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ANALYTICAL AND NUMERICAL SOLUTIONS OF SOME ENTRANCE REGION FLOW PROBLEMS

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Taipei, Formosa, 1958

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

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NOMENCLATURE

a	Channel half-height
A	Cross-sectional area of the tube
B	Magnetic induction, $B = \mu_e H$
C	Friction factor defined in equation (29), Chapter 5.
D	Tube diameter
D_e	Equivalent diameter, 4 x cross-sectional area/perimeter
e	Electric field magnitude factor, $E_0 = e\bar{u}B_0$
E	Electric field strength
F	Body force vector
F_x, F_y, F_z	x, y, z components of body force
F_r, F_ϕ, F_x	r, ϕ, x components of body force
g_x	x component of the acceleration of gravity
H	Magnetic field intensity
K	Constant defined in equation (29), Chapter 5
J	Electric current density
M	Hartmann number, $M = \mu_e H_0 a(\sigma_e / \rho\nu)^{1/2}$
p	Fluid pressure
p_0	Pressure at channel mouth
p_∞	Pressure at far upstream where fluid velocity is zero
Δp	Total pressure drop in the entrance region
P	Dimensionless pressure, $\dfrac{(p - p_0)}{\rho\bar{u}^2}$
R	Dimensionless radial coordinate, $\dfrac{r}{r_0}$

Re_a Reynolds number, $\frac{\rho \bar{u} a}{\mu}$

Re_D Reynolds number, $\frac{\rho \bar{u} D}{\mu}$

Re_s Reynolds number, $\frac{\rho \bar{u} s}{\mu}$

Re_r Reynolds number, $\frac{\rho \bar{u} r_0}{\mu}$

r_0 Tube radius

r, ϕ, x Cylindrical coordinates

s Channel height, $s = 2a$

t Time

u,v,w x, y, z components of the velocity field

$\bar{u}$ Average velocity

u_0 Center line velocity

U Dimensionless velocity in x-direction, $\frac{u}{\bar{u}}$

u_f Velocity at fully developed condition

u* Deviation velocity

v_r, v_ϕ, v_x r, ϕ, x components of the velocity field

$\vec{V}$ Velocity vector

V Dimensionless velocity in y-direction, $\frac{av\rho}{\mu}$

x,y,z Rectangular coordinates

X Dimensionless x-coordinate, $\frac{\mu x}{\rho a^2 \bar{u}} = \frac{(x/a)}{Re_a}$

X' Dimensionless coordinate, $\frac{\mu x}{\rho s^2 \bar{u}} = \frac{X}{4}$

x_e Entrance length

CHAPTER 1. INTRODUCTION

When a viscous fluid enters a conduit (a circular tube or a flat duct) the velocity profile of the flow changes as the flow progresses from the entrance to the interior of the conduit because the wall of the conduit tends to retard the flow. Eventually, the velocity profile will assume a form which is invariant with respect to the direction of flow. The governing equations for the flow in this region of changing velocity profile, the entrance region, are the continuity equation and the equation of motion given below (1, 2).

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho \vec{V}) = 0 \tag{1}$$

and

$$\rho \frac{D\vec{V}}{Dt} = -\text{grad } p - (\vec{\nabla} \cdot \vec{\tau}) + \vec{F}. \tag{2}$$

If the flow is incompressible, the above systems of equations are simplified. When temperature variations are small, the viscosity may be taken to be constant. For Newtonian flow equations (1) and (2) then become

$$\text{div } \vec{V} = 0 \tag{3}$$

and

$$\rho \frac{D\vec{V}}{Dt} = -\text{grad } p + \mu \nabla^2 \vec{V} + \vec{F}. \tag{4}$$

In rectangular coordinates, equations (3) and (4) assume the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x}$$
$$+ \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x \tag{6a}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y}$$
$$+ \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + F_y \tag{6b}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}$$
$$+ \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_z \tag{6c}$$

where u, v, and w are velocity components and F_x, F_y, and F_z are body forces in the direction of x, y, and z, respectively.

In cylindrical coordinates they are

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_x}{\partial x} = 0 \tag{7}$$

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{\partial r} + v_x\frac{\partial v_r}{\partial x}\right) = F_r - \frac{\partial p}{\partial r}$$
$$+ \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial^2 v_r}{\partial x^2}\right) \tag{8a}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + v_x\frac{\partial v_\phi}{\partial x}\right) = F_\phi - \frac{1}{r}\frac{\partial p}{\partial \phi}$$

$$+ \mu\left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r}\frac{v_\phi}{\partial r} - \frac{v_\phi}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} + \frac{\partial^2 v_\phi}{\partial x^2}\right) \qquad (8b)$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_r\frac{\partial v_x}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_x}{\partial \phi} + v_x\frac{\partial v_x}{\partial x}\right) = F_x - \frac{\partial p}{\partial x}$$

$$+ \mu\left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r}\frac{\partial v_x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_x}{\partial \phi^2} + \frac{\partial^2 v_x}{\partial x^2}\right) \qquad (8c)$$

where v_r, v_ϕ, and v_x are velocity components, and F_r, F_ϕ, and F_x are body forces in the direction of r, ϕ, and x, respectively.

For two-dimensional flow under the absence of body forces, equations (5) and (6) become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (9)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (10a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad (10b)$$

in a rectangular coordinate system. (When the body force exists, replace p by $P = p - \Omega$, where $\vec{F} = \nabla\Omega$.)

