TWO-PERSON ZERO-SUM GAME THEORY

by

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INTRODUCTION

The theory of games of strategy may be described as a mathematical theory of decision-making by participants in a competitive environment. Some common examples of games of strategy are such parlor games as chess, bridge, and poker, where the players make use of their ingenuity to outwit each other. The theory of games is gaining importance because of its general applicability to real-life situations which involve conflicting interests in which the outcome is partially controlled by one side and partially by the opposing side of the conflict. Military attack and defense of targets against attack, and economic price competition between two sellers, are real-life games of strategy.

Game theory does not describe how a game should be played but rather what strategy a player should select assuming that his opponent chooses his best possible strategy. The theory of games assumes that a player attempts to select a strategy which maximizes his smallest gain (security level).

Games of chance have been studied mathematically for many years. The mathematical theory of probability has resulted from such study. The French mathematician Emile Borel, in 1921, made one of the first attempts to abstract games of strategy into a mathematical theory of strategy with the formulation of what is now part of the minimax theory, but he was held back by his failure to prove this theorem (8).

John von Neumann, on December 7, 1926, gave a talk to the Mathematical Society in Göttingen in which he proved parts of the minimax theorem (8). However, it was not until 1944, with the publication of Theory of Games of Economic Behavior, by von Neumann and Oskar Morgenstern, that the mathematical theory of games received much attention (4). Dr. Morgenstern is a professor of economics (at Princeton); some chapters of the book stress economic significance of the results of game theory.

Terminology and Classification

A game of strategy is described by its set of rules which specify clearly what each person called a "player" is allowed to do under all possible circumstances. The rules of any game must specify in advance which moves, known as "information sets," are indistinguishable to the players. When an information set consists of a single move, the player is totally informed. When all the moves are of this type the game is said to have perfect information. Ticktacktoe and chess are examples of games whose rules result in perfect information.

The word "play" will be employed to denote the number of times a particular game is played. The word "move" will mean a point in the game at which one of the players selects one of a set of alternatives. The word "choice" will mean the alternative selected. The following example uses this terminology.

> Black won the third play of the (chess) game by a clever choice on his tenth move.

A game is finite if each player has a finite number of moves and a finite number of choices available at each move; other games are called infinite. Games are classified according to the number of players, i.e., as 2-person, 3-person, etc. It is convenient to distinguish between games whose pay-offs are zero-sum and those which are not. Consider a play of an n-person game with players $P_1, P_2, \ldots, P_n$ and let $\phi_i (i=1, 2, \ldots, n)$ be the payment made to P_i at the end of the play. Then if

$$\sum_{i=1}^{n} \phi_i = 0$$

we call the play zero-sum. If every possible play of a game is zero-sum, the game itself is called a zero-sum game. All other games are called non-zero-sum games.

Mathematical Formulation

To simplify the mathematical description of a game, each player formulates in advance a plan for playing the game from the beginning to the end, instead of making his decision at each move. Such a complete plan of play is called a strategy of that player. A strategy must be complete and cover all possible conditions that may arise in the play.

Suppose that one of the players, Blue, has m strategies, which correspond to the numbers

$$i = 1, 2, \ldots, m.$$

Suppose the other player, Red, has n strategies, which correspond to

$$j = 1, 2, \ldots, n.$$

Every pair of strategies, one strategy for each player, determines a play of the game. Thus a play of a game consists of each player making one decision, the selection of a strategy. These two choices determine a play of the game and a pay-off to the two players. Let a_{ij} be the pay-off to Blue. The pay-off to Red is $-a_{ij}$ in a two-person zero-sum game.

The game is thus determined by Blue's pay-off matrix.

$$A = (a_{ij})_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

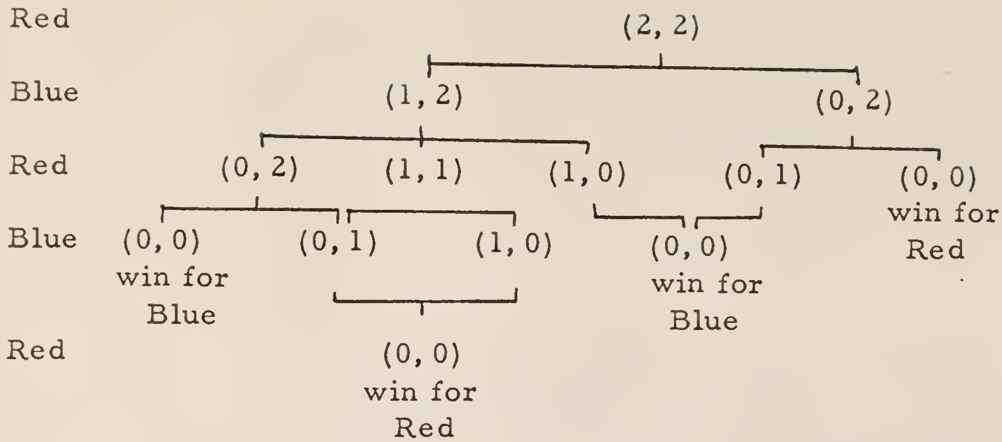
In this matrix each Blue strategy is represented by a row; each Red strategy is represented by a column. If Blue chooses the i^{th} strategy or row i and Red chooses the j^{th} strategy or column j, then Red is to pay Blue the amount a_{ij}. Blue wants a_{ij} to be as large as possible, but he controls only the choice of his strategy i. Red wants a_{ij} to be as small as possible, but he controls only the choice of j. Hence, we have a conflict; Blue maximizing a_{ij} by his choice of i, and Red minimizing a_{ij} by his choice of j. Blue will be referred to as the maximizing player and Red the minimizing player.

As an illustration of a game of strategy, consider the following simplified version of the game known as NIM, for which general rules and description can be found in many places including (1, pp. 36-38).

Let two piles P_1 and P_2 of two items each be given. The player Red takes either one or two items from one pile. Let this pile be called P_1. Player Blue then draws one or two items from either pile. The drawing continues until the player picking up the last item loses.

The following game tree shows the possible succession of moves of Red and Blue. Here, the notation, (x, y) means that the first pile contains x items and the second pile contains y items. On Red's first move he is faced with (2, 2) which means there are two items in each pile. After Red's first move, Blue is faced with either (1, 2), one in the pile P_1 and two in the pile P_2 or (0, 2), two in the pile P_2.

Move of:



Each player decides, in advance, what he will do in any possible situation and all of these decisions together form a strategy.

The player's strategies are given in the following tables. The column headed Move Number is the number of the move by the player. The column headed by Condition Before Move lists all the possible conditions which the player might meet on the particular move in the first column. The column under the heading Condition After Move lists in each column the condition in which the player leaves the game if he uses the strategy above the column. If such a position is not possible using a certain strategy, this is shown by dashes (--).

Move Number	Condition Before Move	Condition After Move				
		Red's Strategies				
		1	2	3	4	5
1	(2, 2)	(1, 2)	(1, 2)	(1, 2)	(1, 2)	(0, 2)
2	(0, 2)	(0, 1)	(0, 0)	(0, 1)	(0, 0)	--
	(1, 1)	(0, 1)	(0, 1)	(1, 0)	(1, 0)	--
	(1, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	--
	(0, 1)	--	--	--	--	(0, 0)

For example, Red's first strategy consists of the following moves: To begin with, he is faced with position (2, 2). Red will take one item from P_1 thus changing the position to (1, 2). At Red's second move he may be faced with (0, 2) if Blue has taken the remaining item from P_1. Red may be faced with (1, 1) if Blue has taken one item from P_2, with (1, 0) if Blue has taken two items from P_2, but Red will not be faced with (0, 1) shown by the dashes in the strategy 1 column. Red decides that if he were faced with (0, 2) he would reduce this to (0, 1), if with (1, 1) to (0, 1) and if with (1, 0) he would reduce this to (0, 0). Red's other strategies are similarly given in the table.

Blue strategies are tabulated as follows:

Move Number	Condition Before Move	Condition After Move					
		Blue's Strategies					
		1	2	3	4	5	6
1	(1, 2)	(0, 2)	(0, 2)	(1, 1)	(1, 1)	(1, 0)	(1, 0)
	(0, 2)	(0, 0)	(0, 1)	(0, 0)	(0, 1)	(0, 0)	(0, 1)
2	(0, 1)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	--	--
	(1, 0)	--	--	(0, 0)	(0, 0)	--	--

The reduction of a game's structure to strategies is called normalization. There exists an elaborate theory referring to games in their extensive form (i.e., taking account of the succession of moves and, in particular, of the pattern of information). This paper will not discuss this aspect of game theory.

The pay-off matrix to Blue is as follows (where 1 denotes a win and -1 a loss by Blue).

$$
\begin{array}{ccc}
 & & \text{Red} & \\
 & \text{Strategy} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} & \text{Row Min.} \\
\text{Blue} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{matrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{matrix} \\
 & \text{Col. Max.} & \begin{matrix} 1 & 1 & 1 & 1 & 1 \end{matrix} &
\end{array}
$$

In addition to the pay-off matrix an extra column and row have been added giving the row minimum and column maximum respectively. This represents the least amount a player can receive from a strategy and is called the security level of that strategy. For Red, the column maximum represents the worst that could happen if he uses each strategy; a loss might occur. For Blue the row minimum for Blue's sixth strategy is +1 so the worst that could happen if Blue uses his sixth strategy is that he would win. Thus, Blue maximizes his security level by selecting strategy six.

The game just represented is a game of perfect information because every previous move is at all times known to each of the players. This game possesses a special feature in that there was an entry in the pay-off matrix which was the smallest in its row and at the same time the largest in its column. Such an element is called a saddle point. If Blue were to announce in advance that he planned to play strategy six, Red could not take advantage and reduce Blue's pay-off. Similarly, if Red were to announce which strategy he was using, Blue could not increase his own pay-off.

