TWO ANALYTIC MODELS FOR HYPERBOLIC GEOMETRY

by

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B. S., Kansas State University, 1962

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

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INTRODUCTION

This report is primarily a summarization of "A new analytic approach to hyperbolic geometry" by Wanda Szmielew and Appendix III of "Grundlagen der Geometrie" by David Hilbert. Also, a brief comparison of the two articles is made. In each of these articles the author develops a field and bases an analytic geometry for hyperbolic geometry on that field. However, the method of developing the fields is quite different in each case. Szmielew defines a much smaller system and shows how to extend this system to the field; Hilbert simply defines a set of elements with certain operations and shows that this system forms a field. The analytic geometry developed by Hilbert has apparently acted as a basis or starting point for other writers on this subject. He was the first one to construct a field in plane hyperbolic geometry without the axiom of continuity and to develop an analytic geometry over it (4, p. 129). However, the set of elements over which the field is defined does not include all of the elements necessary to give a sufficient basis for an analytic geometry for hyperbolic geometry. More will be said about this point later.

SUMMARY OF AN ARTICLE BY SZMIELEW

An ordered field $\overline{\mathcal{S}}=(\overline{S}, +., *, <)$ is constructed in plane hyperbolic geometry by Szmielew. This field is generated by the algebraic system $\mathcal{S}=(S, +., *, <)$ in which the elements of S are segments. The operations +. and * are defined in terms of the Lambert quadrangle[1] and the right triangle, while the relation $<$ coincides with the usual less-than relation for segments. She then shows how a rectangular coordinate system can be constructed over $\mathfrak{F}$.

The problem of constructing an ordered field is reduced to that of construcing a unit interval algebra. A system $\mathfrak{S}=(S, +, \cdot, <)$ is a unit interval algebra if and only if it satisfies the following postulates (4, p. 130):

1. If $x \in S$, then $x \not< x$.

2. If $x, y \in S$, then $x=y$ or $x<y$ or else $y<x$.

3. If $x, y, x+y \in S$, then $x+y=y+x$.

4. If $x, y, z, x+y, (x+y)+z \in S$, then $y+z \in S$ and $(x+y)+z=x+(y+z)$.

5. If $x, z \in S$, then $x<z$ iff[2] $x+y=z$ for some $y \in S$.

6. If $x, y \in S$, then $x \cdot y \in S$ and $x \cdot y=y \cdot x$.

7. If $x, y, z \in S$, then $(x \cdot y) \cdot z=x \cdot (y \cdot z)$.

8. If $x, z \in S$, then $z<x$ iff $z=x \cdot y$ for some $y<S$.

9. If $x, y, z, x+y \in S$, then $(x+y) \cdot z=x \cdot z+y \cdot z$.

It can easily be seen that for any arbitrary ordered field $\mathfrak{F}$ with the zero element 0 and the unit element 1, the open interval (0,1) forms a unit interval algebra. Szmielew shows that any unit interval algebra can be extended to an ordered field.

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[1]The Lambert quadrangle is a quadrangle with three right angles. In hyperbolic geometry, this means the fourth angle is acute.

[2]iff is used for if and only if

It can be shown (4, p. 131) that for every x in S there is a unique element x' in S such that $x \cdot t + x' \cdot t = t$ for every t in S. The element x' is called the complement of x. It can also be shown that the equation $x = x'$ has a single solution in S. The element that satisfies this is denoted by $\frac{1}{2}$; hence, it follows that for any x in S, $\frac{1}{2} \cdot x + \frac{1}{2}$ is in S since for every $y < x'$, $x + y$ is in S. Also, if x and y are in S, then $\frac{1}{2} \cdot x + \frac{1}{2} \cdot y$ is in S.

If in addition to the postulates 1-9 the following statement holds, then $\mathfrak{S}$ is said to be a Euclidean unit interval algebra.

10. If $x \in S$, then $x = y \cdot y$ for some $y \in S$.

The equation $x = y \cdot y$ then has a unique solution for y for a given x. It is denoted by $\sqrt{x}$.

Then if a new operation $+'$ is introduced by putting $x +' y = z$ if and only if $\sqrt{x} + \sqrt{y} = \sqrt{z}$, the algebraic system $\mathfrak{S}' = (S', +', \cdot, <)$ is referred to as the square root derivative of $\mathfrak{S}$. It can be shown that the function $f(x) = \sqrt{x}$ maps the system $\mathfrak{S}$ isomorphically onto the system $\mathfrak{S}'$ (4, p. 133). The following theorem is a result of this.

<u>Theorem</u> 1. The square root derivative $\mathfrak{S}'$ of a Euclidean unit interval algebra $\mathfrak{S}$ is again a Euclidean unit interval algebra.

The following theorem is then proved (4, p. 133). It establishes the fact that any unit interval algebra can be extended to an ordered field.

Theorem 2. (I) Any unit interval algebra $\mathfrak{S}=(S, +'', \cdot, <)$ can be embedded in a commutative ordered field $\bar{\bar{\mathfrak{S}}}=(\bar{\bar{S}}, +., *, <)$ in such a way that S consists of all those elements $\bar{\bar{x}}$ in $\bar{\bar{S}}$ for which $0 < \bar{\bar{x}} < 1$, where 0 is the zero element and 1 is the unit element of the field $\bar{\bar{\mathfrak{S}}}$. (II) In fact, $\bar{\bar{\mathfrak{S}}}$ is up to an isomorphism uniquely determined by $\mathfrak{S}$ in the following sense: if $\bar{\bar{\mathfrak{S}}}_1$ and $\bar{\bar{\mathfrak{S}}}_2$ are two ordered fields generated by $\mathfrak{S}$, then there is an isomorphic mapping of $\bar{\bar{\mathfrak{S}}}_1$ onto $\bar{\bar{\mathfrak{S}}}_2$ which leaves all elements of $\mathfrak{S}$ unchanged. (III) In addition, if $\mathfrak{S}$ is Euclidean then $\bar{\bar{\mathfrak{S}}}$ is also Euclidean.

As an example of this theorem one may consider the unit interval algebra $\mathfrak{S}=(S, +, \cdot, <)$ consisting of the set of rationals greater than 0 and less than 1 with the operations of ordinary addition and multiplication and the usual relation of less-than. Then consider ordered pairs (a,b), a is in S, b is in S, where (a,b) is interpreted as the quotient a/b. By proper definitions for a new addition and multiplication and a congruence relation $\equiv$ and a less-than relation on the ordered pairs, one can form a quotient algebra $\mathfrak{S}/\equiv$. The system $\mathfrak{S}/\equiv$ will be isomorphic to the system $\bar{\bar{\mathfrak{S}}}$ consisting of the set of all positive rationals with the usual operations of addition and multiplication and the relation of less-than. Also, there is a subsystem of $\mathfrak{S}/\equiv$ isomorphic to $\mathfrak{S}$. In other words, the rationals between 0 and 1 can be extended to all positive rationals. In a similar manner (i.e., constructing ordered pairs and properly defining operations and relations) one can extend all positive rationals to all rationals. Actually one extends $\overline{\mathfrak{Z}}$ to the ordered field $\overline{\mathfrak{Z}}/\sim$ which is isomorphic to the ordered field of rationals $\overline{\overline{\mathfrak{Z}}}=(\overline{\overline{S}}, +, \cdot, <)$. This could be represented graphically as follows.

