A STUDY OF LINEAR SYSTEMS WITH RANDOMLY-VARYING PARAMETERS

by

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CHAPTER I

INTRODUCTION

A random linear system (henceforth designated as RLS) is a linear system whose parameter variations are random processes with known probability density functions. If the probability density functions are independent of time the RLS is said to be stationary. The input-output relationship of a RLS may be described by a linear differential equation with stochastic coefficients.

Recent interests in the study of many varied phenomena, such as propagation through randomly-varying media, stability of radio guided vehicles with randomly-varying loop gain, and reflection from fluctuating targets have placed an ever-increasing importance on the methods of analysis of RLS. This report is a survey of the various techniques that have been developed to analyze stationary RLS. On account of the complexity of the problem on the whole the restriction to the stationary case has been found to be essential in all the published papers.

A complete representation of the output of a random linear system requires a knowledge of all the multivariate probability density functions of the output. This problem, however, is so difficult that one has to be satisfied with a limited amount of information in the form of the first probability density functions and the various order moments of the output. Even this information can be obtained in a concrete form for specific cases only. The various techniques developed to analyze these special cases have been discussed in Chapter II.

The limited description of the output of a RLS is often quite sufficient to an engineer since he is primarily concerned with the stability of such systems. In Chapter III it is shown that sufficient criteria for stability of stationary RLS can be established in terms of the available information at the output. The concept of stability of random linear systems, however, is quite different from the same concept for deterministic systems. Chapter III brings out the idea of stability "in the stochastic sense" and attempts to define it in terms of familiar notions of convergence of random variables. It is shown that this permits a very precise definition of stability of random linear systems.

The possibility of further investigation of random linear systems has been discussed in the Conclusion.

CHAPTER II

STATISTICS OF THE OUTPUT

2.1 Introduction

The output of a random linear system is given by the solution of the stochastic differential equation describing the system. A complete representation of the output requires a knowledge of all multivariate probability density functions of the output. This problem is extremely difficult and no solution, even for the simplest case, has yet appeared in literature. From an engineering point-of-view, however, a partial description of the output, involving the first probability density function, is quite sufficient. In most cases, a knowledge of the first and second order moments of the first probability density function, and the autocorrelation function of the output, gives a reasonable idea about the system behavior.

Although a general method of analysis has not yet been developed, effective analytical procedures for special cases, have been formulated. In particular, the case, in which the input is a random, white noise, Gaussian process and in which the parameter variation is also, a Gaussian, white noise process, has received wide attention (1-9). In all the referenced discussions, the objective has been to establish conditions for system stability.

The analysis of first order networks, having a binary random parameter variation for the underlying purpose of obtaining the spectral properties of the output, has been carried out by Redman and Lampard (10).

A review of the various special cases, mentioned above, will be presented in the following sections.

2.2 First Order Systems

A first order system is characterized by the differential equation

$$\frac{d}{dt} y(t) + k(t)\, y(t) = f(t) \tag{2.1}$$

where f(t) and k(t) are (in general) correlated, stationary random processes; and y(t) is the output of the system. A number of physical systems described by the differential equation 2.1 are shown in Table 2.1.

The general solution of equation 2.1 can be written as

$$y(t) = y_1 + y_2 = y_o \epsilon^{-\int_{to}^{t} k(u)du} + \int_{to}^{t} f(\tau)\epsilon^{-\int_{\tau}^{t} k(u)du} d\tau \tag{2.2}$$

where the initial conditions are $y = y_o$ at $t = t_o$; y_1 is the general solution of the homogeneous equation and y_2 is a particular solution of the inhomogeneous equation.

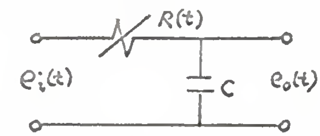
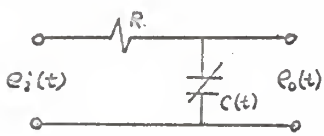
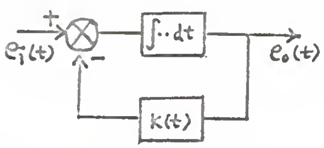
2.2.1 First Order Systems with Gaussian Random Variation of Parameters Subjected to a Gaussian Random Input

2.2.1.1 General Case

A very general analysis of first order systems, in which the parameter variation is a stationary and correlated random process, was given by Tichonov (2). He used an effective tech-

TABLE 2.1

LINEAR FIRST ORDER SYSTEMS

Differential Equation: $\dot{y}(t) + k(t)y(t) = f(t)$				
No.	Type	y(t)	k(t)	f(t)
1(a)		$e_o(t)$	$\frac{1}{R(t)C}$	$e_i(t)k(t)$
1(b)		$\frac{e_o(t)}{k(t)}$	$\frac{1}{RC(t)}$	$e_i(t)$
1(c)		$e_o(t)$	$k(t)$	$e_i(t)$

nique to overcome the difficulty encountered by the dependence of k(t) and f(t).

The solution of the differential equation

$$\frac{dz(t)}{dt} + k(t)\, z(t) = \epsilon^{-\mathcal{L}\, f(t)} \tag{2.3}$$

for the initial conditions $z = y_o$ at $t = t_o$ is given by

$$z(t) = z_1(t) + z_2(t)$$

$$= y_o \epsilon^{-\int_{to}^{t} k(u)du} + \int_{to}^{t} \epsilon^{[-\alpha f(\tau) - \int_{\tau}^{t} k(u)du]} d\tau \quad (2.4)$$

It is observed that y(t), as given by equation 2.2, can also be obtained from equation 2.4. That is,

$$y(t) = z_1(t) - \left[\frac{\partial}{\partial \alpha} z_2(t)\right]_{\alpha=0} \quad (2.5)$$

The first moment of y(t) may be written as

$$y(t) = y_o \left\langle \epsilon^{-\int_{to}^{t} k(u)du} \right\rangle - \left[\frac{\partial}{\partial \alpha} \left\langle z_2(t) \right\rangle\right]_{\alpha=0} \quad (2.6)$$

The auto-correlation function of y(t) may be obtained by averaging the product of y(t) at times t_1 and t_2. Thus

$$\phi(t_1, t_2) = \left\langle y(t_1)\, y(t_2) \right\rangle$$

$$= \left\langle z_1(t_1)\, z_1(t_2) \right\rangle - \left[\frac{\partial}{\partial \alpha} \left\langle z_1(t_1)\, z_2(t_2) \right\rangle\right]_{\alpha=0}$$

$$- \left[\frac{\partial}{\partial \alpha} \left\langle z_1(t_2)\, z_2(t_1) \right\rangle\right]_{\alpha=0}$$

$$+ \left[\frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} \left\langle z_2(t_1)\, z_2(t_2) \right\rangle\right]_{\alpha_1=\alpha_2=0} \quad (2.7)$$

Tichonov showed that the evaluation of the various averages, indicated in the above equations, follows from the use of the n-dimensional characteristic function of a normal random process. He obtained expressions for the mean and the auto-correlation function of the output. These expressions, though quite cumbersome, are important because they have been derived for the most general case, and reduce to convenient forms for some special cases.

Tichonov also considered the asymptotic behavior of the mean and the auto-correlation for various cases, and established conditions for stability.

2.2.1.2 Homogeneous First Order Systems

The case of homogeneous first order systems, with Gaussian parameter variation, was analyzed by Caughey and Dienes (6). Expressions for the first order probability density function were obtained for both white and non-white noise processes. In the former case, the density function was obtained as the solution of the Fokker-Planck equation. This technique has been outlined in section 2.3. The case in which the parameter variation is not necessarily a white noise process is discussed next.

We substitute,

$$\begin{aligned} k(t) &= a + b(t) \\ f(t) &= 0 \end{aligned} \tag{2.8}$$

in equation 2.1 to obtain

$$\frac{dy(t)}{dt} + [a + b(t)]\, y(t) = 0 \tag{2.9}$$

where b(t) is an ergodic, stationary, Gaussian random process with zero mean, having a spectral density $\Phi(w)$ and $a = \langle k(t) \rangle$. From equation 2.2, for initial conditions $y = y_o$ at $t = 0$, we can write

$$y(t) = y_o \in^{-\int_0^t k(u)du} = y_o \in^{-at} \in^{-\int_0^t b(u)du} \tag{2.10}$$

we define a random variable

$$\gamma(t) = \int_0^t b(u)du \tag{2.11}$$

Since $\gamma(t)$ is the result of a linear operation on b(u), a Gaussian process, it is also a Gaussian process with the following characteristics

$$\langle \gamma(t) \rangle = \int_0^t \langle b(u) \rangle \; du = 0 \tag{2.12}$$

$$\sigma_\gamma^{\;2} = \langle \gamma^2(t) \rangle = \int_0^t \int_0^t \langle b(u_1) b(u_2) \rangle \, du_1 du_2 \tag{2.13}$$

From our assumption that b(u) is stationary, we can write

$$\langle b(t_1) \; b(t_2) \rangle = \phi(t_1 - t_2) \tag{2.14}$$

which is the auto-correlation function of b(t).

