OPTIMAL PRODUCTION SCHEDULING AND INVENTORY CONTROL BY THE DISCRETE MAXIMUM PRINCIPLE

by 149

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TABLE OF CONTENTS

		Page
1.	INTRODUCTION	1
2.	THE DISCRETE MAXIMUM PRINCIPLE STATEMENT OF THE ALGORITHM	4
	PROCESS WITH MEMORY IN DECISION	7
3.	CASE STUDIES	9
	CASE 1. A PRODUCTION SCHEDULING PROBLEM--ILLUSTRATION OF A PROCESS WITH MEMORY IN DECISION	9
	CASE 2. A PERSONNEL AND PRODUCTION SCHEDULING PROBLEM	26
	CASE 3. APPLICATION OF THE DISCRETE MAXIMUM PRINCIPLE TO LABOR ASSIGNMENTS AS A DYNAMIC CONTROL PROBLEM	46
	REFERENECES	66
	ACKNOWLEDGMENTS	67

INTRODUCTION

The objective of this report is to demonstrate the applications of the discrete maximum principle to production scheduling and inventory control problems in industrial management. The three cases presented in this report are: Case 1 is a production scheduling problem and is an illustration of a process with memory in decision. Case 2 is also a production scheduling problem but in this case backlogging is permitted. Case 3 deals with the labor assignment as a dynamic control problem. Nelson (8) has optimized labor assignment in a labor machine limited production system by the continuous maximum principle. In this report the problem is extended for a non-linear cost function and for discrete time intervals.

The rapid growth of modern technology has played a remarkable role in the increasing interest in problems of dynamic optimization. Various optimization techniques are now available for analysing systems optimization. One such method is Pontryagin's maximum principle. Originally it was developed in 1956 for continuous processes and has been mainly applied in the field of optimum system control (7).

The first attempt to extend the maximum principle to the optimization of stagewise processes was made by Rozoner in 1959. The various versions of the discrete maximum principle were proposed by Chang, Katz, and Fan and Wang (3). Not many papers have been published on the applicability of the maximum principle to management and operation research problems (4, 5, 6). However, the maximum principle proves to be a powerful technique for solving optimization problems.

A multi-stage decision process may be considered as an abstract notion by which large number of human activities can be represented. A stage may represent any real or abstract entity (a space unit, a time period, etc.) in which a transformation takes place. Those variables which are transformed in each stage are called state variables. The desired transformation of the state variables is achieved through manipulation of decision variables which remain, or may be considered to remain, constant within each stage of the process. The equations which completely describe the transformation at each stage are called performance equations. Any process whose performance equations are linear in state variables is called a linear process. A process which is not linear is called a non-linear process. The basic algorithm of the discrete maximum principle is first stated along with the extension considering the memory in decision. After that, case studies are presented.

Ruiz (9) presented the formulation of the problems of Cases 1 and 2. The computation procedure for these cases is presented in this report.

Case 1 deals with the production scheduling problem. In this case, sales forecast for given periods are stated and the objective is to fulfill the sales requirements as well as to minimize the production cost. This problem is solved by using the extension of the basic algorithm known as "memory in decision". Case 2 also deals with the production scheduling problem and here, too, sales requirements for a given period are known but backlogging is permitted. The iterative procedure for optimum solution is solved by the exhaustive search technique with a computer.

Case 3 deals with the labor assignment as a dynamic control problem in a multifacility network. The system considered has limiting labor resources and the objective here is to allocate the labor force in an optimum way to minimize the non-linear in-process inventory cost function. In the original model, analyzed by Nelson (8), the work pieces are assumed to arrive at the machine center at a continuous rate and hence the continuous maximum principle is employed in optimizing labor assignment. The model considered by Bantwal (2) assumed that the work pieces arrive at discrete time intervals and has a linear in-process inventory cost function. However, the model considered in this report is an extension of the discrete model and assumes that the work pieces arrive at discrete time intervals and the in-process inventory cost is non-linear and hence the discrete version of the maximum principle is used.

THE DISCRETE MAXIMUM PRINCIPLE STATEMENT OF THE ALGORITHM

The following is an outline of the general algorithm of the discrete maximum principle (3, 5).

A multi-stage decision process consisting of N stages in sequence is schematically shown in Fig. 1. The state of the process stream denoted by an s-dimensional vector, $x = (x_1, x_2, \ldots, x_s)$, is transformed at each stage according to an r-dimensional decision vector, $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, which represents the decision made at that stage. The transformation of the process stream at the n^{th} stage is represented by a set of performance equations.

$$x_i^n = T_i^n(x_1^{n-1}, x_2^{n-1}, \ldots, x_s^{n-1}; \theta_1^n, \theta_2^n, \ldots, \theta_r^n), \quad x_i^o = \alpha_i,$$

$$i = 1, 2, \ldots, s; \quad n = 1, 2, \ldots, N.$$

or in vector form

$$x^n = T^n(x^{n-1}; \theta^n), \quad n = 1, 2, \ldots, N; \qquad (1)$$
$$x^o = \alpha$$

A typical optimization problem associated with such a process is to find a sequence of θ^n, $n = 1, 2, \ldots, N$, subject to constraints

$$\psi_i^n \left[\theta_i^n, \theta_2^n, \ldots, \theta_r^n . \right] \leq 0 \qquad n = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, r, \qquad (2)$$

which makes a function of the state variable of the final stage N

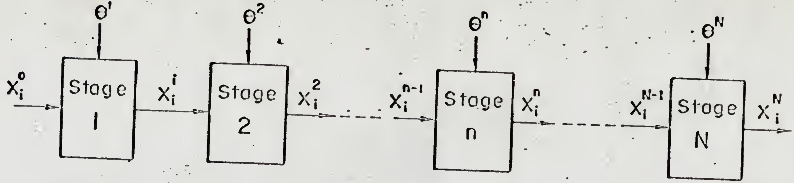

Fig. 1. Multistage decision process

$$S = \sum_{i=1}^{s} c_i x_i^N , \quad c_i = \text{constant} \tag{3}$$

an extremum when the initial condition $x^o = \alpha$ is given. The function S which is to be maximized (or minimized) is the objective function of the process.

The procedure for solving such an optimization problem by the discrete maximum principle is to introduce an s dimensional adjoint vector z^n and a Hamiltonian function H^n satisfying the following relations.

$$H^n = (z^n)^T x^n = \sum_{i=1}^{s} z_i^n \, T_i^n(x^{n-1};\theta^n), \; n = 1, 2, \ldots, N, \tag{4}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^n} , \; i = 1, 2, \ldots, s; \quad n = 1, 2, \ldots, N, \tag{5}$$

and

$$z_i^N = c_i , \quad i = 1, 2, \ldots, s. \tag{6}$$

If the optimum decision vector function θ^n, which makes the objective function S an extremum (maximum or minimum), is interior to the set of admissible decisions, Θ^n, then the set given by equation (2), a necessary condition for S to be a (local) extremum with respect to θ^n is

$$\frac{\partial H^n}{\partial \theta^n} = 0, \quad n = 1, 2, \ldots, N. \tag{7}$$

If θ^n is at a boundary of the set, it can be determined from the condition that H^n is (locally) extremum.

For the optimization problems in which some of the final values of state variables, x_i^N, are preassigned, such as $x_a^N = c_1$, and $x_b^N = c_2$, and the objective function is specified as

$$S = \sum_{\substack{i=1 \\ i \neq a \\ i \neq b}}^{s} c_i \, x_i^N$$

the basic algorithm represented by equations (4) through (7) is still applicable, except that equation (6) is replaced by

$$z_i^N = c_i \,, \qquad i = 1, 2, \ldots, s, \quad i \neq a, b. \tag{9}$$

PROCESSES WITH MEMORY IN DECISIONS (3)

If the transformation at a stage is not only a function of the decision variable θ^n but also of θ^{n-1}, that is, the previous decision has an effect on the subsequent stage, we write

$$x^n = T^n(x^{n-1}; \theta^n; \theta^{n-1}), \quad n = 1, 2, \ldots, N, \tag{10}$$

where the initial decision vector θ^n is a r – dimensional constant vector k, that is,

$$\theta^o = k \,. \tag{11}$$

We are to choose the sequence of θ^n to maximize or minimize the objective function of the process.

To solve such a problem it is necessary to introduce a new state vector χ such that

$$\chi^n = \theta^n, \; n = 0, 1, 2, \ldots, N, \tag{12}$$

and to introduce a new decision vector to satisfy

$$w^n = \theta^n - \theta^{n-1} \; , \quad n = 1, 2, \ldots, N. \tag{13}$$

Substituting equations (12) and (13) into equation (10), we obtain

$$x^n = T^n(x^{n-1}; \; \chi^{n-1} + w^n; \; \chi^{n-1}), \quad n = 1, 2, \ldots, N. \tag{14}$$

It is obvious that the new state vector χ satisfies the performance equation.

$$\chi^n = \chi^{n-1} + w^n \; , \quad n = 1, 2, \ldots, N. \tag{15}$$

Equations (14) and (15) are in the general form of equation (1), although the dimension of the state vector is increased to $(s + r)$. Thus we obtain an enlarged system process with $(s + r)$ state variables and r decision variables. The $(s + r)$ performance equations at each stage are provided by equation (14) and (15). Equation (11) gives the initial conditions for the r new state variables.

3. CASE STUDIES

CASE 1. A PRODUCTION SCHEDULING PROBLEM--ILLUSTRATION OF A PROCESS WITH MEMORY IN DESCISION

DESCRIPTION OF THE PROBLEM:

This case deals with the types of problems where the sales forecast is known in advance and the management wishes to schedule their production so as to obtain lowest production cost.

The following data are sales forecasts, initial inventory and production. The cost function is given in a situation in which the management desires to obtain the lowest production cost.

Q^i = Sales in period i, i = 1, 2, ..., N (sales must be satisfied),

I^o = Initial inventory,

P^o = Initial production,

I^N = Inventory desired at the end of the production run.