Unit interval algebra; set: rationals between 0 and 1	Form ordered pairs → and define operations and relations	Quotient algebra isomorphic to second system; set: all positive rationals Subsystem isomorphic to original unit interval algebra

Form ordered pairs → and define operations and relations	Second quotient algebra isomorphic to ordered field; set: all rationals Subsystem isomorphic to system over all positive rationals; hence, ordered field is extension of original unit interval algebra

Statement (I) of Theorem 2 is proved as followss: Consider the algebraic system $\overline{\mathfrak{Z}}_1=(\overline{S}_1, +_1, \cdot_1, <_1)$ defined in the following manner. The set $\overline{S}_1$ is the Cartesian product S x S of S, and the operations $+_1$, $\cdot_1$ and the relation $<_1$ for any two elements (a,b) and (c,d) of $\overline{S}_1$ are defined by the following formulas (the operations + and $\cdot$ are the operations in $\mathfrak{Z}$, and similarly for the relation $<$).

$(a,b)+_1(c,d)=(\frac{1}{2}\cdot(a\cdot d)+\frac{1}{2}\cdot(b\cdot c),\frac{1}{2}\cdot(b\cdot d))$, where $\frac{1}{2}$ is that element in S satisfying the equation $x=x'$,

$(a,b)\cdot_1(c,d)=(a\cdot c,b\cdot d)$,

$(a,b) <_1 (c,d)$ iff $a\cdot d < b\cdot c$.

Also, the relation $\equiv$ is defined as follows:

$(a,b) \equiv (c,d)$ iff $a \cdot d = b \cdot c$.

One can show that $\equiv$ is a congruence relation in $\bar{\mathscr{S}}_1$. Also, the set $\bar{S}_1$ forms a commutative group under the operation of $\cdot_1$. The operation $+_1$ on $\bar{S}_1$ is closed, associative and commutative, and the operation $\cdot_1$ distributes with respect to addition. The system $\bar{\mathscr{S}}_1/\equiv$ will be called a quotient algebra.

The unit element of the group $(\bar{S}_1, \cdot_1)/\equiv$ coincides with the coset consisting of all elements (a,b) with $a=b$. Let S_1 be a subset of $\bar{S}_1$ consisting of all elements (a,b) with $a<b$. Then (4, p. 134) the algebraic subsystem $(S_1, +_1, \cdot_1, <_1)/\equiv$ of $\bar{\mathscr{S}}_1/\equiv$ is isomorphic to the unit interval algebra $\mathscr{S}$. The correspondence is $(a,b) \longleftrightarrow c$ where c is that element of S such that $a=b \cdot c$ (see postulate 8 for unit interval algebra). The element c is unique for a given a and b. Also, $\bar{\mathscr{S}}_1/\equiv$ can be modified to an isomorphic system $\bar{\mathscr{S}}=(\bar{S}, +, \cdot, <)$; hence, $\bar{\mathscr{S}}$ is an extension of $\mathscr{S}$. The set S consists then of all those elements $\bar{x}$ of $\bar{S}$ for which $\bar{x}<1$.

By an analogous procedure one constructs an ordered field $\bar{\bar{\mathscr{S}}}=(\bar{\bar{S}}, +, \cdot, <)$ which is an extension of $\bar{\mathscr{S}}$ such that $\bar{S}$ consists of all positive elements of $\bar{\bar{S}}$. This is accomplished in the following manner. Consider the system $\bar{\bar{\mathscr{S}}}'=(\bar{\bar{S}}', +', \cdot', <')$. The set $\bar{\bar{S}}'$ is the Cartesian product $\bar{S} \times \bar{S}$ of $\bar{S}$, and the operations $+'$, $\cdot'$ and the relation $<'$ are defined as follows for any two elements (a,b) and (c,d):

$(a,b)+'(c,d)=(a+c,b+d)$,

$(a,b)\cdot'(c,d)=(a\cdot c+b\cdot d,a\cdot d+b\cdot c)$,

$(a,b) <' (c,d)$ iff $a+d < b+c$.

The relation $\sim$ is defined by the next statement.

$(a,b) \sim (c,d)$ iff $a+d=b+c$.

The relation $\sim$ is a congruence relation is $\overline{\overline{\mathfrak{Z}}}'$. The set $\overline{\overline{S}}'$ forms a commutative group under the operation of $+'$. The set $\overline{\overline{S}}'$ minus the element (a,a) forms a commutative group under the operation of $\cdot'$, and $\cdot'$ distributes with respect to $+'$. With the order relation, the system $\overline{\overline{\mathfrak{Z}}}'/\sim$ is an ordered field.

The zero element of the group $(\overline{\overline{S}}', +')/\sim$ coincides with the coset consisting of all elements (a,a) while the unit for $(\overline{\overline{S}}', \cdot',)/\sim$ is the coset consisting of all elements of the form $(a+1,a)$ where 1 is unity in $\overline{\mathfrak{Z}}$. Since the elements in $\overline{S}$ have inverses with respect to multiplication, the inverse for non-zero elements of $\overline{\overline{S}}'$ with respect to $\cdot'$ are defined as follows: If $b < a$, then the inverse of (a,b) is $((d+1)/c, d/c)$ where c is that element of $\overline{S}$ such that $b+c=a$. If $a < b$, the inverse of (a,b) is $(d/c, (d+1)/c)$ where c is that element of $\overline{S}$ such that $a+c=b$. In either case d is any element of $\overline{S}$. Since $a=b$ implies (a,b) is the zero element, this defines the inverse for all non-zero elements. If S' is the subset of $\overline{\overline{S}}'$ consisting of all elements (a,b) with $a > b$, then the algebraic subsystem $(S', +', \cdot', <')/\sim$ is isomorphic to the system $\overline{\mathfrak{Z}}$. The correspondence is $(a,b) \longleftrightarrow c$ where c is that element such that $a=b+c$. Also, $\overline{\overline{\mathfrak{Z}}}'/\sim$ can be modified to an isomorphic system $\overline{\overline{\mathfrak{Z}}}=(\overline{\overline{S}}, +, \cdot, <)$; hence, $\overline{\overline{\mathfrak{Z}}}$ is an extension of $\overline{\mathfrak{Z}}$. The set $\overline{S}$ consists of all those elements $\overline{\overline{x}}$ of $\overline{\overline{S}}$ for which $\overline{\overline{x}} > 0$. Therefore, the ordered field $\overline{\overline{\mathscr{S}}}$ is an extension of $\mathscr{S}$ such that for every $\overline{\overline{x}}$ in $\overline{\overline{S}}$, $\overline{\overline{x}}$ is in S if and only if $0 < \overline{\overline{x}} < 1$, where 0 is the zero element and 1 is the unit element of $\overline{\overline{\mathscr{S}}}$. This completes the proof of statement (I).

