FLEXURAL RESISTANCE OF REINFORCED CONCRETE SLABS

763

by

GUANG SHI LIN

B. S., National Taiwan University, 1964

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

Approved by:

Kenneth H. Rosebraugh
Major Professor

TABLE OF CONTENTS

SYNOPSIS........ 1
INTRODUCTION........ 3
Purpose........ 3
Scope........ 4
THEORY........ 5
Plastic Behavior of Concrete Members........ 5
Yield Moment of Reinforced Concrete Slabs........ 6
Development of Yield-Line Theory........ 10
DESIGN BY YIELD-LINE THEORY........ 12
Yield Criterion........ 12
Yield-Line Patterns........ 12
Concentrated Loads........ 22
Orthogonally Anisotropic Reinforcement........ 25
PRACTICAL DESIGN METHODS........ 27
One-Way Slabs........ 27
Simple-Span Slab........ 27
Continuous Slab........ 28
Two-Way Slabs........ 29
Triangular Slab........ 29
Slabs Supported on Four Sides........ 31
Slabs Supported on Three Sides........ 37
Floor Slab........ 43
Comparison with Test Results........ 45
CONCLUSIONS........ 50

ACKNOWLEDGMENT.. 53
BIBLIOGRAPHY.. 54

FLEXURAL RESISTANCE OF
REINFORCED CONCRETE SLABS

by

GUANG SHI LIN

SYNOPSIS

An outline of the yield-line theory is presented in this report in order to review the basis and technique of slab analysis in this setting. The yield-line theory, developed by Professor K. W. Johansen in Denmark, is based on the full plasticity being present in a pattern of yield lines. The locations of yield lines depend on support and loading conditions.

The yield-line pattern may be deduced logically, from geometry, from model or full-scale tests. With some experience a designer will generally assume a yield pattern the first or second time which gives a yield moment only a few percent in error. There are two methods for the solution of slabs using the yield-line theory. The first is based on equilibrium of each section and the second makes use of the method of virtual work. In the virtual work method, the external work done by the loads to cause a small, arbitrary virtual deflection must equal the internal work done as the slab rotates at the yield lines to accomodate this deflection. Both of these two methods are upper-bound solutions.

However, most authorities are agreed that the theoretical ultimate flexural strengths obtained by the yield-line analysis

are in good agreement with experimental results, and generally on the conservative and safe side.[1*]

Yield-line analysis can provide rational answers to problems of complex slab design with limited mathematical effort. It is thus seen to be a powerful analytical tool for structural engineers.

*The numbers refer to items in the list of references.

INTRODUCTION

It seems proper to base the design of most reinforced concrete structures primarily on their ultimate load-carrying capacity, since carrying a load safely is their primary objective. The elastic theory is unsatisfactory for estimating the ultimate flexural strength of many reinforced concrete structures, while the plastic theory can recognize the true characteristics of a material. The term "Yield-Line Theory" is used for the application of plastic design to flat slabs. It is based on plastic behavior occurring in a pattern of yield lines. The location of these yield lines depends on loading and boundary conditions. By means of the yield line analysis, the relations between the applied loads and the ultimate resisting moments of the slab can be obtained. The final design of cross sections may be carried out by ultimate strength design, or by straight line theory.

Purpose

The purpose of this report is to present an outline of yield-line theory in order to review the basis and technique of slab anslysis in this setting. Some practical design examples in slabs of irregular as well as rectangular shapes for a variety of support conditions and loading are illustrated by the use of the theory. The theoretical strengths obtained by the yield-line theory are compared with the test results of this country in very recent years. These investigations showed that most of the theoretical strengths are larger than the values tested, which means the theory yields results on the conservative and safe side. By means of the yield-line theory, the ultimate flexural strength may be found for reinforced concrete slabs without using complex differential equations resulting from the use of the theory of elasticity. It is thus seen to be an extremely powerful analytical tool for structural engineers.

However, the tremendous scope of the relevant material dictates the character of this brief review which, of necessity, is one in breadth rather than in depth. It has been the aim of this broad review to facilitate acquaintance with the present state of our knowledge in this field, so that this kind of design method can be utilized with understanding and discretion.

Scope

This report will begin with a general discussion of plastic behavior and ultimate strength of reinforced concrete members. Attention is then focused on the principle and application of the yield-line theory.

Numerical examples are thoroughly worked out to illustrate the theory and to compare with the test results. The theoretical strengths obtained are in good agreement with experimental results.

The last chapter presents the conclusions that can be reached.

THEORY

Plastic Behavior of Concrete Members

The moment and curvature relation curve shown in Fig. 1 is a typical example of plastic behavior for a simply supported slab with about 0.5 percent reinforcement. At loads below those which initiate yielding in the reinforcement the slab will behave essentially in an elastic manner, although some inelasticity will result

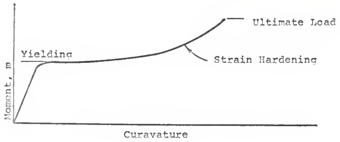


Fig. 1--Moment-Curvature Relation

from flexure cracks in the concrete. A further increase in the loads will cause additional curvature of the member. The curvature at which strain hardening becomes significant, and the curvature at which crushing of the concrete takes place, are about 10 and 15 times that corresponding to first yielding, respectively.[1] In a uniformly loaded simple-span slab near failure, the curvature will be concentrated in a region of maximum moment, a line across the slab at midspan. Such a line identified by a concentration of cracking and curvature is referred to as a yield line. For a determinate slab the formation of one yield line is all that is needed for failure, because the slab is able to rotate about the supports without increasing the load. Indeterminate slabs can maintain equilibrium even after the formation of one or more yield lines. Most slabs can sustain the rotation and deformation necessary to insure free moment redistribution, but this is highly dependent on the reinforcement ratio, and it can be realized only at large deflections. Lightly reinforced members can generally undergo considerable rotation, while overreinforced members may fail by crushing of concrete before much rotation occurs.

However, in structural engineering there is a steadily increasing realization of the importance of the plastic behavior and ultimate flexural strength of reinforced concrete structures.

Yield Moment of Reinforced Concrete Slabs

For all practical purposes, the ultimate strength of a concrete section, based on the yield point stress of the steel, may be said to be the plastic moment. In determining the ultimate strength of a section it is desirable to consider the moment per unit width at the yield level, neglecting any effects of strain hardening. According to the ACI Standard Building Code, this moment is

$$m = M_p/b = d^2 \cdot p.f_{yp}(1 - \tfrac{1}{2}q) \quad \text{ft-lbs/ft or lbs.} \qquad (1)$$

where

m = moment per unit width
M_p = total plastic moment

b = width
d = effective depth
p = percentage of reinforcing
f_{yp} = yield point tensile stress
f_c' = compressive strength of concrete
q = the tension reinforcement index = $p \cdot f_{yp}/f_c'$.

In one-way simply supported slabs, $m = wL^2/8$, where w is a uniform load per unit area, and m is the yield moment according to equation 1.

In two-way slabs reinforced in two perpendicular directions for moments m_x and m_y, the moment resistance of the slab in any direction inclined at an angle α with either m_x or m_y is

$$m_n = m_x \cos^2\alpha + m_y \cos^2(\pi/2-\alpha) = m_x \cos^2\alpha + m_y \sin^2\alpha \quad . \qquad (2)$$

This equation has been verified by experiments.[2] If the reinforcement is isotropic, $m_x = m_y = m$, then Eq. 2 gives

$$m_n = m \cos^2\alpha + m \sin^2\alpha = m \quad .$$

Hence, the yield moment is the same in all directions if the reinforcement is isotropic. This is generally referred to as the square yield criterion, where the yield moment in one direction is independent of the yield moment supplied by the reinforcement at right angles.