MATRIX GAMES

A two-person zero-sum game Γ consists of a pair of sets I and J and a real valued function $\emptyset$ defined on the pairs (i, j) where $i \in I$ and $j \in J$. The elements $i \in I$ and $j \in J$ are called the strategies for players Blue and Red respectively. The function $\emptyset$ is called the pay-off function. The pay-off function is represented by the game matrix $A = (a_{ij})$.

Consider the general pay-off matrix $A = (a_{ij})$ for any strategy i which Blue chooses; he can be sure of receiving at least

$$\min_{j \in J} a_{ij} ,$$

where the minimum is taken over all of Red's strategies. Since Blue is at liberty to choose i, he can make his choice in such a way as to insure that he receives at least

$$\max_{i \in I} \min_{j \in J} a_{ij} ,$$

called the maxmin of Γ. Similarly, for any strategy j which Red may choose, he can be sure Blue gets no more than

$$\max_{i \in I} a_{ij} .$$

Since Red is at liberty to choose j, he can choose it in such a way that Blue gets at most

$$\min_{j \in J} \max_{i \in I} a_{ij} ,$$

called the minmax of Γ. These two quantities in general are different, and a relationship between them is given by the theorem:

Theorem 1:

If $\emptyset$ is a function of x and y, for which

$$\min_{y \in Y} \max_{x \in X} \emptyset (x, y)$$

and

$$\max_{x \in X} \min_{y \in Y} \emptyset (x, y)$$

exist, then

$$\max_{x \in X} \min_{y \in Y} \emptyset (x, y) \leq \min_{y \in Y} \max_{x \in X} \emptyset (x, y) .$$

Proof:

From the definition of minimum, given any $x \in X$, one has

$$\min_{y \in Y} \phi(x, y) \leq \phi(x, y) .$$

From the definition of maximum, given any $y \in Y$, one has

$$\phi(x, y) \leq \max_{x \in X} \phi(x, y) .$$

Hence, combining the above two inequalities,

$$\min_{y \in Y} \phi(x, y) \leq \phi(x, y) \leq \max_{x \in X} \phi(x, y) .$$

Since the right-hand side of the preceding inequalities is independent of x,

$$\max_{x \in X} \min_{y \in Y} \phi(x, y) \leq \max_{x \in X} \phi(x, y) .$$

Since the left-hand side of the same inequalities is independent of y,

$$\max_{x \in X} \min_{y \in Y} \phi(x, y) \leq \min_{y \in Y} \max_{x \in X} \phi(x, y) .$$

Games with Saddle-Points

If the preceding inequality becomes an equality,

$$\max_{i \in I} \min_{j \in J} a_{ij} = \min_{j \in J} \max_{i \in I} a_{ij} = v,$$

then Blue can choose a strategy so as to receive at least the common value and Red can keep Blue from getting more than v. The strategies for Blue and Red respectively i^*, j^* which guarantee this value v are referred to as optimal strategies. This pair of strategies is called the solution of the game.

Suppose $\emptyset$ is a real valued function such that $\emptyset(i, j)$ is defined $i \in I$ and $j \in J$; then a point a_{i^*, j^*} where $i^* \in I$ and $j^* \in J$ is called a saddle-point of $\emptyset$ if

1). $\emptyset(i, j^*) \leq \emptyset(i^*, j^*)$ for all $i \in I$ and

2). $\emptyset(i^*, j^*) \leq \emptyset(i^*, j)$ for all $j \in J$.

A necessary and sufficient condition for a game to have a saddle-point is that there exists an element of the pay-off matrix which is simultaneously the minimum of its row and the maximum of its column. A game may have several saddle-points. In such a case all the saddle-points have the same value.

Theorem 2:

The equality

$$\max_{i \in I} \min_{j \in J} \emptyset(i, j) = \min_{j \in J} \max_{i \in I} \emptyset(i, j)$$

holds if, and only if, $\emptyset$ has a saddle-point.

Proof:

If (i^*, j^*) is a saddle-point, then for all $j \in J$ and $i \in I$

$$\emptyset (i, j^*) \leq \emptyset (i^*, j^*), \tag{1}$$

$$\emptyset (i^*, j^*) \leq \emptyset (i^*, j). \tag{2}$$

Since the inequality (1) is true for all $i \in I$,

$$\max_{i \in I} \emptyset (i, j^*) \leq \emptyset (i^*, j^*).$$

Since the inequality (2) is true for all $j \in J$,

$$\emptyset (i^*, j^*) \leq \min_{j \in J} \emptyset (i^*, j).$$

Combining the above inequalities, one has

$$\max_{i \in I} \emptyset (i, j^*) \leq \emptyset (i^*, j^*) \leq \min_{j \in J} \emptyset (i^*, j). \tag{3}$$

From the definition i^* and j^*, it follows that

$$\min_{j \in J} \max_{i \in I} \emptyset (i, j) \leq \max_{i \in I} \emptyset (i, j^*)$$

and

$$\min_{j \in J} \emptyset (i^*, j) \leq \max_{i \in I} \min_{j \in J} \emptyset (i, j).$$

Therefore,

$$\min_{j \in J} \max_{i \in I} \phi(i,j) \leq \phi(i^*, j^*) \leq \max_{i \in I} \min_{j \in J} \phi(i,j). \tag{4}$$

But by theorem 1, the left member of (4) is not less than $\max_{i \in I} \min_{j \in J} \phi(i,j)$.

Hence, all three members are equal.

$$\min_{j \in J} \max_{i \in I} \phi(i,j) = \phi(i^*, j^*) = \max_{i \in I} \min_{j \in J} \phi(i,j).$$

Conversely, let $i^* \in I$ and $j^* \in J$ such that,

$$\max_{i \in I} \min_{j \in J} \phi(i,j) = \min_{j \in J} \phi(i^*, j)$$

and

$$\min_{j \in J} \max_{i \in I} \phi(i,j) = \max_{i \in I} \phi(i, j^*).$$

Since

$$\min_{j \in J} \max_{i \in I} \phi(i,j) = \phi(i^*, j^*) = \max_{i \in I} \min_{j \in J} \phi(i,j),$$

the equations above lead to the result:

$$\min_{j \in J} \phi(i^*, j) = \max_{i \in I} \phi(i, j^*). \tag{5}$$

From the definition of a minimum,

$$\min_{j \in J} \emptyset\,(i^*, j) \leq \emptyset\,(i^*, j^*) \tag{6}$$

and a maximum,

$$\emptyset\,(i^*, j^*) \leq \max_{i \in I} \emptyset\,(i, j^*). \tag{7}$$

Substituting into equation (5), the definitions from (6) and (7), one has

$$\emptyset\,(i^*, j^*) \leq \min_{j \in J} \emptyset\,(i^*, j)$$
$$\emptyset\,(i^*, j^*) \geq \max_{i \in I} \emptyset\,(i, j^*).$$

Therefore,

$$\emptyset\,(i^*, j^*) \leq \emptyset\,(i^*, j) \text{ for all } j \in J$$

$$\emptyset\,(i^*, j^*) \geq \emptyset\,(i, j^*) \text{ for all } i \in I$$

satisfy the definition of a saddle-point.

The game corresponding to the matrix

$$\begin{array}{cc} & \text{Row Min.} \\ \begin{bmatrix} 2 & 6 & 1 & 1 \\ 3 & 5 & 2 & 4 \end{bmatrix} & \begin{array}{c} 1 \\ 2^* \end{array} \\ \text{Col. Max.} \quad 3 \quad 6 \quad 2^* \quad 4 & \end{array}$$

has a saddle-point. The element in the second row and third column is a saddle-point for the game.

Games with Perfect Information

In order to prove the next theorem, it is convenient to introduce the notation of the truncation of a game of perfect information.

Truncations of a game are those games which arise from a given game if the first move is deleted. The number of truncations of a game is the number of alternatives available at the first move. The strategy of a truncation of a game picks out the same alternatives at the branch points as does the original strategy.

Theorem 3:

If the game Γ with matrix $\emptyset$ (i, j) is a game of perfect information, then Γ has a saddle-point.

Proof:

The proof is by induction on the length of the game. If Γ is of length one (i. e., only one move) the theorem is obvious.

Suppose the theorem is true for all games of length less than K. Let Γ be a game of length K. Suppose that there are r alternatives on the first move, and let Γ_1, Γ_2, ..., Γ_r be the r truncations of Γ. For each of the games Γ_u let $\emptyset_u$ be the corresponding pay-off function and I_u and J_u be the set of pure strategies for Red and Blue respectively.

By the induction hypothesis, there is an equilibrium point in each of the sets I_u and J_u. For each Γ_u, let $(i_u{}^*, j_u{}^*)$ be a saddle-point.

Then,

$$\emptyset_u(i_u, j_u^*) \leq \emptyset_u(i_u^*, j_u^*) \leq \emptyset_u(i_u^*, j_u) \tag{8}$$

for u = 1, 2, ..., r and $i_u \in I_u$, $j_u \in J_u$.

The game Γ has two cases: (1), the first move is made by chance; or (2), the first move is made by one of the players.

Case 1. First move made by chance.

If q is a branch point of the truncated game Γ_u, and corresponds to a move made by Blue or Red, set

$$i^*(q) = i_u^*(q)$$

$$j^*(q) = j_u^*(q) \text{ respectively.}$$

Since the first move is made by chance, i* is defined over branch points of Γ. Thus i* corresponds to a move made by Blue and is a member of I. Similarly j* is a member of J. It is sufficient to show (i*, j*) is a saddle-point of Γ.

