"CONSTANT AREA, CONSTANT IMPULSE FUNCTION AND CONSTANT STAGNATION TEMPERATURE COMPRESSIBLE FLUID FLOW."

by

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TABLE OF CONTENTS

NOMENCLATURE

A- cross-sectional area, ft^2

A_w- wall area, ft^2

C- speed of sound, ft/sec.

C_p'- specific heat at constant pressure, Btu/lbm ^oR

C_p- specific heat at constant pressure, ft lbf/slug ^oR

D- pipe diameter, ft.

G- mass velocity, w'/A lbm/sec ft^2

g- gravitational acceleration, 32.2 ft/sec^2

g_c- gravitational constant 32.2 lbm ft/lbf sec^2

f- friction coefficient

h- enthalpy per unit mass, ft-lbf/slug

J- mechanical equivalent heat, 778 ft.lbf/B

k- ratio of specific heats

M- Mach Number, $V/\sqrt{kRT}$

P- pressure, lbf/ft^2

Q- heat, Btu

R- gas constant, Btu/lbm ^oR

R'- gas constant, ft.lbf/lbm ^oR

T- temperature, ^oR

w'- mass rate of flow, lbm/sec.

w- mass rate of flow, slug/sec.

V- velocity, ft/sec.

Z- elevation, ft.

V_g'- velocity component of gas injection in X-direction, ft/sec.

v- specific volume, ft^3/slug.

V_g- velocity of gas injection, ft/sec.

Greek Letters

ρ' density, lbm/ft^3

ρ density, $slug/ft^3$

τ_w wall shearing stress, lbf/ft^2

(o) signifies the stagnation state

$(\ )^*$ signifies state at which the Mach Number is unity

Δ difference between values

INTRODUCTION

The usual treatment of the compressible fluid flow is to select one of the flow parameters as the independent variable, and to determine how the dependent variables change when the independent variable is allowed to vary. The usual independent parameters for negligible elevation effects are area, stagnation temperature, friction (which causes the impulse function to change), and the mass of the fluid flowing through the duct. By allowing only one independent parameter to be changed, different kinds of the compressible flow will be determined.

In the case of simple area change with constant stagnation temperature, constant mass flow through the duct and negligible friction at the wall of the duct, the flow will approximate a reversible adiabatic process. Here it is said the flow will approximate a reversible adiabatic process because in actual practice, if the variable-area duct is long and poorly insulated, the process will not be adiabatic. Where there is a poorly insulated pipe and one that is sufficiently long the surface area will be large enough to cause heat flow, which is the case for natural-gas pipelines. In the case of a well insulated long pipe, or an uninsulated short pipe, the flow will approximate the reversible and adiabatic process. Because of insulation there will be negligible heat flow in a long pipe, and in the case of an uninsulated short pipe the area for heat transfer will be small and the heat flow will be of a negligible amount. In both cases the pipe is assumed to be perfectly smooth. The treatment of reversible, adiabatic, horizontal, compressible fluid flow is usually found in text books on gas dynamics.

Compressible fluid flow in ducts having constant area, in which friction is present, and having constant stagnation temperature and constant mass flow of the fluid through the duct, will give rise to the Fanno-Line process. In this type of flow there is negligible heat transfer and the system approximates an adiabatic process. In the case of a long natural-gas pipeline there is sufficient area to cause heat transfer, which will give rise to a diabatic process. For short ducts the area for heat transfer is small, and flow of heat will be of negligible amount. For the actual pipeline which is well insulated the flow will approximate the Fanno-Line process. The limiting case for the long duct with frictional flow is one in which there is heat flow and there is constant temperature along the duct. Actually in the Fanno-Line process the one independent parameter is length, and the friction, which results from this independent parameter, causes the impulse function to decrease. This type of flow is discussed in most of the text books on gas dynamics.

The case of diabatic, constant area, constant mass flow, horizontal flow without friction is termed the Rayleigh-Line process. Such ideal conditions do not exist in practice, because the frictional effects in heat exchangers are not negligible. If the stagnation temperature is raised by some chemical combustion process, there will be a change in the chemical composition of the gas flowing through the duct. If this situation is overlooked, and if the flow is analysed by considering the simple stagnation temperature changes, some important conclusions for practical significance will be drawn. This type of flow is called the Rayleigh-Line process, and it is treated in most of the books on gas dynamics, in which the effects of elevation are not considered.

In this report analyses were made on flows in which there is no heat transfer and in which the cross-sectional area, the impulse function and the stagnation temperature are all kept constant. In the first section an analysis was made in order to establish the relations of pressure, density, temperature and mass flow rate in terms of Mach Number. These relations are independent of elevation changes and friction. From these formulas a table and a graph of the properties have been prepared for this fluid flow.

In an actual case of this kind of flow, the frictional effects along the wall of the duct should be considered. This actual fluid flow was treated in the second section of this report, and the effects of Mach Number on the length of the duct were investigated and a table and a graph have been prepared. In this investigation the gas was injected with its forward component of velocity equal to that of the free stream.

The final investigation was made considering the effects of elevation and neglecting the fluid friction at the wall of the duct. First an analytical investigation was made, and this investigation was checked with numerical calculations. The results of these calculations were plotted on graphs.

The fundamental equations of gas dynamics and thermodynamics were used in this investigation. The differential form of the equations provides the information required to show the significance of elevation changes. For the flow having constant area, constant impulse function, and constant stagnation temperature all possible states are realized by the use of the continuity equation, energy equation, equation of state and momentum equation.

SECTION I

Derivations of Pressure, Density, Temperature and Mass Rate Flow in Terms of Mach Number for Constant Area, Constant Impulse Function and Constant Stagnation Temperature Fluid Flow

For this analysis the stagnation temperature is maintained constant by (1) adiabatic flow with the stagnation temperature of the injected gas equal to the stagnation temperature of the main flow, or (2) diabatic flow with the stagnation temperature of the injected gas adjusted to maintain a constant stagnation temperature along the duct.

The injected gas is assumed to mix completely and instantaneously with the main stream flow, i.e., the mixing distance along the duct is zero. In the case of horizontal flow with no friction the injected gas, in order to maintain a constant value of the impulse function, must enter the region of the main flow perpendicularly. In the case of elevation changes the main flow must be upward as will be shown in Section III, and the injected gas must have a component of velocity in the upward direction. In the case of horizontal flow with friction the injected gas must have a velocity component in the direction of the main stream flow, as that will be shown in Section II. No restrictions are placed on the transverse velocity of the injected gas by the flow of heat. However, the main stream flow is assumed to be one-dimensional, i.e., any cross-section of the pipe has uniform properties, and the mixing length along the pipe, as stated before, will be assumed zero.

