh)

In trigonometry the sine law may be developed from the circumscribed circle and the tangent law from the inscribed circle.

as $2R = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = \frac{a}{\sin\alpha}$

and $\tan\frac{1}{2}\alpha = \frac{r}{s-a}$

$\tan\frac{1}{2}\beta = \frac{r}{s-b}$

$\tan\frac{1}{2}\gamma = \frac{r}{s-c}$

THE TRIANGLE AND ITS CIRCLE FROM THE STANDPOINT OF MODERN GEOMETRY

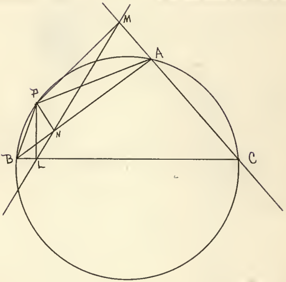


Fig. 1.

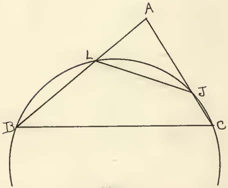


Fig.2.

Pedal or Simson Line

If from any point P on the circumcircle, perpendiculars PL, PM, PN are drawn to the sides of a triangle, then LMN is a straight line and is called the Pedal or Simson line.

In Fig. 1, page 7. From P any point on the circumcircle perpendiculars PL, PM, PN are drawn to BC, CA, AB respectively.

To Prove: LMN is a straight line. In the quadrilateral BLNP, BNP and BLP are right angles.

Then B, L, N, P are concyclic points
and LNP and LBP are supplementary angles.
Also since PNAM are concyclic points
angle PNM = PAM (measured by one-half arc PM)
angle CAP = LNP (both supplementary to LBP)
Now angle CAP and PAM are supplementary
Then angle LNP and PNM are supplementary
Therefore LNM is a straight line.

Antiparallels

In the triangle ABC (Figure 2, page 7) the line JL is drawn making the angle LJA = ABC.

Then angle ALJ equals ACB.

And LJ is antiparallel to BC. This term "antiparallel" is due to Leibniz.

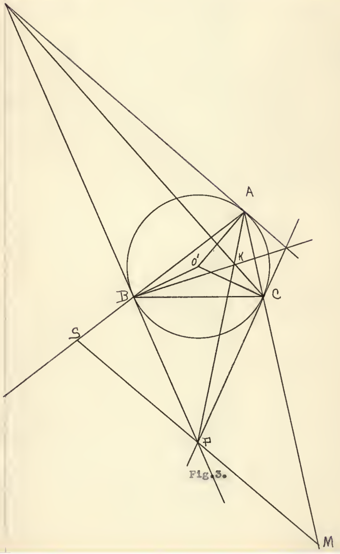


Fig.3.

An interesting fact to note is that L, J, C. B, are concyclic points as the opposite angles in the quadrilateral BCJL are supplementary.

Symmedian Lines and Symmedian Point

A line drawn through the vertex of a triangle so as to bisect the antiparallels to the base is called a symmedian line. The point of concurrence of the three symmedians is called the symmedian point. In 1873 Emile Lemoine called attention to this point which since has been referred to as point de Lemoine in French works; Grebe's point in German works. In this paper the English term symmedian point will be used and will be indicated by K. Its distances from the sides of the triangle are proportional to the lengths of the corresponding sides. In Fig. 3, page 9.

If PQR be the triangle formed by tangents to the circumcircle at A, B, C then AP, BQ, CR, bisect all the antiparallels to BC, CA, AB respectively and are concurrent.

First: To Prove: AP, BQ, CR are concurrent.

PB = CP QC = AC RA = BR

multiplying (PB) (QC) (RA) = (BR)(AQ) (CP)

Therefore AP, BQ, CR are concurrent (Ceva's Theorem)

This point of concurrence is the symmedian point.

Second: To Prove: AP, BQ, CR bisect all antiparallels to BC, CA, AB.

Draw the antiparallel MS to BC through P

S PM is parallel to QAR (angle PMA = MAQ = CBA)

Angle PSB = RAB = RBA = SBP

Therefore PS = PB

Similarly PM = PC   Now PC = PB

Therefore SP = PM and P is mid point of MS and all antiparallels to BC would be bisected by BQ; all antiparallels to BA would be bisected by CR.

AP, BQ, CR are called symmedian lines.

It is important to note that all antiparallels to BC are parallel to tangent at A; all antiparallels to CA are parallel to tangent at B; all antiparallels to AB are parallel to tangent at C.

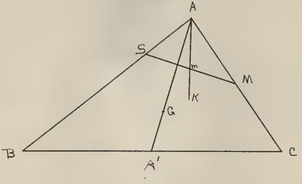


Fig. 4.

Isogonal Conjugates

If from two points lines are drawn to the same vertex of a triangle and these lines make equal angles with the adjacent sides of the triangle these two points are called isogonal conjugates.

Fig. 4, page 12.

In the triangle ABC, G the centroid, and K, the symmedian point are found.

To prove K and G are isogonal conjugates.

If it can be proven that angle GAB = KAC than K and G will be isogonal conjugates.

Draw any antiparallel MS to BC.

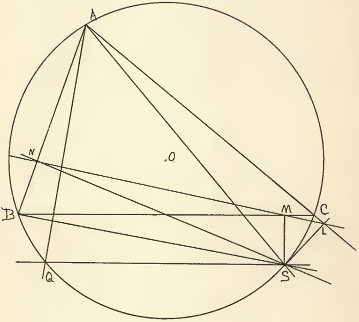


Fig. 5.

AK bisects MS in M. Since triangles ASM and ABC are similar

AB: BC = AM: MS also AB: BA' = AM: Mm.

Then the triangle ABA' is similar to AMm

Therefore angle BAA' = MAm or angle GAB = KAC making K and G isogonal conjugates.

Exercise I. The isogonal conjugate with regard to an angle of a triangle of a line through the vertex of that angle is perpendicular to the Simson line of the second intersection of the given line with the circumscribed circle.

In Fig. 5, page 13.

Given the triangle ABC in circumscribed circle.

Draw a line AS and find Simson line of S which is LMN.

Then draw AQ the isogonal conjugate of AS with regard to angle A.

Then angle BAQ = SAL

To prove: AQ is perpendicular to Simson line LN.

Angle SBC = SAC (measured by $\frac{1}{2}$ arc CS)

B, N, M, S are concyclic points.

Therefore angle SNM = SBM

But angle BAQ = SAC (isogonal conjugates with respect to angle A.

Then angle SBC = BAQ and angle SNM = BAQ

Then since SNA is a right angle and angle SNM = BAQ

Therefore NL is perpendicular to AQ.

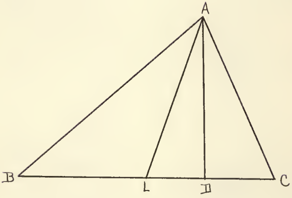


Fig. 6.

To Prove: $\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{AL}^2 + 2\,\overline{BL}^2$ when L is mid point of BC.

Draw AD perpendicular to BC.

Let L be the mid point of BC.

In triangle ABL

$$\overline{AB}^2 = \overline{AL}^2 + \overline{BL}^2 - 2\,(BL)\,(DL)$$

In triangle ALC

$$\overline{AC}^2 = \overline{AL}^2 + \overline{LC}^2 + 2\,(CL)\,(DL)$$

But BL = LC.