Let the probabilities assigned to the r alternatives at the first move be a_1, a_2, ..., a_r; then

$$\emptyset\,(i, j) = \sum_{u=1}^{r} a_u \emptyset_u(i_u, j_u).$$

In particular, since $i_1^*, i_2^*, \ldots, i_r^*$ are truncations of i* and $j_1^*, j_2^*, \ldots, j_r^*$ are truncations of j*,

$$\emptyset\,(i, j^*) = \sum_{u=1}^{r} a_u \emptyset_u(i_u, j_u^*),$$

$$\emptyset\,(i^*, j) = \sum_{u=1}^{r} a_u \emptyset_u(i_u^*, j_u),$$

and

$$\emptyset\,(i^*, j^*) = \sum_{u=1}^{r} a_u \emptyset_u(i_u^*, j_u^*).$$

From (8) it follows that

$$\sum_{u=1}^{r} a_u \emptyset_u(i_u, j_u^*) \leq \sum_{u=1}^{r} a_u \emptyset_u(i_u^*, j_u^*) = \emptyset(i^*, j^*),$$

$$\sum_{u=1}^{r} a_u \emptyset_u(i_u^*, j_u) \geq \sum_{u=1}^{r} a_u \emptyset_u(i_u^*, j_u^*) = \emptyset(i^*, j^*).$$

Hence (i^*, j^*) is a saddle-point of Γ.

Case 2. The first move made by one of the players.

The first move may be made by either player since the proof for Red's move is analogous to the proof for Blue's move and vice versa. Assume the first move, q_0, of Γ is made by Blue.

Let

$$\max_{u \leq r} \emptyset_u(i_u^*, j_u^*) = \emptyset_m(i_m^*, j_m^*). \tag{9}$$

Define a function i^* by setting

$$i^*(q_0) = u \text{ and } i^*(q) = i_u^*(q)$$

for any point q in the truncated games Γ_u which corresponds to a move made by Blue. The j^* is defined as in the previous case. The strategies i^* and j^* thus defined are strategies of Γ. It will now be shown these strategies yield a saddle-point of Γ.

If $i \in I$ for Blue in Γ and i_m is its truncation to Γ_m, then

$$\phi(i^*, j) = \phi_m(i_m^*, j_m).$$

In particular,

$$\phi(i^*, j^*) = \phi_m(i_m^*, j_m^*). \tag{10}$$

Thus, $j \in J$ for Red in Γ, and if j_m is its truncation to Γ_m,

$$\phi(i^*, j^*) = \phi_m(i_m^*, j_m^*) \leq \phi_m(i_m^*, j_m) = \phi(i^*, j).$$

Suppose that

$$i(q_0) = u.$$

Let i_u be the truncation of i to Γ_u. Then, if $j \in J$ and j_u is its truncation to Γ_u,

$$\phi(i, j) = \phi_u(i_u, j_u).$$

In particular,

$$\phi(i, j^*) = \phi_u(i_u, j_u^*)$$

Now from (9),

$$\phi_m(i_m^*, j_m^*) \geq \phi_u(i_u^*, j_u^*).$$

Hence, using equation (10),

$$\phi(i^*, j^*) = \phi_m(i_m^*, j_m^*) \geq \phi(i_u^*, j_u^*) \geq \phi_u(i_u, j_u^*) = \phi(i, j^*).$$

Thus (i^*, j^*) is a saddle-point.

The existence of a saddle-point in the game of chess follows from the fact that it is a game of perfect information. It all the possible strategies for chess were enumerated, optimal strategies could be found. However, because of the large number of strategies for chess, saddle-points have not been computed. The game ticktacktce has a small number of strategies, and so an optimal strategy can be found.

Games without Saddle-Points

Consider games whose pay-off matrix is such that

$$\max_{i \in I} \min_{j \in J} a_{ij} < \min_{j \in J} \max_{i \in I} a_{ij}$$

The left-hand side of the inequality represents Blue's minimum security level (the least amount Blue can receive) and the right-hand side represents the negative of Red's minimum security level (the most Blue can receive).

The game defined by the pay-off matrix

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

does not have a saddle-point for

$$\min_{j \in J} \max_{i \in I} a_{ij} = 3$$

$$\max_{i \in I} \min_{j \in J} a_{ij} = 2$$

Since the game matrix has no saddle-point, previous methods do not determine optimum ways for Blue and Red to play. If Red can discover Blue's optimal strategy, Red can drive his winning down to 1 if the strategy is his first or 2 if the strategy is the second. Therefore, if Blue's strategies were discovered, his winnings would be 1 or 2. However, Blue is trying to get either 3 or 4.

Thus, in a game without a saddle-point, the player's strategy will depend on his opponent's choice. Therefore, each player's strategy should be kept unknown to his opponent. One way to do this is to play certain strategies by using a random device for selecting a strategy. He may choose a probability distribution over his set of strategies and then an associated random probability distribution over the whole set of strategies for the play of the game. Such a probability distribution over the whole set of strategies of a player is called a mixed strategy.

Suppose in the above game, Blue plays strategy 1 with frequency x and plays strategy 2 with frequency 1-x, and suppose Red plays 1 with frequency y and plays 2 with frequency 1-y.

		Blue	
		y	1-y
Red	x	1	3
	1-x	4	2

The mathematical expectation of Blue is

$$E(xy) = 1xy + 3x(1-y) + 4(1-x)y + 2(1-x) \cdot (1-y)$$

$$= 4xy + x + 2y + 2$$

$$= -4(x-\frac{1}{2})(y-\frac{1}{4}) + 5/2$$

When E(xy) is written in the above form, it is easily seen that if Blue takes x = 1/2 he can insure that the expectation will be at least 5/2. Red can insure the expectation of Blue will be no more than 5/2 by playing y = 1/4.

Since Blue's maxmin is 2, he will settle for 5/2 and play x = 1/2. Similarly Red's minmax is -3 and he will reconcile himself to getting -5/2 and play y = 1/4.

Thus the optimal mixed strategy for Blue is to play strategy 1 with probability 1/2 and strategy 2 with probability 1/2. Red's optimal strategy is to play strategy 1 with probability 1/4 and strategy 2 with probability 3/4. The value of the game is 5/2. The two optimal strategies are called the solution of the game.

Mixed strategies will be represented by column matrices. Let x_i be the probability of selecting strategy i. Then a mixed strategy, or probability distribution X, for Blue, may be represented by the row vector

$$X' = \left[x_1, x_2, \ldots, x_m\right]$$

where

$$x_i \geq 0 \qquad i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_i = 1.$$

Similarly, if y_j is the probability of selecting strategy j, then a mixed strategy or probability distribution Y, for Red, is a column vector

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}.$$

where

$$y_j \geq 0 \quad j = 1, 2, \ldots, n$$

and

$$\sum_{j=1}^{n} y_j = 1.$$

If $x_i = 1$ for some i, then X is called a pure strategy.

Suppose Blue chooses strategy i and Red chooses mixed strategy y; the expected pay-off to Blue is

$$h_i = \sum_{j=1}^{n} a_{ij} y_j$$

which is the i^{th} component of the column vector

$$H = AY = \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ \cdot \\ \cdot \\ h_m \end{bmatrix}$$

If Red uses strategy j and Blue uses mixed strategy x, the expected pay-off to Red is

$$k_j = \sum_{i=1}^{m} a_{ij} x_i$$

which is the j^{th} component of the row vector K', where

$$K' = X'A = (k_1, k_2, \ldots, k_n).$$

If Blue and Red use mixed strategies X and Y respectively, then the expected pay-off to Blue is

$$v = X'AY = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_iy_j = K'Y = X'H.$$

The following example of the Colonel Blotto Game taken from Dresher (3, pp. 7-8) demonstrates the application of mixed strategies.

Example: Colonel Blotto Game.

> Colonel Blotto and his enemy each try to occupy two posts by distributing their forces suitably. Let us assume that Colonel Blotto has 4 regiments and the enemy has 3 regiments which are to be divided between the two posts. Define the pay-off to Colonel Blotto at each post as follows: If Colonel Blotto has more regiments than the enemy at the post, Colonel Blotto receives the enemy's regiments plus one (the occupation of the post is equivalent to capturing one regiment); if the enemy has more regiments than Colonel Blotto at the post, then Colonel Blotto loses one plus his regiments at the post; if each side places the same number of regiments, it is a draw and each side gets zero. The total pay-off is the sum of the pay-offs at the two posts.
>
> Colonel Blotto has 5 strategies, or five different ways of dividing 4 regiments between the two posts. The enemy has 4 strategies, or four different ways of dividing his 3 regiments. There are, therefore, twenty ways for the two sides to distribute their forces.

It is evident that if Colonel Blotto places 3 regiments at the first post and 1 at the second, and if the enemy places 2 regiments at the first post and 1 at the second, then Blotto wins what amounts to 3 regiments. However, if Colonel Blotto places 2 regiments at each post and the enemy places all of his 3 regiments at either post, then Colonel Blotto loses 2 regiments. The following pay-off matrix summarizes the payment to Colonel Blotto for each of the twenty possible distributions:

Colonel Blotto Pay-off

Enemy Strategies

$$\text{Colonel Blotto Strategies} \quad \begin{array}{c} \\ (4,0) \\ (0,4) \\ (3,1) \\ (1,3) \\ (2,2) \end{array} \begin{array}{c} \begin{array}{cccc} (3,0) & (0,3) & (2,1) & (1,2) \end{array} \\ \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 3 \\ -2 & -2 & 2 & 2 \end{bmatrix} \end{array}$$

In the Colonel Blotto pay-off matrix, if the enemy uses the mixed strategy

$$Y = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix}$$

and Colonel Blotto uses a pure strategy, Blotto's expectation for each of his pure strategies is H = AY, or

$$H = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 3 \\ -2 & -2 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 2\ 1/4 \\ 1 \\ 1\ 3/4 \\ 1/2 \\ 1 \end{bmatrix},$$

where the components of H represent Colonel Blotto's receipts corresponding to each one of his five pure strategies.