With this theorem as a basis, one is then ready to construct the algebraic system $\mathscr{S}=(S, +., *, <)$. As was mentioned earlier, the ordered field $\overline{\mathscr{S}}=(\overline{S}, +., *, <)$ follows from this, and finally the rectangular coordinate system based on the ordered field. It will actually turn out that the system $\mathscr{S}$ is a Euclidean unit interval algebra. Then it will follow immediately that $\overline{\mathscr{S}}$ is an ordered field by Theorem 2.

The definition of the elements of S is as follows: These are the free segments. By a segment is meant any non-ordered pair p, q of distinct points of the hyperbolic plane. Then the set of all segments congruent to a given segment pq is called the free segment determined by pq and is denoted (pq). Free segments will be represented by the variables A, B, C, X, Y, Z, ..., with subscripts at times. The order relation $<$ is extended to free segments by the following condition.

(i) $X < Y$ iff q is between p and r, X=(pq), Y=(pr) for some distinct points p, q, r.

Statement (i) implies the next three statements.

(ii) $X \nless X$;

(iii) $X=Y$ or $X < Y$ or else $Y < X$;

(iv) if $X < Y$ and $Y < Z$, then $X < Z$.

An angle is defined as any non-ordered pair $\overline{GH}$ of half lines $\overline{G}$ and $\overline{H}$ which are supposed to be non-collinear and to have a common origin. The set of all angles congruent to a given angle $\overline{GH}$ is called the free angle determined by $\overline{GH}$ and is denoted $(\overline{GH})$. Free angles will be represented by variables α, β, γ, δ, ... The relation of less-than for free angles is completely analoguous to that for free segments.

Consider any line $\overline{\overline{L}}$ and any point p on $\overline{\overline{L}}$. Let $\overline{M}$ be a half-line extending from p and not coinciding with $\overline{\overline{L}}$. The point p separates $\overline{\overline{L}}$ into two half-lines; call these $\overline{L}_1$ and $\overline{L}_2$. Then $\overline{L}_1$ and $\overline{M}$ determine $(\overline{L}_1\overline{M})$, and $\overline{L}_2$ and $\overline{M}$ determine $(\overline{L}_2\overline{M})$. If $(\overline{L}_1\overline{M})$ $(\overline{L}_2\overline{M})$, then $(\overline{L}_1\overline{M})$ is said to be an acute free angle, and $(\overline{L}_2\overline{M})$ is said to be an obtuse free angle. If $(\overline{L}_1\overline{M})=(\overline{L}_2\overline{M})$, then $(\overline{L}_1\overline{M})$ is called the free right angle. The free right angle will be denoted by ρ. All other free angles are either acute or obtuse. Also, the operation of addition for free segments and free angles is defined as one would expect. For example, X+Y=Z if and only if q is between p and r, X=(pq), Y=(qr), Z=(pr) for some distinct points p, q, r.

It is well-known in hyperbolic geometry that every free segment determines a unique angle of parallelism. This can be seen in the following manner. Take an oriented line l and a point p not on l. Then let q be the perpendicular projection of p onto l. The points p and q determine a free segment X. Then let $\overline{G}$ be the half-line parallel to l and with origin p, and let $\overline{H}$ be the half-line containing X and with origin p. The free angle $(\overline{GH})$ is called the angle of parallelism for the free segment X. It is easily seen then that each free segment determines a unique free angle, and this free angle will always be acute. This establishes a one-to-one correspondence between free segments and acute free angles. This correspondence will be denoted by P(X), and for convience the following notation will be used: $P(A)=\alpha$, $P(B)=\beta$, ... It may also be noted that as the free segment increases in magnitude the associated free angle decreases. Then the condition $P(A)+P(A')=\rho$ determines a unique free segment A' called the complement of A. If $A < B$, then $A' > B'$, and it also follows that $A''=A$.

The definitions of a free right triangle and the free Lambert quadrangle are still needed before one is ready to construct the unit interval algebra. The free right triangle is defined as follows: Let p, q and r be any points such that pq, qr and pr form a right triangle with pq and qr as legs and pr as hypotenuse. Call the acute angles α and β. Then the set of right triangles congruent to triangle pqr will be denoted $X\alpha Z\beta Y$ where X, Y and Z denote the free segments (pq), (qr), and (pr), respectively. The set $X\alpha Z\beta Y$ will be called the free right triangle $X\alpha Z\beta Y$. This will be written symbolically as $T(X\alpha Z\beta Y)$. If there exists $T(X\alpha Z\beta Y)$, then any two of the five terms X, α, Z, β, Y determine the other three uniquely. This is easily seen for the various cases. For example, consider $T(X\alpha Z\beta Y)$ and assume another right triangle with X and Y as legs, i.e. $T(X\sigma V\gamma Y)$. It is true in hyperbolic, as well as Euclidean geometry that if two legs of one right triangle are equal to two legs of a second right triangle, then the two triangles are congruent. This means $T(X\alpha Z\beta Y)$ and $T(X\sigma V\tau Y)$ are congruent; hence, α, Z and β are unique.

The free Lambert quadrangle is defined very much the same as the free right triangle. Let p, q, r and s be any points such that the segments pq, qr, rs and ps form a Lambert quadrangle with the acute angle at s. Call the acute angle β. Then the set of Lambert quadrangles congruent to pqrs will be denoted $XAZ\beta Y$ where X, A, Z and Y denote the free segments (pq), (qr), (rs) and (ps), respectively. The set $XAZ\beta Y$ will be called the free Lambert quadrangle $XAZ\beta Y$. The free Lambert quadrangle will be written symbolically as $Q(XAZ\beta Y)$. If there exists $Q(XAZ\beta Y)$, then any two of the five terms X, A, Z, β, Y determine uniquely the remaining three. A theorem due to Liebmann expresses an equivalence between the free right triangle and the free Lambert quadrangle. It is: There exists $T(X\alpha Z\beta Y)$ iff there exists $Q(XA'Z\eta B)$. Some consequences of this theorem are: There exists $Q(XA'Z\eta B)$ iff there exists $Q(A'XB\eta Z)$ iff there exists $T(A'\xi'B\gamma Y)$; hence, the next statement follows.

(a) There exists $T(X\alpha Z\beta Y)$ iff there exists $T(A'\xi'B\gamma Y)$. Statements (b) through (e) are listed without intermediate steps (4, p. 140).

(b) There exists $T(X\alpha Z\beta Y)$ iff there exists $T(A'\beta X'\eta'Z')$;

(c) for every X and β there exists α, Z, Y such that $T(X\alpha Z\beta Y)$ exists;

(d) if $Z > Y$, then there exists X, α, β such that $T(X\alpha Z\beta Y)$ exists;

(e) if $\alpha + \beta < \rho$, then there exists X, Y, Z such that $T(X\alpha Z\beta Y)$ exists.

