A STUDY OF THE ANALOGY OF WATER TABLE FLOW TO TRANSONIC GAS FLOW

by

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NOMENCLATURE

c : velocity of water wave

c' : velocity of sound

T : local absolute temperature

k : specific heat ratio of gas

R : gas constant

g : gravitational acceleration

ρ : density of water

ρ' : density of gas

ρ_o' : stagnation density of gas

λ : wave length of water wave

V : total velocity of water

V' : total velocity of gas

h : local water depth

p : local pressure of water

p' : local pressure of gas

p_d : dynamic pressure of water

p_s : static pressure of water

p_o' : stagnation pressure of gas

T_o : stagnation temperature of gas

h_o : total head of water

t : surface tension of water

u : x-component velocity of water

v : y-component velocity of water

u' : x-component velocity of gas

v' : y-component velocity of gas

M : Mach number of water

M' : Mach number of gas

M_1 : Mach number before the hydraulic jump

c^* : the velocity of sound at the sonic line

$\tilde{u}$: x-component perturbation velocity

$\tilde{v}$: y-component perturbation velocity

INTRODUCTION

A water table is a device for the study of the two-dimensional non-viscous, isentropic flow of a perfect gas by means of an analogous shallow-water flow in an open channel.

It is well known that water is usually considered to be incompressible. Yet, theoretically, there exists an analogy between the two flows.

A successful derivation of the theory was made by Preiswerk; see the corresponding NACA report [1]. In his report, Preiswerk concluded that the analogy existed only between water flow and the flow of a fictitious gas for which the specific heat ratio, k, was equal to 2.

Several recent publications [2],[3],[4], written by Enrique J. Klein also indicate that the value of k cannot be other than 2 if the two-dimensional analogy is to exist. Although efforts have been made to obtain an analogy when the ratio of the specific heats for the gas involved in the gas flow is other than 2, it has always been shown that,in this case, the analogy is applicable only for one-dimensional flow.

Preiswerk's derivation only showed that the restriction, $k = 2$, was a sufficient condition; yet, it can be proved that this restriction is also a necessary condition. A proof of the necessity is shown in Section 1.6.

The water table analogy is used mainly for the study of transonic flow. The derivation of the analogy is based on an assumption, namely, that the loss of energy due to hydraulic jump in the water table flow is negligible. For Mach numbers of the water greater than $\sqrt{2}$, the analogy fails because of the non-negligible energy losses associated with the large hydraulic jumps which can occur [6]. The effects of hydraulic jump with respect to the analogy have been discussed in a paper co-authored by Gilmore, Plesset and Crossley [7].

Usually, transonic wind tunnel testing is difficult owing to the influence of wave reflection from the walls. The same difficulty occurs in the water table experiment. Therefore, a slotted wall arrangement is necessary in the test sections of the wind tunnel or analogous water table under these conditions.

The water table in the K.S.U. Mechanical Engineering Laboratory, originally designed and built nearly twenty years ago, has been redeveloped. A new illumination system has also been established. The redeveloped table has given very satisfactory test results.

I. THEORY OF THE WATER TABLE ANALOGY

1.1. The Velocity of Sound and the Velocity of a Water Wave

In the case of compressible flow, the velocity of sound is written as

$$C' = \sqrt{K g R T}$$

the wave velocity of water can be expressed as: [6]

$$C = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi t}{\rho\lambda}\right)\tanh\frac{2\pi h}{\lambda}}$$

$$= \sqrt{gh}\left[1 + 2\frac{\pi^2 t}{\rho g \lambda^2} + O\left(\frac{h^2}{\lambda^2}\right)\right] \qquad (1)$$

where O indicates the order of magnitude of the most significant terms in the remainder.

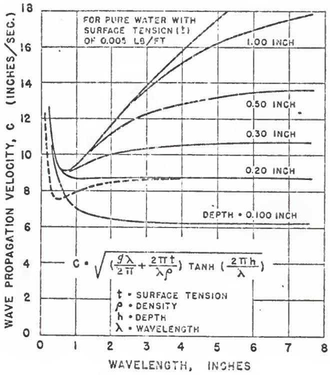


Fig. 1 Variation of wave velocity with wave length from Eq.(1). Dotted line shows effect of decreasing t by 50 per cent. (From Reference [6])

Fig. 1. is a plot of C vs. λ for pure water with surface tension $t = 0.005\ ^{lbf}/_{ft}$. It is seen that, for a water depth of the order of 0.2 in. and $\lambda \geq 1$ in., C becomes a function of depth only.

Equation (1) shows that, with $t = 0.005\ ^{lbf}/_{ft}$ and $h = 0.2^{in}$ the second and third terms can be neglected and the velocity of the water wave can be written as:

$$C = \sqrt{gh}\ .$$

Thus, within the range shown above, the local water depth, h, has an effect on c which is similar to the effect of the local absolute temperature T on the velocity of sound in the gas flow; i.e.,

$$C' \sim \sqrt{T} \qquad \text{and} \qquad C \sim \sqrt{h}\ .$$

These relationships then imply that, if $h \sim T$, then $c \sim c'$.

