MATRIX ANALYSIS OF CONTINUOUS PARABOLIC ARCHES

by

CHAO-SHYONG HSU

Diploma, Taipei Institute of Technology, 1963

A MASTER'S REPORT

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

Approved by:

[Signature]

Major Professor
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>SYNOPSIS</td>
<td>1</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>2</td>
</tr>
<tr>
<td>SINGLE FIXED ARCH THEORY</td>
<td>4</td>
</tr>
<tr>
<td>MATRIX ANALYSIS OF CONTINUOUS ARCHES BY THE DISPLACEMENT METHOD</td>
<td>9</td>
</tr>
<tr>
<td>NUMERICAL EXAMPLE</td>
<td>16</td>
</tr>
<tr>
<td>CONCLUSIONS</td>
<td>30</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>31</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>32</td>
</tr>
<tr>
<td>NOTATION</td>
<td>33</td>
</tr>
<tr>
<td>APPENDIX A FLOW DIAGRAM FOR THE DISPLACEMENT METHOD</td>
<td>35</td>
</tr>
<tr>
<td>APPENDIX B CHECK RESULTS OF EXAMPLE USING ENERGY METHOD</td>
<td>48</td>
</tr>
<tr>
<td>APPENDIX C SINGLE ARCH INFLUENCE LINES</td>
<td>55</td>
</tr>
</tbody>
</table>
SYNOPSIS

The purpose of this report is to present a method of analyzing continuous arches with varying cross section on slender piers by obtaining influence lines of the redundants. The analysis is based on the displacement method.

First, the single fixed arch theory is presented. Then, the continuous arch is analyzed by applying the displacement method. Finally, a numerical example consisting of two unsymmetrical parabolic continuous arches with a central slender pier is given to illustrate the use of the method.

Influence lines for the redundants are drawn. The influence coefficients are checked by using the energy method. The comparison shows that the results agree closely with each other.

A detailed flow diagram is given to demonstrate the solution of the continuous arch problem by a digital computer. The flow diagram is based on the following assumptions: (1) it applies to any number of arch spans, with interior arch joints on piers; (2) the equation of the centroidal axis of each arch can be expressed as \( y = ax^2 + bx + c \) with origin at either end; (3) the ratio of the moment of inertia at any section of the arch to the moment of inertia at the crown is \( \sec \theta \), where \( \theta \) is the angle between the tangent to the arch and the horizontal axis; and (4) the ratio of the moment of inertia of the pier to that of the crown is constant.
INTRODUCTION

Matrix analysis is a relatively new approach to structural analysis. The main advantage of analyzing a structural system by the matrix method is that the analyses can be performed by a computer conveniently. The matrix method is particularly easy to handle if a structure must be analyzed for the effects of several loading patterns, such as that used in determining influence lines.

Fig. 1 Typical continuous arch on slender piers.

"An arch is a girder (beam or truss) usually curved in form, that develops reactions with inwardly directed horizontal components under the action of vertical loads alone."\(^1\) A typical continuous arch on slender piers is shown in Fig. 1. The effect of slender piers is, in general, to decrease the horizontal thrusts, to increase the crown moments and to throw stress onto adjacent arch spans and onto other piers. The slender piers are a necessary provision for a large span.

The matrix method presented in this report is the displacement method. Basically, it makes use of the single fixed arch theory, namely the method of analyzing a single fixed arch. The analysis requires a knowledge of the equations of the centroidal axis of each single arch and of the relative moments of inertia.

This method also involves application of the Müller-Breslau Principle and numerical integration. Although it is quite tedious to use for manual computations, the basic idea of each method is simple. It should be introduced whenever a computer is available.
SINGLE FIXED ARCH THEORY

An unsymmetrical fixed arch is shown in Fig. 2(a). It is statically indeterminate to the third degree. Figure 2(c) indicates the statically determinate base structure obtained by making a free end at point 0.

![Diagram of fixed arch](image)

Fig. 2 Fixed arch analysis by superposition.

In the following discussion, stresses are assumed to be below the elastic limit, that is, Hooke's law applies.

Superposition can be used as long as Hooke's law applies. From Fig. 2, the following three equations exist:

\[ x_1 \delta_{11} + x_2 \delta_{12} + x_3 \delta_{13} = \Delta_{10} , \]
\[ x_1 \delta_{21} + x_2 \delta_{22} + x_3 \delta_{23} = \Delta_{20} , \]
\[ x_1 \delta_{31} + x_2 \delta_{32} + x_3 \delta_{33} = \Delta_{30} . \]

In matrix notation,
\[
\begin{pmatrix}
\delta_{11} & \delta_{12} & \delta_{13} \\
\delta_{21} & \delta_{22} & \delta_{23} \\
\delta_{31} & \delta_{32} & \delta_{33}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
=
\begin{pmatrix}
\Delta_{10} \\
\Delta_{20} \\
\Delta_{30}
\end{pmatrix}
\cdots \cdots \cdots \cdots \cdots (1)
\]

where

- \(x_1\) = horizontal force at origin point 0,
- \(x_2\) = vertical force at point 0,
- \(x_3\) = moment at point 0,
- \(\delta_{ij}\) = displacement in base structure at point 0 in \(X_i\) direction, due to \(X_j\) = unit acting only,
- \(\Delta_{10}\) = displacement in base structure at point 0 in \(X_1\) direction, due to all external loads acting.

The \(\delta\) terms can be determined by using the dummy unit load method,

- \(\delta_{11} = \int_{\text{Area}} \frac{m_1^2}{EI} \ ds = \int_{A} \frac{y^2}{EI} \ ds = \int_{0}^{L} \frac{y^2}{EI} \cos \theta \ dx \),
- \(\delta_{22} = \int_{\text{Area}} \frac{m_2^2}{EI} \ ds = \int_{A} \frac{x^2}{EI} \ ds = \int_{0}^{L} \frac{x^2}{EI} \cos \theta \ dx \),
- \(\delta_{33} = \int_{\text{Area}} \frac{1}{EI} \ ds = \int_{0}^{L} \frac{dx}{EI} \cos \theta \),
- \(\delta_{12} = \delta_{21} = \int_{\text{Area}} \frac{m_1 m_2}{EI} \ ds = \int_{A} \frac{xy}{EI} \ ds = \int_{0}^{L} \frac{xy}{EI} \cos \theta \),
- \(\delta_{13} = \delta_{31} = \int_{\text{Area}} \frac{m_1}{EI} \ ds = \int_{A} \frac{y}{EI} \ ds = \int_{0}^{L} \frac{y}{EI} \cos \theta \),
- \(\delta_{23} = \delta_{32} = \int_{\text{Area}} \frac{m_2}{EI} \ ds = \int_{A} \frac{x}{EI} \ ds = \int_{0}^{L} \frac{x}{EI} \cos \theta \).
Defining

\[ \int_0^L \frac{y^2}{EI} \cos \theta \, dx = I_x, \quad \int_0^L \frac{x^2}{EI} \cos \theta \, dx = I_y, \quad \int_0^L \frac{dx}{EI} \cos \theta = A, \]

\[ \int_0^L \frac{xy}{EI} \cos \theta \, dx = I_{xy}, \quad \int_0^L \frac{y}{EI} \cos \theta \, dx = A_y, \quad \int_0^L \frac{x}{EI} \cos \theta = A_x. \]

Eq. (1) becomes,

\[
\begin{bmatrix} I_x & I_{xy} & A_y \\ I_{xy} & I_y & A_x \\ A_y & A_x & A \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} \Delta_{10} \\ \Delta_{20} \\ \Delta_{30} \end{bmatrix} \]

(2)

\(X_1, X_2, \) and \(X_3,\) which are obtained by setting \(\Delta_{10} = 1, \Delta_{20} = 0,\) and \(\Delta_{30} = 0,\) will cause a unit displacement in the \(X_1\) direction only when \(X_1, X_2, \) and \(X_3\) are applied simultaneously at 0. In other words, this is the way to determine the reactions at the origin point 0 due to a unit horizontal displacement at point 0.

