GENERAL SOLUTIONS FOR THE MOMENTS IN CONTINUOUS BEAMS

by

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CONTENTS

LIST OF TABLES AND FIGURES

I. TABLES:

II. FIGURES:

GENERAL SOLUTIONS FOR THE MOMENTS IN CONTINUOUS BEAMS

SYNOPSIS

Formulas for final end moments of prismatic and non-prismatic continuous multi-span (up to four spans) beams were derived in this report, based on the moment distribution theory invented by Hardy Cross in 1932. The moment distribution method is a process of performing a series of cycles of "locking", "releasing", and "balancing" of the unbalanced moments at joints of an indeterminate structure whose moments are to be found, and the results are given at each completion of cycle-performing, only at different accuracy, for the higher the number of cycles performed the higher the accuracy of the results; therefore the accuracy of the results depend entirely upon how many cycles are performed by the designing engineer. In this report, each of the formulas derived represents the result of an infinite number of cycles of "locking", "releasing", and "distributing" of the unbalanced moment at the joint. Therefore the accuracy now is at its maximum; i.e., the actual results are obtained.

Since the moment distribution method is not self-checked, so in its long and tedious calculation process, it is difficult to keep from introducing some erroneous values into the process of calculation; and, once this error is induced, the results will be all wrong. Yet just like accuracy of the results, the higher number of cycles performed, the higher the possibility of errors being introduced. But as for these formulas represented in this report, each requires only the amount of labor for evaluation approximately equivalent to that required for one cycle of the "locking", "releasing", and "distributing" process; i.e., it brings the possibility of getting erroneous results to its minimal degree.

When the moment distribution method is applied to solve for

moments of certain structure, the properties of the structure should be known beforehand. The properties of the structure are several constants such as fixed end moment, stiffness factor and carry-over factor, etc. These are available in a fairly wide range in the pamphlet, "Handbook of Frame Constants", published by the Portland Cement Association. Therefore, with the aid of this pamphlet, a great variety of continuous beams (up to four-span) can be solved accurately and easily by employing these formulas. As for continuous beams with span number over four, although the formulas for moments were not derived in the report, they are readily to be derived based on formulas for four-span continuous beams available in this report and introducing imaginary joints in the manner presented in this report.

One of the most important things to a designing engineer of a continuous structure (usually bridges) is the influence line of the structure, for it shows the stress distribution (so is the requirement of distribution of material) throughout the whole structure, thus it renders the engineer economizing his structure of adequate safety possible to the degree desired. The formulas represented in this report are found of great use in constructing the influence line; as mentioned before, they require less labor for calculation and give exact results.

There are three numerical examples in this report illustrating the application of these formulas, both finding the final moments of the beams and drawing the influence line for the whole structure. The first example is a two-span continuous beam of variable I throughout the structure. Second and third are three and four-span continuous beams of variable I throughout the structure, and again the beams in these examples are haunched near the supports.

INTRODUCTION

Introduced by Prof. Hardy Cross's classic paper "Analysis of Continuous Frames by Distribution of Fixed-End Moments" in 1932, the moment distribution method is now a favorite tool of structural engineers in structural design. This method is used for finding moments in continuous beams and frames by succesive approximations. That is, it's practical a process of successively "locking", "releasing", and "balancing" the unbalanced moments at each joint of an indeterminate structure. Therefore, the degree of accuracy of final moments found by this method depends solely upon the number of cycles of operation of this approximation process. This operation may be discontinued at any number of cycles or continued to any degree of precision that the engineer may desire. When the process of successively operating the unbalanced moments at joints progresses to quite a few numbers of cycles, an interesting fact is discovered that the sum of the distribution moments at any joint is in the form of an infinite series:

$$a + ar + ar^2 + \ldots\ldots\ldots\ldots + ar^n = \frac{a}{1-r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

as this principle is introduced into the method of moment distribution, the general solutions of final moments in continuous beams, both prismatic and non-prismatic, are obtained. Each general solution derived will be able to give the final moments, which is equivalent performed. Since these general solutions are functions of the fixed-end moments, the carry-over factors, and the stiffness factors of the member, then whenever those factors are available, the final moments of the member can easily be found by means of these general solutions.

An influence line is a curve whose ordinate (or abscissa) at any point represents the value of some particular function due to a unit load acting at that point. The drawing of an influence line for a continuous structure subjected to moving live load is considered vitually indispensable to structural engineers for design and analysis of stress of the structure. As for drawing the influence lines, these general solutions are found valuable on account of labor and time saving.

Before the general solutions to be derived, some basic terms and values indispensable for the moment distribution method which the derivation is based on, have to be introduced and defined in the following chapter.

PHYSICAL PROPERTIES OF PRISMATIC AND NON-PRISMATIC BEAMS

Introduction

It can be said that the moment distribution method is a secondary method, as it can't be employed without the stiffness factors, carry-over factors, and the fixed-end moments of the structure being determined beforehand.

As for the determination of these constants, a method called "moment-area method" is usually employed. After these constants have been evaluated, the moment distribution method can then be applied to solve for the final moments.

Stiffness Factor, Carry-Over Factor, and Fixed-End Moments of Prismatic Members

The stiffness of a beam is the moment which can rotate an angle of one radian of the simply supported end of a beam whose other end is fixed (Fig. Ia). As the prismatic beam is of constant cross section stiffness it is

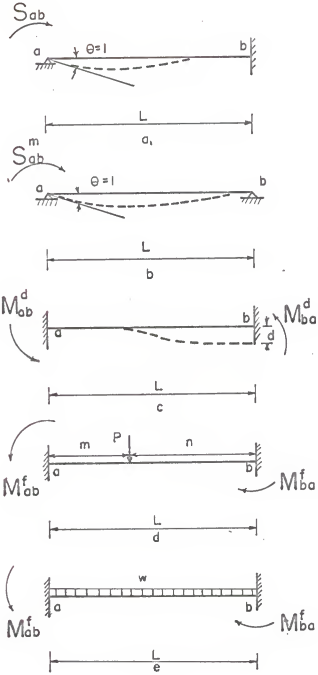
$$S = \frac{4EI}{L} \qquad \text{(2)}$$

and its carry-over factor, or the ratio of the moments induced at the fixed end to the moment producing rotation at the simply supported end, is

$$C = \tfrac{1}{2} \qquad \text{(3)}$$

The moment that can rotate one end an angle of one radian of a simply supported beam is called modified stiffness (Fig. Ib)