Costs:

$C(P^n - P^{n-1})^2$ = Cost due to change in production level from n^{th} stage to (n-1)st stage,

$D(E - I^n)^2$ = Inventory cost,

where C, D, and E are constants and are greater than zero. Find the optimum production level at each period (stage) to minimize total cost.

FORMULATION AND SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

The problem is a general N-period (stage) problem and the process stream can be represented as shown in Figure 1. Let us define each period of production as a stage and let

θ^n = production at the n^{th} stage, P^n,

x_1^n = inventory at the end of n^{th} stage,

x_2^n = cost up to and including the n^{th} stage.

The material balance at each stage gives:

$$I^{n-1} + P^n = Q^n + I^n.$$

Then the performance equations are:

$$x_1^n = x_1^{n-1} + \theta^n - Q^n, \quad n = 1, 2, \ldots, N, \qquad (1)$$

$$x_1^o = I^o \quad \text{(initial inventory)} \qquad (1a)$$

$$x_1^N = I^N \quad \text{(final inventory)} \qquad (1b)$$

$$x_2^n = x_2^{n-1} + C(\theta^n - \theta^{n-1})^2 + D(E - x_1^n)^2, \qquad (2)$$

$$x_2^o = 0. \qquad (2a)$$

Substitution of equation (1) into equation (2) yields.

$$x_2^n = x_2^{n-1} + C(\theta^n - \theta^{n-1})^2 + D(E - x_1^{n-1} - \theta^n + Q^n) \qquad (3)$$

$$= T_2^n(x^{n-1}; \theta^n; \theta^{n-1}), \quad n = 1, 2, \ldots, N. \qquad (3a)$$

Equation (3) shows that the transformation at each stage is not only a function of the decision variable θ^n but also of θ^{n-1}, that is, the previous decision has an effect on the subsequent stages. This type of process is defined as a process with memory in decision (2). To solve this problem by the discrete maximum principle the following transformations are required.

Let

$$x_3^n = \theta^n , \qquad n = 1, 2, \ldots, N, \tag{4}$$

be a new state variable and

$$w^n = \theta^n - \theta^{n-1} , \qquad n = 1, 2, \ldots, N, \tag{5}$$

be a new decision variable, which satisfies

$$x_3^n = x_3^{n-1} + w^n , \qquad n = 1, 2, \ldots, N, \tag{6}$$

$$x_3^0 = P^o \text{ , the initial production.} \tag{6a}$$

The performance equations can be modified by substituting equations (4), (5), and (6) into equations (1) and (3), which gives

$$x_1^n = x_1^{n-1} + x_3^{n-1} + w^n - Q^n , \; n = 1, 2, \ldots, N, \tag{7}$$

$$x_2^n = x_2^{n-1} + C(w^n)^2 + D(E-x_1^{n-1}-x_3^{n-1}-w^n+Q^n) , \tag{8}$$

$$n = 1, 2, \ldots, N.$$

The objective function to be minimized is

$$S = \sum_{i=1}^{s} c_i \, x_i^N = x_2^N \tag{9}$$

where x_2^N is the total cost incurred and therefore:

$$c_1 = 0, \tag{9a}$$

$$c_2 = 1, \tag{9b}$$

$$c_3 = 0. \tag{9c}$$

Introducing the Hamiltonian function, H^n, and the adjoint variables, z_i^n, gives:

$$H^n = z_1^n \, (x_1^{n-1} + x_3^{n-1} + w^n - Q^n) + z_2^n \Big[x_2^{n-1} + C(w^n)^2 + D(E - x_1^{n-1} - x_3^{n-1} - w^n + Q^n)^2 \Big] + z_3^n \, (x_3^{n-1} + w^n),$$

$$n = 1, 2, \ldots, N. \tag{10}$$

Therefore the recurrence relation of the adjoint variables is

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n - z_2^n \Big[2D(E - x_1^{n-1} - x_3^{n-1} - w^n + Q^n) \Big],$$

$$n = 1, 2, \ldots, N, \tag{11}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n \, , \quad n = 1, 2, \ldots, N, \tag{11a}$$

$$z_3^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_1^n - 2Dz_2^n(E - x_1^{n-1} - x_3^{n-1} - w^n + Q^n) + z_3^n ,$$

$$n = 1, 2, \ldots, N. \tag{11b}$$

From equations (11), (11a), (11b) and from equations (9a), (9b), and (dc) we obtain

$$z_1^N \neq c_1 = 0 \quad \text{because } x_1^N \text{ is fixed,} \tag{12a}$$

$$z_2^N = c_2 = 1, \tag{12b}$$

$$z_3^N = c_3 = 0. \tag{12c}$$

Combining equations (11a) and (12b) gives

$$z_2^n = 1 , \quad n = 1, 2, \ldots, N. \tag{13}$$

We apply the necessary condition of optimality according to the maximum principle which states that the optimal choice of the decision variable will be found where

$$\frac{\partial H^n}{\partial \theta^n} = 0. \tag{14}$$

Therefore, applying the condition of equation (14) to equation (10) yields:

$$\frac{\partial H^n}{\partial w^n} = 0 = z_1^n \frac{\partial x_1^n}{\partial w^n} + z_2^n \frac{\partial x_2^n}{\partial w^n} + z_3^n \frac{\partial x_3^n}{\partial w^n} . \tag{15}$$

We take partial derivatives of equations (6), (7), and (8) with respect to w^n, and insert the respective derivatives into equation (15) which yields

$$z_1^n + z_2^n \left[2C(w^n) - 2D(E-x_1^{n-1}-x_3^{n-1}-w^n+Q^n) \right] + z_3^n = 0,$$

$$n = 1, 2, \ldots, N. \qquad (16)$$

Rearranging equation (16) results in

$$z_1^n = z_2^n \left[2D(E-x_1^{n-1}-x_3^{n-1}-w^n+Q^n) - 2C(w^n) \right] -z_3^n. \qquad (17)$$

Combining equations (11), (13), and (17) gives

$$z_3^n = 2D(E-x_1^{n-1}-x_3^{n-1}-w^n+Q^n) - 2C(w^n) + 2C(w^{n+1}) + z_3^{n+1}. \qquad (18)$$

Combining equations (11b), (17), and (18) gives

$$z^{n+1} = 2C(w^n) - 4C(w^{n+1}) - 2D(E-x_1^{n-1}-x_3^{n-1}-w^n+Q^n) ,$$

$$n = 1, 2, \ldots, N. \qquad (19)$$

Inserting equation (19) into equation (18) gives the following recurrence relation for optimal conditions.

$$2C(w^{n-1}) - 4C(w^n) - 2D(E-x_1^{n-2}-x_3^{n-2}-w^{n-1}+Q^n) + 2C(w^{n+1}) = 0.$$

This can be written as

$$(2C+2D)\ w^{n-1} - 4C(w^n) - 2D(E-x_1^{n-2}-x_3^{n-2}+Q^n) + 2C(w^{n+1}) = 0,$$

$$n = 1, 2, \ldots, N. \qquad (20)$$

The optimal sequence of w^n, $n = 1, 2, \ldots, N$, can be obtained by utilizing the recurrence equation (20) and the performance equations (6), (7), and (8). In the systematical search for w^n, $n = 1, 2, \ldots, N$, we assume that there exists an optimal sequence of w^n, $n = 1, 2, \ldots, N$, which satisfies equation (20) with permissible error.

CALCULATION PROCEDURE:

Step 1. Assume a value of w^1.

Step 2. Assume $w^2 = 1.0$.

Step 3. Calculate x_1^1 and x_3^1 from equations (6) and (7).

Step 4. Calculate x_1^2 and x_3^2 from equations (6) and (7).

Step 5. With the values of w^1 and w^2, calculate w^3 from equation (20).

Step 6. Calculate x_1^n, and x_3^n for $n = 3, 4, \ldots, N$ from equations (6) and (7), and w^n, $n = 4, 5, \ldots, N$ from equation (20).

Step 7. Compare the calculated value of x_1^N with the given value of x_1^N. If the error, ER, defined as $ER = (x_1^N)_{\text{calculated}} - (x_1^N)_{\text{given}}$ is equal to zero or the value of ER is less than the permissible error, $(ER)_{MAX}$, the optimal solution is obtained.

Step 8. However, if $ER < 0$ and $|ER| > (ER)_{MAX}$, w^2 is replaced by

$$w^2 = w^2 + 1.0$$

and the calculation is repeated from step 3. If $ER > 0$ and $|ER| > (ER)_{MAX}$, then w^1 is replaced by

$$w^1 = w^1 + 1.0$$

and the calculation is repeated from Step 2.

Step 9. If the optimal solution is obtained, x_2^N, is obtained from equation (8).

This problem is solved in an IBM 1410 computer. A flow chart of calculation procedure is given in Fig. 2, and the FORTRAN program is given in Table 1. The symbol table for the computer program is also given in Table 2.

NUMERICAL EXAMPLES:

A three stage system and a six stage system are considered. However, the computer program and the numerical method developed are for systems with an arbitrary number of stages.

THREE STAGE SYSTEM:

Data given:

I^o = initial inventory = 12,

I^N = final inventory desired = 10,

P^o = initial production = 15.

Cost functions:

$\$100\ (P^n - P^{n-1})^2$ = Cost due to change in production level,

$\$20\ (10 - I^n)^2$ = Inventory cost per period.

Requirements:

Sales forecast, Q^n, n = 1, 2, 3 periods.

$Q^1 = 30$,

$Q^2 = 10$,

$Q^3 = 40$.

The optimum solution obtained from equations (20), (6), (7), and (8) for three stage system is as follows (see Table 3a):

Difference in production level w^n:

$w^1 = 6$,

$w^2 = 5$,

$w^3 = 5$.

Inventory level at the end of each stage is:

$x_1^1 = 3$,

$x_1^2 = 19$,

$x_1^3 = 10$.

Production at each stage is:

$x_3^1 = 21$,

$x_3^2 = 26$,

$x_3^3 = 31$.