Impulse function. The impulse function is defined as

$$F = PA + \rho V^2 A \text{ - - - - - - - - - } (1)$$

For a perfect gas, and from the equation of state,

$$V^2 = \frac{P}{RT} V^2 \quad \text{and } kRT = C^2$$

$$\rho V^2 = PkM^2 \text{ - - - - - - - - } (2)$$

From equations (1) and (2)

$$F = PA + PkM^2 A = PA(1+kM^2)$$

$$F^* = P^*A^*(1+kM^{*2}), \quad A = A^*, \quad M^* = 1, \text{ and } F^* = F$$

$$\frac{P}{P^*} = \frac{1+k}{(1+kM^2)} \text{ - - - - - } (3)$$

Temperature Relationship. From the steady flow energy equation and the requirement of constant stagnation temperatures,

$$T_o = T_o^* = T(1 + \frac{k-1}{2} M^2) = T^*(1 + \frac{k-1}{2} M^{*2})$$

$$= T^*(\frac{k+1}{2}), \quad M^* = 1$$

$$\frac{T}{T^*} = \frac{\frac{k+1}{2}}{(1+ \frac{k-1}{2} M^2)} \text{ - - - - - - - } (4)$$

Density Ratios. From the equation of state

$$\frac{\rho}{\rho^*} = \frac{P/RT}{P^*/RT^*} = \frac{PT^*}{P^*T} \text{ - - - - } (5)$$

From equations (3), (4), and (5)

$$\frac{\rho}{\rho^*} = \frac{k+1}{1+kM^2} \times \frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}} = \frac{2(1+\frac{k-1}{2}M^2)}{(1+kM^2)} \text{ - - - - - } (6)$$

<u>Velocity</u> <u>Ratio</u>.

$$\frac{V}{V^*} = \frac{MC}{M^*C^*}, \quad M^* = 1$$

$$= \frac{M\sqrt{kRT}}{(kRT^*)^{\frac{1}{2}}} = M\sqrt{\frac{T}{T^*}} \text{ - - - - - - - - - - - - - - } (7)$$

From equations (4) and (7)

$$\frac{V}{V^*} = M\sqrt{\frac{\frac{k+1}{2}}{(1+\frac{k-1}{2}M^2)}} \text{ - - - - - - - - - - - - - - - - } (8)$$

<u>Mass</u> <u>Rate</u> <u>Flow</u> <u>Ratio</u>. From the equation of continuity,

$$\frac{w}{w^*} = \frac{\rho VA}{\rho^* V^* A^*}, \qquad A = A^*$$

$$= \frac{\rho V}{\rho^* V^*} \text{ - - - - - - - - - - - - - - - - } (9)$$

From equations (9), (8), and (6)

$$\frac{w}{w^*} = \frac{(1+k)}{(1+kM^2)} \cdot \frac{(1+\frac{k-1}{2}M2)}{(\frac{k+1}{2})} .M. \sqrt{\frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2}}$$

$$\frac{w}{w^*} = \frac{(1+k)M}{(1+kM^2)} \cdot \sqrt{\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}}$$

$$\frac{w}{w^*} = \frac{M\sqrt{2(k+1)\ (1+\frac{k-1}{2}M2)}}{1+kM^2} \text{ - - - - - - - - - - - - } (10)$$

In order to determine the manner in which the Mach Number may change, the relation between entropy change and Mach Number change will be investigated.

<u>Entropy Relationship</u>. The entropy change from ( )*, the state where $M = 1{\cdot}0$, to a given state is

$$S^* - S = C_p \ln \frac{T_o^*}{T_o} - R \ln \frac{P_o^*}{P_o}$$

$$T_o = T_o^* \quad \text{(by assumption)}$$

$$S - S^* = -R \ln \frac{P_o}{P_o^*}$$

$$\frac{S-S^*}{R} = \ln \frac{P_o^*}{P_o} \quad \text{- - - - - - - - - - - - - - - - - - (11)}$$

From the isentropic pressure relationship,

$$\frac{P_o}{P} = \left(1 + \frac{k-1}{2} M^2\right)^{k/k-1} \quad \text{- - - - - - - - - - - - - (12)}$$

$$\frac{P_o^*}{P^*} = \left(1 + \frac{k-1}{2} M^2\right)^{k/k-1}$$

$$= \left(1 + \frac{k-1}{2}\right)^{k/k-1}, \quad M^* = 1$$

$$= \left(\frac{k+1}{2}\right)^{k/k-1} \quad \text{- - - - - - - - - - - - - - - - - - (13)}$$

From equations (12) and (13)

$$\frac{P_o}{P_o^*} = \frac{P \left(1 + \frac{k-1}{2} M^2\right)^{k/k-1}}{P^* \left(\frac{1+k}{2}\right)^{k/k-1}} \quad \text{- - - - - - - - - - - - - - - - -(14)}$$

From equations (14) and (3)

$$\frac{P_o}{P_o^*} = \frac{(1+k)}{(1+k M^2)} \left[\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}\right]^{k/k-1} \quad \text{- - - - - - (15)}$$

From equations (11) and (15)

$$S-S^* = R \quad \ln \frac{(1+kM^2)}{(1+k)} \left[\frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2} \right]^{k/k-1}$$

$$\frac{S-S^*}{R} = \ln \frac{(1+k\,M^2\,)}{(1+k)} \left[\frac{\frac{1+k}{2}}{1+\frac{k-1}{2}M^2} \right]^{\frac{k}{k-1}} \quad ----- \quad (16)$$

All the derived relations in this section are expressed in terms of Mach Number only, and thus are independent of elevation changes and friction.

Differentiating equation (16),

$$\frac{ds}{R} = \frac{(k+1)}{(1+k\,M^2)} \cdot \left[\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}} \right]^{k/k-1} d\left[\frac{1+kM^2}{1+k}\right]\left[\frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2} \right]^{k/k-1}$$

$$= \frac{k+1}{1+kM^2} \left[\frac{(\frac{k+1}{2})}{\frac{k+1}{2}} \right]^{k/k-1} \cdot \frac{1}{1+k} \cdot \left[1+\frac{k-1}{2}M^2 \right]^{k/k-1}$$

$$d\left[\frac{1+k\,M^2}{(1+\frac{k-1}{2}M^2\,)} \;k/k-1 \right]$$

$$= \frac{1}{1+k\,M^2} \left[1+\frac{k-1}{2}M^2 \right]^{k/k-1} \left[\frac{(1+\frac{k-1}{2}M^2)^{k/k-1} \quad (k)\;d(M)^2}{(1+\frac{k-1}{2}M^2)^{\frac{2k}{k-1}}} \right.$$

$$\left. - \left[\frac{(1+kM^2)\frac{k}{k-1}\;(1+\frac{k-1}{2}M^2)^{1/k-1}}{(1+\frac{k-1}{2}M^2)^{2k/k-1}} \cdot \frac{k-1}{2} \right] d\,(M)^2 \right.$$

$$\frac{ds}{R} = \frac{k}{1+kM^2}\; d(M)^2 - \frac{k}{2} \cdot \frac{1}{1+\frac{k-1}{2}M^2}\; d\,(M)^2$$

$$\frac{ds}{R} = \frac{k\;(1-M^2)\;d(M)^2}{2(1+k\,M^2)\;(1+\frac{k-1}{2}M^2)} \quad ----------- (17)$$

From the second law of thermodynamics an adiabatic process can never be accomplished with a decrease in entropy. An explanation of equation (17) shows that

(i) In the case of $M<1$, the value of $d(M)^2$ must be positive.

(ii) In the case of $M>1$, the value of $d(M)^2$ must be negative.

It is concluded that for subsonic flow the Mach Number must increase, and that for supersonic flow the Mach Number must decrease.