$$S^m = \frac{3EI}{L} \qquad \text{(4)}$$



$$S_{ab} = \frac{4EI}{L}$$

$$S^{m}_{ab} = \frac{3EI}{L}$$

$$M^{d}_{ab} = M^{d}_{ba} = -\frac{6EId}{L^{2}}$$

$$M^{f}_{ab} = \frac{P_{mn^2}}{L^{2}}$$

$$M^{f}_{ab} = \frac{P_{m^2n}}{L^{2}}$$

$$M^{f}_{ab} = M^{f}_{ba} = \frac{WL^{2}}{12}$$

FIG. 1 PROPERTIES OF PRISMATIC BEAM

When one end of a beam moves a distance d normal to the original axis, the moment that can hold the end tangent in its original direction (Fig. Ic) is

$$M = \frac{6EId}{L^2} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (5)$$

If both ends of a beam are completely fixed against rotation and translation, moments will be induced at ends when the beam is under-loading. These moments are called fixed-end moments, and can be evaluated by the moments-area method. The fixed end moments induced both by concentrated and uniform loads for beams of constant cross section and straight axis are shown in Fig. Id and Ie.

Stiffness Factors, Carry-Over Factors, and Fixed-End Moments of Non-prismatic Members

The moments diagram of the continuous prismatic beams under uniform loading shows that the negative moments at the supports are greater than the positive moments near the center of the spans. Also the shears are greater near the supports than elsewhere in the span. For this reason many continuous beams and girder bridges are haunched, i.e., shallower near mid-span and deeper toward the supports.

When analyzing by method of moment distribution, the haunched, continuous (or non-prismatic) beams can be treated in the same manner as a prismatic beam, only the fixed-end moments, stiffness factors and carry-over factors are of different values. The modified stiffness for a simply supported non-prismatic beam is

$$S^m = S(1-C_{ab}C_{ba}) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (6)$$

The fixed-end moments, stiffness factors and carry-over factors can be evaluated by the moment-area method, but it is usually a tedious and time-consuming task. Thanks are due to the structural engineers of the Portland Cement Association for their kind contribution of the "Handbook of Frame Constants" which offers a wide range of those constants. With the aid of this publication, a great variety of problems of non-prismatic members can be easily solved by the moment distribution method.

THE GENERAL SOLUTIONS FOR THE FINAL MOMENTS OF CONTINUOUS BEAMS

Two-Span Continuous Beams

The general solutions for final moments at supports of a two-span continuous beam subjected to any arbitrary loading are easily obtained as shown in Fig. 2. When the end of the continuous beam is hinged, its fixed-end moment, just like the stiffness of the beam to be modified, should be cancelled and distributed before the general solutions are applied.

Example 1. Determine the bending moments at supports of the continuous beam with parabolic haunches loaded as shown in Fig. 3.

Solution: from the "Handbook of Frame Constants", the following values are obtained:

$C_{ab} = C_{ba} = 0.694 \qquad C_{bc} = 0.334 \qquad C_{cb} = 0.910$

$K_{ab} = K_{ba} = 12.03\frac{EI}{L} \qquad K_{bc} = 14.62\frac{EI}{L} \qquad K_{cb} = 5.36\frac{EI}{L}$

$M^f_{ab} = M^f_{ba} = 0.1025WL^2 \qquad M^f_{bc} = 0.2138PL \qquad M^f_{cb} = 0.0742PL$

$$S^M_{bc} = \frac{14.62}{20}(1 - 0.334 \times 0.910) = 0.507EI$$

$D_{ba} = 0.442 \qquad D_{bc} = 0.558$

$C_{ab} = C_{ba} = 0.694 \qquad C_{bc} = 0.334 \qquad C_{cb} = 0.910$

$$M^f_{ab} = -M^f_{ba} = 0.1025WL^2 = 0.1025 \times 1 \times 900 = 92.25^{k-1}$$

$$M^f_{bc} = 0.2138 \times 20 \times 20 - C_{cb}M^f_{cb} = 85.52 - 0.910 \times (-29.68) = 112.54$$

$$M^f_{cb} = -0.0742 \times 20 \times 20 = -29.68^{k-1}$$

$$M^f_b = M^f_{ba} + M^f_{bc} = -92.25 + 112.54 = 20.29$$

After the above values are substituted into the general solutions for two-span continuous beam, the final moments at supports are:

$$M_{ab} = M^f_{ab} - M^f_b D_{ba} C_{ba} = 92.25 - (20.29)(0.442)(0.694) = 86.04$$

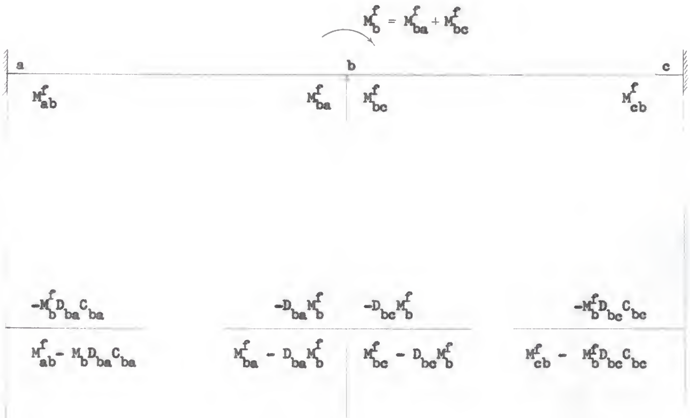


FIG. 2 MOMENT DISTRIBUTION FOR TWO-SPAN CONTINUOUS BEAM

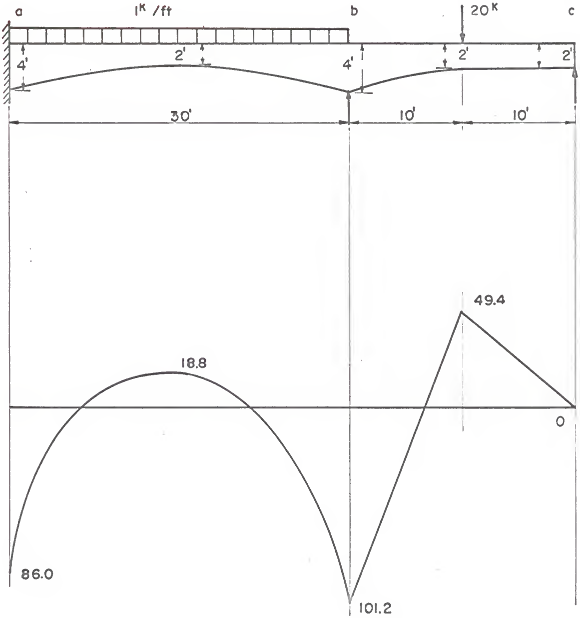


FIG. 3 MOMENT DIAGRAM OF THE TWO-SPAN CONTINUOUS NON-PRISMATIC BEAM EXAMPLE 1

$$M_{ba} = M^f_{ba} - D_{ba}M^f_b \quad = -92.25 - (0.442)(20.29) \quad = -101.2$$