Using Wiener's theorem, $\phi(t_1 - t_2)$ can be written in terms of the spectral density of b(t). Thus

$$\phi(u_1-u_2) = \int_0^\infty \Phi(w) \text{ Cos } w(u_1-u_2) \, dw \tag{2.15}$$

Substituting in 2.13 we get

$$\sigma_\gamma^2 = \int_0^t \int_0^t \int_0^\infty \Phi(w) \text{ Cos } w(u_1-u_2) \, dw \, du_1 \, du_2 \tag{2.16}$$

Changing the order of integration and integrating with respect to u_1 and u_2 successively, we obtain

$$\sigma_\gamma^2 = \int_0^\infty \Phi(w) \frac{[1 - \text{Cos } wt]}{w^2} \, dw \tag{2.17}$$

The probability density function of γ is, thus, completely characterized and we may write

$$P(\gamma) = \frac{1}{\sqrt{2\pi}\,\sigma_\gamma} \, \epsilon^{-\frac{\gamma^2}{2\sigma_\gamma^2}} \tag{2.18}$$

The output moments can now be obtained using equations 2.10 and 2.18; thus,

$$\langle y^p \rangle = \int_{-\infty}^{\infty} y^p \, P(\gamma) \, d\gamma \tag{2.19}$$

$$\text{or } \langle y^p \rangle = \int_{-\infty}^{\infty} y_o^p \, \epsilon^{-apt - p\gamma} \frac{1}{\sqrt{2\pi}\,\sigma_\gamma} \, \epsilon^{-\frac{\gamma^2}{2\sigma_\gamma^2}} \, d\gamma$$

$$\langle y^p \rangle = y_o^p \, \epsilon^{-p(at - \frac{1}{2}p\,\sigma_\gamma^2)} \tag{2.20}$$

It is observed that, if $\sigma_\gamma{}^2$ is bounded as $t \to \infty$, then $\langle y^p \rangle$ tends to zero asymptotically.

The probability density function P(y) of the output can be obtained using 2.18 and the following theorem.

Theorem 2.1. Let η and ξ be two random variables related by the linear transformation $\eta = f(\xi)$ which maps ξ into η uniquely. Then

$$P_\eta(y) = P_\xi(\gamma) \left| \frac{d\gamma}{dy} \right|_{\gamma = f^{-1}(y)} \tag{2.21}$$

where γ and y are the range variables of ξ and η respectively.

We have

$$y = y_o \, \epsilon^{-at - \gamma(t)} = f(\gamma) \tag{2.22}$$

$$\frac{dy}{d\gamma} = -y_o \, \epsilon^{-at - \gamma} = -y \quad \therefore \left| \frac{d\gamma}{dy} \right| = \frac{1}{y} \tag{2.23}$$

Substituting, 2.23 and 2.18 in 2.21, we obtain

$$P(y) = \frac{1}{\sqrt{2\pi}\ \sigma_\gamma} \, \epsilon^{-\gamma^2/2\sigma_\gamma{}^2} \left. \frac{1}{y} \right|_{\gamma = f^{-1}(y)} \tag{2.24}$$

From 2.22, we have

$$\text{Ln} \frac{y}{y_o} = -at - \gamma$$

$$\therefore \quad \gamma = - [at + \text{Ln}\ y/y_o] \tag{2.25}$$

Substituting 2.25 in 2.24, we get

$$P(y) = \frac{1}{y\sqrt{2\pi}\ \sigma_\gamma} \quad {}^{\frac{-(at + \text{Ln}\ y/yo)^2}{2\sigma_\gamma{}^2}} \tag{2.26}$$

The probability of exceeding a value y', can be found by integrating 2.20 from y' to ∞. Thus

$$\Pr\,(y > y') = \int_{y'}^{\infty} P(y)\,dy$$

It has been established (6) that the probability of exceeding y' tends to zero asymptotically. This leads to rather interesting conclusions regarding the stability of the system. This will be discussed in the chapter on stability.

2.2.1.3 Step Response of a First Order System

The probability density function of the output of a first order system subjected to a step input was obtained by Rosenbloom (1). He considered a first order system of the type 1a, shown in Table 2.1 which is described by equation 2.1. For zero initial conditions, the output is given by

$$y = \int_0^t e_i(\tau)\, k(\tau)\, \epsilon^{-\int_\tau^t k(u)du}\, d\tau \tag{2.28}$$

For an unit step input, 2.28 reduces to

$$y = \int_0^t k(\tau)\, \epsilon^{-\int_\tau^t k(u)du}\, d\tau \tag{2.29}$$

we observe that

$$\frac{d}{d\tau}\,\epsilon^{-\int_{\tau}^{t} k(u)du} = \epsilon^{-\int_{\tau}^{t} k(u)du}\,\frac{d}{d\tau}\left[-\int_{\tau}^{t} k(u)du\right]$$

$$= -\epsilon^{-\int_{\tau}^{t} k(u)du}\left[k(t)\,\frac{dt}{d\tau} - k(\tau)\,\frac{d\tau}{d\tau} - \int_{\tau}^{t}\frac{dk(u)}{d\tau}\,du\right]$$

$$= k(\tau)\,\epsilon^{-\int_{\tau}^{t} k(u)du}$$

We can, therefore, write

$$y = \int_0^t \frac{d}{d\tau}\left(\epsilon^{-\int_{\tau}^{t} k(u)du}\right) d\tau = \epsilon^{-\int_{\tau}^{t} k(u)du}\Bigg|_0^t$$

or

$$y = 1 - \epsilon^{-\int_0^t k(u)du} \tag{2.30}$$

Putting $k(t) = a + b(t)$ where $b(t)$ is a stationary Gaussian process with zero mean and $a = \langle k(t) \rangle$, we have

$$y = 1 - \epsilon^{-at}\,\epsilon^{-\int_0^t b(u)du} \tag{2.31}$$

As was done, for the case of homogeneous first order systems, we define a new random variable

$$\gamma = \int_0^t b(u)du$$

which is the same as equation 2.11. It has been shown that γ is a Gaussian process, with a probability density function given by equation 2.18; that is,

$$P(\gamma) = \frac{1}{\sqrt{2\pi}\ \sigma_\gamma} \epsilon^{-\frac{\gamma^2}{2\sigma_\gamma{}^2}}$$

where

$$\sigma_\gamma{}^2 = \int_0^\infty \Phi(w) \frac{[1 - \operatorname{Cos} wt]}{w^2} dw$$

The pth output moment can be evaluated by solving the integral

$$\langle y^p \rangle = \int_{-\infty}^{\infty} y^p\ P(\gamma)\ d\gamma \tag{2.32}$$

where

$$y = 1 - \epsilon^{-at} \epsilon^{-\gamma}$$

from 2.31,

$$\text{or} \quad \langle y^p \rangle = \int_{-\infty}^{\infty} \frac{(1 - \epsilon^{-at}\ \epsilon^{-\gamma})^p}{\sqrt{2\pi}\ \sigma_\gamma} \epsilon^{-\frac{\gamma^2}{2\sigma_\gamma{}^2}} d\gamma \tag{2.33}$$

To get the output probability density function, we use theorem 2.1. We have, therefore,

$$P(y) = P(\gamma) \left| \frac{d\gamma}{dy} \right|_{\gamma = f(y)} \tag{2.34}$$

From $y = 1 - \epsilon^{-at} \epsilon^{-\gamma}$ [eq: 2.31] we get,

$$\gamma = - [at + \operatorname{Ln} (1 - y)] \tag{2.35}$$

and

$$\frac{dy}{d\gamma} = \epsilon^{-at} \epsilon^{-\gamma} = 1 - y\ . \tag{2.36}$$

Substituting, 2.35 and 2.36 in 2.34, we get

$$P(y) = \frac{1}{\sqrt{2\pi}\ \sigma_\gamma\ (1 - y)}\ \epsilon^{-\frac{[at + Ln\ (1 - y)]^2}{2\sigma_\gamma^2}} \tag{2.37}$$

Rosenbloom's underlying aim in the analysis was to study the step response of first order systems. A discussion of the results is presented in the chapter on stability.