An alternative theory involves imagining the reinforcement dragged at right angles to the opening yield lines (Fig. 2). With this kinking of the reinforcement the yield moment on the yield line is

$$m_n = m_x \cos\alpha + m_y \sin\alpha \quad . \qquad (3)$$

When $m_x = m_y = m$, the maximum moment $m_n = 1.414m$ occurs at $\alpha = 45$. When $m_y = (1/2)m$, the yield moment increases to 1.12m at $\alpha = 26^{\circ}$, afterward decreasing to 0.5m at $\alpha = 90^{\circ}$.

The kinking theory is more optimistic than the square yield criterion. But the correct procedure for isotropically reinforced slabs would appear to be somewhere in between the square and the kinking theories.[2]

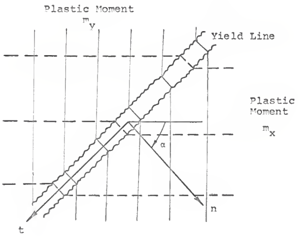


Fig. 2. Kinking of reinforcement on an inclined yield line.

It is now possible to consider the yield criterion for orthotropic reinforcement as shown in Fig. 3. In rectangular slabs it is well known that slabs tend to span in the short direction, making square mesh reinforcing uneconomical. The whole subject of orthotropic reinforcement has never been pursued with the rigor of other plastic analyses. The difficulty is to find a suitable yield criterion which is physically compatible with the classical theory of thin plates. The theory in common use now is derived from the concept that a yield line may be considered to form in small steps, these steps being at right angles to the reinforcement which emerges in two directions. The steps indicate that the reinforcement m exerts a total moment of $m \cdot L\cos\alpha$, represented by a vector AB, while the βm - reinforcement exerts a moment $\beta m \cdot L\sin\alpha$, represented by BC, where the coefficient of orthotropy β graphically is BC/BD. Hence the moment m_n on the yield line is computed from the effects of the moments on all the steps[2]

$$m_n . L = mL\cos\alpha . \cos\alpha + \beta mL\sin\alpha\sin\alpha$$

or

$$m_n = m\cos^2\alpha + \beta m\sin^2\alpha \qquad . \qquad (4)$$

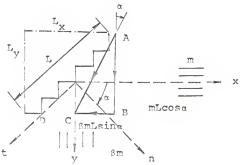


Fig. 3. Yield criterion for orthotropic reinforcement

Development of Yield-Line Theory

Flexure of plates is a problem on which scientists and engineers have been working for nearly two hundred years. The earliest studies were in the area of vibrations of plates, particularly those producing sounds. In 1788 Jacques Bernouilli tried to explain the results of Chaladrie's experiments on vibrating plates, and he treated the square slabs as if they consisted of cross beams or strips. In 1811 Lagrange developed his equations for the flexure and vibration of plates. Navier solved Lagrange's equation for a simply supported rectangular plate in 1820. In 1850 an elastic theory of plates was formulated by Kirchoff. In 1890 a simplified theory of failure in plates was advanced by Bach, and a corresponding method for reinforced concrete slabs was given by Suenson. In 1821 Age Ingerslev introduced a simplified yield-line theory, but because it was based on an assumption that was occasionally incorrect it could not be applied in all cases. In 1931 K. W. Johansen presented a simplified yield-line theory which was applicable to all cases. Strengthened as to the reality of yield-line theory by tests on small experimental plates, Johansen developed a mathematical yield-line theory for thin plates in 1934.[3] He assumed that a reinforced concrete slab, similar to a continuous beam or frame of a perfectly plastic material, will develop yield hinges under overload, but will not collapse until a mechanism is formed. In 1953, Eivind Hognestad's "Yield-Line Theory for the Ultimate

Strength of Reinforced Concrete Slabs" was the first chief source of information on yield-line theory published in English. Actually, Hognestad's paper is a summary of Johansen's work.

In the past 25 years many tests have been conducted in many European countries which verify that the inelastic behavior of concrete structures. The favorable results of these tests have become an accepted part of design practice. In Russia and Britain non-linear relationships are used to determine the moments in indeterminate structures as well as using the ultimate strength of sections. In Denmark and Norway the inelastic moment redistribution is used but the design of sections is based on the allowable working stress. During the past twenty years experimentation and analytical studies of the inelastic behavior of structural concrete in this country have been purposely directed toward ultimate strength design. Relatively little emphasis has been placed on limit design studies and yield-line theory. At present the development of the yield-line theory is guided principally by the ACI-ASCE Committee on Design of Reinforced Concrete Slabs.[4] However, it is clear that the engineers of various countries are taking different paths to the same eventual full consideration of plasticity in design.

DESIGN BY YIELD-LINE THEORY

Yield Criterion

There have been some questions raised as to the validity of the square yield criterion for finding the resisting moment along a yield line at any angle. The more optimistic kinking theory has been proposed by some investigators. But the truth would appear for isotropic reinforcement to be somewhere between the square criterion and the kinking criterion. The former mistakenly supposes that the steel remains in line when any yield line opens. The latter overestimates the benefits attributed to kinking since the tests have shown that crushing of the concrete straightens out the tendons slightly.

Yield-Line Pattern

When a slab is overloaded, yielding will begin in regions of high moment, and as the loading continues to increase yield lines will form and spread into a pattern referred to as a yield-line pattern. When the yield lines have spread to the slab edges, the loading capacity of the slab will be exhausted and the slab will reach a state of neutral equilibrium. The yield lines divide the slab into several parts and a heavy concentration of curvature takes place in the yield lines since the plastic deformations are much larger than the elastic ones. Near the ultimate load it is reasonable to assume that the individual slab parts are plane, and that all deformations take place in the yield lines. It then follows that the yield lines must be straight, and that deformations of the slab may be considered as rotations of the slab parts about axes at their supports. Also a yield line between two slab parts must pass through the intersection of the axes of rotation of the two parts. The axis of rotation must lie in a line of support and must pass through the columns. The general nature of the possible yield-line patterns may be determined in this manner. There are two methods, the equilibrium method and the virtual work method, for the solution of slabs using yield-line theory. The former is based on equilibrium conditions of the individual slab parts, and the latter is carried out with the aid of the principle of virtual work. However, since the yield moments are principal moments, twisting moments are zero in the yield lines, and in most cases the shearing forces also are zero. Thus, only the moment m per unit length of yield-line acts perpendicular to these lines, and the total moment may be represented by a vector in the direction of the yield line with the magnitude m times the length of the line. The resulting moment for an individual slab part is then found by vector addition. At a free or simply supported edge, both bending and twisting moments should theoretically be zero. Thus, when a yield line meets a free edge, it must do so perpendicularly since yield line moments are maximum moments and so cannot have torsional moments acting parallel or perpendicular to them. Experiments indicate that the yield lines generally turn only quite close to the edge. It is usually more convenient for computations to assume that the yield line continues straight on to the edge. If this assumption is to be made, a correction factor becomes necessary.