Formulas For Numerical Calculation

$$\frac{w_1}{w_2} = \frac{\rho_1 V_1 A_1}{\rho_2 V_2 A_2} \quad \text{but } A_1 = A_2$$

$$= \frac{\rho_1 V_1}{\rho_2 V_2} = \frac{P_1}{RT_1} \cdot \frac{RT_2}{P_2} \cdot \frac{M_1 C_1}{M_2 C_2}$$

$$= \frac{P_1}{T_1} \cdot \frac{T_2}{P_2} \cdot \frac{M_1}{M_2} \cdot \sqrt{\frac{T_1}{T_2}}$$

$$= \frac{P_1}{P_2} \cdot \frac{M_1}{M_2} \cdot \sqrt{\frac{T_2}{T_1}}$$

$$\frac{P_1}{P_2} = \frac{P_1}{P*} \cdot \frac{P*}{P_2}$$

(The values of the ratios of $P/P*$ can be taken from Rayleigh-Line tables because the impulse function is constant in the Rayleigh-Line process.)

$$\frac{T_1}{T_2} = \frac{T_1}{T_o} \cdot \frac{T_o}{T_2}$$

(The values of the ratios can be taken from the isentropic tables.)

Substituting these values of $\frac{T_1}{T_2}$ and $\frac{P_1}{P_2}$ in the formulas for w_1/w_2,

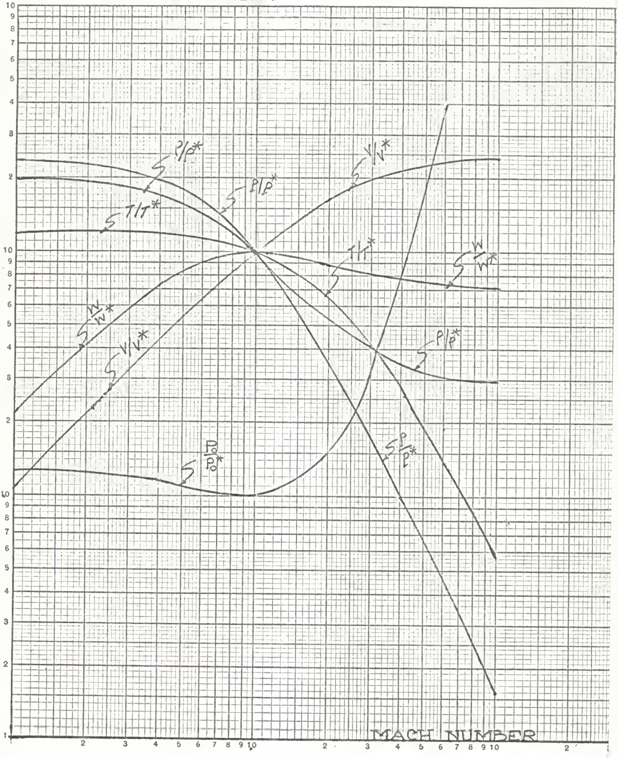
$$\frac{w_1}{w_2} = \frac{P_1}{P*} \cdot \frac{P*}{P_2} \cdot \sqrt{\frac{T_1}{T_o} \cdot \frac{T_o}{T_2}} \cdot \frac{M_1}{M_2}$$

Table 1. Properties for constant area, constant impulse function and constant stagnation temperature fluid flow as function of Mach Number.*

Mach Number	P/P^*	T/T^*	ρ/ρ^*	w/w^*	P_o/P_o^*	V/V^*
0.1	2.3668	1.1976	1.9763	0.21628	1.2591	0.1092
0.2	2.2727	1.1904	1.9090	0.416597	1.2345	0.2180
0.4	1.9607	1.1620	1.6862	0.72734	1.1565	0.4320
0.6	1.5957	1.1194	1.4255	0.90494	1.07525	0.6348
0.8	1.2658	1.0638	1.1898	0.98180	1.0193	0.8272
1.0	1.0000	1.0000	1.0000	1.00000	1.0000	1.0000
2.0	0.36363	0.66666	0.54545	0.89072	1.5030	1.6230
3.0	0.17647	0.42857	0.41176	0.80868	3.4244	1.9620
4.0	0.10256	0.28462	0.35890	0.76751	8.2268	2.1400
6.0	0.04669	0.14634	0.31906	0.73234	38.4950	2.2980
8.0	0.02649	0.08675	0.30458	0.71861	136.6200	2.3536
10.0	0.01702	0.05714	0.29787	0.71205	381.6100	2.3900

*These values are independent of elevation changes and frictional effects.

Fig. 1. Properties for constant area, constant stagnation temperature and constant area fluid flow.



SECTION II

Analysis of Horizontal Flow With Wall Friction

In this analysis, the effects of wall friction are investigated through the introduction of the friction parameter. In this analysis a step-by-step numerical method will be used, as no exact method of analysis could be found. The steps will be divided in the increment of 0·1 Mach Number.

Assume: 1. Flow to be horizontal, with wall friction.

2. One dimensional and steady flow.

3. Continuous injection of gas, with complete and instantaneous mixing of the injected gas.

4. Constant area.

5. Constant impulse function.

6. Constant stagnation temperature.

7. The gas is perfect.

8. $V_{g'}$ is the forward component of the velocity V_g with which the gas is injected. $V_{g'} = V$, the main stream velocity.

Sample calculations to illustrate the step-by-step method of analysis:

$M_1 = 0{\cdot}5$, $A = 1\ ft^2$, $P_1 = 100$ psia, $T_1 = 1000\ ^{o}R$, $4f = {\cdot}02$ i.e., $f = {\cdot}005$

$$\rho_1 = \frac{P_1}{RT_1} = \frac{100}{0.37043 \times 1000} = 0{\cdot}26995\ lbm/ft^3$$

$$V_1 = M_1 C_1 = \sqrt{kRT_1}\cdot M_1\ ,\quad k = 1{\cdot}4,\quad R = 53{\cdot}342$$

$$V_1 = 0{\cdot}5 \times 49{\cdot}0172\sqrt{1000} = 775{\cdot}08\ fps$$

$$w_1 = A_1\rho_1 V_1,\quad A_1 = 1\ ft^2$$

$= 1 \times 0.26995 \times 775.08$

$= 209.23$ lbm/sec.

$M_2 = 0.6$

$\frac{T_2}{T_o} = 0.93284$ (from isentropic tables), $T_o = 1050\ ^oR$

$T_2 = 1050 \times 0.93284 = 979.48\ ^oR$

$V_2 = M_2C_2 = M_2\sqrt{kRT_2} = 0.6 \times 49.017 \times \sqrt{979.48}$

$= 920.25$ ft/sec.

$\frac{P_2}{P*} = 1.5975$ (Rayleigh Line tables)

$\frac{P_1}{P*} = 1.7778, \quad P* = \frac{100}{1.7778} = 56.24$ psia.

$P_2 = 1.5975 \times 56.24$

$= 89.757$ psia.

Equation of state,

$$\rho_2 = \frac{P_2}{RT_2} = \frac{89.759}{0.3705 \times 979.48} = 0.24739 \text{ lbm/ft}^3$$

Continuity Equation,

$w_2 = A_2\rho_2V_2, \quad A_2 = 1 \text{ ft}^2$

$= 0.24739 \times 920.25$

$w_2 = 227.66$ lbm/sec.