Three operations are now defined in terms of the free right triangle and the free Lambert quadrangle. The first of these three operations, denoted by $\odot$, is defined as follows: Given three free segments X, Y and Z, $X \odot Y = Z$ if and only if there exists $T(X\alpha Z\beta Y)$ for some α and β (Fig. 1). It follows from the previous discussion that for any given X and Y there exists a unique Z such that $X \odot Y = Z$. Also, if X is in S, then $X = Y \odot Y$ for a unique Y in S.

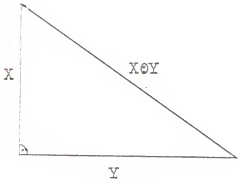

Fig. 1

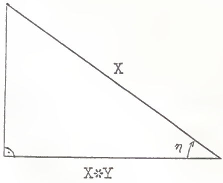

Fig. 2

The second operation, denoted by $*$, is defined as follows: $X * Y = Z$ if and only if there exists $T(A\beta X\eta Z)$ for some A and β (Fig. 2). The definition of $*$ implies the following statements.

(v) If $X, Y \in S$, then $X * Y \in S$;

(vi) $X * Y = Z$ implies $X > Z$.

Also, with the help of the Liebmann theorem (4, p. 141), the next two statements follow:

(vii) If $X > Z$, then $X*Y=Z$ for some $Y \in S$;

(viii) $X*Y=Y*X$; i.e., the operation $*$ is commutative.
The associativity of $*$ is expressed by the next statement.

(ix) $(X*Y)*Z=X*(Y*Z)$.

The proof of this is a good exercise in working with the operation, but it will not be reproduced here (4, p. 142). From (b) and the definitions of $\odot$ and $*$ the next statement follows:

(f) $X \odot Y=Z$ iff $X'*Y'=Z'$.

This implies $X*Y=Z$ if and only if $X' \odot Y'=Z'$. Statement (f) is proved as follows. From the definition of $*$, $X'*Y'=Z'$ is equivalent to $T(A'\beta X'\eta'Z')$. But $T(A'\beta X'\eta'Z')$ is equivalent to $T(X\alpha Z\beta Y)$ by statement (b), and the latter implies $X \odot Y=Z$, and conversely. Then the systems $(S, *)$ and $(S, \odot)$ are isomorphic; hence,

(x) if $X \in S$, then $X=Y*Y$ for a unique $Y \in S$.

The third operation, denoted by $\oplus$, is defined as follows: Given three free segments X, Y and Z. $X \oplus Y=Z$ if and only if $X=A*Z$, $Y=A'*Z$ for some A in S. This means there exists a free Lambert quadrangle with X and Y as two adjacent sides and with its acute angle opposite the vertex determined by X and Y. Then Z is the segment joining this vertex with the one at the acute angle. It can be seen that $X \oplus Y$ is not defined for every X and Y. As an example consider the case where α' corresponds to the angle of parallelism for Y. Then there would not exist a free Lambert quadrangle satisfying the conditions.

The definition of $\oplus$ implies $A*Z\oplus A'*Z=Z$, which may be treated as a particular case of the distributive law.

(xi) If $X\oplus Y \in S$, then $X*Z\oplus Y*Z=(X\oplus Y)*Z$.

The distributive law can be proved as follows: Let $X\oplus Y=B$. Then $X=A*B$ and $Y=A'*B$ for some A in S, and $(X\oplus Y)*Z=B*Z$ $=A*(B*Z)\oplus A'*(B*Z)=(A*B)*Z\oplus(A'*B)*Z=X*Z\oplus Y*Z$. The definition of $\oplus$ implies the following statements.

(xii) If $X\oplus Y \in S$, then $X\oplus Y=Y\oplus X$;

(xiii) $X\oplus Y=Z$ implies $X < Z$.

The definition of $\oplus$ and (vii) imply the converse of (xiii):

(xiv) If $X < Z$, then $X\oplus Y=Z$ for some $Y \in S$.

Also, the associative law can be developed:

(xv) If $X\oplus Y$, $(X\oplus Y)\oplus Z \in S$, then $Y\oplus Z \in S$ and $(X\oplus Y)\oplus Z=X\oplus(Y\oplus Z)$.

If formulas (i)-(xv) are compared to the postulates for a unit interval algebra, it will be seen that they imply the following theorem.

<u>Theorem</u> 3. The system $\not{S}_o=(S, \oplus, *, <)$ is a Euclidean unit interval algebra.

If the operation of $\oplus$ is modified by putting $X+.Y=Z$ if and only if $\sqrt{X}\oplus\sqrt{Y}=\sqrt{Z}$, then the resulting system is the square root derivative of $\not{S}_o$. Using Theorem 1, one obtains the following theorem.

<u>Theorem</u> 4. The system $\not{S}=(S, +., *, <)$ is a Euclidean unit interval algebra.

The next theorem follows from Theorem 2.

<u>Theorem</u> 5. The system $\not{S}=(S, +., *, <)$ can be embedded in

a commutative Euclidean field $\overline{\mathcal{S}}=(\overline{S}, +., *, <)$, with the zero element 0 and the unit element 1, such that the following statement is true.

(1) For every $X \in \overline{S}$, $X \in S$ iff $0 < X < 1$.

The field $\overline{\mathcal{S}}$ is uniquely determined up to an isomorphism by $\mathcal{S}$ in the sense of Theorem 2.

It will be assumed from now on that the field has been fixed. The operations $+.$ and $*$ and the relation $<$ are now understood to be defined for arbitrary elements of the field and not only for free segments. Also, the variables A, B, C, ... range over all elements of $\overline{S}$. The operations of addition, subtraction, and scalar product on the elements of the Cartesian product $\overline{S} \times \overline{S}$ are defined as follows: Let (X_1,X_2) and (Y_1,Y_2) be in $\overline{S} \times \overline{S}$. Then

$$(X_1,X_2)+.(Y_1,Y_2)=(X_1+.Y_1,X_2+.Y_2),$$
$$(X_1,X_2)-(Y_1,Y_2)=(X_1-Y_1,X_2-Y_2),$$
$$(X_1,X_2)*(Y_1,Y_2)=X_1*Y_1+.X_2*Y_2,$$
$$(X_1,X_2)^2=(X_1,X_2)*(X_1,X_2).$$

Expressions will now be developed for ordinary addition and subtraction in terms of $+.$ and $*$. Since $\sqrt{X}\oplus\sqrt{Y}=\sqrt{Z}$ if and only if $X+.Y=Z$, this implies $X\oplus Y=Z$ if and only if $X^2+.Y^2=Z^2$. If this is applied to $A*Z\oplus A'*Z=Z$, one obtains $(A*Z)^2+.(A'*Z)^2=Z^2$ for any arbitrary A and Z since the first equation holds for any arbitrary A and Z. This implies $A^2+.A'^2=1$; hence, $A'=\sqrt{1-A^2}$. Since $X\ominus Y=Z$ if and only if $X'*Y'=Z'$ and $X''=X$, it is true that $X\ominus Y=\sqrt{X^2+.Y^2-X^2*Y^2}$. This is shown as follows: $X\odot Y=Z=Z''=(X'*Y')'=\sqrt{1\_(X'*Y')^2}$ $=\sqrt{1\_(\sqrt{1\_X^2}*\sqrt{1\_Y^2})^2}=\sqrt{1\_(1\_X^2)*(1\_Y^2)}=\sqrt{X^2+.Y^2\_X^2*Y^2}$. Since $Q(XAZ\beta Y)$ implies $X\oplus A=A\odot Z$, it follows that if there exists $Q(XAZ\beta Y)$, then $X=\sqrt{1\_A^2}*Z$.