1.2. The Energy Equations:

Consider water flowing out of an infinite basin onto a horizontal bottom. (Fig. 2). If it is assumed that the flow is two-dimensional, open-channel flow without friction, along the flow filament A-B, the energy or Bernoulli equation can be written in the following way:

$$P_d + \frac{\rho}{2} V^2 + P_s + \rho g z = P_o + \rho g z_o$$

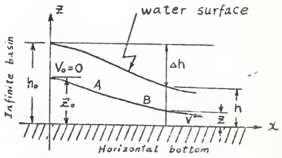


Fig. 2 The parameters in Sec. 1.2.

where

p_d : dynamic pressure

p_s : static pressure,

but, following Streeter [8] this equation can be rewritten as

$$P + \frac{\rho}{2} V^2 + \rho g z = P_o + \rho g z_o$$

or
$$V^2 = 2g(Z_o - Z) + 2(P_o - P)/\rho \quad . \tag{2}$$

The static pressure at a position on the filament depends linearly on the vertical distance under the free surface at that position, or

$$P_o = \rho g(h_o - Z_o) \quad , \tag{3a}$$

$$P = \rho g(h - Z) \quad . \tag{3b}$$

Substituting Equations (3) into Equation (2) gives

$$V^2 = 2g(h_o - h) = 2g\Delta h \quad .$$

The maximum attainable velocity is, thus,

$$V_{max}^2 = 2gh_o \quad .$$

In the case of perfect gas flow,

$$V'^2 = 2gc_p(T_o - T) = 2gc_p\Delta T$$

and

$$V'_{max.} = 2gc_pT_o \quad . \quad [9] \quad .$$

The energy equations for the two flows may thus be written

$$\left(\frac{V}{V_{max.}}\right)^2 = \frac{2g\Delta h}{2g\,h_o} = \frac{h_o - h}{h_o}$$

and

$$\left(\frac{V'}{V'_{max.}}\right)^2 = \frac{2gc_p\Delta T}{2gc_pT_o} = \frac{T_o - T}{T_o} \quad .$$

Therefore, if

$$\frac{V}{V_{max.}} = \frac{V'}{V'_{max.}} \quad ,$$

then

$$\frac{h_o - h}{h_o} = \frac{T_o - T}{T_o}$$

or

$$\frac{h}{h_o} = \frac{T}{T_o} \quad .$$

Conversely, if $$\frac{h}{h_o} = \frac{T}{T_o} ,$$

then $$\frac{V}{V_{max.}} = \frac{V'}{V'_{max.}} .$$

In addition to the result obtained in Section (1.1), there exist, therefore, two analogies, one with respect to c and c' and another with respect to V and V' if $h \sim T$. In other words,

$$c \sim c' \quad \text{and} \quad V \sim V' \qquad \text{if} \quad h \sim T .$$

1.3. The Continuity Equations.

The continuity equation for two-dimensional, incompressible, open-channel flow can be written as [1]

$$\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0 . \tag{4}$$

A similar equation describes the continuity of a two-dimensional gas flow.

$$\frac{\partial(\rho' u')}{\partial x} + \frac{\partial(\rho' v')}{\partial y} = 0 . \tag{5}$$

By comparison of Equations (4) and (5), it can be seen that the relation, $h = C\rho'$ (where C is a constant), is a sufficient condition for similarity of the two different flows defined by Equations (4) and (5).

Let $C = \frac{h_o}{\rho'_o}$; then the continuity equations provide another analogy between the two flows. With respect to the water depth and the density of gas, the Equation, $\frac{\rho'}{\rho'_o} = \frac{h}{h_o}$ is obtained. For isentropic flow,

$$\frac{\rho'}{\rho'_o} = \left(\frac{T}{T_o}\right)^{\frac{1}{K-1}} ,$$

so that

$$\frac{h}{h_o} = \left(\frac{h}{h_o}\right)^{\frac{1}{K-1}}$$

if the two previously developed sufficient conditions for an exact analogy between the two flows are to be satisfied. This last equation is satisfied only for k = 2.

1.4. The Pressures

For a perfect gas

$$\frac{p'}{p_o'} = \left(\frac{\rho'}{\rho_o'}\right)\left(\frac{T}{T_o}\right) ,$$

Since $\frac{T}{T_o} = \frac{h}{h_o}$ and $\frac{\rho'}{\rho_o'} = \frac{h}{h_o}$, the relation between water depth and gas pressure becomes

$$\frac{p'}{p_o'} = \left(\frac{h}{h_o}\right)^2 .$$

No direct analogy exists between the static pressure below the surface of the water and any property of the gas flow.

1.5. The Mach Numbers

The Mach number of the water flow is defined as

$$M = \frac{V}{c} = \frac{V}{\sqrt{gh}} .$$

For the gas flow, the Mach number is defined as

$$M' = \frac{V'}{c'} = \frac{V'}{\sqrt{kgRT}} . \qquad (6)$$

If

$$\frac{V}{V_{max.}} = \frac{V'}{V'_{max.}} , \qquad (7)$$

then

$$\frac{h}{h_o} = \frac{T}{T_o} .$$

From Equations (6) and (7),

$$\frac{M}{M'} = \frac{V_{max.}}{V'_{max.}} \cdot \frac{\sqrt{kgRT}}{\sqrt{gh}}$$

$$= \sqrt{\frac{2g\,h_o}{2g\,C_p\,T_o}} \cdot \sqrt{\frac{(C_p - C_v)\,k \cdot T_o \cdot h}{h_o \cdot h}}$$

$$= \sqrt{\frac{(C_p - C_v)\,k}{C_p}} = \sqrt{\frac{C_p - C_v}{C_v}} = \sqrt{k-1}$$

In the case of a gas for which k = 2, the Mach numbers become identical, i.e., M = M' provided that

$$\frac{V}{V_{max.}} = \frac{V'}{V'_{max.}} \qquad \text{or} \qquad \frac{h}{h_o} = \frac{T}{T_o} .$$

1.6. The Necessity for the Value of k to be Equal to 2 with Respect to a Two-dimensional Analogy

In the present discussion, the term "analogy" is applied whenever the properties of a two-dimensional isentropic gas flow and an open-channel water flow are everywhere similar. Moreover, under these conditions, the geometries of the stream lines of the two flows are similar. Actually, study of only this kind of close analogy brings a useful result. For similar flows with different fluids, different velocities and with geometrically similar patterns, similarity requires that the ratio of the forces acting on the fluid particles at geometrically similar points be equal at every instant of time [5]. In the present case, the analogy deals with compressible flow and an incompressible, open-channel flow. If one considers the relationship of the total velocity vectors of the fluid particles rather than the forces acting on them, the analogy may be more clear with respect to the physical picture.

