ANALYSIS AND DESIGN OF BEAM-COLUMNS

by

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TABLE OF CONTENTS

SYNOPSIS

An "exact" method and three approximate methods for the analysis of the behavior of members subjected to bending moments and axial compression are reviewed on the basis of elastic theory. Sample problems that are frequently encountered in steel structures are treated. The effects of lateral and torsional buckling are not included. Two design methods that are adopted in the AISC (14) and AASHO (15) Specifications are described and a design example is worked in order to illustrate the design procedure according to the AISC Specifications (14).

INTRODUCTION

This paper contains a review of analysis and design procedures for steel members subjected to combined bending and compression. The bending may arise from transverse forces and/or from known eccentricities of the axial loads at one or both ends. These kinds of members are generally referred to as beam-columns. Beam-columns are commonly analyzed and designed as isolated members, whereas in practice they are usually parts of a frame. In framed steel structures, there are three categories of individual members:

(1) Beams. Beams are members subjected to forces which predominantly produce bending.

(2) Columns. Columns are members axially loaded predominantly in compression.

(3) Beam-columns. Beam-columns are members whose loads are a combination of the loads on beams and columns as stated above.

When the bending is small compared with the axial force, the bending can often be neglected and the member can be treated as a centrally loaded column. On the other hand, when the axial force is small compared with the bending, the axial force may be negligible and the problem becomes that of beam analysis.

The basic equation for analyzing a beam-column problem is based on the relationship between the moment and the curvature of a beam, i.e., $EIy'' = -M$, and takes into account the compressive forces. In the equation $EIy'' = -M$, E, I, y'', and M are modulus of elasticity, moment of inertia, curvature, and moment, respectively. Professor N. M. Newmark (1)* has developed a numerical approach which provides a means for obtaining an approximate but satisfactory solution for beam-columns. The energy approach is also an approximate solution, but it is a convenient method when the loadings are complicated.

Early solution of beam-column problems have been reviewed by Bleich (2). Ketter, Kaminsky, and Beedle (3) have presented a method for determining the plastic behavior of laterally supported wide-flange shapes under combined axial forces and moments. Galambos (4) and Ketter (5) have presented numerically determined interaction curves for the maximum strength of beam-columns under various end conditions, including the effect of residual stress. The effect of end restraint on the eccentrically loaded column has been explored by Bijaard, Winter, Fisher and Mason (6). Two approximate methods for estimating the strength of beam-columns have been reviewed by B. G. Johnston (7), the first method is based on the concept that the load which produces the initiation of yielding in the fiber subjected to the

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*Numbers in parentheses refer to the numbered references in the bibliography.

maximum stress provides a lower bound of the failure load, and the second method is based upon the interaction equations.

METHODS OF ANALYSES

The purpose of the following analyses is to show the procedures for finding the maximum deflection and maximum bending moment of a member subjected to moment and axial compression simultaneously. The general method is an "exact" method, while the numerical method and the energy method are approximate methods.

GENERAL METHOD

By neglecting the effects of shearing deformations and axial shortening of the beam the differential equation (8) of the axis of the beam-column as shown in Fig. 1 can be expressed in the following alternate forms

$$EIy'' = -M , \tag{1}$$

$$EIy''' + Py' = -V , \tag{2}$$

$$EIy^{IV} + Py'' = q . \tag{3}$$

The differential equations can be solved by use of the given boundary conditions, and consequently the deflections, bending moments, and end rotations may be obtained. Three examples of the use of this method are given in the following sections.

The bending moments and forces shown in Fig. 1 are assumed to be positive in the directions shown unless otherwise noted.

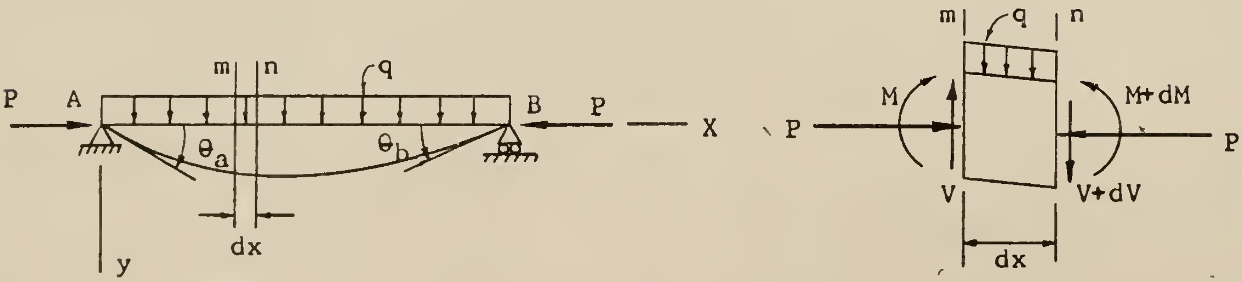

Fig. 1. Sign conventions.

Beam-column with Uniform Lateral Loads

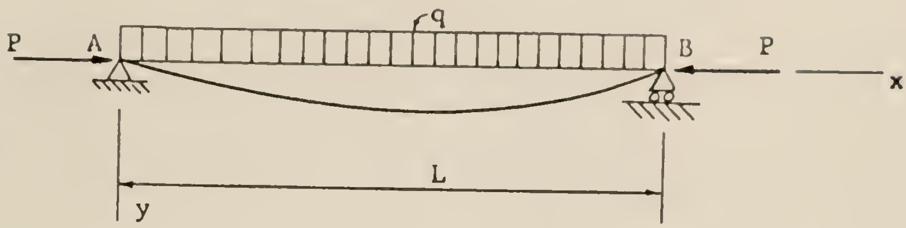

Fig. 2. Beam-column with uniform lateral loads.

The governing differential equation for the behavior of a beam subjected to transverse and axial loads is

$$EI\, y^{IV} + Py'' = q\ . \tag{3}$$

The general solution of this differential equation is

$$y = A\cos kx + B\sin kx + C\,x + D + \frac{q\,x^2}{2\,P}\ , \tag{4}$$

in which

$$k^2 = \frac{P}{EI}\ . \tag{5}$$

The arbitrary constants A, B, C, and D are evaluated from the boundary conditions of the beam. Since the deflection and bending moment are zero at the ends of the beam, the conditions are

$$y = y'' = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L\ .$$

From the two conditions at $x = 0$, we obtain

$$A = -D = \frac{q}{k^2\,P}\ ,$$

and the conditions at $x = L$ give

$$B = \frac{q}{k^2 P}\,\frac{1-\cos kL}{\sin kL}\ , \qquad C = -\frac{qL}{2P}\ .$$

Substituting these values of the constants into Eq. 4, the equation for the deflection curve is found to be

$$y = \frac{q}{EIk^4}\left[\frac{\sin kx + \sin k(L-x)}{\sin kL} - 1\right] - \frac{q}{2EIk^2}\,x(L-x) \ . \tag{6}$$

Letting x=L/2 in Eq. 6, the deflection at the center-line of the beam is found to be

$$\Delta = (y)_{x=L/2} = \frac{5qL^2}{384\ EI}\,f_1(u) \ , \tag{7}$$

where

$$u = kL/2 \ , \tag{8}$$

and

$$f_1(u) = \frac{12(2\sec u - 2 - u^2)}{5\ u^4} \ . \tag{9}$$

By taking the first derivative of Eq. 6, we obtain an equation for the slope at any point on the beam. Letting x=0 yields

$$\theta_a = \theta_b = (y')_{x=0} = \frac{qL^3}{24EI}\,f_2(u) \ , \tag{10}$$

where

$$f_2(u) = \frac{3(\tan u - u)}{u^3} \ . \tag{11}$$

The bending moment at the center of the beam is given by

$$M = -EI(y'')_{x=L/2} = \frac{qL^2}{8}\,f_3(u) \ , \tag{12}$$

where

$$f_3(u) = \frac{2(1-\cos u)}{u^2\cos u} \ . \tag{13}$$

The first factors on the right-hand sides of Eqs. 7, 10, and 12 are the values produced by a uniform load acting alone, and the second factors* $f_1(u)$, $f_2(u)$, and $f_3(u)$ represent the effects of the axial compression forces.

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*Tables of these functions have been provided in the Appendix of Reference 8.

Beam-column with a Concentrated Lateral Load

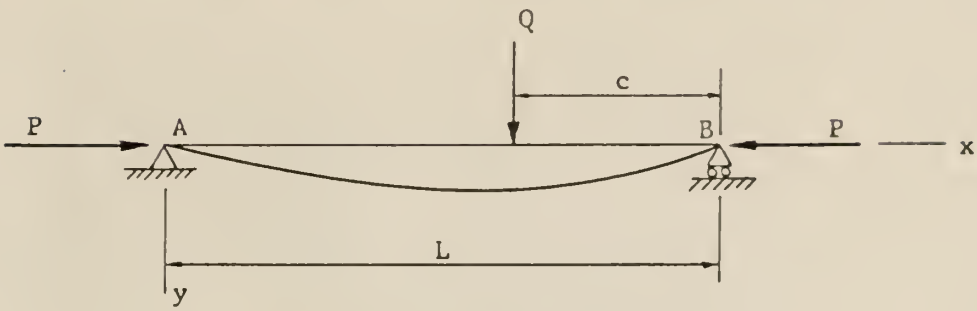

Fig. 3. Beam-column with a concentrated load.