2.2.2 First Order Systems with White Noise Input and a Random Binary Parameter Variation

Redman and Lampard (10) obtained expressions for the auto-correlation function of the output of first order systems shown in Fig. 2.1, under the following assumptions:

(i) One parameter is switched randomly between two values, the variation being a stationary binary process.

(ii) The lengths of successive time intervals are statistically independent.

(iii) The output process is ergodic.

For zero initial conditions, the solution of equation 2.1 is given by

$$y(t) = \int_0^t e_1(v)\ k(v)\ \epsilon^{-\int_v^t k(u)du}\ dv \tag{2.38}$$

The auto-correlation function of the output, expressed as a time average, is

$$\Phi_{oo}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} y(t)\ y(t+\tau)\ dt \tag{2.39}$$

Since, the output process is assumed ergodic, this time average is equal to the ensemble average given by

$$\phi_{oo}(\tau) = \langle y(t)\ y(t+\tau)\rangle \tag{2.40}$$

Thus,

$$\phi_{oo}(\tau) = \int_0^t \int_0^{t+\tau} \langle e_i(v_1)\ e_i(v_2)\rangle \Big\langle k(t)\ k(t+\tau)$$

$$\exp: \left[-\int_{v_1}^{t} k(u)du - \int_{v_2}^{t+\tau} k(u)du \right] \Big\rangle\ dv_1\ dv_2$$

$$\text{for } \tau \geqslant 0\ . \tag{2.41}$$

We can write,

$$k(t) = a + b(t) \tag{2.42}$$

where b(t) is a symmetric binary random process $\pm m$ with $a > m > 0$. For a white noise input process, which is also ergodic,

$$\langle e_1(v_1)\ e_1(v_2)\rangle = \phi_{ff}(v_1 - v_2)$$

$$= \frac{S}{2}\ \delta(v_1 - v_2) \tag{2.43}$$

where S is the power spectral density of the white noise process. Substituting equation 2.43 and 2.42 in 2.41 and simplifying, we get

$$\phi_{00}(\tau) = \frac{S}{2} \epsilon^{-at} \int_0^t \epsilon^{-2a(t-v)} \Big\langle \Big[a + b(t)\Big] \Big[a + b(t+\tau)\Big]$$

$$\text{exp:} \left[-2\int_v^t b(u)du - \int_t^{t+\tau} b(u)du \right] \Big\rangle \, dv \; ; \; \tau \geqslant 0. \qquad (2.44)$$

To evaluate the above integral Redman and Lampard used results of an earlier paper (11) in which they had devised a procedure to obtain the multidimensional transition probability density and the associated characteristic function of a random process defined by

$$\gamma(t) = \int_0^t \eta(u)du \qquad (2.45)$$

where $\eta(u)$ is a symmetric binary random process.

The auto-correlation function for a Poisson distribution was shown by Redman and Lampard to be

$$\phi_{00}(\tau) = \frac{S}{4} \epsilon^{-(a+\mu)|\tau|}$$

$$\left\{ \left[a + \frac{\mu m^2}{a^2 + a\mu - m^2} \right] \text{Cosh} |\tau| \sqrt{\mu^2 + m^2} \right.$$

$$\left. + \frac{1}{\sqrt{\mu^2 + m^2}} \left[\mu a - m^2 - \frac{\mu m^2 (a + \quad)}{a^2 + a\mu - m^2} \right] \text{Sinh} |\tau| \sqrt{\mu^2 + m^2} \right.$$

$$-\infty < \tau < \infty \qquad (2.46)$$

where

μ = mean rate of Poisson parameter variation.

If we set m = 0, equation 2.46 reduces to

$$\phi_{00}(\tau)_{m=0} = \frac{S}{4}\, a\, \epsilon^{-a|\tau|} \tag{2.47}$$

This is the expression for the auto-correlation function of the output of a time-invariant RC network, time constant 1/a with a white noise input process.

For low switching rates, 2.46 reduces to

$$\lim_{\frac{\mu}{a} \to 0} \phi_{00}(\tau) = \frac{S}{4}\, \epsilon^{-a|\tau|} \left\{ a \operatorname{Cosh} m|\tau| \; - m \operatorname{Sinh} m|\tau| \right\} \tag{2.48}$$

This relation is seen to be the arithmetic mean of the two auto-correlation functions obtained for two separate RC network, with time constants

$$\frac{1}{a+m} \quad \text{and} \quad \frac{1}{a-m} .$$

For very high switching rates, the identity with the two possible states of the network is lost, and the auto-correlation function becomes

$$\lim_{\mu/a \to \infty} \phi_{00}(\tau) = \frac{S}{4}\, a\, \epsilon^{-a|\tau|} \tag{2.49}$$

Redman and Lampard also obtained an expression for the output spectral density using the expression for the auto-correlation and Wiener's relation relating the two. They also verified the theoretical results by experimental measurements.

2.3 Higher Order Systems

The analysis of higher order systems was first carried out by Samuels and Eringen (3) and Samuels (4) who obtained an expression for the auto-correlation of the output in the form of an integral equation which could be evaluated in the closed form for some special cases. Samuels considered random linear systems in which all the parameters but one are constants. He obtained expressions for the mean square of the output for the cases where the random parameter varied as (i) a white noise process and (ii) a narrow band process. Samuels (5) later extended his work to include the case where any number of parameters vary as a white noise process.

In later works (6,7,8) various authors have used the Fokker-Planck equation to derive relations for the output moments of a random linear system subjected to a white noise input process and in which any number of parameters may vary as white noise processes. This technique permits the evaluation of higher order moments in terms of lower order moments and simplifies the study of moment-stability. The Fokker-Planck equation technique has been discussed in section 2.3.1.

The same technique was also used by Ariaratnam and Graefe (8,9) to study the behavior of discrete linear systems in which the parameters vary as Wiener processes (integral of white noise processes).

2.3.1 Random Linear Systems with a White Noise Input Process and with the Parameters varying as White Noise Processes (9)

We consider a general linear system governed by n first order state equations:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = -\begin{bmatrix} a_{11} & \cdot\cdot & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdot\cdot & a_{nn} \end{bmatrix}\begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} - \begin{bmatrix} \alpha_{11} & \cdot\cdot & \alpha_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_{n1} & \cdot\cdot & \alpha_{nn} \end{bmatrix}\begin{bmatrix} y_1 \\ \\ \\ \\ \\ y_n \end{bmatrix} + \begin{bmatrix} \alpha_{10} \\ \alpha_{20} \\ \\ \\ \\ \alpha_{n0} \end{bmatrix} \qquad (2.50)$$

where $\underline{Y} = [y_1 \ . \ . \ . \ y_n]^T$ is the state vector; $[a_{ij}] = \underline{A}$ is a nxn matrix of constant elements; $[\alpha_{ij}] = \underline{\alpha}$ is an nxn matrix and $[\alpha_{no}] = \underline{\alpha}_o$ is an nx1 matrix of Gaussian white noise processes.

It can be shown (9) that the trajectory of the state point in the n-dimensional state space is a Markov process and the first probability density of the response is governed by the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i}(A_i P) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial y_i \, \partial y_j}(B_{ij}P) \qquad (2.51)$$

where

$$A_i = \lim_{t \to o} \frac{\langle \Delta y_i \rangle}{\Delta t} \quad \text{and} \quad B_{ij} = \lim_{t \to o} \frac{\langle \Delta y_i \Delta y_j \rangle}{\Delta t} \qquad (2.52)$$

In order to evaluate the coefficients A_i and B_{ij} we shall first formulate the state equations.

The state of a system, described by n state variables y_1 . . . y_n, can be represented at every instant by a point in the n-dimensional state space. In the absence of an external disturbance, let the system be described by the equation

$$\frac{dY}{dt} = - \underline{A}\ \underline{Y} \tag{2.53}$$

where $\underline{Y} = (y_1 \ . \ . \ . \ y_n)^T$ is the state vector and $\underline{A} = [a_{ij}]$ is a constant nxn matrix.

If the state point is at a position p at the instant t, then in a finite interval Δt, the state point will be shifted by

$$\underline{Y} = - \underline{A}\ \underline{Y} \Delta t + O(\Delta t) \tag{2.54}$$

where $O(\Delta t)$ means that there are higher order terms.