Adding equations

$$\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{AL}^2 + 2\,\overline{BL}^2$$

also

$$\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{AL}^2 + \tfrac{1}{2}\,\overline{BC}^2$$

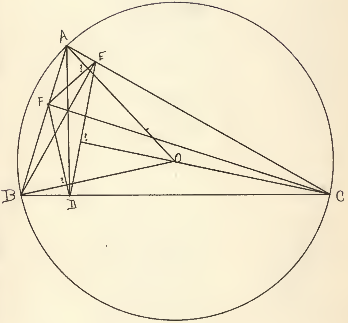


Fig. 7.

Circles

Pedal Circle. The pedal circle of a triangle is the circle which circumscribes the pedal or orthocentric triangle.

The pedal triangle of a triangle is formed by connecting the feet of the perpendiculars from each vertex to the opposite side of the original triangle.

In Fig. 7, page 17, DEF is the pedal or orthocentric triangle.

If a pedal triangle of a point is designated then perpendiculars from this point to the sides are drawn and the lines connecting the feet of these perpendiculars form the pedal triangle of the point. The pedal triangle of a triangle is of course the pedal triangle of the orthocentric point. The pedal circle of a point is the circumscribed circle of the pedal triangle of that point.

Coolidge, gives this interesting theorem, Two isogonally conjugate points have the same pedal circle[4].

Exercise II. The lines joining the circumcentre to the vertices of a triangle are perpendicular to the sides of the pedal triangle. The pedal triangle is the orthocentric triangle DEF. BFEC are concylic points since BEC and CFB are right angles.

FE is antiparallel to BC

In like manner

DE is antiparallel to AB

FD is antiparallel to AC

The antiparallels are parallel to the tangents at A, B and C.

Since the radius of circumscribed circle is perpendicular to the tangents then the radius is perpendicular to each antiparallel.

∴ the lines joining the circumcentre to the vertices of a triangle are perpendicular to the sides of the pedal or orthocentric triangle.

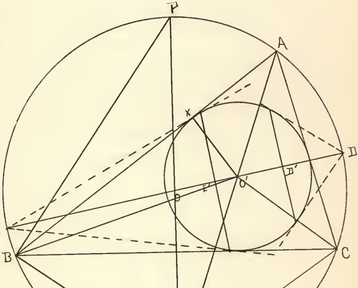


Fig.8.

<u>Relation between Circles Circumscribed and Inscribed to a Triangle</u>. Two triangles are so related that a triangle can be inscribed in one and circumscribed to the other.

To find the relation between their radii.

In Fig. 8, page 20.

Let R be the radius of the circumscribed circle

r be the radius of the inscribed circle

d be the distance between their centers O, O'.

Let OO' meet the circumscribed circle in DE.

The inverses of D and E in the inscribed circle are D' and E' respectively.

Then $(O'D')\ (O'D) = r^2$

$$O'D' = \frac{r^2}{O'D} = \frac{r^2}{R-d}$$

Similarly $O'E' = \frac{r^2}{O'E} = \frac{r^2}{R+d}$

Adding $O'D' + O'E' = \frac{r^2}{R-d} + \frac{r^2}{R+d} = r \quad (d^2 = R^2 - 2Rr)$

Dividing through by r^2

$$\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r}$$

Therefore the radii of the circles circumscribed and inscribed to a triangle are connected by this equation, where d is the distance of their centres.

$$\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r}$$

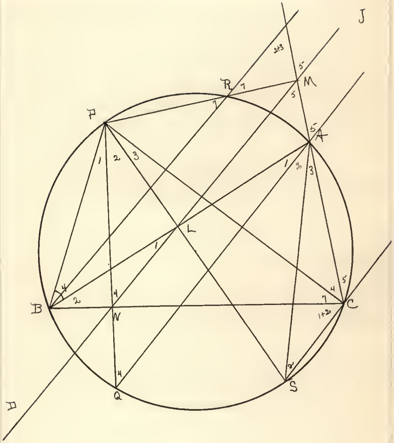


Fig. 9.

Exercise III. If the perpendiculars from any point on the circumcircle to the sides of a triangle meet the circle again in Q, R, S respectively then AQ, BR, CS are parallel to Simson's line of that point.

Figure 9, page 22.

Let P be any point on the circumcircle and draw its Simson line DJ.

To Prove: AQ, BR, CS are parallel to DJ.

BNLP are concyclic points (Angles BNP and BLP are right angles).

Then angle BPN = BLN (both measured by $\frac{1}{2}$ arc BN).

In circumscribed circle: angle BAQ = BPQ (both measured by $\frac{1}{2}$ arc BQ).

Therefore angle BLN = BAQ

Then AQ is parallel to DJ (exterior interior-angles are equal)

Now angle QPS = NBL (since B, N, LP are concyclic points)

Angle QPS = QAS (both measured by $\frac{1}{2}$ arc QS)

Angle ASC = ABC (both measured by $\frac{1}{2}$ arc AC)

Therefore angle ASC = QAS

And SC is parallel to QA (alternate interior angles are equal)

Angle 7 + angle 2 + angle 3 = 90° (right angle at PNC)

Angle (1+2) + angle 7 + angle 4 + angle 5 = 180°

(straight angle through C)

Angle (1+2) + angle 4 = 90° (right triangle BPN)

Subtracting: angle 7 + angle 5 = 90°

Therefore angle 5 = angle (2 + 3)

Then DJ is parallel to BR (alternate interior angles are equal)

Therefore AQ, BR, CS are parallel to Simpson's line of Point P.

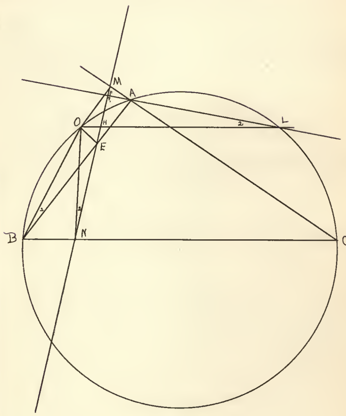


Fig. 10.

Exercise IV. Show that if O by any point on the circumcircle of the triangle ABC and OL be drawn parallel to BC to meet the circumcircle in L, then will LA be perpendicular to the pedal line of O with respect to the triangle.

Figure 10, page 25.

Draw the pedal line MEN of O.

To Prove: FL is perpendicular to MEN

Angle FHL = HNC (exterior-interior angles of OL and BC)

Since OENB are concyclic points

angle OBE = angle ONE

Also angle OBE = ALO (measured by $\frac{1}{2}$ arc AO)

Then angle ONE = ALO

Angles ONE plus HNC = 90°

Substituting angle ALO plus FHL = 90°

Therefore angle HFL must equal 90°

and line LA is perpendicular to pedal line MEN of point O.

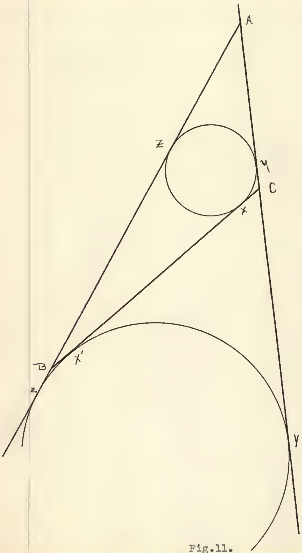


Fig.11.

<u>Inscribed and Escribed Circles</u>. The distances measured from the vertices of a triangle to the points of tangency of the inscribed and escribed circles on each side of the triangle are equal.

In Figure 11, page 27.