Now, if the two players' strategies are

$$X = \begin{bmatrix} 1/4 \\ 0 \\ 0 \\ 1/4 \\ 1/2 \end{bmatrix}, \qquad Y = \begin{bmatrix} 1/4 \\ 0 \\ 1/2 \\ 1/4 \end{bmatrix},$$

then Blotto's expectation is $E = X'AY = X'H$, or

$$(1/4,\ 0,\ 0,\ 1/4,\ 1/2) \begin{bmatrix} 2\ 1/4 \\ 1 \\ 1\ 3/4 \\ 1/2 \\ 1 \end{bmatrix} = 19/16.$$

Colonel Blotto's expectation for other combinations of strategies can be evaluated in a similar fashion.

Graphical Representation of Mixed Strategies

It is possible to represent graphically the expectation of a player as a function of his mixed strategies.

If one player has two strategies and the other has any number of strategies, it is possible to solve the game graphically in two dimensions.

Consider a game with pay-off matrix

$$\begin{array}{ll} & \text{Red Strategies} \\ & \begin{array}{cc} R_1 & R_2 \end{array} \\ \text{Blue Strategies} \begin{array}{c} B_1 \\ B_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} . \end{array}$$

Any randomized strategy $X = (x_1, x_2)$ for Blue can be identified with a point (x_1, x_2) on the segment of length one as in Figure 1. If Blue chooses (x_1, x_2) and Red chooses strategy R_1, the pay-off to Blue is

$$\phi(X, R_1) = a_{11}x_1 + a_{12}x_2 .$$

Geometrically, $\phi(X, R_1)$ may be represented as the R_1 line joining a_{11} and a_{12}. Blue's security level may be represented as the vertical height from (x_1, x_2) to the R_1 line.

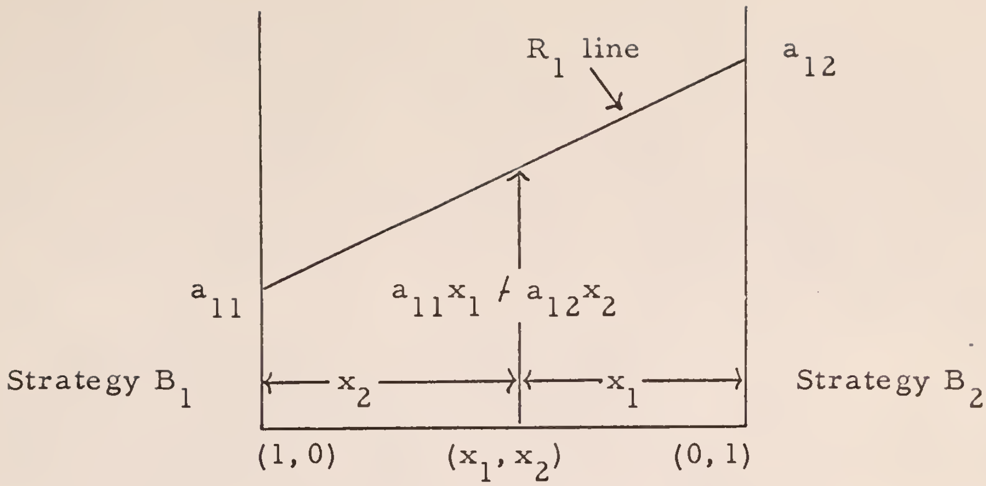


Figure 1.

Similarly if (x_1, x_2) is used against R_2, Blue's security level may be represented as the vertical height $a_{21}x_1 + a_{22}x_2$.

Blue may maximize his security level against Red's best play by playing $X^* = (x_1^*, x_2^*)$.

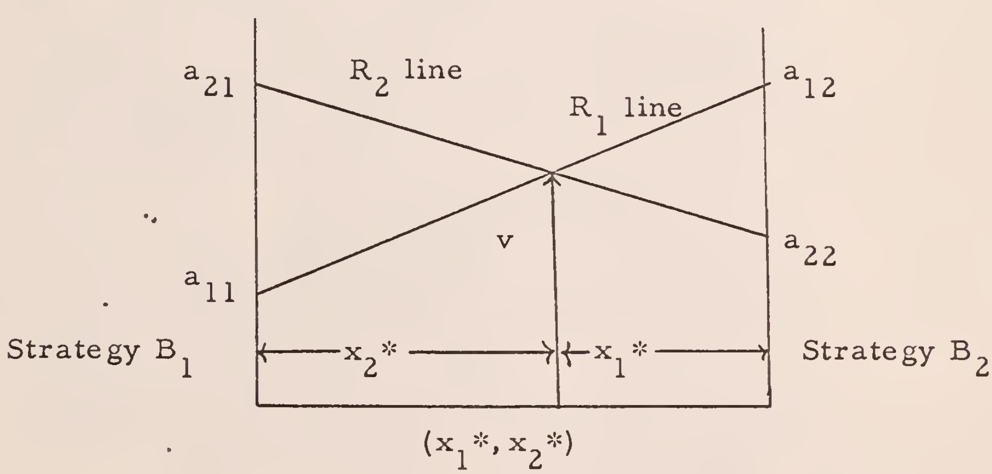


Figure 2.

Since the two lines intersect at the point $([x_1, x_2], v)$ then

$$a_{11}x_1^* + a_{12}x_2^* = v$$

$$a_{21}x_1^* + a_{22}x_2^* = v. \tag{11}$$

Since $x_1^* = (1 - x_2^*)$ the system (11) above may be reduced to

$$a_{11}(1-x_2^*) + a_{12}x_2^* = a_{11} + (a_{12} - a_{11})x_2^* = v$$

$$a_{21}(1-x_2^*) + a_{22}x_2^* = a_{21} + (a_{22} - a_{21})x_2^* = v. \tag{12}$$

The system (12) has solution

$$x_1^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}, \qquad x_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad \text{and}$$

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} . \tag{13}$$

To minimize Blue's pay-off, Red must use a mixed strategy $Y = (y_1, y_2)$. The pay-off from these mixed strategies is

$$\phi(X, Y) = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2$$

$$= y_1(a_{11}x_1 + a_{21}x_2) + y_2(a_{12}x_1 + a_{22}x_2). \tag{14}$$

The minimax theorem states if both Red and Blue use optimal strategies, the pay-off is the value of the game v, i.e.,

$$v = a_{11}x_1{}^*y_1{}^* + a_{12}x_1{}^*y_2{}^* + a_{21}x_2{}^*y_1{}^* + a_{22}x_2{}^*y_2{}^*. \tag{15}$$

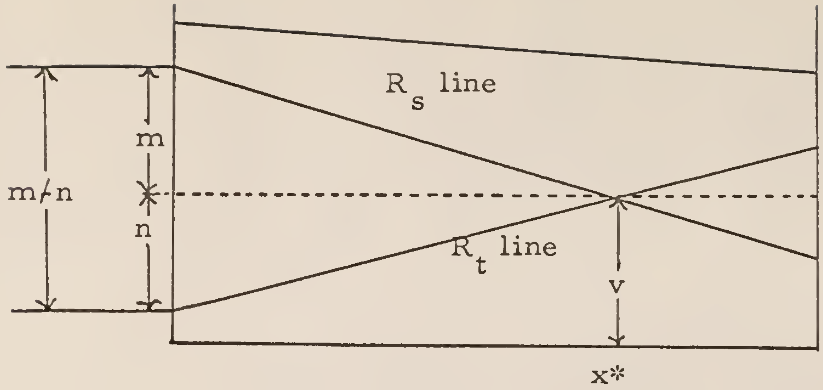
The values of $x_1{}^*$, $x_2{}^*$, and v have already been found (13); hence, equation (15) has the solution

$$y_1 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \quad \text{and} \quad y_2 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} .$$

Equation (14) above may be represented by a line which is a weighted average of the lines R_1 and R_2 in our diagram. Since $y_1 + y_2 = 1$, the line must always be between R_1 and R_2 and hence must pass through the point ($[x_1{}^*, x_2{}^*]$, v). For all members of this family of lines except the horizontal, if Blue chooses either pure strategy B_1 or B_2 his return will exceed v. This is impossible by the minimax theorem, hence, Red's optimal strategy may be represented geometrically as the horizontal line through a minimum ordinate of the intersections of two of Blue's pure strategies.

Red's optimal strategy $Y^* = (y_1{}^*, y_2{}^*)$ is a weighted average of two intersecting pure strategy lines R_s and R_t which have the minimum ordinate at the point of their intersection. Therefore,

$$v = y_s{}^*R_s + y_t{}^*R_t.$$



To find y_s* and y_t* let m and n be given by the equations

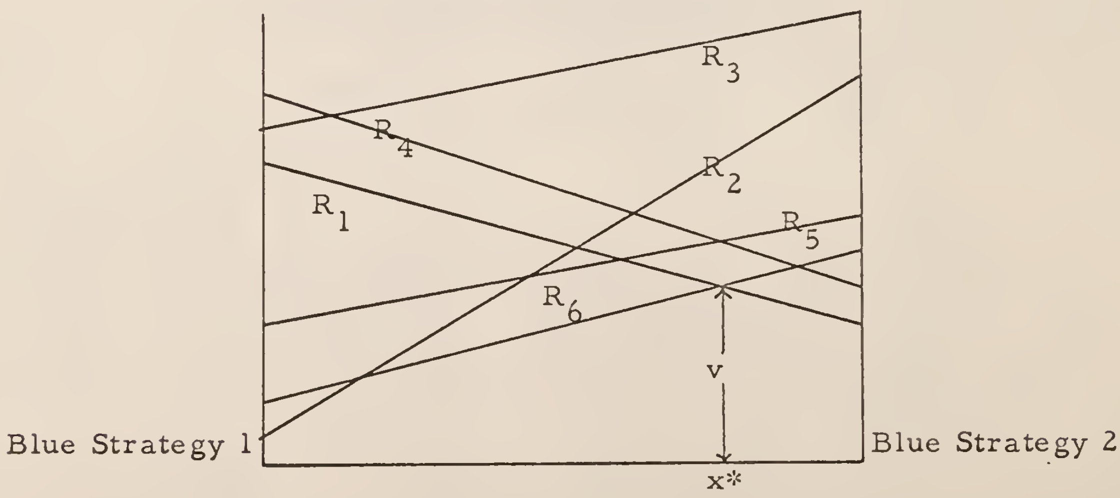
$$m = |R_s - v| \qquad \text{and} \qquad n = |R_t - v| \ .$$

Then

$$y_s^* = \frac{n}{m + n} \qquad \text{and} \qquad y_t^* = \frac{m}{m + n} \ .$$

An extension of this graphical analysis to games where one player has more than two strategies is extremely simple. Consider the case where Blue has two strategies (B_1, B_2) and Red has six (R_1, R_2, . . . , R_6).