Total optimal cost obtained:

$x_2^3 = \$11,619.20$.

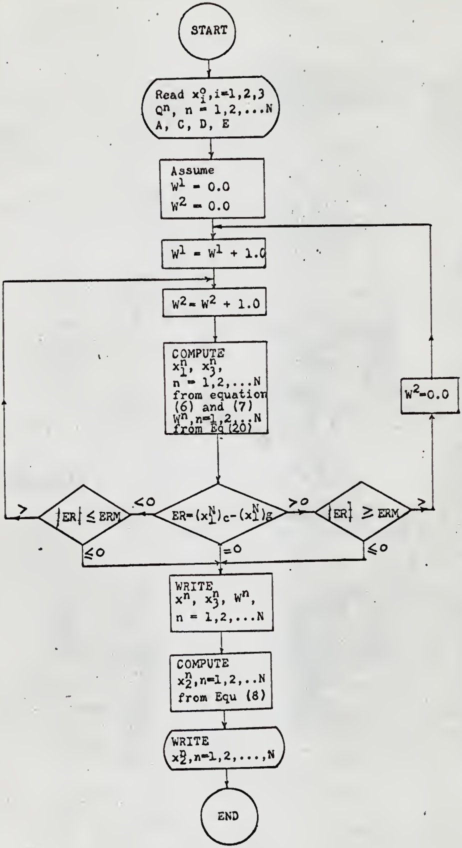

Fig. 2 Flow Chart

Table 1. SYMBOL TABLE

ITEM	PROGRAM SYMBOL
STAGE	N
x_i^n	X(I,N)
w^n	W(N)
$n = 0, 1, 2, \ldots, N$	N = 1, 2, ..., N
$(x_1^N)_c - (x_1^N)_g$	ER
MAXIMUM PERMISSIBLE ERROR	ERM
Sales Requirement	Q^i, $i = 1, 2, \ldots, N$

Table 2 Computer Program

```
      DIMENSION W(50),X(3,50),S(50),Y(50)
    5 FORMAT (I3)
   10 FORMAT (3F12.2)
   11 FORMAT (6F12.2)
   12 FORMAT (4F12.2)
   21 FORMAT (F12.4)
   52 FORMAT (1X,F8.4,4X,22HMINIMUM POSSIBLE ERROR)
   54 FORMAT (/1X,1HN,6X,6HX(1,N),6X,6HX(3,N),8X,4HW(N))
   55 FORMAT (/,I2,3F12.0)
   72 FORMAT (/4X,8HX(2,N)=$,F12.2)
      READ (1,5)N
      READ (1,10) (X(J,1),J=1,3)
      READ(1,11)(S(I),I=2,N)
      READ(1,12)A,C,D,E
      P=2.*D
      Q=4.*C
      R=2.*C+2.*D
      T=2.*C
      S(1)=0.0
      W(2)=0.0
      W(3)=0.0
   15 W(2)=W(2)+1.0
   20 W(3)=W(3)+1.0
      DO 35 I=2,N
      W(I+2)=(P*(E-X(1,I-1)-X(3,I-1)+S(I))+Q*W(I+1)-R*W(I))/T
      X(3,I)=X(3,I-1)+W(I)
   35 X(1,I)=X(1,I-1)+X(3,I-1)+W(I)-S(I)
      ER=X(1,N)-A
      WRITE(3,21)ER
      IF(ER)19,50,16
   19 IF(ER+.5)20,50,50
   16 IF(ER-.5)50,50,45
   45 W(3)=0.0
      GO TO 15
   50 WRITE(3,52)ER
      WRITE(3,54)
      WRITE(2,54)
      N2=1
      N1=2
   56 WRITE(3,55)N2,X(1,N1),X(3,N1),W(N1)
      WRITE(2,55)N2,X(1,N1),X(3,N1),W(N1)
      N2=N2+1
      N1=N1+1
      IF(N2-(N-1))56,56,76
   76 IF(N1-N)56,56,58
```

```
58 DO 60 I=2,N
   X(2,I)=X(2,I-1)+C*W(I)**2+D*(E-X(1,I-1)-X(3,I-1)-W(I)+S(I))**2
60 CONTINUE
   WRITE(3,72)X(2,N)
   WRITE(2,72)X(2,N)
   STOP
   END
```

Table 3a Results (Three Stage Process)

INPUT DATA

N=NO. OF STAGES=3

X(1,0)=12.0	Q(1)=30.0	A=10.0
X(2,0)=0.0	Q(2)=10.0	C=100.0
X(3,0)=15.0	Q(3)=40.0	D=20.0
		E=10.0

OUTPUT

N	X(1,N)	X(3,N)	W(N)
1	3.	21.	6.
2	19.	26.	5.
3	10.	31.	5.

X(2,N)=$ 11619.20

SIX STAGE SYSTEM:

Data given:

I^o = initial inventory = 12,

I^N = final inventory desired = 13,

P^o = initial production = 15.

Cost functions:

\$100 $(P^n - P^{n-1})^2$ = Cost due to change in production level,

\$20 $(10 - I^n)^2$ = Inventory cost per period.

Requirements:

Sales forecast, Q^n, n = 1, 2, ..., 6 periods.

$Q^1 = 30$,

$Q^2 = 10$,

$Q^3 = 40$,

$Q^4 = 20$,

$Q^5 = 15$,

$Q^6 = 25$.

The optimum results obtained from the solution of equation (20), (6), (7) and (8) for a six stage system is as follows (see Table 3b):

Difference in production level at each stage w^n

$w^1 = 8$,

$w^2 = 5$,

$w^3 = 3$,

$w^4 = -1$,

$w^5 = -7$,

$w^6 = -15$.

Table 3b Results (Six Stage Process)

INPUT DATA

N=NO. OF STAGES=6

X(1,0)=12.0	Q(1)=30.0	A=13.0
X(2,0)=0.0	Q(2)=10.0	C=100.0
X(3,0)=15.0	Q(3)=40.0	D=20.0
	Q(4)=20.0	E=10.0
	Q(5)=15.0	
	Q(6)=25.0	

OUTPUT

N	X(1,N)	X(3,N)	W(N)
1	5.	23.	8.
2	23.	28.	5.
3	14.	31.	3.
4	23.	29.	-1.
5	30.	22.	-7.
6	13.	7.	-15.

X(2,N)=$ 54335.32

Inventory level at the end of each stage is

$$x_1^1 = 5,$$

$$x_1^2 = 23,$$

$$x_1^3 = 14,$$

$$x_1^4 = 23,$$

$$x_1^5 = 30,$$

$$x_1^6 = 13.$$

Production at each stage is

$$x_3^1 = 23,$$

$$x_3^2 = 28,$$

$$x_3^3 = 31,$$

$$x_3^4 = 29,$$

$$x_3^5 = 22,$$

$$x_3^6 = 7.$$

Total minimum optimal cost obtained is

$$x_2^6 = \$54,335.32.$$

CASE 2. A PERSONNEL AND PRODUCTION SCHEDULING PROBLEM

DESCRIPTION OF THE PROBLEM:

This case mainly deals with the type of problems where the management knows exactly the requirements of their products in the future periods and they are interested in planning their production so as to meet the known market requirements with minimum operating cost of manufacturing (1).

It is required to plan operations in a situation in which initial conditions, costs, and market requirements are given as follows:

P^o = Initial production,

W^o = Initial work force,

I^o = Initial inventory,

I^N = Final inventory (after the production run is completed),

K = Production units per worker per period in regular time.

Costs:

$\$ \; G \; (W^n - W^{n-1})^2$ = Cost due to change in work force,

$\$ \; V \; (P^n)$ = Production cost,

$\$ \; C \; (P^n - KW^n)^2$ = Overtime cost,

$\$ \; D \; (E - I^n)^2$ = Inventory cost,

where G, V, C, D, and E are positive constants.

Requirements:

Q^i = Sales in the i^{th} period, $i = 1, 2, \ldots, N$.

Back order is permitted.

The management desires to make those plans which will result in the lowest operating cost in meeting the above requirements to solve for an optimum schedule.

FORMULATION AND SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

The problem is a general N stage problem and each period is represented as a stage

Let

$\theta_1^n = P^n - P^{n-1}$ = difference in production between the n^{th} stage (period) and previous stage (period).

$x_1^n = P^n$ = Production rate during the n^{th} stage (period),

$\theta_2^n = W^n - W^{n-1}$ = Difference in work force between the n^{th} stage (period) and the previous stage (period),

$x_2^n = W^n$ = Work force during the n^{th} stage (period),

$x_3^n = I^n$ = Inventory at the end of n^{th} stage (period),

x_4^n = Sum of the cost up to and including the n^{th} stage (period).

Therefore we can write the following performance equations.