Consider the total velocity vectors of the two flows with a relationship $V' = \alpha V$ (where α is a scalar constant) which holds everywhere at every instant of time. Then the flowing directions of the particles at every geometrically similar point and at every instant of time are the same, and the magnitudes of the displacement are different with a common scale factor, α. In other words, the slopes and the curvatures of the stream lines are the same at every instant of time. Hence, the flow patterns are similar. For $V' = \alpha V$, since V' and V are vectors, necessarily and sufficiently, $u' = \alpha u$ and $v' = \alpha v$ where u, v and u', v' are x, y component velocities of V and V', respectively. For the present analogy, one more factor is involved, namely, the wave angles of the discontinuities, since, if a weak shock is considered (this is the usual case), the Mach numbers uniquely determine the wave angles in the two-dimensional gas flow and on the surface of the water flow. Now the wave angles, together with upstream Mach numbers, determine the directions of the flowing particles of the fluids. Hence, for similar flows, necessarily $M = M'$. By the definition of Mach number, $M = M'$ implies that

$$\frac{\frac{T_o}{T} - 1}{\frac{h_o}{h} - 1} = K - 1 \qquad (8)$$

Since $\frac{T_o}{T}$ and $\frac{h_o}{h}$ are functions of x and y and k is independent of x and y, no other independent relationship between $\frac{T_o}{T}$ and $\frac{h_o}{h}$ with k as a parameter can be established. Otherwise, with two unknowns in the two equations, $\frac{T_o}{T}$ and $\frac{h_o}{h}$ can be determined and can be expressed as functions of k. But this is in contradiction to the fact that both $\frac{T_0}{T}$ and $\frac{h_0}{h}$ are functions of x and y. This implies that the constant, α, in $V' = \alpha V$ is not arbitrary. Consider the transformation, $\frac{\rho'}{\rho_0'} = \xi \frac{h}{h_0}$, where now ξ is not yet proved to be constant; then, the isentropic relation requires that

$$\frac{\rho'}{\rho_0'} = \left(\frac{T}{T_0}\right)^{\frac{1}{K-1}} = \xi \frac{h}{h_0}.$$

From Equation (8),

$$\frac{h}{T} \cdot \left(\frac{T_0 - T}{h_0 - h}\right) = K - 1$$

or,

$$\frac{h}{T} \cdot \frac{V'^2}{2g\,c_p} \cdot \frac{2g}{V^2} = K - 1$$

or

$$\frac{h}{T} \cdot \frac{\alpha^2}{c_p} = K - 1.$$

Hence,

$$\frac{h}{T} = \frac{c_p(K-1)}{\alpha^2}$$

or

$$\left(\frac{h}{T}\right)^{\frac{1}{K-1}} = \left[\frac{c_p(K-1)}{\alpha^2}\right]^{\frac{1}{K-1}},$$

If $\xi = f_1(x,y)$, then

$$\frac{\rho'}{h} = \xi \frac{\rho_0'}{h_0} = \frac{\rho_0'}{h_0} \cdot f_1(x,y)$$

or

$$\frac{\rho'}{h^{\frac{1}{K-1}}} = \frac{\rho_0'}{h_0} f_1(x,y) \cdot h^{\frac{K-2}{K-1}} = f_2(x,y).$$

Therefore,

$$\frac{\rho'}{h^{\frac{1}{K-1}}} \cdot \frac{h^{\frac{1}{K-1}}}{T^{\frac{1}{K-1}}} = \frac{\rho'}{T^{\frac{1}{K-1}}} = f_2(x,y)\left[\frac{c_p(K-1)}{\alpha^2}\right]^{\frac{1}{K-1}}.$$

But from the isentropic relation, in the case of the gas flow,

$$\frac{\rho'}{T^{\frac{1}{k-1}}} = \frac{\rho_o'}{T_o^{\frac{1}{k-1}}} = constant \quad .$$

Hence, $f_2(x,y)$ is necessarily a constant. This implies that, (i) $f_1(x,y)$ is a constant and $k = 2$ or (ii) $f_1(x,y) = C h^{\frac{2-k}{k-1}}$ where C is a constant. For case (i), since $k = 2$, from Equation (8), one can conclude that $\frac{T_o}{T} = \frac{h_o}{h}$ and, hence, that $\frac{V'}{V'_{max.}} = \frac{V}{V_{max.}}$. Thus,

$$\alpha = \frac{V'_{max.}}{V_{max.}}$$

In the possible case that $f = c\, h^{\frac{2-k}{k-1}}$, consider the continuity equation for the isentropic gas flow: By substituting the transformation into the continuity equation of the gas flow, one can obtain the following expression:

$$\left[\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} \right] + \left(\frac{2-k}{k-1}\right)\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = 0 \quad .$$

Since, according to the continuity equation of the two-dimensional open-channel water flow,

$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \neq 0 \quad ,$$

again it is necessary that $k = 2$ in order to obtain the expression,

$$\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} = 0 \quad .$$

1.7. Summary of the Analogy Between the Two Flows

Usually, in two-dimensional, non-viscous, gas dynamic problems, there exist three independent variables, e.g., p', T, and V'. However, for isentropic flows, only two of these variables are independent. The conditions imposed on the water flow in order to provide an analogy to the gas flow allow the equations describing the phenomena of both flows to be satisfied simultaneously.