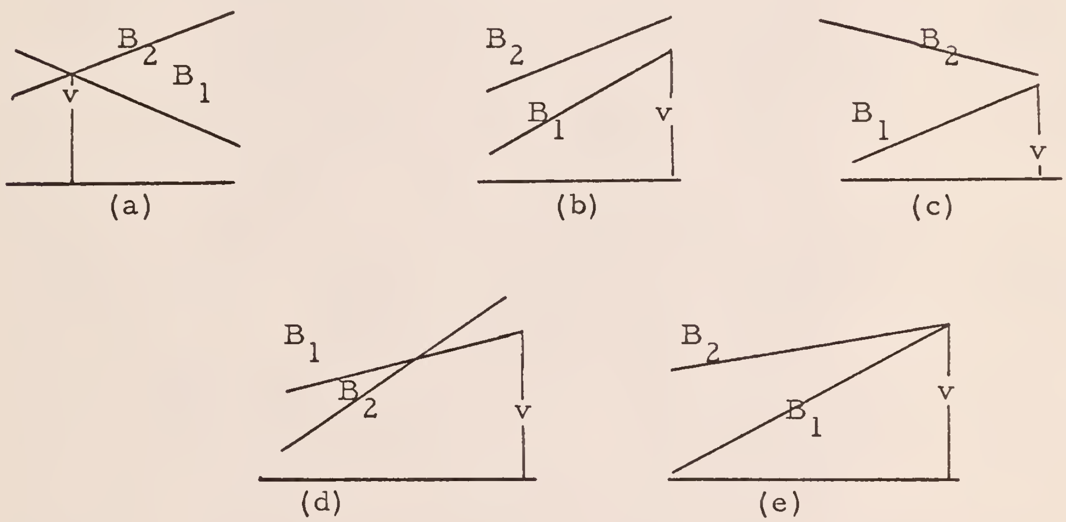
Each Red strategy is represented in the diagram below.



If Red wishes to hold Blue down to at most v, he must use a randomized strategy involving only R_1 and R_6, and thus this case is reduced to the case previously studied.

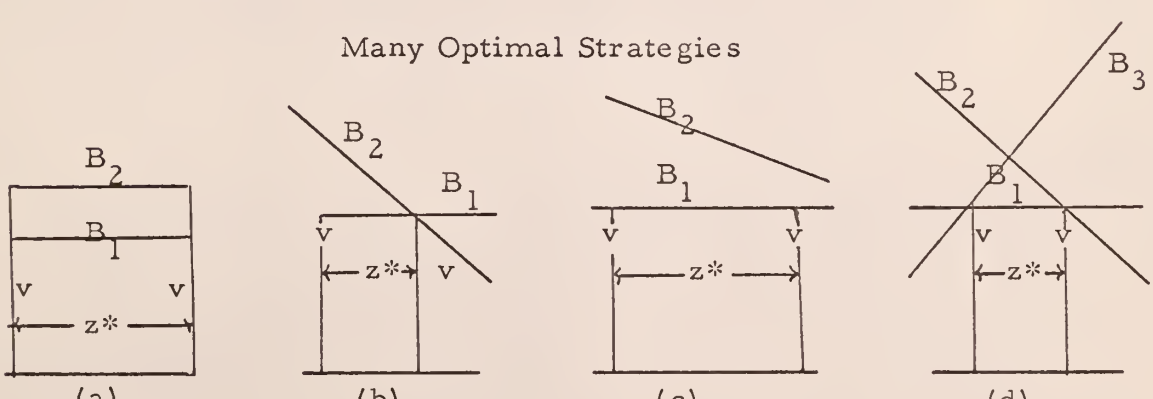
Lest the reader assume that all games have the same graphical representation as the one just analyzed, the following examples taken from Luce and Raiffa (6) present different features that might occur.

A Unique Optimal Strategy



In (a), Red has a unique optimal mixed strategy. In cases (b) and (c) strict dominance by strategy B_2 is shown.

Many Optimal Strategies



In these cases, Blue's optimal strategy may fall anywhere within the z^* interval.

Bennion (2) and Vajda (12) give examples of graphical solutions of games in three dimensions. Luce and Raiffa (6) present an alternate method of geometrical representation.

THE FUNDAMENTAL THEOREM OF GAME THEORY

Before proving the minimax theorem the previous discussion is formalized by the generalization of any two-person zero-sum finite game Γ as follows:

i. There are two players, Blue and Red. Both players are malevolent (i.e., each is concerned with maximizing his own gains, or minimizing his own losses).

ii. Blue has a set $I = (i_1, i_2, \ldots, i_m)$ of m pure strategies.

iii. Red has a set $J = (j_1, j_2, \ldots, j_n)$ of n pure strategies.

iv. Associated with each pair of strategies (i, j) is the pay-off $\phi(i, j)$ units from Red to Blue. $\phi(i, j)$ is abbreviated by a_{ij}.

v. Both players are aware of, and intelligent enough to evaluate accurately, the pay-offs associated with both players' alternative strategies.

vi. Blue may adopt a mixed strategy by employing i_1, with probability x_m where

$$\sum_{i=1}^{m} x_i = 1 \text{ and } x_i \geq 0 \text{ for } i = 1, 2, \ldots, m.$$

Such a strategy is represented by the row vector

$$X' = \left[x_1, x_2, \ldots, x_m\right] .$$

The set of all randomized strategies for Blue is designated by X_m.

vii. Similarly, Red's mixed strategy is denoted by a column vector

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

where

$$\sum_{j=1}^{n} y_j = 1 \qquad y_j \geq 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

The set of all randomized strategies for Red are designated by Y_n.

viii. For each mixed strategy pair (X, Y), the pay-off $\emptyset(X, Y)$ is defined to be

$$\emptyset(X, Y) = X'AY = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

ix. The pure strategy game Γ may be denoted by the triplet $(\emptyset, I, J)$ which designates the two pure strategy spaces I and J and the pay-off function $\emptyset$.

The extension of Γ to spaces of mixed strategies is denoted by the triplet $(\emptyset, X_m, Y_n)$.

x. Blue's objective is to select a mixed strategy X from X_m so as to maximize his security level (return). This strategy called the optimal strategy is denoted as X*. Because the game is zero-sum, Red's objective is to minimize Blue's return by the selection of a strategy Y from Y_n. This strategy denoted as Y* is the optimal strategy for Red. The set X* Y* is called a solution for the game Γ.

The following proof of the minimax theorem makes use of a theorem from Glickman ((5) Lemma 2.5, p. 31).

Lemma:

Let

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

Then either

(i) there exists an element $X' = (x_1, \ldots, x_m)$ of X_m such that $a_{1j}x_1 + a_{2j}x_2 + \ldots + a_{mj}x_m \geq 0$ for $j = 1, \ldots, n$, or

(ii) there exists an element $Y = (y_1, \ldots, y_n)$

$$a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{in}y_n \leq 0 \text{ for } i = 1, \ldots, m.$$

Theorem 4:

If X and Y are mixed strategies of the game Γ, then,

$$\min_{Y} \max_{X} \phi(X, Y) = v = \max_{X} \min_{Y} \phi(X, Y)$$

Proof:

If condition (i) of Lemma 2.5 holds, there is an element $(x_1, \ldots, x_m) \in X_m$ such that

$$a_{1j}x_1 + a_{2j}x_2 + \ldots + a_{mj}x_m \geq 0 \quad \text{for } j = 1, 2, \ldots, n,$$

and hence for every $Y \in Y_n$

$$\phi(X, Y) = \sum_{j=1}^{n} (a_{1j}x_1 + a_{2j}x_2 + \ldots + a_{mj}x_m) \, Y_j \geq 0 \tag{16}$$

Since (16) holds for every $Y \in Y_n$,

$$\min_{Y \in Y_n} \phi(X, Y) \geq 0$$

and, hence,

$$\max_{X \in X_m} \min_{Y \in Y_n} \phi(X, Y) \geq 0$$

If condition (ii) of Lemma 2.5 holds, there is an element $(y_1, \ldots, y_n) \in Y_n$ such that

$$a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{in}y_n \leq 0 \text{ for } i = 1, 2, \ldots, m,$$

and hence there exists for every $X \epsilon X_m$

$$\emptyset(X, Y) = \sum_{i=1}^{m} (a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{in}y_n) \cdot X_i \leq 0 \tag{17}$$

Since the above holds for every $X \epsilon X_m$,

$$\max_{X \epsilon X_m} \emptyset(X, Y) \leq 0$$

and, hence,

$$\min_{Y \epsilon Y_n} \max_{X \epsilon X_m} \emptyset(X, Y) \leq 0 \tag{18}$$

Since either condition (i) or condition (ii) of Lemma 2.5 holds, then at least one of the inequalities (17) or (18) must hold, and hence the following cannot be true

$$\max_{X \epsilon X_m} \min_{Y \epsilon Y_n} \emptyset(X, Y) < 0 < \min_{Y \epsilon Y_n} \max_{X \epsilon X_m} \emptyset(X, Y) \ . \tag{19}$$