$$M_{bc} = M^f_{bc} - D_{bc}M^f_b \quad = 112.54 - (0.558)(20.29) \quad = 101.2$$

$$M_{cb} = 0$$

Three-Span Continuous Beams

Derivation of the general solution for a three-span continuous beam is complicated. The process of derivation in detail is presented in Fig. 4 and 5. In this process, it is seen that all the distributed moments are in the form of a simple infinite series:

$$M^b_{bc} = M^f_{bc} - M^f_b D_{bc}(1 + F_{bc} + F^2_{bc} + \ldots\ldots\ldots\ldots + F^n_{bc} \ldots\ldots\ldots)$$
$$+ M^f_b F_{bc}(1 + F_{bc} + F^2_{bc} + \ldots\ldots\ldots\ldots\ldots\ldots + F^n_{bc} + \ldots\ldots\ldots)$$

where M^b_{bc} is the final moment at end "b" of span bc due to the unbalanced moment M^f_b at support "b". Since F is less than unity, when n approaches to infinite, F^n approaches to zero. This means the above series is convergent. Therefore

$$M^b_{bc} = M^f_{bc} - M^f_b D_{bc}\frac{1}{1-F_{bc}} + M^f_b F_{bc}\frac{1}{1-F_{bc}} = M^f_b - M^f_b \frac{D_{bc}-F_{bc}}{1-F_{bc}} \quad \ldots\ldots (6)$$

In the same manner, other general solutions are obtained and presented in Fig. 4.

As for M^f_c, the unbalanced moment at support "c" is distributed in the same way as for M^f_b. The general solutions for final moments due to M^f_c are presented in Fig. 5.

Example 2. For the non-prismatic continuous beam shown in Fig. 6.

Find 1. The influence line for M_{bc}

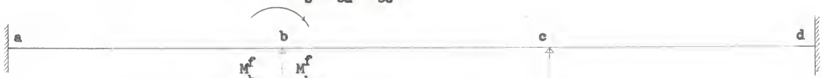
2. The influence line for M_e

3. The influence line for R_b

Solution: from the "Handbook of Frame Constants"

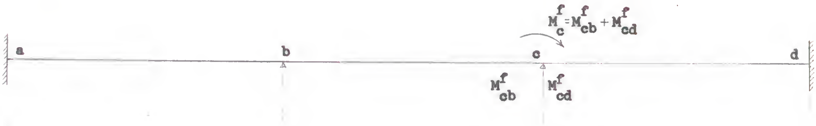
$C_{ab} = 0.829$	$C_{dc} = 0.766$	$C_{bc} = 0.628$
$C_{ba} = 0.648$	$C_{cd} = 0.611$	$C_{cb} = 0.678$

$$M^f_b = M^f_{ba} + M^f_{bc}$$



	M^f_{ba}	M^f_{bc}			
$-M^f_b E_{ba}$ $\longleftarrow$	$-M^f_b D_{ba}$	$-M^f_b D_{bc}$ $\longrightarrow$	$-M^f_b D_{bc} C_{bc} = -M^f_b E_{bc}$		
		$M^f_b F_{bc} = M^f_b E_{bc} D_{cb} C_{cb}$ $\longrightarrow$	$M^f_b D_{cb} E_{bc}$	$M^f_b D_{cd} E_{bc}$ $\longrightarrow$	$M^f_b E_{bc} E_{cd}$
$-M^f_b F_{bc} E_{ba}$ $\longleftarrow$	$-M^f_b F_{bc} D_{ba}$	$-M^f_b F_{bc} D_{bc}$ $\longrightarrow$	$-M^f_b F_{bc} E_{bc}$		
		$M^f_b F^2_{bc}$ $\longrightarrow$	$M^f_b F_{bc} E_{bc} D_{cb}$	$M^f_b F_{bc} E_{bc} D_{cd}$ $\longrightarrow$	$M^f_b F_{bc} E_{bc} E_{cd}$
$- - - -$	$- - - -$	$- - - -$	$- - - -$	$- - - -$	$- - - -$
$M^b_{ab} = -M^f_b \dfrac{E_{ba}}{1-F_{bc}}$	$M^b_{ba} = M^f_{ba} -M^f_b \dfrac{D_{ba}}{1-F_{bc}}$	$M^b_{bc} = M^f_{bc} -M^f_b \dfrac{D_{bc}-F_{bc}}{1-F_{bc}}$	$M^b_{cb} = -M^f_b \dfrac{E_{bc}(1-D_{cb})}{1-F_{bc}}$	$M^b_{cd} = M^f_b \dfrac{E_{bc} D_{cd}}{1-F_{bc}}$	$M^b_{dc} = M^f_b \dfrac{E_{bc} E_{cd}}{1-F_{bc}}$

FIG.4 MOMENT DISTRIBUTION FOR THREE-SPAN CONTINUOUS BEAM (ASSUMING $M^f_c = 0$)



$M^c_{ab} = M^f_c \frac{E_{cb} E_{ba}}{1-F_{bc}}$

$M^c_{ba} = M^f_c \frac{E_{cb} D_{ba}}{1-F_{bc}}$

$M^c_{bc} = -M^f_c \frac{E_{cb}(1-D_{bc})}{1-F_{bc}}$

$M^c_{cb} = M^f_{cb} - M^f_c \frac{D_{cb} - F_{bc}}{1-F_{bc}}$

$M^c_{cd} = M^f_{cd} - M^f_c \frac{D_{cd}}{1-F_{bc}}$

$M^c_{dc} = -M^f_c \frac{E_{cd}}{1-F_{bc}}$

FIG. 5 MOMENT DISTRIBUTION FOR THREE-SPAN CONTINUOUS BEAM (ASSUMING $M^f_b = 0$)