$w_2 - w_1 = 18.23 \text{ lbm} = \triangle w_1$

Assume the length of the pipe for this change of Mach Number is

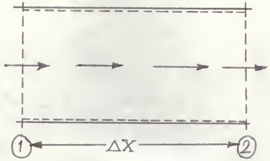


Fig. 2

X from section (1) to section (2) then the wall shearing stress is (Fig. 2)

$$\tau_w = f\rho\frac{V^2}{2} \text{ - - - - - - - - - - - - - - (18)}$$

In the present analysis an averaging method will be used, therefore, the above shearing stress will be reduced to an average shearing stress between section (1) and section (2), and it is assumed that the co-efficient of friction is constant throughout the flow.

$$\tau_w = f\,\overline{\frac{\rho V^2}{2}}$$

$$\overline{\frac{\rho V^2}{2}} = \frac{\rho_1 V_1{}^2 + \rho_2 V_2{}^2}{2\text{x}2}$$

$$= \frac{0.26995 \times (775.08)^2 + 0.24739(920.25)^2}{4\text{x}32.174}$$

$$= 2,890 \text{ - - - - - - - - - - - - - - - - - - - (19)}$$

τ_w = .005 x 2890 (substituting the value from (19) in (18)

= 14.45 lby/ft^2

The wall friction between Section (1) and Section (2).

F_w = frictional force, = shearing stress x wall area.

$= A_w \cdot \tau_w$

A_w = wetted area of the wall.

$= \pi D\Delta X \qquad A = 1\ ft^2$

$D = 1.129$ ft.

$A_w = \pi \times 1.129\Delta X.$

The wall frictional force is set equal to the axial component of the momentum of the weight of the gas injected to get the length required for this kind of change of flow properties.

$$\Delta F_w = \Delta_w V_{g'} \qquad V_{g'} = \bar{V}$$

$$\Delta F_w = \Delta_w \bar{V} \quad \text{- -} \quad (20)$$

$$\bar{V} = \frac{775.08 + 920.24}{2} = 847.66 \text{ ft/sec.}$$

$$\Delta w = \frac{18.43}{32.174} = 0.573 \text{ slug/sec.}$$

$$F_w = \pi \times 1.129 \times 14.45\Delta X_1 = 51.251\ X_1$$

From equation (20)

$$\Delta X_1 \times 51.25 = 0.573 \times 847.66$$

$$\Delta X_1 = \frac{0.573 \times 847.66}{51.25}$$

$$= 9.465 \text{ ft.}$$

$$\frac{4fX_1}{D} = \frac{4 \times .005 \times 9.465}{1.129} = 166 \times 10^{-3}$$

These calculations were repeated for other increments of Mach Numbers. The results are plotted on Fig. 3, in which the "X" in the expression $\frac{4fX}{D}$ represents the distance along the pipe from a particular section to the section where Mach Number, M = 1.0.

From the analysis presented in Section I, the flow will always occur in such a manner that the Mach Number tends toward unity, as shown by the arrows on the graph.

Fig. 3. Dimensionless pipe length parameter for horizontal flow with wall friction. $\frac{4fX_{max}}{D}$ vs Mach Number.

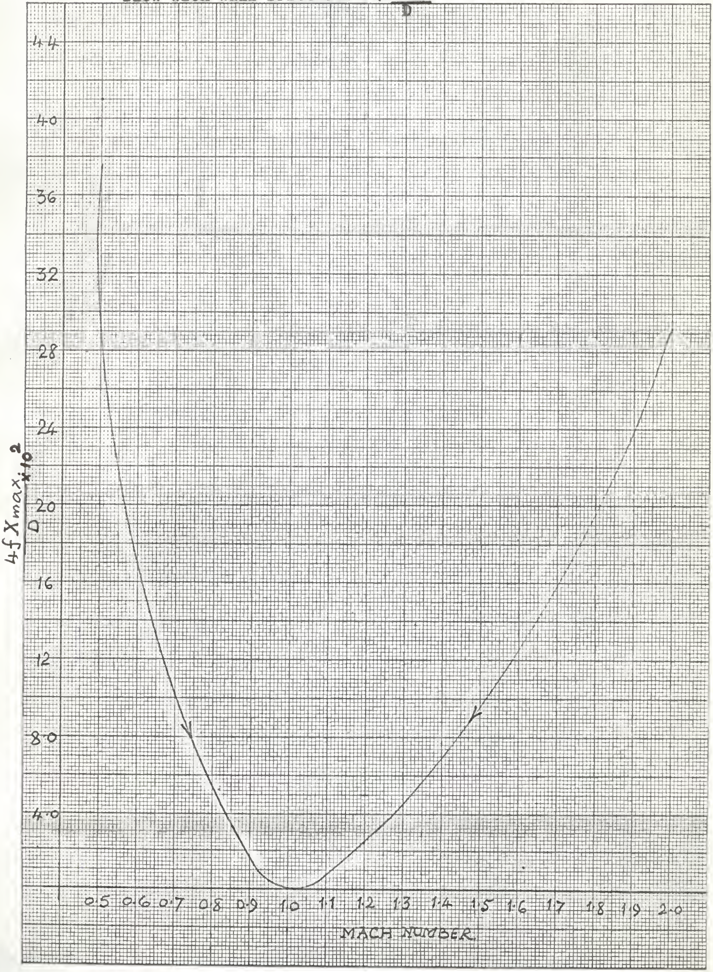


Table 2. Dimensionless pipe length parameters for horizontal flow with wall friction.

M	ΔX	X	$\frac{4fX_{max}}{D} \times 10^2$
1.0	0	0	0
1.1	0.343	0.343	0.686
1.2	0.873	1.216	2.432
1.3	1.22	2.436	4.872
1.4	1.405	3.841	7.682
1.5	1.515	5.356	10.712
1.7	3.47	8.826	17.652
2.0	5.71	14.536	29.072
1.0	0	0	0
0.9	0.448	0.448	0.896
0.8	1.69	2.138	4.276
0.7	3.25	5.488	10.976
0.6	5.53	11.018	22.036
0.5	9.465	20.483	40.966

SECTION III

Mathematical Analysis of Vertical Flow Without Friction

Vertically Upward Flow.

Assume:

1. Constant area.
2. Constant impulse function
3. Constant stagnation temperature.
4. Frictionless flow.
5. Perfect gas.
6. Continuous gas injection with complete and instantaneous mixing of the injected gas.
7. One dimensional, steady fluid flow.

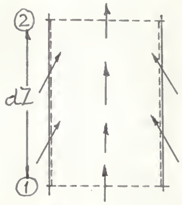


Fig. 4.

Consider the flow in a duct between two sections an infinitesimal distance dZ apart. In this element of duct length, dZ, gas is injected into the stream at mass rate of flow dw.

Equation of State.

$$P = \rho RT$$

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad \text{- -} \quad (21)$$

<u>Definition of Mach Number</u>.

$$M^2 = \frac{V^2}{kRT}$$

$$\frac{d(M)^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T} \quad \text{-------------------} (22)$$

<u>Equation of Continuity.</u> The injected gas flow is assumed to be injected continuously along the length of duct.