The bending moments in the left- and right-hand portions of the beam in Fig. 3 are, respectively,

$$M_L = \frac{Qc}{L} x + Py , \qquad M_R = \frac{Q(L-c)}{L}(L-x) + Py .$$

Using Eq. 1, we have

$$EIy''_L = -\frac{Qc}{L} x - Py , \quad \text{and}$$

$$EIy''_R = -\frac{Q(L-c)}{L}(L-x) - Py .$$

The general solutions of these equations are

$$y_L = A \cos kx + B \sin kx - \frac{Qc}{PL} x , \quad \text{and} \tag{14}$$

$$y_R = C \cos kx + D \sin kx - \frac{Q(L-c)(L-x)}{PL} . \tag{15}$$

By use of the end conditions $y = 0$ at $x = 0$ and at $x = L$, we obtain

$$A = 0 , \qquad C = -D \tan kL .$$

Similarly, using the fact that the slopes and deflections of each segment must be equal at the point of application of the load Q, i.e. $y_L = y_R$ and $y'_L = y'_R$ at $x = L - c$, we obtain

$$B = \frac{Q \sin kc}{Pk \sin kL}, \qquad D = -\frac{Q \sin k(L-c)}{Pk \tan kL}.$$

Substituting these values of the constants into Eqs. 14 and 15, we obtain the following equations for the two portions of the deflection curve:

$$y_L = \frac{Q \sin kc}{Pk \sin kL} \sin kx - \frac{Qc}{PL} x, \qquad 0 \leq x \leq L-c \tag{16}$$

$$y_R = \frac{Q \sin k(L-c)}{Pk \sin kL} \sin k(L-x) - \frac{Q(L-c)(L-x)}{PL}. \qquad L-c \leq x \leq L \tag{17}$$

By taking the first and second derivatives of Eqs. 16 and 17, the expressions (8) for slope and curvature can be obtained. In the particular case of a load applied at the center of the beam, the maximum deflection, end rotation, and the maximum bending moment are obtained by substituting c=L/2 and the appropriate values of x into Eq. 16. This procedure gives

$$\Delta = (y)_{x=L/2} = \frac{QL^3}{48EI} \cdot f_2(u), \tag{18}$$

$$\theta_a = \theta_b = (y')_{x=0} = \frac{QL^2}{16EI} \cdot f_3(u), \tag{19}$$

$$M = -EI(y'')_{x=L/2} = \frac{QL}{4} \frac{\tan u}{u}. \tag{20}$$

Again, the second factors on the right-hand sides of Eqs. 18 to 20, $f_2(u)$, $f_3(u)$, and $(\tan u)/u$ are the effects of the axial compression on the beam.

Beam-column with Couples

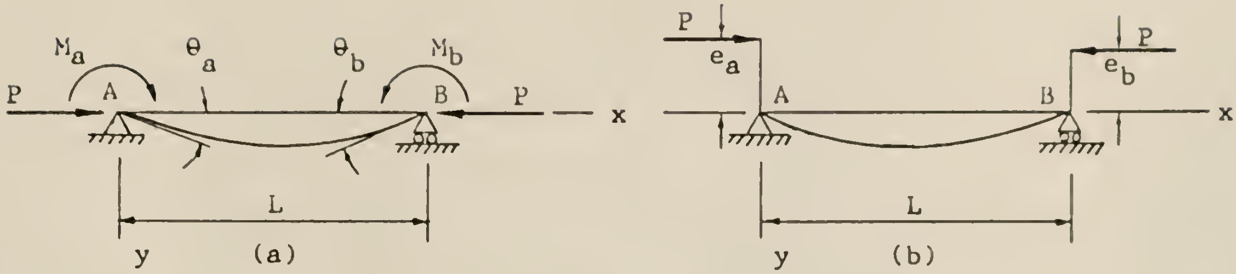

Fig. 4. Beam-column with couples.

If two couples M_a and M_b are applied at the ends A and B respectively, the bending moment at any section in the beam is

$$M = M_a - \frac{M_a - M_b}{L} x + Py . \tag{21a}$$

Using Eq. 1, we obtain

$$EIy'' = - M_a + \frac{M_a - M_b}{L} x - Py .$$

The general solution of this differential equation is

$$y = A \cos kx + B \sin kx + \frac{M_a - M_b}{EIk^2 L} x - \frac{M_a}{EIk^2} . \tag{21b}$$

By using the condition y = 0 at x = 0 and x = L, we obtain

$$A = \frac{M_a}{P} , \qquad B = \frac{1}{P \sin kL} (M_b - M_a \cos kL) .$$

By substituting these values of A and B into Eq. 21b, we obtain

$$y = \frac{M_a}{P} \left[\frac{\sin k(L-x)}{\sin kL} - \frac{L-x}{L} \right] + \frac{M_b}{P} \left[\frac{\sin kx}{\sin kL} - \frac{x}{L} \right] . \tag{22}$$

The bending moments M_a and M_b may arise from eccentrically applied compressive forces P acting as shown in Fig. 4b. By substituting $M_a = P\,e_a$ and $M_b = P\,e_b$ in Eq. 22, we obtain

$$y = e_a \left(\frac{\sin k(L-x)}{\sin kL} - \frac{L-x}{L}\right) + e_b \left(\frac{\sin kx}{\sin kL} - \frac{x}{L}\right) . \qquad (23)$$

The end rotations θ_a and θ_b in Fig. 4a are obtained by taking the first derivative of Eq. 22 and letting x=0 and x=L, respectively. The rotations may be expressed as

$$\theta_a = (y')_{x=0} = \frac{M_a L}{3EI} f_4(u) + \frac{M_b L}{6EI} f_5(u) , \qquad (24)$$

$$\theta_b = -(y')_{x=L} = \frac{M_a L}{6EI} f_5(u) + \frac{M_b L}{3EI} f_4(u) , \qquad (25)$$

where

$$f_4(u) = \frac{3}{2u}\left(\frac{1}{2u} - \frac{1}{\tan 2u}\right) , \qquad (26)$$

$$f_5(u) = \frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right) . \qquad (27)$$

Again, the factors* $f_4(u)$ and $f_5(u)$ in Eqs. 26 and 27 are the results of the effects of the axial compressive forces on the beam. In the case of two equal couples $M_a = M_b = M$, we obtain from Eq. 22

$$y = \frac{ML^2}{8EI} \frac{2}{u^2 \cos u} \left[\cos\left(u - \frac{2ux}{L}\right) - \cos u\right] . \qquad (28)$$

The deflection at the center of the beam is obtained by substituting x=L/2 in Eq. 28; this gives

$$\Delta = (y)_{x=L/2} = \frac{ML^2}{8EI} f_3(u) . \qquad (29)$$

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*Tables of these functions have been furnished in the Appendix of Reference 8.

The end rotations are obtained by taking the first derivative of Eq. 28 and substituting x=0; this yields

$$\theta_a = \theta_b = (y')_{x=0} = \frac{ML}{2EI} \cdot \frac{\tan u}{u} . \tag{30}$$

The maximum bending moment is obtained by taking the second derivative of Eq. 28 and substituting x=L/2, from which

$$M_{\mathrm{C\!\!\!L}} = - EI(y'')_{x=L/2} = M \cdot \sec u . \tag{31}$$

From the examples given above, it may be seen that the deflections, bending moments, and end rotations are linear functions of lateral loadings or applied couples. This indicates that the principle of superposition, which is widely used when lateral loads act alone on a beam, can also be applied in the case of the combined axial and lateral loads, but with the restraint that the axial forces should act with lateral load superposed.

The center deflections under three conditions of loading (i.e. uniform load, one concentrated load at the center-line, and two equal end moments) are given in Eqs. 7, 18, and 29 respectively. In each case, the deflection is equal to the product of two factors, the first factor being the deflection without axial load and the second factor being an amplification factor which is the effect of the axial compression and thus depends on the value of "P/P_{cr}". An approximate expression (8) for the amplification factors $f_1(u)$, $f_2(u)$, $f_3(u)$ is

$$\frac{1}{1 - P/P_{cr}} ,$$

where P_{cr} is the critical load. For a simply supported beam, the critical load is

$$P_{cr} = \frac{\pi^2 EI}{L^2} .$$

This approximate expression for the amplification factor can be used with good accuracy, in place of $f_1(u)$, $f_2(u)$, and $f_3(u)$. For values of P/P_{cr} less than 0.6, the error is less than 2 per cent (8).

ENERGY METHOD

The energy method is widely used in structural mechanics; it represents a unique mathematical application of the basic law of conservation of energy. In applying this method to beam-column problems, it is necessary to assume a deflection curve for the beam-column which satisfies the end conditions of the beam. A deflection curve often used is expressed by

$$y = \sum_n a_n \sin \frac{n\pi x}{L} , \tag{32}$$

where the a_n are the undetermined maximum amplitudes of the corresponding sine curves. Assuming a virtual deflection da_n of the beam, the strain energy of the beam will increase an amount equal to ΔU while the external loads do work equal to ΔT. Letting

$$\Delta U = \Delta T , \tag{33}$$

where
$$U = \int_0^L \frac{M^2\,dx}{2EI} = \frac{EI}{2} \int_0^L (y'')^2\,dx , \tag{34}$$

the coefficient a_n can be determined, and consequently the deflections, end rotations, and bending moments can be found. The three examples given before are treated by this method in the following sections.

Beam-column with Uniform Lateral Loads

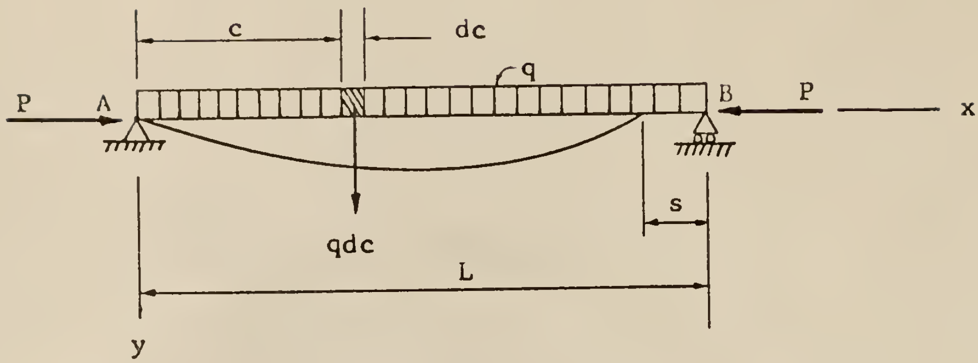


Fig. 5. Beam-column with uniform lateral loads

Using Eq. 32, the strain energy of bending of a beam is

$$U = \frac{EI}{2}\int_0^L (y'')^2\,dx = \frac{EI}{2}\int_0^L \left[-\sum_n a_n \frac{n^2\pi^2}{L^2}\sin\frac{n\pi x}{L}\right]^2 dx ,$$

or

$$U = \frac{EI\pi^4}{4L^3}\sum_n n^4 a_n^2 . \tag{35}$$

The strain energy increase due to the virtual change da_n in a_n is

$$\Delta U = \frac{\partial U}{\partial a_n}\,da_n = \frac{EI\pi^4}{2L^3}\cdot n^4 a_n\,da_n . \tag{36}$$

The horizontal displacement (9) of end B, which is equal to the difference between the length of the deflection curve and the length of the initial straight form, is

$$s = \frac{1}{2}\int_0^L (y')^2\,dx = \frac{\pi^2}{4L}\sum_n a_n^2 n^2 . \tag{37}$$

The change in s due to the virtual change of da_n in a_n is

$$ds = \frac{\partial s}{\partial a_n} da_n = \frac{\pi^2}{2L} \cdot n^2 a_n da_n . \quad (38)$$

Work done by the axial force P is

$$\Delta T_1 = P \cdot ds = \frac{P\pi^2}{2L} \cdot n^2 a_n da_n . \quad (39)$$

Work done by the uniform lateral load is

$$\Delta T_2 = \int_0^L (q\ dc)(\Delta y)_{x=c} = \frac{2qL}{\pi} \frac{1}{n} da_n , \quad \text{for odd values of } n. \quad (40)$$

Substituting the values of ΔU and ΔT (sum of ΔT_1 and ΔT_2) in Eq. 33 gives

$$a_n = \frac{4qL^4}{\pi^5 EI} \frac{1}{n^3(n^2 - \alpha)} , \quad \text{for odd values of } n, \quad (41)$$

where $\alpha = P/P_{cr}$.