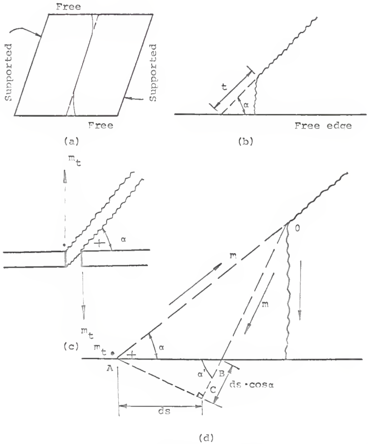


Fig. 4. Conditions at a boundary

If the yield line shown in Fig. 4 is simplified by extending it as a straight line to the edge, a pair of the statical equivalent of twisting moments and shearing forces m_t must be introduced at the corner. The magnitude of the force m_t may be established by considering the equilibrium of the triangle AOB as shown in Fig. 4d. Then with $\alpha \doteq \alpha'$, taking moments about the BO-axis,

$$m \cdot AO - m \cdot BO = m_t \cdot ds \cdot \sin\alpha$$

$$m \cdot ds \cdot \cos\alpha = m_t \cdot ds \cdot \sin\alpha$$

or

$$m_t = m \cdot \cot\alpha \quad . \tag{5}$$

The effects of the forces m_t will drop out when the total work done by the loads on the entire structure is calculated, since they are equal and opposite and move through the same distance. This boundary condition was first introduced into the yield-line theory by Johansen in 1931.

The solution of equations of equilibrium may at times be simplified by application of the principle of virtual work. The positions of the axes of rotation for the various slab segments can be known if the yield-line pattern is assumed to be known by introducing some parameters. A virtual displacement may be chosen so that rotations take place only in the yield lines. The virtual work of the pairs of concentrated shears m_t is then zero for the slab as a whole. The virtual work of the yield moments for each individual slab part is the scalar product of two vectors, a rotation $\overline{\theta}$ and a resultant $\overline{M}$ of the moments in the yield lines. For the slab as a whole, this work of the internal forces plus the work of the loads must equal zero, and the work of the reactions are zero as they act along the axes of rotation. Therefore,

$$\Sigma \overline{M}\, \overline{\theta} + \Sigma \iint \omega \cdot \delta \cdot dxdy = 0 \tag{6}$$

in which the summation is over the entire slab, and the integration is over the individual slab parts.[1]

The moment across the yield lines being a maximum value, the value of $\overline{M}$ is proportional to the unit yield moment m. Thus, Eq. 6 can be used to determine the correct yield-line pattern and to solve for m for a given load w if the yield-line pattern is known.

If a type of pattern is assumed in accordance with the support conditions and characterized by a number of unknown parameters x_1, x_2, x_3,...., Eq. 6 may be written

$$m = F\,(x_1,\ x_2,\ x_3, \ldots,\ w) \quad . \tag{7}$$

The correct yield pattern then is found by the maximum criteria

$$\frac{\partial F}{\partial x_1} = 0; \qquad \frac{\partial F}{\partial x_2} = 0; \qquad \frac{\partial F}{\partial x_3} = 0; \ldots\ldots, \tag{8}$$

and the final yield moment m is determined by substituting the corresponding parameter values into Eq. 7.[1]

It may be shown that application of the virtual work principle to yield-line patterns which do not differ considerably from the correct pattern will result in yield moments only slightly smaller than the correct ones. A yield-line pattern is in accord with the support conditions, and the necessary yield mement m for a given load w is computed from the virtual work equations. A rule of thumb to keep in mind when locating yield-line patterns is that simply supported or free edges as well as openings "attract" yield lines, while fixed edges "repel" yield lines. Fig. 5 shows some typical yield-line patterns.

The yield-line theory is a non-linear theory, and the principle of superposition is thus not strictly applicable. However, its use gives results still on the safe side as the sum of the yield moments corresponding to a number of individual loadings is greater than or equal to the yield moment corresponding to the sum of the loadings. These conditions were verified by Johansen in 1943.[1,3]

As a yield line approaches the corner between two intersecting supported sides, the crack usually spreads out into a Y form. A pair of shearing forces is then necessary to prevent the corner from lifting. If the slab is fastened to its support, the Y will be closed by a crack from A to B. If the slab is not fastened down, there will be no crack from A to B as shown in Fig. 6. The slab part I is referred to as a corner lever. The corner lever gives a higher necessary yield moment than the single yield line, and the Y-shaped yield pattern is therefore more dangerous.

To find the length h for a simply supported slab with a uniformly distributed load, w, per unit area, equilibrium of moments for triangle II may be taken about axis AB which gives

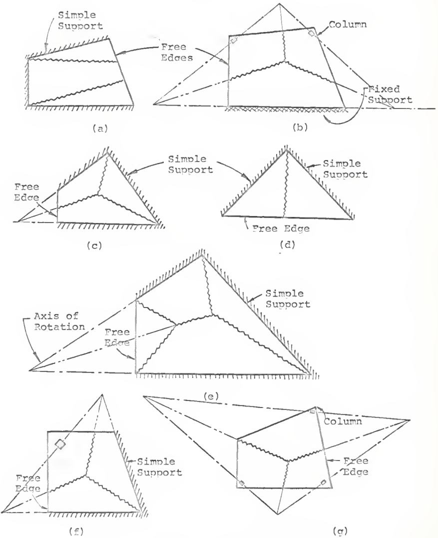


Fig. 5. Typical yield-line patterns.

$$1/2\ wh\sqrt{2}\cdot c.1/3h - \sqrt{2}\cdot c(m_1 + m_2) = 0$$

or

$$h = \sqrt{\frac{6(m_1+m_2)}{w}} \qquad (c_1 = c_2 = c) \qquad (9)$$

The value of h is independent of c. If h is found to be greater than the distance from AB to where the corner crack intersects the longitudinal crack, it means that the Y-shaped yield pattern will not form.

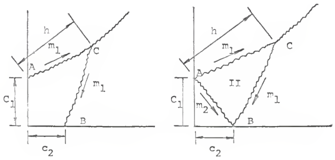


Fig. 6. Conditions at a corner.

The corner cracking tends to reduce the carrying capacity of the slab by about 10 percent. It is usually not possible to give a direct solution for an entire slab including the corner cracking. In some cases, a trial-and-error procedure must be used. According to Johansen it is most expedient in practical design to disregard the corner levers and then later apply corrections.[2] The analysis becomes considerably more complicated if the possibility of corner levers is introduced, and the error made by neglecting them is usually very small. The general belief is that acute-angled corners, absence of top-reinforcement in corners, presence of restraining moments on nearby edges, and point loads, all contribute to a reduction of the ultimate load. Table I gives some of percentage reduction to allow for corner effects.[2]

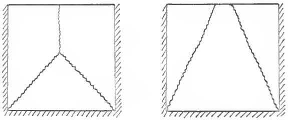
Table I. Percentage Reduction to Allow for Corner Effects

Shape of Slab	Nature of Loading	Top reinforcement m/M	Reduction
Square slab supported on all four sides	Distributed	0.5	- 8.5%
	"	1.0	- 2.0%
	"	1.0	0
	"	1.0*	- 9.5%
Square slab supported on all four sides	Central point load	0	-21.5%
		1.0	0
	"	1.0*	-21.5%
Square slab with one free edge	Distributed	0	-10%
	"	1.0	-11%
3:1 Rect. with one long free edge	Distributed	0	-11%
	"	1.0*	- 9%

*with restraint moment -M on edges.

For right angle corners, the solution obtained by neglecting the corner effects is sufficiently accurate. When the corner is acute, the error may become very serious.[2]

The rectangular slab with three edges supported and the other free may be formed in either of two mechanisms as shown in Fig. 7. According to Wood, case (b) is impossible in isotropic reinforced concrete square slabs; however, with unequal or orthotropic reinforcement this mode could occur.[2]



(a) (b)

Fig. 7. Yield patterns of slabs with free edge.