Let A_v be the matrix which arises from A by subtracting v from each element of A:

$$A_v = \begin{bmatrix} a_{11} - v, & \dots, & a_{1n} - v \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{m1} - v, & \dots, & a_{mn} - v \end{bmatrix}.$$

Let ϕ_v be the expectation function for A_v, so that for any X and Y that are members of X_m and Y_n respectively,

$$\begin{aligned} \phi_v(X, Y) &= \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} - v)\, X_i Y_j \\ \phi_v(X, Y) &= \phi(X, Y) - v. \end{aligned} \tag{20}$$

Since the inequalities (19) do not hold for A, the following conditions do not hold for A_v.

$$\max_{X \in X_m} \min_{Y \in Y_n} \phi_v(X, Y) < 0 < \min_{Y \in Y_n} \max_{X \in X_m} \phi_v(X, Y) \tag{21}$$

Thus, from (20) and (21) the following conditions do not hold:

$$\max_{X \in X_m} \min_{Y \in Y_n} \phi(X, Y) - v < 0 < \min_{Y \in Y_n} \max_{X \in X_m} - v$$

Hence, for every v the following do not hold:

$$\max_{X \in X_m} \min_{Y \in Y_n} \phi(X, Y) < v < \min_{Y \in Y_n} \max_{X \in X_m} \phi(X, Y) \tag{22}$$

Since inequalities (22) are false for every v, the relations

$$\max_{X \in X_m} \min_{Y \in Y_n} \phi(X, Y) \geq v \geq \min_{Y \in Y_n} \max_{X \in X_m} \phi(X, Y) \tag{23}$$

are true for every v.

From Theorem 3,

$$\max_{X \in X_m} \min_{Y \in Y_n} \phi(X, Y) \leq \min_{Y \in Y_n} \max_{X \in X_m} \phi(X, Y) \tag{24}$$

Therefore, it follows from (22) and (24) that

$$\max_{X \in X_m} \min_{Y \in Y_n} \phi)X, Y) = \min_{Y \in Y_n} \max_{X \in X_m} \phi(X, Y) = v. \tag{25}$$

SOLVING MATRIX GAMES

From the minimax theorem, it follows that each player has an optimal strategy. Using an optimal strategy, a player can expect to win (lose) a fixed amount regardless of the strategy selected by his opponent, and this fixed amount is as large (small) as it is strategically possible. A player may win more (lose less) than this fixed amount from (to) his opponent if his opponent does not use an optimal strategy. A pair of optimal strategies, one for each player, is the solution for the game. The value of a game is the average amount v that one player must pay his opponent if both use their optimal strategies.

The computations required for the solution of any but the simplest games are so extensive that it would be virtually impossible to obtain solutions without the use of automatic computers. The advent of automatic computers has made it possible to obtain an answer to some problems in a reasonable length of time. This paper will discuss the solution of games without specific reference to the solution by automatic computers.

Since the amount of computation required to obtain a solution depends upon the number of strategies, it is important to reduce their number whenever possible. Sometimes it is possible to tell by direct inspection of the matrix that certain strategies will always have probability zero in an optimal strategy. A poor strategy of a player is defined as some pure strategy which appears with probability zero in every optimal mixed strategy of that player. If a player has a poor strategy, that strategy may be eliminated from the set of pure strategies and the resulting game will have the same solution.

Poor strategies are found by examining the pay-off matrix for dominances. Suppose some row of the pay-off matrix $A = (a_{ij})$ is such that

$$a_{ij} > a_{kj} \qquad\qquad j = 1, 2, \ldots, n,$$

i.e., the elements of some row i are larger than the corresponding elements of another row k. Then the strategy i of Blue is said to dominate strictly strategy k of Blue and strategy k is a poor strategy. However, for Red, if the elements of a column r are larger than the corresponding elements of some column s

$$a_{ir} > a_{is} \qquad i = 1, 2, \ldots, m,$$

then strategy r strictly dominates strategy s and strategy r is a poor strategy.

For example, if

$$\begin{bmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix}$$

is the pay-off matrix of some game, then no optimal strategy for Blue should assign a positive probability to the third row. No matter what Red does, Blue can improve his pay-off by choosing the second row rather than the third row. In a similar manner, since every element of the first column of the above matrix is less than the corresponding element in the third column and since Red wants to minimize the pay-off, then the third column may be eliminated obtaining

$$\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix}$$

for the simplified game matrix.

The solution of the original game may be obtained by solving the simplified game. The optimal mixed strategy of the original game is obtained by assigning probability zero to the poor strategies and the remaining strategies are assigned the same probability as in the solution of the simplified game. Hence, the value of the original game is the same as the value of the simplified game.

If some strategy k is dominated by a convex linear combination of strategies r and s, i.e.,

$$a_{ik} < c \cdot a_{ir} + (1-c) \cdot a_{is} \qquad 0 \leq c \leq 1 \text{ for all } i = 1, 2, \ldots, m,$$

then strategy k is a poor strategy and may be eliminated simplifying the computations required to solve the game. Similarly columns may also exhibit convex linear dominance and be eliminated.

For example, consider a game whose pay-off matrix is

$$\begin{bmatrix} 24 & 0 \\ 0 & 8 \\ 4 & 5 \end{bmatrix}$$

Notice that

$$4 < 1/4 \cdot (24) + 3/4 \cdot (0) \quad \text{and} \quad 5 < 1/4 \cdot (0) + 3/4 \cdot (8).$$

Hence, Blue would never be wise to play strategy three for he could always do better by dividing between the first two strategies any probability that he might consider assigning to the third strategy. Thus the game might be reduced to a simpler game whose matrix is

$$\begin{bmatrix} 24 & 0 \\ 0 & 8 \end{bmatrix} .$$

The following sections will be devoted to the solution of matrix games by different methods. The first method consists of the algebraic solution of a large system of inequalities and equalities. The second method, which is the most common and is applicable to automatic computers, is linear programming. An approximation method of solving games by fictitious play concludes the section.

For the reader interested in different methods of solving matrix games, Dresher (3), McKinsey (7), and Williams (14) present a matrix solution. Dresher (3) discusses a mapping method and Luce and Raiffa (6) illustrate the use of differential equations for the solution of matrix games.

To solve a game it suffices to find vectors X and Y whose elements satisfy the following conditions.

$$
\begin{aligned}
&x_1 + x_2 + \ldots + x_m = 1 \qquad y_1 + y_2 + \ldots + y_n = 1 \\
&x_i \geq 0 \quad i = 1, 2, \ldots, m \qquad y_j \geq 0 \quad j = 1, 2, \ldots, n \\
&a_{1j}x_1 + a_{2j}x_2 + \ldots + a_{mj}x_m \geq v \text{ for all } j = 1, 2, \ldots, n \\
&a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{in}y_n \leq v \text{ for all } i = 1, 2, \ldots, m
\end{aligned}
\tag{26}
$$

Algebraic Solution

The usual methods of elementary algebra do not suffice to solve systems like (26) above containing inequalities as well as equalities. The minimax theorem guarantees that there is a solution to this system and the algebraic method enables one to find this solution by separately considering all possible cases that arise when a $\geq$ sign is replaced by an = or $>$ sign and the $\leq$ sign is replaced by an = or $<$ sign. The two following examples from McKinsey (7) illustrate how this method is applied.

Example 1:

To find the value and optimal strategies for the game whose pay-off matrix is

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 3 \\ -1 & 2 & 1 \end{bmatrix},$$

find x_1, x_2, x_3, y_1, y_2, y_3 and v which satisfy the following conditions

$$\begin{array}{ll} x_1 + x_2 + x_3 = 1 & y_1 + y_2 + y_3 = 1 \\ 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1,\ 0 \leq x_3 \leq 1 & 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1,\ 0 \leq y_3 \leq 1 \\ (1)\ x_1 + (-1)\ x_2 + (-1)\ x_3 \geq v & (1)\ y_1 + (-1)\ y_2 + (1)\ y_3 \leq v \\ (-1)\ x_1 + (-1)\ x_2 + (2)\ x_3 \geq v & (-1)\ y_1 + (-1)\ y_2 + (3)\ y_3 \leq v \\ (1)\ x_1 + (3)\ x_2 + (1)\ x_3 \geq v & (-1)\ y_1 + (2)\ y_2 + (1)\ y_3 \leq v. \end{array} \qquad (27)$$

A solution can be found by separately considering the 2^6 cases that arise when the $\leq$ sign is replaced by an = sign or $<$ sign and the $\geq$ sign is replaced by an = sign or $>$ sign.

To solve the above game, replace the last six inequalities by equalities and the result obtained using elementary algebra is

$x_1 = 6/13$, $x_2 = 3/13$, $x_3 = 4/13$, $y_1 = 6/13$, $y_2 = 4/13$, $y_3 = 3/13$, and $v = 1/3$.

Example 2:

To find the value and optimal strategies of the game whose pay-off matrix is

$$\begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}.$$

Again it suffices to find vectors X and Y which satisfy the conditions of (27) and which also satisfy

$$(3)\,x_1 + (-1)\,x_2 + (2)\,x_3 \geq v \qquad (3)\,y_1 + (-2)\,y_2 + (4)\,y_3 \leq v$$

$$(-2)\,x_1 + (4)\,x_2 + (2)\,x_3 \geq v \qquad (-1)\,y_1 + (4)\,y_2 + (2)\,y_3 \leq v$$

$$(4)\,x_1 + (2)\,x_2 + (6)\,x_3 \geq v \qquad (2)\,y_1 + (2)\,y_2 + (6)\,y_3 \leq v.$$

However, considering the first case where all six inequalities are replaced by equalities, no solution for these equations exist which simultaneously makes x_1, x_2, x_3, y_1, y_2, and y_3 all non-negative.