Substituting this value of a_n into Eq. 32, yields the equation for deflection curve.

$$y = \frac{4qL^4}{\pi^5 EI} \sum_{n=odd} \frac{1}{n^3(n^2 - \alpha)} \sin \frac{n\pi x}{L} . \quad (42)$$

Using only the first term of Eq. 42, the deflection at the center of a beam is

$$\Delta = (y)_{x=L/2} = \frac{4qL^4}{\pi^5 EI} \frac{1}{(1 - \alpha)} = \frac{5.03\ qL^4}{384\ EI} \frac{1}{(1 - \alpha)} . \quad (43)$$

The bending moment at the center of the member is

$$M = \frac{qL^2}{8} + P\Delta = \frac{qL^2}{8} + \frac{5.03\ qL^4}{384\ EI} \frac{P}{(1 - \alpha)} . \quad (44)$$

Beam-column with a Concentrated Load

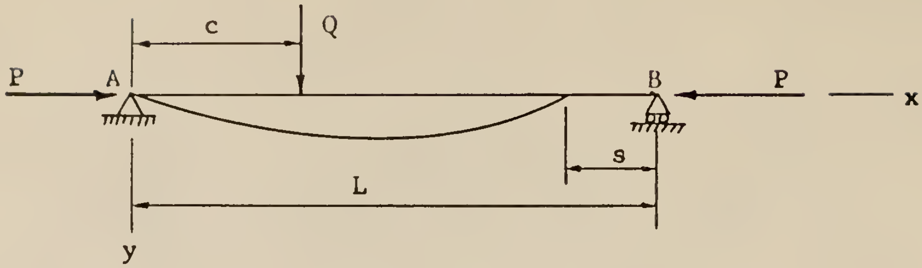


Fig. 6. Beam-column with a concentrated load.

Assuming that the deflection curve is as expressed in Eq. 32, the change in strain energy of the beam due to the change da_n in a_n is

$$\Delta U = \frac{EI\pi^4}{2L^3} \cdot n^4 a_n da_n . \tag{36}$$

Work done by the external forces P and Q due to the change da_n in a_n is given by the following equation:

$$\Delta T = P \cdot ds + Q \cdot (\Delta y)_{x=c} = \frac{P\pi^2}{2L} n^2 a_n da_n + Q \cdot \sin\frac{n\pi c}{L} da_n .$$

By use of the fact that $\Delta U = \Delta T$, we obtain

$$a_n = \frac{2QL^3}{\pi^4 EI} \cdot \frac{1}{n^2 (n^2 - \alpha)} \sin\frac{n\pi c}{L} . \tag{45}$$

Substituting this value into Eq. 32 we obtain

$$y = \frac{2QL^3}{\pi^4 EI} \sum_n \frac{1}{n^2 (n^2 - \alpha)} \sin\frac{n\pi c}{L} \sin\frac{n\pi x}{L} . \tag{46}$$

The center deflection and center moment respectively are given by the following expressions:

$$\Delta = (y)_{x=L/2} = \frac{2QL^3}{\pi^4 EI} \frac{1}{1-\alpha} = \frac{.986\ QL^3}{48\ EI} \frac{1}{1-\alpha} , \tag{47}$$

$$M = \frac{QL}{4} + P\Delta = \frac{QL}{4} + \frac{.986\ QL^3}{48\ EI} \frac{P}{(1-\alpha)} . \tag{48}$$

Beam-column with Couples

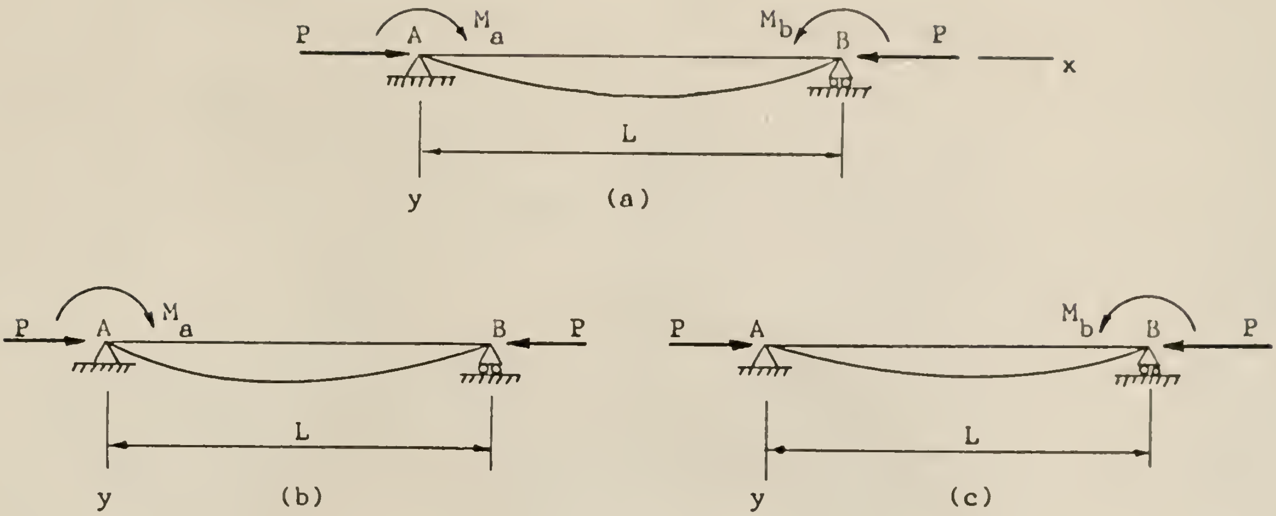


Fig. 7. Beam-column with couples.

In order to simplify the problem, we may divide the problem of Fig. 7a into two problems as shown in Fig. 7b and 7c by applying the principle of superposition. In solving the problem of Fig. 7b, we use Eq. 32 as the assumed deflection curve; then the change in strain energy due to the virtual change da_n in a_n is the same as that given in Eq. 36. Work done by the axial force P and the applied couple M_a due to the virtual change da_n in a_n is expressed as

$$\Delta T = P \cdot ds + M_a \cdot d\theta_a \ .$$

The term ds has been given in Eq. 38 and $d\theta_a$ is the angular displacement due to the virtual change da_n in a_n; this may be expressed as

$$d\theta_a = \frac{\partial (y')}{\partial a_n} \cdot da_n \Bigg|_{x=0} = \frac{n\pi}{L} da_n \ , \qquad (49)$$

from which the following equation is obtained

$$\Delta T = \frac{M_a n\pi}{L} da_n + \frac{P n^2 \pi^2}{2L} \cdot a_n da_n . \tag{50}$$

Setting $\Delta U = \Delta T$, we obtain

$$a_n = \frac{2M_a L^2}{\pi^3 EI} \frac{1}{n(n^2 - \alpha)} . \tag{51}$$

Using this value in Eq. 32, the following result is obtained

$$y = \frac{2M_a L^2}{\pi^3 EI} \sum_n \frac{1}{n(n^2 - \alpha)} \sin \frac{n\pi x}{L} . \tag{52}$$

The deflection curve of the beam in Fig. 6c is obtained by using (L-x) instead of x and M_b instead of M_a in Eq. 54, which gives

$$y = \frac{2M_b L^2}{\pi^3 EI} \sum_n \frac{1}{n(n^2 - \alpha)} \sin \frac{n\pi (L-x)}{L} . \tag{53}$$

By applying the principle of superposition, the deflection curve of the beam in Fig. 6a is found to be

$$y = \frac{2M_a L^2}{\pi^3 EI} \sum_n \frac{1}{n(n^2 - \alpha)} \sin \frac{n\pi x}{L} + \frac{2M_b L^2}{\pi^3 EI} \sum_n \frac{1}{n(n^2 - \alpha)} \sin \frac{n\pi (L-x)}{L} . \tag{54}$$

For the case of two equal moments $M_a = M_b = M_o$, the deflection curve is

$$y = \frac{4M_o L^2}{\pi^3 EI} \sum_{n=odd} \frac{1}{n(n^2 - \alpha)} \sin \frac{n\pi x}{L} . \tag{55}$$

Since the series is a rapidly converging one we can obtain reasonable approximations to the center deflection and center moment by taking the first term of the series, as indicated in the following equations

$$\Delta = (y)_{x=L/2} = \frac{4 M_o L^2}{\pi^3 EI} \frac{1}{1 - \alpha} = \frac{1.03 M_o L^2}{8EI} \frac{1}{1 - \alpha} , \tag{56}$$

$$M = M_o + P \cdot \Delta = M_o + \frac{1.03 M_o L^2}{8EI} \frac{P}{1 - \alpha} . \tag{57}$$

NUMERICAL METHOD

By applying the conjugate-beam method, Newmark (1) has developed a numerical procedure for the determination of deflections and moments of a beam-column with lateral load. This procedure eliminates much of the mathematics of the "exact" differential equation approach or the finite difference approximations. When the bar has a cross section which varies along the span or has a complicated distributed loading, a numerical procedure of successive approximations is quite useful. The purpose of this section is to compute the deflections and the moments of a beam-column by use of Newmark's numerical procedure.

The relationship between the distributed load q, shear force V, moment M, slope θ, and deflection y of beams are well known. They can be expressed as follows:

$$
\begin{aligned}
q &= \frac{dv}{dx} \quad \text{or} \quad V = \int q\,dx \,, \\
V &= \frac{dM}{dx} \quad \text{or} \quad M = \int v\,dx \,, \\
\frac{M}{EI} &= \frac{d\theta}{dx} \quad \text{or} \quad \theta = \int \frac{M}{EI}\,dx \,, \\
\theta &= \frac{dy}{dx} \quad \text{or} \quad y = \int \theta\,dx \,.
\end{aligned}
\tag{58}
$$

From the above equations, the shears, moments, slopes and deflections of a beam can be determined by successive numerical integration.