Under these conditions, there is a perfect correspondence between certain properties of the gas flow and corresponding properties of the water flow. Thus, the water table analogy is established, and experimental results from the water table can be used to check theoretical results pertaining to a corresponding gas flow for which the specific heat ratio of the gas is equal to 2. For instance, if the dimensionless water depth at point A in the channel is identical with the dimensionless temperature at A' in the gas flow, the identity of Mach numbers at A and A' proves the correctness of the theoretical gas dynamic equations for the particular case that k = 2.

The reverse statement, i.e., "If M = M', then $\frac{h}{h_o} = \frac{T}{T_o}$ ", is also useful, because, if it is true, one can compare the wave patterns of the two flows first and, then check the correspondence of the dimensionless water depth to the dimensionless temperature.

The proof of the statement is as follows:

If M = M', then

$$\frac{\dfrac{V}{\sqrt{gh}}}{\dfrac{V'}{\sqrt{kgRT}}} = \sqrt{k-1} = 1 \qquad \text{since } k = 2.$$

This implies that

$$\frac{\sqrt{2g(h_o-h)}}{\sqrt{2gC_p(T_o-T)}} = \frac{\sqrt{gh}}{\sqrt{kgRT}} \cdot \frac{\sqrt{2gh_o}}{\sqrt{2gC_pT_o}} \cdot \frac{\sqrt{RkT_o}}{h_o}$$

or

$$\frac{h_o-h}{T_o-T} = \frac{h}{T}$$

or

$$\frac{\dfrac{h_o}{h}-1}{\dfrac{T_o}{T}-1} = 1 .$$

Thus,

$$\frac{T}{T_o} = \frac{h}{h_o}$$

II. WATER TABLE REDEVELOPMENT

In order to obtain a laminar velocity profile in the vertical, streamwise (x-z)plane with mean x-component velocities within the desired test range, the water table in the Mechanical Engineering Laboratory as shown in Fig. 3 has been releveled and refinished. Also, three screens have been placed in the upstream water reservoir to stablize the flow. It was found that the unstable water source and the roughness of the spillway were the main reasons for the undesirable turbulence which was so troublesome before the table was modified. The slope of the water table was also readjusted to provide a uniform flow of the desired depth at the test section.

The new lighting system includes a high-intensity light source and its associated power supply, a Fresnel lens, a ground glass, and a viewing mirror. The arrangement of this system is shown in Fig. 4.

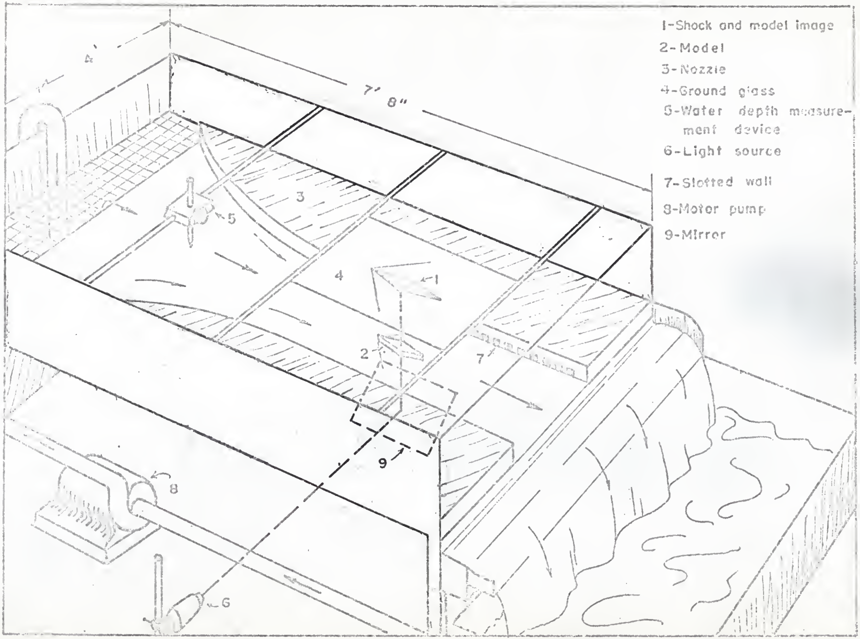


Fig. 3 The redeveloped water table.

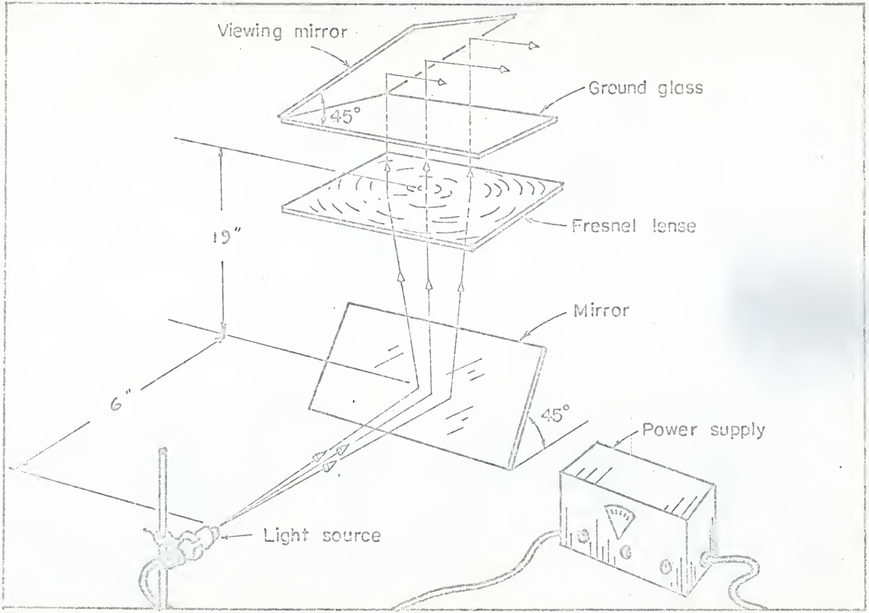


Fig. 4 Lighting system of the water table .