To Prove: $Bx_1 = Cx$

First prove $Bx_1 = S - C$ and $Cx = s - c$

Now $Bx_1 = Bz_1 = Az_1 - c$

$Ay_1 + Az_1 = AC + Cy_1 + AB + BZ_1$

$= AC + Cx_1 + AB + Bx_1$

$= AC + AB + BC$

But $Ay_1 = Az_1$

Therefore $2\ AZ_1 = 2\ S$

$AZ_1 = S$

Now $BZ_1 = AZ_1 - C$

Therefore $Bz_1 = S - C$ or $Bx_1 = S - C$

Now $Ay + Cy + Cx + Bx + Bz + ZA = AB + AC + BC$

transposing $Cx + Cy = AB + AC + BC - Ay - Bx - Bz - ZA$

$Cx + Cy = AC + BC - BZ + ZA)$

$C\bar{x} + Cy = AC + BC - AB$

$Cx + Cy = b + a + c$

$Cx = Cy$

Then $2\ C\ x = 2\ S - 2\ C$

Therefore $Cx = S - C$

Therefore $Bx_1 = Cx$

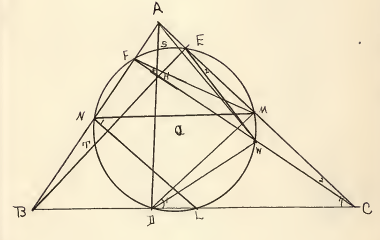


Fig. 12.

In a similar proof with the other escribed circles $Cy = Ay_1$ $AZ = BZ_1$

Nine-point Circle. The middle points of the sides of a triangle, the feet of the altitudes and the points half-way from the orthocentre to the vertices lie on a circle which is called the nine-point circle of the triangle. There are several independent discoverers of this circle. In 1822, Feuerbach, professor of gymnasium in Erlangen published a pamphlet in which he arrives at the nine-point circle, and proves that it touches the incircle and excircles(2). The Germans called it Feuerbach's Circle. Feuerbach's theorem will be discussed later in this thesis.

Figure 12, page 29.

In the triangle ABC, erect the perpendiculars AD, BE, CF and mark the mid-points of BC, CA, AB to be L, M, N respectively. STW are the mid-points of HA, HB, HC respectively.

To Prove: D, L, W, M, E. S, F, N, T are concyclic points

Pass a circle through L, M, N cutting the sides of the triangle ABC in D, E, F. Then to prove D, E, F are the feet of the altitudes.

MN is parallel and equal to $\frac{1}{2}$ BC

NL is parallel and equal to $\frac{1}{2}$ AC

Therefore LCMN is a parallelogram

Angle LNM = LCM (opposite angles of a parallelogram)

Angle CDM = LNM (measured by $\frac{1}{2}$ arc LM)

Then angle MDC = MCD

Therefore DCM is an isosceles triangle

DM = CM = MA

Therefore ADL is a right angle and AD is altitude from A to BC

In a similar proof BE is altitude from B to CA

CF is altitude from C to AB

Therefore the circle passes through the feet of the medians and feet of the altitudes which are six of the nine points.

To Prove: The circle passes through W the mid-point of HC.

H, D, C, E are concyclic points(opposite angles supplementary)

Then HW = EW = WC = DW

Therefore EWC is an isosceles triangle

In the triangle FCM

Angle WFM = WEM (measured by $\frac{1}{2}$ arc WM)

Also angle WFM = WCM (both equal WEC)

Therefore FCM is an isosceles triangle equiangular to isosceles triangle EWC

Then angle EWC = FMC

angle FWE = FME(supplements of equal angles)

Therefore W is on circumference of FME.

In a similar proof S and T are on the circumference LMN.

Therefore this nine-point circle passes through the feet of the medians, feet of the altitudes and the mid-points between the orthocentre and vertices of the triangle.

Radius of Nine-point Circle.

Since the nine-point circle of ABC is the circumcircle of LMN (Figure 12, page 29) and the sides of the triangle LMN are respectively equal to half the sides of the triangle ABC, then the radius of the nine-point circle is one-half the radius of the circumcircle. Consequently in this dissertation the radius of the nine-point circle will be denoted by $\frac{1}{2}$ R.

As the nine-point circle is frequently quoted in relation to other circles, reference will be made to this circle again in this paper.

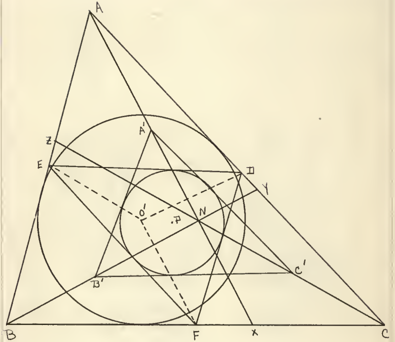


Fig. 13.

P Circle. There is another circle much less known than the nine-point circle but possessing a number of analogous properties. This circle is inscribed in the triangle whose vertices are the middle points of the sides of the original triangle, and is called the P circle.

Fig. 13, page 33.

To construct the P circle.

Draw any triangle ABC, calling the mid points DEF. The circle inscribed in DEF is the P circle.

To find the Nagel Point of the triangle ABC.

Connect each vertex with the point of tangency of the escribed circle to the opposite side. The intersection of these three lines is the Nagel point. In the figure AX, BY, CZ meet in N, the Nagel point.

P is the center of the P circle

O' is the center of inscribed circle.

A',B',C' are mid points of AN, BN, CN.

These terms will be used in pointing out analogies between the nine-point circle and the P circle.

Some Interesting Analogies Between the Nine-point Circle and P Circle (3)

Nine-point Circle	P Circle
1. Circumscribed to the triangle whose vertices are the middle points of the sides.	1. Inscribed in the triangle whose vertices are the middle points of the sides.
2. Radius one-half that of circumscribed circle.	2. Radius one-half that of inscribed circle.
3. Nine-point circle passes through half-way from the orthocentre to the vertices of the triangle.	3. P circle touches the sides of the triangle whose vertices lie half-way between the Nagel point and the vertices of the given triangle.
4. Nine-point circle cuts the sides of the triangle where they meet the corresponding altitudes.	4. P circle touches the sides of the middle point triangle where they meet the lines from the Nagel point to the corresponding vertices of the given triangle.
5. Nine-point circle meets the lines through the points mid-way from the orthocentre to the given vertices parallel to the corresponding side-lines where they meet the perpendicular bisectors of the given sides.	5. Touches the sides of the triangle, whose vertices are half-way from the Nagel point to the given vertices, at the points where each meets the line from the centre of the inscribed circle to the middle point of the corresponding sides of the original triangle.

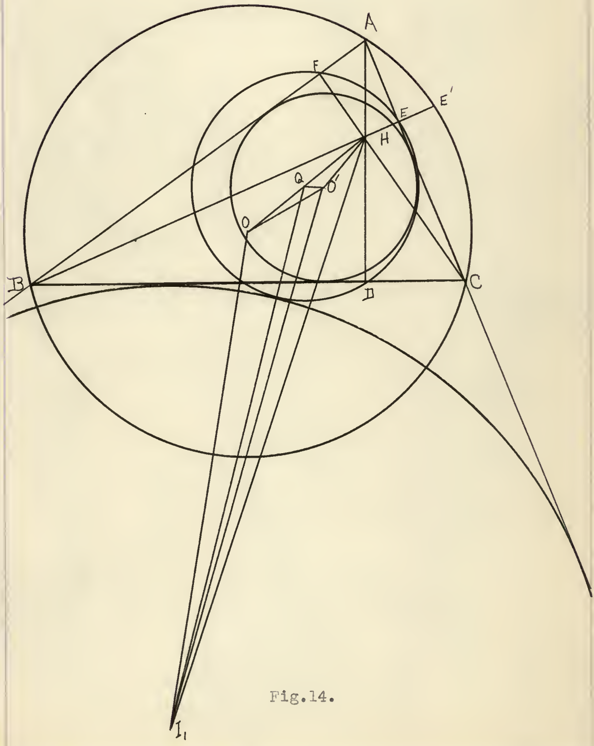


Fig.14.