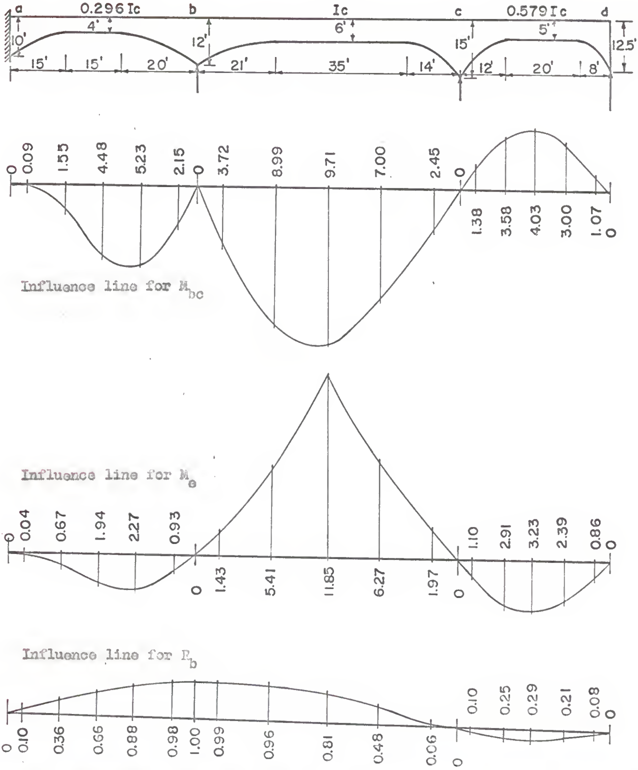


FIG. 6 INFLUENCE LINE FOR M_{bc}, M_e AND R_b OF THE NON-PRISMATIC CONTINUOUS BEAM (FOR EXAMPLE 2)

$S_{ab} = 11.07\frac{EI}{L}$ $S_{bc} = 7.92\frac{EI}{L}$ $S_{cd} = 9.96\frac{EI}{L}$

$S_{ba} = 14.15\frac{EI}{L}$ $S_{cb} = 7.33\frac{EI}{L}$ $S_{dc} = 7.94\frac{EI}{L}$

Since support "d" is hinged, therefore $S^m_{cd} = S_{cd}(1- C_{dc}C_{cd}) = 0.531S_{cd}$

Substituting I and L into stiffness S, the following values are obtained:

$S_{ab} = 0.065\ I_c$ $S_{bc} = 0.112\ I_c$ $S_{cd} = 0.077\ I_c$

$S_{ba} = 0.084\ I_c$ $S_{cb} = 0.1037\ I_c$

$D_{ba} = 0.429$ $D_{bc} = 0.571$ $D_{cb} = 0.931$ $D_{cd} = 0.069$

$E_{bc} = D_{bc}C_{bc} = 0.359$ $E_{cb} = D_{cb}C_{cb} = 0.631$

$F_{bc} = E_{bc}E_{cb} = 0.2265$ $1-F_{bc} = 0.7735$

$D_{bc} - F_{bc} = 0.345$ $D_{cb} - F_{bc} = 0.705$

The ordinates of influence line for M_{bc}, M_{cb}, M_e and R_b, obtained by substitution of the fixed-end moments from "Handbook of Frames Constants" into the general solutions presented in Fig. 4 and 5, are shown in Table I, II, III and IV.

The influence lines are drawn and presented in Fig. 6.

Four-Span Continuous Beams

The derivation of general solutions for the final moments of the four-span continuous beam will be very complicated without some assumption made beforehand. Here an imaginary rigid joint at "d" is introduced, which makes rotation of the joint impossible (Fig. 7). Then the four-span continuous beam becomes a three-span continuous beam a-b-c-d and a fixed end single-span beam d-e.

Three-span continuous beam a-b-c-d can be treated with the general solutions for three-span continuous beam already derived (Fig. 4 and 5). Firstly M^f_b is applied and distributed throughout the beam a-b-c-d. Then the imaginary rigid joint at support "d" is removed and introduced to support "b". Now the unbalanced moment

$$M_b^f \frac{E_{bc}E_{cd}}{1-F_{bc}} = M_b^f G_{dc}$$

store at support "d", due to distribution of M_b^f, is distributed in this new three-span continuous beam b-c-d-e. Then the imaginary rigid joint is transferred from "b" back to "d" again, and the unbalanced moment

$$M_b^f G_{dc} \frac{E_{dc}E_{cb}}{1-F_{cd}} = M_b^f G_{dc} H_{bc}$$

stored at "b", due to the distribution of $M_b^f G_{dc}$, is again distributed ...... etc. In this manner the supports "d" and "b" are introduced the imaginary rigid joint alternately, then the unbalanced moments at "b" and "d" are distributed accordingly. This process is shown in Fig. 7, 8 and 9. It is seen that the sum of moments distributed is also in the form of an infinite series

$$\begin{aligned} M_b^b &= M_{bc}^f + M_{bc1} + M_{bc2} + M_{bc3} + \dots\dots\dots\dots M_{bcn} + \dots\dots\dots \\ &= M_{bc}^f - M_b^f G_{bc} \left(1 + G_{dc}H_{bc} + G_{dc}^2 H_{bc}^2 + \dots\dots\dots G_{dc}^n H_{bc}^n + \dots\dots\right) \\ &\quad + M_b^f G_{dc} H_{bc} \left(1 + G_{dc}H_{bc} + G_{dc}^2 H_{bc}^2 + \dots\dots\dots\dots G_{dc}^n H_{bc}^n + \dots\right) \end{aligned}$$

When n approaches infinite, M_{bc}^b will be

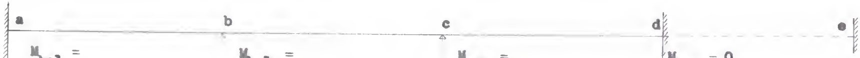
$$M_{bc}^b = M_{bc}^f - M_b^f \frac{G_{bc} - G_{dc} H_{bc}}{1 - G_{dc}H_{bc}}$$

after substitution of the values of G_{dc}, G_{bc} and H_{bc} made in the above expression, simplified, it becomes

$$M_{bc}^b = M_{bc}^f - M_b^f \frac{D_{bc}(1 - F_{cd}) - F_{bc}}{1 - F_{bc} - F_{cd}} \quad \dots\dots\dots\dots\dots\dots\dots (7)$$

In the same manner the general solutions for other moments due to M_b^f are obtained and presented in Fig. 10.