Assume $X+Y=Z$ where $+$ is now ordinary addition of segments. Then $X<Z$; hence, $\sqrt{X}<\sqrt{Z}$ and $\sqrt{X}=A*\sqrt{Z}$ for some A in S. Then $\sqrt{X*Z}=A*Z$ and $X=A*\sqrt{X*Z}$. Therefore, there exists a right triangle $a_1b_1c_1$ with the right angle at c_1 (the angle at a_1 is α) such that $(a_1b_1)=Z$, $(a_1c_1)=\sqrt{X*Z}$, and $(a_1d_1)=X$, provided d_1 is the perpendicular projection of c_1 on a_1b_1. Then $(b_1d_1)=Y$. By a similar argument there is a second right triangle $a_2b_2c_2$ such that $(a_2b_2)=Z$, $(b_2c_2)=\sqrt{Y*Z}$, $(b_2d_2)=Y$, provided d_2 is the perpendicular projection of c_2 on a_2b_2. It is easily seen that triangles $a_1b_1c_1$ and $a_2b_2c_2$ are congruent; hence, $(b_1c_1)=\sqrt{Y*Z}$. Therefore, $\sqrt{X*Z}\odot\sqrt{Y*Z}=Z$, which is equivalent to $X+.Y\_X*Y*Z=Z$ by the preceeding argument. It follows from this that $X+Y=(X+.Y)/(1+.X*Y)$ and $Z-X$ $=(Z\_X)/(1\_X*Z)$ for $X<Z$ (where $1/X$ is the inverse of X with respect to $*$). Then $X+Y<X+.Y$ for every two free segments X and Y.

The distance between two points is defined as follows: Given two arbitrary points p and q,

$$d(p,q)=\begin{cases}0 & \text{if } p=q\\ (pq) & \text{if } p\neq q\end{cases},$$

and the element $d(p,q)$ is the distance between the points p and q. It is noted that $d(p,q)$ is always an element of $\overline{S}$, and $d(p,q) < 1$ for any two points p and q.

The rectangular coordinate system is the next consideration. Consider an arbitrary point p and two arbitrary oriented lines L_1 and L_2 perpendicular to each other at a point $\underline{a}$. Let p_i be the perpendicular projections of p upon L_i $(i=1,2)$. Then for $i=1,2$,

$$X_i^p = \begin{cases} 0 \text{ if } p_i = \underline{a}, \\ (ap_i) \text{ if } \underline{a} \text{ falls before } p_i \text{ on } L_i, \\ -(ap_i) \text{ if } p_i \text{ falls before } \underline{a} \text{ on } L_i. \end{cases}$$

Then X_1^p and X_2^p are elements of $\overline{S}$. The function $\phi(p)=(X_1^p,X_2^p)$ is defined for every point p, and it is called the rectangular coordinate system with the axes L_1 and L_2. The elements X_1^p and X_2^p are the first and second coordinates of the point p in the system ϕ. The point $\underline{a}$ is called the origin of ϕ.

Every rectangular coordinate system ϕ establishes a one-to-one correspondence between the points p of the hyperbolic plane and the elements (X_1,X_2) of the Cartesian product $\overline{S} \times \overline{S}$ satisfying the condition $X_1^2 +. X_2^2 < 1$. For every point p, $\phi(p)*\phi(p) < 1$, or $\phi^2(p) < 1$. Since $\phi^2(p) < 1$, $\phi^2(q) < 1$, it follows that $(\phi(p)*\phi(q))^2 \leq \phi^2(p)*\phi^2(q) < 1$. Then the following definition for $F(p,q)$ correlates with every two points p and q an element of $\overline{S}$, and it is easily seen that $0 < F(p,q) \leq 1$: $F(p,q)=((1-\phi^2(p))*(1-\phi^2(q)))/(1-\phi(p)*\phi(q))^2$. Since $\phi(a)=(0,0)$, $\phi^2(a)=0$, then $F(a,p)=F(p,a)=1-\phi^2(p)$ for every point p.

To develop an expression for the distance between two points p and q, let $\emptyset(p)=(X_1^p,X_2^p)$ and $\emptyset(q)=(X_1^q,X_2^q)$. Assume that $0<X_1^p<X_1^q$ and $0<X_2^p<X_2^q$. Denote the perpendicular projections of points p and q upon L_1 and L_2 by p_i, q_i $(i=1,2)$. Let r be the perpendicular projection of p upon the line qq_1 (Fig. 3). It should be noted that figures aq_1qq_2 and qp_1pp_2 are Lambert quadrangles. Since $Q(XAZ\beta Y)$ implies $X=\sqrt{1\_A^2}*Z$, then $Q(X_2^pX_1^p(pp_1)\gamma(pp_2))$ implies $(pp_1)=X_2^p/\sqrt{1\_(X_1^p)^2}$ where

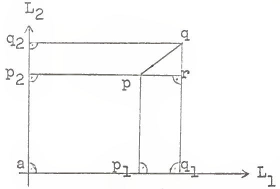

Fig. 3

γ is the angle at p. In the same manner, $(qq_1)=X_2^q/\sqrt{1\_(X_1^q)^2}$. Also, from $Z\text{-}X=(Z\_X)/(1+.X*Z)$ it follows that $(p_1q_1)=(X_1^q\_X_1^p)/(1\_X_1^q*X_1^p)$. In the same manner as for (pp_1) and (qq_1), it follows that $(q_1r)=\sqrt{1\_(p_1q_1)^2}*(pp_1)$ and $(pr)=(p_1q_1)/\sqrt{1\_(q_1r)^2}$, and as for (qq_1), $(qr)=((qq_1)\_(q_1r))/(1\_(qq_1)*(q_1r))$. Since $X\ominus Y=\sqrt{X^2+.Y^2\_X^2*Y^2}$, then $(pq)=\sqrt{(pr)^2+.(qr)^2\_(pr)^2*(qr)^2}$. Szmielew indicates that by use of these formulas, an analytic formula for distance can be obtained. It is $d(p,q)=\sqrt{1\_F(p,q)}$, and the formula holds for the general case; i.e., without the original assumption as to the relative positions of p and q.