The same argument applies to the other two cases, that is, \(\Delta_{10} = 0, \Delta_{20} = 1, \Delta_{30} = 0\) and \(\Delta_{10} = 0, \Delta_{20} = 0, \Delta_{30} = 1.\)

Substituting for \(\Delta_{10}, \Delta_{20}, \) and \(\Delta_{30}\) in Eq.(2),

\[
\begin{bmatrix} I_x & I_{xy} & A_y \\ I_{xy} & I_y & A_x \\ A_y & A_x & A \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
then,

\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix} = \begin{pmatrix}
    I_x & I_{xy} & \overline{A_y} \\
    I_{xy} & I_y & \overline{A_x} \\
    \overline{A_y} & \overline{A_x} & A
\end{pmatrix}^{-1} \begin{pmatrix}
    H_1 \\
    H_2 \\
    H_3
\end{pmatrix} = \begin{pmatrix}
    V_1 \\
    V_2 \\
    V_3
\end{pmatrix}
\]

\[
\begin{pmatrix}
    M_1 \\
    M_2 \\
    M_3
\end{pmatrix}
\]

where

$H_1$, $V_1$, and $M_1$ will cause a unit displacement at 0 in the $X_1$ direction,
$H_2$, $V_2$, and $M_2$ will cause a unit displacement at 0 in the $X_2$ direction,
$H_3$, $V_3$, and $M_3$ will cause a unit displacement at 0 in the $X_3$ direction.

To obtain the influence lines, Müller-Breslau’s Principle plays an important role. "The ordinates of the influence line for any stress element (such as axial force, shear, moment, or reaction) of any structure are proportional to those of the deflection curve which is obtained by removing the restraint corresponding to that element from the structure and introducing in its place a corresponding deformation into the primary structure which remains."\(^2\)

Furthermore, "In the case of an indeterminate structure, this principle is limited to structures the material of which is elastic and follows Hooke’s law."\(^2\)
In other words, Müller-Breslau’s Principle states that "an influence line may be drawn by producing artificially a unit displacement corresponding to the ‘stress’ for which the influence line is desired. The term ‘stress’ includes reaction, thrust, moment or shear, as the case may be."\(^3\)

---


\(^3\) P. C. A. Concrete Information ST 41-2, "Concrete Building Frames Analyzed by Moment Distribution," p.2.
Thus, the influence line for the horizontal force $X_1$ is the deflection curve caused by $H_1$, $V_1$, and $M_1$; the influence line for the vertical force $X_2$ is the deflection curve caused by $H_2$, $V_2$, and $M_2$; and the influence line for the moment $X_3$ is the deflection curve caused by $H_3$, $V_3$, and $M_3$.

To find the deflection curves, the conjugate beam method is used. The deflection curve of the real beam is obtained by double integration of the area of the elastic weight on the conjugate beam. That is

$$\text{defl'n} = \int_A \int \frac{M}{EI} \, ds,$$

where

$$M = M_1 + V_1 x + H_1 y,$$

or

$$M = M_2 + V_2 x + H_2 y,$$

or

$$M = M_3 + V_3 x + H_3 y.$$

This can be performed by a digital computer. The Gaussian 5-point Integration Formula is used for the first integration and Simpson's Rule is used for the second integration.

A detailed flow diagram for single fixed arch theory is given in the first part of the flow diagram of the displacement method in Appendix A, to illustrate the computer approach.

---

MATRIX ANALYSIS OF CONTINUOUS ARCHES BY THE DISPLACEMENT METHOD

Fig. 3 The choice of redundants.

Figure 3 shows a series of arches on slender piers. The arches may or may not be identical and symmetrical. It is assumed that the equation of the centroidal axis of each arch is given and that the relative moments of inertia are known. It is also assumed that the material obeys Hooke's law.

The structure shown in Fig. 3 is statically indeterminate to the ninth degree. In other words, nine redundants must be removed to obtain a statically determinate structure, or base structure. Among numerous base structures, the one shown in Fig. 3 is suggested. The reason for this selection is that with this base structure identical redundant patterns can be assigned to each single arch.

The redundants for the continuous arches can be analyzed by the displacement method once the stiffness values are known. These values can be conveniently determined by applying single fixed arch theory.
Figure 4 shows the four degrees of freedom of the structure shown in Fig. 3. The degrees of freedom at the joints are indicated by arrows \( d_1, d_2, d_3, \) and \( d_4 \). The positive sense of rotations, displacements, moments, and forces is as indicated by these arrows. The \( q \) terms are the unbalanced joint moments or unbalanced horizontal forces due to the external load \( P \).

The displacement method of analysis may be divided into the following operations:

(a) Analysis of each single arch assuming that they are fixed ended and determination of the end moments and thrusts due to \( d_i = 1, i=1, 2, 3, 4 \).

This can be done by applying the single fixed arch theory discussed in the previous section. Furthermore, influence lines for the reactions of each single arch can also be obtained.

(b) Determination of the moments and horizontal forces on the pier due to \( d_i = 1, i=1, 2, 3, 4 \).
Figure 5 shows a pier subjected to a unit rotation, with no translation and with the far end fixed.

Then, by the slope-deflection method,

\[
\begin{align*}
M_{BC} &= \frac{2EI}{L} (2\phi_B + \phi_C) = \frac{2EI}{L} (2) = \frac{4EI}{L}, \\
M_{CB} &= \frac{2EI}{L} (\phi_B + 2\phi_C) = \frac{2EI}{L}, \\
H_{BC} &= -\frac{1}{L} (M_{BC} + M_{CB}) = -\frac{6EI}{L^2}.
\end{align*}
\]

Figure 6 shows a pier subjected to a unit translation.
Figure 6 shows a pier subjected to a unit translation, with no rotation and the far end fixed.

Also, by the slope-deflection method,

\[ v_{BC} = \frac{2EI}{L} \left( -3 \frac{1}{L} \right) = -\frac{6EI}{L^2}, \]

\[ M_{CB} = \frac{2EI}{L} \left( -3 \frac{1}{L} \right) = -\frac{6EI}{L^2}, \]

\[ H_{BC} = -\frac{1}{L} (M_{BG} + M_{CB}) = \frac{12EI}{L^3}. \]

(c) Determination of \([ S ]\), the stiffness matrix for a single arch and pier; and of \([ K ]\), the stiffness matrix of the whole structure.

The values of the S terms are found in step (a). For instance, \( S_{ij} \) is the value of end moment or thrust for a single arch in \( X_1 \) direction due to \( d_j = 1 \). Figure 7 shows the determination of \( S_{ij} \) due to \( d_j = 1, j=1, 2, 3, 4 \).

(a) \( d_1 = 1 \).

Fig. 7 Determination of \([ S ]\).
Fig. 7 Determination of $[S]$. (Continued)
The stiffness coefficients of $\mathbf{K}$ at the joints are then obtained by combination of the $S_{ij}$ values and the stiffnesses of the piers obtained in step (b). In other words, $K_{ij}$ is the value of moment or force in the $q_i$ direction due to $d_j = 1$. For example,

$$K_{11} = S_{21} + S_{41} + \left( \frac{4EI}{L} \right)_{\text{pier}},$$

$$K_{21} = S_{31} + S_{61} + \left( \frac{6EI}{L^2} \right)_{\text{pier}}.$$

(d) Determination of the resultant reactions, $\{FT\}$.

Since $\mathbf{K}$ is obtained in step (c), the unknown rotations and displacements can be found from $\{D\} = [\mathbf{K}]^{-1}\{q\}$. With the rotations and displacements known, correction moments and forces at the ends of the members may be obtained, making use of the single arch stiffness $\mathbf{S}$. The reactions, due to the correction of $\{D\}$, are given by $\{FD\} = [\mathbf{S}]\{D\}$. Then the final moments and forces are obtained by $\{FT\} = \{FF\} + \{FD\}$, where $\{FF\}$ is the fixed end moment or force due to the external load $P$.

Influence coefficients of the redundants are determined by varying the position of $P$, where $P$ is a unit load. A numerical example will be given to demonstrate the method of analysis.