It is assumed that the injected gas and the main stream have mixed perfectly at the down stream boundary of the control volume. The mass flow of the main gas stream may be expressed as follows:

$$w = \rho AV , \quad A = A^* = \text{constant.}$$

$$\frac{dw}{w} = \frac{d\rho}{\rho} + \frac{dV}{V} \quad \text{-------------------} (23)$$

<u>Energy Equation</u>. The energy equation for the flow through the control surface of figure may be written, assuming the gravity effects,

$$w'(h + \frac{V}{2g_cJ} + \frac{gZ}{g_cJ}) + dw' \left[h + \frac{V2}{2g_cJ} + \frac{g}{g_cJ}(Z) \right]$$

$$= (w' + dw')(h + dh) + \frac{(V+dV)^2}{2g_cJ}(w' + dw')$$

$$+ \frac{g}{g_cJ}(w' + dw')(Z+dZ)$$

$$w'h + w' \frac{V2}{2g_cJ} + \frac{gZ}{g_cJ} w' + dw'h + dw' \frac{V^2}{2g_cJ} + \frac{g}{g_c} \frac{dw'Z}{J}$$

$$= w'h + dw'h + dh.w' + dwdh$$

$$+ \frac{V^2w'}{2g_cJ} + \frac{2VdVw'}{2g_cJ} + \frac{dw'V^2}{2g_cJ} + \frac{2VdV}{2g_cJ} dw'$$

$$+ \frac{g}{g_cJ}(w'Z+dw'Z+w'dZ+dwdZ)$$

$$dh.w' + \frac{2w'VdV}{2g_cJ} + \frac{g}{g_cJ} w'dZ = 0 \quad \text{------} (24)$$

Momentum Equation. The net force acting on the material within the differential control volume is equal to the increase of momentum flux of the streams flowing through the control volume.

The momentum equation may be written,

$$PA - (P+dP)A - AdZg\,(\rho + \frac{d\rho}{2})$$

$$= (w+dw)\,(V+dV) - V_{g'}\,dw - wV$$

$V_{g'}$ is the forward velocity component of velocity V_g with which the injected gas crosses the control surface. After simplifying,

$$-dPA - AdZ(\rho + \frac{d\rho}{2})\;g = wdV + Vdw - V_{g'}dw$$

Neglecting the terms of second order

$$-AdP - AdZ\;g + V_{g'}\;dw = wdV - Vdw \quad \text{- - - - - - - - - - -} \quad (25)$$

Definition of impulse function:

$$F = PA + wV$$

For constant impulse function and constant area

$$-AdP = wdV + Vdw$$

Momentum equation reduces to

$$-A \cdot g\;dZ + Vg'\;dw = 0 \quad \text{- - - - - - - - - - - - - - - - -} \quad (26)$$

Development of dw in Terms of Mach Number.

$$T_o = T(1 + \frac{k-1}{2}M^2) \quad \text{(isentropic relationship)}$$

From the equation of state and from equation (3)

$$\rho = \frac{P}{RT} = \frac{P*(1+k)}{(1+k\;M^2)RT}$$

Substituting the value of T from the stagnation temperature relationship into the isentropic temperature relationship.

$$= \frac{P^*(1-k)}{(1-k\ M^2)RT_o} \cdot \frac{(1+\frac{k-1}{2}M^2)}{1} \quad \text{- - - - - - - - - -} \quad (27)$$

$$= C_1 \frac{(1+\frac{k-1}{2}M^2)}{1+k\ M^2} \quad , \quad C_1 = \frac{P^*(1+k)}{RT_o} \quad \text{- - - -} \quad (28)$$

Now

$$w = \quad AV = \rho AM\sqrt{kRT}$$

$$= \rho AM\sqrt{\frac{kRT_o}{(1+\frac{k-1}{2}M^2)}} \qquad \text{Substituting the value } \rho$$

$$= C_1 \frac{1+\frac{k-1}{2}M^2}{1+k\ M^2} \quad x \quad A.M\sqrt{\frac{kRT_o}{1+\frac{k-1}{2}M^2}}$$

$$w = \frac{C_2\ A(M^2)^{\frac{1}{2}}(1+\frac{k-1}{2}M^2)^{\frac{1}{2}}}{1+k\ M^2} \quad \text{- - - - - - - - - - - - -} \quad (29)$$

Where

$$C_2 = C_1\ \ kRT_o \quad \text{- -} \quad (30)$$

$$\frac{dw}{w} = \frac{\frac{1}{2}\ d(M)^2}{M^2} + \frac{\frac{1}{2}\ \frac{k-1}{2}\ d(M)^2}{1+\frac{k-1}{2}M^2} - \frac{k\ d(M)^2}{1+k\ M^2} = N$$

$$dw = w \cdot N$$

Substituting the value of w from the above expression.

$$dw = \left[\frac{\frac{1}{2}\ d(M)^2}{M^2} + \frac{\frac{1}{2}\ \frac{k-1}{2}\ d(M)^2}{1+\frac{k-1}{2}M^2} - \frac{k\ d(M)^2}{1-k\ M^2}\right]$$

$$X \quad \frac{C_2\ A(M^2)^{\frac{1}{2}}\ (1+\frac{k-1}{2}M^2)^{\frac{1}{2}}}{1+k\ M^2} \quad \text{- - - - - - - - - - - - - -} \quad (31)$$

Coming back to the original values of C_1 and C_2.

$$dw = \frac{P^*(1+k)}{(1+kM^2)RT_o} \cdot \left(1+\frac{k-1}{2}M^2\right) AM \sqrt{\frac{kRT_o}{1+\frac{k-1}{2}M^2}}$$

$$X \left[\frac{\frac{1}{2}}{M^2} + \frac{\frac{\frac{1}{2}(k-1)}{2}}{1+\frac{k-1}{2}M^2} - \frac{k}{1+k\,M^2}\right] d(M)^2$$

$$dw = \frac{P^*(1+k)}{(1+k\,M^2)} \quad \frac{\sqrt{k}}{RT_o} \sqrt{\frac{1+\frac{k-1}{2}M^2}{1}} \quad AM\left[\frac{1}{2M^2} - \frac{k-1}{4(1+\frac{k-1}{2}M^2)} - \frac{k}{1+k\,M^2}\right] d(M)^2 \text{ - - - - - - - - - - - -(32)}$$

<u>Development of Dimensionless Elevation Parameter</u>.

Now equation (26) is written,

$Vg'\ dw = A\rho g dZ$

From equation (26), (27), and (32)

$$Ag \quad \frac{P^*(1+k)\left(1+\frac{k-1}{2}M^2\right)}{(1+k\,M^2)RT_o} \quad dZ$$

$$= Vg' \quad \frac{P^*(1+k)}{(1+k\,M^2)RT_o} \cdot AM\left(1+\frac{k-1}{2}M^2\right) \cdot \sqrt{\frac{kRT_o}{1+\frac{k-1}{2}M^2}}$$

$$X \quad \left[\frac{1}{2\,M^2} + \frac{\frac{k-1}{4}}{1+\frac{k-1}{2}M^2} - \frac{k}{1+k\,M^2}\right] d(M)^2$$

$$\frac{gdZ}{d(M)^2} = \left[\frac{M\ kRT_o}{(1+\frac{k-1}{2}M^2)^{\frac{1}{2}}}\right] \left[\frac{1}{2M^2} + \frac{\frac{k-1}{4}}{1+\frac{k-1}{2}M^2} - \frac{k}{1+k\,M^2}\right] Vg'$$

$$\frac{\frac{gdZ}{\sqrt{kRT_o}}}{} \cdot d(M)^2 \cdot \frac{1}{Vg'} = \frac{M}{(1+\frac{k-1}{2}M^2)^{\frac{1}{2}}}\left[\frac{1}{2M^2} - \frac{\frac{k-1}{4}}{1+\frac{k-1}{2}M^2} - \frac{k}{1+k\,M^2}\right] \text{---(33)}$$

In order to determine the possible types of flow, it is necessary to analyze the system through consideration of the second law of thermodynamics. The changes in entropy must be positive because the flow is (i) adiabatic by assumption (ii) irreversible by virtue of the mixing process of the injected gas and the main-stream flow.