III. TRANSONIC NOZZLE AND TEST SECTION DEVELOPMENT

3.1. Nozzle Design

The nozzle of the water table was designed according to Sauer's approximate method [10].

In this method, the velocity components u', v', in the x, y directions respectively, are expressed in terms of perturbation velocity components $\tilde{u}$ and $\tilde{v}$ as:

$$\frac{u'}{c^*} = 1 + \tilde{u}$$

and

$$\frac{v'}{c^*} = \tilde{v}$$

where c^* is the velocity of sound at the sonic line. This representation of the velocity components is possible since, near the throat of the nozzle, v' is small and u' is nearly equal to c^*. Hence, $\tilde{u}$ and $\tilde{v}$ are small compared with unity and

$$\frac{c^* - u'}{c^*} << 1 , \qquad \frac{v'}{c^*} << 1$$

and

$$u' \sim c^* .$$

By substitution into the equation of motion, the two-dimensional, steady, irrotational, isentropic, perturbation equation of motion can be obtained: Thus,

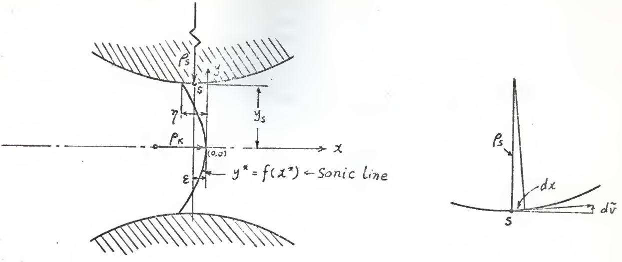
$$(c'^2 - u'^2)\frac{\partial u'}{\partial x} - 2u'v'\frac{\partial u'}{\partial y} + (c'^2 - v'^2)\frac{\partial v'}{\partial y} = 0 ,$$

can be approximated by

$$(K+1)\,\tilde{u}\,\frac{\partial \tilde{u}}{\partial y} - \frac{\partial \tilde{v}}{\partial y} = 0 \qquad (9)$$

where k is the specific heat ratio.

The flow is symmetric about the x-axis (Fig. 5).



(a) (b)

Fig. 5 Sonic line parameters.

The dimensionless perturbation velocities are assumed as follows:

$$\tilde{u} = f_0'(x) + y^2 f_2'(x) + y^4 f_4'(x) + \cdots \tag{10 a}$$

and

$$\tilde{v} = 2y\, f_2(x) + 4y^3 f_4(x) + \cdots \tag{10 b}$$

where f_0, f_2, f_4, ... are terms of the perturbation velocity potential function; the perturbation velocity potential function is defined by

$$\frac{1}{c^*}\varphi = f_0(x) + y^2 f_2(x) + y^4 f_4(x) + \cdots .$$

Since, near $x = 0$, $y = 0$, from Equation(10a),

$$f_0' = \left[\tilde{u}(x)\right]_{y=0} \approx \left(\frac{d\tilde{u}}{dx}\right)_0 x \quad . \tag{11}$$

From Equations (9), (10) and (11), the dimensionless perturbation velocities can be expressed in terms of $\left(\frac{d\tilde{u}}{dx}\right)_0$ near $x = 0$, $y = 0$, i.e.,

$$\tilde{u} = \left(\frac{d\tilde{u}}{dx}\right)_0 x + \frac{\kappa+1}{2}\left(\frac{d\tilde{u}}{dx}\right)_0^2 y^2 + \cdots \tag{12 a}$$

and

$$\tilde{v} = (\kappa+1)\left(\frac{d\tilde{u}}{dx}\right)_0^2 x y + \frac{(\kappa+1)^2}{6}\left(\frac{d\tilde{u}}{dx}\right)_0^3 y^3 + \cdots \tag{12b}$$

At the sonic line,

$$u'^2 + v'^2 = c^{*2}$$

or

$$(1+\tilde{u})^2 + \tilde{v}^2 = 1 .$$

Since $\tilde{u} = 0$, from Equation (12a),

$$x^* = -\frac{\kappa+1}{2}\left(\frac{d\tilde{u}}{dx}\right)_0 y^{*2} . \tag{13}$$

Equation (13) shows that the sonic line is a parabola.

By definition of the curvature,

$$\frac{1}{\rho_\kappa} = \frac{\frac{d^2x^*}{dy^{*2}}}{\sqrt{1+\left(\frac{dx^*}{dy^*}\right)^2}} = (\kappa+1)\left(\frac{d\tilde{u}}{dx}\right)_0 . \tag{14}$$

Since, at s, (Fig. 5) $\tilde{v}_s = 0$, from Equation (12b)

$$\varepsilon = \frac{\kappa+1}{6}\left(\frac{d\tilde{u}}{dx}\right)_0 y_s^2 . \tag{15}$$

At the wall, in vicinity of the throat,

$$y^* \approx y_s .$$

Thus, from Equations (13) and (15),

$$\eta = \frac{\kappa+1}{3}\left(\frac{d\tilde{u}}{dx}\right)_0 y_s^2 = 2\varepsilon . \tag{16}$$