A complete flow diagram for determining the influence coefficient of the redundants by the displacement method is given in Appendix A. This flow diagram can be used to analyze any number of arches with interior arch joints on piers. The flow diagram has been formulated based on the following assumptions: (1) the equation of the centroidal axis of every single arch can be
expressed in the form $y = \alpha x^2 + \beta x + \gamma$, with origin at each end point;

(2) the ratio of $I$, the moment of inertia at any section of a single arch, to $I_c$, the moment of inertia at the crown of the arch, is equal to $\sec \theta$, where $\theta$ is the angle between the horizontal axis and the tangent to the arch at the corresponding section; (3) the ratio of $I_p$, the moment of inertia of the pier, to $I_c$ is constant.
NUMERICAL EXAMPLE

Fig. 8 Numerical example.

Given: A two-span continuous parabolic arch on a slender pier is given in Fig. 6.

For arch AB: \( y_1 = -0.008x_1^2 + 1.12x_1 \) origin at pt. A
\( y_2 = -0.008x_2^2 + 0.8x_2 \) origin at pt. B

For arch BC: \( y_3 = -0.008x_3^2 + 0.8x_2 \) origin at pt. B
\( y_4 = -0.008x_4^2 + 0.48x_4 \) origin at pt. C

Required: Influence lines for \( M_{AB}^*, M_{BA}^*, H_{AB}, M_{BC}^*, M_{CB}^* \), and \( H_{BC}^* \).
Solution:

Step 1. Determination of the end moment and thrust for each arch due to $d_1 = 1$, and $d_2 = 1$.

Fig. 9 End moments and forces of each arch due to $d_1 = 1$, $d_2 = 1$. (in terms of $EI_o$)
The moment and forces of each arch at support B due to $d_1 = 1$ or $d_2 = 1$ are determined by applying Eq. (3) from the single fixed arch theory. The moment at the opposite end, such as support A or support C, is then calculated from the equations of statics. The results are shown in Fig. 9.

Step 2. Determination of the end moment and horizontal force for the pier due to $d_1 = 1, d_2 = 1$. The results are illustrated in Fig. 10.

Fig. 10 End moments and horizontal forces due to $d_1 = 1, d_2 = 1$. (in terms of $EI_c$)

Horizontal force due to $d_1 = 1$, is

$$H = - \frac{6 EI}{L^2} = - \frac{6 E(5I_c)}{30^2} = -0.033333 EI_c.$$ 

Moment due to $d_1 = 1$, is

$$M = \frac{4 EI}{L^3} = \frac{4 E(5I_c)}{30} = 0.666667 EI_c.$$ 

Horizontal force due to $d_2 = 1$, is

$$H = \frac{12 EI}{L^2} = \frac{12 E(5I_c)}{30^2} = 0.002222 EI_c.$$
Moment due to $d_2 = 1$, is

$$M = -\frac{6EI}{L^2} = -\frac{6E(5I)}{30^2} = -0.033333 \frac{EI}{c}.$$ 

The values determined in Step 1 and Step 2 are listed in Table I.

<table>
<thead>
<tr>
<th></th>
<th>$d_1 = 1$</th>
<th>$d_2 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ARCH AB</strong></td>
<td>$M_{AB}$: $-0.0250000914$</td>
<td>$-0.002170142256$</td>
</tr>
<tr>
<td></td>
<td>$M_{BA}$: $+0.075000525$</td>
<td>$+0.002170152$</td>
</tr>
<tr>
<td></td>
<td>$H_{BA}$: $+0.002170152$</td>
<td>$+0.0001130281$</td>
</tr>
<tr>
<td><strong>ARCH BC</strong></td>
<td>$M_{CB}$: $-0.03749994224$</td>
<td>$-0.007324270436$</td>
</tr>
<tr>
<td></td>
<td>$M_{BC}$: $+0.11249952$</td>
<td>$+0.0073241912$</td>
</tr>
<tr>
<td></td>
<td>$H_{BC}$: $+0.0073241912$</td>
<td>$+0.0008583088$</td>
</tr>
<tr>
<td><strong>PIER BD</strong></td>
<td>$M_{BD}$: $+0.666667$</td>
<td>$-0.033333$</td>
</tr>
<tr>
<td></td>
<td>$H_{BD}$: $-0.033333$</td>
<td>$+0.0022222$</td>
</tr>
</tbody>
</table>

Step 3. Determination of the stiffness for single arches and pier.

They are determined in Step 1 and Step 2, namely,

$$[S] = \begin{pmatrix} 
-0.0250000914 & -0.002170142256 \\
+0.075000525 & +0.002170152 \\
+0.002170152 & +0.0001130281 \\
+0.11249952 & +0.0073241912 \\
-0.03749994224 & -0.007324270436 \\
+0.0073241912 & +0.0008583088 \end{pmatrix}$$

$$K_{11} = 0.075000525 + 0.11249952 + 0.666666667 = 0.851166712 \text{EI}_c,$$

$$K_{21} = 0.002170152 + 0.0073241912 - 0.033333333 = -0.0238389901 \text{EI}_c,$$

$$K_{12} = 0.002170152 + 0.0073241912 - 0.033333333 = -0.0238389901 \text{EI}_c,$$

$$K_{22} = 0.000113028h3 + 0.00085830888 + 0.002222222 = 0.00319355953 \text{EI}_c.$$

Or,

$$[K] = \text{EI}_c \begin{bmatrix} 0.851166712 & -0.0238389901 \\ -0.0238389901 & 0.00319355953 \end{bmatrix}$$

Then,

$$[K]^{-1} = \frac{1}{\text{EI}_c} \begin{bmatrix} 14.7881852 & 11.038946 \\ 11.038946 & 395.53273 \end{bmatrix}$$

Step 5. Set up $[q]$ and $[FF]$.

By applying single fixed arch theory, the influence lines for $M_{AB}$, $M_{BA}$, $H_{BA}$ in single arch AB and influence lines for $M_{BC}$, $M_{CB}$, $H_{BC}$ in single arch BC are obtained in Appendix C.