In using this method, a beam is divided into segments and the distributed loading is reduced to equivalent concentrated loads at each division point, or station along the beam. The equivalent concentrated loads at stations are equal and opposite to the reactions at the stations when the segments of the beam between the stations are treated as simply supported beams. The variation of the distributed load (10) can be assumed to be:

(1) Linear between stations (Fig. 8a, b),

(2) A second-degree Parabola (Fig. 8c, d).

An example is given in Table 1 to illustrate the application of Newmark's numerical method. The steps for computing deflections and moments in a beam-column are as follows:

1. Compute deflections and moments due to lateral loads alone:

   a. Obtain the values of moment at each station.

   b. Compute the M/EI values which are the fictitious loadings in the conjugate beam.

   c. Compute equivalent concentrated loads "R" at each division point by using Fig. 8c and d.

   d. Calculate "Average Slopes" which are the fictitious shearing forces in the conjugate beam. Due to symmetry, the reaction at the left support is equal to one-half of the equivalent concentrated load "R", i.e.,

$$(6.84 + 8.54 + 7.42) + \frac{1}{2}(7.92) = 26.76 .$$

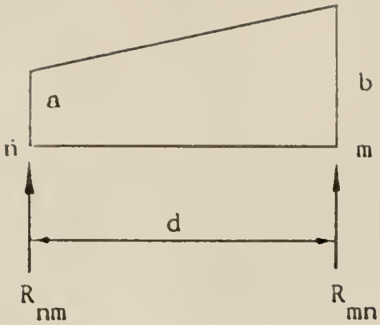
   e. Compute deflections, "y_i", which are the fictitious moments in the conjugate beam due to the lateral loads alone and are obtained by the following equation:

$$M_j = M_{j-1} + V_k \cdot d ,$$

   where

   d = segment length,

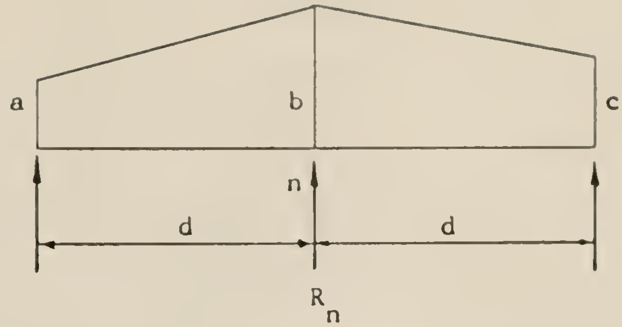
   V_k = Fictitious shearing forces in conjugate beam between station "j" and "j-1", and



$$R_{nm} = \frac{d}{6}(2a+b)$$

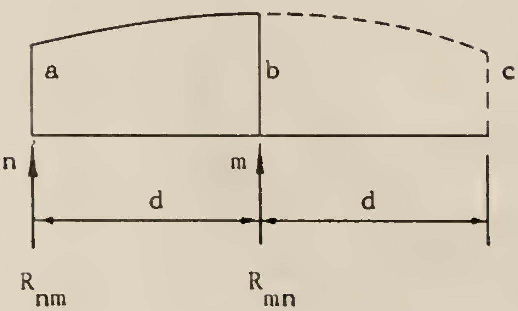
$$R_{mn} = \frac{d}{6}(2b+a)$$

(a)



$$R_n = \frac{d}{6}(a+4b+c)$$

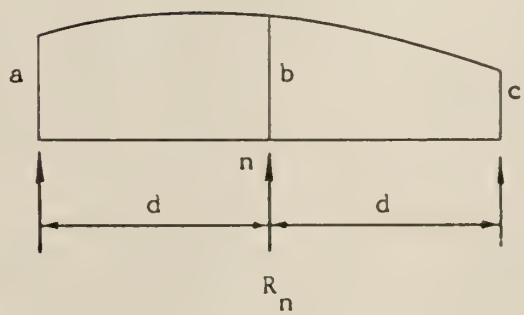
(b)



$$R_{nm} = \frac{d}{24}(7a+6b-c)$$

$$R_{mn} = \frac{d}{24}(3a+10b-c)$$

(c)



$$R_n = \frac{d}{12}(a+10b+c)$$

(d)

Fig. 8. Equivalent concentrated load.

j = Station number.

2. Calculation of buckling load:

a. The procedure is to estimate a reasonable trial deflected configuration and calculate the trial bending moments produced by the axial load. Due to symmetry, the values of y_{1c} selected in Table 1 are ordinates to a sine curve. The deflections produced by these bending moments are then computed (same procedures as stated in Step 1 above) and compared with the estimated trial deflections. The ratios of the assumed deflections y_{1a} to the computed deflections y_{1c} must equal one for the stable configuration to exist (11). Considering the maximum and minimum values of the ratios, we see that the upper and lower limits for P_{cr} are

$$\frac{15.35\ EI}{L^2} < P_{cr} < \frac{17\ EI}{L^2} .$$

b. The results can be improved by repeating the cycle of calculations. The second cycle begins with the deflections y_{2a}, which are proportional to the deflections y_{1c} found from the first cycle of computations. These values can be multiplied by any constant factor in order to adjust the order of magnitude of the figures. In this case, they are multiplied by 100/376 in order to give a_2 as the deflection at the center. The results of the second cycle show that the load P_{cr} is between the values

$$\frac{16.4\ EI}{L^2} < P_{cr} < \frac{16.7\ EI}{L^2} .$$

To obtain a more accurate result, we can calculate average values (8) of the deflections y_{2a} and y_{2c} as follows:

$$(y_{2a})_{av} = \frac{1}{L}\int_0^L (y_{2a})\,dx \ , \quad \text{and}$$

$$(y_{2c})_{av} = \frac{1}{L}\int_0^L (y_{2c})\,dx \ . \qquad (60)$$

Replacing the above equations by summations, we obtain

$$(y_{2a})_{av} = \frac{1}{L}\Sigma\,(y_{2a})\cdot\Delta x \ , \quad \text{and}$$

$$(y_{2c})_{av} = \frac{1}{L}\Sigma\,(y_{2c})\cdot\Delta x \ .$$

Since the segment length Δx is constant, we have

$$\frac{(y_{2a})_{av}}{(y_{2c})_{av}} = \frac{2(42 + 74 + 93) + 100}{2(164 + 287 + 358) + 383} \cdot \frac{EI}{Pd^2} = \frac{16.58\ EI}{L^2}$$

Equating the average values of y_{2a} and y_{2c} gives

$$P_{cr} = \frac{16.58\ EI}{L^2} \ .$$

For beams subjected to both lateral and axial loads, the deflections due to lateral load alone will be amplified in the case of axial compression or reduced in the case of axial tension. An approximation of the amplification factor (8, 10, 11) is given by

$$\frac{1}{1 - P/P_{cr}} \ ,$$

while the reduction factor is

$$\frac{1}{1 + T/P_{cr}} \ ,$$

where T is axial tension.

3. Calculation of total deflections and moments:

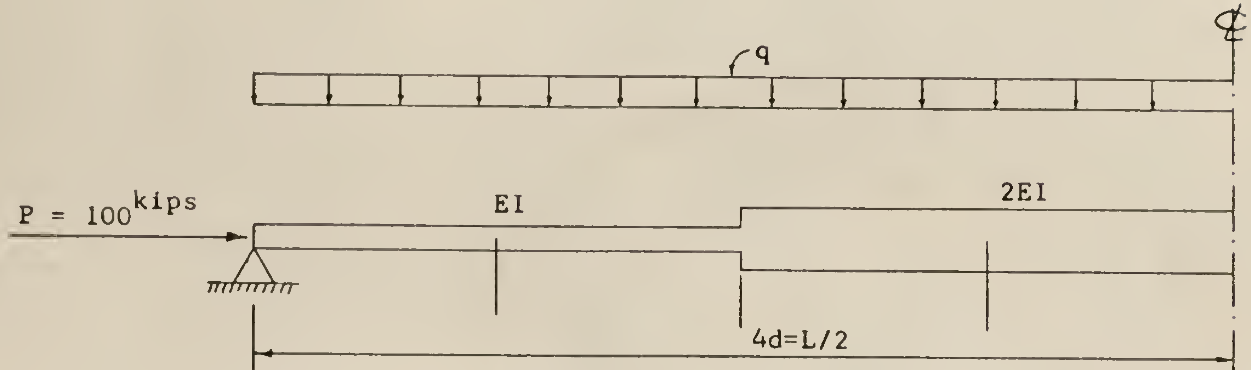
   a. Estimate total deflection y, and thus additional deflections ${}_1y_a$ due to axial load, as shown in Table 1.

b. Compute the additional deflections ${}_1y_c$ due to axial load, and check with the estimated ones from Step a.

c. If the computed additional deflections ${}_1y_c$ do not agree within an allowable limit with the estimated ${}_1y_a$, then the computed deflections are used as estimated ones to repeat Step b until close agreement is reached.

d. The total moments are equal to the sums of the moments due to lateral loads alone and the moments due to axial forces with total deflections.

Table 1. Deflections and moments of beam-column with uniform lateral load



q = 1 lb/ft.

$E = 30\times10^{6}$ psi

$I = 10\ in^{4}$

L = 10 ft.