If the slab is reinforced in the top and this steel is not carried far enough into the slab a localized form of failure as shown in Fig. 8 might occur. In both types of failure it is close enough to assume that the yield line within the span forms just beyond the point where the top steel has been discontinued.

It is necessary to investigate all reasonable failure mechanisms in each case in order to establish that the correct solution has been found. The correct location of yield lines is a unique location for a given loading.

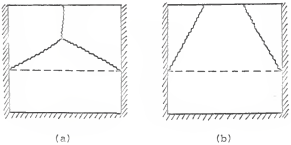


Fig. 8. Localized forms of failure.

Concentrated Loads

Concentrated loads or reactions usually cause a circular failure area with many radial cracks as shown in Fig. 9.[5] Such a curved negative yield-line pattern may be considered as consisting of an infinite number of infinitesimal triangles. In a slab with

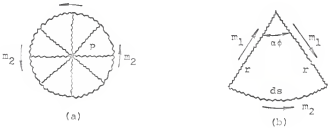


Fig. 9. Concentrated loads.

fixed edges, the radius of the circle will be tangent to the nearest edge. Assuming that P is applied over a small circle of radius ρ and that the concrete cannot crack under the load, the virtual work equation for a deflection of the load P may be written as

$$\int (m_1 + m_2)\, ds \frac{\delta}{r-\rho} = P \cdot \delta$$

where $ds = r\,d\phi$.

Integrating from 0 to 2π gives

$$m_1 + m_2 = \frac{P}{2\pi}\left(1 - \frac{\rho}{r}\right) \quad . \tag{10}$$

If the load is applied in such a way that the concrete can crack under the load,[5]

$$\int_0^{2\pi} (m_1+m_2)\, r\, d\phi \frac{\delta}{r} = \int_0^{2\pi} \frac{P}{\pi\rho^2} \cdot \frac{1}{2}\rho^2 d\phi \cdot \frac{r-2/3\rho}{r} \cdot \delta$$

or

$$m_1 + m_2 = \frac{P}{2\pi}\left(1 - \frac{2\rho}{3r}\right) \tag{11}$$

in which (m_1+m_2) increases with r.

When a slab is fixed on all edges, the yield circle will be tangent to the slab circumference. This will result in the largest possible r and a maximum value of (m_1+m_2).

In the case of concentrated loads as well as in the case of point supports, the failure of the slab is most likely to be due to a combination of shear and bending and then an analysis of this type may not be valid. Fig. 10 shows some typical yield-line patterns for concentrated loads on various supports.

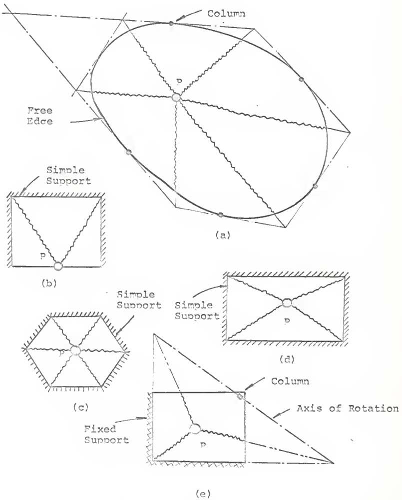


Fig. 10. Typical yield-line patterns for concentrated loads.

Orthogonally Anisotropic Reinforcement

Orthotropic reinforcement is a useful phrase employed by the Polish team led by Olszak and Sawczuk, and refers to the placing of unequal reinforcement in two directions at right angles. It is one of the simplest ways of saving steel. Since the resisting moments in two perpendicular directions are not the same in a slab with the type reinforcement the yield-line methods outlined heretofore are not directly applicable. Johansen has shown that the analysis of anisotropically reinforced slabs may be reduced to the case of isotropic reinforcement by adjusting the various dimensions and loads.

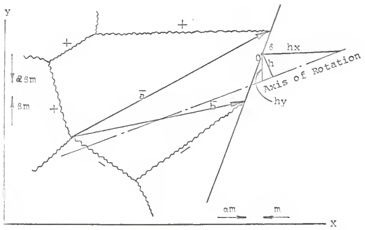


Fig. 11. Slab with orthogonally anisotropic reinforcement.

If the slab part shown in Fig. 11 rotates a virtual angle about the arbitrary axis of rotation indicated so that a point C is given a virtual deflection δ, then the component rotations are

$$\theta_x = \frac{\delta}{h_y} \qquad ; \qquad \theta_y = \frac{\delta}{h_x} \qquad .$$

Then the equation of virtual work becomes

$$(ma_x + \alpha mb_x)\frac{\delta}{h_y} + (\beta ma_y + \alpha\beta mb_y)\frac{\delta}{h_x} = \iint wzdxdy \qquad . \qquad (12)$$

Considering now a slab with isotropic reinforcement whose dimensions in the y-direction and load per unit area are the same as in the slab above, but whose dimensions in the x-direction are r times those above, the equation of virtual work becomes

$$r(ma_x + \alpha mb_x)\frac{\delta}{h_y} + (ma_y + \alpha mb_y)\frac{1}{rh_x} = \iint wzrdxdy \qquad . \qquad (13)$$

By dividing Eq. 13 by r, it may be seen that Eq. 12 and 13 coincide if $\beta = \frac{1}{r^2}$.

Therefore, the anisotropically reinforced slab with the yield moments m, αm and βm, $\alpha\beta$m may be analyzed as an equivalent isotropically reinforced slab by dividing the linear dimensions in the x-direction by $\sqrt{\beta}$ and using m and αm as the moments in both directions. The load w per unit area remains the same.

PRACTICAL DESIGN METHOD

The design problem in yield-line analysis is usually to estimate the necessary yield moment m for a slab subject to given ultimate loads and with given dimensions and support conditions. The correct value of m must be a maximum value resulting from the correct yield-line pattern and satisfying the equations of equilibrium.

The discussion of the theory behind the case to be studied will begin with a one-way simple-span slab and proceed to the cases of floor slabs.

One-way Slabs

Simple-Span Slab. A simply supported slab is considered subject to a uniform load w as shown in Fig. 12.

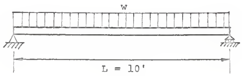


Fig. 12. Simple-span slab

The flexural strength of the slab may be expressed for design purposes by the equation

$$m = wL^2/8 \quad , \quad \text{where } L = 10'$$

$$m = 12.5w$$

or

$$w = 0.08m \quad .$$

The units of m are ft-lb per ft or lb.

Continuous Slab. A continuous slab is considered to sustain a uniformly distributed load of w psf. The slab has a 10' span length and is reinforced according to the theory of elasticity, as shown in Fig. 13.

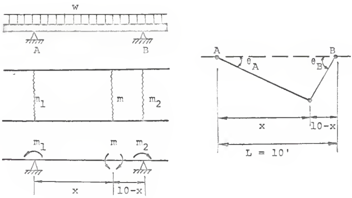


Fig. 13. Continuous one-way slab.

Taking the left segment of the slab as a free body, the equation for moment equilibrium becomes

$$m + m_1 = 1/2wx^2 \quad .$$

Similarly, for the right slab segment,

$$m + m_2 = 1/2w(L - x)^2 \quad .$$

Solution of these two equations gives, for L = 10',

$$m = \frac{50w}{\left(\sqrt{1+\frac{m_1}{m}} + \sqrt{1+\frac{m_2}{m}}\right)^2} \quad .$$

If the fixed slab is reinforced according to the theory of elasticity, then

$$\frac{m_1}{m} = 2 \qquad , \qquad \frac{m_2}{m} = 2$$

and

$$m = 25/6w \qquad m_1 = m_2 = 25/3w$$

or $$w = 0.24m = 0.12m_1 = 0.12m_2 \quad .$$

Two-Way Slabs

Triangular Slab. Figure 14 shows a uniformly loaded triangular slab simply supported on two edges with the third edge free.