$[q]$, the matrix of unbalanced joint moments and unbalanced horizontal forces, is set up from the influence coefficients of $N_{BA}$, $M_{BC}$, $H_{BA}$, $H_{BC}$; while $[FF]$, the matrix of moments or thrust due to fixed ends, is set up from the influence coefficients of $M_{AB}$, $M_{BA}$, $H_{BA}$, $M_{BC}$, $M_{CB}$, $H_{BC}$. $[q]$ and $[FF]$ are given in Table II and Table III respectively.
Table II  Matrix $a_{ij}$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+0.00</td>
<td>+0.00</td>
</tr>
<tr>
<td>2</td>
<td>+0.663706</td>
<td>+0.060493907</td>
</tr>
<tr>
<td>3</td>
<td>+2.1570424</td>
<td>+0.2086422</td>
</tr>
<tr>
<td>4</td>
<td>+3.840006</td>
<td>+0.40000031</td>
</tr>
<tr>
<td>5</td>
<td>+5.214817</td>
<td>+0.5975311</td>
</tr>
<tr>
<td>6</td>
<td>+5.925919</td>
<td>+0.771605</td>
</tr>
<tr>
<td>7</td>
<td>+5.759987</td>
<td>+0.900000</td>
</tr>
<tr>
<td>8</td>
<td>+4.6459</td>
<td>+0.9679008</td>
</tr>
<tr>
<td>9</td>
<td>+2.65475</td>
<td>+0.9678999</td>
</tr>
<tr>
<td>10</td>
<td>-0.00009</td>
<td>+0.8999978</td>
</tr>
<tr>
<td>11</td>
<td>-2.96309</td>
<td>+0.7716017</td>
</tr>
<tr>
<td>12</td>
<td>-5.73645</td>
<td>+0.5975266</td>
</tr>
<tr>
<td>13</td>
<td>-7.68019</td>
<td>+0.399996</td>
</tr>
<tr>
<td>14</td>
<td>-8.01205</td>
<td>+0.208637</td>
</tr>
<tr>
<td>15</td>
<td>-5.8076</td>
<td>+0.060488</td>
</tr>
<tr>
<td>16</td>
<td>+0.00</td>
<td>+0.00</td>
</tr>
<tr>
<td>17</td>
<td>+4.85994</td>
<td>-0.189866</td>
</tr>
<tr>
<td>18</td>
<td>+5.11994</td>
<td>-0.600026</td>
</tr>
<tr>
<td>19</td>
<td>+2.93993</td>
<td>-1.033621</td>
</tr>
<tr>
<td>20</td>
<td>-0.00005</td>
<td>-1.3500243</td>
</tr>
<tr>
<td>21</td>
<td>-2.500032</td>
<td>-1.464664</td>
</tr>
<tr>
<td>22</td>
<td>-3.840012</td>
<td>-1.3500151</td>
</tr>
<tr>
<td>23</td>
<td>-3.780003</td>
<td>-1.0336038</td>
</tr>
<tr>
<td>24</td>
<td>-2.56</td>
<td>-0.6000052</td>
</tr>
<tr>
<td>25</td>
<td>-0.8999988</td>
<td>-0.18984518</td>
</tr>
<tr>
<td>26</td>
<td>+0.00</td>
<td>+0.00</td>
</tr>
</tbody>
</table>
### Table III Matrix $F^a_i$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>2</td>
<td>-5.80789</td>
<td>-0.663706</td>
<td>-0.2086422</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>3</td>
<td>-8.01236</td>
<td>-2.1570424</td>
<td>-0.0604939</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>4</td>
<td>-7.68051</td>
<td>-3.640006</td>
<td>-0.4000031</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>5</td>
<td>-5.73677</td>
<td>-5.214817</td>
<td>-0.5975311</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>6</td>
<td>-2.96343</td>
<td>-5.925919</td>
<td>-0.771605</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>7</td>
<td>-0.00044</td>
<td>-5.759987</td>
<td>-0.9000</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>8</td>
<td>+2.55448</td>
<td>-1.6459</td>
<td>-0.9679008</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>9</td>
<td>+1.64567</td>
<td>-2.65475</td>
<td>-0.9678999</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>10</td>
<td>+5.759802</td>
<td>+0.00009</td>
<td>-0.899978</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>11</td>
<td>+5.925783</td>
<td>+2.96309</td>
<td>-0.7716017</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>12</td>
<td>+5.214722</td>
<td>+5.73645</td>
<td>-0.5975266</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>13</td>
<td>+3.83995</td>
<td>+7.68019</td>
<td>-0.399996</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>14</td>
<td>+2.1570159</td>
<td>+8.01205</td>
<td>-0.208637</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>15</td>
<td>+0.6636984</td>
<td>+5.8076</td>
<td>-0.060488</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>16</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
<tr>
<td>17</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>-4.85994</td>
<td>-0.900001</td>
<td>+0.189866</td>
</tr>
<tr>
<td>18</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>-5.11994</td>
<td>-2.559999</td>
<td>+0.60026</td>
</tr>
<tr>
<td>19</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>-2.93993</td>
<td>-3.779982</td>
<td>+1.033621</td>
</tr>
<tr>
<td>20</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.00005</td>
<td>-3.839956</td>
<td>+1.3500243</td>
</tr>
<tr>
<td>21</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+2.500032</td>
<td>-2.499904</td>
<td>+1.464864</td>
</tr>
<tr>
<td>22</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+3.840012</td>
<td>+0.00015</td>
<td>+1.350015</td>
</tr>
<tr>
<td>23</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+3.780003</td>
<td>+2.94021</td>
<td>+1.0336038</td>
</tr>
<tr>
<td>24</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+2.56</td>
<td>+5.12026</td>
<td>+0.600052</td>
</tr>
<tr>
<td>25</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.8999988</td>
<td>+4.86028</td>
<td>+0.18984518</td>
</tr>
<tr>
<td>26</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
<td>+0.</td>
</tr>
</tbody>
</table>
Step 6. Determination of $[ FT ]$, the matrix of influence coefficients for $M_{AB}$, $M_{BA}$, $H_{BA}$, $M_{BC}$, $M_{CB}$, and $H_{BC}$.

$[ FT ]$ is obtained in the following manner.

$$[ FT ] = [ FF ] + [ FD ]$$

$$= [ FF ] + [ S ] [ D ]$$

$$= [ FF ] + [ S ] [ K ]^{-1} [ q ]$$

The results of $[ FT ]$ are drawn in Fig. 11 through Fig. 16.

These results are checked in Appendix B using the energy method. A comparison of the influence coefficients obtained from the two methods shows excellent agreement (see Figs 11-16).
<table>
<thead>
<tr>
<th>Energy Method</th>
<th>Displacement Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>+0.27015</td>
<td>+0.27018</td>
</tr>
<tr>
<td>+0.83647</td>
<td>+0.83658</td>
</tr>
<tr>
<td>+1.40257</td>
<td>+1.40276</td>
</tr>
<tr>
<td>+1.76510</td>
<td>+1.76533</td>
</tr>
<tr>
<td>+1.81372</td>
<td>+1.81397</td>
</tr>
<tr>
<td>+1.53117</td>
<td>+1.53139</td>
</tr>
<tr>
<td>+0.99320</td>
<td>+0.99335</td>
</tr>
<tr>
<td>+0.36862</td>
<td>+0.36869</td>
</tr>
<tr>
<td>-0.08074</td>
<td>-0.08073</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>+0.94874</td>
<td>+0.94892</td>
</tr>
<tr>
<td>+2.40805</td>
<td>+2.40850</td>
</tr>
<tr>
<td>+3.85338</td>
<td>+3.85415</td>
</tr>
<tr>
<td>+4.88537</td>
<td>+4.88643</td>
</tr>
<tr>
<td>+5.22975</td>
<td>+5.23106</td>
</tr>
<tr>
<td>+4.73745</td>
<td>+4.73891</td>
</tr>
<tr>
<td>+3.38650</td>
<td>+3.38600</td>
</tr>
<tr>
<td>+1.27209</td>
<td>+1.27350</td>
</tr>
<tr>
<td>-1.37346</td>
<td>-1.37224</td>
</tr>
<tr>
<td>-4.20065</td>
<td>-4.19974</td>
</tr>
<tr>
<td>-6.73289</td>
<td>-6.73229</td>
</tr>
<tr>
<td>-8.36842</td>
<td>-8.36820</td>
</tr>
<tr>
<td>-8.38033</td>
<td>-8.38045</td>
</tr>
<tr>
<td>-5.91658</td>
<td>-5.91695</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
</tbody>
</table>

Fig. 11 Influence line for MAB.
<table>
<thead>
<tr>
<th>Energy Method</th>
<th>Displacement Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>-0.444115</td>
<td>-0.444152</td>
</tr>
<tr>
<td>-1.35694</td>
<td>-1.35705</td>
</tr>
<tr>
<td>-2.25258</td>
<td>-2.25276</td>
</tr>
<tr>
<td>-2.79418</td>
<td>-2.79442</td>
</tr>
<tr>
<td>-2.80712</td>
<td>-2.80737</td>
</tr>
<tr>
<td>-2.27632</td>
<td>-2.27654</td>
</tr>
<tr>
<td>-1.34632</td>
<td>-1.34648</td>
</tr>
<tr>
<td>-0.32121</td>
<td>-0.32130</td>
</tr>
<tr>
<td>+0.33531</td>
<td>+0.33528</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>+5.12633</td>
<td>+5.12634</td>
</tr>
<tr>
<td>+7.28354</td>
<td>+7.28330</td>
</tr>
<tr>
<td>+7.31942</td>
<td>+7.31889</td>
</tr>
<tr>
<td>+5.97126</td>
<td>+5.97039</td>
</tr>
<tr>
<td>+3.86576</td>
<td>+3.86461</td>
</tr>
<tr>
<td>+1.51910</td>
<td>+1.51774</td>
</tr>
<tr>
<td>-0.66308</td>
<td>-0.66455</td>
</tr>
<tr>
<td>-2.38566</td>
<td>-2.38715</td>
</tr>
<tr>
<td>-3.46609</td>
<td>-3.46548</td>
</tr>
<tr>
<td>-3.82435</td>
<td>-3.82555</td>
</tr>
<tr>
<td>-3.50295</td>
<td>-3.50389</td>
</tr>
<tr>
<td>-2.64695</td>
<td>-2.64759</td>
</tr>
<tr>
<td>-1.51395</td>
<td>-1.51429</td>
</tr>
<tr>
<td>-0.47209</td>
<td>-0.47218</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
</tbody>
</table>