Feuerbach's Theorem. The nine-point circle touches the incircle and the excircles. This property is usually called Feuerbach's Theorem.

Figure 14, page 36.

O' is center and r the radius of inscribed circle

Q is center and $\frac{1}{2}$ R the radius of nine-point circle

OQ is the distance between the centers

If it can be proven that the difference of the radii of the nine-point circle and the incircle is equal to the distance between their centers, then the circles touch internally.

To Prove: $O'Q = \frac{1}{2} R - r$

In the triangle O'OH since Q is mid-point of OH

$\overline{O'H}^2 + \overline{O'O}^2 = 2\,\overline{O'Q}^2 + 2\,\overline{OQ}^2$ (Appolonius' Theorem)

$\overline{O'H}^2 + \overline{O'O}^2 = 2\,\overline{O'Q}^2 + \frac{1}{2}\,\overline{OH}^2$

Substituting these values: $\overline{O'H}^2 = 2r^2 - (AH)(HD)$

$$\overline{O'O}^2 = R^2 - 2Rr$$

$$\overline{OH}^2 = R^2 - 2(AH)(HD)$$

Then $2r^2 - (AH)(HD) + R^2 - 2Rr = 2\,\overline{O'Q}^2 + \frac{R^2}{2} - (AH)(HD)$

Collecting and clearing of fractions

$$4\,\overline{O'Q}^2 = R^2 - 4Rr + 4r^2$$

Then $2O'Q = R - 2r$ (extracting square root)

Therefore $O'Q = \frac{1}{2} R - r$ and the nine-point circle and inscribed circle tough internally.

If it can be proven that the distance between the centres of the nine-point circle and escribed circle is equal to the sum of their radii then the nine-point circle touches the escribed circle externally.

Figure 14, page 36.

I_1 is the center of the escribed circle tangent to BC

r_1 is the radius of this escribed circle.

To Prove $I_1Q = \frac{1}{2} R + r_1$

In the triangle I_1OH

$\overline{I_1H}^2 + \overline{I_1O}^2 = 2\,\overline{I_1Q}^2 + \frac{1}{2}\,\overline{OH}^2$ (Appolonius' Theorem)

Substituting these values: $\overline{I_1H}^2 = 2\,r_1^2 - (AH)\,(HD)$

$\overline{I_1O}^2 = R^2 + 2R\,r_1$

$\overline{OH}^2 = R^2 - 2(AH)\,(HD)$

Then $2\,r_1^2 - (AH)\,(HD) + R^2 + 2Rr_1 = 2\,\overline{I_1Q}^2 + \frac{1}{2}R^2 - (AH)\,(HD)$

Collecting and simplifying $I_1Q = \frac{1}{2} R + r_1$

In a similar way it is proven that the nine-point circle touchss the other two escribed circles externally.

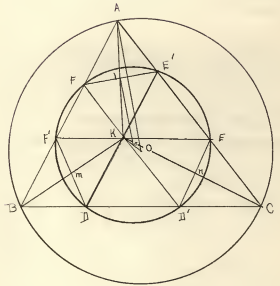


Fig. 15.

Lemoine's First Circle or the Triplicate Ratio Circle.

If through the symmedian point K parallels F'KE, D'KF, E'KD be drawn to the sides of a triangle ABC, the six points D, D', E, E', F, F' all lie on a circle whose center o is the mid-point of KO. This circle is called Lemoine's First Circle.

Figure 15, page 39.

To Prove D, D', E, E', F, F' are concyclic points.

A FKE' is a parallelogram and Ak bisects FE'.

Therefore FE' is antiparallel to BC or EF'.

Since angles FE'E and FF'E are supplementary angles

Then F'EE'F are concyclic points

Since DF' is antiparallel to CA then angle BF'D = C

Since F'E is parallel to BC then angle E'EF' = C

angle BF'D = F'DE' (alternate interior angles)

Therefore angle F'DE' = E'EF'

Then D lies on circumference passing through F'E'E

Since D'E is antiparallel to Ab then angle D'EC =F'BD

Since F'E is parallel to BC then angle FF'E = F'BD

angle ED'F = D'EC (alternate interior angles)

Therefore angle ED'F = FF'E

Then D' lies on circumference passing through FE'E

Therefore the six points D, D', E, E', F,F' are concyclic.

To Prove: o is centre of circle.

Let AK intersect FE' in l then join lo

lo is parallel to AO because l is mid-point of AK and o is mid-point of KO;

Now AO is perpendicular to FE' because FE' is parallel to tangent at A and AO is perpendicular to tangent at A.

Therefore lo is perpendicular to FE'.

In a similar proof no and mo are perpendicular to and bisect the chords D'E and F'D respectively.

Therefore o is the center of the circle passing through the six points.

Since DD': EE' : FF' = a^3: b^3: c^3 this circle has been called the Triplicate Ratio Circle.

To Prove: DD': EE' = a^3: b^3 (Figure 15, page 39)

In the similar triangles (similar because equiangular)

DKD', KD'C and KEE'

DD': D'K = a: b

CE : D'C = a: b

KE : EE' = a: b

Multiplying

$$\frac{DD' \times CE \times KE}{D'K \times D'C \times EE'} = a^3 : b^3$$

But CE = D'K KE = D'C

Therefore DD': EE' = a^3: b^3

In similar triangles

F'KF, KAE' and EKE'(Similar because equiangular)

KF: FF' = b: c

FA: AE' = b: c

EE': KE' = b: c

Multiplying

$$\frac{KF \times FA \times EE'}{FF' \times AE' \times KE'} = b^3 : c^3$$

But KF = AE'   FA= KE'

Therefore EE': FF' = $b^3 : c^3$

As DD': EE' = $a^3 : b^3$

Therefore DD':EE': FF' = $a^3 : b^3 : c^3$

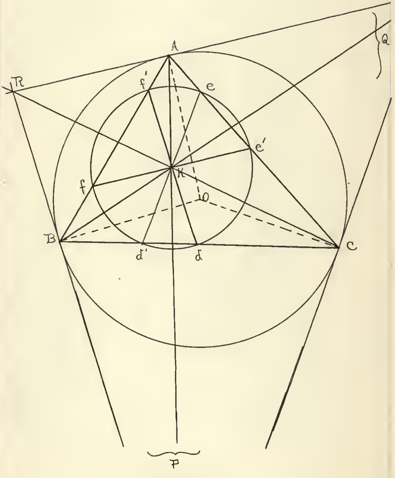


Fig. 16.

Lemoine's Second Circle or Cosine Circle. If through the symmedian point of a triangle antiparallels to the sides be drawn they meet the sides in six points which lie on a circle called Lemoine's Second Circle. As this circle cuts on each side of the triangle a segment proportional to the cosine of the opposite angle, it has also been called a Cosine Circle.