Also similarly, the general solutions for final moments due to the unbalanced moment M_d^f at support "d", are obtained and presented in Fig. 11. As for M_c^f, the unbalanced moment support "c", the general solutions for final moments are derived with the four-span continuous beam restrained as shown in Fig. 12. Make use of the general solution



$$M_{ba1} = -M_b^f \frac{D_{ba}}{1-F_{bc}} = -M_b^f G_{ba}$$

$$M_{bc1} = -M_b^f \frac{D_{bc}-F_{bc}}{1-F_{bc}} = -M_b^f G_{bc}$$

$$M_{cd1} = M_d^f \frac{F_{bc} D_{cd}}{1-F_{bc}} = M_b^f G_{cd}$$

$$M_{de1} = 0$$

$$M_{cb1} = -M_b^f \frac{F_{bc}(1-D_{bc})}{1-F_{bc}} = -M_b^f G_{cb}$$

$$M_{dc1} = M_b^f \frac{E_{bc} E_{cd}}{1-F_{bc}} = M_b^f G_{dc}$$

$$M_{ba2} = 0$$

$$M_{bc2} = M_b^f G_{dc} \frac{E_{dc} E_{cb}}{1-F_{cd}} = M_b^f G_{dc} H_{bc}$$

$$M_{cb2} = M_b^f G_{dc} \frac{E_{dc} D_{cb}}{1-F_{cd}} = M_b^f G_{dc} H_{cb}$$

$$M_{cd2} = -M_b^f G_{dc} \frac{E_{dc}(1-D_{cd})}{1-F_{cd}} = -M_b^f G_{dc} H_{dc}$$

$$M_{dc2} = -M_b^f G_{dc} \frac{D_{dc}-F_{cd}}{1-F_{cd}} = -M_b^f G_{dc} H_{dc}$$

$$M_{de2} = -M_b^f G_{dc} \frac{D_{de}}{1-F_{cd}} = -M_b^f G_{dc} H_{de}$$

FIG. 7 MOMENT DISTRIBUTION OF FOUR-SPAN CONTINUOUS BEAM WITH IMAGINARY RIGID JOINT INTRODUCED AT JOINTS "d" AND "b" ALTERNATELY

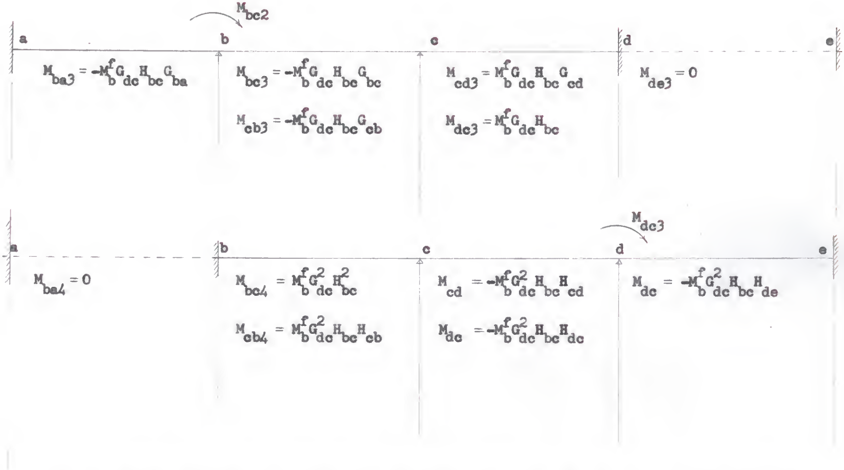


FIG. 8 MOMENT DISTRIBUTION OF FOUR-SPAN CONTINUOUS BEAM WITH IMAGINARY RIGID JOINT INTRODUCED AT JOINTS "d" AND "b" ALTERNATELY

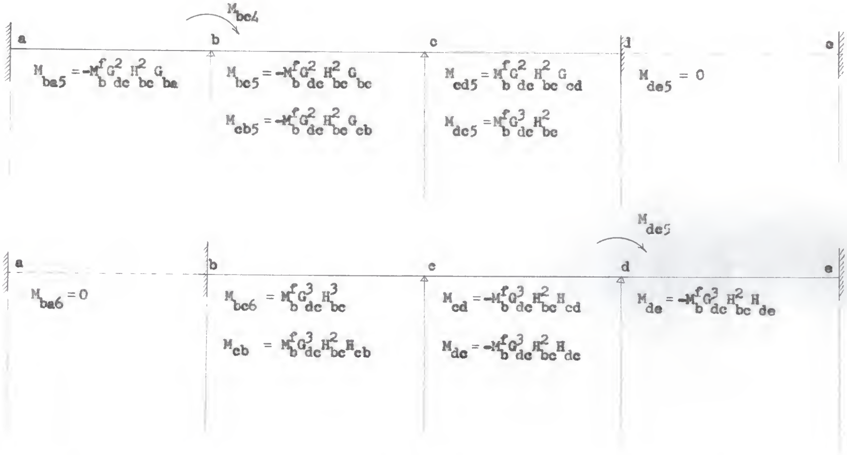
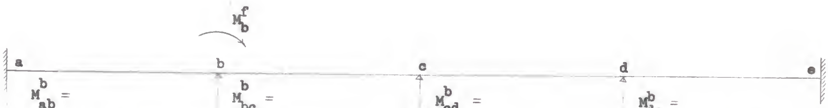


FIG. 9 MOMENT DISTRIBUTION OF FOUR-SPAN CONTINUOUS BEAM WITH IMAGINARY RIGID JOINT INTRODUCED AT JOINTS "d" AND "b" ALTERNATELY



$$M_{ab}^{b} = -M_b^f \frac{E_{ba}(1-F_{cd})}{1-F_{bc}-F_{cd}}$$

$$M_{bc}^{b} = M_{bc}^{f} - M_b^f \frac{D_{bc}(1-F_{cd})-F_{bc}}{1-F_{bc}-F_{cd}}$$

$$M_{cd}^{b} = M_b^f \frac{E_{bc}(D_{cd}-F_{cd})}{1-F_{bc}-F_{cd}}$$

$$M_{de}^{b} = -M_b^f \frac{E_{cd}E_{bc}D_{de}}{1-F_{bc}-F_{cd}}$$

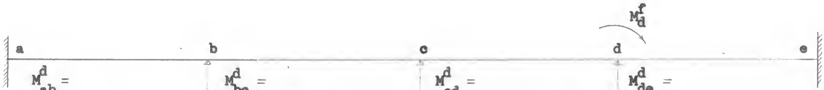
$$M_{ba}^{b} = M_{ba}^{f} - M_b^f \frac{D_{ba}(1-F_{cd})}{1-F_{bc}-F_{cd}}$$