Fig. 12 Influence line for M_{BA}.  

\[ \text{G} \]
<table>
<thead>
<tr>
<th>Energy Method</th>
<th>Displacement Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>-0.01705</td>
<td>-0.01705</td>
</tr>
<tr>
<td>-0.05260</td>
<td>-0.05261</td>
</tr>
<tr>
<td>-0.08781</td>
<td>-0.08782</td>
</tr>
<tr>
<td>-0.10980</td>
<td>-0.10981</td>
</tr>
<tr>
<td>-0.11171</td>
<td>-0.11172</td>
</tr>
<tr>
<td>-0.09269</td>
<td>-0.09270</td>
</tr>
<tr>
<td>-0.05786</td>
<td>-0.05787</td>
</tr>
<tr>
<td>-0.01838</td>
<td>-0.01838</td>
</tr>
<tr>
<td>+0.00863</td>
<td>+0.00862</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>-0.08222</td>
<td>-0.08222</td>
</tr>
<tr>
<td>-0.23000</td>
<td>-0.23002</td>
</tr>
<tr>
<td>-0.40673</td>
<td>-0.40673</td>
</tr>
<tr>
<td>-0.58201</td>
<td>-0.58207</td>
</tr>
<tr>
<td>-0.73176</td>
<td>-0.73183</td>
</tr>
<tr>
<td>-0.83812</td>
<td>-0.83820</td>
</tr>
<tr>
<td>-0.88952</td>
<td>-0.88961</td>
</tr>
<tr>
<td>-0.88065</td>
<td>-0.88074</td>
</tr>
<tr>
<td>-0.81245</td>
<td>-0.81253</td>
</tr>
<tr>
<td>-0.69214</td>
<td>-0.69221</td>
</tr>
<tr>
<td>-0.53321</td>
<td>-0.53326</td>
</tr>
<tr>
<td>-0.35538</td>
<td>-0.35542</td>
</tr>
<tr>
<td>-0.18468</td>
<td>-0.18470</td>
</tr>
<tr>
<td>-0.05338</td>
<td>-0.05338</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
</tbody>
</table>

Fig. 13 Influence line for $\gamma_{BA} (=H_{AB})$.  

---
<table>
<thead>
<tr>
<th>Energy Method</th>
<th>Displacement Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>-0.10825</td>
<td>-0.10823</td>
</tr>
<tr>
<td>-0.55625</td>
<td>-0.55620</td>
</tr>
<tr>
<td>-1.43247</td>
<td>-1.43240</td>
</tr>
<tr>
<td>-2.69689</td>
<td>-2.69680</td>
</tr>
<tr>
<td>-4.18093</td>
<td>-4.18085</td>
</tr>
<tr>
<td>-5.58755</td>
<td>-5.58749</td>
</tr>
<tr>
<td>-6.49113</td>
<td>-6.49111</td>
</tr>
<tr>
<td>-6.33757</td>
<td>-6.33761</td>
</tr>
<tr>
<td>-4.44424</td>
<td>-4.44430</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>-1.18541</td>
<td>-1.18539</td>
</tr>
<tr>
<td>-1.11738</td>
<td>-1.11721</td>
</tr>
<tr>
<td>-0.219351</td>
<td>-0.219316</td>
</tr>
<tr>
<td>+1.05438</td>
<td>+1.05491</td>
</tr>
<tr>
<td>+2.46031</td>
<td>+2.46100</td>
</tr>
<tr>
<td>+3.72411</td>
<td>+3.72492</td>
</tr>
<tr>
<td>+4.66140</td>
<td>+4.66228</td>
</tr>
<tr>
<td>+5.15365</td>
<td>+5.15453</td>
</tr>
<tr>
<td>+5.14810</td>
<td>+5.14892</td>
</tr>
<tr>
<td>+4.65783</td>
<td>+4.65854</td>
</tr>
<tr>
<td>+3.76173</td>
<td>+3.76228</td>
</tr>
<tr>
<td>+2.60148</td>
<td>+2.60185</td>
</tr>
<tr>
<td>+1.39659</td>
<td>+1.39679</td>
</tr>
<tr>
<td>+0.41440</td>
<td>+0.41445</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
</tbody>
</table>

Fig. 14: Influence line for MBC.
<table>
<thead>
<tr>
<th>Energy Method</th>
<th>Displacement Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>+5.61124</td>
<td>+5.61152</td>
</tr>
<tr>
<td>+7.45553</td>
<td>+7.45579</td>
</tr>
<tr>
<td>+6.87745</td>
<td>+6.87766</td>
</tr>
<tr>
<td>+4.99324</td>
<td>+4.99340</td>
</tr>
<tr>
<td>+2.69084</td>
<td>+2.69096</td>
</tr>
<tr>
<td>+0.62983</td>
<td>+0.62991</td>
</tr>
<tr>
<td>-0.75855</td>
<td>-0.75845</td>
</tr>
<tr>
<td>-1.27132</td>
<td>-1.27123</td>
</tr>
<tr>
<td>-0.93390</td>
<td>-0.93381</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>+0.59136</td>
<td>+0.59135</td>
</tr>
<tr>
<td>+0.40143</td>
<td>+0.40132</td>
</tr>
<tr>
<td>-0.27726</td>
<td>-0.27750</td>
</tr>
<tr>
<td>-1.19609</td>
<td>-1.19646</td>
</tr>
<tr>
<td>-2.15036</td>
<td>-2.15084</td>
</tr>
<tr>
<td>-2.97926</td>
<td>-2.97983</td>
</tr>
<tr>
<td>-3.56591</td>
<td>-3.56653</td>
</tr>
<tr>
<td>-3.83732</td>
<td>-3.83794</td>
</tr>
<tr>
<td>-3.76440</td>
<td>-3.76499</td>
</tr>
<tr>
<td>-3.36199</td>
<td>-3.36249</td>
</tr>
<tr>
<td>-2.68882</td>
<td>-2.68922</td>
</tr>
<tr>
<td>-1.64754</td>
<td>-1.64760</td>
</tr>
<tr>
<td>-0.98466</td>
<td>-0.98483</td>
</tr>
<tr>
<td>+0.29072</td>
<td>+0.29076</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
</tbody>
</table>

Fig. 15 Influence line for MCB.
<table>
<thead>
<tr>
<th>Energy Method</th>
<th>Displacement Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>+0.09176</td>
<td>+0.09177</td>
</tr>
<tr>
<td>+0.29581</td>
<td>+0.29582</td>
</tr>
<tr>
<td>+0.52237</td>
<td>+0.52238</td>
</tr>
<tr>
<td>+0.70456</td>
<td>+0.70457</td>
</tr>
<tr>
<td>+0.79834</td>
<td>+0.79836</td>
</tr>
<tr>
<td>+0.78253</td>
<td>+0.78255</td>
</tr>
<tr>
<td>+0.65883</td>
<td>+0.65885</td>
</tr>
<tr>
<td>+0.45176</td>
<td>+0.45178</td>
</tr>
<tr>
<td>+0.20873</td>
<td>+0.20874</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
<tr>
<td>-0.09251</td>
<td>-0.09250</td>
</tr>
<tr>
<td>-0.07501</td>
<td>-0.07499</td>
</tr>
<tr>
<td>+0.01215</td>
<td>+0.01218</td>
</tr>
<tr>
<td>+0.13463</td>
<td>+0.13468</td>
</tr>
<tr>
<td>+0.26410</td>
<td>+0.26417</td>
</tr>
<tr>
<td>+0.37823</td>
<td>+0.37830</td>
</tr>
<tr>
<td>+0.46067</td>
<td>+0.46076</td>
</tr>
<tr>
<td>+0.50110</td>
<td>+0.50119</td>
</tr>
<tr>
<td>+0.49519</td>
<td>+0.49527</td>
</tr>
<tr>
<td>+0.49460</td>
<td>+0.49467</td>
</tr>
<tr>
<td>+0.35701</td>
<td>+0.35706</td>
</tr>
<tr>
<td>+0.24608</td>
<td>+0.24611</td>
</tr>
<tr>
<td>+0.13148</td>
<td>+0.13150</td>
</tr>
<tr>
<td>+0.03890</td>
<td>+0.03891</td>
</tr>
<tr>
<td>+0.0</td>
<td>+0.0</td>
</tr>
</tbody>
</table>
CONCLUSIONS

The following conclusions can be made from the work completed in this report.