If the angle pqr is a right angle, then the figure pqr is a right triangle; hence,

$$(pq)\ominus(qr)=(pr)=\sqrt{(pq)^2+.(qr)^2\_(pq)^2*(qr)^2}.$$

For $p\neq q$, $(pq)=d(p,q)$; therefore,

$\sqrt{(pq)^2 +. (qr)^2 \_ (pq)^2 * (qr)^2} = \sqrt{1\_F(p,q)*F(q,r)} = \sqrt{1\_F(p,r)}$.
The result is then that the angle pqr is a right angle if and only if $F(p,q)*F(q,r)=F(p,r)$. If p and a (the origin) coincide, then by use of the formula for $F(p,q)$ and from the fact that $F(a,p)=F(p,a)=1\_\phi^2(p)$, it follows that angle aqr is a right angle if and only if $\phi(q)*(\phi(r)\_\phi(q))=0$. If the points q and a coincide, $\phi(p)*\phi(r)=0$ is the condition for perpendicularity.

The last problem to be considered is an analytic formula for collinearity. Let K be an arbitrary straight line. If the line K does not go through a, then the perpendicular projection of a upon K is denoted b, and if the line does go through a, then b is an arbitrary point different from a and lying on the perpendicular to K at a. Then for $\phi(b)=(B_1,B_2)$ and an arbitrary point p with $\phi(p)=(X_1,X_2)$, p will lie on K if and only if $\phi(b)*(\phi(p)\_\phi(b))=0$ in the first case, and $\phi(b)*\phi(p)=0$ in the second case. Then the straight line K has the equation $B_1*X_1+.B_2X_2+.B_3=0$ if it is agreed that $B_3=\_\phi^2(b)$ if K goes through a and that $B_3=0$ if K does not go through a. Since $\phi^2(b)=B_1^2+.B_2^2<1$ if $\phi(b)=(B_1,B_2)$, then $\_B_3=B_1^2+.B_2^2>B_3^2$. Also, every linear equation $B_1*X_1+.B_2*X_2+.B_3=0$ with $B_1^2+.B_2^2>B_3^2$ describes a line K. Then if $\phi(p)=(X_1^p,X_2^p)$, $\phi(q)=(X_1^q,X_2^q)$, $\phi(r)=(X_1^r,X_2^r)$, p, q, and r will be collinear if and only if

$$\begin{vmatrix} 1 & X_1^p & X_2^p \\ 1 & X_1^q & X_2^q \\ 1 & X_1^r & X_2^r \end{vmatrix} = 0.$$

This consludes the summary of the article by Szmielew. There are several points that might be noted. First of all, unity (1) in the field developed corresponds to a free segment of infinite length. This can be seen by considering the definition of the operation * and noting that as the length of a free segment becomes infinite, its corresponding angle of parallelism approaches the zero angle. Also, the zero (0) element corresponds to a free segment of length 0. Hence, it is convenient to think of the set S as consisting of the absolute value of the lengths of all possible finite segments. Then the set $\bar{S}$ consists of the lengths (direction taken into account) of all possible segments, including one of infinite length in the negative direction and one of infinite length in the positive direction.

An interesting problem is how to construct the sum X+.Y, given X and Y. It will be recalled that this operation was defined as follows: X+.Y=Z if and only if $\sqrt{X} \oplus \sqrt{Y} = \sqrt{Z}$. This means one must determine $\sqrt{X}$ and $\sqrt{Y}$ given X and Y. Since $0 < X < 1$, it is noted that $X < \sqrt{X}$ and $0 < \sqrt{X} < 1$. There does not appear to be any formal way of determining $\sqrt{X}$, but it is not hard to devise a method to approximate $\sqrt{X}$. Once $\sqrt{X}$ and $\sqrt{Y}$ have been determined, $\sqrt{Z} = \sqrt{X} \oplus \sqrt{Y}$ can be determined by constructing the Lambert quadrangle with $\sqrt{X}$ and $\sqrt{Y}$ as adjacent sides and the acute angle opposite the angle formed by $\sqrt{X}$ and $\sqrt{Y}$. The diagonal to the acute angle is $\sqrt{Z}$. Then one constructs a right triangle with hypotenuse $\sqrt{Z}$ and one acute angle equal to $P(\sqrt{Z})$, and the side adjacent to $P(\sqrt{Z})$ is $Z=X\dotplus Y$.

SUMMARY OF AN ARTICLE BY HILBERT

Hilbert begins by stating four sets of axioms upon which he constructs his foundation for hyperbolic geometry. These four sets are axioms of connection, axioms of order, axioms of congruence, and axioms of intersecting and non-intersecting lines. It is noted that any point of a line will divide the line into two half-lines or halves; hence, to determine a half-line one simply picks a point and extends a ray from this point. Then if two different half-lines are extended from the same point, they determine an angle. Also, it is possible to define what is meant by the interior of an angle with the help of his axioms, and it agrees with one's intuitive concept of interior points. These interior points form the angle space of the angle.

The following axiom for hyperbolic geometry, as Hilbert states it, corresponds to the parallel axiom in Euclidean geometry.

Axiom. If b is an arbitrary line and A a point not lying on it, then there are always two half-lines a_1, a_2 through A, which do not make one and the same line and do not intersect the line b, whereas all the half-lines that lie on the angle space made by a_1, a_2 and extend from A intersect line b.

The two half-lines a_1 and a_2, and any two lines of which

a_1 and a_2 are respectively half-lines, are said to be parallel to line b. Any line not containing either a_1 or a_2 and having no point in common with the angle space formed by a_1 and a_2 is termed non-intersecting with respect to b. Any line lying within the angle space is termed intersecting with respect to b.

Hilbert also constructs an ordered field on which to base his analytic geometry. Let this field be denoted $\not{E}=(E, +, \cdot, <)$. Then the set E consists of what he calls ends. These ends are defined as follows: Each half-line determines an end. This simply means that, for example, each half-line extending from a point A determines an end, and if two half-lines are parallel, they determine the same end. In fact, all half-lines parallel to one another determine the same end. Some authors consider these ends as points at infinity, and it is helpful to think of them in this manner. Also, the fact that a line has two ends is obvious from the definition of an end. These ends will be denoted by $\underline{a}$ (or a when no confusion can arise), b, c, ... , and a half-line extending from A with the end $\underline{a}$ will be denoted by (A,a). A line whose ends are $\underline{a}$ and b will be denoted by (a,b).

The concept of mirror-images is needed for the definitions of addition and multiplication, and it is defined as would be expected. That is, the mirror image of a point in a line is that point which lies on the extended perpendicular from the point to the line and at the same distance from the line as the original point.

Hilbert then establishes a series of five theorems which he uses for the construction of his analytic geometry. The proofs of all of these theorems will not be given here.

Theorem 1H. If two lines intersect a third line under equal corresponding angles, then they are not parallel to one another.

Theorem 2H. If there exists two lines a_1, b_1 such that they are not parallel nor intersect one another, then there exists a line which is perpendicular to both of them.