1. Deflection due to lateral load alone:

Station number	0		1		2		3		4	Common factor
Moment	0		7		12		15		16	$L^2/128$
M/EI	0		7	12		6	7.5		8	$L^2/128EI$
R	0		6.84		8.54		7.42		7.92	$dL^2/128EI$
Average slope		26.76		19.92		11.38		3.96		"
y_i	0		26.76		46.68		58.06		62.02	$d^2L^2/128EI$

Table 1. (Continued)

2. Calculation of buckling load:

First Cycle

Station number	0		1		2		3		4	Common factor
Assumed y_{1a}	0		38		71		92		100	$a_1/100$
$P(y_{1a})/EI$	0		38	71		35.5	46		50	$Pa_1/100EI$
R			38		50		46		49	$Pda_1/100EI$
Average slope		158.5		120.5		70.5		24.5		"
Computed y_{1c}	0		158.5		279		349.5		376	$Pd^2a_1/100EI$
y_{1a}/y_{1c}	0		15.35		16.3		16.85		17.0	EI/PL^2

Second Cycle

y_{2a}	0		42		74		93		100	$a_2/100EI$
y_{2c}	0		164		287		358		383	$Pd^2a_2/100EI$
y_{2a}/y_{2c}	0		16.4		16.5		16.6		16.7	EI/PL^2

Table 1. (Continued)

$$\frac{(y_{2a})_{av}}{(y_{2c})_{av}} = 16.58\,\frac{EI}{PL^2}\,, \quad P_{cr} = 16.58\,\frac{EI}{L^2} = 345.4 \text{ kips}\,, \quad \frac{P}{P_{cr}} = 0.296$$

$$\text{Amplification factor} = \frac{1}{1 - 0.298} = 1.42\,, \quad \text{Approx. } y_a = 1.42y_i - y_i = 0.42y_i\,.$$

3. Calculation of total deflections and moments:

First Cycle

Station number	0		1		2		3		4	Common factor
${}_1y_a = 0.42y_i$	0		11.2		19.7		24.4		26.1	$d^2L^2/128EI$
$y = y_i + {}_1y_a$	0		37.96		66.38		82.46		88.12	"
Py/EI	0		37.96		66.38, 33.19		41.23		44.06	$Pd^2L^2/128(EI)^2$
R			37.2		46.8		40.7		43.6	$Pd^3L^2/128(EI)^2$
Average slope		146.5		109.3		62.5		21.8		"
${}_1y_c$	0		146.5		255.8		318.3		340.1	$Pd^4L^2/128(EI)^2$

Converting to the same unit as ${}_1y_a$

${}_1y_c$	0	11.0	19.18	23.85	25.50	$d^2L^2/128EI$

Table 1. (Concluded)

Station number	0	1	2	3	4	Common factor
<u>Second Cycle</u>						
${}_2y_a$	0	11.0	19.18	23.85	25.50	$d^2L^2/128EI$
${}_2y_c$	0	10.92	19.10	23.75	25.40	"
<u>Third Cycle</u>						
${}_3y_a$	0	10.92	19.10	23.75	25.40	"
${}_3y_c$	0	10.91	19.06	23.69	25.35	"
$y = y_i + {}_3y_c$	0	37.67	65.71	81.74	87.34	$d^2L^2/128EI$
y	0	3.176	5.540	6.894	7.356	inches

The maximum moment at center $= 16 \frac{L^2}{128} + 100 \frac{7.356}{12} = 73.9$ ft-kips

MOMENT DISTRIBUTION

The axial compression in a slender member modifies the stiffness factors, carry-over factors, and fixed-end moments as used in the standard moment distribution method. When the effect of axial force is considered in the analysis, it does not alter the basic procedures used in standard moment distribution; all that is necessary is to replace the usual values for the stiffness factors, carry-over factors, and fixed-end moment coefficients by factors which have been determined for a beam-column. For example, a compressive force will reduce the rotational stiffness of a beam as compared to the same beam with no axial force. As the compressive force increases, the stiffness continues to decrease and when the axial force reaches the buckling load the beam will have no resistance to rotation at one end and the stiffness will have become zero.

Sign Convention

A special sign convention will be used in this section. The bending moments, end rotations, and axial forces shown in Fig. 9 are assumed to be positive.

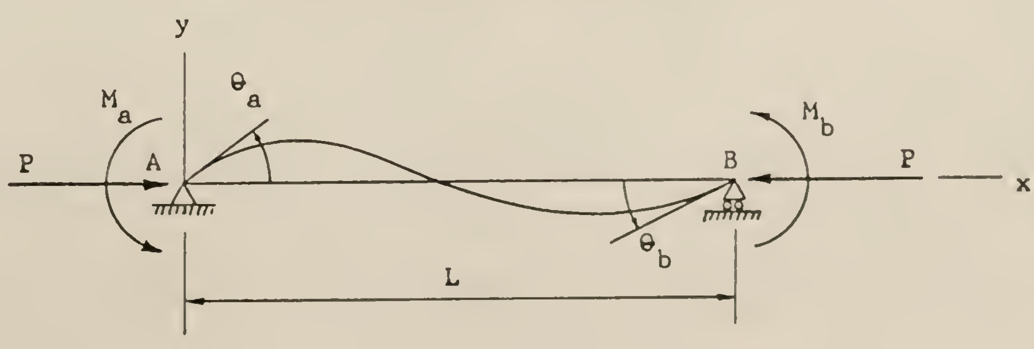


Fig. 9. Sign convention.

Stiffness and Carry-over Factors

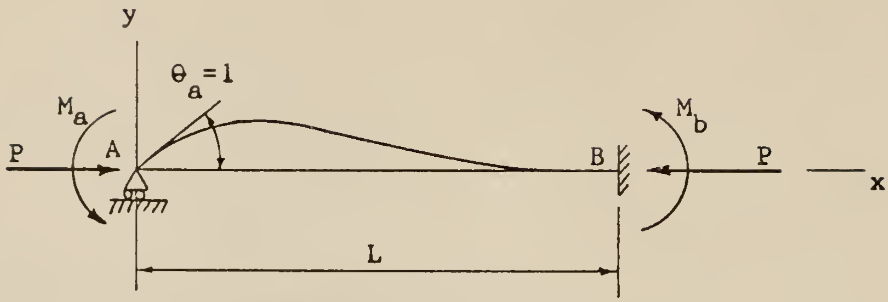


Fig. 10. Beam-column with far end fixed.

The stiffness K is defined as the value of the moment required to rotate the near end of the beam through a unit angle when the far end is fixed. The "near end" of the member is the end of the member where the joint is being balanced and the "far end" is the opposite end (12).

Eqs. 24 and 25 can be modified to suit the sign convention used in this section by changing the signs of θ_a and M_a. The modified equations are

$$\theta_a = \frac{M_a L}{3EI} f_4(u) - \frac{M_b L}{6EI} f_5(u) , \quad \text{and}$$

$$\theta_b = -\frac{M_a L}{6EI} f_5(u) + \frac{M_b L}{3EI} f_4(u) . \tag{61}$$

In order to determine the stiffness factor for a beam with the far end fixed (Fig. 10), all that is necessary is to substitute $\theta_a = 1$ and $\theta_b = 0$ into Eq. 61 and then solve for the moment M_a. This moment, which is equal to the stiffness K_{ab}, is given by the expression

$$K_{ab} = \frac{4EI}{L} \; \frac{3 f_4(u)}{4[f_4(u)]^2 - [f_5(u)]^2} . \quad \text{(Far end fixed)} \tag{62}$$

Also, the ratio of the moment, M_b, at the fixed end to the moment, M_a, at the near end gives the carry-over factor

$$C_{ab} = \frac{1}{2}\ \frac{f_5(u)}{f_4(u)} . \quad \text{(For far end fixed)} \tag{63}$$

When the axial force P becomes zero, the values of $f_4(u)$ and $f_5(u)$ become unity and the above two expressions for K_{ab} and C_{ab} reduce to 4EI/L and 1/2, respectively. Theoretically, when the axial force P reaches the critical buckling value for a beam column with one end fixed and the other end simply supported (Fig. 10), the stiffness factor becomes zero and the carry-over factor approaches infinity (12).

If the far end of the beam-column is simply supported instead of fixed, the end rotations can be obtained by letting $M_b = 0$. Then Eq. 61 gives

$$\theta_a = \frac{M_a L}{3EI}\ f_4(u) ,$$

and (64)

$$\theta_b = -\frac{M_a L}{6EI}\ f_5(u) .$$

The stiffness factor can be obtained by setting $\theta_a = 1$ in Eq. 64 and solving for M_a, which gives

$$K_{ab} = \frac{3EI}{L}\ \frac{1}{f_4(u)} . \quad \text{(Far end simply supported)} \tag{65}$$

When the axial force is zero (u = 0) the stiffness is 3EI/L and when it is equal to the critical buckling load for a beam-column with simply supported ends ($u = \pi/2$) the stiffness is zero.

Fixed-end Moments

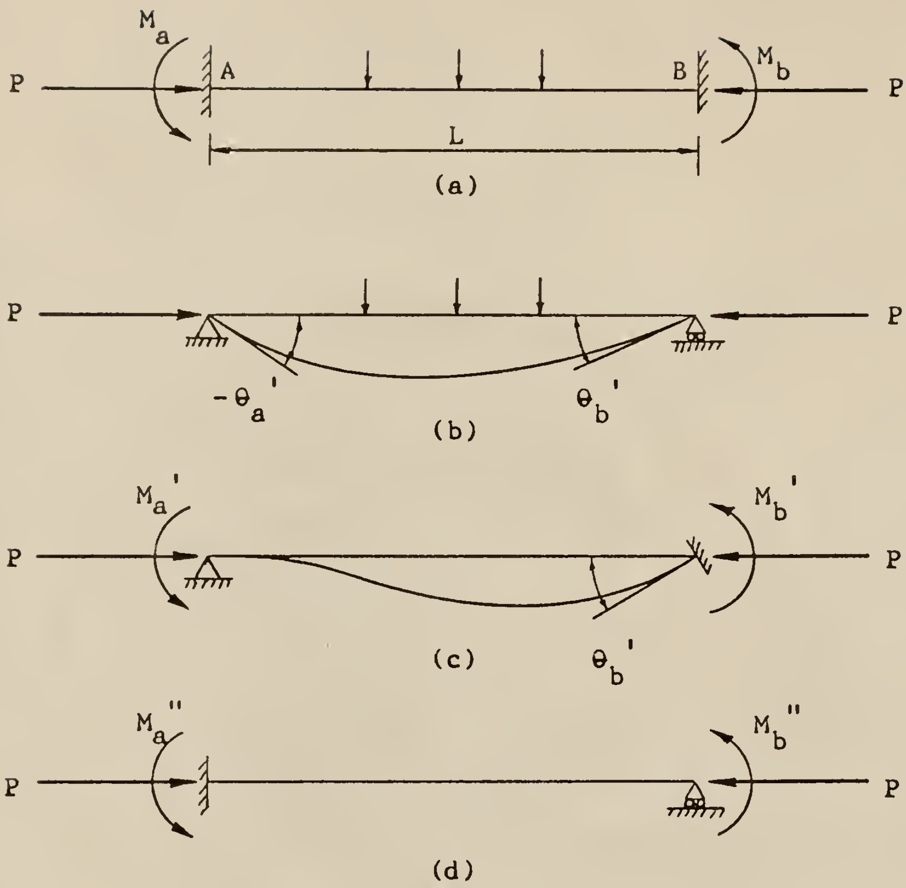


Fig. 11. Fixed-end beam with axial load.