If the system receives a continual random disturbance

$$\underline{F}(t) = - \left[\underline{\alpha}(t)\ \underline{Y}(t) - \underline{\alpha_o}(t) \right] \tag{2.55}$$

the system is shifted by

$$\underline{Y} = - \underline{A}\ \underline{Y} \Delta t + O(\Delta t) + \int_t^{t+\Delta t} \underline{F}(\tau)\, d\tau$$

or,

$$\underline{Y} = - \underline{A}\ \underline{Y} \Delta t + O(\Delta t) - \int_t^{t+\Delta t} [\underline{\alpha}(\tau)\ \underline{Y}(\tau) - \underline{\alpha_o}(\tau)]\, d\tau \tag{2.56}$$

In the limit as $\Delta t \rightarrow 0$, equation 2.56 becomes

$$\frac{d\underline{Y}}{dt} = - \underline{A}\ \underline{Y} - \underline{\alpha}(t)\ \underline{Y}(t) - \alpha_o(t)$$

which corresponds to equation 2.50.

The random processes $\alpha_{ij}(t)$ and $\alpha_{io}(t)$ have the following statistical properties:

$$\left\langle \alpha_{ij}(t) \right\rangle = 0$$

$$\left\langle \alpha_{ij}(t_1)\, \alpha_{rs}(t_2) \right\rangle = 2\, D_{ijrs}\, \delta(t_1 - t_2) \qquad (2.57)$$

where $i, r = 1, 2 \ldots n$

$j, s = 0, 1 \ldots n$

and $\langle --- \rangle$ represents the ensemble average.

After considerable mathematical manipulation it can be shown that the coefficients of the Fokker-Planck equation are given by:

$$A_i = -\sum_{r=1}^{m} \left\{ a_{ir} y_r(t) + 2\, D_{irro} - 2 \sum_{s=1}^{n} D_{irro}\, y_s(t) \right\} \qquad (2.58)$$

$$B_{ij} = 2 \left\{ D_{iojo} - \sum_{r=1}^{n} \left[-D_{irjo}\, y_r(t) + \sum_{s=1}^{n} (-D_{iojs} y_s(t) \right. \right.$$

$$\left. \left. + D_{irjs}\, y_r(t) y_s(t) \right] \right\} \qquad (2.59)$$

Although, no general method of solution of the Fokker-Planck equation is available, it is possible to obtain differential equations governing the moments of the system response, which may be solved recursively for the various moments.

Let the mixed moments of order N be given by

$$m_N\,(k_1 k_2 \ldots k_n) = \left\langle y_1^{k_1}\, y_2^{k_2} \ldots y_n^{k_n} \right\rangle$$

$$N = 1, 2 \ldots$$

where $k_1, k_2 \ldots k_n$ are positive integers satisfying

$$\sum_{i=1}^{n} k_i = N.$$

Multiplying 2.51 by $y_1^{k_1} y_2^{k_2} \ldots y_n^{k_n}$ and integrating by parts over the entire state space, we get the moment equations:

$$\frac{d}{dt} m_N(k_1, k_2 \ldots k_n) = - \sum_{i=1}^{n} \sum_{r=1}^{n} a_{ir} k_i m_N(k_1, \ldots k_i - 1, k_r + 1, \ldots k_n)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} \begin{Bmatrix} k_i k_j ; (i \neq j) \\ k_i(k_i - 1); (i = j) \end{Bmatrix} D_{irjo}\; m_N(k_1, \ldots k_i - 1, \ldots k_j - 1 \ldots k_r + 1, \ldots k_s + 1 \ldots k_n)$$

$$- 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} \begin{Bmatrix} k_i k_j ; (i \neq j) \\ k_i(k_i - 1); (i = j) \end{Bmatrix} D_{irjo}\; m_{N-1}(k_1, \ldots k_i - 1, \ldots k_j - 1, \ldots k_r + 1, \ldots k_n)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \begin{Bmatrix} k_i k_j ; (i \neq j) \\ k_i(k_i - 1); (i = j) \end{Bmatrix} D_{iojo}\; m_{N-2}(k_1, \ldots k_i - 1, \ldots k_j - 1, \ldots k_n)$$

$$- \sum_{i=1}^{n} \sum_{r=1}^{n} D_{irro}\, k_i\, m_{N-1}(k_1 \ldots, k_i - 1 \ldots k_n) \tag{2.60}$$

$$+ \sum_{i=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} D_{irrs}\, k_i\, m_N(k_1 \ldots, k_i - 1, \ldots k_s + 1, \ldots k_n)$$

The evaluation of the last term of each summations warrants special attention. It is possible that more than one k with the same subscript l arises in the argument of m. In this case the power of y in the moment equation should be taken as k plus the algebraic sum of numbers added to all the k appearing in the moment term. This becomes clear in the examples to appear.

We see that the Nth order moments are only related to the moments of order less than N, and for a particular N the number of equations are equal to the number of mixed moments of that order. Once the first moments are determined, the higher order moments may be determined recursively.

2.3.1.1 Example First Order Systems

A first order system is defined by the differential equation

$$\frac{dy}{dt} + (a_{11} + \alpha_{11})\, y = \alpha_{01} \qquad (2.61)$$

where a_{11} = constant

α_{11}, α_{01} are white noise processes.

Let $y_1 = y$ (2.62)

The state equation may be written as

$$\frac{d}{dt}\,\underline{Y} = -\,\underline{A}\,\underline{Y} - \underline{\alpha}\,\underline{Y} + \underline{\alpha}_o \qquad (2.63)$$

where,

$$\underline{Y} = y_1,\ \underline{A} = a_{11},\ \underline{\alpha} = \alpha_{11}$$

and $$\underline{\alpha}_o = \alpha_{o1}.$$

Using equation 2.60, and after some mathematical manipulation, we obtain the moment equation:

$$\frac{d}{dt}\left\langle y_1^N \right\rangle + N\,(a_{11} - ND_{1111})\left\langle y_1^N \right\rangle = -\,N\,(2N-1)D_{1110}\left\langle y_1^{N-1} \right\rangle$$

$$+\,N(N-1)\,D_{1010}\left\langle y_1^{N-2} \right\rangle \qquad (2.64)$$

The first and second moments (N = 1 and N = 2, respectively) may be obtained as

$$\frac{d}{dt}\left\langle y_1 \right\rangle + (a_{11} - D_{1111})\left\langle y_1 \right\rangle = -\,D_{1110} \qquad (2.65)$$

and

$$\frac{d}{dt}\left\langle y_1^2 \right\rangle + 2(a_{11} - 2\,D_{1111})\left\langle y_1^2 \right\rangle = -\,6\,D_{1110}\left\langle y_1 \right\rangle + 2\,D_{1010} \qquad (2.65)$$

These equations may be used very conveniently to study the moment stability of the system.

<u>2.3.1.2 Second Order Systems</u>

The system is defined by the differential equation

$$\frac{d^2y}{dt^2} + (a_{22} + \alpha_{22})\frac{dy}{dt} + (a_{21} + \alpha_{21})\,y = \alpha_{20}. \qquad (2.67)$$

We let the state variables be $y_1 = y$

and $$y_2 = \frac{dy}{dt} = y_1 \qquad (2.68)$$

From equation 2.67, we obtain

$$y_2 = - a_{22}\, y_2 - \alpha_{22}\, y_2 - a_{21} y_1 - \alpha_{21} y_1 + \alpha_{20}$$

Thus,

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = - \begin{bmatrix} 0 & -1 \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_{20} \end{bmatrix} \qquad (2.69)$$

Comparison of equation 2.69 with equation 2.50 shows that

$$a_{11} = 0 \quad a_{12} = -1 \quad \alpha_{11} = \alpha_{12} = \alpha_{10} = 0.$$

The various moment equations may be obtained by using equations 2.60.

Two equations for the first moment ($N = 1$) are obtained corresponding to $k_1 = 1$, $k_2 = 0$ and $k_1 = 0$, $k_2 = 1$. The expressions simplify if we note that the D_{irjs} corresponding to α_{11}, α_{12} and α_{10} are zero.

Similarly, three equations for the second moment ($N = 2$) are obtained corresponding to ($k_1 = 2$, $k_2 = 0$), ($k_1 = 1$, $k_2 = 1$), and ($k_1 = 0$, $k_2 = 2$).