When the flow is axisymmetrical the corresponding equations in cylindrical coordinates are

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_x}{\partial x} = 0 \qquad (11)$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + v_x\frac{\partial v_r}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$+ \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial x^2}\right) \qquad (12a)$$

$$\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + v_x \frac{\partial v_x}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$+ \nu \left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} + \frac{\partial^2 v_x}{\partial x^2}\right) \quad . \tag{12b}$$

It is often considered that the change of the velocity profile is caused by the formation of a boundary layer whose thickness increases to the radius of the tube toward the downstream direction. Then the following Prandtl's boundary layer assumption can be used to simplify the equation of motion.

(1) $\dfrac{\partial^2 u}{\partial x^2}$ $\left(\text{or } \dfrac{\partial^2 v_x}{\partial x^2}\right)$ is very small compared with $\dfrac{\partial^2 u}{\partial y^2}$ $\left(\text{or } \dfrac{\partial^2 v_r}{\partial r^2}\right)$ so it can be neglected.

(2) The transverse velocity v (or v_r) is small in comparison with u (or v_x).

(3) Consequently the pressure gradient $\dfrac{\partial p}{\partial x}$ is a function of x alone.

Therefore equations (10) and (12) are reduced to the following boundary layer equations. In rectangular coordinates

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{13}$$

and in cylindrical coordinates

$$\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + v_x \frac{\partial v_x}{\partial x} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r}\right) . \tag{14}$$

If the flow is steady, the terms $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial v_x}{\partial t}$ are zero and the governing equations in the rectangular coordinate systems are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{15}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad . \tag{16}$$

The corresponding equetions in the cylindricel coordinate system are

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_x}{\partial x} = 0 \tag{17}$$

$$v_r \frac{\partial v_x}{\partial r} + v_x \frac{\partial v_x}{\partial x} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r}\right) \quad . \tag{18}$$

A study of the velocity field and friction factor in the entrance region of tubes and ducts is of practical importance and has been a subject of investigation for meny years.

In this entrance region we wish to obtein expressions for (1) the pressure drop between any two sections, (2) the velocity distribution et eny section, and (3) the value of x at which the fully developed flow is ettained.

The two simultaneous equations, equations (15) end (16) or equations (17) and (18) with appropriete boundery conditions, ere sufficient to solve the two unknowns, u and v (or v_x and v_r). The pressure ceased to be an unknown function since it can now be evaluated from the potential flow in the central core by the aid of the Bernoulli equation.

Although equations (15) and (16) or equetions (17) and (18) are much simpler than the original equations representing governing equetions, their exact analytical solutions have not been found so far. However, various approximetion methods of solution have been employed to solve the entrance region problem. The approximate solutions may be classified in four general categories--the momentum integral method, linearization method, matching method, and finite difference method.

In the momentum integral method the flow is divided into a boundary layer part near the wall and a potential flow part in the central core. A parabolic velocity profile (or any other similar velocity profile) is assumed in the boundary layer and is joined with the center core velocity profile which is assumed to be a straight line. A momentum integral equation is derived based on the momentum conservation principle. This approach was devised and applied by Schiller (3) for flow in a circular tube and is similar to the Karman-Pohlhausen momentum integral method applied to a flat plate.

In the second category of solutions, the inertia terms of boundary layer equations are linearized. This category of solutions is capable of providing continuous solutions for the velocity of distribution and pressure drop in the entrance region. This class of solutions for circular tubes is mainly due to the work of Langhaar (4) and Targ (5). Langhaar's approach was extended by Han (6) to obtain solutions for rectangular ducts, and Targ's approach was used by Lundgren and Sparrow _et al_ (7, 8) to obtain solutions for channels of parallel plates and for ducts of arbitrary cross section.

The third group of solutions is constructed by matching together the boundary layer solutions which are valid near the entrance and the perturbations of the fully developed solutions which are valid for downstream. This class of solutions was originally given by Schlichting (1, 9) for a parallel-plate channel and was also applied to circular tubes by Atkinson-Goldstein (10).

The fourth approach involves reduction of the continuity and momentum equations to finite difference equations which are solved numerically on an electronic digital computer. This method has been applied by Bodoia and Osterle (11), Hwang and Fan (12), and Wang and Longwell (13) for flow in a parallel-plate channel. However, it should be indicated that the solutions in References (11, 12) were obtained by solving the boundary layer equation, equations (15) and (16), while the results in Reference (13) were obtained from the solution of the continuity equation and the Navier-Stokes equation, equations (9), (10a), and (10b), in which the time dependent terms are omitted for the steady-state condition.

The purpose of this report is to present in detail the solutions of the various methods discussed above. Moreover, the finite difference method is applied to solve the entrance region problem of MHD flow in a parallel-plate channel with a non-magnetically fully developed velocity profile at the entry. The results are then compared with those obtained by means of the momentum integral method (14).

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CHAPTER 2. SCHILLER'S METHOD

Schiller's method is also known as Karman-Pohlhausen momentum-integral method. When a viscous fluid enters a circular tube with a uniform velocity profile across the cross section, a thin boundary layer is often considered to have been formed. This is mentioned in the previous chapter. The thickness of this layer increases with the distance from the entry until it is equal to the radius of the pipe. Outside the boundary layer, the flow is considered to be frictionless and accelerated. In this method the velocity profile is assumed to be a curve build-up by two parabolas and a straight line (see Fig. 1); the vertices of the parabolas lie on the border of the boundary layer. The parabolic velocity in the boundary layer is assumed to be

$$\frac{u}{u_0} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad (1)$$

where $y = r_0 - r$

and u_0 is the velocity of the central core and is a function of x; δ is the thickness of the boundary layer.