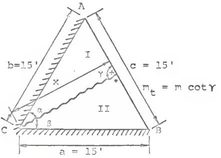


Fig. 14. Triangular slab.

Equilibrium Method

The equilibrium equation of moments for slab part I about support CA gives

$$m \cdot x \cdot \cos\alpha = m_t \cdot x \cdot \sin\alpha + 2.5 \cdot w \cdot x^2 \cdot \sin^2\alpha \quad .$$

The equilibrium equation of moments for slab part II about support BC gives

$$m \cdot x \cdot \cos\beta = -m_t \cdot x \cdot \sin\beta + 2.5 w \cdot x^2 \cdot \sin^2\beta$$

in which

$$x = \frac{15 \sin(\alpha+\gamma)}{\sin \gamma} \quad .$$

From these two equations, it is possible to determine the value of m or w. It is found that

$$m_t = 0$$

and $\quad m = 9.375w \quad$ or $\quad w = 0.107m.$

Virtual Work Method

A unit displacement is assumed at the intersection of the free edge AB and yield line. The virtual work equation then gives

$$m(\cot\alpha + \cot\beta) = 2.5 \cdot w \cdot x(\sin\alpha + \sin\beta)$$

or $$m = 37.5 \; w \cdot \sin\alpha \cdot \sin\beta \quad .$$

Let $\frac{\partial m}{\partial \beta} = 0 \qquad$ then $\angle\alpha = \angle\beta$

and $m = 9.375w$

or $w = 0.107m$.

Slabs Supported on Four Sides

Square Slab

A square slab subject to a uniform load w is considered simply supported on all four sides as shown in Fig. 15. In this symmetrical slab, yield lines must form along the slab diagonals.

Equilibrium Method

The moment equilibrium of any one of the identical slab parts gives

$$\frac{w \cdot a^2}{4} \cdot \frac{a}{6} = 2 \cdot \frac{m \cdot a}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \quad \text{where } a = 10'$$

$$m = 4.15\ w$$

or $$w = 0.24\ m \quad .$$

Virtual Work Method

The equation of virtual work gives

$$\frac{1}{4} \cdot w \cdot a^2 \cdot \frac{a}{6} \theta = m \cdot a \cdot \theta$$

$$m = 4.15\ w$$

or $$w = 0.24\ m \quad ,$$

which agrees with the previous results.

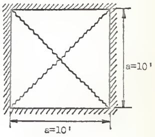


Fig. 15 Square slab

Rectangular Slab

A fixed rectangular slab subject to a uniform load w is considered in Fig. 16.

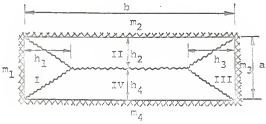


Fig. 16. Rectangular slab.

Different amounts of reinforcement may be provided at the four sides so that $m_1 = pm$, $m_2 = qm$, $m_3 = rm$, $m_4 = sm$, where m is the positive yield moment in the slab.

The equilibrium equations of the four slab parts become

$$\text{I.} \quad m \cdot a \cdot (1+p) = 1/6 \cdot w \cdot a \cdot h_1^{\,2}$$

$$\text{II.} \quad m \cdot b \cdot (1+q) = 1/6 \cdot w \cdot h_1 \cdot h_2^2 + 1/6 \cdot w \cdot h_3 \cdot h_2^{\,2} + 1/2 \cdot w \cdot (b-h_1-h_3) \cdot h_2^{\,2}$$

$$\text{III.} \quad m \cdot a \cdot (1+r) = 1/6 \cdot w \cdot a \cdot h_3^{\,2}$$

$$\text{IV.} \quad m \cdot b \cdot (1+s) = 1/6 \cdot w \cdot h_1 \cdot h_4^{\,2} + 1/6 \cdot w \cdot h_3 \cdot h_4^{\,2} + 1/2 \cdot w \cdot (b-h_1-h_3) \cdot h_4^{\,2}$$

Solution of these four equations gives

$$m = \frac{w\ a^2}{24}\left[\sqrt{3 + (\frac{a_r}{b_r})^2} - \frac{a_r}{b_r}\right]^2 \qquad (14)$$

In which a_r and b_r are given by the equations

$$a_r = \frac{2\ a}{\sqrt{1+q} + \sqrt{1+s}} \qquad (15)$$

$$b_r = \frac{2\ b}{\sqrt{1+p} + \sqrt{1+r}} \qquad (16)$$

This general solution was developed by Ingerslav in 1921.

For a simply supported rectangular slab, $p = q = r = s = 0$, $a_r = a$, and $b_r = b$.

For a simply supported square slab, $a_r = b_r = a = b$, and Eq. 14 reduces to $m = w \cdot a^2/24$, which agrees with the previous result.

Square Slab with Corner Levers

A simply supported and uniformly loaded square slab, when analyzed for the assumed yield pattern with no corner levers, required a moment capacity of $w \cdot L^2/24$ as shown in the previous example. If it is assumed that the corners are held down and that symmetrical corner levers exist as shown in Fig. 17, the moment equilibrium of slab part II then gives

$$x \cdot \sqrt{2} \cdot m = w \cdot x \cdot \sqrt{2} \cdot \frac{h^2}{6} \qquad \text{(Let } m' = 0\text{)}$$

or $$h = \sqrt{\frac{6 \cdot m}{w}} \quad .$$

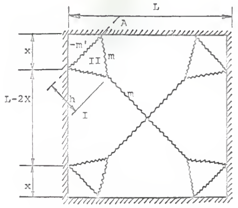


Fig. 17. Square slab with corner levers.

The moment equilibrium of slab part I gives

$$y = m'(L - 2x) - w\left[\frac{a^3}{24} - \frac{x}{3}(x/2 + h/\sqrt{2})^2\right] = 0 \quad .$$

Let $\frac{\partial y}{\partial x} = 0$, then

$$x = 2/3\sqrt{\frac{m}{w}}\ (\sqrt{21} - 2\sqrt{3})$$

and $\quad m = wL^2/22 \quad .$

Such an analysis results in a required moment of $wL^2/22$, an increase of about 9 percent as compared with the results of an analysis neglecting corner levers.

Orthotropic Slab

Consider the 12' x 12' square slab of Fig. 18 with anisotropic reinforcement and yield moments m in one direction and βm in an orthogonal direction. It is simply supported on all four sides and carries a uniformly distributed ultimate load of w psf.

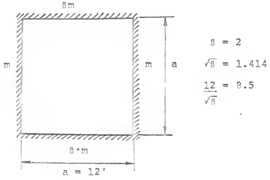


Fig. 18. Orthotropic slab.

This slab may be analyzed as an equivalent isotropically reinforced slab by dividing the linear dimensions in the m direction by $\sqrt{\beta}$ and using m as the yield moments in both directions as indicated in Fig. 19. The load w per unit area remains the same.

Now let the equivalent slab with isotropic reinforcement, for which the linear dimensions in the m direction are $1/\sqrt{\beta}$ times those of the anisotropically reinforced square slabs, as shown in Fig. 19 be considered.