In matrix notation the moment equations may be written as

$$\frac{d}{dt}\begin{bmatrix} \langle y_1\rangle \\ \langle y_2\rangle \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_{21}-D_{2221} & -a_{22}-D_{2222} \end{bmatrix}\begin{bmatrix} \langle y_1\rangle \\ \langle y_2\rangle \end{bmatrix} + \begin{bmatrix} 0 \\ D_{2220} \end{bmatrix} \qquad (2.70)$$

and

$$\frac{d}{dt}\begin{bmatrix}\langle y_1^2\rangle\\ \langle y_1y_2\rangle\\ \langle y_2^2\rangle\end{bmatrix} = \begin{bmatrix}0 & 2 & 0\\ D_{2221}-a_{21} & D_{2222}-a_{22} & +1\\ 2\,D_{2121} & 6D_{2122}-2a_{21} & 4D_{2222}-2a_{22}\end{bmatrix}\begin{bmatrix}\langle y_1^2\rangle\\ \langle y_1y_2\rangle\\ \langle y_2^2\rangle\end{bmatrix}$$

$$-\begin{bmatrix}0 & 0\\ 4\,D_{2120} & 6\,D_{2220}\\ D_{2220} & 0\end{bmatrix}\begin{bmatrix}\langle y_1\rangle\\ \\ \langle y_2\rangle\end{bmatrix} \qquad (2.71)$$

These two equations may be used conveniently to study the moment stability of the system. This is discussed in Chapter III.

2.4 Remarks

Sections 2.2 and 2.3 reveal the complexity of the problem of determining the probability density functions of the output of a random linear system. Concrete results can be obtained for certain first order systems only while for higher order systems one has to be content with determinations of moments. As mentioned in the introduction, this knowledge is often quite sufficient from an engineering point of view since one is usually concerned with the problem of stability of such systems.

It must be noted that for higher order systems we have confined ourselves to the case where the parameter variations are Gaussian white noise processes. For other types of variations the problem is highly complicated and no general solution has appeared in the papers scanned.

CHAPTER III

STABILITY OF RANDOM LINEAR SYSTEMS

3.1 Introduction

The problem of establishing general stability criteria of random linear systems is very complicated. The most appropriate way to deal with the problem is to extend to random linear systems the stability concepts of deterministic systems. As the RLS can exist in an ensemble of configurations, determined only by probabilistic considerations, it is best to consider the stability of the system as a whole and, if possible, explain it in terms of the stability of a particular configuration that the system can exist in.

Before establishing any criteria it is necessary to define exactly what we mean by the stability of a random linear system. It is essential that this definition satisfy the intuitive concept of stability and therefore it seems logical that a definition based on the one ordinarily associated with deterministic systems is in order. A linear time-invariant system, which is initially in equilibrium is said to be stable if and only if the system returns to equilibrium after a finite disturbance. In the case of non-random systems, this return to equilibrium is a purely deterministic process. The return of a random system to equilibrium can only be described probabalistically. The most precise definition of stability of random linear systems can be given in terms of the convergence of random variables.

Definition 3.1

A random linear system, initially in equilibrium, is "strictly" stable, if and only if, the random variables representing its response to a finite disturbance, converge "almost certainly" (Appendix).

This definition, although very desirable, is rather difficult to adhere to in practice. The sufficiency conditions, stated in the following section, in general ao not imply almost certain convergence. It is, therefore, necessary to reexamine the stability concept in terms of the less rigorous convergence-in-probability notion.

Definition 3.2

A random linear system is (asymptotically) stable if the random variables, representing the response to a finite disturbance, converge asymptotically in probability.

We can now establish sufficiency conditions for stability of random linear systems.

Theorem 3.1. (Bershad and DeRusso (12)) For a stationary RLS to be stable it is sufficient, though not necessary, that the condition

$$\lim_{t \to \infty} \langle H^2(t) \rangle = 0$$

be satisfied. $H(t)$ is the random response of the system to a unit impulse applied at $t = 0$ and $\langle --- \rangle$ represents the ensemble averaging operator.

Proof of Theorem 3.1. In order to prove the theorem we make use of the following Lemma.

Lemma 1. If $\langle H^2(t)\rangle$ is bounded, so is $\langle H(t)\rangle$.

Proof of Lemma. Consider the ensemble of random functions $\{H(t)\}$ with probability density functions $P_H(\eta)$ where η is the range variable of H. Then

$$\langle H^2(t)\rangle = \int_{-\infty}^{\infty} H^2(t)\, dP(\eta)$$

and

$$\langle H(t)\rangle = \int_{-\infty}^{\infty} H(t)\, dP(\eta)$$

From Schwarz inequality we have

$$\left[\int_{-\infty}^{\infty} H(t)\, G(t)\, dP(\eta)\right]^2 \leq \int_{-\infty}^{\infty} H^2(t)\, dP(\eta) \int_{-\infty}^{\infty} G^2(t)\, dP(\eta)$$

For G = 1, this becomes

$$\langle H(t)\rangle^2 \leq \langle H^2(t)\rangle .$$

Therefore, if $\langle H^2(t)\rangle$ is bounded, so is $\langle H(t)\rangle$.

The proof of theorem 3.1 follows from Lemma 1 and Chebyshev's inequality,

$$\text{Prob}\left\{\left|H(t) - \langle H(t)\rangle\right| \geq K\sigma_H\right\} \leq \frac{1}{K^2}$$

where $K > 0$

and $\sigma_H^2 = \langle H^2(t)\rangle - \langle H(t)\rangle^2$.

Assume, $\lim_{t\to\infty} \langle H^2(t)\rangle = 0$

Since $\langle H(t)\rangle^2 \leq \langle H^2(t)\rangle$

σ_H^2 tends to zero as t tends to infinity. We let $k \to \infty$ as $t \to \infty$ such that $k\sigma_H \to \epsilon$ where $\epsilon > 0$. Thus,

$$\lim_{\substack{t \to \infty \\ k \to \infty}} \text{Prob}\left\{ \left| H(t) - \langle H(t)\rangle \right| \geq \epsilon \right\} = 0$$

If ϵ is made arbitrarily small, the above equation means that H(t) converges asymptotically to $\langle H(t)\rangle$ in probability where $\langle H(t)\rangle$, which is less than $\langle H^2(t)\rangle$, tends to zero. Therefore H(t) converges to zero in probability.

This means that "almost all" H(t) of the ensemble $\{H(t)\}$ converge to zero asymptotically.

As stated before, convergence in probability, in general, does not imply almost certain convergence. However, in certain special cases, it can be shown (12, 13) that the sufficient condition stated in theorem 3.1 does guarantee almost certain convergence (Appendix).

A similar sufficiency condition can be established in terms of the response of a RLS to a unit step input.

An alternate way to describe the stability of a random linear system is in terms of the convergence of distribution concept. This is not uncommon in probability theory (14) and can be extended to explain the stability of first order systems discussed later. This concept is stated in the form of a definition.

<u>Definition 3.3</u>

A stationary random linear system is stable if

$$\lim_{t \to \infty} P(y;t) = \delta(y_o)$$

where $P(y;t)$ is the probability density function at time t of the random variable $y(t)$ representing the response of the system to a finite disturbance.

This definition is equivalent to definition 3.2 since the area of an unit impulse function is unity. This means that $y(t)$ converges asymptotically to y_o in probability.

Definition 3.3 has very restrictive use because of the obviously difficult job of determining the probability density function of the output.

3.2 Examples

3.2.1 Impulse Response of a First Order System

We consider first order systems described by a differential equation:

$$\frac{dy(t)}{dt} + k(t)\, y(t) = e_1(t)\, k(t)$$

where $e_1(t)$ is the input to the system,

$y(t)$ is the output of the system

and $k(t)$ is the open loop gain (Table 2.1)

The solution of this equation for zero initial conditions is given by equation 2.28, which is repeated for convenience

$$y(t) = \int_0^t e_1(\tau)\, k(\tau)^{\,-\int_\tau^t k(u)du}\, d\tau \qquad (3.1)$$

If $e_1(t)$ is a dirac-delta function at $t = 0$, equation 3.1

reduces to

$$H(t) = k_0 \, \epsilon^{-\int_0^t k(u)du} \tag{3.2}$$

where k_0 is the value of $k(t)$ at $t = 0$ and is assumed to be known, and $H(t)$ is the impulse response of the system.

Equation 3.2 is similar to equation 2.10 of the homogeneous first order system with y_0 and $y(t)$ replaced by k_0 and $H(t)$ respectively.