Rewriting equation (17)

$$\frac{ds}{R} = \frac{k\,(1-M^2)\,d(M)^2}{2(1+k\,M^2)\,(1 + \frac{k-1}{2}M^2)}$$

From equation (17), with $\frac{ds}{R}$ necessarily always positive.

For $M<1$, $d(M)^2$ must be positive.

For $M>1$, $d(M)^2$ must be negative.

If the expression for dw is examined by the above criteria, it will be observed that dw is always positive for all values of the Mach Number.

For the analysis of expression (33) for vertical flow, the second law of thermodynamics imposes the following conditions.

(i) $d(M)^2$ is always positive in subsonic flow.

(ii) $d(M)^2$ is always negative in supersonic flow.

(iii) dw is positive for subsonic as well as supersonic flow.

After analysing expression (17) for the above conditions, it is concluded that in case of subsonic flow the only possible flow is vertically upward flow with increasing Mach Number. Similarly in the case of supersonic flow, the only possible flow is vertically upward flow with decreasing Mach Number. The limiting value of the Mach Number is unity in subsonic as well as supersonic flow.

Numerical Analysis for Vertical Flow Without Friction

The numerical analysis is done with the help of the momentum equation (26). In the analysis for upward flow, to overcome the gravity effects, the injected gas must have a forward component of velocity.

The integral form of equation (26) is,

$$\int g\rho dZ = \int V_{g'}\, dw$$

Assume $V_{g'}$ = Constant.

The integration of the above equation results in the dimensionless parameter, $\frac{\rho^* g(Z^* - Z)}{w^* V_{g'}}$, which is a function of Mach Number and was used in the preparation of Table 3, Fig. 5, 6.

Development of $\frac{\rho^* g(Z^*-Z)}{V_{g'} w^*}$

From equations (26), (27), and (32)

$$g \cdot \frac{P^*(1+k)}{(1+k\, M^2)RT_o} \cdot \frac{1+\frac{k-1}{2}M^2}{1} \cdot dZ$$

$$= AMV_{g'}\ \frac{P^*(1+k)}{(1+k\, M^2)}\sqrt{\frac{k}{RT_o}} \cdot \sqrt{\frac{1+\frac{k-1}{2}M^2}{1}}$$

$$X\left[\frac{1}{2M^2} + \frac{k-1}{4\left(1+\frac{k-1}{2}M^2\right)} - \frac{k}{1+k\, M^2}\right] d(M)^2 \quad ----- \quad (33)$$

Now

$$\rho^* = \frac{P^*}{RT^*}$$

$$= \frac{P^* T_o}{R\frac{(k+1)}{2}} \quad -------- \quad (33a), \qquad T^* = \frac{T_o}{\frac{1+k}{2}}$$

And $w^* = \rho^* A^* V^*$

$$= \rho^* A \; M^* \quad kRT^*$$

$$= \rho^* \cdot A \cdot \sqrt{\frac{kRT_o}{\frac{k-1}{2}}} \text{ - - - - - - - - - - - - - - - - (33b)}$$

Substituting equation (33a) in (33b)

$$w^* = \frac{P^* T_o}{R(\frac{k+1}{2})} \cdot A \cdot \sqrt{\frac{kRT_o}{\frac{k+1}{2}}} \text{ - - - - - - - - - - - - (33c)}$$

Substituting equations (33a), (33c) in equation (33')

$$g \cdot \frac{\rho^* R (\frac{k+1}{2})^{(\frac{k+1}{2})} (k+1)}{T_o (1+k\,M^2)\; RT_o} \cdot \frac{1+\frac{k-1}{2} M^2}{1} \quad dZ$$

$$= AMVg' \; \frac{w^* R \cdot \frac{k+1}{2} \sqrt{\frac{k+1}{2}}}{AT_o \quad kRT_o} \; \frac{(1+k)\sqrt{k} \cdot \sqrt{(1+\frac{k-1}{2} M2)}}{(1+k\,M^2) \quad RT_o \quad\quad 1}$$

$$X \left[\frac{1}{2M^2} + \frac{k-1}{4(1+\frac{k-1}{2} M2)} - \frac{k}{1+k\,M^2} \right] d(M)^2$$

After simplification and transposing the required terms on the left hand side to get the required form,

$$\frac{g\rho^* dZ}{w^* V g'} = \frac{M}{\sqrt{2(k+1)\;(1+\frac{k-1}{2}M^2)}}$$

$$X \left[\frac{1}{2M^2} + \frac{k-1}{4(1+\frac{k-1}{2}M^2)} - \frac{k}{(1+k\,M^2)} \right] d(M)^2$$

If the above expression is integrated between M and M*.

$$\int_M^{M^*} \frac{g\rho^* dZ}{w^* V g'} = \int_M^{M^*} \frac{M}{\sqrt{2(k+1)\;(1+\frac{k-1}{2}M^2)}}$$

$$X \left[\frac{1}{2M^2} + \frac{k-1}{4(1+\frac{k-1}{2}M^2)} - \frac{k}{(1+kM^2)} \right] d(M)^2$$

The integration of the left hand side is $\dfrac{g\rho^*(Z^* - Z)}{w^*Vg'}$

The author was not able to integrate the integral on the right hand side of the equation, and a step-by-step method of numerical calculation was used.

Numerical Analysis of Elevation Effects Without Friction

$M_1 = 0.5$

$M_2 = 0.6$

$T_1 = 1000^oR$

$P_1 = 100$ psia. $\qquad T_o = 1050^oR$

$\frac{T_2}{T_o} = 0.95238 \qquad T_2 = .95238 \times 1050 = 979.48^oR$ (isentropic tables)

$\rho_1 = \frac{P_1}{RT_1} = \frac{100}{0.37043 \times 1000} = 0.26995 \text{ lbm/ft}^3$

$V_1 = 49.0172 \times 0.5\sqrt{1000} = 49.0172 \times 31.625 \times 0.5$

$= 1550.1 \times 0.5 = 775.08$ f.p.s.

$w_1 = \rho_1 V_1 = 0.26995 \times 775.08$

$= 209.23$ lbm/sec.

$T_2 = T_o \times 0.95238 = 979.48^oR$. (isentropic tables)

$V_2 = 49.0172 \times 0.6 \times\sqrt{979.48}$

$= 31.29 \times 49.0172 \times 0.6$

$= 1533.7 \times 0.6 = 920.25$ f.p.s.

$\frac{P_2}{P^*} = 1.5975$, $P^* = 56.24$ psia (Rayleigh-Line Tables)

$P_2 = 1.5975 \times 56.24 = 89.757$ psia

$$\rho_2 = \frac{89.757}{0.37043 \times 979.48} = \frac{89.76}{362.825} = 0.24739 \text{ lbm/ft}^3$$

$$w_2 = \rho_2 V_2 = 920.248 \times 0.24739$$

$$= 227.66 \text{ lbm}$$

$$\bar{\rho} = 0.25867 \text{ lbm/ft}^3$$

$$w_2 - w_1 = 18.43 \text{ lbm}$$

Table 3. Dimensionless elevation parometer for subsonic vertical flow without friction.