Figure 16, page 43.

Draw the antiparallels, f, e', d'e and df', through K the symmedian point.

To Prove: d, d', e, e', f, f' are concyclic points.

Since K is the symmedian point, it is the mid-point of the antiparallels.

Then Kf' = Kd

In the triangle f'Kf

angle Kff' = C (f'd is antiparallel to CA)

angle Kff' = C (fe' is antiparallel to BC)

Therefore Kf' = Kf

Then Kf' = Kf = Kd

In a similar proof

Ke' = Ke = Kf

Kd' = Kd = Kf'

Therefore a circle with radius Kf' and center at K passes through the six points d, d', e, e', f, f'.

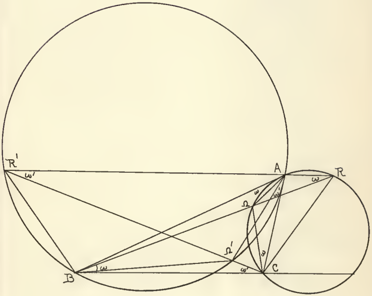


Fig. 17.

Brocard Circle. a. Brocard Points. b. Brocard Angles. c. Brocard Circle. d. Brocard's First Triangle.

a. Brocard Points. In the triangle ABC(Figure 17, page 45)if a circle be described tangent to AB at A and passing through C, and AR be the chord parallel to BC, then BR meets the circle again in a point Ω. This point is called the positive Brocard point.

Then if a circle be described tangent to AC at A and passing through B and R'A be the chord parallel to BC then R'C cuts the circle again in Ω'. This point is called the negative Brocard point.

b. Brocard Angles. In the triangle ABC

angle $\Omega AB = \Omega CA = \Omega RA$ (measured by $\frac{1}{2}$ arc ΩA)

angle $\Omega RA = \Omega BC$ (alternate-interior angles)

Therefore $\Omega AB = \Omega BC = \Omega CA$

Each of these angles is called the Brocard angle of the triangle and is denoted by ω.

Now using the negative Brocard point

angle $\Omega' CB = \Omega R'A$(alternate-interior angles)

angle $\Omega' BA = \Omega' AC =$ 'R'A(measured by $\frac{1}{2}$ arc $A\Omega'$)

Therefore $\Omega' CB = \Omega' AC = \Omega' BA = \omega'$

In the quadrilateral R'BCR, BC is parallel to R'R

angle ARC = BAC (measured by $\frac{1}{2}$ arc AC)

angle BR'A = BAC (measured by $\frac{1}{2}$ arc BA)

Then angle ARC = BR'A and R'BCR is an isosceles trapezoid. Then triangle BΩ'C ≅ BΩC

Therefore angle Ω'CB = ΩBC or $\omega = \omega'$

Then Ω and Ω' are isogonal conjugates as they are equally inclined to the sides of the triangle ABC.

It would perhaps be interesting to notice the position of the Brocard angles with respect to the sides of the triangle without the confusing lines of the figure needed in

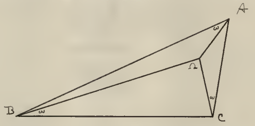


Fig. 18. Positive Brocard Angles.

Fig. 19. Negative Brocard Angles.

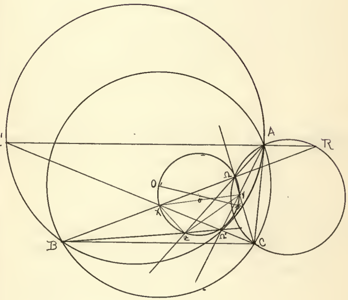


Fig. 2 . Brocard Circle.

the original construction and proof of the equality of the Brocard angles.

Unfortunately, the names of geometricians which have been attached to certain notable points, lines, angles and circles are not always the men who first discovered their properties. Crelle and Jacobi first studied the remarkable facts concerning the Brocard points and Brocard angles. However, the Brocard circle which will be discussed next is Brocard's own creation.

c. Brocard Circle (Figure 20, page 48).

The circle described upon OK as a diameter passes through the Brocard points and is called the Brocard Circle.

d. Let x be the intersection of BΩ and CΩ'

y be the intersection of AΩ', and CΩ

z be the intersection of BΩ', and AΩ

then the triangle xyz is called Brocards first triangle and the vertices are on Brocard's circle.

It shall next be proven that the Brocard circle passes through seven notable points.

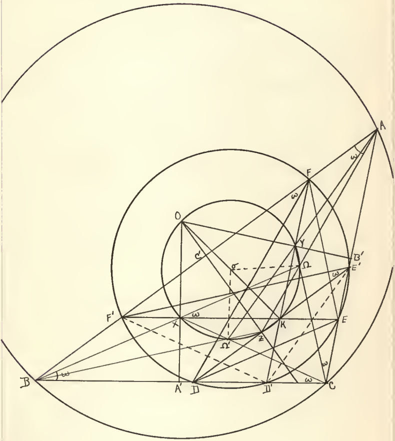


Fig. 21.

Brocard Circle.

The Brocard circle passes through the Brocard points Ω and Ω' and the three points of intersection of $A\Omega$, $B\Omega$, $C\Omega$, with $B\Omega'$, $C\Omega'$, $A\Omega'$ respectively, also angle ΩoK = $\Omega' oK = 2\omega$.

Figure 21, page 50.

Draw the Triplicate Ratio Circle through D, D', E, E', F, F'.

Also locate K, the symmedian point, Ω and Ω', the Brocard points.

Join $A\Omega$, $B\Omega$, $C\Omega$.

Angle DFF' = DE'F'

DE'F' = ω since K is the Brocard point of the triangle D'E'F'

$\omega = \Omega AB$ since Ω is positive

Brocard point of the triangle ABC

$A\Omega$ is parallel to FD.

Similarly $B\Omega$ is parallel to DE

$C\Omega$ is parallel to EF

Then parallels to D'E', E'F' F'D' through A,B,C will pass through Ω'.

Let $B\Omega$ meet F'E in x Join Cx

Then F'K = BD = xE = XK + KE

F'X = KE = D'C

Therefore F'X = KE = D'C and F'X is parallel to D'C

Therefore CX is parallel to D'F and CX passes through Ω'

Similarly

$C\Omega$, $A\Omega'$ meet in Y on D'F

$A\Omega$, $B\Omega'$ meet in Z on DE'

Since angle XBC = ω = XCB

Therefore XB = XC

Then XA' the perpendicular to BC bisects BC and passes through O.

Therefore OXK is a right angle.

Therefore the Brocard Circle passes through X and in like manner through Y and Z.

In quadrilateral $Y\Omega Z$ O:

Since Z, O, Y, are on the circle, if it could be proven that angle $Z\Omega Y$ is a supplement of angle ZOY then Ω would be on the circle.

Angle $Y\Omega Z$ = $A\Omega C$

$A\Omega C$ is supplement of A because $\Omega CA = \omega$ and $\Omega AF = \omega$

Now angle OLC' plus angle B'OC' = 90 degrees

angle A plus angle B'LA = 90 degrees

[illegible] angle OLC' = angle B'LA (vertical angles)

Therefore angle A = angle B'OC'

Then $A\Omega C$ is supplement of B'OC'

but angle $A\Omega C$ = angle $Y\Omega Z$

Therefore $Y\Omega Z$ is supplement of B'OC'

Therefore Ω is on the Brocard circle and in like manner Ω' is proven to be on circle.