Substituting the following into equation 3.2,

$$k(t) = a + b(t) \tag{3.3}$$

where $\langle b(t) \rangle = 0$ and $\langle k(t) \rangle = a$,

we obtain

$$H(t) = k_0 \, \epsilon^{-at} \, \epsilon^{-Z} \tag{3.4}$$

where

$$Z = \int_0^t b(u)du \tag{3.5}$$

If $b(t)$ is a stationary, ergodic Gaussian process the first probability density of the impulse response is given by equation 2.26 with slight modification. Thus,

$$P(H) = \frac{1}{H(t)\sqrt{2\pi\,\sigma_Z^2}} \, \epsilon^{\frac{-(at + \mathrm{Ln}\, H(t)/k_0)^2}{2\sigma_Z^2}} \tag{3.6a}$$

where

$$\sigma_Z^2 = \int_0^\infty \frac{1-\mathrm{Cos}\ wt}{w^2} \, \Phi(w)\, dw \tag{3.6b}$$

and $\Phi(w)$ = spectral density of $b(t)$.

P(H), with b(t) a white noise process, is shown in Fig. 3.1 as a function of H(t) and t.

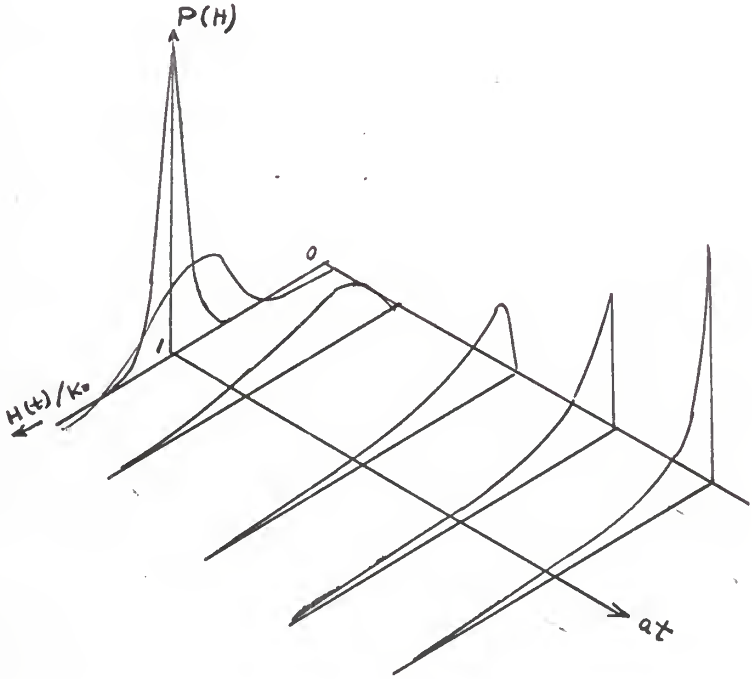


FIGURE 3.1

IMPULSE RESPONSE OF A FIRST ORDER SYSTEM

It can be shown that P(H) tends to a delta function asymptotically. From Definition 3.3 this implies that the system is stable.

It is interesting to note that the stability of the moments is not necessary for the system to be stable according to Definition 3.2. This is seen from the following analysis.

An expression similar to equation 2.20 can be written for the pth moment of the impulse response. With necessary modifications of equation 2.20 we have

$$\langle H^p(t)\rangle = k_0^p \, \epsilon^{-p(at - \frac{1}{2}p\sigma_Z^2)} \tag{3.7}$$

From equation 3.7 we have

$$\sigma_Z^2 = \int_0^{\infty} \frac{1 - \text{Cos } wt}{w^2} \, \Phi(w)dw$$

For b(t), a white noise process, $\Phi(w) = \frac{2D}{\pi}$ which when substituted in 3.6b yields

$$\sigma_Z^2 = 2Dt \tag{3.8}$$

Substituting 3.8 in 3.7 we obtain

$$\langle H^p(t)\rangle = k_0^p \, \epsilon^{-p(at - pDt)} \tag{3.9}$$

If $a > pD$ the pth moment is stable.

If $a < D$, all moments become unstable. This, however, does not effect the nature of the probability density function. Thus the system is stable according to Definition 3.3 although its moments become unbounded with time.

This interesting deduction can be easily explained by probability concepts, although it is difficult to give a precise physical reasoning. It has been shown in theorem 3.1 that the stability of the mean square implies convergence in probability. The reverse, however, is not true in general. Thus, it is quite possible for a sequence of random variables to converge in probability, while not converge in the mean square.

A plausible reason for the instability of the moments is not obvious. The instability of the moments is often attributed to the fact that there is always a finite probability of the output becoming $+\infty$, and the moments may be emphasizing the large positive value of the output. However, this reasoning implies that the moments are always unbounded, which contradicts our results since we know that the pth output moment can be made stable if $a > pD$, without affecting the nature of the probability density curves.

It will be noted that if $a > 2D$, $\langle H^2(t) \rangle$ is always positive and an exponentially decaying function. From theorem A.1 (Appendix) it follows that $a > 2D$ is a sufficient condition for the system to be stable according to the stronger Definition 3.1. That is, the system is "almost certainly" stable or is stable almost everywhere in time.

3.2.2 Step Response of First Order Systems

Rosenbloom (1) obtained an expression for the first probability density of the step response of a first order network and discussed the stability problem of such systems. Rosenbloom's analysis has been summarized in section 2.2. Some of the expressions will be rewritten for convenience:

$$P(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Z\,(1-y)} \, e^{\frac{-[at + \mathrm{Ln}\,(1-y)]^2}{2\sigma_Z^2}} \tag{3.10}$$

If the b(t) is a white noise process, the first probability density is as shown in Fig. 3.2.

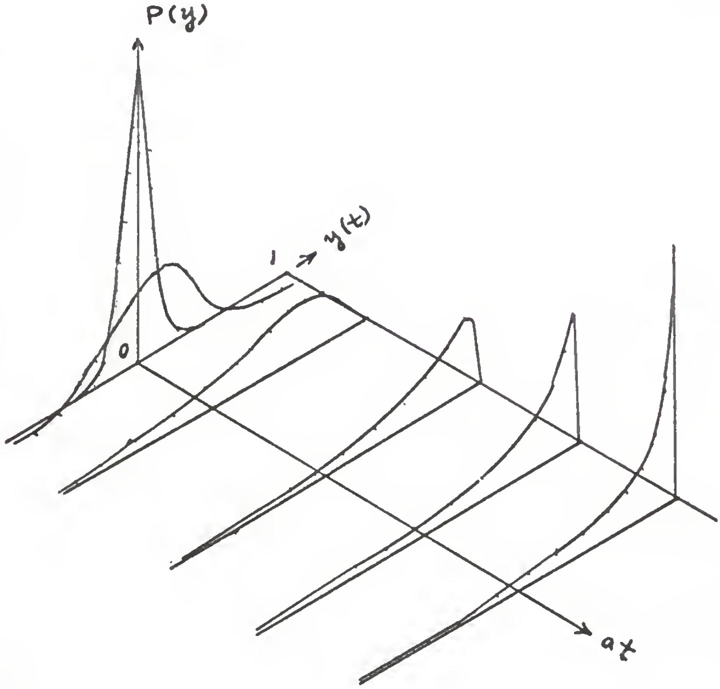


FIGURE 3.2

STEP RESPONSE OF A FIRST ORDER SYSTEM

It is seen that the probability density functions converge to a delta function at y = 1, as t tends to infinity. From theorem 3.2, this implies that the system is stable.

From equation 2.3, we can determine the first and second moments as follows:

$$\langle y \rangle = 1 - \epsilon^{-(a-D)t}$$

$$\langle y^2 \rangle = 1 - \epsilon^{-2(a-2D)t} - 2\,\epsilon^{-(a-D)t}$$

If $a < 2D$ both the first and second moments tend asymtotically to $-\infty$. This phenomena is similar to that encountered in the impulse response. As stated earlier convergence in mean square implies convergence in probability although the reverse is not true. Thus, it is quite possible for the system to be stable according to Definition 3.3 while have moments which do not converge. As before, a plausible physical explanation for the instability of the moments has not been given in the literature.