The equation of continuity and the equation of motion (the momentum equation) give two relations among u_0, x, and δ. Elimination of δ will lead to a relation between u_0 and x.

The momentum theory is used here for the x direction to obtain a momentum integral equation representing the equation of motion. The control surface is formed by two planes a-b and c-d, spaced apart from each other by the small distance dx and

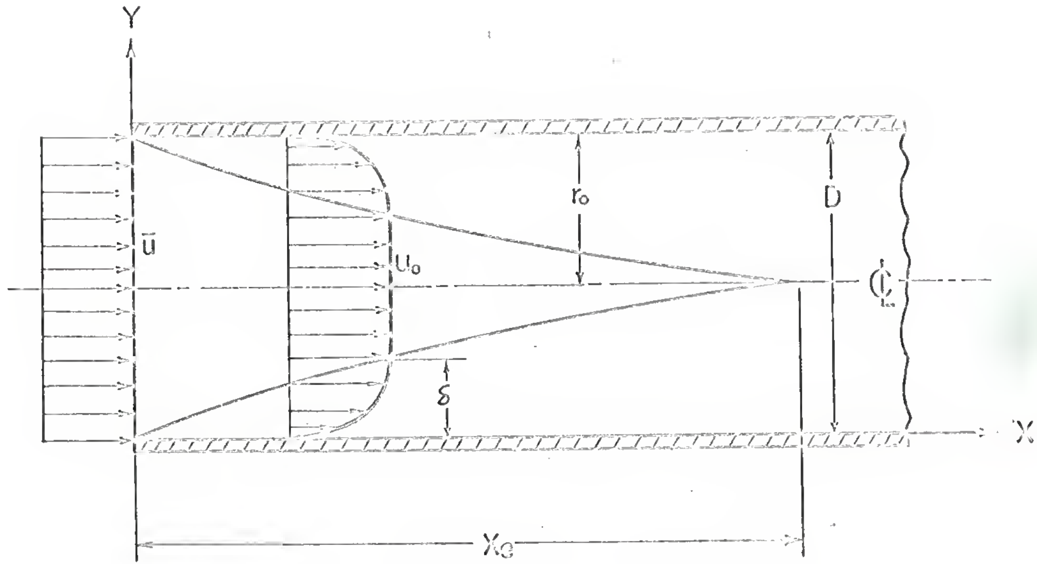

Fig. 1. Velocity profile in a entrance region of a circular tube.

by the wall (see Fig. 2). By means of the momentum balance on this control surface the following momentum integral equation is obtained (see Appendix I for detail).

$$\frac{d}{dx}\left(\int_0^{\delta} u^2(r_0 - y)\,dy\right) - u_0 \frac{d}{dx}\left(\int_0^{\delta} u(r_0 - y)\,dy\right)$$

$$= -2\nu u_0\left(\frac{r_0}{\delta}\right) + \frac{1}{2} r_0^2 \left(2\frac{\delta}{r_0} - \left(\frac{\delta}{r_0}\right)^2\right) u_0 \frac{du_0}{dx}. \tag{2}$$

Substituting the assumed velocity profile expressed by equation (1) into the integral form of continuity equation and carrying out the integration, we obtain a relation between the boundary layer thickness, δ, and the center line velocity, u_0, as shown below:

$$\frac{\delta}{r_0} = 2 - \sqrt{4 - 6\left(1 - \frac{\bar{u}}{u_0}\right)} \tag{3}$$

where $\bar{u}$ is the average velocity.

Substituting equations (1) and (3) into equation (2), and carrying out the integration, we obtain a relation between u_0 and x as follows:

$$\frac{x/D}{Re_D} = \frac{1}{16}\left[\frac{58}{15}\eta - \frac{22}{5}\ln(1+\eta) - \frac{17}{15}\sqrt{4 + 2\eta - 2\eta^2}\right.$$

$$- \frac{16}{5}\sqrt{\frac{4 - 2\eta}{1+\eta}} - \frac{37\sqrt{2}}{10}\sin^{-1}\left(\frac{2\eta - 1}{3}\right)$$

$$\left. - \frac{37\sqrt{2}}{10}\sin^{-1}\left(\frac{1}{3}\right) + \frac{26}{3}\right] \tag{4}$$

where

$$\eta = \frac{u_0}{\bar{u}} - 1 .$$

Therefore the velocity profile in the entrance region is defined

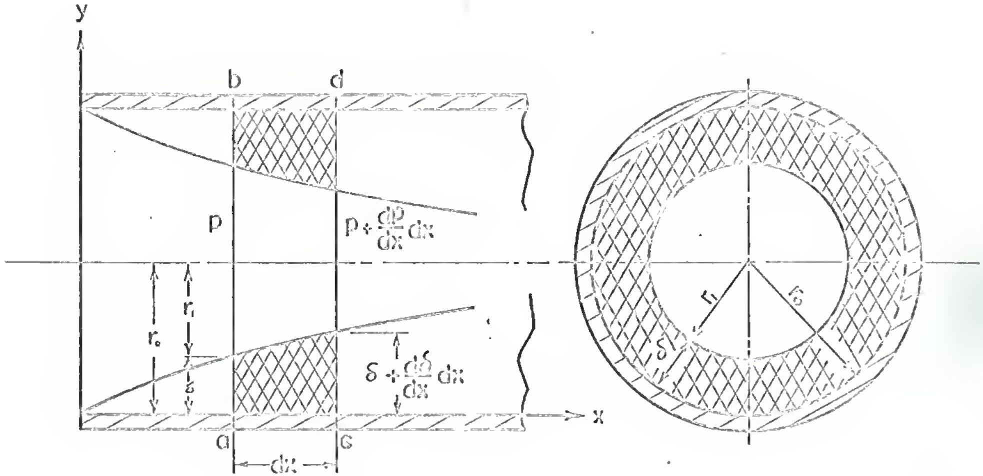

Fig. 2 Region for deriving momentum integral equation.