Finally since o is center of circle

angle $\Omega oK = 2\Omega ZK = 2\Omega AB = 2\omega$

angle $\Omega' oK = 2\Omega XK = 2\omega$

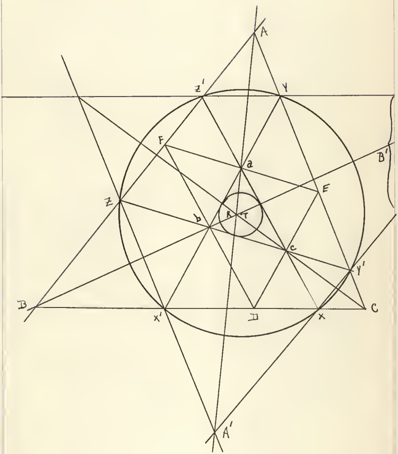


Fig. 22.

Taylor Circle.

Figure 22, page 54.

In the triangle ABC draw DEF the orthocentric triangle and find a, b, c, the middle points of the sides of the triangle DEF.

Then if

bc meet CA, AB in Y'Z respectively

ca meet AB, BC in Z'X respectively

ab meet BC, CA in X'Y respectively

the six points X, X', Y, Y', Z, Z' will lie on a circle called the Taylor Circle, whose center coincides with the incenter of the triangle abc.

Now the sides of the orthocentric triangle are always antiparallel to the sides of the original triangle.

Since FD is antiparallel to CA, angle DFZ = C

Since ZY' is parallel to FE and FE is antiparallel to BC.

Then ZY' is antiparallel to BC, angle Y'ZF = C

Therefore ZbF is an isosceles triangle and bZ = bF

In like manner bx' = bD

Therefore the circle on FD as diameter and bF as radius passes through Z and X'.

Since x'D FZ are concyclic points ZX' is antiparallel to FD

ZX' is antiparallel to XZ' (XZ' being parallel to FD)

Therefore Z, Z', X, X' are concyclic points,

In a similar proof Z'Y is parallel to BC

Angle Z'YX' = YX'X (alternate-interior angles)

Since YX' is antiparallel to AB (angle YX'X = A)

Since Z'X is antiparallel to CA (angle X'XZ = A)

Therefore Y lies on the circle through X'XZ'

In a similar proof Y' is on the circumference passing through X, X', Y, Z, Z'.

Therefore the Taylor Circle passes through the six points X, X', Y, Y', Z, Z'.

To Prove: T, the center of the Taylor circle is also the incenter of the triangle abc.

Since bZ = bX' the perpendicular to ZX' through its middle point bisects the angle abc and passes through the incenter of the triangle abc. Then the perpendicular bisector of XY' will pass through the center of the Taylor Circle and also bisect the angle bca (since XC = Y'C). Finally the perpendicular bisector of Z'Y bisects the angle bac (since aZ' = aY) and passes through the incenter of the triangle abc.

Therefore T the center of the Taylor Circle is also the incenter of the triangle abc.

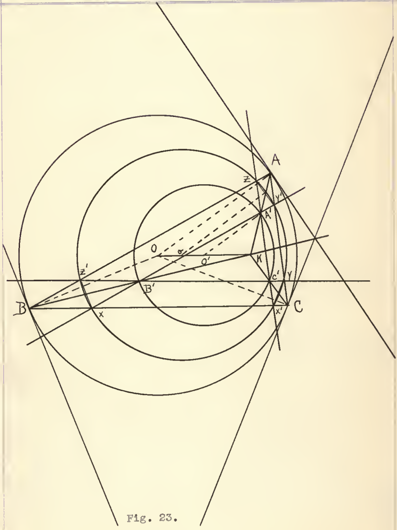


Fig. 23.

Tucker Circles.

If K be the symmedian point of the triangle ABC and points A'B'C' are taken on KA, KB, KC so that

KA': KA = KB': KB = KC' : KC = $\frac{1}{2}$ or any constant ratio

Then if B'C' meet CA, AB in YZ' respectively

C'A' meet AB, BC in ZX' respectively

A'B' meet BC, CA in XY' respectively

the points X, X', Y, Y', Z, Z' lie on a circle whose center bisects the line joining the circumcentres of ABC, A'B'C'. This circle is called the Tucker Circle.

Figure 23, page 57.

A'B' is parallel to AB and A'C' is parallel to AC

Then AZA'Y' is a parallelogram and ZY' and AA' bisect each other.

Since AA' is a symmedian line ZY' is antiparallel BC and parallel to the tangent to the circumcircle at A.

The triangle ABC is similar to triangle A'B'C'(sides parallel).

The triangles ABC and A'B'C' are similarly situated with respect to K so the tangents to the circumcircles at A and A', B and B', C and C' must be parallel.

If O and O' are the circumcentres the perpendicular bisector to ZY' is parallel to AO and A'O' and passes through the mid point of OO'.

Similarly the perpendiculars to XZ' and YX' through their middle points pass through o.

Now to prove X, X', Y, Y', Z, Z' are concyclic points.

Since ZY' is antiparallel to BC or Z'Y, Z, Z', Y, Y' are concyclic points.

Since Z'X is antiparallel to AC angle BZ'X = C

Also since Z'Y is parallel to BC angle Y'YZ' = C

Angle BZ'X = Z'XY' (alternate-interior angles)

Then angle Z'XY' = Z'YY'

Therefore X is on the circumference of Z', Y', Y.

In a similar proof X' is on the same circumference.

Therefore the one of the Tucker circles passes through XX' YY' ZZ'. By changing the constant ratio other Tucker circles could be found with the center always the mid-point of OO'.

The circumcircle, Lemoin's first circle, Lemoine's Second circle and Taylor circles are particular cases of Tucker circles.

As the ratio KA': KA changes, the locus of the centre of the Tucker circle is the line OK. When KA': KA = 1 Then A'B'C' coincides with A,B,C respectively, and the Tucker circle becomes the circumscribed circle.

When KA': KA = O A'B'C' coincide with K and the Tucker circle becomes Lemoine's First Circle or the Triplicate Ratio Circle.

When KA': KA = -1 the Tucker circle becomes the cosine circle for the middle point of AA' will coincide with K and the antiparallels ZY', YX', XZ' will all pass through K since ZY' bisects AA', Z'X bisects BB' and X'Y bisects CC'.

The Taylor Circle is also a Tucker circle for in Figure 22, page 54, it was proven YZ', ZX', XY' are parallel to BC, CA, AB respectively. If they are produced to meet in A'B'C' then BB' is a diagonal of the parallelogram BXB'Z' and therefore bisects Z'X, which is antiparallel to CA.

Therefore BB' is a symmedian and passes through K. Similarly AA' and CC' pass through K and by parallels Then KA': KA = KB': KB = KC': KC which is the criterion of the Tucker circle.

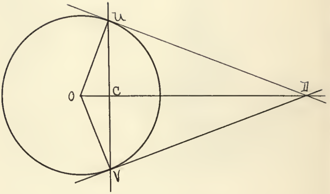


Fig. 24.

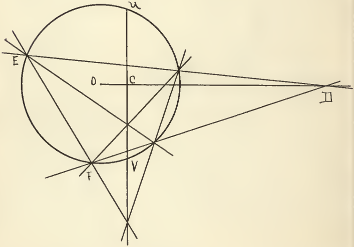


Fig. 25.

Poles and Polars of a Triangle with Respect to a Circle.

To find the polar of a given point with respect to a circle.

Figure 24, page 61.