Equilibrium Method

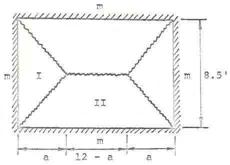


Fig. 19. Equivalent isotropically reinforced slab.

Yield lines should form as shown in Fig. 19. Equilibrium of slab part I gives

$$m = w \cdot a^2/6 \quad .$$

Equilibrium of slab part II gives

$$12m = 1/3w \cdot a \cdot 4.25^2 + w(6 - a)4.25^2 \quad .$$

Solution of these two equations gives

$$a = 4.95'$$

and $$w = 0.246\ m \quad .$$

Virtual Work Method

The virtual work equation gives

$$12m(1/4.25)2 + 8.5m(1/4.95)2$$

$$= w \cdot 2 \cdot 12 \cdot 4.25 \cdot \tfrac{1}{2} \cdot 2 + w \cdot 4.95 \cdot 4.25 \cdot 1/3 \cdot 2 + w(4.95 \cdot 4.25)2/3$$

or $$w = 0.246\ m$$

which coincides with the previous result.

Slabs Supported on Three Sides

Square Slab

A uniformly loaded square slab simply supported on three edges with the third edge free is shown in Fig. 20.

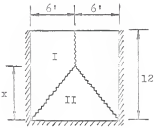


Fig. 20. Square slab with three edges simply supported and one free.

Equilibrium Method

Equilibrium of slab part I gives

$$12 \cdot m = 6(12-x)w \cdot 3 + 1/2 \cdot x \cdot 6 \cdot w \cdot 2 \quad .$$

Equilibrium of slab part II gives

$$12 \cdot m = 1/2 \cdot 12 \cdot x \cdot w \cdot x/3 \quad .$$

Solution of these two equations gives

$$w = 0.098 \text{ m} \qquad \text{and} \qquad x = 7.8' \quad .$$

Virtual Work Method

The virtual work equation gives

$$2 \cdot 12 \cdot m \cdot 1/6 + 12 \cdot m(1/7.8)$$

$$= 2 \cdot 4.2 \cdot 6 \cdot w \cdot 1/2 + 2 \cdot 1/2 \cdot 7.8 \cdot 6 \; w \cdot 1/3$$

or $\qquad w = 0.0984 \text{ m}$

which agrees well with the previous result.

Square Slab with Corner Levers

Figure 21 shows a simply supported on three sides and uniformly loaded square slab. If it is assumed that the corners are held down and that symmetrical corner levers exist; negative yield lines may be present across the corners.

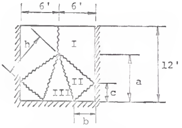


Fig. 21. Square slab with corner levers.

Equilibrium Method

Equilibrium of the three slab parts gives

$$\text{I.} \quad (12 - c)m = (12 - a)6 \cdot w \cdot 3 + 1/2(a - c)6 \cdot w \cdot 2 \quad ,$$

$$\text{II.} \quad m = w \cdot h^2/6 \quad ,$$

$$\text{III.} \quad m = w \cdot a^2/6 \quad .$$

With the aid of geometrical conditions, these equations give

$$a = 7.8' \quad , \quad b = 2.6' \quad , \quad c = 2' \quad , h = 7.8' \quad ,$$

and $\quad w = 0.09 \text{ m}.$

In reality concrete has some flexural strength, and the actual influence of the levers will be somewhat smaller than the theoretical value.

Virtual Work Method

The virtual work equation gives

$$2(10 \cdot m \cdot 1/6) + 2(3.25 \cdot m \cdot 1/8.25) + 6.8 \cdot m \cdot 1/7.8$$

$$= 2(4.2 \cdot 6)w \cdot 1/2 + (1/2 \cdot 5.8 \cdot 6)w \cdot 1/3 + 2(1/2 \cdot 3.25 \cdot 8.25)w/3$$

$$+ (1/2 \cdot 6.8 \cdot 7.8)w/3$$

or $\quad w = 0.09 \text{ m}$

which coincides with the previous result.

Rectangular Slab

Consider a uniformly loaded 18' x 12' rectangular slab simply supported on three sides with one edge free as shown in Fig. 22.

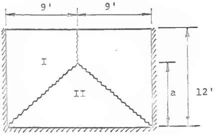


Fig. 22. Rectangular slab with three edges simply supported and one edge free.

Equilibrium Method

Yield lines should form as indicated in Fig. 22 and equilibrium of three slab parts give

$$\text{I.} \quad 12 \cdot m = (12 - a) \cdot w \cdot 9 \cdot 4.5 + 1/2 \cdot a \cdot 9 \cdot w \cdot 3 \quad ,$$

$$\text{II.} \quad 18 \cdot m = 1/2 \cdot 18 \cdot a \cdot w \cdot a/3 \quad .$$

Solution of these two equations gives

$$w = 0.0573\ m \qquad \text{and} \qquad a = 10.25' \quad .$$

Virtual Work Method

The virtual work equation gives

$$2 \cdot 12 \cdot m \cdot 1/9 + 18 \cdot m \cdot 1/10.25$$

$$= 2 \cdot 1.75 \cdot 9 \cdot w/2 + 1/2 \cdot 10.25 \cdot 9 \cdot w/3 + 1/2 \cdot 18 \cdot 10.25 \cdot w/3$$

or $$w = 0.0573\ m$$

which agrees with the previous result.

Orthotropic Slab

A uniformly loaded square slab with yield moments m in one direction and $\beta \cdot m$ in an orthogonal direction is simply supported on three edges and the third edge is free as shown in Fig. 23.

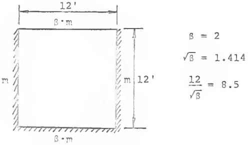


Fig. 23. Orthotropic slab with three edges simply supported and one edge free.

Considering now a slab with isotropic reinforcement whose dimensions in the $\beta \cdot m$ direction and load per unit area are the same as in the given slab shown in Fig. 23, but whose dimensions in the m direction are $1/\sqrt{\beta}$ times those above. The ultimate load can be computed by the methods previously described in this report.

Equilibrium Method

The equations of equilibrium conditions for the three slab parts become

$$12 \cdot m = 4.25 \cdot a \cdot w \cdot 2.125 + 1/2 \cdot a \cdot 4.25 \cdot w \cdot 4.25/3$$

or $$12 \cdot m = 9.04 \cdot a + 3.01 \cdot a \cdot w$$

and $$8.5 \cdot m = 1/2 \; 8.5 \cdot a \cdot w \; a/3$$

or $$m = w \; a^2/6 \quad .$$

The solution of these two equations gives

$$a = 6' \qquad \text{and } w = 0.166 \; m \quad .$$

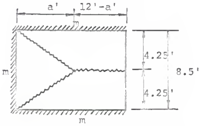


Fig. 24. Equivalent isotropically reinforced slab with the moments m and the same uniform load w as in the given slab shown in Fig. 23.

Virtual Work Method

The virtual work equation gives

$$2 \cdot 12 \cdot m(1/4.25) + 8.5 \; m(1/6)$$

$$= 2.6(4.25 \cdot w \cdot 1/2) + 1/2 \cdot 6 \cdot 4.25 \cdot w \cdot 1/3 + 1/2 \cdot 8.5 \cdot 6 \cdot w/3$$

or $$w = 0.166 \text{ m}$$

which is the same as the previous result, and at the same times shows how much simpler this method is than the elastic method.

Floor Slab. Consider the 28' x 16' fixed or continuous slab of Fig. 25, which has a 10' x 7' hole with free edges provided for a staircase.