by equations (1), (3), and (4). Given a value of u_0, the values of x and δ are known from equations (4) and (3). The velocity distribution at the corresponding section is defined by equation (1). The flow is fully developed when $\delta = r_0$ and $u_0 = 2\bar{u}$. Hence $\eta = 1$. Then from equation (4), the entrance length is found to be

$$x_e = 0.0288\ DRe_D.$$

Differentiating both sides of equation (4) with respect to x, we see that $\frac{d\eta}{dx}$, that is $\frac{du_0}{dx}$, does not vanish at $\eta = 1$. This means that the center line velocity does not asymptotically approach its final value, $2\bar{u}$, at the fully developed section.

For the calculation of pressure drop in the entrance region, the Bernoulli equation is applied since it is assumed that a potential flow in the central core exists throughout the entrance region. Therefore the pressure distribution is assumed to be dependent only on the potential flow. The pressure drop between any two locations in this region is found as

$$p_1 - p_2 = \frac{\rho}{2}\left[(u_0)_2{}^2 - (u_0)_1{}^2\right] = \frac{\rho}{2}\,\bar{u}^2\left[(\eta_2{}^2 - \eta_1{}^2) + 2(\eta_2 - \eta_1)\right].$$

Since $x = 0$ and $\eta_0 = 0$ at the entry, and $x = x_e$ and $\eta = 1$ at the fully developed section, the total pressure drop in the entrance region is

$$\Delta p = \frac{3}{2}\,\rho\,\bar{u}^2 .$$

The pressure drop in the entrance region is larger than that in the fully developed region of equal length because of the increased friction loss and the fact that the kinetic energy of the fluid increases as it passes downstream. For the evaluation of total pressure drop between a section in the entrance region and a section in the fully developed region, it is more convenient to treat the entrance region as a fully developed one and the pressure drop thus calculated is modified by a correction factor. The equation for calculating the total pressure drop in the fully developed section is found to be

$$\frac{p_0 - p}{\frac{\rho \bar{u}^2}{2}} = \frac{64}{Re_D D} x + 1.16$$

where 1.16 is the correction factor. If the fluid enters the pipe from a region where the pressure is p_∞ and the velocity is negligible, there is a pressure drop of $\frac{1}{2}\rho \bar{u}^2$, and then the total pressure drop is

$$\frac{p_\infty - p}{\frac{\rho \bar{u}^2}{2}} = \frac{64x}{Re_D D} + 2.16 .$$

The derivation and solution are given in detail in Appendix I.

CHAPTER 3. LANGHAAR'S LINEARIZING APPROXIMATE METHOD

The difficulty in obtaining an exact analytical solution of the boundary layer equations stems from the fact that the convective terms are nonlinear. Langhaar (1) introduced an assumption that the convective terms in the boundary layer equation for the axial direction, x, (see Fig. 1) are equal to the product of the kinematic viscosity ν, the axial velocity u, and a parameter β. This parameter is considered to be a function of x only. Langhaar's assumption can be written as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nu \beta^2 u \tag{1}$$

and gives rise to the following linearized form of the boundary layer equation

$$\nu \beta^2 u = g_x - \frac{1}{\rho} \frac{dp}{dx} + \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2}$$

where g_x is the x component of the acceleration of gravity.

In order to solve the problem for a flow in a circular tube, equation (2) is transformed into an expression with cylindrical coordinates as

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \beta^2 u = \alpha \tag{3}$$

where

$$\alpha = - \frac{1}{\nu} \left(g_x - \frac{1}{\rho} \frac{dp}{dx}\right)$$

which is a function of x alone.

Equation (3) is regarded as a velocity function generating equation. The solution of equation (3) is expressed in terms of a modified Bessel function as

$$\frac{u}{\bar{u}} = \frac{I_0(\gamma) - I_0(\gamma \frac{r}{r_0})}{I_2(\gamma)} \tag{4}$$

where $\gamma = \beta r_0$, r_0 is the radius of the tube, and $\bar{u}$ is the average velocity.

Therefore, if the relation between γ and x is known, equation (4) will define the velocity profiles everywhere in the entrance region. It is worth noting that when $\beta = 0$, (that is, $\gamma = 0$), the acceleration terms are zero. The solution then assumes the form of a fully developed parabolic profile. When $\beta \rightarrow \infty$, (that is, $\gamma \rightarrow \infty$), the solution assumes a uniform profile at the entry (see Appendix II for details). Thus the solution provides a smooth transition from the entry where $\gamma \rightarrow \infty$ to the fully developed section where $\gamma = 0$.