$$M_{cb}^{b} = -M_b^f \frac{E_{bc}(1-D_{cb}-F_{cd})}{1-F_{bc}-F_{cd}}$$

$$M_{dc}^{b} = M_b^f \frac{E_{cd}E_{bc}(1-D_{dc})}{1-F_{bc}-F_{cd}}$$

$$M_{ed}^{b} = -M_b^f \frac{E_{cd}E_{bc}E_{de}}{1-F_{bc}-F_{cd}}$$

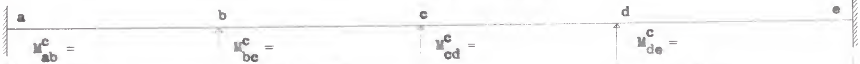
FIG. 10 MOMENT DISTRIBUTION FOR GENERAL SOLUTIONS FOR FOUR-SPAN CONTINUOUS BEAM



$M_{ab}^d = -M_d^f \dfrac{E_{cd}E_{bc}E_{ba}}{1-F_{bc}-F_{cd}}$	$M_{bc}^d = M_d^f \dfrac{E_{cd}E_{bc}(1-D_{bc})}{1-F_{bc}-F_{cd}}$	$M_{cd}^d = -M_d^f \dfrac{E_{dc}(1-D_{cd}-F_{cd})}{1-F_{bc}-F_{cd}}$	$M_{de}^d = M_{de}^f - M_d^f \dfrac{D_{de}(1-F_{bc})}{1-F_{bc}-F_{cd}}$
$M_{ba}^d = -M_d^f \dfrac{E_{cd}E_{bc}D_{ba}}{1-F_{bc}-F_{cd}}$	$M_{cb}^d = M_d^f \dfrac{E_{dc}(D_{cb}-F_{bc})}{1-F_{bc}-F_{cd}}$	$M_{dc}^d = M_{dc}^f - M_d^f \dfrac{D_{dc}(1-F_{bc})-F_{cd}}{1-F_{bc}-F_{cd}}$	$M_{ed}^d = -M_d^f \dfrac{E_{de}(1-F_{bc})}{1-F_{bc}-F_{cd}}$

FIG. 11 MOMENT DISTRIBUTION FOR GENERAL SOLUTION FOR FOUR-SPAN CONTINUOUS BEAM

$$M^f_c = M^f_{cb} + M^f_{cd}$$



$$M^c_{ab} = M^f_c \frac{E_{cb}E_{ba}}{1-F_{bc}-F_{cd}}$$

$$M^c_{bc} = -M^f_c \frac{E_{cb}(1-D_{bc})}{1-F_{bc}-F_{cd}}$$

$$M^c_{cd} = M^f_{cd} - M^f_c \frac{D_{cd}-F_{cd}}{1-F_{bc}-F_{cd}}$$

$$M^c_{de} = M^f_c \frac{E_{cd}D_{de}}{1-F_{bc}-F_{cd}}$$

$$M^c_{ba} = M^f_c \frac{E_{cb}D_{ba}}{1-F_{bc}-F_{cd}}$$

$$M^c_{cb} = M^f_{cb} - M^f_c \frac{D_{cb}-F_{bc}}{1-F_{bc}-F_{cd}}$$

$$M^c_{dc} = -M^f_c \frac{E_{cd}(1-D_{dc})}{1-F_{bc}-F_{cd}}$$

$$M^c_{ed} = M^f_c \frac{E_{cd}E_{de}}{1-F_{bc}-F_{cd}}$$

FIG. 12 MOMENT DISTRIBUTION FOR GENERAL SOLUTIONS FOR FOUR-SPAN CONTINUOUS BEAM for three-span continuous beam, the moment stored at the imaginary rigid joint "b" is:

$$M^c_{bc} = -M^f_c \frac{E_{cb}}{1-F_{cd}}$$

and this moment is then distributed by means of the general solutions for four-span continuous beam for M^f_b, as presented in Fig. 12. The result, after simplification, is

$$M^c_{bc} = -M^f_c \frac{E_{cb}(1-D_{bc})}{1-F_{bc}-F_{cd}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

similarly, the moment at any support now is the sum of moment in the restrained three-span continuous beam due to M^f_c plus the moment in four-span continuous beam due to the moment stored at the imaginary rigid joint "b", such as

$$M^c_{cb} = (M^f_{cb} - M^f_c \frac{D_{cb}}{1-F_{cd}}) + M^f_c \frac{E_{cb}}{1-F_{cd}} \frac{E_{bc}(1-D_{cb}-F_{cd})}{1-F_{bc}-F_{cd}}$$

$$= M^f_{cb} - M^f_c \frac{D_{cb} - F_{bc}}{1-F_{bc}-F_{cd}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

In the same manner, other general solutions are obtained and presented in Fig. 12.

Example 3. Draw an influence line for M_b of the beam shown in Fig. 13. Solution: from the "Handbook of Frame Constants", the following values are obtained:

$C_{ab} = 0.829$	$C_{ba} = 0.648$	$C_{bc} = 0.628$	$C_{cb} = 0.678$
$C_{cd} = 0.611$	$C_{dc} = 0.766$	$C_{de} = 0.615$	$C_{ed} = 0.917$
$K_{ab} = 11.07\frac{EI}{L}$	$K_{ba} = 14.15\frac{EI}{L}$	$K_{bc} = 7.92\frac{EI}{L}$	$K_{cb} = 7.33\frac{EI}{L}$
$K_{cd} = 9.96\frac{EI}{L}$	$K_{dc} = 7.94\frac{EI}{L}$	$K_{de} = 17.82\frac{EI}{L}$	$K_{ed} = 11.96\frac{EI}{L}$
$S_{ab} = 0.065$	$S_{ba} = 0.084$	$S_{bc} = 0.112$	$S_{cb} = 0.104$
$S_{cd} = 0.144$	$S_{dc} = 0.115$	$S_{de} = 0.172$	$S_{ed} = 0.115$

TABLE I - INGLUENCE LINE ORDINATES FOR M_{bc} (FOR EXAMPLE 2)