(1) Since the influence lines for the redundants are obtained, the moment, shear and thrust at any section due to any loading condition on the structure can be calculated.

(2) The matrix analysis presented in this report can be performed conveniently by a digit computer rather than by hand computations.

(3) The method, presented in detail in the flow diagram, may be applied to any continuous arch, symmetrical or unsymmetrical.

(4) The flow diagram, given in appendix A, is based on the assumptions that the arches are parabolic; that the ratio of $I$, the moment of inertia at any section of the arch, to $I_c$, the moment of inertia at the crown, is sec9. However, with minor modifications the method is applicable to any other shape of arches and applicable to the case where the ratio of $I$ to $I_c$ is a function of $x$.

(5) In this study, the influence coefficients of the redundants, obtained by the displacement method and by the energy method, have been shown to agree very closely. It is evident that any structure may be analyzed by one method, and then independently checked by the other.
ACKNOWLEDGMENT

The writer wishes to express his sincere gratitude and deep appreciation to his major professor, Dr. Peter B. Cooper, for his valuable advice, criticism and suggestions during the preparation of this report.
REFERENCES


NOTATION

E modulus of elasticity of the material.
I moment of inertia at any section of arch.
I_c moment of inertia at crown.
I_p moment of inertia of pier.
θ the angle between the tangent at any section of the arch and horizontal axis.
A total area of an arch divided by EI.
X̄ the distance from centroid to y-axis.
Ȳ the distance from centroid to x-axis.
I_x moment of inertia of the area A with respect to the x-axis.
I_y moment of inertia of the area A with respect to the y-axis.
I_xy product of inertia of the area A with respect to x- and y-axes.
X_1 horizontal force at origin point O.
X_2 vertical force at origin point O.
X_3 moment at origin point O.
δ_{mn} displacement in base structure at O in X_m direction, due to X_n unit acting only.
Δ_{10} displacement in base structure at O in X_1 direction, due to all external loads acting.
m_1 moment anywhere in the structure due to X_1 unit.
m_2 moment anywhere in the structure due to X_2 unit.
d_j degree of freedom.
α, β, γ arbitrary constants.
unbalanced joint moment or horizontal force in $d_1$ direction.

$X_1, X_2, \ldots, X_i$ redundants.

$S_{ij}$ the moment or force in $X_i$ direction due to $d_j = 1$.

$K_{ij}$ the moment or force required at joint in $q_i$ direction to cause $d_j = 1$.

$P$ external load.

$D$ displacement of joint.

$FF$ moments or forces due to fixed ends.

$FD$ moments or forces due to displacement $D$.

$FT$ total moments or forces, sum of $FF$ and $FD$.

$[ \ ]$ matrix notation.

$\{ \}$ column matrix.

$[ S ]$ stiffness matrix of single arch.

$[ X ]$ stiffness matrix of whole structure.
I. Single Arch Analysis

Number of arches & interval length

Constants of Gaussian 5-pt. Integration Formula

<table>
<thead>
<tr>
<th>$t_5$</th>
<th>$t_4$</th>
<th>$t_3$</th>
<th>$a_5$</th>
<th>$a_4$</th>
<th>$a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-t_1$</td>
<td>$-t_2$</td>
<td>0</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
</tr>
<tr>
<td>0.90617985</td>
<td>0.53846931</td>
<td>0.56888889</td>
<td>0.23692689</td>
<td>0.47862869</td>
<td>0.58888889</td>
</tr>
</tbody>
</table>

Arch span length
Initialize [ FF ]

A

\[ j = 1 \]

\[ FF_{1,j} = 0 \]

\[ j = N3 \]

Yes

No \( j = j + 1 \)

\[ i = 3m \]

Yes

No \( i = i + 1 \)

\[ l = 1 \]

\[ N1 = \frac{n_k}{n} + 1 \]

\[ k = 2l - 1 \]

READ \( \beta_k, \theta_k, \gamma_k \)

1

2

3

\[ A = 0 \quad I_x = 0 \quad HA(1) = 0 \quad HI_x(1) = 0 \]

\[ \Delta T = 0 \quad I_y = 0 \quad HA_x(1) = 0 \quad HI_y(1) = 0 \]

\[ \Delta T_y = 0 \quad I_{xy} = 0 \quad HA_y(1) = 0 \quad HI_{xy}(1) = 0 \]

\[ j = 2 \]

HA(\(j\)) = 0 \quad HA_x(\(j\)) = 0 \quad HA_y(\(j\)) = 0

HI_x(\(j\)) = 0 \quad HI_y(\(j\)) = 0 \quad HI_{xy}(\(j\)) = 0

B
Gaussian integrating process

$\begin{align*}
  x_a &= n_x - (j - 2)h \\
  x_b &= n_x - (j - 1)h \\
  i &= 1 \\
  X &= t_i h/2 + (x_a + x_b)/2 \\
  Y &= \alpha_k x^2 + \beta_k X + \gamma_k \\
  G_A(i) &= 1. \quad G_{I_x}(i) = Y^2 \\
  G_X(i) &= X \quad G_{I_y}(i) = X^2 \\
  G_Y(i) &= Y \quad G_{I_{xy}}(i) = XY \\
  i &= 1 \\
  \text{No} \quad i &= i + 1 \\
  \text{Yes} \\
  i &= 5 \\
  \text{No} \quad i &= i + 1 \\
  \text{Yes} \\
  \text{Yes} \\
  HA(j) &= HA(j) + a_i G_A(i) h/2 \\
  HA_x(j) &= HA_x(j) + a_i G_X(i) h/2 \\
  HA_y(j) &= HA_y(j) + a_i G_Y(i) h/2 \\
  H_{I_x}(j) &= H_{I_x}(j) + a_i G_{I_x}(i) h/2 \\
  H_{I_y}(j) &= H_{I_y}(j) + a_i G_{I_y}(i) h/2 \\
  H_{I_{xy}}(j) &= H_{I_{xy}}(j) + a_i G_{I_{xy}}(i) h/2 \\
  i &= 5 \\
  \text{No} \quad i &= i + 1 \\
  \text{Yes} \\
  \end{align*}$
Single arch properties

\[
\begin{align*}
A &= A + HA(j) \\
A_x &= A_x + HA_x(j) \\
A_y &= A_y + HA_y(j) \\
A_{xy} &= A_{xy} + HA_{xy}(j)
\end{align*}
\]

\[
\begin{align*}
I_x &= I_x + HI_x(j) \\
I_y &= I_y + HI_y(j) \\
I_{xy} &= I_{xy} + HI_{xy}(j)
\end{align*}
\]

\[
\begin{align*}
\mathbf{AC}_k &= 
\begin{pmatrix}
I_x & I_{xy} & A_{xy} \\
I_{xy} & I_y & A_x \\
A_{xy} & A_x & A
\end{pmatrix}_k
\end{align*}
\]

\[
\begin{align*}
\mathbf{AC}_k^{-1} &= 
\begin{pmatrix}
H_{1k} & H_{2k} & H_{3k} \\
V_{1k} & V_{2k} & V_{3k} \\
M_{1k} & M_{2k} & M_{3k}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
AH(1) &= 0 \\
SH(1) &= 0 \\
AV(1) &= 0 \\
SV(1) &= 0 \\
AM(1) &= 0 \\
SM(1) &= 0
\end{align*}
\]

\[
N_2 = (N_1 + 1)/2
\]
Simpson's rule

\[
\begin{align*}
AH(j) &= AH(j) + AH(j-1) \\
AV(j) &= AV(j) + AV(j-1) \\
AM(j) &= AM(j) + AM(j-1)
\end{align*}
\]

\[
\begin{align*}
SH(i) &= SH(i-1) + (AH(j) + 4AH(j+1) + AH(j+2)) \frac{h}{3} \\
SV(i) &= SV(i-1) + (AV(j) + 4AV(j+1) + AV(j+2)) \frac{h}{3} \\
SM(i) &= SM(i-1) + (AM(j) + 4AM(j+1) + AM(j+2)) \frac{h}{3}
\end{align*}
\]