The fixed-end moments can be obtained by superposing the solutions of the problems shown in Fig. 11b, c, and d. In order to cancel the effects of the end rotations, moments must be applied at the ends. In Fig. 11c, a moment M_a' is applied at the left end to rotate the axis of the member to the horizontal and the right end of the member is assumed to be held against rotation thus, inducing a moment M_b'. These two moments can be expressed as

$$M_a' = -K_{ab}\theta_a' ,$$

and

$$M_b' = M_a' C_{ab} = -K_{ab} C_{ab} \theta_a' .$$

In Fig. 11d, the left end of the member is held against rotation and a moment M_b'' is applied at the right end to remove the angle θ_b'. This gives

$$M_b'' = -K_{ba} \theta_b' ,$$

and

$$M_a'' = M_b'' C_{ba} = -K_{ba} C_{ba} \theta_b' .$$

By superposing the results of Fig. 11b, c, and d, the fixed-end moments are

$$M_a = M_a' + M_a'' = -K_{ab} \theta_a' - K_{ba} C_{ba} \theta_b' ,$$

and (66)

$$M_b = M_b' + M_b'' = -K_{ba} \theta_b' - K_{ab} C_{ab} \theta_a' .$$

If the member is assumed to have constant cross-section, the stiffness and carry-over factors at each end of the member will be the same and the above expressions can be written as (12)

$$M_a = -K_{ab} (\theta_a' + C_{ab} \theta_b') , \quad \text{and}$$

(67)

$$M_b = -K_{ab} (\theta_b' + C_{ab} \theta_a') .$$

Uniform Load

For the case of a fixed-ended beam-column with a uniform load the angles of rotation of the ends must be found for the simply supported beam-column as shown in Fig. 2 and Eq. 10. These angles are

$$\theta_a = -\theta_b = -\frac{qL^3}{24EI} f_2(u) . \tag{68}$$

Substituting Eqs. 62, 63 and 68 into Eq. 67, the fixed-end moments (12) are

$$M_a = -M_b = \frac{qL^2}{12} f_6(u) , \quad (69)$$

where

$$f_6(u) = \frac{3}{u^2} (1 - \frac{u}{\tan u}) . \quad (70)$$

Concentrated Load at Center of Beam-column

In the case of this loading, the end rotations of the simply supported beam-column are (refer to Eq. 19)

$$\theta_a = -\theta_b = -\frac{QL^2}{16EI} f_3(u) . \quad (71)$$

Using Eqs. 62, 63, 71 and 67, the fixed-end moments become (12)

$$M_a = -M_b = \frac{QL}{8} f_7(u) , \quad (72)$$

where

$$f_7(u) = \frac{2}{u} (\frac{1-\cos u}{\sin u}) . \quad (73)$$

Fixed-end Moments Due to Joint Translation

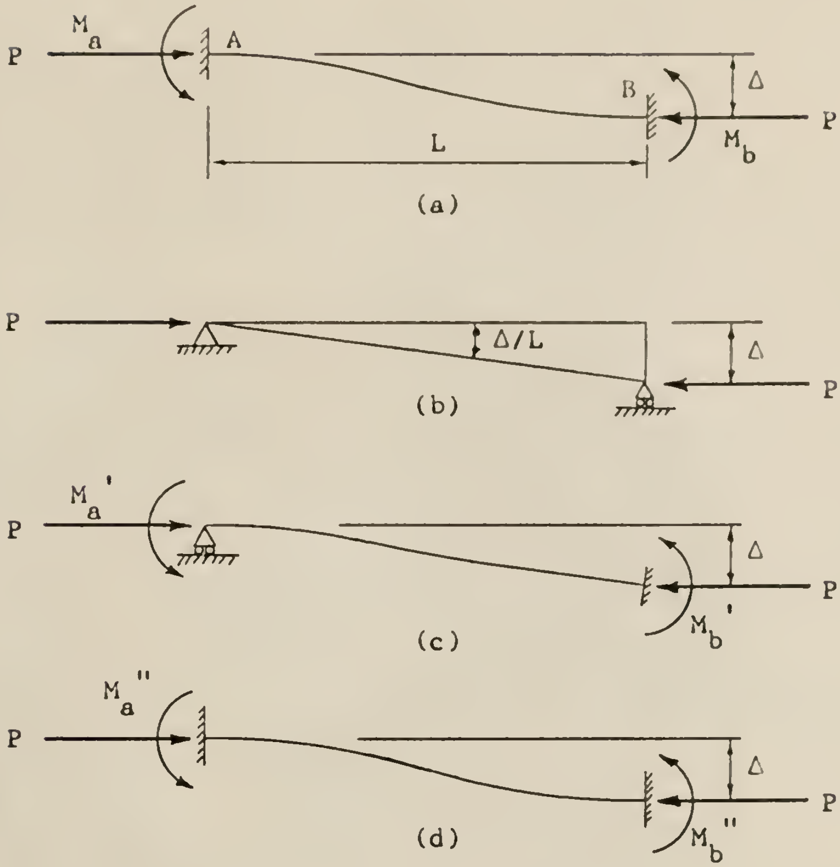


Fig. 12. Fixed-end moments due to joint translation.

By using the same procedures as in deriving "Fixed-end Moments" discussed above, the fixed-end moments due to joint translation shown in Fig. 12a can be obtained by superposing the results of Fig. 12b, c, and d. The equations, which are necessary for this case, can be expressed as

$$M_a' = K_{ab} \frac{\Delta}{L}, \qquad M_b' = K_{ab} C_{ab} \frac{\Delta}{L};$$

$$M_b'' = K_{ba} \frac{\Delta}{L}, \qquad M_a' = K_{ba} C_{ba} \frac{\Delta}{L};$$

and

$$
\begin{aligned}
M_a &= M_a' + M_a'' = (K_{ab} + K_{ba}C_{ba}) \frac{\Delta}{L} , \\
M_b &= M_b' + M_b'' = (K_{ba} + K_{ab}C_{ab}) \frac{\Delta}{L} .
\end{aligned}
\tag{74}
$$

If the member is of constant cross-section, the stiffness and carry-over factors are the same at each end, and Eq. 74 is reduced to

$$M_a = M_b = K_{ab} (1 + C_{ab}) \frac{\Delta}{L} . \tag{75}$$

After substitution of the relationships given in Eqs. 62 and 63 for the stiffness and carry-over factors, Eq. 75 becomes

$$M_a = M_b = \frac{6EI\Delta}{L^2} f_8(u) ,$$

where

$$f_8(u) = \frac{u^2 \sin u}{3(\sin u - u \cos u)} . \tag{76}$$

The same general procedure as for standard moment distribution, but modified to make use of the superposition method, can be used in analyzing a structure with sidesway.

Following the superposition technique of analysis, the moments in the frames shown in Fig. 13a and b are determined first. The frame in Fig. 13b is supported against joint translation and subjected to the same loads as acting on the original frame. The frame in Fig. 13c is subjected to an arbitrary horizontal displacement Δ' and then held against further joint translation. Both of these frames are analyzed by moment distribution considering the beam-column effect in the members. Therefore, it is necessary to make an initial estimate of the axial forces in each member.

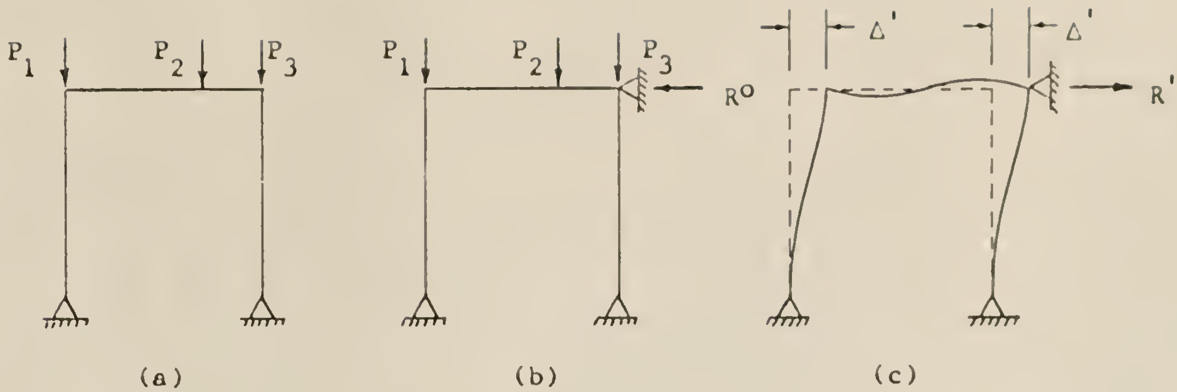


Fig. 13. Rigid frame with sidesway.

This estimate, which can be made by an approximate analysis of the original structure, gives a value of the factor "kL" for each member of the frame, and the factor "kL" is held constant throughout the analyses of the frames shown in Fig. 13b and c. The superposition equations to be used are the same as in the standard moment distribution calculations. For the notations shown in Fig. 13, the equation

$$R^o - aR' = 0 \tag{77}$$

determines the constant of proportionality "a" and the equation

$$M = M^o + aM' \tag{78}$$

determines the moments M in the original structure, assuming that M^o and M' represent moments in the frames of Fig. 13b and c, respectively. After determining the moments in the original frame, the values of the axial forces in the members are revised to more accurate values and the entire process repeated if necessary.

DESIGN METHODS

Two simple approximate methods have been developed for estimating the strength of beam-columns. The first method, generally referred to as the secant formula method, is based upon the concept that the load which produces initiation of yielding in the fibers subjected to maximum stress provides a lower bound to the failure load. The second method is to consider the member as a cross between a beam under pure bending and a column under pure axial load, its two limiting cases (13).

SECANT FORMULA METHOD

As stated above, the secant formula method specifies that the maximum stress in the beam-column, modified by a factor of safety, may not exceed the yield stress of the material. The formulas developed for this procedure apply only to members that fail by bending in the plane of the applied loads. The procedure applies best for material having a linear elastic stress-strain relationship.

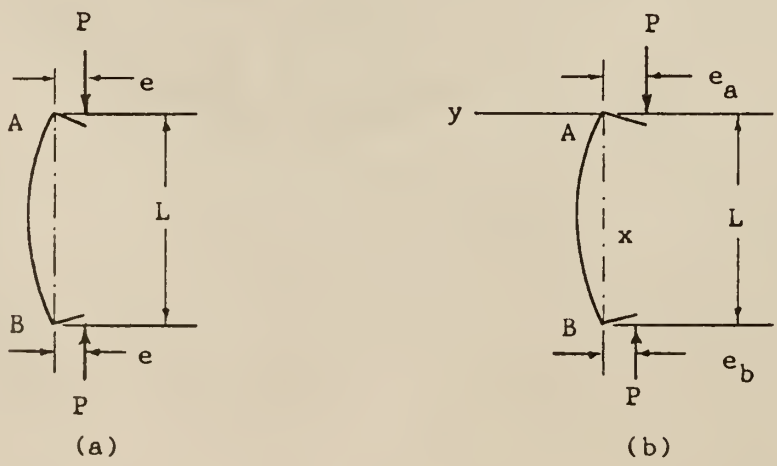


Fig. 14. Notation for eccentrically loaded members.