3.3 Moment Stability of Higher Order Systems

In this section sufficient conditions for stability of certain higher order random linear systems (described in section 2.3) will be established. It has been shown that for general systems in which the parameter variations are white noise processes the transition probability density is given by a partial differential equation of the Fokker-Planck type. Using this equation it is possible to obtain moment generating functions (equation 2.60). These moment generating functions can, in general, be described by the matrix differential equation

$$\frac{d}{dt}\,\underline{M}_n = \underline{A}_n\,\underline{M}_n + \sum_{i=1}^{n-1} \underline{B}_i\,\underline{M}_i \tag{3.11}$$

where $\underline{M}_n$ is a column matrix with various mixed moments of order n as components, $\underline{A}_n$ is a constant matrix and

$$\sum_{i=1}^{n-1} \underline{B}_i \, \underline{M}_i$$

represents other matrices of lower order moments.

To consider the stability of $\underline{M}_n$ it is sufficient to consider the "homogeneous" equation

$$\frac{d}{dt} \underline{M}_n = \underline{A}_n \underline{M}_n \qquad (3.12)$$

provided the lower order moments are bounded. The necessary and sufficient condition for stability of $\underline{M}_n$ can be stated as a theorem.

<u>Theorem 3.2</u>. If the nth order mixed moments are given by the matrix differential equation

$$\frac{d}{dt} \underline{M}_n = \underline{A}_n \underline{M}_n + \sum_{i=1}^{n-1} \underline{B}_i \, \underline{M}_i \qquad (3.13)$$

where the $\underline{M}_i$ are bounded, then the necessary and sufficient condition for $\underline{M}_n$ to be stable is that the eigenvalues of $\underline{A}_n$ lie in the left half of the complex plane.

The proof for this theorem is straight forward and is available in any standard book on state variables.

<u>3.3.1 Example</u>

In section 2.3 the generating matrix differential equation for the second moments of a second order system has been derived (eq: 2.71). According to theorem 3.2 it is sufficient to consider the homogeneous portion of equation 2.71 for stability of the second moments.

According to the notation of theorem 3.2, we have

$$A_2 = \begin{bmatrix} 0 & 2 & 0 \\ D_{2221}-a_{21} & D_{2222}-a_{22} & 1 \\ 2D_{2121} & 6D_{2122}-2a_{21} & 4D_{2222}-2a_{22} \end{bmatrix}$$

For simplicity we shall assume that α_{21} and α_{22} are not correlated. Therefore, $D_{2122} = D_{2221} = 0$.

The characteristic equation $|\lambda \underline{I} - \underline{A}_2| = 0$ results in the following equation:

$$\lambda^3 + E_2 \lambda^2 + E_1 \lambda + E_0 = 0$$

where,

$$E_2 = 3a_{22} - 5D_{2222}$$

$$E_1 = 4D_{2222}^2 - 6a_{22} D_{2222} + 2a_{22}^2 + 4a_{21}$$

and

$$E_0 = 4a_{21}a_{22} - 8D_{2222}a_{21} - 4D_{2121}$$

The necessary and sufficient conditions for the roots of the characteristic equation to lie in the left half of the complex plane can be obtained from the Routh-Hurwitz criterion. Applying this criterion we obtain the following conditions:

$$E_2 > 0$$

$$E_0 > 0$$

$$E_2E_1 > E_0 .$$

If only one coefficient varies randomly we have the following special cases.

Case 1. $\alpha_{22} = 0 \qquad (D_{2222} = 0)$

The characteristic equation reduces to

$$\lambda^3 + 3a_{22}\lambda^2 + (2a_{22}^2 + 4a_{21})\lambda + (4a_{21}a_{22} - 4D_{2121}) = 0$$

The Routh-Hurwitz criteria yields the following necessary and sufficient conditions for stability:

$$a_{22} > 0$$

$$a_{21}a_{22} > D_{2121}.$$

and

$$6a_{22}(a_{22}^2 + 2a_{21}) > 4(a_{21}a_{22} - D_{2121})$$

Case 2. $\alpha_{21} = 0 \qquad (D_{2121} = 0)$

The characteristic equation reduces to

$$\lambda^3 + F_2\lambda^2 + F_1\lambda + F_0 = 0$$

where

$$F_2 = 3a_{22} - 5D_{2222}$$

$$F_1 = 4D_{2222}^2 - 6a_{22}D_{2222} + 2a_{22}^2 + 4a_{21}$$

$$F_0 = 4a_{21}a_{22} - 8a_{21}D_{2222}$$

For stability the necessary and sufficient conditions are

$$F_2 > 0$$

$$F_0 > 0$$

$$F_2F_1 > F_0.$$

3.4 Remarks

It has been shown in this chapter that the stability of a RLS can be defined precisely in terms of the convergence of the random variable representing the output. It is difficult to use these definitions directly to determine the stability of a RLS since the output cannot be described completely. It is, therefore, necessary to establish sufficiency conditions in terms of a quantity which can be evaluated without major difficulty. In most cases the mean-square of the output can be obtained quite easily and therefore sufficiency conditions in terms of the mean-square would be most useful. It has been shown in theorem 3.1 that the stability of the mean-square always guarantees asymptotic stability. Under certain conditions the mean-square stability also guarantees stability almost everywhere in time (Theorem A.1).

Another phase of the stability problem is the time required for a RLS to settle down to its equilibrium value. This problem has not been investigated theoretically although some experimental work has been carried out (15) to determine conditions for a small settling time.

CHAPTER IV

CONCLUSION

The study of random linear systems is essential for a better understanding of a number of phenomena, as, for example, the propagation through randomly-varying media, stability of radio guided vehicles with randomly-varying loop gain, and reflection from fluctuating targets. It has been pointed out in Chapter II that the investigations of the output of a RLS are often limited by the mathematical complexity of the problem. Fortunately, the complete description of the output is in most cases, unnecessary since one is usually concerned with the overall behavior of the system. In particular, the stability of such systems is of interest. In Chapter III it has been shown that the stability of random linear systems is best described in terms of the convergence of the random variables representing the output. The stability can be conveniently studied in terms of the mean square of the output. In most of the published works on the subject the investigations have been directed towards obtaining the mean-square of the output since this statistic is relatively easy to evaluate. These investigations have been fruitful in specific cases only.

It has been shown in Chapter III that for certain systems, in which the variation of parameters is a white noise process, the transition probability density function is given by a partial differential equation of the Fokker-Planck type. A general solution of this equation is very difficult and is unnecessary if the object is to study the stability of the system. It may be possible, however, to solve this equation numerically. Such a solution would be helpful in the investigation of the stability theory since it would give a firmer foundation to the study of stability in terms of the convergence of distribution concept.

The contents of this report indicate that the analysis of RLS is limited to a large extent by the mathematical complexity of the problem. Although solutions for specific cases can be obtained under certain simplifying assumptions, no general procedure has been formulated at this stage. In particular, the behavior of non-stationary RLS has hardly been studied. The investigations of RLS, at the present level, have been successful only to a limited extent and there is a need for further research, both theoretical and experimental.

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A P P E N D I X

The stability of random linear systems can be conveniently discussed in terms of convergence of random variables. In probability theory this convergence is defined in a number of ways depending on the "strength" of the convergence. In particular, the following concepts are commonly employed.

<u>Convergence in Probability</u> (weak convergence)

A random process $\{H(t)\}$ converges, to a stochastic limit $H(\tau)$, in probability if, for any positive number ϵ,

$$\lim_{t \to \tau} P\left\{ \left| H(t) - H(\tau) \right| > \epsilon \right\} = 0$$

A necessary and sufficient condition for such convergence (Bartlett) is that for any positive ϵ and η there is a t' such that

$$P\left\{ \left| H(t) - H(\tau) \right| > \epsilon \right\} < \eta$$

for all $t, \tau \geqslant t'$.

<u>Convergence in the Mean-Square Sense</u>

A random process $\{H(t)\}$ is said to converge, to a stochastic limit $H(\tau)$, in the mean-square sense if

$$\lim_{t \to \tau} \left\langle \left[H(t) - H(\tau) \right]^2 \right\rangle = 0$$

where $\langle --- \rangle$ represents the ensemble averaging operator.

The necessary and sufficient condition for mean square convergence is (Bartlett) that for any positive ϵ there is a t' such that

$$\left\langle \left[H(t) - H(\tau) \right]^2 \right\rangle \qquad \text{for all } t, \tau \geqslant t'$$

In the previous definitions we have been considering the random process $\{H(t)\}$. We have an ensemble of functions H(t) as shown in Fig. A.1; at time t' the random variable is H(t').

Let us regard each function H(t), for all time t, as a random variable and denote that particular function as $H_k(t)$. Thus H(t) has realized values $H_1(t)$, $H_2(t)$ . . . .