Mach Number	$\frac{\rho^* g (Z^* - Z)}{w^* \, Vg'}$	$\Delta Z_{ft.}$	$Z_{ft.}$
0.5	123.74×10^{-3}		0
0.6	74.49×10^{-3}	1550.1	1550.1
0.7	39.24×10^{-3}	1113.0	2663.1
0.8	12.24×10^{-3}	724.77	3387.8
0.9	3.94×10^{-3}	392.86	3780.7
1.0	0	120.05	4000.7

$$w^* = 251.57 \text{ lbm/sec.}$$

$$\rho^* = 0.17351 \text{ lbm/ft}^3$$

$$Vg' = 700 \text{ ft/sec.}$$

$$g = 32.17 \text{ ft/sec.}^2$$

Fig. 5. Dimensionless elevation parameter for subsonic vertical flow without friction.

$$\frac{\rho^* g(Z - Z)}{w^* V_g'} \text{ vs Mach Number}$$

Table 4. Dimensionless elevation parameter for supersonic vertical upward flow without friction.

Mach Number	$\frac{g\rho^*(Z^*-Z)}{w^* V_{g'}}$	$\Delta Z_{ft.}$	$Z_{ft.}$
2.0	177.0×10^{-3}	0	0
1.7	99.08×10^{-3}	2455.3	2455.3
1.5	59.10×10^{-3}	1257.1	3712.4
1.4	40.5×10^{-3}	590.41	4302.8
1.3	24.8×10^{-3}	499.69	4802.5
1.2	12.0×10^{-3}	402.21	5204.7
1.1	2.05×10^{-3}	271.80	5475.5
1.0	0.0	99.158	5574.6

$w^* = 251.57$ lbm/sec.

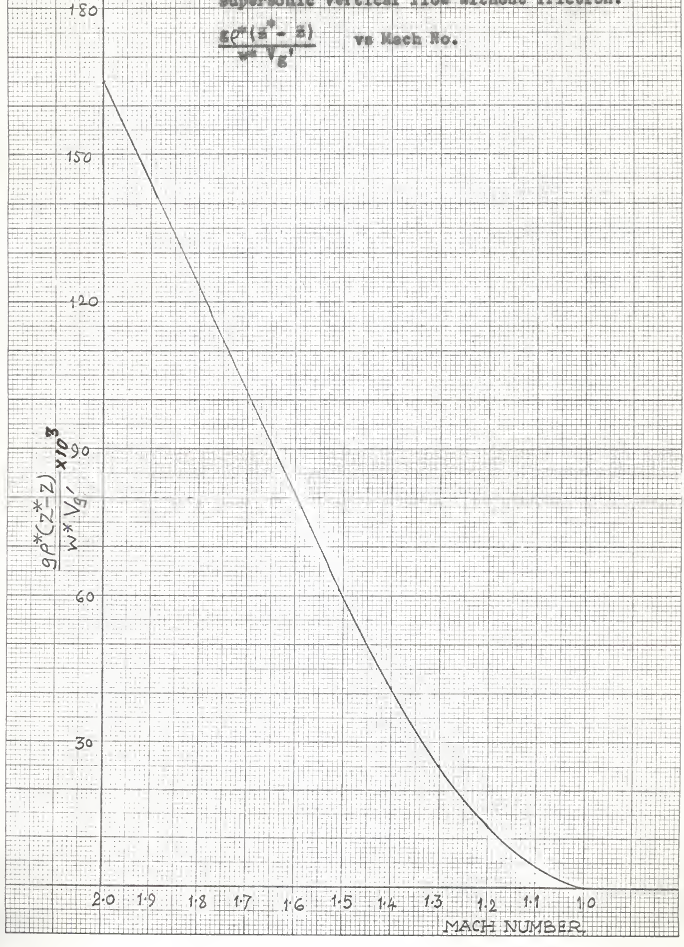
$\rho^* = 0.17351$ lbm/ft^3

$V_{g'} = 700.0$ ft/sec.

$g = 32.17$ ft/sec.$^{-2}$

Fig. 6. Dimensionless elevation parameter for supersonic vertical flow without friction.

$\dfrac{g\rho^*(z^* - z)}{w^* V_g'}$ vs Mach No.



$$A\rho z = \frac{V_{g'}}{g_c}(w_2 - w_1)$$

$$V_{g'} = 700 \text{ f.p.s.}$$

$$1 \times 0.25887 \times z_1 = \frac{700}{32.174}\ (18.43)$$

$$= 21.756\ (18.43)$$

$$= 400.96$$

$$z_1 = \frac{400.96}{0.25887}$$

$$= 1550.1 \text{ ft.}$$

$$z_1 = 0$$

$$z_2 = 1550.1 \text{ ft.}$$

Comparison of the Fanno Line and the Rayleigh Line With Flow Having Constant Area, Constant Impulse Function, and Constant Stagnation Temperature Fluid Flow

In the Fanno Line, the stagnation temperature and the mass flow are constant but the impulse function decreases. In the Rayleigh-Line process the impulse function and the mass flow are constant but the stagnation temperature varies. In the case of the flow under analysis the impulse function as well as the stagnation temperature are constant, but the mass flow rate varies through the duct. In all three flows the area of cross-section of the duct is constant. All these three flows are compared by plotting the following graphs:

(i) P/P^* against v/v^* (Where v is specific volume)

(ii) T/T^* against $\frac{S-S^*}{R}$

Table 5. Comparison of the Rayleigh Line and the Fanno Line with flow having Constant A, Constant F, and Constant T_o.

Mach Number	Fanno Line		Rayleigh Line		Const. A, Const. F, and Const. T_o	
	P/P*	v/v*	P/P*	v/v*	P/P*	v/v*
0.1	10.94	0.1094	2.366	0.0236	2.366	0.5060
0.2	5.455	0.2182	2.272	0.0900	2.272	0.5240
0.4	2.695	0.4313	1.960	0.3137	1.960	0.5935
0.6	1.763	0.6348	1.595	0.5744	1.595	0.7020
0.8	1.289	0.8251	1.265	0.8101	1.265	0.8410
1.0	1.000	1.0000	1.000	1.0000	1.000	1.0000
2.0	0.4082	1.6330	0.3636	1.454	0.3636	1.834
3.0	0.2182	1.9640	0.1764	1.588	0.1764	2.438
4.0	0.1336	2.1380	0.1025	1.641	0.1025	2.784
6.0	0.0637	2.2950	0.0466	1.680	0.0466	3.136
8.0	0.0368	2.3590	0.0264	1.6954	0.0264	3.284
10.0	0.0239	2.3900	0.01702	1.702	0.01702	3.360