Flowchart:

1. \( j = N1 \)
   - Yes: \( j = j + 1 \)
   - No: \( i = 2 \)

2. \( j = 2i - 3 \)

3. \( i = N2 \)
   - Yes: \( i = i + 1 \)
   - No:

4. \( k = 2j \)
   - Yes:
     - \( NNI = 0 \)
       - \( i = 1 \)
       - \( NNI = NNI + n_i \)
       - \( i = i + 1 \)
     - Yes: \( i = 1 \)
   - No:
     - \( NNI = NNI + n_i \)
       - \( i = i + 1 \)

5. \( NNI = NNI + n_i - 1 \)
     - Yes: \( i = 1 \)
     - No:

6. \( E \)
II. Continuous Arch Analysis

Degrees of freedom

Initialize \([ q ]\)
Set up \([ q \)]

\[
q_{i,j} = \frac{FF_{3i+1}}{2}, j + \frac{FF_{3i+5}}{2}, j \\
q_{i+1,j} = \frac{FF_{3i+3}}{2}, j + \frac{FF_{3i+9}}{2}, j
\]

Height of pier & ratio of \(I\) of pier to \(I_c\)

Initialize \([ S \)]
Set up [S]

\[
\begin{align*}
S_{i,j} &= M_{3,j+1} + n_k V_{3,j+1} \\
&\quad + (d_j n_k^2 + \beta_j n_k + \gamma_j) H_{3,j+1} \\
S_{i+1,j} &= M_{3,j+1} \\
S_{i+2,j} &= H_{3,j+1} \\
S_{i+3,j} &= M_{3,j+2} \\
S_{i+4,j} &= H_{3,j+2} \\
S_{i+5,j} &= M_{3,j+2} + n_k V_{3,j+2} \\
&\quad + (d_j n_k^2 + \beta_j n_k + \gamma_j) H_{3,j+2}
\end{align*}
\]

\[
\begin{align*}
S_{i,j+1} &= M_{1,j+1} + n_k V_{1,j+1} \\
&\quad + (d_j n_k^2 + \beta_j n_k + \gamma_j) H_{1,j+1} \\
S_{i+1,j+1} &= M_{1,j+1} \\
S_{i+2,j+1} &= H_{1,j+1} \\
S_{i+3,j+1} &= M_{1,j+2} \\
S_{i+4,j+1} &= H_{1,j+2} \\
S_{i+5,j+1} &= M_{1,j+2} + n_k V_{1,j+2} \\
&\quad + (d_j n_k^2 + \beta_j n_k + \gamma_j) H_{1,j+2}
\end{align*}
\]
Pier properties

Initialize \([K]\)

Set up \([K]\)

\[\begin{align*}
MP_{11} &= 4d_1/p_1 & HP_{11} &= -6d_1/p_1^2 \\
MP_{12} &= -6d_1/p_1^2 & HP_{12} &= 12d_1/p_1^3
\end{align*}\]

\[\begin{align*}
X_{11} &= S_{21} + S_{41} + MP_{11} & K_{12} &= S_{22} + S_{42} + MP_{12} \\
X_{21} &= S_{31} + S_{61} + HP_{11} & K_{22} &= S_{32} + S_{62} + HP_{12}
\end{align*}\]
\[ K_{31} = S_{51} \quad K_{32} = S_{52} \]
\[ K_{41} = S_{61} \quad K_{42} = S_{62} \]

\[ j = f - 1 \]

\[ i = \frac{3i + 1}{2} \]

\[ K_{j-2,j} = S_{i-1,j} \]
\[ K_{j-1,j} = S_{1+1,j} \]
\[ K_{j,j} = S_{1,j} + S_{1+2,j} + \frac{1 + \text{MP}}{2},1 \]
\[ K_{j+1,j} = S_{1+1,j} + S_{1+4,j} + \frac{1 + \text{MP}}{2},1 \]

\[ f = 4 \quad \text{Yes} \]
\[ f = 4 \quad \text{No} \]

\[ j = 3 \]

\[ i = \frac{3i + 1}{2} \]
Influence coefficients of redundants of continuous arch

<table>
<thead>
<tr>
<th>Equation</th>
<th>Equation</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{j-2, j} = S_{i-1, j} )</td>
<td>( K_{j-2, j+1} = S_{i-1, j+1} )</td>
<td>( K_{j-1, j} = S_{i+1, j} )</td>
</tr>
<tr>
<td>( K_{j-1, j} = S_{i+1, j} )</td>
<td>( K_{j-1, j+1} = S_{i+1, j+1} )</td>
<td>( K_{j, j} = S_{i, j} + S_{i+2, j} )</td>
</tr>
<tr>
<td>( K_{j, j} = S_{i, j} + S_{i+2, j} + \frac{MP}{2} )</td>
<td>( K_{j, j+1} = S_{i, j+1} + S_{i+2, j} + \frac{MP}{2} )</td>
<td>( K_{j+1, j} = S_{i+1, j+1} + S_{i+3, j+1} + \frac{HP}{2} )</td>
</tr>
<tr>
<td>( K_{j+1, j} = S_{i+1, j+1} + S_{i+3, j+1} + \frac{HP}{2} )</td>
<td>( K_{j+2, j+1} = S_{i+3, j+1} )</td>
<td>( K_{j+3, j+1} = S_{i+4, j+1} )</td>
</tr>
</tbody>
</table>

Flowchart:

1. \( j = x - 3 \)  
2. \( j = j + 2 \)  
3. Inverse of matrix \( K \)  
4. \( [D] = [K]^{-1} \times [q] \)  
5. \( [FD] = [S] \times [D] \)  
6. \( [TP] = [TF] + [FD] \)  
7. Punch \([TF]\)  
8. Stop

Diagram:

- Flowchart showing the calculation steps for the influence coefficients.
The energy method may be divided into the following operations:

1) Six redundants are treated as external loads acting on a statically determinate base structure. The energy of arch AB, \( U_1 \); of arch BC, \( U_2 \); and of pier BD, \( U_3 \); is then calculated. Thus,

\[
U_1 = U_1(x_1, x_2, x_3),
\]

\[
U_2 = U_2(x_4, x_5, x_6),
\]

\[
U_3 = U_3(x_2, x_3, x_4, x_6).
\]

The total energy is given by \( U = U_1 + U_2 + U_3 \). Thus,

\[
\begin{bmatrix}
\frac{\partial U}{\partial x_1}
\end{bmatrix} = [C]\begin{bmatrix}
x_1
\end{bmatrix} \quad i = 1, 2, \ldots, 6.
\]

2) \( \begin{bmatrix} x_1 \end{bmatrix} = \left[C^T\right]^{-1}\begin{bmatrix} \frac{\partial U}{\partial x_1} \end{bmatrix} \quad i = 1, 2, \ldots, 6. \) Let \([B]\) denotes \( \left[C^T\right]^{-1}\), then

\[
\begin{bmatrix} x_1 \end{bmatrix} = [B]\begin{bmatrix} \frac{\partial U}{\partial x_1} \end{bmatrix}.
\]