Applying the approximate geometry of Fig. 5(b),

$$\frac{1}{\rho_s} \approx \frac{\partial\tilde{v}}{\partial x} ,$$

and using Equations (10),

$$\frac{1}{\rho_s} = (\kappa+1)\left(\frac{d\tilde{u}}{dx}\right)_0^2 y_s = \frac{1}{\rho_\kappa}\left(\frac{\partial\tilde{u}}{\partial x}\right)_0 y_s . \tag{17}$$

Solving Equations (13) through (17) for $(\frac{d\tilde{u}}{dx})_o$, one can obtain the sonic-line parameters in the following form:

$$x^* = -\frac{k+1}{2y_s} \cdot \frac{\rho_K}{\rho_s} \cdot y^{*2} , \tag{18a}$$

$$\rho_K = \sqrt{\frac{1}{k+1} \cdot \rho_s \cdot y_s} , \tag{18b}$$

and

$$\eta = 2\varepsilon = \sqrt{\frac{k+1}{9} \cdot \frac{y_s^3}{\rho_s}} . \tag{18c}$$

In the present nozzle design, some difficulties arise owing to the fact that the water is incompressible. This implies that the open-channel water flow is not two-dimensional. However, neglecting the small change of depth, Equations (18) can be used to obtain the contour of the sonic line. In conformity with the water table analogy, k = 2 was used in the design. The values of y_s and ρ_s were chosen as 6" and 51", respectively.

From Equations (18),

$$\rho_K = \sqrt{(\tfrac{1}{3})(51)\cdot(6)} = 10.1 \text{ in.} ,$$

$$x^* = \frac{(3)(10.1)}{(2)(6)(51)} y^{*2} = 0.0495\, y^{*2} ,$$

and

$$\eta = 2\varepsilon = \sqrt{\frac{(3)(216)}{(9)(51)}} = \sqrt{1.41} = 1.19 \text{ in} .$$

Fig. 6 shows the result of the design.

3.2. The Adjustment of Mach numbers at the Test Section

As indicated before, the actual water flow is not two-dimensional. Therefore, the area ratio of the nozzle does not uniquely determine the Mach number of the water. The desired Mach number at the test section can be obtained by adjusting the slope of the table over which the water flows and/or by adjusting the discharge valve downstream of the water pump.

3.3. Slotted Wall Development

It was found during experiments that the solid walls of the test section reflected the water waves. The resulting waves of the pattern disturbed each other and resulted in non-uniform flow behind the first incident wave on the wall.

In order to eliminate the reflected waves, the walls of the test section were slotted as shown in Fig. 3. It is known that the wave is reflected from a straight solid wall in like sense (i.e. the compression wave is reflected as compression wave and vise versa), while the wave is reflected from a boundary of constant pressure in unlike sense. Hence, by using a slotted wall at the test section, one can obtain a condition midway between that of wave reflection in like sense from a solid wall and reflection in unlike sense from a constant-pressure boundary. It is well known that a compression wave and an expansion wave tend to cancel each other. The objective of reducing the reflected waves is thus achieved by a straight side wall which consists of alternate solid segments and slots.

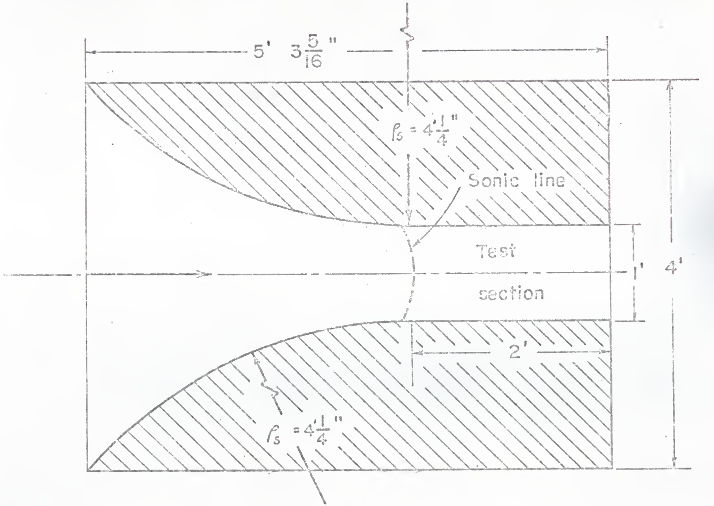


Fig. 6 The designed nozzle for the water table .

IV. MEASUREMENTS IN A TRANSONIC SECTION

4.1. Effect of Capillarity

As shown in Fig. 1, Section 1.1 a water wave of short length does not obey the equation, $C \approx \sqrt{gh}$. This Figure shows that, with a water depth of the order of 0.2 inches, the propagation velocity of the wave of short length becomes a function of both depth and wavelength. The second term of Equation (1) in Section 1.1 shows that, with small wavelength, the surface tension should be reduced in order that the propagation velocity become a function of depth only, or $C \approx \sqrt{gh}$.

Experimentally, the wave of short length propagates as a so-called capillary wave in front of the gravitational wave. Since the capillary wave does not obey the equation $C \approx \sqrt{gh}$, there is no analogy between it and the wave of gas flow. Moreover, if the capillary wave is too strong, one might not be able to distinguish it from the wave of interest.

In the present experiment, n-propanol was added to the water to reduce the surface tension. The boiling point and density of n-propanol are similar to those of water, and n-propanol was found to be effective in reducing the surface tension of the resulting solution. With the addition of eight pints of n-propanol to the water contained by the water table apparatus (approximately 90 gallons) the surface tension was decreased about 50 per cent as compared with the surface tension of pure water.