Production rate:

$$x_1^n = x_1^{n-1} + \theta_1^n = T_1^n\ (x^{n-1};\ \theta^n)\ ,\ n = 1,\ 2,\ \ldots,\ N, \tag{1}$$

$$x_1^0 = P^o\ . \tag{1a}$$

Work force:

$$x_2^n = x_2^{n-1} + \theta_2^n = T_2^n\ (x^{n-1};\ \theta^n),\ n = 1,\ 2,\ \ldots,\ N, \tag{2}$$

$$x_2^0 = W^o\ . \tag{2a}$$

Inventory:

$$x_3^n = x_3^{n-1} + x_1^n - Q^n. \tag{3}$$

Substituting equation (1) into equation (3) yields

$$x_3^n = x_3^{n-1} + x_1^{n-1} + \theta_1^n - Q^n = T_3^n\ (x^{n-1};\ \theta^n),$$

$$n = 1,\ 2,\ \ldots,\ N, \tag{4}$$

$$x_3^0 = I^0\ , \tag{4a}$$

$$x_3^N = I^N\ . \tag{4b}$$

Costs:

$$x_4^n = x_4^{n-1} + G(\theta_2^n)^2 + V\ x_1^n + C(x_1^n - Kx_2^n)^2 + D(E - x_3^n)^2,$$

$$n = 1,\ 2,\ \ldots,\ N, \tag{5}$$

$$x_4^0 = 0\ . \tag{5a}$$

Substituting equations (1), (2), and (4) into equation (5) yields

$$x_4^n = x_4^{n-1} + G(\theta_2^n)^2 + V(x_1^{n-1}+\theta_1^n) + C(x_1^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n)^2$$

$$+ D(E-x_3^{n-1}-x_1^{n-1}-\theta_1^n+Q^n)^2$$

$$= T_4^n(x^{n-1};\ \theta^n)\ ,\ n = 1,\ 2,\ \ldots,\ N. \tag{6}$$

Objective function:

$$\text{Minimize}\quad S = \sum_{i=1}^{4} c_i\ x_i^N = x_4^N\ , \tag{7}$$

$$\text{therefore } c_i = 0,\ i = 1,\ 2,\ 3, \tag{7a}$$

$$\text{and } c_4 = 1. \tag{7b}$$

The Hamiltonian Function is

$$H^n = z_1^n(x_1^{n-1}+\theta_1^n) + z_2^n(x_2^{n-1}+\theta_2^n) + z_3^n(x_3^{n-1}+x_1^{n-1}+\theta_1^n-Q^n)$$

$$+ z_4^n\Big[x_4^{n-1}+G(\theta_2^n)^2 + V(x_1^{n-1}+\theta_1^n)$$

$$+ C(x_1^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n)^2 + D(E-x_3^{n-1}-x_1^{n-1}-\theta_1^n+Q^n)^2\Big]\ ,$$

$$n = 1,\ 2,\ \ldots,\ N\ . \tag{8}$$

The adjoint variables are:

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n + z_3^n + z_4^n \Big[V + 2C(x_1^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n)$$

$$- 2D(E-x_3^{n-1}-x_1^{n-1}-\theta_1^n+Q^n) \Big] , \ n = 1, 2, \ldots, N, \quad (9)$$

$$z_1^N = c_1 = 0, \quad (9a)$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n - 2CK\, z_4^n \,(x^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n) ,$$

$$n = 1, 2, \ldots, N, \quad (10)$$

$$z_2^N = c_2 = 0, \quad (10a)$$

$$z_3^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_3^n - 2Dz_4^n \,(E-x_3^{n-1}-x_1^{n-1}-\theta_1^n+Q^n),$$

$$n = 1, 2, \ldots, N, \quad (11)$$

$$z_3^N \neq c_3 = 0 \quad \text{(End point fixed)} , \quad (11a)$$

$$z_4^{n-1} = \frac{\partial H^n}{\partial x_4^{n-1}} = z_4^n , \ n = 1, 2, \ldots, N, \quad (12)$$

$$z_4^N = c_4 = 1. \quad (12a)$$

The combination of equations (12) and (12a) gives

$$z_4^n = 1, \quad n = 1, 2, \ldots, N. \quad (12b)$$

The necessary condition for the optimum decision variables, θ_i^n, is

$$\frac{\partial H^n}{\partial \theta_i^n} = 0 , \quad n = 1, 2, \ldots, N; \quad i = 1, 2.$$

Therefore differentiating equation (8) partially with respect to θ_1^n and θ_2^n gives

$$\frac{\partial H^n}{\partial \theta_1^n} = 0 = z_1^n + z_3^n + z_4^n \Big[V+2C(x_1^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n)$$

$$- 2D(E-x_3^{n-1}-x_1^{n-1}-\theta_1^n+Q^n) \Big] , \tag{13}$$

$$\frac{\partial H^n}{\partial \theta_2^n} = 0 = z_2^n + 2z_4^n G(\theta_2^n) - 2z_4^n CK(x_1^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n) \quad . \tag{14}$$

Combination of equation (10), (12b) and (14) gives

$$\theta_1^n = \frac{G}{CK}(\theta_2^n) - \frac{G}{CK}(\theta_2^{n+1}) - (x_1^{n-1}-Kx_2^{n-1}-K\theta_2^n),$$

$$n = 1, 2, \ldots, N. \tag{15}$$

Equation (15) gives one of the optimality conditions for the multistage process under consideration. Another recurrence relation for other optimality conditions can be found by combining equations (9), (11), and (13), and substituting equation (15) into the resulting equation yielding:

$$2DE - 2Dx_3^{n-1} - 2DKx_2^{n-1} + 2DQ^n - \frac{2D(G/C) + 2DK^2 + 2C(G/C)}{K} \theta_2^n$$

$$+ \frac{2C(G/C) + 2C(G/C) + 2D(G/C)}{K} \theta_2^{n+1} - \frac{2C(G/C)}{K} \theta_2^{n+2} = 0,$$

$$n = 1, 2, \ldots, N. \tag{16}$$

thus the optimization problem can be solved by the following procedure utilizing the optimality conditions given by equations (15) and (16), together with the set of performance equations (1) through (5). In the systematical search for θ_2^n, n = 1, 2, ..., N we assume that there exists an optimal sequence which satisfies equations (15) and (16) with permissible error.

CALCULATION PROCEDURE:

Step 1. Assume $\theta_2^1 = 0$ and $\theta_2^2 = 0$.

Step 2. Compute θ_2^n, θ_1^n, x_1^n, x_2^n, x_3^n, for n = 1, 2, ..., N from equations (16), (15), (1), (2), and (4) respectively.

Step 3. Compare x_3^N calculated in Step 2 with x_3^N given.

$$ER = (x_3^N)_c - (x_3^N)_g$$

ER will be in either one of the following situations (a) equal to zero (b) greater than zero (c) less than zero. If it is (a) then we reached the optimal stage, go to Step 11, if it is (b) go to Step 8, if it is (c) go to Step 4.

Step 4. Decrease the value of θ_2^1 by 10 let ER calculated in Step 3 be equal to ER 1, and go to Step 2. Compute ER again, then go to Step 5.

Step 5. Compare $(|ER1| - |ER|)$.
If this is greater than zero go to Step 4 until

ER is greater than or equal to zero, then go to Step 7. If ($|ER1| - |ER|$) is less than zero, go to Step 6.

Step 6. Increase the value of θ_2^1 by 10, take ER1 = 0.0, and go to Step 2 until ER is greater than or equal to zero, then go to Step 8.

Step 7. Increase the value of θ_2^1 by 1 and go to Step 2 until ER is less than or equal to zero, then go to Step 9.

Step 8. Decrease the value of θ_2^1 by 1 and go to Step 2 until ER is less than or equal to zero. Then go to Step 10.

Step 9. Decrease the value of θ_2^1 by 0.1 and go to Step 2 until ER has achieved minimum possible value, then go to Step 11.

Step 10. Increase the value of θ_2^1 by 0.1 and go to Step 2 until ER has achieved minimum possible value, then go to Step 11.

Step 11. Solution has reached optimum stage. Compute x_4^n, $n = 1, 2, \ldots, N$.

Step 12. Increase the value of θ_2^2 by 10 and reinitialize $\theta_2^1 = 0$, then go to Step 2.

Step 13. Compare the values of x_4^N computed before with the value of x_4^N computed after Step 11.

$$ER2 = (x_4^N)_{old} - (x_4^N)_{new}$$

ER2 will be in either one of the following situations. (a) greater than zero (b) less than or equal to zero. If it is (a) then go to Step 2, if it is (b) then go to Step 14.

Step 14. Decrease the value of θ_2^2 by 50 and take $\theta_2^1 = 0$, go to Step 2 until ER2 is greater than zero, then go to Step 15.

Step 15. Decrease the value of θ_2^2 by 1 and go to Step 2, until ER2 again becomes less than zero. Then go to Step 16.

Step 16. Stop the iterative procedure.

Step 17. Plot the curve between x_4^N and θ_2^2 with various values of x_4^N and θ_2^2 from the above computations. The one which gives minimum cost will be the optimum value.

A flow diagram, a symbol table and a computer program for IBM 1410 is given in Fig. 3, Tables 4 and 5, respectively.

NUMERICAL EXAMPLES

A three stage and a five stage system are considered. However, the computer program and the numerical method developed are for a system with an arbitrary number of stages.

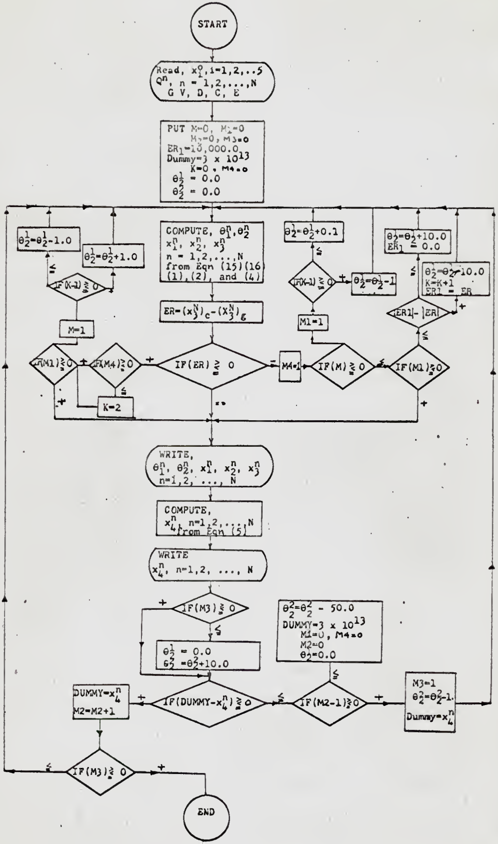

Fig. 3. Flow Chart

Table 4 SYMBOL TABLE

ITEM	PROGRAM SYMBOL
STAGE	N
$n = 0, 1, 2, \ldots, N$	$N = 1, 2, \ldots, N$
x_i^n	X(I,N)
Sales requirement	S(I)
$(x_3^N)_c - (x_3^N)_g$	ER
x_3^N	Ax