The first step in developing the secant formula is the analysis of maximum combined stress produced by axial load, applied bending moment, and bending moment due to deflection. Consider the case of two eccentrically applied compressive forces P with unequal eccentricities as shown in Fig. 14b, the deflection curve has been derived in Eq. 23.

$$y = e_a \left(\frac{\sin k(L-x)}{\sin kL} - \frac{L-x}{L}\right) + e_b \left(\frac{\sin kx}{\sin kL} - \frac{x}{L}\right) , \qquad (23)$$

in which e_a denotes the larger eccentricity and e_b, the smaller one. By letting $t = \frac{e_b}{e_a}$, the deflection curve is reduced to

$$y = e_a \left(\frac{t - \cos kL}{\sin kL} \sin kx + \cos kx + \frac{1-t}{L} x - 1\right) . \qquad (79)$$

Substituting Eq. 79 into Eq. 21a, we find that the expression for bending moment at any section in the member, after some rearrangement, is

$$M = P e_a \left[\frac{t - \cos kL}{\sin kL} \sin kx + \cos kx\right] . \qquad (80)$$

Taking the first derivative of M with respect to x and setting dM/dx = 0, we find the location of the maximum moment as

$$x = \frac{1}{k} \arctan \frac{t - \cos kL}{\sin kL} .$$

Substituting this back into Eq. 80, after some manipulation we obtain the maximum moment,

$$M_{max} = P e_a \left(\frac{\sqrt{t^2 - 2t \cos kL + 1}}{\sin kL}\right) . \qquad (81)$$

The maximum fiber stress in the member is equal to

$$f_{max} = \frac{P}{A} + \frac{M_{max}\, c}{I} ,$$

$$f_{max} = \frac{P}{A} \left(1 + \frac{e_a c}{r^2} \frac{\sqrt{t^2 - 2t \cos kL + 1}}{\sin kL}\right) . \tag{82}$$

where c is the distance from the neutral axis to the extreme fiber in compression. It is this stress which is to be set equal to the yield point stress f_y to determine the yield point load P_y, thus

$$\frac{P_y}{A} \left(1 + \frac{e_a c}{r^2} \frac{\sqrt{t^2 - 2t \cos kL + 1}}{\sin kL}\right) = f_y . \tag{83}$$

The limiting average stress F_a for design use can be obtained from Eq. 83 by replacing P_y with nF_aA (where n is the factor of safety) and then solving for F_a, thus

$$F_a = \frac{f_y/n}{1 + e_a c/r^2 \sqrt{t^2 - 2t \cos kL + 1} \csc kL} . \tag{84}$$

In Appendix C of the AASHO Specifications (1965 edition) we find formula A, which is almost identical with Eq. 84

$$f_s = \frac{f_y/n}{1 + (0.25 + e_a c/r^2)\, B \operatorname{cosec} \phi} = \frac{P}{A} , \tag{85}$$

where B is the factor similar to $\sqrt{t^2 - 2t \cos kL + 1}$ and ϕ replaces kL. The term 0.25 which is added to the nondimensional factor $e_a c/r^2$ is to account for the effects of all imperfections, such as initial crookedness, nonhomogeneity, and residual stress.

Equations 84 and 85 are quite difficult to use for design, and their main use is in plotting column curves such as those in Appendix C of the AASHO Specifications. Such curves are reasonably easy to use for the trial and error design of beam columns.

For the particular case in which the eccentricity of the load is a constant value e, we set t = 1, and after some slight manipulation Eq. 82 reduces to the well-known secant formula:

$$f_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2} \sec \frac{kL}{2}\right] , \tag{86}$$

and the corresponding expression for allowable stress for design use is

$$F_a = \frac{f_y/n}{1 + (ec/r^2) \sec (kL/2)} . \tag{87}$$

The secant formula suggested by Column Research Council (7) is

$$f_{max} = \frac{P}{A}\left[1 + \left(\frac{ec}{r^2} + \frac{e_o c}{r^2}\right) \sec \frac{L}{2r} \sqrt{\frac{P}{AE}}\right] , \tag{88}$$

and the corresponding expression for allowable stress is

$$F_a = \frac{f_y/n}{1 + (ec/r^2 + e_o c/r^2) \sec (L/2r) \sqrt{nF_a/E}} , \tag{89}$$

where e_o is the assumed equivalent eccentricity representing the effects of all defects.

A good approximation for the maximum moment in a beam-column can be found from the following equation:

$$M_{max} = M_o + \frac{P\Delta_o}{1 - P/P_{cr}} , \tag{90}$$

where M_o and Δ_o are the moment and deflection, respectively, without regard to the added moment caused by deflection. Eq. 90 can conveniently be written as:

$$M_{max} = M_o \left(\frac{1 + \psi P/P_{cr}}{1 - P/P_{cr}}\right) , \qquad (91)$$

in which, for a simply supported constant-section member,

$$\psi = \frac{\pi^2 \Delta_o EI}{M_o L^2} - 1 .$$

If the approximate maximum moment given by Eq. 91 is used to determine the maximum combined stress in a beam-column, the following formula is obtained:

$$f_{max} = \frac{P}{A} + \left(\frac{1 + \psi\alpha}{1 - \alpha}\right) \frac{M_o c}{I} , \qquad (92)$$

where $\alpha = P/P_{cr}$.

Since formulas for lateral deflection Δ_o under different loading conditions are available in handbooks, Eq. 92 greatly simplifies calculation of maximum stress in beam-columns. Several values of ψ for various conditions are given (7) in Table 2.

Table 2. Parameter ψ in Eqs. 91 and 92.

Loading condition*	
Constant moment	+ 0.233
Concentrated lateral load at midspan	− 0.178
Uniform lateral load	+ 0.028

*Each loading condition shown in Table 2 is assumed to act simultaneously with axial compression forces.

If Eq. 92 is used for the same design case as that of Eq. 90 (equal eccentricities e, M_o = Pe, and $\psi = + 0.233$), the allowable stress becomes:

$$F_a = \frac{f_y/n}{1 + \left(\frac{1 + 0.233 n\alpha}{1 - n\alpha}\right)\frac{ec}{r^2}} \quad (93)$$

This equation gives results in close agreement with those obtained from Eq. 89. The results are within 1% for $n\alpha$ less then 0.8, and within 2.5% for $0.8 < n\alpha < 0.95$ (7).

INTERACTION FORMULA METHOD

This method is based on the use of interaction formulas for beam-columns subjected to bending in the weak direction, and for beam-columns subjected to strong-direction bending provided that they are adequately braced against lateral buckling. Interaction formulas have a simple form, are convenient to use, and have a wide scope of application. Allowable stresses determined from interaction formulas vary continuously and in a smooth transition from stresses for concentrically loaded columns at one limit to stresses for beams at the other. Formulas for combined stresses used in AISC Specifications are based on this method.

The strength of members subjected to bending combined with compression forces can be expressed conveniently by interaction formulas in terms of the ratios P/P_u and M/M_u, where

P = compression forces at actual failure,

P_u = ultimate load for the centrally loaded column for buckling in the plane of the applied moment,

M = maximum bending moment at actual failure, and

M_u = ultimate bending moment in the absence of axial load.

The following equation is the basis for several such interaction formulas:

$$\frac{P}{P_u} + \frac{M}{M_u} \leq 1 . \qquad (94)$$

In the elastic range, an approximation of the maximum bending moment for beam-columns subjected to combined bending and compression producing maximum moment at or near the center of the member may be obtained from Eq. 91 by setting $\psi = 0$.

$$M_{max} = \frac{M_o}{1 - (P/P_{cr})} , \qquad (95)$$

where P = applied axial load,

P_{cr} = elastic critical load for buckling in the plane of applied moment, and

M_o = maximum applied moment, not including contribution of axial load interacting with deflections.

Substituting Eq. 95 into Eq. 94 gives

$$\frac{P}{P_u} + \frac{M_o}{M_u \left[1 - (P/P_{cr})\right]} \leq 1 . \qquad (96)$$

For eccentrically loaded columns having equal end eccentricities e at both ends, Eq. 96 may be expressed as

$$\frac{P}{P_u} + \frac{P\,e}{M_u \left[1 - (P/P_{cr})\right]} \leq 1 . \qquad (97)$$

Galambos and Ketter (4) presented dimensionless interaction curves for the ultimate strength of typical wide-flange beam-columns bent in the strong direction and having (a) equal axial-load eccentricities at both ends, and (b) eccentricity at one end only. The parameters used for the maximum-strength formulas are P/P_y and M/M_u, and the interaction formulas take the following form:

$$A\,\frac{M}{M_u} + B\,\frac{P}{P_y} + C\,\left(\frac{P}{P_y}\right)^2 \leq 1 , \qquad (98)$$

where A, B, and C are empirical coefficients that are functions of L/r and of loading conditions, and P_y is the column axial load at the fully yielded condition ($P_y = A\,f_y$).

The AISC Specifications (14) use a complete tabulation of the

Galambos-Ketter coefficients of Eq. 98 for beam columns loaded as described in the preceding paragraph.

For the case of strong-direction bending, where a beam-column is bent in double curvature by moments producing plastic hinges at both ends, the AISC recommends a strength formula that can be written as

$$0.85 \frac{M_o}{M_p} + \frac{P}{P_y} \leq 1 .$$

This equation is independent of L/r, and M_p is the plastic moment ($M_p = Z\, f_y$, where Z is the plastic modulus).

For designing a beam-column subjected to unequal end moments, it may be over conservative to use the maximum of these in an interaction formula for design of the member, especially where the end moments are of opposite sign. This is because the interaction formula assumes the maximum moment to be at or near the center of the span. The AISC Specifications present an approximate expression of "equivalent uniform moment," which can be expressed as follows:

$$\frac{M_{eq}}{M_a} = 0.6 + 0.4 \frac{M_b}{M_a} \geq 0.4 , \tag{99}$$

where M_{eq} is the equivalent uniform moment, and M_b/M_a is the ratio of the smaller to larger moments at the ends of the member.