(a) If from a point, D, without a circle, two tangents be drawn, the line UV passing through the points of contact is the polar of the given line.

UV is perpendicular to OD

Then triangle OUC is similar to ODU

OD: OU = OU: OC

Therefore $\overline{OU}^2 = OC \cdot OD$

Then AB is divided harmonically in the points C and D and UV is the polar of D.

(b) To find the polar by another method.

Figure 25, page 61.

From the point D, without the circle,

Draw two secants DE and DF

Draw the diagonals of quadrilateral formed by these 4 points of intersection

Then draw UV through the intersection of the diagonals and through the intersection of the two sides of the quadrilateral(not the secants) produced. Where UV cuts the line OD is C the harmonic of conjugate of D. By same proof as in (a) UV is polar of D.

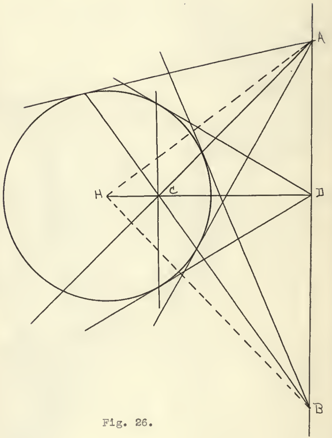


Fig. 26.

The Polar Circle.

If a triangle be such that each side is the polar of the opposite vertex with respect to a given circle, the triangle is said to be self-polar or self-conjugate with respect to the circle; and the circle is said to be polar with respect to the triangle.

Theorem concerning Self-Conjugate Triangle.

If two sides of a triangle are the polars of the opposite vertices with respect to a circle, the third side is the polar of the opposite vertex with respect to the same circle.

Figure 26, page 63.

A is the pole of BC

B is the pole of AC

Therefore C is the pole of AB

It is also true that the altitudes of a self-reciprocal triangle pass through the centre of the polar circle, that is the centre of a circle, polar with respect to a given triangle is the orthocentre of the triangle.

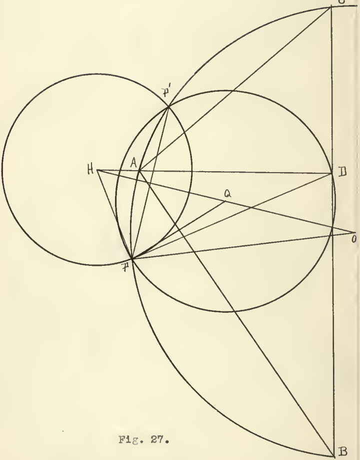


Fig. 27.

a. The centre of the circle to which a given triangle is self-conjugate is at the orthocentre of the triangle, and its radius is a mean proportional between HA and HD.

Figure 27, page 65.

Since the centre of the circle lies in the line through the pole perpendicular to the polar, the centre of the circle to which the triangle ABC is self-conjugate must be H, since A is inverse of D $\overline{HP}^2 = (HA)(HD)$

Therefore the radius is a mean proportional between HA and HD.

b. The circumcircle of a triangle, the nine-point circle and the circle to which the triangle is self-conjugate are co-axal.

In a system of intersecting co-axal circles, the circles all have the same common chord.

To Prove, then that the circumcircle, nine point circle and the circle to which the triangle is self-conjugate are the same common chord.

In Figure 27, page 65.

Given the triangle ABC with its polar circle, center H; its nine-point circle, center Q; the circumscribed circle, center O.

To Prove: The circles are co-axal.

Let P be one of the points of intersection of the circumcircle and the polar circle

Q is the mid-point of OH (center of nine-point circle)

In the triangle HPO

$$\overline{PH}^2 + \overline{PO}^2 = 2\,\overline{PQ}^2 + \tfrac{1}{2}\,\overline{OH}^2$$

Now $\overline{PH}^2 = (HA)\,(HD)$ and $\overline{PO}^2 = R^2$

Substituting

$$(HA)\,(HD) + R^2 = 2\,\overline{PQ}^2 + \tfrac{1}{2}\,\overline{OH}^2$$

Now $\overline{OH}^2 = R^2 + 2\,(HA)\,(HD)$

Then $(HA)\,(HD) + R^2 = 2\,\overline{PQ}^2 + \tfrac{1}{2}\,R^2 + (HA)\,(HD)$

Collecting and clearing of fractions

$$4\,\overline{PQ}^2 = R^2$$

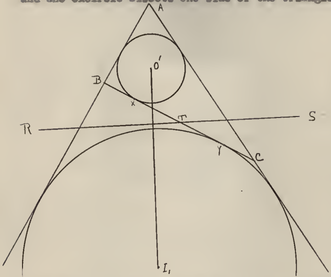
$$PQ = \tfrac{1}{2}\,R$$

Therefore P is on circumference of nine-point circle with center Q.

The same proof holds for P'

Therefore PP' is a common chord of the three circles and they are co-axal.

Exercise V. Prove that the radical axis of the incircle and one excircle bisects one side of the triangle.



Given: Triangle ABC with inscribed circle O' and escribed circle I_1; O, I_1 the line of centers; RS the radical axis cutting the side BC in T.

To Prove: BT = TC

TX = TY (Tangents from any point on radical axis to circles are equal).

BX = CY (Points of tangency of escribed and inscribed circles are equidistant from the two vertices of the triangle on the included side)

Adding TX + BX = TY + CY

Therefore BT = TC

Fuhrmann's Circle.

The circle on the segment from the Nagel point to the orthocentre as diameter passes through those three points on the altitudes whose distances from the corresponding vertices are equal to the diameter of the inscribed circle. This circle is called Fuhrmann's circle.

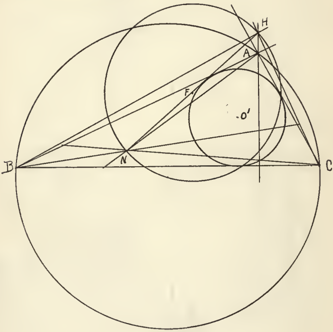


Fig. 28.

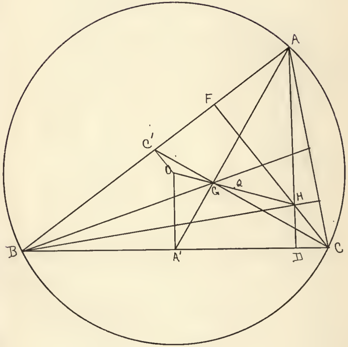


Fig. 20.

Euler Line of Triangle.

The circumcentre, centroid, orthocentre and nine-point centre of a triangle lie on a straight line called the Euler line of a triangle.

In Figure 28, page 69, locate the circumcentre O, and the centroid G. Then produce OG to a point H so that OG = $\frac{1}{2}$ GH.

To Prove: H is the orthocentre

Produce the join of AH to meet BC in D

Since A'G = $\frac{1}{2}$ GA Then $\frac{A'G}{GA} = \frac{OG}{GH}$

Angle AGH = OGA' (vertical angles)

Therefore the triangle A'OG is similar to the triangle AGH (Two sides including equal angles are proportional)

Then OA' is parallel to AD (alternate-interior angles are equal)

Since OA' is perpendicular to BC then

AD is perpendicular to BC

Similarly CH is perpendicular to AB.

Therefore H is the orthocentre.

Now A'D is a chord of the nine-point circle

The perpendicular bisector of A'D would pass through the centre of the nine-point circle and also through the mid-point of OH.