Table 5 Computer Program

```
      DIMENSION THETA(2,50),X(4,50),S(50),Z(50),H(50),Y(50)
   10 FORMAT (4F12.2)
   12 FORMAT (6F12.2)
   25 FORMAT (6F10.2)
   33 FORMAT(1X,F14.4)
   34 FORMAT (1X,1HN,1X,10HTHETA(1,N),1X,10HTHETA(2,N))
   36 FORMAT (1X,I2,2F10.0)
   37 FORMAT(1X,F12.4,5X,22HMINIMUM POSSIBLE ERROR)
   42 FORMAT (1X,1HN,8X,6HX(1,N),7X,6HX(2,N),7X,6HX(3,N))
   55 FORMAT (1X,I2,1X,3F12.0)
   70 FORMAT (1X,8HX(4,N)=$,E20.8)
      N=6
      READ(1,10)(X(J,1),J=1,4)
      READ(1,12)(S(I),I=2,N)
      READ(1,25)G,V,C,D,E,O
      M=0
      M1=0
      M2=0
      M3=0
      ER1=10000.0
      ER=0.0
      K=0
      AX=300.0
      DUMMY=30000000000000.0
      A=G/(C*O)
      B=G/(C*O)
      P=(2.*D*E*O)/(2.*G)
      Q=(2.*D*O)/(2.*G)
      R=(2.*D*O)/(2.*G)
      F=(2.*D*(O)**2)/(2.*G)
      T=(2.*D*(G/C)+2.*D*(O)**2+2.*C*(G/C))/(2.*G)
      U=(2.*C*(G/C)+2.*D*(G/C)+2.*C*(G/C))/(2.*G)
      THETA(2,2)=0.0
      THETA(2,3)=0.0
      GO TO 38
   30 THETA(2,2)=THETA(2,2)+10.
      ER1=0.0
      GO TO 38
   39 IF(ABS(ER1)-ABS(ER))30,30,46
   46 THETA(2,2)=THETA(2,2)-10.0
      ER1=ER
      K=K+1
   38 DO 40 I=2,N
      Z(I)=X(1,I-1)-O*(X(2,I-1))-O*THETA(2,I)
      THETA(1,I)=A*THETA(2,I)-B*THETA(2,I+1)-Z(I)
      H(I)=U*THETA(2,I+1)
      THETA(2,I+2)=P-Q*X(3,I-1)+R*S(I)-F*X(2,I-1)-T*THETA(2,I)+H(I)
      X(1,I)=X(1,I-1)+THETA(1,I)
      X(2,I)=X(2,I-1)+THETA(2,I)
      X(3,I)=X(3,I-1)+X(1,I-1)+THETA(1,I)-S(I)
```

```
 40 CONTINUE
    ER=X(3,N)-AX
    WRITE(3,33)ER
    IF(ER)100,45,32
 32 IF(M4)41,41,43
 41 K=2
 43 IF(M1)500,500,45
500 M=1
    IF(K-1)501,501,502
501 THETA(2,2)=THETA(2,2)-1.0
    GO TO 38
502 THETA(2,2)=THETA(2,2)+1.0
    GO TO 38
100 M4=1
    IF(M)300,300,200
300 IF(M1)39,39,45
200 M1=1
    IF(K-1)201,201,202
201 THETA(2,2)=THETA(2,2)+0.1
    GO TO 38
202 THETA(2,2)=THETA(2,2)-0.1
    GO TO 38
 45 WRITE(3,37)ER
    N2=1
    N1=2
    WRITE(3,42)
    WRITE(2,42)
 50 WRITE(3,55)N2,X(1,N1),X(2,N1),X(3,N1)
    WRITE(2,55)N2,X(1,N1),X(2,N1),X(3,N1)
    N2=N2+1
    N1=N1+1
    IF(N2-(N-1))50,50,76
 76 IF(N1-N)50,50,56
 56 WRITE(3,34)
    WRITE(2,34)
    N3=1
    N4=2
 58 WRITE(3,36)N3,THETA(1,N4),THETA(2,N4)
    WRITE(2,36)N3,THETA(1,N4),THETA(2,N4)
    N3=N3+1
    N4=N4+1
    IF(N3-(N-1))58,58,59
 59 IF(N4-N)58,58,61
 61 DO 65 I=2,N
    Y(I)=C*(X(1,I)-O*X(2,I))**2+D*(E-X(3,I))**2
    X(4,I)=X(4,I-1)+G*THETA(2,I)**2+V*X(1,I)+Y(I)
 65 CONTINUE
    WRITE(3,70)X(4,N)
    WRITE(2,70)X(4,N)
```

```
      M=0
      M1=0
      M4=0
      ER1=100000.0
      K=0
      IF(M3)81,81,82
   81 THETA(2,3)=THETA(2,3)+10.0
      THETA(2,2)=0.0
   82 IF(DUMMY-X(4,N))75,75,80
   80 DUMMY=X(4,N)
      M2=M2+1
      IF(M3)38,38,990
   75 IF(M2-1)700,700,800
  700 THETA(2,3)=THETA(2,3)-50.0
      DUMMY=30000000000000.0
      M2=0
      THETA(2,2)=0.0
      GO TO 38
  800 THETA(2,3)=THETA(2,3)-1.0
      M3=1
      DUMMY=X(4,N)
      GO TO 38
  990 STOP
      END
```

THREE STAGE SYSTEM

Data given:

P^o = initial production = x_1^0 = 2000,

W^o = initial work force = x_2^0 = 600,

I^o = initial inventory = x_3^0 = 300,

I^N = final inventory desired = x_3^3 = 300.

K = production units per worker per period in regular time = 3.

Cost:

Cost due to change in workforce = \$200 $(W^n - W^{n-1})^2$,

Production cost = \$50 (P^n),

Overtime cost = \$25 $(P^n - KW^n)^2$,

Inventory cost = \$20 $(500 - I^n)^2$.

Requirements:

Sales force cost Q^n, n = 1, 2, 3, is

Q^1 = 3000,

Q^2 = 1800,

Q^3 = 2400.

The optimum solution obtained from equations (15), (16), (1), (2), and (4) for a three stage system is as follows (see Table 5a)

Production at each stage is:

x_1^1 = 2686,

Table 6a Three Stage System

INPUT DATA

N=NO. OF STAGES

X(1,0)=2000	Q(1)=3000	AX=300
X(2,0)=600	Q(2)=1800	G=200
X(3,0)=300	Q(3)=2400	V=50
X(4,0)=0		C=25
		D=20
		E=500

OUTPUT

N	X(1,N)	X(2,N)	X(3,N)
1	2686.	756.	-13.
2	2276.	756.	463.
3	2239.	753.	302.

N	THETA(1,N)	THETA(2,N)
1	686.	156.
2	-410.	.
3	-37.	-2.

X(4,N)=$ 15703839.00

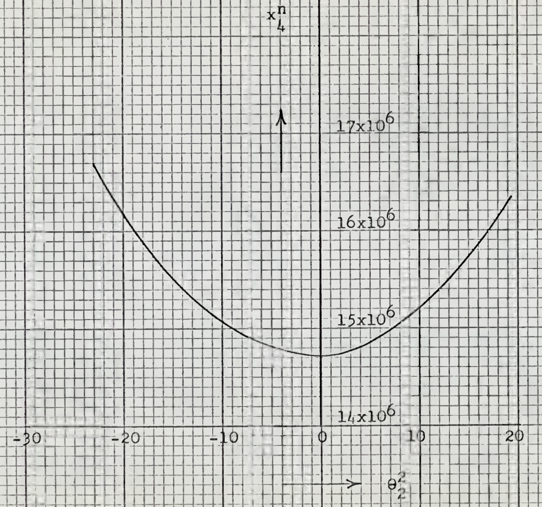

Fig. 4 Curve between Cost and Change of Labor Force

$$x_1^2 = 2276,$$
$$x_1^3 = 2239.$$

Work force:

$$x_2^1 = 756,$$
$$x_2^2 = 756,$$
$$x_2^3 = 753.$$

Final inventory level = $x_3^3 = 300$.

The total optimal cost obtained, which is the minimum point in Fig. 4a between x_4^n and θ_2^2, is:

$$x_4^3 = \$\ 15{,}703{,}839.00.$$

Simulation shows that this is the optimal solution.

FIVE STAGE SYSTEM

In this case except for the sales forecase the rest of the data is the same as that of the three stage system.

Sales requirements:

Q^n, $n = 1, 2, 3, 4, 5$, is

$$Q^1 = 3000,$$
$$Q^2 = 1800,$$
$$Q^3 = 2400,$$
$$Q^4 = 2000,$$
$$Q^5 = 2400.$$

The optimum solution obtained from equations (15), (16), (1) (2), and (4) for a five stage system is as follows (see Table 6b).

Table 6b Five Stage System

INPUT DATA

N=NO. OF STAGES

X(1,0)=2000	Q(1)=3000	AX=300
X(2,0)=600	Q(2)=1800	G=200
X(3,0)=300	Q(3)=2400	V=50
X(4,0)=0	Q(4)=2000	C=25
	Q(5)=2400	D=20
		E=500

OUTPUT

N	X(1,N)	X(2,N)	X(3,N)
1	2045.	643.	-654.
2	1122.	643.	-1331.
3	566.	946.	-3165.
4	1098.	2101.	-4066.
5	6766.	5208.	300.

N	THETA(1,N)	THETA(2,N)
1	45.	43.
2	-923.	.
3	-556.	302.
4	532.	1155.
5	5667.	3106.

X(4,N)=$ .57816230E 10

Production at each stage

$$x_1^1 = 2045,$$
$$x_1^2 = 1122,$$
$$x_1^3 = 566,$$
$$x_1^4 = 1098,$$
$$x_1^5 = 6766.$$

Work force used at each stage

$$x_2^1 = 643,$$
$$x_2^2 = 643,$$
$$x_2^3 = 946,$$
$$x_2^4 = 2101,$$
$$x_2^5 = 5208.$$

Final inventory level = $x_3^5 = 300$.