For determining function $\gamma(x)$, the momentum integral equation is used. Integrating the boundary layer equation in cylindrical coordinates over the cross section, we obtain

$$2 \frac{d}{dx} \int_0^{r_0} u^2 r dr = g_x r_0^2 - \frac{1}{\rho} \frac{dp}{dx} r_0^2 + 2 \nu \int_0^{r_0} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}\right) r dr. \tag{5}$$

The term $g_x r_0^2 - \frac{1}{\rho} \frac{dp}{dx} r_0$ can be eliminated by applying the Navier-Stokes equation in the center line of the tube. After the substitution of equation (4) and simplification, the following relation is obtained.

$$X = \int_{Z}^{1/2} Y dZ \tag{6}$$

where

$$X = \frac{x}{r_0 \ Re_r} ,$$

$$Y = \frac{I_2(\gamma)}{2\gamma I_1(\gamma) - \gamma^2} ,$$

and

$$Z = \frac{4I_0(\gamma)I_2(\gamma) - \left[I_0(\gamma) - 1\right]^2 - 2I_1^2(\gamma)}{2I_2^2(\gamma)} .$$

Equation (6) provides a relation between x and γ. The integration can be carried out numerically; the results are presented in Table 1 of Reference (1).

The entrance length is defined as the distance from the entry to a section where the center line velocity is 99 per cent of the fully developed one. This definition differs from that given by Schiller. When the value of 99 per cent is reached, the ratio of the center line velocity to the average velocity is 1.98. From equation (4),

$$\frac{u_0}{\bar{u}} = \frac{I_0(\gamma) - 1}{I_2(\gamma)} .$$

Hence, when $\frac{u_0}{\bar{u}} = 1.98$, the corresponding value of γ (and therefore the value of x) can be determined. The entrance length is found to be

$$x_e = 0.0568 \ D \ Re_D .$$

In calculating the pressure drop in the entrance region, Langhaar applied the mechanical energy balance instead of using the Bernoulli equation because in the latter part of the entrance length, the Bernoulli equation ceases to be applicable. The energy loss due to viscous dissipation was considered. Equating the work of pressure force and frictional force to the change in kinetic energy gives the energy equation

$$\bar{u}A(p_0 - p) = \mu \int_0^x dx \int_A \left(\frac{\partial u}{\partial r}\right)^2 dA + \frac{1}{2}\rho \int_A (u^3 - \bar{u}^3)\, dA. \quad (7)$$

Substituting equation (4) into equation (7) and simplifying, gives the total pressure drop between a supply reservoir and the section x;

$$\frac{p_\infty - p}{\frac{1}{2}\rho \bar{u}^2} = 1 + 2\int_0^X \frac{\gamma^2}{I_2^{\,2}} (I_1^{\,2} - I_0 I_2)\, dX + 2\int_0^1 (U^3 - 1) R\, dR \quad (8)$$

where

$$U = \frac{u}{\bar{u}} \quad \text{and} \quad R = \frac{r}{r_0}.$$

The integrations in equation (8) can be carried out numerically and the results are presented in Table 2 of Reference (1). Hence the pressure drops in the entrance region are defined.

To calculate the pressure drop between the entrance of the tube and a location in the fully developed region, the entrance region is treated as if it were a fully developed one and the pressure drop thus calculated is modified by a correction factor as follows:

$$\frac{p_\infty - p}{\frac{1}{2}\rho \bar{u}^2} = \frac{64\ x}{DRe_D} + 2.28$$

where 2.28 is the correction factor.

The solution in detail is given in Appendix II.

Reference

1. H. L. Langhaar, "Steady Flow in the Transition Length of a Straight Tube," Trans. of ASME, J. of Applied Mechanics, 64, A-55, (1942); also see Ph.D. Thesis by H. L. Langhaar, Lehigh University, (1940).

CHAPTER 4. SCHLICHTING'S MATCHING METHOD

Let us consider the flow in a parallel plate channel with a uniform velocity distribution at the entry (Reference 1). It is assumed that the flow is incompressible and two-dimensional. The governing boundary layer equation and continuity equation are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \; . \tag{2}$$

In this method the flow field is divided into two sections, the upstream and the downstream. In the upstream section, the boundary layers develop in a similar way to the flow over a flat plate with a pressure gradient in the direction of flow. The stream function and center line velocity are expanded in power series, as in the case of the flat plate. The solution is found after the coefficients of the series are determined by substituting the assumed series into equation (1). For the downstream section, the velocity is assumed to be the sum of the fully developed parabolic velocity distribution and a deviation velocity which approaches zero as $x \longrightarrow \infty$. This deviation velocity is obtained by a series expansion in the upstream direction. Having obtained both solutions in the form of a series expansion, they are joined at a point where both solutions are valid. In this way an approximate description of the flow field in the entire entrance region is obtained.

For the solution of the upstream section, the center line velocity is assumed to be in the following form

$$u_0(x) = \bar{u}(1 + K_1 \epsilon + K_2 \epsilon^2 + \ldots) \qquad (3)$$

where $\epsilon = \sqrt{\dfrac{\nu x}{a^2 \bar{u}}}$; and K_1, K_2 . . . are constants to be determined.

Substituting equation (3) into the pressure gradient at the center line in the flow direction, we obtain

$$-\frac{1}{\rho}\frac{dp}{dx} = u_0 \frac{du_0}{dx} = \frac{\bar{u}^2\epsilon}{2x}\Big[K_1 + (K_1^2 + 2K_2)\epsilon + 3(K_1K_2 + K_3)\epsilon^2 + \ldots\Big] \qquad (4)$$

It is also assumed that the stream function, $\Psi(x, y')$, can be expanded into a series in ϵ. This series is similar in form to that of u_0 but with coefficients which are functions of x and y' as shown below.

$$\Psi(x, y') = \bar{u}a\left[\epsilon f_0(\eta) + \epsilon^2 f_1(\eta) + \ldots\right]$$

where $\eta = \sqrt{\dfrac{\bar{u}}{\nu x}}\; y'$.