Span	Item	0.1L	0.3L	0.5L	0.7L	0.9L
ab	$M^f_b = -M^f_{ba}$	0.21	3.48	10.05	11.73	4.83
	M^*_{bc}	0.09	1.55	4.48	5.32	2.15
bc	M^f_{bc}	6.54	13.83	11.05	4.16	0.20
	$-M^f_{cb}$	0.28	3.79	10.27	13.44	6.69
	$0.554M^f_{bc}$	3.62	7.66	6.12	2.30	0.11
	$-0.350M^f_{cb}$	0.10	1.33	3.59	4.70	2.34
	M^{**}_{bc}	3.72	8.99	9.71	7.00	2.45
cd	M^f_{dc}	0.07	1.50	5.20	7.40	3.81
	$M^f_{dc}C_{dc}$	0.05	1.15	3.98	5.67	2.92
	M^f_{cd}	3.88	9.09	7.35	2.87	0.14
	$-M^{***}_{bc}$	1.38	3.58	4.03	3.00	1.07

* $M_{bc} = 0.446\ M^f_b$

** $M_{bc} = 0.554\ M^f_{b\cancel{c}} - 0.350\ M^f_c$

*** $M_{bc} = 0.350\ M^f_c$

TABLE II – INFLUENCE LINE ORDINATES FOR M_{cb} (FOR EXAMPLE 2)

Span	Item	0.1L	0.3L	0.5L	0.7L	0.9L
ab	$M_b^f = -M_{ba}^f$	0.21	3.48	10.05	11.73	4.83
	M_{cb}^{*}	0.01	0.21	0.60	0.70	0.29
bc	M_{bc}^f	6.54	13.83	11.05	4.16	0.20
	M_{cb}^f	0.28	3.79	10.27	13.44	6.69
	$0.089M_c^f$	0.03	0.34	0.91	1.20	0.60
	$0.06M_b^f$	0.39	0.83	0.66	0.25	0.00
	M_{cb}^{**}	0.42	1.17	1.57	1.45	0.60
cd	M_{dc}^f	0.07	1.50	5.20	7.40	3.81
	$M_{dc}^f C_{dc}$	0.05	1.15	3.98	5.67	2.92
	M_{cd}^f	3.88	9.09	7.53	2.87	0.14
	$-M_c^f$	3.93	10.24	11.51	8.54	3.06
	M_{cb}^{***}	3.58	9.33	10.49	7.78	2.79

* $M_{cb} = -0.06M_b^f$

** $M_{cb} = -0.089M_c^f - 0.06M_b^f$

*** $M_{cb} = -0.911M_c^f$

TABLE III - INFLUENCE LINE ORDINATES FOR M_e (FOR EXAMPLE 2)

Span	Item	0.1L	0.3L	0.5L	0.7L	0.9L
	$0.5M_{bc}$	0.05	0.78	2.24	2.62	1.08
ab	$0.5M^{*}_{cb}$	0.01	0.11	0.30	0.35	0.15
	M_e	-0.04	-0.67	-1.94	-2.27	-0.93
	M^{**}_{oe}	3.5	10.5	17.5	10.5	3.5
	$0.5M_{bc}$	-1.86	-4.50	-4.86	-3.50	-1.23
bc	$0.5M_{cb}$	-0.21	-0.59	-0.79	-0.73	-0.30
	$-M_e$	1.43	5.41	11.85	6.27	1.97
	$0.5M_{bc}$	0.69	1.76	2.02	1.50	0.54
cd'	$0.5M^{*}_{cb}$	1.79	4.67	5.25	3.89	1.40
	$-M_e$	1.10	2.91	3.23	2.39	0.86

* M_{cb} is obtained from Table II.

** M_{oe} is the moment of simply supported beam bc, M^f_b and M^f_c are assumed to be redundants. So only when load moves in span bc M_{oe} is not zero.

TABLE IV – INFLUENCE LINE ORDINATES FOR R_b (FOR EXAMPLE 2)

Span	Item	0.1L	0.3L	0.5L	0.7L	0.9L
	R^*_{cb}	0.10	0.30	0.50	0.70	0.90
	$M_{ba}/50$	0.00	0.03	0.09	0.11	0.04
ab	$(M_{bc} + M^{**}_{cb})/70$	0.00	0.03	0.07	0.08	0.04
	R_b	0.10	0.36	0.66	0.88	0.98
	R_{cb}	0.90	0.70	0.50	0.30	0.10
	$M_{ba}/50$	0.07	0.16	0.19	0.10	0.04
bc	$(M_{bc} - M_{cb})/70$	0.02	0.10	0.10	0.08	0.01
	R_b	0.99	0.96	0.81	0.48	0.06
	$-M_{ba}/50$	0.03	0.07	0.08	0.06	0.02
cd	$-(M_{bc} + M_{cb})/70$	0.07	0.18	0.21	0.15	0.06
	$-R_b$	0.10	0.25	0.29	0.21	0.08

* R_{cb} is obtained by assuming that M^f_a, M^f_b and M^f_c are redundants.

** M_{cb} is obtained from Table II.

TABLE V - INFLUENCE LINE ORDINATES FOR M_{ba} (FOR EXAMPLE 3)

Span	Item	0.1L	0.3L	0.5L	0.7L	0.9L
ab	$-M^{f}_{ba}$	0.21	3.48	10.05	11.73	4.83
	$-M_{ba}$	0.11	1.79	5.18	6.05	2.49
	M^{f}_{bc}	6.53	13.82	11.05	4.15	0.20
	$-M^{f}_{cb}$	0.28	3.79	10.25	13.43	6.69
bc	$0.515M^{f}_{bc}$	3.36	7.12	5.69	2.14	0.10
	$-0.154M^{f}_{cb}$	0.04	0.58	1.58	2.07	1.03
	$-M_{ba}$	3.40	7.70	7.27	4.21	1.13
	M^{f}_{cd}	3.88	9.09	7.53	2.88	0.14
	$-M^{f}_{dc}$	0.07	1.50	5.20	7.40	3.82
cd	$0.154M^{f}_{cd}$	0.60	1.40	1.16	0.44	0.02
	$-0.069M^{f}_{dc}$	0.00	0.10	0.36	0.51	0.26
	M_{ba}	0.60	1.50	1.52	0.95	0.28
dc	M^{f}_{da} e	5.62	13.60	12.66	4.10	0.16
	$-M_{ba}$	0.39	0.95	0.87	0.28	0.01