The total optimum cost obtained, which is the minimum point in Fig. 4b between x_4^5 and θ_2^2, is:

$$x_4^5 = \$\ 578,162,300.00.$$

CASE 3. APPLICATION OF THE DISCRETE MAXIMUM PRINCIPLE TO LABOR ASSIGNMENT AS A DYNAMIC CONTROL PROBLEM WITH NON-LINEAR COST FUNCTION

In this case a problem of labor assignment in a labor and machine limited production system is formulated as a dynamic control problem. The criterion function employed here is to minimize the total in-process inventory cost over a given time span. This problem with linear cost function was first solved to obtain necessary and sufficient conditions for the optimal control by the continuous maximum principle by Nelson (8). Bantwal (2) solved the problem by the discrete maximum principle. In this case, an attempt has been made to consider the problem with a non-linear cost function by employing a discrete version of the maximum principle.

The problem can be stated as follows:

Consider a production system consisting of L laborers and m machine centers. Each machine center $i = 1, 2, \ldots, m$ consists of f_i identical machines. We assume,

$$L < \sum_{i=1}^{m} f_i$$

so that labor is a limiting resource.

Let

λ = rate of arrival of work units to the machine center in work units per period,

μ_i = service rate in work units per period for each machine in machine center i when there is a laborer assigned to the machine, $i = 1, 2, \ldots, m$,

x_i^n = queue length at machine center i at the n^{th} period measured in work units, $i = 1, 2, \ldots, m$,

K_i = inventory cost per work unit per period at machine center i, $i = 1, 2, \ldots, m$.

The following assumptions are made. A job lot is a block of successively arriving work characterized by identical processing requirements. Each job lot requires processing at a completely ordered sequence of machine centers. Both the job routings and service time requirements are known in advance. The work force is completely homogenous and flexible; i.e., every laborer is equally efficient at any given machine center. Only one laborer can work on a machine at one time.

Work is processed at each machine center at discrete time intervals. The service rate of the machine center at any period is proportional to the number of laborers assigned to the machine center at that period. The queue discipline is arbitrary except that only one job lot can be processed in any machine center at one time. The portion of a job lot that has been processed instantaneously enters the appropriate queue for its succeeding operation.

The problem is to find a labor assignment procedure that minimizes total in-process inventory costs over the n time periods. The cost of carrying the inventory over a period is the inventory cost times the square of the queue length at the machine center i at the n^{th} period, $i = 1, 2, \ldots, m$. Denote the system state vector of queue length in the n^{th} period by

$$x = (x_1^n, x_2^n, \ldots, x_m^n).$$

Introducing a decision vector $\theta = (\theta_1^n, \theta_2^n, \ldots, \theta_m^n)$ where θ_i^n is the number of laborers assigned to machine center i in the n^{th} period. Considering the decision vector 0 which satisfies the following constraints (a) through (e) belongs to the set of admissible vectors.

a) If $\sum_{i=1}^{m} \theta_i^n < L$, then there cannot exist i such that $\theta_i^n < f_i$ and $x_i^n > 0$ for $0 < n < N$, where N is total number of periods under consideration.

b) $\theta_i^n = 0$ whenever $x_i^n = 0$, for $i = 1, 2, \ldots, m$, and $0 < n < N$.

c) θ_i^n = an integer, for $i = 1, 2, \ldots, m$ and $0 < n < N$.

d) $0 \leqslant \theta \leqslant f_i$, for $i = 1, 2, \ldots, m$ and $0 < n < N$.

e) $\sum_{i=1}^{m} \theta_i^n \leqslant L$, for $0 < n < N$.

The meaning of constraint (a) is that as many as possible of the laborers will be used at any given period. This is necessary to reflect the principle goal of producing finished products for income. Constraint (b) states that laborers are to be assigned only to machine centers that have work to be performed at any given period. Constraint (c) gives an indication of the indivisibility of a single laborer. Constraint (d) signifies the limitations of the machine centers to absorb labor productively. Constraint (e) assures that the total size of the labor force is not to exceed the given number.

As stated before the cost of carrying the inventory over a period is the inventory cost times the square of the queue length. This cost function follows the curve shown in Fig. 5, which shows that in-process inventory cost in this case is less (when the queue length is small) than that in the case of linear relation between queue length and the function of in-process inventory cost. But when the queue length exceeds a particular value the non-linear inventory cost function curve increases rapidly.

Therefore the objective of problem is to minimize

$$\sum_{n=1}^{N} \sum_{i=1}^{m} K_i (x_i^n)^2 .$$

AN ALGORITHM BASED ON THE DISCRETE MAXIMUM PRINCIPLE

The performance equations are given by

$$x_i^n = x_i^{n-1} + p_{oi}^n \lambda + \sum_{\substack{j=1 \\ j \neq i}}^{m} p_{ji}^{n-1} \mu_j \, \theta_j^{n-1} - \mu_i \, \theta_i^n, \; i = 1, 2, \ldots, m, \tag{1}$$

$$x_i^o = \alpha ,$$

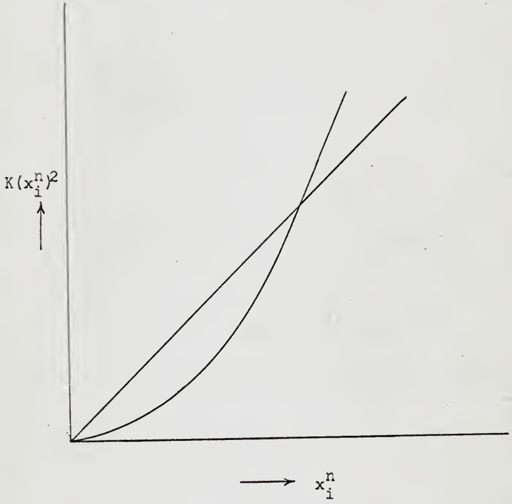

Fig. 5. NON-LINEAR COST FUNCTION

where

P_{ij}^{n} = represents the transition of work units from machine center i to j in the n^{th} period. This is equal to one if work is transferred and zero otherwise.

Similarly

P_{oi}^{n} = represents the transition of work units from outside to machine center i in the n^{th} period. This is equal to one if work is transferred and zero otherwise.

The second, third, and fourth terms on the right handside of equation (1) represents changes in queue length caused by work units arriving from outside the system, work units arriving from other machine centers, and work units completed and departing for subsequent processing. We introduce a new state variable x_{m+1}^{n} to represent cost, i.e.,

$$x_{m+1}^{n} = x_{m+1}^{n-1} + \sum_{i=1}^{m} K_i (x_i^n)^2 \qquad (2)$$

where x_{m+1}^{n} is the total cost up to and including the n^{th} stage and $\sum_{i=1}^{m} K_i (x_i^n)^2$ is the cost incurred at the n^{th} stage.

The objective function to be minimized is

$$S = \sum_{i=1}^{m} c_i \, x_i^N + c_{m+1} \, x_{m+1}^N = x_{m+1}^N , \qquad (3)$$

$$c_i = 0, \quad i = 1, 2, \ldots, m, \tag{4}$$

$$c_{m+1} = 1. \tag{4a}$$

The Hamiltonian function and (m+1) dimensional adjoint variables which satisfy the following relation are

$$H^n = \sum_{i=1}^{m} z_i^n \, x_i^n + z_{m+1}^n \, x_{m+1}^n$$

$$= \sum_{i=1}^{m} z_i^n (x_1^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j \neq i}}^{m} P_{ji}^{n-1} \mu_j \, \theta_j^{n-1} - \mu_i \, \theta_i^n) +$$

$$z_{m+1}^n (x_{m+1}^{n-1} + \sum_{i=1}^{m} K_i (x_i^n)^2 \tag{5}$$

Substituting the value of x_i^n from equation (1) in equation (5) yields

$$H^n = \sum_{i=1}^{m} z_i^n (x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j \neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1} - \mu_i \theta_i^n) + z_{m+1}^n \Big[\, x_{m+1}^{n-1}$$

$$+ \sum_{i=1}^{m} K_i (x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j \neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1} - \; {}_i\theta_i^n)^2 \Big] \; , \tag{6}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}} = z_i^n + z_{m+1}^n (2K_i (x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j \neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1}$$

$$- \mu_i \theta_i^n)), \; n = 1, 2, \ldots, N; \; i = 1, 2, \ldots, m, \tag{7}$$

$$z_i^N = 0 \; , \quad i = 1, 2, \ldots, m,$$

$$z_{m+1}^{n-1} = \frac{\partial H^n}{\partial x_{m+1}^{n-1}} = z_{m+1}^n, \quad n = 1, 2, \ldots, N, \tag{8}$$

$$z_{m+1}^N = c_{m+1} = 1. \tag{8a}$$

Combining equations (8) and (8a) gives

$$z_{m+1}^n = 1, \quad n = 1, 2, \ldots, N. \tag{8b}$$

Substituting equation (8b) in equation (6) and rearranging the terms yields

$$\begin{aligned} H^n = {} & \sum_{i=1}^m z_i^n x_i^{n-1} + \sum_{i=1}^m z_i^n P_{oi}^n \lambda + \sum_{i=1}^m z_i^n \sum_{\substack{j=1 \\ j\neq i}}^m P_{ji}^{n-1} \mu_j \theta_j^{n-1} - \sum_{i=1}^m z_i^n \mu_i \theta_i^n \\ & + x_{m+1}^{n-1} + \sum_{i=1}^m K_i (x_i^{n-1})^2 + \sum_{i=1}^m K_i (P_{oi}^n \lambda)^2 + (\sum_{\substack{j=1 \\ j\neq i}}^m P_{ji}^{n-1} \mu_j \theta_j^{n-1})^2 \\ & + \sum_{i=1}^m K_i (\mu_i \theta_i^n)^2 + \sum_{i=1}^m 2K_i (P_{oi}^n \lambda\, x_i^{n-1}) + \sum_{i=1}^m 2K_i x_i^{n-1} \sum_{\substack{j=1 \\ j\neq i}}^m \\ & \cdot P_{ji}^{n-1} \mu_j \theta_j^{n-1} + \sum_{i=1}^m 2K_i P_{oi}^n \lambda \sum_{\substack{j=1 \\ j\neq i}}^m P_{ji}^{n-1} \mu_j \theta_j^{n-1} - \sum_{i=1}^m 2K_i \\ & \cdot (x_i^{n-1} \mu_i \theta_i^n) - \sum_{i=1}^m 2K_i (P_{oi}^n \lambda\, \mu_i \theta_i^n) - \sum_{i=1}^m 2K_i\, \mu_i \theta_i^n \sum_{\substack{j=1 \\ j\neq i}}^m \\ & \cdot P_{ji}^{n-1} \mu_j \theta_j^{n-1}) \,. \end{aligned} \tag{9}$$

S is minimum when H^n is a minimum. Equation (9) shows that H^n is non-linear with respect to θ_i^n.