Then

$$u(x, y') = \frac{\partial \Psi}{\partial y'} = \bar{u}\left[f_0'(\eta) + \epsilon f_1'(\eta) + \ldots\right], \qquad (5)$$

$$v(x, y') = -\frac{\partial \Psi}{\partial x} = -\frac{\bar{u}a}{2x}\left[\epsilon f_0 + 2\epsilon^2 f_1 + 3\epsilon^2 f_2 + \ldots\right] + \frac{\bar{u}a\,\epsilon\eta}{2x}\left[f_0' + \epsilon f_1' + \epsilon^2 f_2' + \ldots\right]. \qquad (6)$$

Inserting equations (4), (5), and (6) into equation (1) and collecting terms in powers of ϵ, we obtain differential equations for f_0, f_1, ... The equation of the first approximation (for the zero power of ϵ) is

$$f_0 f_0'' + 2\, f_0''' = 0 \; . \tag{7}$$

This is identical to the differential equation of flow over a flat plate given and solved by Blasius. The second approximation (for the first power of ϵ) is determined by the equation

$$2f_1''' + f_0 f_1'' - f_0' f_1' + 2\, f_0'' f_1 = -K_1 \tag{8}$$

which is a linear differential equation of third order. The boundary conditions are $f_1 = f_1' = 0$ at $\eta = 0$ and $f_1' = K_1$ at $\eta = \infty$.

These differential equations can be solved numerically. With the values of f_0, f_1, . . ., and K_1, K_2, . . . being found, the solution of this upstream section is complete.

For the solution of the downstream section, we assume

$$u(x, y) = u_f(y) - u^*(x, y) \tag{9}$$

where

$$u_f(y) = \frac{3}{2}\, \bar{u}\left(1 - \frac{y^2}{a^2}\right)$$

and u^* is the deviation velocity, which is assumed to be in the form

$$\frac{u^*}{\bar{u}} = C_1 e^{-\lambda_1 \epsilon^2}\, \psi_1' \left(\frac{y}{a}\right) \; . \tag{10}$$

Substituting equations (9) and (10) into equation (1) yields the following differential equation for ψ_1:

$$\psi_1^{iv} + 3\lambda_1\left[\frac{1}{2}\left(1 - \frac{y^2}{a^2}\right)\psi_1'' + \psi_1\right] = 0 \tag{11}$$

with λ_1 as its eigenvalue. The boundary conditions are

$$\psi_1 = \psi_1'' = 0 \qquad \text{at} \qquad y = 0$$

$$\psi_1 = \psi_1' = 0 \qquad \text{at} \qquad y = \pm a$$

$$\psi_1' = 1 \qquad \text{at} \qquad y = 0 .$$

The last boundary condition is chosen arbitrarily. This is possible since C_1 is still free. The solution for ψ_1 in equation (11) is obtained by the power series method. Assume

$$\psi_1\left(\frac{y}{a}\right) = \sum_{n=0}^{\infty} A_n \left(\frac{y}{a}\right)^n . \tag{12}$$

By substituting equations (12) into (11), the recursion formula for A_n is obtained.

$$A_{n+4} = \frac{3\lambda_1\left[(n-2)(n+1)A_n - (n-1)(n+2)A_{n+2}\right]}{2(n+1)(n+2)(n+3)(n+4)}$$

The values of A_n and λ_1 can be obtained from this recursion formula together with boundary conditions. Thus the deviation velocity is defined if C_1 is known, which can be determined in the process of matching both solutions.

The pressure drop in the upstream section is obtained from Euler's equation

$$-\frac{1}{\rho}\frac{dp}{dx} = u_0 \frac{du_0}{dx} . \tag{13}$$

Since the boundary layer thickness is very thin in this section and the flow is mostly irrotational in the central core. Integrating equation (13) gives the pressure drop between any two locations in the upstream section. On the other hand, the pressure drop in the downstream section involves the perturbation velocity, $u^*(x, y)$. Substituting the assumed velocity distribution, equation (9), into the boundary layer equation (1)and noting that the pressure varies with x only, we obtain

$$\frac{1}{\rho}\frac{dp}{dx} = \nu\left[-\frac{3\bar{u}}{a^2} - \frac{\bar{u}C_1}{a^2} e^{-\lambda_1\epsilon^2}\psi_1'''(1)\right]. \tag{14}$$

Integrating equation (14) from the joining section, $\epsilon*$ to ϵ gives the pressure drop in the downstream section

$$\frac{p - p*}{\frac{1}{2}\rho\bar{u}^2} = -6(\epsilon^2 - \epsilon*^2) + \frac{2C_1}{\lambda_1}\psi_1'''(1)(e^{-\lambda_1\epsilon^2} - e^{-\lambda_1\epsilon*^2}).$$

For the complete determination of the velocity distributions as well as the pressure drops in the entire entrance region, only C_1 and $\epsilon*$, the location of the matching section, remain to be determined. This is done when the upstream and downstream solutions are matched. The conditions that the center line velocities of both solutions be equal and that the pressure gradient $\frac{dp}{dx}$ be identical at the matching section provide two simultaneous equations for C_1 and $\epsilon*$ as follows:

$$1 + K_1\epsilon + K_2\epsilon^2 + K_3\epsilon^3 + K_4\epsilon^4 = \frac{3}{2} - C_1 e^{-\lambda_1\epsilon^2}, \tag{15}$$

$$\frac{1}{2\epsilon}(1 + K_1\epsilon + K_2\epsilon^2 + K_3\epsilon^3 + K_4\epsilon^4)(K_1 + 2K_2\epsilon + 3K_3\epsilon^2 + 4K_4\epsilon^3)$$
$$= 3 + C_1 e^{-\lambda_1\epsilon^2}\psi_1'''(1). \tag{16}$$

Therefore C_1 and $\epsilon*$ are determined from equations (15) and (16). The solution is then complete.