It should be pointed out that when there are holes in slabs, equilibrium methods can result in considerable difficulty. The trouble is that when a yield line is near a corner of a hole a small displacement of the line can cause the line to intersect a different edge of the hole. The effect is to cause a pronounced discontinuity in $m \cot \alpha$, both in magnitude and in sign. Again, with equilibrium methods, it is possible to have a yield line which just intersects a corner of a hole such that a slight change of location can cause it to miss the hole entirely. In that event the edge forces suddenly vanish.[2] In evaluating either the equilibrium of separate slab parts, or alternatively the external work done on the system, it will be found more convenient to divide the slab into separate triangles, rectangles, etc., rather than to attempt to subtract for the hole.

The assumed yield-line pattern is shown in Fig. 25. The assumption is suggested by the guidelines previously described in this report. It is well known that simply supported or free edges as well as openings "attract" yield lines, while fixed edges "repel" lines.

Equilibrium Method

The yield moment m is first computed for each slab part separately. It should be noted that the pair of concentrated shearing forces m_t as defined on page 15 should act down in the corner.

Part A: $16 \cdot 2m = (120/6)16 \cdot 11^2 + m(11/7)11 - m(11/9)11$

$$28.15\ m = 38,720$$

or $$m = 1,375 \text{ lb.}$$

Part B: $28 \cdot 2m = (120/6)11 \cdot 7^2 + (120/6)7 \cdot 7^2$

$$+ (120/2)10 \cdot 7^2 - m(7/7)7 + 1610 \cdot 7 \cdot 7/2$$

$$63.00\ m = 86,640$$

or $$m = 1,375 \text{ lb.}$$

Part C: $16 \cdot 2m = (120/6)16 \cdot 7^2 + m(7/7)7 + m(7/9)7$

$$+ 1070 \cdot 7 \cdot 7/2$$

$$30.56\ m = 41,880$$

or $$m = 1,375 \text{ lb.}$$

Part D: $28 \cdot 2m = 120(11 \cdot 9^2/6 + 7 \cdot 9^2/6 + 10 \cdot 2^2/6)$

$$+ m(11/9)9 + m(7/9)9 + 580 \cdot 7 \cdot 5 \cdot 5.5$$

$$38.00\ m = 52,200$$

or $$m = 1,375 \text{ lb.}$$

In this case the four m-values are all equal and m = 1,375 lb. is a satisfactory design value.

Virtual Work Method

A virtual deflection of unity along the yield line ab gives the rotations:

$$\theta_A = 1/11 \quad ; \quad \theta_B = 1/7 \quad ;$$

$$\theta_C = 1/7 \quad ; \quad \theta_D = 1/9 \quad .$$

The equilibrium equations above are written in such a manner that the virtual work equations can be established easily.

$$m\left(\frac{28.15}{11} + \frac{63}{7} + \frac{30.56}{7} + \frac{38}{9}\right)$$

$$= \frac{38,720}{11} + \frac{86,640}{7} + \frac{41,880}{7} + \frac{52,200}{9}$$

$$20.15\ m = 27,680$$

or $$m = 1,375 \text{ lb}$$

which is the same as the previous result.

Comparison with test results. The yield-line theory has been applied by some investigators to most existing test data pertaining to reinforced concrete slabs failing in flexure. It may be generally stated that tests verify that the yield-line patterns are a reality, and that the ultimate loads predicted by the theory are on the average 80 to 90 percent of those observed in tests.

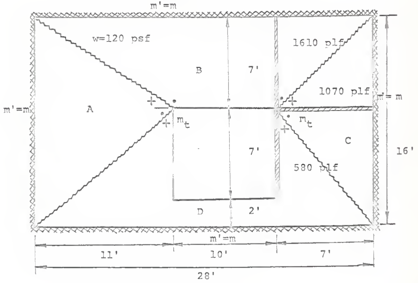


Fig. 25. A typical floor slab.

As an example, a total of 22 slabs were tested at Arizona State University during the past two years.[6] Table II shows the number, size, support condition, average compressive strength, and reinforcing location for all of the slabs. Table III gives results of the slabs tested. The actual load found from the tests on the slabs, and ratio of the actual load to the theoretical load are also shown. Looking at the load ratios in column four of Table III it will be noticed that for slab series A they are generally around two, with close agreement between identical slabs. It should be noted that the load ratio for slabs A11 and A12 are much closer to unity than the other A series slabs. It might be due to the higher percentage of reinforcement in these two slabs.

The actual loads, on the whole, were much larger than the predicted loads. It is believed that a large amount of the extra load carried was accomplished by membrane action and strain hardening. This idea has been given by most authorities in the studies of flat plates, but the study of membrane action itself is a recent development. The membrane effect is greater in slabs with fixed edges.

However, it seems satisfactory for design purposes to base predictions of flexural strength on the slightly conservative values resulting from the yield-line theory.

Table II

Slab Properties

Slab No.	Slab Size inches	Reinf. Loc.	Support Condition	Avg. Compressive Strength, psi
A1	12 x 12	B*	Simple	4430
A6	12 x 12	B	Simple	3680
A2	12 x 12	T&B**	Simple	No Data
A3	12 x 12	T&B	Simple	5600
A4	12 x 12	T&B	Corner***	5220
A5	12 x 12	T&B	Corner	4620
A7	12 x 12	T&B	Fixed	5010
A8	12 x 12	T&B	Fixed	5010
B3	12 x 18	T&B	Corner	3240
B4	12 x 18	T&B	Corner	5010
B1	12 x 18	T&B	Fixed	4190
B2	12 x 18	T&B	Fixed	4100
C1	12 x 24	T&B	Corner	3110
C2	12 x 24	T&B	Fixed	4100
D2	12 x 30	T&B	Corner	4190
D1	12 x 30	T&B	Fixed	3110
E1	12 x 36	T&B	Corner	3680
E2	12 x 36	T&B	Fixed	3240
A10	12 x 12	$\beta=1\frac{1}{2}$	Corner	4400
A9	12 x 12	$\beta=1\frac{1}{2}$	Fixed	4400
A12	12 x 12	$\beta=2$	Corner	4920
A11	12 x 12	$\beta=2$	Fixed	4920

*Isotropic reinforcement in the bottom only

**Isotropic reinforcement in the top and bottom

***Corners clamped

Table III

Test Results and Comparisons

Slab No.	m lbs from Ten Test	Theor. Load W lbs.	Actual Load W_t lbs.	W_t/W	m lbs from Beam	Theor. Load W' lbs	W_t/W'
A1	70.5	930	2100	2.26	102	1350	1.55
A6	79.2	1045	1920	1.84	116	1530	1.25
A2	75	990	2040	2.06	102	1350	1.51
A3	75	990	2160	2.18	102	1350	1.60
A4	75	1060	2340	2.20	138	1950	1.20
A5	75	1060	2340	2.20	118	1670	1.40
A7	75	2120	4560	2.15	128	3620	1.26
A8	75	2120	4200	1.98	128	3620	1.16
B3	75	928	900	1.07	92	1140	0.87
B4	75	928	990	1.07	128	1580	0.63
B1	75	1856	1530	0.82	66	1630	0.94
B2	75	1856	2070	1.12	82	2030	1.02
C1	75	900	1080	1.02	100	1200	0.90
C2	75	1800	1740	0.97	82	1970	0.88
D2	75	975	862	0.88	66	858	1.00
D1	75	1870	900	0.48	100	2490	0.36
E1	75	1010	1125	1.11	116	1565	0.72
E2	75	1940	1215	0.63	92	2380	0.51
A10	50	785	2040	2.60	68	1070	1.91
A9	50	1570	3360	2.14	68	2140	1.57
A12	75	1270	2100	1.65	113	1910	1.10
A11	75	2540	4500	1.77	113	3820	1.18

CONCLUSIONS

The yield-line theory is a plastic theory based on the full plasticity of reinforced concrete slabs. By means of yield-line analysis, a rational solution for the failure load may be found for slabs of any shape, with various boundary conditions, and for concentrated loads as well as distributed and partially distributed loads. The analysis of complex slabs has been made possible with a minimum of mathematical effort. Yield-line theory is thus seen to be a powerful analytical tool for the structural engineer.