In equation (9) z_i^n, x_i^{n-1}, x_{m+1}^{n-1}, P_{oi}^n, P_{ji}^{n-1}, K_i, λ, μ_j and μ_i are constants. Therefore, the variable portion of Hamiltonian H^n, H_V^n can be written as

$$H_V^n = -\sum_{i=1}^{m} z_i^n \mu_i \theta_i^n - \sum_{i=1}^{m} 2K_i (x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j\neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1}) \mu_i \theta_i^n$$

$$+ \sum_{i=1}^{m} K_i (\mu_i)^2 (\theta_i^n)^2. \tag{10}$$

H_V^n is non-linear with respect to the decision variable θ_i^n. The necessary condition for the optimality is

$$\frac{\partial H_V^n}{\partial \theta_i^n} = - z_i^n \mu_i - 2K_i (x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j\neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1}) \mu_i$$

$$+ 2K_i \ (\mu_i)^2 \ \theta_i^n = 0. \tag{11}$$

This can be written as

$$z_i^n \mu_i = - 2K_i (x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j\neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1}) \mu_i + 2K_i (\mu_i)^2 \theta_i^n. \tag{12}$$

Dividing both sides by μ_i gives

$$z_i^n = - 2K_i (x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j\neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1}) + 2K_i \mu_i \theta_i^n . \tag{13}$$

Substituting the value of z_i^n given by equation (13) into equation (7) yields

$$z_i^{n-1} = 2K_i(x_i^{n-1}+P_{oi}^n\lambda + \sum_{\substack{j=1\\j\neq i}}^{m} P_{ji}^{n-1}\mu_j\theta_j^{n-1}) - 2K_i\,\mu_i\theta_i^n$$

$$- 2K_i(x_i^{n-1} + P_{oi}^n\lambda + \sum_{\substack{j=1\\j\neq i}}^{m} P_{ji}^{n-1}\mu_j\theta_j^{n-1}) + 2K_i\mu_i\theta_i^n$$

which gives

$$z_i^{n-1} = 0, \qquad n = 1, 2, \ldots, N. \tag{14}$$

Substituting this value of $z_i^n = 0$ from equation (14) into equation (13) yields

$$2K_i\mu_i\theta_i^n = 2K_i(x_i^{n-1}+P_{oi}^n\lambda + \sum_{\substack{j=1\\j\neq i}}^{m} P_{ji}^{n-1}\mu_j\,\theta_j^{n-1}) .$$

This can be written as

$$\theta_i^n = \frac{1}{\mu_i}\left[x_i^{n-1} + P_{oi}^n\lambda + \sum_{\substack{j=1\\j\neq i}}^{m} P_{ji}^{n-1}\mu_j\theta_j^{n-1}\right] , \tag{15}$$

$$i = 1, 2, \ldots, m;\ n = 1, 2, \ldots, N.$$

Equation (15) gives the relation from which the optimum value of decision variable θ_i^n can be computed. It satisfies the constraints given (a) through (e). But there is one danger, namely if the service rate of the next machine in line is low, this decision will not constitute optimal policy. In order to avoid this undesirable solution and also to satisfy the constraint (e) on the decision variable, a time dependent priority π_i^n is calculated. Let

μ_i^* = service rate in work units per period for the machine center to which work being processed at machine center i at the n^{th} period, that is the service rate of machine center in line.

K_i^* = inventory charges per unit period for the next machine center in line.

f_i^* = number of machines in the next machine center in line.

Then, the time dependent priority, π_i^n, is given by

$$\pi_i^n = (f_i^* \mu_i^* K_i - f_i \mu_i K_i^*), \quad i = 1, 2, \ldots, m. \qquad (16)$$

Optimum policy is to allot in any period n, as many of L laborers as given by equation (15) for which $x_i^n > 0$ in order of decreasing values of π_i^n.

NUMERICAL EXAMPLE:

A four machine center system is considered, so that m = 4. Work pieces are processed first on machine center 1, then on 2, 3 and finally on machine center 4. The constants assigned for this problem are given in Table 7.

Table 7. DATA

λ = work piece arrival rate = 60 units per hour

Machine Center m	Number Of identical machines r	Service Rate in units per hour, μ	Inventory Cost in $ per Unit per hour K
1	6	10	$ 1
2	15	5	0.40
3	6	12	0.60
4	5	15	0.75
5	1	60	0.70
(Inspection Station)			

The maximum number of laborers available is 25. Each laborer is assumed to be equally competent to work in any machine center. We also assume that there is no initial in-process inventory. The cost of carrying inventory over a period is the inventory cost for that period times the square of the queue length of units waiting to be processed at a given period. Determine how many laborers should be assigned to each machine center every hour, to minimize in-process inventory cost for a time span of 8 hours.

To solve this problem of four machine center and one inspection station, we can assume the inspection station to be the fifth machine center, because this will simplify the calculation procedure.

First calculate the time dependent priority for each machine center. In this case it may be noted that

$$K_i^* = K_{i+1} \text{ and } \mu_i^* = \mu_{i+1}\ , \quad i = 1, 2, 3, 4.$$

Hence we can write the π_i^n expression in equation (16), that is,

$$\pi_i^n = (f_i^* \mu_i^* K_i - f_i \mu_i K_i^*).$$

Therefore

$$\pi_1^n = (15 \times 5 \times 1 - 6 \times 10 \times 0.40) = 51,$$

$$\pi_2^n = (6 \times 12 \times 0.40 - 15 \times 5 \times 0.60) = -16,$$

$$\pi_3^n = (5 \times 15 \times 0.60 - 6 \times 72 \times 0.75) = -9,$$

$$\pi_4^n = (1 \times 60 \times 0.75 - 5 \times 15 \times 0.70) = -7.$$

Therefore, if there is a queue length of work pieces at all four machine centers, each machine center has a priority of allocation of labor force, i.e., machine center 1 has a priority for maximum labor assigned to it over the rest of the machine centers. Similarily the next priority goes to machine center 4 and so on. This can be written as:

$$\pi_1^n > \pi_4^n > \pi_3^n > \pi_2^n .$$

The assignments can be computed by using equations (15) and equation (1), that is,

$$\theta_i^n = \frac{1}{\mu_i} (x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j \neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1} , \quad (15)$$

$$x_i^n = x_i^{n-1} + P_{oi}^n \lambda + \sum_{\substack{j=1 \\ j \neq i}}^{m} P_{ji}^{n-1} \mu_j \theta_j^{n-1} - \mu_i \theta_i^n . \quad (1)$$

Hence for n = 1:

$$i = 1, \ \theta_1^1 = \frac{1}{10} (0 + 60 + 0) = 6,$$

$$i = 2, \ \theta_2^1 = \frac{1}{5} (0 + 0 + 0) = 0,$$

$$i = 3, \ \theta_3^1 = \frac{1}{5} (0 + 0 + 0) = 0,$$

$$i = 4, \ \theta_4^1 = \frac{1}{15} (0 + 0 + 0) = 0,$$

$$x_1^1 = (0 + 60 + 0 - 6 \times 10) = 0,$$
$$x_2^1 = (0 + 0 + 0 - 0) = 0,$$
$$x_3^1 = (0 + 0 + 0 - 0) = 0,$$
$$x_4^1 = (0 + 0 + 0 - 0) = 0,$$
$$x_5^1 = (0 + 0 + 0 - 0) = 0 \quad \text{(inspection station)}.$$

For n = 2:

$$i = 1, \ \theta_1^2 = \frac{1}{10} (0 + 60 + 0) = 6,$$

$$i = 2, \ \theta_2^2 = \frac{1}{5} (0 + 0 + 1 \times 10 \times 6) = 12,$$

$$i = 3,\ \theta_3^2 = \frac{1}{12}\ (0 + 0 + 0) = 0,$$

$$i = 4,\ \theta_4^2 = \frac{1}{15}\ (0 + 0 + 0) = 0,$$

$$x_1^2 = (0 + 60 + 0 - 6 \times 10) = 0,$$
$$x_2^2 = (0 + 0 + 1 \times 10 \times 6 - 5 \times 12) = 0,$$
$$x_3^2 = (0 + 0 + 0 - 0) = 0,$$
$$x_4^2 = (0 + 0 + 0 - 0) = 0,$$
$$x_5^2 = (0 + 0 + 0 - 0) = 0.$$

For $n = 3$:

$$i = 1,\ \theta_1^3 = \frac{1}{10}\ (0 + 60 + 0) = 6,$$

$$i = 2,\ \theta_2^3 = \frac{1}{5}\ (0 + 0 + 1 \times 10 \times 6) = 12,$$

$$i = 3,\ \theta_3^3 = \frac{1}{12}\ (0 + 0 + 1 \times 12 \times 5) = 5,$$

$$i = 4,\ \theta_4^3 = \frac{1}{15}\ (0 + 0 + 0) = 0,$$

$$x_1^3 = (0 + 60 + 0 - 6 \times 10) = 0,$$
$$x_2^3 = (0 + 0 + 1 \times 10 \times 6 - 5 \times 12) = 0,$$
$$x_3^3 = (0 + 0 + 1 \times 12 \times 5 - 12 \times 5) = 0,$$
$$x_4^3 = (0 + 0 + 0 - 0) = 0,$$
$$x_5^3 = (0 + 0 + 0 - 0) = 0.$$