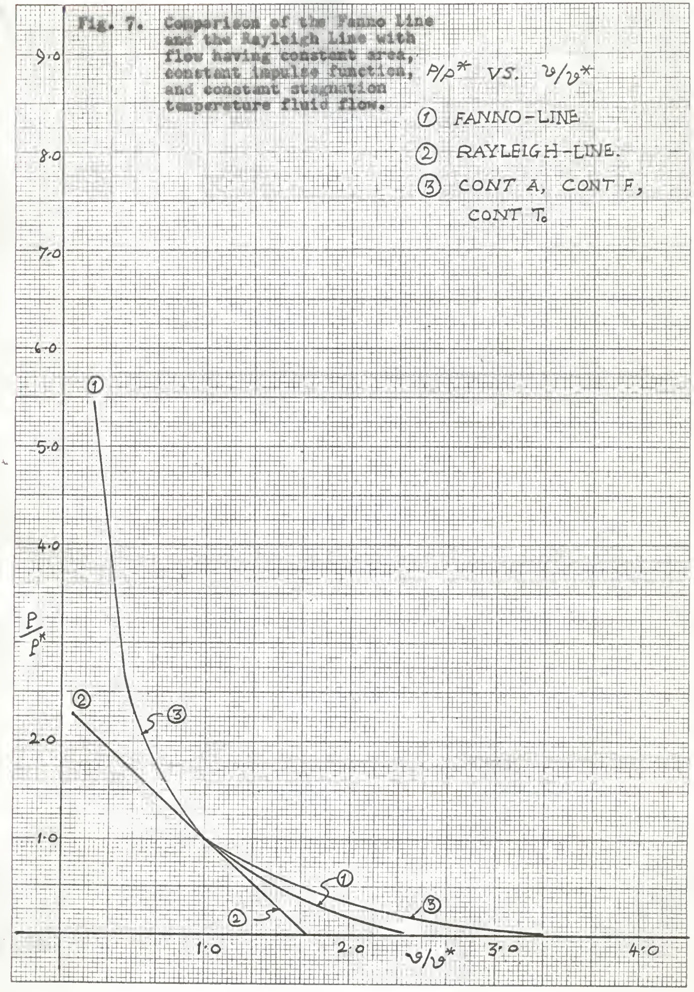
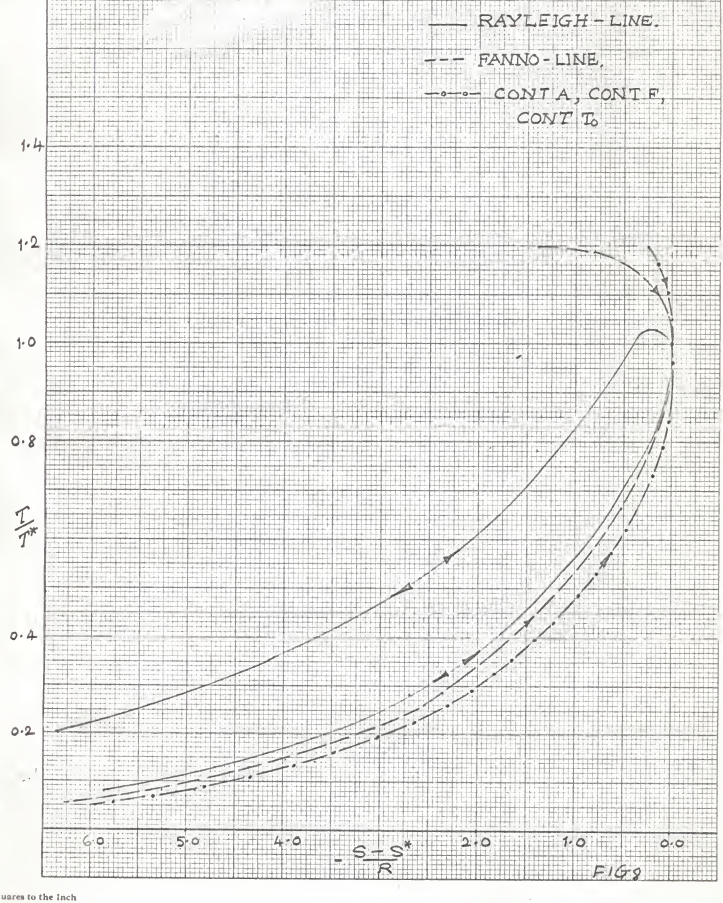


Fig. 7. Comparison of the Fanno line and the Rayleigh Line with flow having constant area, constant impulse function, and constant stagnation temperature fluid flow.

Table 6. Comparison of the Rayleigh Line and the Fanno Line with flow having Constant A, Constant F, and Constant T_o.

Mach Number	Fanno Line		Rayleigh Line		Const. A, Const. F, and Const. T_o	
	T/T^*	$-\frac{S-S^*}{R}$	T/T^*	$-\frac{S-S^*}{R}$	$-\frac{S-S^*}{R}$	T/T^*
0.1	1.197	1.363	0.0560	10.97	0.2311	1.197
0.2	1.190	1.085	0.2066	6.325	0.2070	1.190
0.4	1.162	0.4637	0.6151	1.988	0.1430	1.162
0.6	1.119	0.1739	0.9167	0.770	0.0720	1.119
0.8	1.063	0.0392	1.0250	0.1470	0.0198	1.063
1.0	1.000	0.0000	1.0000	0.0000	0.0000	1.000
2.0	0.6666	0.5247	0.5289	1.2150	0.4054	0.6666
3.0	0.4285	1.4430	0.2802	2.7100	1.2290	0.4285
4.0	0.2846	2.3700	0.1683	3.9500	2.1000	0.2846
6.0	0.1463	3.9740	0.0784	5.8350	3.6600	0.1463
8.0	0.0867	5.2470	0.0449	7.2300	4.9120	0.0867
10.0	0.0571	6.2800	0.0289	8.3000	5.9420	0.0571

Fig. 8. Comparison of the Fanno Line and the Rayleigh Line with flow having constant area, constant impulse function, and constant stagnation temperature fluid flow.



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BIBLIOGRAPHY

1. Keenan, J.H., and Kaye, Joseph. Gas Tables. John Wiley and Sons, New York, 1948, pp. 139-160.

2. Shapiro, A. H. The Dynamics and Thermodynamics of Compressible Fluid Flow. The Ronald Press Company, New York, 1953, pp. 231-248.

"CONSTANT AREA, CONSTANT IMPULSE FUNCTION AND CONSTANT STAGNATION TEMPERATURE COMPRESSIBLE FLUID FLOW."

by

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AN ABSTRACT OF A MASTER'S REPORT

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Constant area, constant stagnation temperature, and constant impulse function compressible fluid flow was investigated in this report. In order to achieve this type of flow it was necessary to have gas injected in the main stream flow. In the first section of the report the ratios of pressure, temperature, velocity, and mass rate flow through the duct were derived as functions of Mach Number. From these formulas a table and a graph of properties for this flow were prepared. These ratios are independent of friction effects as well as elevation effects. In the second section of the report the flow was considered to be horizontal and one in which there was friction at the wall of the duct. In this analysis an averaging method was used and the results were expressed in the form of the dimensionless parameter, $\frac{4fX_{max}}{D}$. This averaging process was necessary as the author could not find an exact method for the determination of the flow properties. In this analysis the gas injection had a forward velocity component which was equal to the main stream gas velocity. The frictional coefficient was assumed to be constant throughout the duct. The third section of the analysis was made with vertical flow without wall friction in which the forward component of injected gas was kept constant throughout the duct. In this analysis the dimensionless parameter $\frac{\rho^*(Z^* - Z)}{w^* V_g'}$ was developed and numerical calculations were done by a step-by-step method, because an exact method could not be found by the author.

In all the above cases it was found that gas was always added to the main stream, and that the flow always tended toward a Mach Number equal to unity.

In the last part of the report a comparison of the Rayleigh Line, the Fanno Line and the flow having constant area, constant stagnation temperature and constant impulse function was made by plotting these processes on the graphs in which coordinates were P/P^* and v/v^*, and T/T^* and $\frac{S-S^*}{R}$.