The numerical values of C_1 and λ_1 given by Schlichting are 0.3485 and 18.75, respectively. Values of K_1, K_2, ..., ψ_1, ψ_1'', ..., and f_0, f_1, ..., are not available. The values calculated by others can be found in References (2, 3).

The entrance length given by Schlichting is

$$x_e = 0.010\ Re_D\ D_e$$

where De is the equivalent diameter.

The solution is given in detail in Appendix III.

References

1. H. Schlichting, Boundary Layer Theory, McGraw-Hill, New York, 1960, p. 169, or H. Schlichting, "Laminare Kanaleinlaufströmung," ZAMM 14, pp. 368, (1934).

2. M. Roidt and R. D. Cess, "An Approximate Analysis of Laminar Magnetohydrodynamic Flow in the Entrance Region of a Flat Duct," J. of Applied Mechanics, March, 1962, pp. 171-176.

3. E. J. Barsness, "Magnetohydrodynamic Effect upon Laminar Flow in the Entrance Region of Perfectly Conducting Parallel Plates," M.S. Thesis, University of Pittsburgh, 1960.

CHAPTER 5. A FINITE DIFFERENCE ANALYSIS OF MAGNETOHYDRODYNAMICS CHANNEL ENTRANCE FLOW WITH PARABOLIC VELOCITY AT THE ENTRY

Summary

The entrance region problem of magnetohydrodynamics flow with nonmagnetically fully developed (parabolic) velocity at the entry is investigated. The flow is in a horizontal channel of parallel and electrically nonconducting plates on which a uniform magnetic field is imposed vertically. The basic equations are transformed to finite difference equations and solved by a digital computer. The variations of velocity profiles, friction factors, and pressure drops for Hartmann numbers of 2.5, 4, 10, and 50 are presented. Values of the entrance length are compared with the only available results by Maciulaitis and Loeffler, who solved the problem by momentum-integral method. The two sets of results appear to be substantially different.

Introduction

This chapter presents a study of magnetohydrodynamic (MHD) flow in the entrance region of a parallel-plate channel. The velocity profile at the entry of a MHD channel is assumed to be a fully developed non-MHD flow (parabolic profile). The velocity profile eventually develops to be a Hartmann velocity profile in the entrance region of a parallel-plste channel where a uniformly distributed magnetic field is applied to the channel. The entrance length of such a MHD channel, the development of the velocity profile, and the pressure drop are studied in this work.

Although the analysis of MHD entrance region problem has been made by many investigators, most of them (1, 2, 3, 4) studied the case of developing velocity from a uniformly distributed one at the entry to Hartmann velocity. There is only one paper (5) published studying the case of development of velocity from parabolic to Hartmann velocity. In this paper the Karman-Pohlhausen integral method with the assumption of Schiller's velocity profile was used. Since there is no other literature available dealing with this specific case, the results could not be compared, although the results obtained for the case of uniform to Hartmann velocity by using the same integral method have agreed well with others (5).

A practical importance in this problem is the length of the entrance region. The length of the entrance region is defined in Reference (5) as the distance required for the friction factor to come within 10 per cent of the fully developed value. Another usual definition of the length of entrance region is the length required for the center line velocity to reach 99 per cent of the fully developed one. The results obtained in the present work are quite different from each other using the different definitions. Also the result in the present work is quite different from that of Reference (5) using the friction factor definition. The entrance length defined according to the comparison with center line velocity is not available in Reference (5).

A finite difference analysis method (2, 3) which yields good results for the developing velocity from uniform to Hartmann velocity is used in the present work.

Basic Equation

Let us consider an electrically conducting fluid entering a horizontal channel of parallel plates which are electrically nonconducting. The velocity profile has been assumed to be nonmagnetically fully developed (parabolic profile) at the entry of the duct (that is, $x = 0$), after which a uniformly distributed magnetic field is applied perpendicularly to the plates. (See Fig. 3.) The following assumptions are made for simplicity: (a) the flow is laminar and steady; (b) all fluid properties (ρ, C_p, k, ν) are constant; (c) permeability, μ_e, and electrical conductivity, σ_e, are constant scalar quantities; (d) rapid oscillations do not exist,and therefore the displacement current is negligible; (e) the effect of gravitational force is negligible; (f) the usual Prandtl boundary layer assumptions apply; (g) variation in the z-direction is negligible, i.e., the flow is two-dimensional; (h) the electric field term measured across the insulated channel walls is zero; and (i) the induced field in x-direction, B_x, is negligible in comparison with the applied field, B_0.