Theorem 3H. If there are any two half-lines not parallel to one another, then there exists one line which is parallel to both half-lines; that is, there exists a line which possesses two assumed ends a and b.

Theorem 4H. Let a_1, b_1 be any two lines parallel to one another and O a point lying in the region of the plane between a_1 and b_1. Let Oa be the image of the point O in a_1 and Ob the image of the point O in b_1 and M the midpoint of OaOb; then there exists a half-line extending from M such that it is parallel to a_1 and b_1 and perpendicular to OaOb at M.

Theorem 5H. If a_1, b_1, c_1 are three lines which possess the same end w, then there exists a straight line d_1 with the same end w, so that the consecutive application of the reflections in the straight lines a_1, b_1, c_1 is equivalent to the reflection in the line d_1. This is expressed through the formula RcRbRa=Rd, where Ra denotes reflection of any figure

a_1 and similarly for Rb, Rc and Rd.

The following is an illustration of a specific case of this theorem. There are various possibilities as to the relative positions of the three lines to one another. The case in which b_1 lies in the interior of the region of the plane between a_1 and c_1 will be considered. If O is a point of b_1, let Oa and Oc be the reflections of O in a_1 and c_1 respectively. Then d_1 is that line joining the midpoint of OaOc with the end w.

The operation of addition is then defined as follows: First pick any line and denote its ends as 0 and ∞. Choose any point Q on this line, and erect the perpendicular to $(0,\infty)$ at Q. Denote the ends of this perpendicular as +1 and -1. Those ends which lie on the same side of $(0,\infty)$ as +1 are denoted as positive and those on the same side of $(0,\infty)$ as -1 are denoted as negative. Also, -$\underline{a}$ will be the reflection of $\underline{a}$ in $(0,\infty)$. Now let $\underline{a}$ and b be any two ends distinct from ∞, Qa be the reflection of Q in (a,∞), and Qb the reflection of Q in (b,∞). Then connect the midpoint of QaQb with ∞, and denote the other end of the resulting line as a+b. The end a+b is called the sum of the ends $\underline{a}$ and b. Then the operation + is closed and commutative from the definition; the identity element is 0 since 0+a=a+0=a for any $\underline{a}$ different from ∞. It would appear that ∞ could serve as the identity, also. However, it should be noted that addition is defined only for those ends different from ∞. Also, a+(-a)=0; hence,

-a is the additive inverse of a for any a different from ∞.

The associativity of addition is not so obvious. It can be proved as follows: Denote the reflections in the lines $(0,\infty)$, (a,∞), (b,∞) as Ro, Ra, Rb respectively. Then, by Theorem 5H, there exists a line (d,∞) such that Rd=RbRoRa. Consider the point Qa (the reflection of Q in (a,∞)) and the operation RbRoRa (reflection in (a,∞) followed by reflection in $(0,\infty)$ followed by reflection in (b,∞)). It is obvious that Qa will pass into point Qb through this operation. Hence, Qb is necessarily the reflection of Qa in (d,∞), and therefore d=a+b. Then R(a+b)=RbRoRa is valid. Now let g be any end different from ∞. Then, by use of the formula just stated, Ra+(b+g)=R(b+g)RoRa=RgRoRbRoRa, and R(a+b)+g=RgRoR(a+b)=RgRoRbRoRa; hence, Ra+(b+g)=R(a+b)+g and a+(b+g)=(a+b)+g. It is pointed out by Hilbert that it is not necessary to start with Qa to develop the formula R(a+b)=RbRoRa. Any point Q' of $(0,\infty)$ distinct from Q can be picked, and then Q'a considered the same way Qa was.

The multiplication of ends is defined in the following manner: Let a, b be any two ends different from 0 and ∞. Then both of the lines (a,-a) and (b,-b) are perpendicular to $(0,\infty)$. Call the intersection of (a,-a) and $(0,\infty)$ A, and the intersection of (b,-b) and $(0,\infty)$ B. Extend the segment QA from B to C; i.e., QA=BC and such that the direction from Q to A is the same as the direction from B to C. Construct the perpendicular at C to the line $(0,\infty)$, and denote the positive or negative end of this perpendicular as the product ab of the ends a, b according as these ends are either both positive and both negative respectively or one positive and one negative.

It is easily seen from the definition of multiplication that the operation is closed and commutative. The product (ab)c would be formed by first finding the product ab from the definition. For example, if the intersection of lines (c,-c) and (0,∞) is denoted by C, the product (ab)c would be found as follows: Extend the segment QA from B to D with the direction of QA taken into account. Then the end of the perpendicular to (0,∞) at D is ab. Extend the segment QD from C to F, once more with direction taken into consideration. The end of the perpendicular to (0,∞) at F is the product (ab)c. It is easily seen that the segment QF is essentially the sum QA+QB+QC with direction of all segments taken into account. If the product a(bc) is formed, a point F' is obtained, but it follows that the segment QF' is also the sum QA+QB+QC. This means a(bc)=(ab)c; hence, the operation of multiplication is associative. It is easily seen that $1\cdot a=a\cdot 1=a$; hence, 1 is the identity of multiplication. To find the multiplicative inverse of a given element $a\neq 0$, it is sufficient to determine a point A' such that A'Q=QA; i.e., Q is the midpoint of AA'. Then the end a' of the perpendicular to (0,∞) at A' is the multiplicative inverse of a. It might be noted that a' is unique since the distance QA and the perpendicular to a line at a point are unique. Also, a and a' must both be positive or both negative.

The only postulate for a field that has not been shown for $\not{E}$ is that multiplication distributes with respect to addition. In order to show this, let a, b and c be any three ends and construct the end b+c. Then the multiplication of any end by the end a is essentially a translation of the line $(0,\infty)$ by a distance equal to QA. That is, the products ab, ac, and a(b+c) all result in ends which are determined by a translation of the lines (c,-c), (b,-b) and ((b+c),-(b+c)) by a distance QA along $(0,\infty)$. Now the sum of two ends can be determined by a construction originating at A (or any point) as well as Q, and the sum of b and c is still b+c. Then the sum of the ends ab and ac will be the same as a(b+c). That is, ab+ac=a(b+c), and multiplication distributes with respect to addition. Hence, $\not{E}=(E, +, \cdot, <)$ is an ordered field. The ordering of the elements is accomplished by agreeing that $a > b$ if a+(-b) is positive.

There are several properties that should be considered before considering the equation of a point. First of all, if the line (a,∞) is reflected in the line (b,∞), the resulting line is $(2b-a,\infty)$. Let P be any point on the line which results from the reflection of (a,∞) in (b,∞), and consider the series of reflections of P: Rb, Ro, R(-a), Ro, Rb. It is obvious that these reflections will leave P unchanged. However, the formula for these reflections is

RbRoR(-a)RoRb=R(2b-a); i.e., that composite operation gives the same results as a reflection in the line $(2b-a,\infty)$. This means the point P necessarily lies on the line $(2b-a,\infty)$.