For $n = 4$:

$$i = 1,\ \theta_1^4 = \frac{1}{10}\ (0 + 60 + 0) = 6,$$

$$i = 2,\ \theta_2^4 = \frac{1}{5}\ (0 + 0 + 1 \times 5 \times 12) = 12,$$

$$i = 3,\ \theta_3^4 = \frac{1}{12}\,(0 + 0 + 1 \times 12 \times 5) = 5,$$

$$i = 4,\ \theta_4^4 = \frac{1}{15}\,(0 + 0 + 1 \times 12 \times 5) = 4.$$

At $n = 4$, the summation of $\sum_{i=1}^{4} \theta_i^4 = 27$, and the maximum force available is only 25. Since machine center 2 has the lowest priority, to make the sum of labor assigned at $n = 4$ to be 25, we allot 10 laborers at machine center 2. That is, $\theta_2^4 = 10$. Then we obtain

$$x_1^4 = (0 + 60 + 0 - 6 \times 10) = 0,$$
$$x_2^4 = (0 + 0 + 1 \times 10 \times 6 - 10 \times 5) = 10,$$
$$x_3^4 = (0 + 0 + 1 \times 5 \times 12 - 12 \times 5) = 0,$$
$$x_4^4 = (0 + 0 + 1 \times 12 \times 5 - 15 \times 4) = 0,$$
$$x_5^4 = (0 + 0 + 0 - 0) = 0.$$

For $n = 5$:

$$i = 1,\ \theta_1^5 = \frac{1}{10}\,(\ 0 + 60 + 0) = 6,$$

$$i = 2,\ \theta_2^5 = \frac{1}{5}\,(10 + 0 + 1 \times 10 \times 6) = 14,$$

$$i = 3,\ \theta_3^5 = \frac{1}{12}\,(0 + 0 + 1 \times 10 \times 5) = 4,$$

$$i = 4,\ \theta_4^5 = \frac{1}{15}\,(0 + 0 + 1 \times 12 \times 5) = 4.$$

Here, as the sum of laborers assigned this hour also exceeds 25, we will allot only 11 laborers on machine center 2, which has the lowest priority. Hence $\theta_2^5 = 11$. Then we obtain

$$x_1^5 = (0 + 60 + 0 - 6 \times 10) = 0,$$
$$x_2^5 = (10 + 0 + 1 \times 10 \times 6 - 11 \times 5) = 15,$$
$$x_3^5 = (0 + 0 + 1 \times 5 \times 10 - 12 \times 4) = 2,$$
$$x_4^5 = (0 + 0 + 1 \times 12 \times 5 - 15 \times 4) = 0,$$
$$x_5^5 = (0 + 0 + 1 \times 15 \times 4 - 60 \times 1) = 0.$$

For $n = 6$:

$$i = 1,\ \theta_1^6 = \frac{1}{10}(0 + 60 + 0) = 6,$$

$$i = 2,\ \theta_2^6 = \frac{1}{5}(15 + 0 + 1 \times 10 \times 6) = 15,$$

$$i = 3,\ \theta_3^6 = \frac{1}{12}(2 + 0 + 1 \times 11 \times 5) = 5,$$

$$i = 4,\ \theta_4^6 = \frac{1}{15}(0 + 0 + 1 \times 12 \times 4) = 3,$$

The sum of laborers assigned to various machine centers exceeds the total available this hour. Therefore we will allot here 12 laborers on machine center 2, which has the lowest priority and 4 laborers on machine center 3 which has the next lowest priority. The positive queue length is taken into account in allocation of labors to machine centers 2 and 3

Hence $\theta_2^6 = 12$ and $\theta_3^6 = 4$. Then we obtain

$$x_1^6 = (0 + 60 + 0 - 6 \times 10) = 0,$$
$$x_2^6 = (15 + 0 + 1 \times 10 \times 6 - 5 \times 12) = 15,$$
$$x_3^6 = (2 + 0 + 1 \times 11 \times 5 - 4 \times 12) = 9,$$
$$x_4^6 = (0 + 0 + 1 \times 12 \times 4 - 15 \times 3) = 3,$$
$$x_5^6 = (0 + 0 + 1 \times 15 \times 4 - 60 \times 1) = 0.$$

For n = 7:

$$i = 1, \ \theta_1^7 = \frac{1}{10} (0 + 60 + 0) = 6,$$

$$i = 2, \ \theta_2^7 = \frac{1}{5} (15 + 0 + 1 \times 10 \times 6) = 15,$$

$$i = 3, \ \theta_3^7 = \frac{1}{12} (9 + 0 + 1 \times 12 \times 5) = 5,$$

$$i = 4, \ \theta_4^7 = \frac{1}{15} (3 + 0 + 1 \times 12 \times 4) = 3,$$

The sum of laborers assigned to various machine centers exceeds the total number available. Hence we will take $\theta_2^7 = 11$. Then we obtain

$$x_1^7 = (0 + 60 + 0 - 10 \times 6) = 0,$$
$$x_2^7 = (15 + 0 + 1 \times 10 \times 6 - 11 \times 5) = 20,$$
$$x_3^7 = (9 + 0 + 1 \times 12 \times 5 - 12 \times 5) = 9,$$
$$x_4^7 = (3 + 0 + 1 \times 12 \times 4 - 15 \times 3) = 6,$$
$$x_5^7 = (0 + 0 + 1 \times 15 \times 3 - 60 \times 1) = -15.$$

For n = 8:

$$i = 1, \ \theta_1^8 = \frac{1}{10} (0 + 60 + 0) = 6,$$

$$i = 2, \ \theta_2^8 = \frac{1}{5} (20 + 0 + 1 \times 10 \times 6) = 16,$$

$$i = 3, \ \theta_3^8 = \frac{1}{12} (9 + 0 + 1 \times 5 \times 11) = 5,$$

$$i = 4, \ \theta_4^8 = \frac{1}{15} (6 + 0 + 1 \times 12 \times 5) = 4.$$

The sum of laborers exceeds the total number available. Hence we will take the one which gives the minimum units waiting to be processed. Again the positive queue length, $x_i^8 > 0$, $i = 1, 2, 3, 4$, is taken into account in the allocation of labors. Therefore,

$$\theta_2^8 = 10 \text{ and } \theta_3^8 = 5,$$

Then we obtain

$$\begin{aligned}
x_1^8 &= (0 + 60 + 0 - 10 \times 6) = 0, \\
x_2^8 &= (20 + 0 + 1 \times 10 \times 6 - 10 \times 5) = 30, \\
x_3^8 &= (9 + 0 + 1 \times 11 \times 5 - 12 \times 5) = 4, \\
x_4^8 &= (6 + 0 + 1 \times 5 \times 12 - 15 \times 4) = 6, \\
x_5^8 &= (0 + 0 + 1 \times 3 \times 15 - 60 \times 1) = -15.
\end{aligned}$$

Here x_5^8 is negative, the reason of this is that the inspection station remains idle for some time because of the lack of units produced at machine center 4. Hence there is no queue at this station. We can compute the total in-process inventory cost from equation (2), which is,

$$x_6^n = x_6^{n-1} + \sum_{i=1}^{m} K_i \ (x_i^n)^2.$$

Hence the total cost will be

$$\begin{aligned}
x_6^8 = {} & 0.40 \ (x_2^4)^2 + 0.40 \ (x_2^5)^2 + 0.60 \ (x_3^5)^2 + 0.40 \ (x_2^6)^2 \\
& + 0.60 \ (x_3^6)^2 + 0.75 \ (x_4^6)^2 + 0.40 \ (x_2^7)^2 + 0.60 \ (x_3^7)^2 \\
& + 0.75 \ (x_4^7)^2 + 0.40 \ (x_2^8)^2 + 0.60 \ (x_3^8)^2 + 0.75 \ (x_4^8)^2
\end{aligned}$$

$$= 0.40 \times 100 + 0.40 \times 225 + 0.60 \times 4 + 0.40 \times 225$$

$$+ 0.60 \times 81 + 0.75 \times 0 + 0.40 \times 400 + 0.60 \times 81$$

$$+ 0.75 \times 36 + 0.40 \times 900 + 0.60 \times 16 + 0.75 \times 36.$$

$$= 40.0 + 90.0 + 2.40 + 90.0 + 48.60 + 6.75 + 160.0$$

$$+ 48.60 + 27.0 + 360.0 + 9.60 + 27.0$$

$$= \$909.95.$$

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OPTIMAL PRODUCTION SCHEDULING AND INVENTORY CONTROL BY THE DISCRETE MAXIMUM PRINCIPLE

by

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AN ABSTRACT OF A MASTER'S REPORT

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The objective of this report is to demonstrate the applicability of the discrete maximum principle to production scheduling and inventory control problems frequently encountered in industrial management. The basic algorithm of the discrete maximum principle along with the extension "memory in decisions" is stated. Several case studies are presented.

Both Case 1 and Case 2 deal with production scheduling, where the objective is to minimize the production cost. However, in Case 1, there is only one decision variable which signifies the production rate at each period. The extension of the basic algorithm known as "memory in decisions" is employed to solve this case. In Case 2 two decision variables are involved. The first decision variable signifies the change in the production rate between the present and the previous periods, where as the second decision variable represents the change in the number of labor force employed between the present and the previous periods. Back logging is also permitted in this case and, therefore, the production cost structure is different from that of Case 1. Case 3 deals with the labor assignment as a dynamic control problem in a multi-facility network. The system considered has a limiting labor resource and the objective is to allocate the labor force in an optimum way so as to minimize the nonlinear in-process inventory cost function.

In each of the cases considered the optimality condition represented by a recurrence relation of the decision variables is obtained. Such a recurrence relation is valid for a multi-stage system. From above cases it can be concluded that the discrete maximum principle is practical for solving industrial management problems.