Almost Certain Convergence (Also called strong convergence or convergence with probability one)

H(t) converges almost certainly if the realized sequence $H_1(t)$, $H_2(t)$ . . . converges to H'(t) in probability. That is, for any positive ϵ,

$$\lim_{k \to \infty} P \left\{ \left| H_k(t) - H'(t) \right| > \epsilon \right\} = 0$$

for almost all t.

A sufficient criterion for a·c convergence is (Bartlett)

$$\int_0^\infty \left\langle \left| H(t) - H(\tau) \right|^p \right\rangle dt < \infty$$

for some $p > 0$.

Convergence in Distribution (Doob)

If the sequence $\{F_n\}$ is the sequence of distribution functions of the sequence of random variables $\{H(t_n)\}$, and if

$$\lim_{n \to \infty} F_n(\gamma) = F(\gamma)$$

at every point of continuity of F, then, the sequence of random variables $\{H(t_n)\}$ is said to converge in distribution.

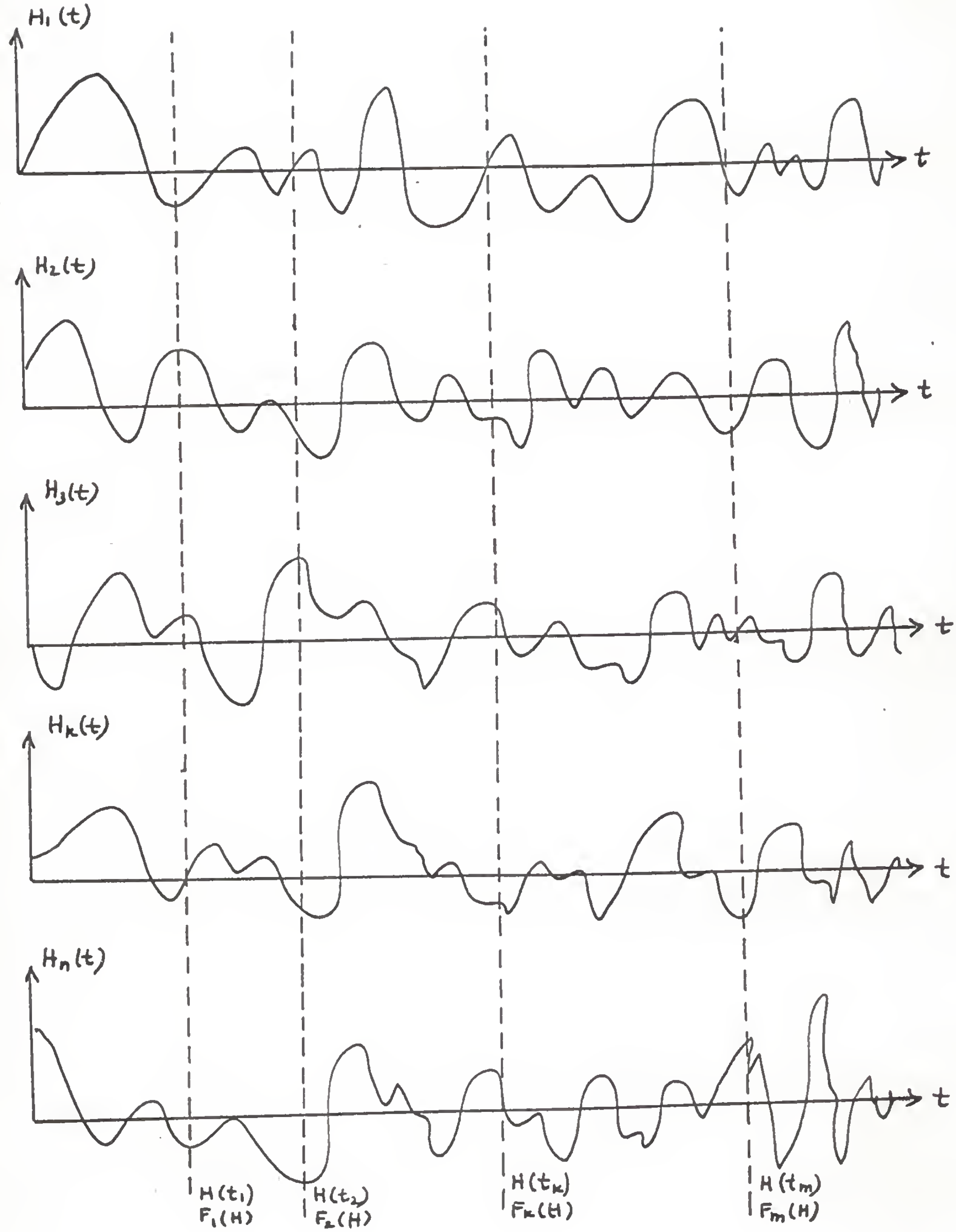

FIGURE A.1

ENSEMBLE OF H(t) : ILLUSTRATION OF CONVERGENCE

It can be shown that almost certain convergence implies convergence in probability although the reverse is not true. It can also be shown that mean square convergence implies convergence in probability. There is, however, no relation in general between mean square convergence and almost certain convergence.

The implications of the three definitions of stability discussed in section 3.1 can be discussed in terms of the definitions of convergences.

Definition 3.1 states that the system reaches equilibrium almost certainly or with probability one and implies that the system is "almost always" stable in time. In other words, the probability of the system going unbounded at any time is of measure zero. This is obviously the most desirable condition.

Definition 3.2, on the other hand, implies that the system is ultimately stable, although it may be unbounded for most of the time. In other words, definition 3.2 is concerned only with the asymptotic behavior of the system. Definition 3.3 is equivalent to definition 3.2.

Under certain conditions, a system which is mean square stable can be stable "almost certainly." A number of such cases have been discussed by Bershad and DeRusso and Bershad. In particular, if $\langle H^2(t)\rangle$ is an exponentially decaying function the system will be "almost certainly" stable. This is stated as a theorem.

<u>Theorem A.1</u>. If H(t) represents the impulse response of a random linear system, and if $\langle H^2(t)\rangle$ is an exponentially decaying

function, then H(t) converges almost certainly to zero (system is stable according to definition 3.1).

Proof. We have stated (without proof) that the sufficient condition for almost certain convergence is that

$$\int_0^\infty \left\langle \left| H(t) - H(\tau) \right|^P \right\rangle dt < \infty$$

for some $P > 0$.

For $\tau \to \infty$ we have $H(\tau) = 0$ as the equilibrium value of H(t). The condition for almost certain convergence (P chosen as 2) becomes

$$\int_0^\infty \left\langle H^2(t) \right\rangle dt < \infty$$

If $\left\langle H^2(t) \right\rangle$ is an exponentially decaying function then the condition is obviously satisfied. Therefore, H(t) converges to zero almost certainly. Therefore the system is stable almost everywhere in time.

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A STUDY OF LINEAR SYSTEMS WITH RANDOMLY-VARYING PARAMETERS

by

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AN ABSTRACT OF A MASTER'S REPORT

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A random linear system (henceforth designated as RLS) is a linear system whose parameter variations are random processes with known probability density functions. If the probability density functions are independent of time the RLS is said to be stationary. The input-output relationship of a RLS may be described by a linear differential equation with stochastic coefficients.

Recent interests in the study of many varied phenomena, such as propagation through randomly-varying media, stability of radio guided vehicles with randomly-varying loop gain, and reflection from fluctuating targets have placed an ever-increasing importance on the methods of analysis of RLS. This report is a survey of the various techniques that have been developed to analyze stationary RLS. On account of the complexity of the problem on the whole the restriction to the stationary case has been found to be essential in all the published papers.

A complete representation of the output of a random linear system requires a knowledge of all the multivariate probability density functions of the output. This problem, however, is so difficult that one has to be satisfied with a limited amount of information in the form of the first probability density functions and the various order moments of the output. Even this information can be obtained in a concrete form for specific cases only. The various techniques developed to analyze these special cases have been discussed in chapter 2.

The limited description of the output of a RLS is often quite sufficient to an engineer since he is primarily concerned with the stability of such systems. In chapter 3 it is shown that sufficient criteria for stability of stationary RLS can be established in terms of the available information at the output. The concept of stability of random linear systems, however, is quite different from the same concept for deterministic systems. Chapter 3 brings out the idea of stability "in the stochastic sense" and attempts to define it in terms of familiar notions of convergence of random variables. It is shown that this permits a very precise definition of stability of random linear systems.

The possibility of further investigation of random linear systems has been discussed in the Conclusion.