4.2. Water Depth

Since, for the water table technique, the Mach number of the water is restricted within the range of $M < \sqrt{2}$, the dimensionless water depth ratio is restricted to $\frac{h_o}{h} < 2$ since $\frac{h_o}{h} = 1 + \frac{1}{2}M^2$. Also, according to Fig. 1

of Section 1.1, the depth of water should be of the order of 0.2 inches. In the present experiment, the water depth at the test section was set to be approximately 1/4 inches.

4.3. The Mach Number of the Water

According to the equation, $M = \frac{V}{\sqrt{gh}}$, the Mach number of the water can be obtained if the surface velocity and the depth of the water are known. For a two-dimensional, open-channel flow, the relation between the surface velocity and the mean velocity is $U = \frac{3}{2}\bar{u}$ [Appendix], where U is the surface velocity. The mean velocity is measured indirectly by measuring the volume flow rate or weight flow rate. The depth of the water was measured by means of a vertical bar with a sharp end as shown in Fig. 3. Several measurements of Mach number are shown in Table 1. Fig. 7 shows the accuracy with which a particular theoretical result has been verified on the modified K.S.U. Mechanical Engineering Laboratory water table. This Figure shows that the maximum combined error of the experimental data with respect to a particular theoretical result is less than 3 per cent.

4.4. Shock Waves

The formation of Mach waves on the water surface is similar to that in a corresponding gas flow. Hence, the relation, $M = \frac{1}{\sin\alpha}$, where α is half of the Mach angle, still holds. Fig. 8 is a picture of a comparatively weak shock wave in a water table flow where $M_1 = 1.21$ and the wedge angle equals 6°. Fig. 10 shows the details of the formation of the image of the shock wave on the ground glass. As is indicated in Fig. 10, the relatively light portion of the screen is the image of the peak of the hydraulic jump. As stated previously, the hydraulic jump corresponds (at least qualitatively) to the shock wave of

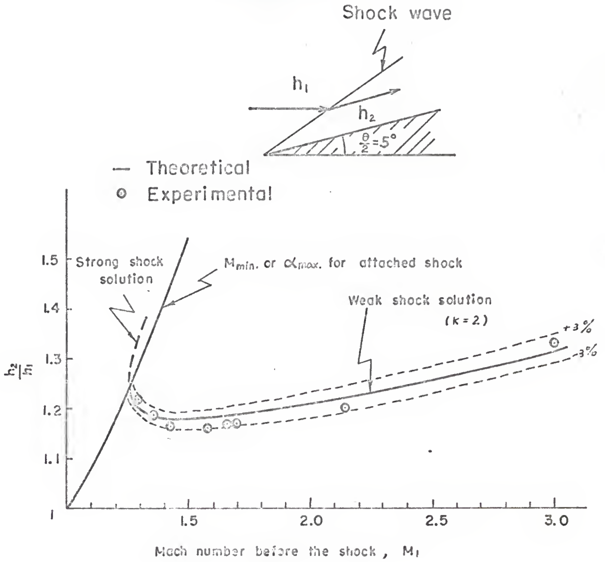


Fig. 7 Comparison of the theoretical and the experimental data for a water flow about a thin wedge ($\theta = 10°$) .

the corresponding gas flow. Fig. 9 shows typical capillary waves and Mach reflections at the upper right corner of the picture. In this same picture, a little n-propanol has been spread on the water surface in the left upper corner. In this latter region, the capillary waves are seen to be greatly diminished.

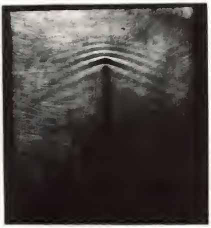


Fig. 8 $M_1 = 1.21$, $\theta = 6°$
$h = 0.21''$.

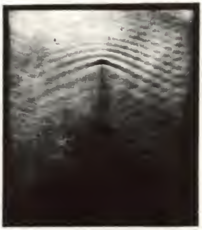


Fig. 9 $M_1 = 1.27$, $\theta = 6°$
$h = 0.26''$.

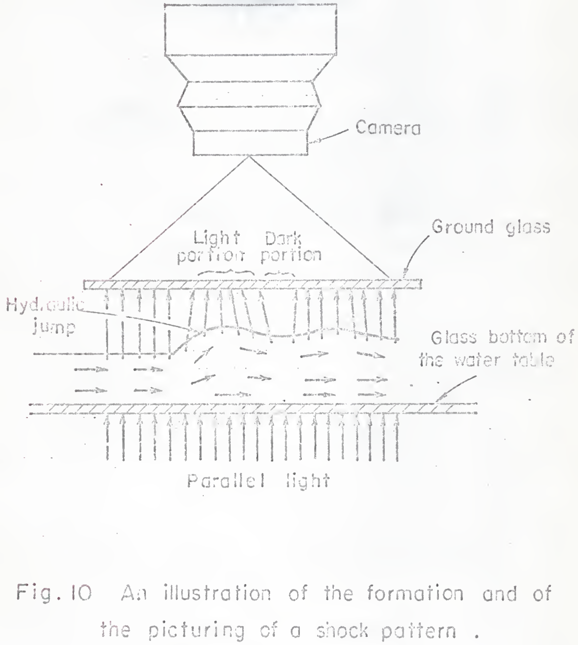
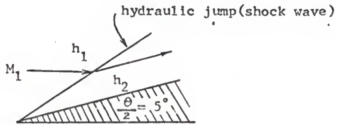


Fig. 10 An illustration of the formation and of the picturing of a shock pattern .