To obtain a solution to the game replace the $\geq$ by $>$ or $=$ and $\leq$ by $<$ or $=$ in the remaining inequalities and solve the resulting system. Continuing in this way by trial and error, finally the case

$$3x_1 - x_2 + 2x_3 = v \qquad 3y_1 - 2y_2 + 4y_3 < v$$

$$-2x_1 + 4x_2 + 2x_3 = v \qquad -y_1 + 4y_2 + 2y_3 = v$$

$$4x_1 + 2x_2 + 6x_3 > v \qquad 2y_1 + 2y_2 + 6y_3 = v$$

$$x_1 + x_2 + x_3 = 1 \qquad y_1 + y_2 + y_3 = 1$$

which has a solution is found.

Since $4x_1 + 2x_2 + 6x_3 > v$, Red will not include y_3 in his optimal strategy. This implies $y_3 = 0$. Since $3y_1 - 2y_2 + 4y_3 < v$, Blue will not include x_1 in his optimal strategy, implying $x_1 = 0$. Solve the remaining system. The set

$$x_1 = 0,\ x_2 = 0,\ x_3 = 1 \quad \text{and} \quad y_1 = 2/5\ ,\ y_2 = 3/5,\ y_3 = 0,$$

and $v = 2$

satisfies all equalities and inequalities and is non-negative. Thus, the optimal strategies are $x = (0, 0, 1)$, $y = (2/5, 3/5, 0)$ and the value of the game is 2.

For one wishing to read further on this topic, reference (7) contains a more complete description and some additional examples worked out completely.

The algebraic method has the disadvantage that the number of possible systems of equations grows extremely large for a matrix game of only medium size. The next method, linear programming, is an adaptation of the algebraic method but it has the advantage that it moves from an infeasible solution toward a more feasible solution in a systematic manner until the solution is determined.

Linear Programming

To show that an arbitrary game may be solved by the methods of linear programming, let a pay-off matrix $A = (a_{ij})$ be given. If Blue chooses the mixed strategy $X' = (x_1, \ldots, x_n)$ then he can be certain of obtaining at least

$$\min_j \sum_{i=1}^{m} a_{ij} x_i = v.$$

Therefore,

$$a_{1j}x_1 + a_{2j}x_2 + \ldots + a_{mj}x_m \geq v \quad \text{for } j = 1, 2, \ldots, n$$

$$x_1 + x_2 + \ldots + x_m = 1 \qquad (28)$$

$$x_1, x_2, \ldots, x_m \geq 0.$$

Blue wants to make v as large as possible. This value is not necessarily positive, but a constant large enough to make v positive may be added to all the entries of the pay-off. This increases the value of the game by the same constant but does not change the solution. Therefore, v may be assumed to be positive and new variables, $x_i' = x_i/v$, may be defined. Dividing the inequalities of (28) by v, then

$$\sum_{i=1}^{m} a_{ij}x_i' \geq 1 \quad \text{for } j = 1, \ldots, n \qquad (29)$$

and

$$\sum_{i=1}^{m} x_i' = 1/v. \qquad (30)$$

The right-hand side of equation (30) must be minimized. Thus, the problem has been reduced to a linear programming problem in the usual form.

Repeating the same argument for Red, the set of inequalities

$$\sum_{j=1}^{n} a_{ij}y_j' \leq 1 \quad \text{for } i = 1, \ldots, m$$

holds, and

$$\sum_{j=1}^{n} y_j' \text{ is to maximized.}$$

These two problems are dual to one another; by solving one of them, the other is solved implicitly. Having found $x_i^{*\prime}$ and $y_j^{*\prime}$ and the minimum of $\sum_{i=1}^{m} x_i^{*\prime}$ which equals the maximum of $\sum_{j=1}^{n} y_j^{*\prime}$ we have the value v of the game;

$$x_i^{*\prime} \cdot v = x_i^{*} \quad \text{and} \quad y_i^{*\prime} \cdot v = y_i^{*}$$

indicate the best strategies, constituting a solution.

Example:

The Colonel Blotto Game (page 19) may be solved by linear programming.

Three units may be added to each element of the pay-off matrix without changing the solution. The resulting pay-off matrix becomes

$$A = \begin{bmatrix} 7 & 3 & 5 & 4 \\ 3 & 7 & 4 & 5 \\ 4 & 2 & 6 & 3 \\ 2 & 4 & 3 & 6 \\ 1 & 1 & 5 & 5 \end{bmatrix}.$$

The value of the game is increased by three but the optimal strategies remain the same.

Denote any (pure or mixed) strategy for Blue (Blotto) by the row vector

$$X' = \left[x_1, x_2, x_3, x_4, x_5\right]$$

so that

$$x_i \geq 0 \text{ for } i = 1, 2, \ldots, 5 \tag{31}$$

and

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1. \tag{32}$$

Blue's expectation against each of Red's four strategies is given respectively by the elements of the column vector

$$A'X = \begin{bmatrix} 7x_1 + 3x_2 + 4x_3 + 2x_4 + 1x_5 \\ 3x_1 + 7x_2 + 2x_3 + 4x_4 + 1x_5 \\ 5x_1 + 4x_2 + 6x_3 + 3x_4 + 5x_5 \\ 4x_1 + 5x_2 + 3x_3 + 6x_4 + 5x_5 \end{bmatrix}$$

Let v denote the smallest element of the vector, or their common value if there is no unique smallest element. Hence,

$$\begin{aligned} 7x_1 + 3x_2 + 4x_3 + 2x_4 + 1x_5 &\geq v \\ 3x_1 + 7x_2 + 2x_3 + 4x_4 + 1x_5 &\geq v \\ 5x_1 + 4x_2 + 6x_3 + 3x_4 + 5x_5 &\geq v \\ 4x_1 + 5x_2 + 3x_3 + 6x_4 + 5x_5 &\geq v \end{aligned} \tag{33}$$

Blue wishes to choose his strategy X so as to maximize v. This can be done by minimizing (1/v). Change notation as follows:

$$x_i' = x_i/v \qquad \text{and} \qquad m = 1/v.$$

Since v is positive, the division of (31) and (33) by v gives

$$x_i' \geq 0 \text{ for } i = 1, 2, \ldots, 5$$

and

$$\begin{aligned} 7x_1' + 3x_2' + 4x_3' + 2x_4' + x_5' &\geq 1 \\ 3x_1' + 7x_2' + 2x_3' + 4x_4' + x_5' &\geq 1 \\ 5x_1' + 4x_2' + 6x_3' + 3x_4' + 5x_5' &\geq 1 \\ 4x_1' + 5x_2' + 3x_3' + 6x_4' + 5x_5' &\geq 1 \ . \end{aligned} \tag{34}$$

Then Blue wants to determine x_i' for i = 1, 2, ..., 5 subject to the constraints of (34) so that

$$x_1' + x_2' + x_3' + x_4' + x_5' = m$$

is a minimum. Therefore, A's problem reduces to a linear programming problem which can be solved by the simplex method, or by other methods. After finding $x_1^{*\prime}$, $x_2^{*\prime}$, $x_3^{*\prime}$, $x_4^{*\prime}$, $x_5^{*\prime}$ and m, Blue may find his optimal strategy X* by

$$x_i^* = x_i^{*\prime}/m \text{ for } i = 1, \ldots, 5.$$

Blue's optimal strategy for the Colonel Blotto Game is

$$X^{*\prime} = [4/9, 4/9, 0, 0, 1/9] \ . \tag{35}$$

Red's problem is the dual of Blue's problem. Denoting Red's strategy by

$$Y' = \left[y_1, y_2, y_3, y_4\right]$$

so that

$$y_i \geq 0 \text{ for } i = 1, 2, 3, 4 \quad \text{and } y_1 + y_2 + y_3 + y_4 = 1.$$

Red's expectation is the negative of the column vector.

$$AY = \begin{bmatrix} 7y_1 + 3y_2 + 5y_3 + 4y_4 \\ 3y_1 + 7y_2 + 4y_3 + 5y_4 \\ 4y_1 + 2y_2 + 6y_3 + 3y_4 \\ 2y_1 + 4y_2 + 3y_3 + 6y_4 \\ y_1 + y_2 + 5y_3 + 5y_4 \end{bmatrix}.$$

Let v denote the largest element of the vector or their common value if there is no unique largest element. Hence,

$$\begin{aligned} 7y_1 + 3y_2 + 5y_3 + 4y_4 &\leq v \\ 3y_1 + 7y_2 + 4y_3 + 5y_4 &\leq v \\ 4y_1 + 2y_2 + 6y_3 + 3y_4 &\leq v \\ 2y_1 + 4y_2 + 3y_3 + 6y_4 &\leq v \\ y_1 + y_2 + 5y_3 + 5y_4 &\leq v \end{aligned}$$

Red wishes to choose his strategy Y so as to minimize v. This can be done by maximizing 1/v. Change the notation again

$$y_j' = y_j/v \qquad \text{and } M = 1/v.$$

Then Red wishes to determine

$$y_1' \geq 0 \quad y_2' \geq 0 \quad y_3' \geq 0 \quad y_4' \geq 0$$

so that

$$\begin{aligned}
7y_1' + 3y_2' + 5y_3' + 4y_4' &\leq 1 \\
3y_1' + 7y_2' + 4y_3' + 5y_4' &\leq 1 \\
4y_1' + 2y_2' + 6y_3' + 3y_4' &\leq 1 \\
2y_1' + 4y_2' + 3y_3' + 6y_4' &\leq 1 \\
y_1' + y_2' + 5y_3' + 5y_4' &\leq 1
\end{aligned}$$

and so that

$$y_1' + y_2' + y_3' + y_4' + y_5' = M$$

is a maximum. Red's problem has been reduced to a linear programming problem and he can find his optimal strategy Y* by

$$y_j^* = y_j^{*\prime}/m \quad \text{for } j = 1,\ 2,\ 3,\ 4.$$

Red's optimal strategy is

$$Y^{*\prime} = \left[1/18,\ 1/18,\ 4/9,\ 4/9\right]. \tag{36}$$

When these optimal mixed strategies are used on the original pay-off matrix then the value of the game is 14/9 to Blotto.

$$[4/9,\ 4/9,\ 0,\ 0,\ 1/9] \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & 0 & 3 \\ -2 & -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1/18 \\ 1/18 \\ 4/9 \\ 4/9 \end{bmatrix} = 14/9 \qquad (37)$$

Bennion (2), Glicksman (5), Tucker (11), and Vajda (12 and 13) have a more complete development of game theory as a linear problem. Luce and Raiffa (6) develop an alternate proof of the minimax theorem as a consequence of the duality theorem of linear programming.