Guidelines for establishing axes of rotation and yield lines may be summarized as follows:

1. Yield lines are generally straight, and deformations of the slab may be considered as rotations of the slab parts about axes in their supports.
2. Axes of rotation must pass through columns.
3. Axes of rotation generally lie in lines of support. The support line may be a real hinge, or it may establish the location of a yield line which acts as a plastic hinge.
4. A yield line between two slab parts passes through the intersection of the axes of rotation of the adjacent parts.

By applying the guidelines to slabs, the general nature of the yield-line patterns may be determined. Final determination of the yield-line pattern may be made with the aid of the equilibrium conditions.

The yield-line analysis is predicated upon necessary rotation capacity being available at the yield lines. It is presumed that earlier failure will not occur due to shear, bond, or other causes. Also, yield-line theory gives no information on stresses, deflection, or severity of cracking under service load conditions. In general, lightly reinforced slabs will have adequate rotation capacity to attain the ultimate loads predicted by yield-line analysis. Most investigators who have carried out practical tests are agreed that yield-line analysis is on the conservative and safe side.

There are no standard designs or unique solutions of slabs by yield-line theory. Most authorities are agreed that the analysis is safe for predicting the collapse load of slabs if a reasonable attempt has been made to find the lowest mode of collapse.

When slabs are restrained in some way around their edges it has been found from tests that the restrained slab carries much more load than that indicated by normal yield-line analysis. This is thought by some investigators to be due to membrane effects.

However, it is well known that the yield-line theory is based on the more complete utilization of the actual properties of both steel and concrete, in contrast to the idealized elastic behavior on which the conventional straight-line theory is based. In spite of some limitations, yield-line analysis provides answers to problems of slab design which cannot be handled by other means, and so will undoubtedly assume a position of increasing importance in engineering practice. Furthermore, in structural engineering there is a steadily increasing realization of the importance of the yield-line analysis in reinforced concrete slab design.

ACKNOWLEDGMENTS

The author wishes to express his deep gratitude to his Major Instructor, Professor Vernon H. Rosebraugh, for his help, guidance, advice, and criticism in the preparation of the study.

Indebtedness is also expressed to Dr. Jack B. Blackburn and Dr. Robert R. Snell for their helpful suggestions and advice during the development of this report.

BIBLIOGRAPHY

1. Hognestad, E.
   "Yield-Line Theory for the Ultimate Flexural Strength of Reinforced Concrete Slabs." Journal of the American Concrete Institute, Vol. 24, No. 7. (March, 1953).

2. Wood, R. H.
   "Plastic and Elastic Design of Slabs and Plates." New York: Ronald Press Company, 1961.

3. Johansen, K. W.
   "Yield-Line Theory." Translated from the Danish. London: Cement and Concrete Association, 1962. Original Copyright 1943.

4. Yu, C. W., E. Hognestad
   "Review of Limit Design for the Structural Concrete." Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Proc. Paper 1878, Vol. 84, No. ST 8. (December, 1958.)

5. Dunham, C. W.
   "Advanced Reinforced Concrete." New York: McGraw-Hill Book Company, 1963.

6. Harper, A. S.
   "Laboratory Tests of Reinforced Concrete Slabs Supported on Three Sides and Free on a Fourth Side." A report on a semester project, Department of Civil Engineering, Arizona State University, 1966.

7. Ferguson, P. M.
   "Reinforced Concrete Fundamentals." New York: John Wiley & Sons, Inc., 1958.

8. Jones, L. L.
   "Ultimate Load Analysis of Reinforced and Prestressed Concrete Structures." New York: Interscience Publishers, 1962.

9. Timoshenko, S., S. Woinowsky-Krieger
   "Theory of Plates and Shells." New York: McGraw-Hill Company, 1959.

10. Winter, G., L. C. Urquhart, C. E. O'Rourke, A. H. Nilson
    "Design of Concrete Structures." New York: McGraw-Hill Book Company, 1964.

11. American Society of Civil Engineers
"Report of ASCE-ACI Joint Committee on Ultimate Strength Design." 1955.

12. American Concrete Institute
"ACI Standard Building Code Requirements for Reinforced Concrete." 1963.

13. Antoni Sawczuk and Thomas Jaeger
"Grenztragfahigkeits-Theorie der Platten." Berlin: Gottingen Heidelberg, 1963.

14. Hernnikoff, A. P.
"Importance of Strain Hardening in Plastic Design." Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 91, No. ST 4 (August, 1965)

15. Whitney, C. S.
"Plastic Theory of Reinforced Concrete Design." Proceedings of the American Society of Civil Engineers. December, 1940.

16. Anderson, B. G.
"Why Design by the Ultimate Strength Theories?" Journal of the American Concrete Institute, Vol. 23, No. 10. (June, 1952)

FLEXURAL RESISTANCE OF
REINFORCED CONCRETE SLABS

by

GUANG SHI LIN

B. S., National Taiwan University, 1964

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

The yield-line theory is a plastic theory for the prediction of the ultimate strength of reinforced concrete slabs. It was first developed by K. W. Johansen. It permits the determination of failure moments in slabs of any shape for a variety of support conditions and loading with a minimum of mathematical effort.

The general crack pattern which the yield lines will form may sometimes be deduced logically, from geometry, and may sometimes be obtained from model or full-scale tests. With some experience a designer will generally assume a yield pattern the first or second time, which gives a yield moment only a few percent in error. After the general pattern of yielding and rotation has been determined, the specific location and orientation of the axes of rotation and the failure load for the slab can be established by either the equilibrium method or the virtual work method. The equilibrium method is based on the equilibrium of the various segments of the slabs. The virtual work method makes use of the principle of equilibrium of internal and external work. All reasonable failure mechanisms for each case should be investigated in order that the correct solution may be found.

The ultimate loads obtained by the yield-line theory are, on the average, 80 to 90 percent of those observed in tests. The additional load carrying capacity of the slabs is generally credited to membrane action and strain hardening. It seems satisfactory for design purposes to base predictions of flexural strength on the slightly conservative values resulting from the yield-line theory.

It is realized that the yield-line theory is predicated upon having available the required rotation capacity at the yield lines. It is presumed that earlier shear or bond failure will not occur. In addition, the theory gives no information on stresses, deflection, or severity of cracking under service load conditions. Therefore, it must be supplied by a check of the conditions under service loads, particularly deflections under sustained loadings. This may be achieved through the practical design specifications of minimum values for reinforcement percentages and slab thickness.

However, by means of the yield-line theory, the ultimate loads can be predicated for slabs of any shapes, supported in a variety of ways, and for concentrated loads as well as distributed and partially distributed loads. It usually can provide a rational solution to problems of slab design which cannot be handled by other means, and so will undoubtedly be a powerful analytic method of increasing importance in structural engineering.

The purpose of this report is to present an outline of yield-line theory in order to review the basis and technique of slab analysis in this setting. The author believes that this report does give positive results toward the achievement of this purpose.