C'F is also a chord in the nine-point circle and the perpendicular bisector of C'F would also pass through the centre of the nine-point circle and the mid-point of OH. Then the mid-point Q of OH is the centre of the nine-point circle. Therefore O, G, Q, H all lie on the Euler line in a triangle.

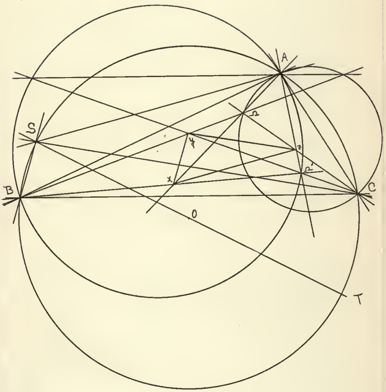


Fig. 30.

Steiner's Point and Tarry's Point.

The lines through the vertices of a triangle parallel to the corresponding side-lines of Brocard's first triangle are concurrent on the circumscribed circle. This point of concurrence is called Steiner's Point. That diametrically opposite is Tarry's Point (6).

To locate Steiner's Point and also Tarry's Point on a circumscribed circle.

Given: The triangle ABC with the circumscribed circle.

Draw the circle touching AB at A and passing through C. Draw the line AR parallel to BC. Then the line BR cuts the circle in the positive Brocard point Ω.

Then draw the circle touching AC at A. Construct the line R'A parallel to BC. Then the line CR' cuts this circle in the negative Brocard point Ω'.

Denote the intersections of $A\Omega$, $B\Omega$, $C\Omega$, with $B\Omega'$, $C\Omega'$, $A\Omega'$ by x, y, z respectively. Then xyz is called the first Brocard triangle.

Now through the vertices A, B, C draw lines parallel to zx, yx, yz respectively. These lines are concurrent in a point S of the circumscribed circle which is called Steiner's Point. The Point T diametrically opposite is called Tarry's Point.

Exercise VI.

a. What is the result of inverting a triangle with respect to its incircle?

b. What is the result of inverting a triangle with respect to its circumcircle?

c. What is the result of inverting a triangle with respect to its escribed circle?

INVERTING A TRIANGLE

(a) With respect to its incircle.

Given: The triangle ABC and the inscribed circle with center O'. As the inscribed circle is the circle of inversion O' is the center of inversion. (Figure 31, page 78).

To invert the side of the triangle CA. Let b be any point on CA. Then draw tangents from b to the circle of inversion. The line joining the points of tangency cuts the join of O'b in the inverse of b which is designated by b'. The point S of course inverts into itself. The point at infinity inverts into the center of inversion. Then the inverse of all points on CA lie on the circumference O'C'S.

In like manner the inverse of any point a on BC is a' on the circle B'RO'.

Consequently the side BC inverts into the circumference B'RO'.

Similarly the side AB inverts into the circle C'TO. It is clearly seen that any line not passing through the center of inversion inverts into a circle passing through the center of inversion.

It is interesting to note

As the sides CA and BC intersect in C,

So the inverses of these sides intersect in C'

As the sides AB and BC intersect in B,

So the inverses of these sides intersect in B'

As the sides CA and AB intersect in A,

So the inverses of these sides intersect in A'.

Thus the through sides of a triangle inverted with respect to the inscribed circle invert into three circles passing through the center of inversion and tangent to the sides of the triangle.

(b) Inverting a triangle with respect to its circumscribed circle.

Given the triangle ABC with circumscribed circle, center O. By the same methods used for (a) in (Figure 31, page 78). The sides of the triangle ABC are inverted with respect to O as the center of inversion (Figure 32, page 79).

Consequently the side CA inverts into the circle OCA

the side AB inverts into the circle BOC'

the side BC inverts into the circle BOC.

The side CA which is in the circle of inversion will invert into the circumference CB'A outside of the circle of inversion.

The part of CA extends outside of the circle of inversion will invert into the circumference AOC with in the circle of inversion.

Thus, the three sides of a triangle inverted with respect to the circumscribed circle invert into three circles passing through the center of inversion and intersecting the circle of inversion in the vertices of the triangle.

(c) Inverting a Triangle with respect to an escribed circle.

Given the triangle ABC with escribed circle tangent to BC and center I (Figure 33, page 80).

By methods of inversion explained on page 75.

Side AB inverts into circumference I'A'B'

Side BC inverts into circumference IC'B'

Side CA inverts into circumference IC'A'

Thus the three sides of a triangle inverted with respect to an escribed circle invert into three circles tangent to one side and to the other two sides produced.

These circles pass through the center of this escribed circle of inversion.

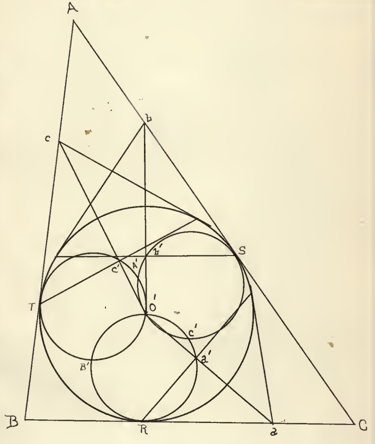


Fig. 31.

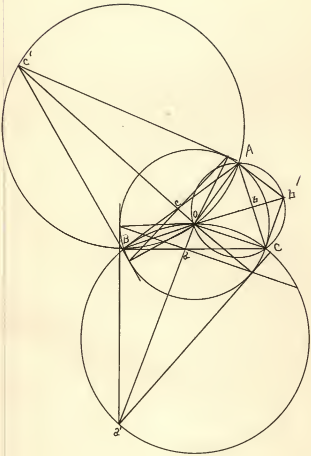


Fig. 32.

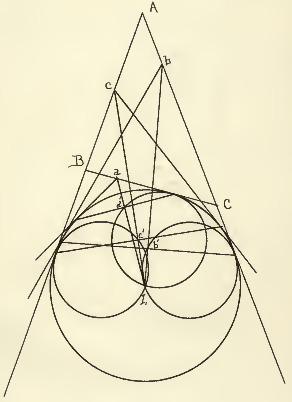


Fig. 33.

ACKNOWLEDGMENT

I wish to acknowledge my indebtedness to Professor A. E. White of the Mathematics Department for his helpful guidance and direction in the work of preparing this thesis.

I also wish to express my appreciation for the kindly interest always manifested by Dr. J. E. Ackert, Chairman of the Graduate Council.

REFERENCES

1. Cajori, Florian. A History of Elementary Mathematics. MacMillan Co., page 260.

2. Cajori, Florian. A History of Elementary Mathematics. MacMillan Co., page 260.

3. Coolidge, Treatise on the Circle and the Sphere. Oxford Press. page 56.

4. Coolidge, Treatise on the Circle and the Sphere, Oxford Press. page 50.

5. Coolidge, Treatise on the Circle and the Sphere, Oxford Press. page 78.

6. Coolidge, Treatise on the Circle and the Sphere, Oxford Press. page 77.

7. Phillips, William H. H. First Principles of Modern Geometry. Sheldon and Company, 1874.

8. Halstead, George Bruce. Metrical Geometry. Ginn, Heath and Company. 1881.

9. Russell, John Wellseley. Pure Geometry. MacMillan Company. 1893.

10. Godfrey and Siddons. Modern Geometry. Cambridge University Press. 1923.

11. Richardson and Ramsey. Modern Plane Geometry. MacMillan Company. 1928.

12. Duvell, C. V. Modern Geometry. MacMillan Company. 1928.