Secondly, some properties of multiplication of ends should be considered. It is fairly obvious that $a\cdot 0=0\cdot a=0$, and that $(-1)a=-a$. Also, if the line (a,b) goes through the point Q, then $ab=-1$; and, conversely. For every positive end p, there is always a positive (and negative) end, whose square will equal p. This positive end will be denoted $\sqrt{p}$.

Hilbert defines coordinates for lines in the plane in terms of the ends of the line. If e, n are the ends of any line, then $u=en$ and $v=\frac{1}{2}(e+n)$ are called the coordinates of the line. The equation of a point is given in terms of lines which pass through that point. If a, b, c are ends, such that the end $4ac-b^2$ is positive, then all of the lines whose coordinates u, v satisfy the equation $au+bv+c=0$ pass through a point. This is proved in the following manner. Construct the ends $x=2a/\sqrt{4ac-b^2}$, $y=b/\sqrt{4ac-b^2}$. Then by operating on the equation $au+bv+c=0$, one arrives at the equation $(xe+y)(xn+y)=-1$. This is done as follows: Multiplying $au+bv+c=0$ by $4a$ and then adding and subtracting b^2 one obtains $4a\ u+4abv+b=-(4ac-b^2)$. Then substituting for u and v in terms of e and n and dividing (or multiplying by the multiplicative inverse) by $4ac-b^2$,

$$(en)(4a^2/(4ac-b^2))+(e+n)(2ab/(4ac-b^2))+(b^2/(4ac-b^2))=-1.$$

Then substituting x, x^2, y, y^2 for their equivalent expressions and factoring, one obtains $(xe+y)(xn+y)=-1$.

Now consider the transformation of an arbitrary variable end w, which is determined through the formula $w'=xw+y$. Actually the two transformations $w'=xw$ and $w'=w+y$ will be considered. According to some previous remarks the multiplication of the arbitrary end w with a constant x is equal to a displacement of the plane along the line $(0,\infty)$ for a fixed distance dependent on x. From the definition of addition of ends, the addition of the end y to the end w can be seen to be equivalent to one fixed displacement of the plane dependent on y. It is a displacement which can be thought of as a rotation of the plane around the end ∞. In order to see this, it is noted that the line (w,∞) passes through reflection in the line $(0,\infty)$ into the line $(-w,\infty)$. By a previous remark again, the line $(-w,\infty)$ passes through reflection in $(y/2,\infty)$ into the line $(w+y,\infty)$. This means the addition of the end y to the end w becomes equivalent to the successively performed reflections in the lines $(0,\infty)$ and $(y/2,\infty)$. Hence, the transformations $e'=xe+y$ and $n'=xn+y$ of the ends e and n result in a certain displacement of the plane along the line (e,n) dependent only on y and x. Then if the line with ends e' and n' passes through a given point p, line (e,n) must pass through a point which is determined by point p. The equation $(xe+y)(xn+y)=-1$ becomes $e'n'=-1$ under the transformation just mentioned. The equation $e'n'=-1$ is the condition that the line (e',n') pass through the point Q. Hence, all lines with ends e and n (e and n are treated as variables) that satisfy the equation $(xn+y)(xe+y)=-1$ pass through a single point.

This completes the summary of the article by Hilbert. Perhaps the most serious objection to the development by Hilbert is the way in which he develops the equation of the point. It should be noted that the set of elements over which the field $\not{Z}$ is defined does not include the end ∞. In other words only lines whose ends are both different from ∞ have coordinates. This means that the linear equation of the point derived by Hilbert does not give a sufficient basis for the foundations of the analytic geometry (4, p. 152).

CONCLUSION

In both of these articles formulas are developed which are sufficient for establishing that a point lies on a line. This is perhaps because it is known that the condition of collinearity can serve as the only primitive notion in hyperbolic geometry (2, p. 87). That is, the whole of hyperbolic geometry can be built on the notion of collinearity. However, the reference to the work supporting this claim is an article published much earlier than the article by Hilbert. At any rate, whether Hilbert actually had proved that collinearity could serve as the only primitive notion or not, he does develop a formula for a line to pass through a point, and he leaves it at this. On the other hand, Szmielew states explicitly that this is the result she intends to arrive at. She also indicates that the relations of betweenness and equidistance could serve the purpose as well (4, p. 135).

There is a modification of Hilbert's method which is made by Paul Szász (3, pp 97-113). He uses essentially the same definitions of addition and multiplication; however, he introduces a distance function before defining multiplication. This distance function is then used in his definition of multiplication, and he also uses it in assigning coordinates to a point. This is a one-to-one function which assigns to every free segment X an end. Szmielew indicates that there actually is an isomorphism between the systems $\mathfrak{Z}$ and $\mathscr{Z}$, and that this can be shown with the help of the development by Szasz. In conclusion, then, it is seen that under the proper modifications (for example, the method of Szász), the systems $\mathfrak{Z}$ and $\mathscr{Z}$ are isomorphic.

ACKNOWLEDGMENT

The writer wishes to express his gratitude to his major advisor, Professor William L. Stamey, for the time and help that he gave in choosing a subject and on problems that arose while working on this report. He also wishes to thank the Ford Foundation and the Department of Mathematics for the financial assistance which helped make this past year of study possible.

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TWO ANALYTIC MODELS FOR HYPERBOLIC GEOMETRY

by

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AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

The purpose of this report is to summarize and explain the results of two articles concerned with analytic models for hyperbolic geometry. The first article treated is "A new analytic approach to hyperbolic geometry" by Wanda Szmielew, and the second article is Appendix III of "Grundlagen der Geometrie" by David Hilbert.

The model constructed by Szmielew is based on an algebraic system. Actually the system that she develops is a field generated by an algebraic system in which the set of elements consists of segments. The operations in the system are defined in terms of the Lambert quadrangle and the right triangle. Finally a rectangular coordinate system is constructed over the field. An analytic condition for a point to lie on a line and a formula for the distance between two points are developed. Other formulas are also developed.

Hilbert's approach is somewhat different than that of Szmielew. He first defines a set of elements and two operations on these elements. Although Hilbert does not use the language of modern algebra, the system that he develops is a field. He also formulates the conditions for a line to pass through a given point. There is one serious objection to the development by Hilbert. It is the fact that his operations are not defined for all elements in the set; hence, the condition that he derives for collinearity does not give a sufficient basis for the analytic geometry. The objection can also be stated by saying that the set does not include all the elements necessary to give a sufficient basis. This objection can be overcome by a modification of his method. This is done in an article by Paul Szász in "The Axiomatic Method."

Although the methods of Szmielew and Hilbert for developing the analytic models appear to be quite different, it is interesting to note that they both tend deliberately toward developing a formula for collinearity. This is to be expected since the notion of collinearity can serve as a complete basis for the analytic geometry. Finally it is noted that even though the two models appear to be quite different, if the modification of Szász is made on Hilbert's model then the two systems can be shown to be isomorphic.