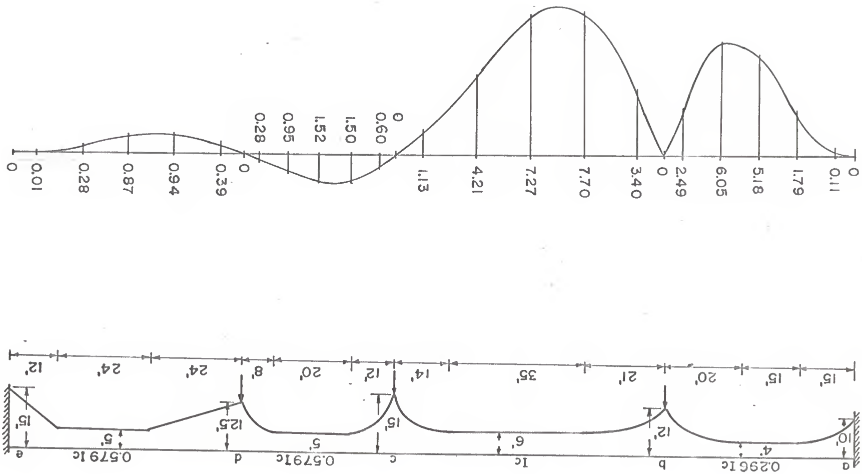


FIG. 13 INFLUENCE LINE M_{ba} OF A NON-PRISMATIC FOUR-SPAN CONTINUOUS BEAM (FOR EXAMPLE 3)

$D_{ba} = 0.429$ $D_{bc} = 0.571$ $D_{cb} = 0.419$ $D_{cd} = 0.581$

$D_{dc} = 0.401$ $D_{de} = 0.599$

$E_{bc} = D_{bc}C_{bc} = 0.278$ $E_{cb} = 0.358$ $E_{dc} = 0.284$

$E_{cd} = 0.355$ $E_{de} = 0.307$ $E_{ed} = 0.369$

$F_{bc} = E_{bc}E_{cb} = 0.102$ $F_{cd} = 0.109$ $1 - F_{bc} - F_{cd} = 0.789$

Substitution of the above values into corresponding general solutions presented in Fig. 10, 11 and 12 gives

$$M_{ba} = M^f_{ba} - 0.515M^f_b - 0.069M^f_d + 0.154M^f_c$$

when load is in span ab,

$$M_{ba} = -0.515M^f_b$$

when load is in span "bc"

$$M_{ba} = -0.515M^f_{bc} + 0.154M^f_{cb}$$

when load is in span "cd"

$$M_{ba} = 0.154M^f_{cd} - 0.069M^f_{dc}$$

when load is in span "de"

$$M_{ba} = -0.069M^f_d$$

The ordinates of the influence line for M_{ba} are obtained and shown in Table V., and the influence line is drawn and shown in Fig. 13.

CONCLUSION

These general solutions presented in this report are derived from the principle of the moment distribution method After a number of cycles of performing "locking", "releasing" and "distribution" operations of moment-distribution, it is found that the distributed moments are in the form of an infinite series:

$$a + ar + ar^2 + \dots\dots\dots\dots\ ar^n + \dots\dots\dots\dots$$

This series will be convergent if r is smaller than unity. Based on this fact, the successively distributing of unbalanced moments means consecutive approximation achieved toward the limits of series, as it is known that r is less than unity in moment distribution process; and when moments at joints are completely balanced, it means that the limit of the series is reached.

It is known that the moment distribution method is not self-checkable; once an error has been introduced during the calculating process, it will then be undetectably distributed throughout the whole structure just as any other moment. Especially when higher accuracy of the final moments is required, the possibility of error-inducing is increased accordingly, since more cycles of distribution are required for higher accuracy results and the possibility of introduction of errors is proportional to the cycles performed. But this possibility is decreased to its minimum when those general solutions are employed; meanwhile the maximum accuracy is required.

For finding the values of final moments and drawing influence lines, these general solutions are found effective for time-saving purposes. They require less than two-thirds of the time for determining the ordinates of an influence line than the conventional method requires, and much less time for finding final moments, for highest accuracy.

All these general solutions are functions of carry-over factors, stiffness factors and fixed-end moments. Therfore when those values are available, the final moments of the subject structure can easily be determined by means of these general solutions. Before these general solutions are employed, a little time in required to determine the values of several terms of the general solutions, but the time required for evaluating the results of general solutions is little. The general solutions therefore are found of special use when the final moments are required for one structure under several different loading conditions.

The general solution for three- and four-span continuous beams can again be used for derivation of that for multi-span continuous beams when the imaginary rigid joints are introduced.

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APPENDIX NOTATIONS

C_{ab}	Carry-over factor from "a" to "b"
D_{ab}	Distribution factor at end "a" of beam "ab", or span "ab"
E	Modulus of Elasticity of a beam
I	Moment of inertia of a beam
L	Span length
M	Bending moment of a beam
M^f_{ab}	Fixed End Moment of beam at end "a" of span "ab"
M^f_b	Sum of the fixed end moments and unbalanced external moments at joint "b"
P	Concentrated load
S_{ab}	Stiffness factor of end "a" of beam "ab" or span "ab"
E_{ab}	$C_{ab} \times D_{ab}$
F_{ab}	$E_{ab} \times E_{ba}$

GENERAL SOLUTIONS FOR THE MOMENTS
IN CONTINUOUS BEAMS

by

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ABSTRACT

This paper is to present a set of general solutions for prismatic and non-prismatic continuous beams, which are in the form of formulas, derived from the principle of the moment distribution method.

The principle of the moment distribution method is to assume each joint (excluding the built-in ends) of the indeterminate structure being considered can be repeatly "locked", then "released", and finally the unbalanced moment accumulated at that joint "distributed". Therefore it is actually a method of successive approximations. But now in this report, these formulas will give exact results, since each formula represents an infinite number of repetitions of performing these "locking", "releasing", and "distributing" operations. Therefore when these formulas are employed, not only considerable labor for calculation will be saved than would be required for moment distribution method, but also the high tendency of introducing errors in computation of the moment distribution method will be brought to its minimum.

Since these general solutions were obtained on the same basis of moment distribution method, the structure constants should be known prior to the application of these formulas. The structure constants are available in a wide range in the pamphlet "Handbook of Frame Constants" published by the Portland Cement Association.

There are also three numerical examples given in this report illustrating the application of these formulas for both finding final moments of the continuous members and determining the influence line ordinates.