Table 1. The test data of the water table experiment with wedge angle, $\theta = 10^\circ$

Test number	d	h_1	$A = dh_1$	$\sqrt{h_1}$	t	V	$M_1 = 0.0764 \cdot \frac{V}{A \cdot t} \cdot \sqrt{\frac{1}{h_1}}$	h_2	h_2/h_1
1	11.7	0.240	2.92	0.500	132	3950	1.57	0.275	1.15
2	11.7	0.175	2.05	0.418	176	4200	2.13	0.210	1.20
3	11.7	0.150	1.76	0.387	480	3710	3.00	0.200	1.33
4	11.7	0.300	3.51	0.547	126	4020	1.27	0.370	1.33
5	11.7	0.220	2.58	0.470	161	4270	1.67	0.258	1.17
6	11.7	0.270	3.16	0.520	116	4120	1.65	0.316	1.17
7	11.7	0.250	2.93	0.500	150	3980	1.35	0.295	1.18
8	11.7	0.290	3.39	0.539	121	4070	1.41	0.336	1.16



d : width of the test section(in.)

h : water depth(in.)

A : cross sectional area of the channel($in.^2$)

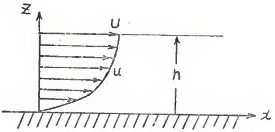
t : time(sec.)

V : volume flow rate($in.^3$)

h_2/h_1 : water depth ratio

APPENDIX

The Relation between the Surface Velocity and the Mean Velocity in a Two-dimensional Open-channel Water Flow (Note: Here the term " two-dimensional water flow " has a different meaning from those used in the other places of this report.)



h: water depth
u: x-component velocity
w: z-component velocity
U: surface velocity
ū: mean velocity
p: pressure
ρ: density
w = 0

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Since $\frac{\partial w}{\partial z} = 0$, then $u = f(z)$.

z-Component Navier-Stokes Equation:

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

where ν is the kinematic viscosity.

Since $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial z} = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial z^2} = 0$

hence, $\frac{\partial P}{\partial z} = 0$, or $P = f(x)$.

x-Component Navier-Stokes Equation:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

Since $\frac{\partial u}{\partial x} = w = \frac{\partial^2 u}{\partial x^2} = 0$, then

$$\frac{dP}{dx} = \frac{d\left(P_{atmosfer} + \rho g h\right)}{dx} = \rho g \frac{dh}{dx} = \mu \frac{d^2 u}{dz^2} = constant \quad (1)$$

where $\mu = \rho\nu$.

Boundary Conditions:

at $z = 0$, $u = 0$ (a)

at $z = h$, (i) $\mu\frac{\partial u}{\partial z} = 0$, hence $\frac{\partial u}{\partial z} = 0$ (ii) $u = U$ (b)

The surface Velocity U:

Integration of Equation(1) yields,

$$\frac{du}{dz} = \frac{\rho g}{\mu}\frac{dh}{dx}\cdot z + C_1 ,$$

where C_1 is a constant.

From boundary condition(b),

$$C_1 = -\frac{\rho g h}{\mu}\cdot\frac{dh}{dx} ,$$

hence,

$$\frac{du}{dz} = \frac{\rho g}{\mu}\frac{dh}{dx}(z-h) .$$

Integration of this last equation yields,

$$u = \frac{\rho g}{\mu}\cdot\frac{dh}{dx}\left(\frac{1}{2}z^2 - hz\right) + C_2$$

where C_2 is a constant.

From boundary condition(a),

$$C_2 = 0 ,$$

then,

$$u = \frac{\rho g}{\mu}\cdot\frac{dh}{dx}\left(\frac{1}{2}z^2 - hz\right) .$$

Since at $z = h$, $u = U$ the surface velocity is then,

$$U = \frac{\rho g}{\mu}\frac{dh}{dx}\left(\frac{1}{2}h^2 - h^2\right) = -\frac{\rho g}{2}\frac{h^2}{\mu}\frac{dh}{dx} .$$

and

$$= \left(-\frac{\rho g}{2}\frac{h^2}{\mu}\frac{dh}{dx}\right)(-2)\left(\frac{1}{h^2}\right)\left(\frac{1}{2}z^2 - z\right) = U\left(\frac{2z}{h} - \frac{z^2}{h^2}\right)$$

$$= U\left[-\left(1-\frac{z}{h}\right)^2 + 1\right] = U\left[1 - \left(1-\frac{z}{h}\right)^2\right]$$

The Mean Velocity:

$$\bar{u} = \frac{1}{h}\int_0^h u\,dz = \frac{1}{h}\int_0^h U\left(2\frac{z}{h} - \frac{z^2}{h^2}\right)dz$$

$$= \frac{U}{h}\left[\frac{z^2}{h} - \frac{1}{3}\frac{z^3}{h^2}\right]_0^h$$

$$= \frac{U}{h}\left(\frac{h^2}{h} - \frac{1}{3}\frac{h^3}{h^2}\right)$$

$$= \frac{2}{3}U .$$

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A STUDY OF THE ANALOGY OF WATER TABLE
FLOW TO TRANSONIC GAS FLOW

by

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B.S., National Taiwan University, 1964

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

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1968

This report contains four parts. The first part is the derivation of the theory of water table analogy. In this part, Preiswerk's derivation has been introduced, and a necessary condition of the analogy, namely $k = 2$, has been shown. The second part is a description of the redevelopment of the original K.S.U. Mechanical Engineering Department water table. The third part reports the design of a transonic nozzle for the water table. In this connection, the use of a slotted-wall at test section has been discussed. The last part of this report presents the test results obtained from experiments with the water table. These test results include the measurement of "Mach number" and an investigation of the accuracy of the results obtained from the water table. Fig. 7 in this last part shows the maximum combined error of the experimental data with respect to a particular theoretical result is less than 3 per cent.