3) The influence line for \( x_4 \) is the deflection curve caused by the \( \{B_{i4}\} \), where \( i = 1, 2, \ldots, 6. \)
Step 1 Set up \( \left\{ \frac{\partial U}{\partial x_i} \right\} = [C] \left\{ x_i \right\} \quad i = 1, 2, \ldots, 6. \)

Fig. B1 Free body of arch AB.

\[
U_1 = \int_0^L \frac{M^2}{2EI} \, ds = \int_0^{120} \frac{M^2}{2EI} \sec \theta \cos \theta \, dx
\]

\[
= \frac{1}{EI} \int_0^{120} \frac{1}{2} (x_1 + V_1 x + x_3 y)^2 \, dx
\]

\[
= \frac{1}{EI} \int_0^{120} \frac{1}{2} [x_1 + \frac{x}{120} (x_1 + x_2 - 19.2 x_3) + x_3 (-0.008 x^2 + 1.12 x)]^2 \, dx.
\]

\[
\frac{\partial U_1}{\partial x_1} = \frac{1}{EI} (40 x_1 + 20 x_2 + 1152 x_3),
\]

\[
\frac{\partial U_1}{\partial x_2} = \frac{1}{EI} (20 x_1 + 40 x_2 + 1152 x_3),
\]

\[
\frac{\partial U_1}{\partial x_3} = \frac{1}{EI} (1152 x_1 + 1152 x_2 + 53085.16 x_3).
\]
Fig. B2 Free body of arch BC.

\[ u_2 = \int_0^L \frac{M^2}{EI} \, ds = \int_0^{80} \frac{M^2}{2EI \sec \theta \cos \theta} \, dx \]

\[ = \frac{1}{EI_c} \int_0^{80} \frac{1}{2} (x_4 + x_6 - x_5 - 12.8x_6 + x_6(-0.006x^2 + 0.8x)) \, dx \]

\[ \frac{\partial u_2}{\partial x_4} = \frac{1}{EI_c} (26.66667x_4 + 13.33333x_5 + 341.33333x_6), \]

\[ \frac{\partial u_2}{\partial x_5} = \frac{1}{EI_c} (13.33333x_4 + 26.66667x_5 + 341.33333x_6), \]

\[ \frac{\partial u_2}{\partial x_6} = \frac{1}{EI_c} (341.33333x_4 + 341.33333x_5 + 6990.50667x_6). \]
Fig. B3 Free body of pier BD.

\[ U_3 = \int_0^L \frac{N}{2EI} \, dx \]

\[ = \frac{1}{EI} \left( \int_0^{30} \frac{1}{10} [X_2 - X_4 + (X_6 - X_3)x]^2 \, dx \right) \]

\[ \frac{\partial U_3}{\partial X_2} = \frac{1}{EI} \left( 6X_2 - 90X_3 - 6X_4 + 90X_6 \right), \]

\[ \frac{\partial U_3}{\partial X_3} = \frac{1}{EI} \left( -90X_2 + 1800X_3 + 90X_4 - 1800X_6 \right), \]

\[ \frac{\partial U_3}{\partial X_4} = \frac{1}{EI} \left( -6X_2 + 90X_3 + 6X_4 - 90X_6 \right), \]

\[ \frac{\partial U_3}{\partial X_6} = \frac{1}{EI} \left( 90X_2 - 1800X_3 - 90X_4 + 1800X_6 \right). \]
The total energy $U$ is the sum of $U_1$, $U_2$, and $U_3$. Thus,

\[
\begin{align*}
\frac{\partial U}{\partial x_1} &= \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_1} + \frac{\partial U_3}{\partial x_1} = \frac{1}{e^I} (40x_1 + 20x_2 + 1152x_3), \\
\frac{\partial U}{\partial x_2} &= \frac{1}{e^I} (20x_1 + 46x_2 + 1062x_3 - 90x_6), \\
\frac{\partial U}{\partial x_3} &= \frac{1}{e^I} (1152x_1 + 1062x_2 + 54885.16x_3 + 90x_4 - 1800x_6), \\
\frac{\partial U}{\partial x_4} &= \frac{1}{e^I} (-6x_2 + 90x_3 + 32.6667x_4 + 13.3333x_5 + 251.3333x_6), \\
\frac{\partial U}{\partial x_5} &= \frac{1}{e^I} (13.3333x_4 + 26.6667x_5 + 341.3333x_6), \\
\frac{\partial U}{\partial x_6} &= \frac{1}{e^I} (90x_2 - 1800x_3 + 251.3333x_4 + 341.3333x_5 + 8790.5067x_6).
\end{align*}
\]

In matrix notation,

\[
\begin{pmatrix}
\frac{\partial U}{\partial x_1} \\
\frac{\partial U}{\partial x_2} \\
\frac{\partial U}{\partial x_3} \\
\frac{\partial U}{\partial x_4} \\
\frac{\partial U}{\partial x_5} \\
\frac{\partial U}{\partial x_6}
\end{pmatrix} =
\begin{pmatrix}
40 & 20 & 1152 & 0 & 0 & 0 \\
20 & 46 & 1062 & -6 & 0 & 90 \\
1152 & 1062 & 54885.16 & 90 & 0 & -1800 \\
0 & -6 & 90 & 32.6667 & 13.3333 & 251.3333 \\
0 & 0 & 0 & 13.3333 & 26.6667 & 341.3333 \\
0 & 90 & -1800 & 251.3333 & 341.3333 & 8790.5067
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{pmatrix}
\]

$$[B] = \begin{pmatrix}
+0.071011512 & +0.017965618 & -0.0019095251 \\
+0.017965618 & +0.061222623 & -0.0016866988 \\
-0.0019095251 & -0.0016866999 & +0.000095585934 \\
+0.015160427 & +0.027521586 & -0.0010042091 \\
+0.010591581 & +0.017407099 & -0.0006699698 \\
-0.0014196713 & -0.0024349915 & +0.000091568307
\end{pmatrix}$$

$$= \begin{pmatrix}
+0.015160427 & +0.010591581 & -0.0014196713 \\
+0.027521586 & +0.017407098 & -0.0024349914 \\
-0.0010042091 & -0.0006699698 & +0.00009156831 \\
+0.054373016 & -0.002085218 & -0.0019610378 \\
-0.002085218 & +0.083138013 & -0.003481017 \\
-0.0019610378 & -0.003481017 & +0.000034879189
\end{pmatrix}$$

Step 3 Find influence lines for $X_i$, $i = 1, 2, \ldots, 6$.

The influence line for $X_j$ is the deflection curve caused by $\{B_{ij}\}$, $i = 1, 2, \ldots, 6$. The results are given in Fig. 11 through Fig. 16.
SINGLE ARCH INFLUENCE LINES
MATRIX ANALYSIS OF CONTINUOUS PARABOLIC ARCHES

by

CHAO-SHYONG HSU

Diploma, Taipei Institute of Technology, 1963

AN ABSTRACT OF A MASTER'S REPORT
submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967
This report presents a matrix analysis of continuous arches on slender piers by obtaining influence lines for the redundants. The analysis is based on the displacement method.

The method consists of two main parts. First, each single arch is analyzed by applying single fixed arch theory. Second, continuous arches are solved by applying the displacement method.

The procedure is demonstrated by a detailed flow diagram, given in Appendix A. The flow diagram is determined based on the assumptions that (1) the arches are parabolic, the equations of the centroidal axis of each arch with the origin at each end are given, (2) the ratio of the moment of inertia at any section of the arch to the moment of inertia at the crown is $\sec\theta$, where $\theta$ is the angle between the tangent to the arch and the horizontal axis; the ratio of the moment of the inertia of the pier to that of the crown is constant, (3) the interior joints are on piers. The method applies to any number of arch spans.

A numerical example consisting of two parabolic arch spans with a central slender pier and two fixed ends is then presented. In addition to analyzing the example by the displacement method, the energy method is used to check the results. A comparison of the results obtained from the two methods shows excellent agreement.

The matrix method of analysis used in the report involves the application of the Müller-Breslau Principle and numerical double integration. In this study, the Gaussian 5-point Integration Formula is used for the first integration and Simpson's Rule is used for the second integration.