In AISC specifications section 1.6.1. the interaction equations are in the form

$$\frac{f_a}{F_a} + \frac{C_m\, f_b}{(1 - \frac{f_a}{F'_e})\, F_b} \leq 1 , \tag{100}$$

and

$$\frac{f_a}{0.6\ F_y} + \frac{f_b}{F_b} \leq 1\ , \tag{101}$$

where f_a and f_b are the axial and bending stresses under the combined loading, F_a and F_b are the allowable stresses for the axially loaded column and beam, respectively, F'_e is the critical stress according to Euler's column theory, and C_m is a coefficient which is expressed as

$$C_m = 0.6 + 0.4\ (M_b/M_a) \geq 0.4\ .$$

In the case of biaxial bending, the design interaction equation is written as

$$\frac{f_a}{F_a} + \frac{C_{mx}\ f_{bx}}{(1 - f_a/F'_{ex})\ F_{bx}} + \frac{C_{my}\ f_{by}}{(1 - f_a/F'_{ey})\ F_{by}} \leq 1\ . \tag{102}$$

where the subscripts, x and y, refer to the x and y axes.

The great usefulness and versatility of interaction formulas arises from the possibility of including a variety of conditions in at least an approximate manner. For instance, if the danger of lateral buckling under applied moment exists, the allowable stress F_b can be reduced accordingly.

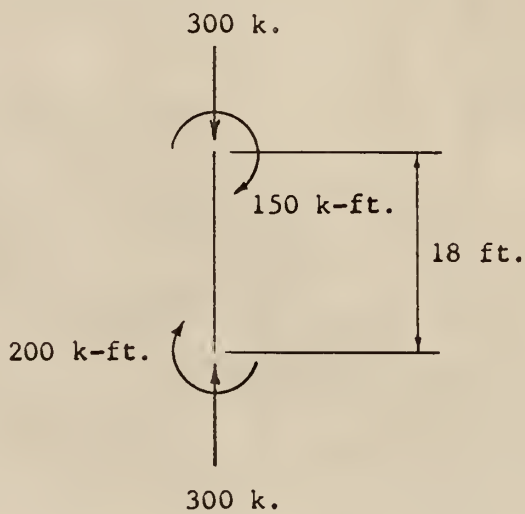
Interaction formulas also allow consideration of end effects such as partial or complete fixity, or sideway. A member with these end effects can be designed safely by replacing the restrained beam-column by an equivalent, hinged-end beam-column having a length equal to the effective length of the real, restrained beam-column, and analyzing this equivalent beam-column for axial compression.

An example illustrating the interaction formula concept is presented in the following section.

EXAMPLE

Problem:

A column in a steel building frame is, according to an analysis, subjected to an axial force of 300 kips and a moment of 200 k-ft at the lower end, 150 k-ft at the upper end. There is no moment about the other axis, and the frame is braced against sidesway buckling in the plane of the loading and in the plane perpendicular to it. Story height is 18 ft. Use A-36 steel and design the member according to AISC Specifications (14).



Solution:

For a quick estimate of the range of member size, consider the moment of 200 kip-ft alone, and compute the required section modulus as

$$S_{req} = \frac{M}{F_b} = \frac{200 \times 12}{22} = 109 \text{ in}^3.$$

Assume that, owing to the presence of axial force, twice this value is reasonable and look for a member with section modulus of about 200 in^3. Entering the Tables on pages 1-12 (AISC), the 14 WF shapes are found to be suitable for this size. Accordingly, consider a 14 WF 127, for which

$$A = 37.33 \text{ in}^2,$$

$$S = 202.0 \text{ in}^3,$$

$$I = 1476.6 \text{ in}^4,$$

$$r_x = 6.29, \text{ and}$$

$$r_y = 3.76.$$

Since sidesway buckling is prevented, assume that the effective length factor K to be 0.65 according to Table C. 1.8.2.

On the basis of these data calculate the slenderness ratios:

$$\frac{L}{r_x} = \frac{18 \times 12}{6.29} = 34.4 \ , \quad \frac{L}{r_y} = \frac{18 \times 12}{3.76} = 57.5 \ .$$

Effective slenderness ratios:

$$\frac{kL}{r_x} = 0.65 \times 34.4 = 23 \ ,$$

$$\frac{kL}{r_y} = 0.65 \times 57.5 = 38 \ .$$

Allowable axial stress:

(Table 1-36)

$$F_{ax} = 20.41 \text{ ksi} \ ;$$

$$F_{ay} = 19.35 \text{ ksi} \ .$$

Axial stress:

$$f_a = \frac{P}{A} = \frac{300}{37.33} = 8.04 \text{ ksi} \ .$$

Reduction factor:

$$C_m = 0.6 + 0.4\left(\frac{M_1}{M_2}\right) = 0.6 + 0.4\left(\frac{150}{200}\right) = 0.3 < 0.4 \ ,$$

$$\text{use } C_m = 0.4$$

Amplification factor:

For $\frac{kL}{r_x} = 23$, $F'_e = 281.88$ ksi ,

$$1 - \frac{f_a}{F'_e} = 1 - \frac{8.04}{281.88} = 0.971 \ .$$

Allowable bending stress: (sec. 1.5.1.4.5)

$$F_b = \frac{12{,}000}{Ld/A_f} = 56 \text{ ksi} > 0.6\ F_y \ , \quad \text{use } F_b = 0.6\ F_y = 22 \text{ ksi} \ .$$

Bending stress:

$$f_b = \frac{M_{max}}{S} = \frac{200 \times 12}{202} = 11.88 \text{ ksi} \ .$$

Check interaction formulas:

$$\frac{f_a}{F_a} = \frac{8.04}{19.35} = 0.416 > 0.15 \ , \quad \text{use Formula (7).}$$

Formula (7a)

$$\frac{8.04}{19.35} + \frac{0.4\ (11.88)}{0.971\ (22)} = 0.416 + 0.222 = 0.638 < 1 \ .$$

Formula (7b)

$$\frac{8.04}{22} + \frac{11.88}{22} = 0.366 + 0.54 = 0.906 < 1.0 \ .$$

This is considerably less than unity and a smaller size is needed.

Following the same procedure shown above, further trials may be carried out in tabular form. From the above results in Formulas (7a) and (7b), it is evident that Formula (7a) controls the case. So, accordingly, calculate only the terms needed for Formula (7a) in the further trials.

Member	A	S	f_a	f_b	F_b	$f_a/22$	f_b/F_b	Sum
14 WF 119	34.99	189.4	8.59	12.7	22	0.39	0.577	0.967
14 WF 111	32.65	176.3	9.19	13.6	22	0.418	0.619	1.037

From these results, it can be seen that 14 WF 111 is overstressed. Therefore the column will be made of 14 WF 119.

CONCLUSION

The foregoing material demonstrates the means of analysis and design of beam-columns. In addition to the four methods for the analysis of the behavior of beam-columns discussed above, there are many other methods for solving a beam-column problem.

The general method which is derived on the basis of the differential equations of beam-columns gives a theoretical and "exact" solution of the problem. From a practical viewpoint, a large number of laterally applied loads or changes in section leads to a solution by the general method which is probably no more precise or applicable than the solution obtained by one of the approximate methods.

The energy method uses a single mathematical expression for the assumed deflection which holds for the entire length of the beam. With this method it is not necessary to discuss separately each portion of the deflection curve between consecutive loads, yet it gives a good approximate solution for the problem. This method of analysis is especially useful in the case of a member with simply supported ends, of a member with complicated loadings, and of a member with various cross-sections.

The numerical method is also an approximate method with a high degree of accuracy. The problem that can be solved by using the numerical procedure as described is almost unrestricted by the type of loading.

The basic principle of the general method may be extended to solve rigid frame problems by using the modified moment distribution method.

The expression for the secant formula is complicated in form and is quite difficult to use for design. The main purpose of the secant formula is to plot column curves such as shown in the AASHO Specifications (15).

Such curves are reasonably easy to use for the design of beam-columns by trial and error.

The interaction formula is easy to use for design and is applicable for a variety of loading conditions.

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NOTATION

a, b, c, d	Numerical coefficients, distances
A	Cross-sectional area
c	Distance from neutral axis to extreme fiber of beam
e_a, e_b, e	Eccentricity
E	Modulus of elasticity
$f_1(u)$, $f_2(u)$	Amplification factors for beam-columns
I	Moment of inertia
k	Axial load factor for beam-column
K	Stiffness factor, effective length factor
L	Length, span
m, n	Integers, numerical factors
M	Bending moment, couple
M_a, M_b	Bending moments at end A and B, respectively
n	Factor of safety
P	Axial force in beam-column
P_{cr}	Critical buckling load
Q	Concentrated force
q	Intensity of distributed load
r	Radius of gyration
R	Shearing force in conjugate beam
R^o, R'	Reactive force
s	Horizontal displacement, distance
S	Section modulus
t	Numerical ratio
T	Work, tension force

u	Axial load factor for beam-columns ($u = kL/2$)
U	Strain energy
V	Shearing force in beam
x, y	Rectangular coordinates
y', y''...	First derivative, second derivative, etc.
Z	Plastic modulus
α	Numerical factor, ratio
Δ	Deflection
e_a, e_b	End slopes, rotations
ψ	Numerical factor

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ANALYSIS AND DESIGN OF BEAM-COLUMNS

by

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AN ABSTRACT OF A MASTER'S REPORT

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Beam-columns are members which are subjected to combined bending and axial compression. The bending may arise from lateral loads, couples applied at any point on the beam, or from end moments caused by eccentricity of the axial loads at one or both ends of the member.

The purposes of this paper are to review the appropriate methods for analyzing beam-column problems, namely, the general method, the energy method, the numerical method, and the moment distribution and to discuss adequate procedures for designing steel beam-columns.

The general method or "exact" method is based on the relationship between moment and curvature of a beam, $EIy'' = -M$, and takes into account the compressive forces. By solving this differential equation with known boundary conditions, the maximum deflection and moment can be obtained.

The energy method is widely used in the field of structural mechanics, and represents a unique mathematical application of the basic law of conservation of energy. By assuming the elastic curve to be described by, $y = \Sigma\, a_n \sin(n\pi x/L)$, the external work done by physically applied external forces or moments may be set equal to the potential energy stored through the action of the internal forces and the elastic strain mechanisms of the beam-column. The coefficients "a_n" can be determined, and the maximum moment obtained.

The numerical method was first presented by N. M. Newmark. It provides a means of obtaining an approximate solution for beam-columns with complex loading.

The theoretical principle of the general method can be extended to calculate the modified stiffness and carry-over factors for use in the method of modified moment distribution. The modified moment distribution method can then be used to solve rigid frame problems.

Two approaches to the design of steel beam-columns are described in this paper. Criteria for these methods are the CRC (Column Research Council) design guide (7), AASHO (15), and AISC (14) Specifications. Three sample problems are analyzed and one design example is given to illustrate the design procedure according to the AISC Specifications.