Iterative Solution of a Game by Fictitious Play

The iterative method can be characterized by the fact that it rests on the traditional statistician's philosophy of basing future decisions on relevant past history. One might expect a statistician, perhaps ignorant of game theory, to keep track of his opponent's past plays and choose at each play the optimal pure strategy against the mixture represented by all of the opponent's past plays.

This method can best be illustrated by an example. Consider the game defined by the pay-off matrix

$$\begin{array}{ll} & \text{Red Strategies} \\ & \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \text{Blue Strategies} \begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix} \end{array}$$

where B_1, B_2, and B_3 represent Blue's strategies and R_1, R_2, and R_3 Red's strategies.

Assume that Blue begins the series of plays by selecting strategy B_1 and Red chooses the pure strategy R_1. At step two, Blue should choose the pure strategy that is best against Red's mixed strategy to this point (i.e., against R_1 Blue chooses B_2). Red likewise chooses the pure mixed strategy that is best against Blue's mixed strategy to this point (i.e., against B_1 Red chooses R_1). The process is then repeated at each step. The player chooses the optimal pure strategy against his opponent's mixed strategy to that point. If this instruction is ambiguous because of non-uniqueness, the player may choose any one of the possible pure strategies which satisfy the requirement.

The column headings for the following table are defined as follows. The number of the play is designated by N and i(N) represents the pure strategy chosen by Blue on the N^{th} play. B_1 is equal to the total receipts of Blue after N plays if Blue uses his B_1 strategy constantly, and similarly for B_2 and B_3. Likewise, j(N) represents the pure strategy chosen by Red on the N^{th} play and R_1 represents the receipts of Red after N plays if Red uses his R_1 strategy constantly. $\underline{v}(N)$ is the least that Blue can expect to receive on the average, after N plays while $\overline{v}(N)$ is the most that Blue can expect to receive on the average after N plays.

$\underline{v}(N) = 1/N \min_j R_j$ $\bar{v}(N) = -1/N \max_i B_i$

N	i(N)	j(N)	Blue Expects B_1	B_2	B_3	Red Expects R_1	R_2	R_3	$\underline{v}(N)$	$\bar{v}(N)$
1	B_1	R_1	1	4	2	- 1	- 2	- 3	1.000	4.000
2	B_2	R_1	2	8	4	- 5	- 2	- 4	1.000	4.000
3	B_2	R_2	4	8	7	- 9	- 2	- 5	.667	2.667
4	B_2	R_2	6	8	10	-13	- 2	- 6	.500	2.500
5	B_3	R_2	8	8	13	-15	- 5	- 6	1.000	2.600
6	B_3	R_2	10	8	16	-17	- 8	- 6	1.000	2.667
7	B_3	R_3	13	9	16	-19	-11	- 6	.857	2.286
8	B_3	R_3	16	10	16	-21	-14	- 6	.750	2.000
9	B_1	R_3	19	11	16	-22	-16	- 9	1.000	2.111
10	B_1	R_3	22	12	16	-23	-18	-12	1.200	2.200
11	B_1	R_3	25	13	16	-24	-20	-15	1.364	2.272
12	B_1	R_3	28	14	16	-25	-22	-18	1.500	2.333
13	B_1	R_3	31	15	16	-26	-24	-21	1.615	2.385
14	B_1	R_3	34	16	16	-27	-26	-24	1.714	2.429
15	B_1	R_3	37	17	16	-28	-28	-27	1.800	2.467
16	B_1	R_3	40	18	16	-29	-30	-30	1.813	2.500
17	B_1	R_1	41	22	18	-30	-32	-33	1.765	2.412
18	B_1	R_1	42	26	20	-31	-34	-36	1.722	2.333
19	B_1	R_1	43	30	22	-32	-36	-39	1.684	2.263
20	B_1	R_1	44	34	24	-33	-36	-42	1.650	2.200

It can be shown that v is the greatest lower bound of $\bar{v}(N)$ and the least upper bound of $\underline{v}(N)$. This fact insures that by carrying the approximation far enough, the value v can be found to any degree of accuracy.

$$v = \lim_{N \to \infty} \bar{v}(N) = \lim_{N \to \infty} \underline{v}(N)$$

By considering the number of times each pure strategy is played in N steps of the above approximation methods, an approximation to an optimal strategy may be found. Thus in the first eight rows of the above table, Blue plays strategy B_1 once, strategy B_2 three times, and strategy B_3 four times; hence, an approximation to an optimal strategy for P_1 is

$$X = (1/8,\ 3/8,\ 4/8).$$

After N steps, an approximation to an optimal strategy will be

$$X(N) = 1/N \sum_{K=1}^{N} i(K), \qquad Y(N) = 1/N \sum_{K=1}^{N} j(K).$$

For the preceding game the approximate optimal strategies are:

$$X(20) = (13/20,\ 3/20,\ 4/20), \quad Y(20) = (6/20,\ 4/20,\ 10/20).$$

It can be verified that the exact optimal strategies are:

$$X^* = (11/20,\ 4/20,\ 5/20), \quad Y^* = (8/20,\ 7/20,\ 5/20).$$

If

$$\lim_{N \to \infty} X(N) \quad \text{and} \quad \lim_{N \to \infty} Y(N) \tag{38}$$

exist, then these limits are a solution of the game. However, the strategies X(N) and Y(N) may not converge. If the strategies fail to converge, the cause is generally the oscillating character of the X(N) and Y(N) around a solution. It can be shown, however, that, in any case, every convergent subsequence of (38) converges to an optimal strategy.

Historically, the method of fictitious play was proposed as a means for actually computing the value of a game. However, as a computational procedure, the method is impractical since the rate of convergence is extremely slow. A number of variants of the method have been proposed which have better convergence properties; however, the method only remains of general theoretical importance. The convergence of this method is proved in Gale (4).

TOPICS FOR FURTHER STUDY

The following sections contain a brief description of topics which the reader may wish to investigate further. References where more details may be found are listed.

N-Person Games

The previous discussion has considered only two-person zero-sum games but the problem may be extended to any number of players. A finite n-person zero-sum game may be thought of as a game in which player, P_i, makes just one choice of a strategy, X_i, from a finite set,

C_i, of possible strategies without being informed about the choice of any of the previous players. After each of the n-players has chosen a strategy, the pay-off for each player P_i is

$$\phi_i(x_1, x_2, \ldots, x_n).$$

Since the game is zero-sum, the pay-off functions $\phi_1, \phi_2, \ldots, \phi_n$ satisfy

$$\sum_{i=1}^{n} \phi_i(x_1, \ldots, x_n) = 0.$$

The theory of n-person games is largely concerned with the questions of what combinations of coalitions will be formed and what payments the players can be expected to make to each other as inducements to join the various coalitions. More than half of von Neumann's book (9) is devoted to this topic.

Non-Zero-Sum Games

Until now all discussion has assumed that the gain of one player is the loss of another (zero-sum), but this is not true in all competitive situations. The bargaining of a labor union and an industrial company over a contract may be considered as a two-person game, but it is not zero-sum for the agreement over a contract is advantageous to both and the shut-down of a plant is disadvantageous to both, but not necessarily to the same extent. Some non-zero-sum games may be simplified by adding an additional player who will act as a "banker" to keep the pay-off zero-sum. Then the game may possibly be solved by n-person game theory. Nash (8) has dealt extensively with this area of game theory.

Once one leaves two-person zero-sum games, however, there are serious theoretic assumptions. A major obstacle in developing n-person and non-zero-sum game theory is the development of a satisfactory theory of coalition formation and the assumptions that are made about communication and collusion among players.

Infinite Games

Not every game situation can be described in terms of only a finite number of strategies. A very simple example is the problem of a manufacturer who is faced with the problem of how much of his product to put into a package to compete favorably with other manufacturers and thus to sell many packages, but he does not want to put so much into the package as not to make a profit.

The solution of an infinite game is not straightforward; in fact, there are infinite games where no solution exists. Since infinite games cannot be treated with the generality of finite games, the solution of infinite games will not be discussed in this paper. Dresher (3) and McKinsey (7) discuss infinite games in some detail.

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TWO-PERSON ZERO-SUM GAME THEORY

by

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A two-person zero-sum game is a conflict of interest which involves two players, hence the name two-person. One player wins what the other loses, thus the sum of their gains is zero. A game is a collection of rules which determines what the players may do.

Two-person zero-sum games are in one sense a special case of linear programming problems. Solving a game amounts to solving a set of equations for non-negative variables in such a way as to maximize (minimize) some function.

The mathematical formulation of a game is illustrated by the reduction of the game NIM from extensive to normal form. The concept of pure strategies is introduced and games with saddle-points are investigated. Games of perfect information are shown to have saddle-points. Games without saddle-points are discussed and mixed strategies are introduced. The Colonel Blotto game illustrates an application of mixed strategies. The geometrical properties of mixed strategies are illustrated by their graphical representation. The generalization of two-person zero-sum games precedes the proof of the minimax theorem.

The solution of matrix games by algebraic methods, linear programming and fictitious play is illustrated. The report concludes with a section of topics for further investigation.