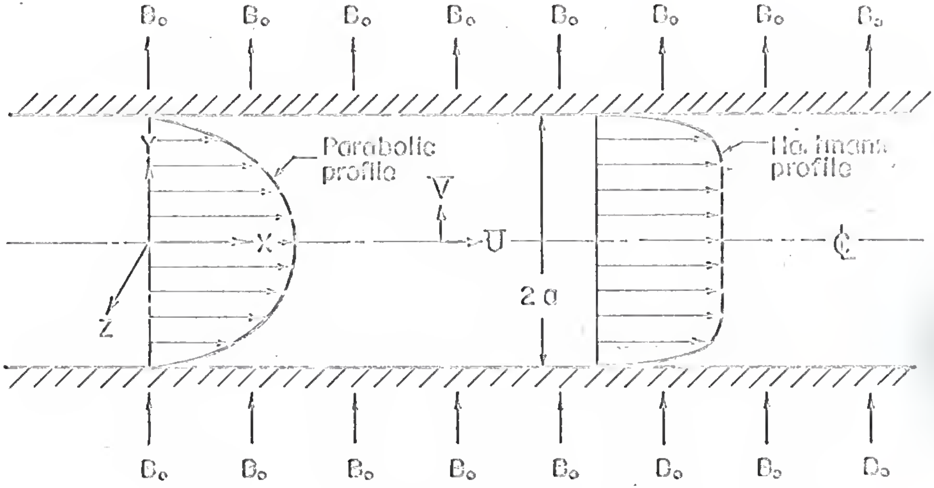

Fig.3. Geometry of parallel plate channel at entrance region with transverse magnetic field and parabolic entrance velocity.

The governing basic equations may be written as follows (1). Maxwell's equations in mks units are

$$\text{curl } \vec{H} = \vec{J}, \qquad \text{div } \vec{J} = 0 \tag{1}$$

$$\text{curl } \vec{E} = -\mu_e \frac{\partial \vec{H}}{\partial t}, \qquad \text{div } \vec{H} = 0 \tag{2}$$

Ohm's law for moving fluids is

$$\vec{J} = \sigma_e \ (\vec{E} + \vec{V} \times \mu_e \vec{H}) \tag{3}$$

where $\vec{V}$ is total velocity in vector form.

The continuity equation is

$$\text{div } \vec{V} = 0 \ . \tag{4}$$

The modified Navier-Stokes equation is

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \text{grad})\vec{V} = -\frac{1}{\rho} \text{grad } p + \nu \nabla^2 \vec{V} + \frac{1}{\rho} (\vec{J} \times \mu_e \vec{H}) . \tag{5}$$

Under the assumptions made above, the basic equations (1) to (5) can be simplified to the following two equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma_e B_0}{\rho} (E_0 + uB_0) \tag{7}$$

In the potential flow, equation (7) may be written

$$u_0 \frac{du_0}{dx} = -\frac{1}{\rho} \frac{dp}{dx} - \frac{\sigma_e B_0{}^2}{\rho} u_0 - \frac{\sigma_e B_0 E_0}{\rho}$$

or

$$-\frac{1}{\rho} \frac{dp}{dx} - \frac{\sigma_e B_0 E_0}{\rho} = u_0 \frac{du_0}{dx} + \frac{\sigma_e B_0{}^2}{\rho} u_0 \tag{8}$$

where u_0 is the center line velocity.

Substituting equation (8) into (7), we have

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_0 \frac{du_0}{dx} + \frac{\sigma_e B_0{}^2}{\rho} (u_0 - u) + \nu \frac{\partial^2 u}{\partial y^2} \quad . \qquad (9)$$

It may be noted that the electric field, E_0, does not appear in equation (9) and thus has no effect on the velocity profile. However, from equation (8), it does have an influence on the pressure gradient required to maintain the velocity distribution in equation (9).

The maximum value for E_0 is obtained by assuming that the side plates are open-circuited. This permits maximum build-up of the electric field and is equivalent to no net current in the z-direction, or

$$\int_{-a}^{a} J_z dy = 0 \ . \qquad (10)$$

Substituting $J_z = \sigma_e(E_0 + uB_0)$ into equation (10), one obtains

$$\int_{-a}^{a} \sigma_e(E_0 + uB_0) dy = \sigma_e E_0 2a + \sigma_e B_0 \int_{-a}^{a} u dy = 0 \ . \qquad (11)$$

Since the continuity equation can be written as

$$\int_{-a}^{a} u dy = 2\bar{u}a \qquad (12)$$

substituting equation (12) into (11) results in

$$E_{0(max)} = -\bar{u}\, B_0 \ .$$

Introducing the electric field factor, e, which varies between zero and one with the external resistance varying from zero to infinity, E_0 may be written as

$$E_0 = -e\bar{u}\, B_0 \ . \qquad (13)$$

Combining equations (13) and (7) gives

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{\sigma_e B_0{}^2}{\rho} (e\overline{u} - u) \ . \tag{14}$$

Introducing the following dimensionless parameters

$$X = \frac{\mu x}{\rho a^2 \overline{u}} = \frac{x/a}{Re_a}$$

$$Y = y/a$$

$$U = u/\overline{u}$$

$$V = \frac{a v \rho}{\mu}$$

$$P = \frac{p - p_0}{\rho \overline{u}^2}$$

$$M = \mu_e H_0 a \sqrt{\frac{\sigma_e}{\rho \nu}} \text{, Hartmann number,}$$

equations (6), (12), and (14) become

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{15}$$

$$\int_0^1 U dY = 1 \tag{16}$$

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = - \frac{dP}{dX} + \frac{\partial^2 U}{\partial Y^2} + M^2 (e - U) \ . \tag{17}$$

The boundary conditions on the flow field are:

at $X = 0$, for $-1 < Y < 1$:

$$U = 1.5(1 - Y^2), \quad V = 0, \quad P = P_0 = 0$$

(18)

for $X \geq 0$ and $Y = 0$:

$$\frac{\partial U}{\partial Y} = 0, \qquad V = 0$$

for $X \geq 0$ and $Y = \pm 1$:

$U = 0, \quad V = 0$.

The first boundary condition shows that the velocity profile is parabolic at the entry (X = 0) after which a uniformly distributed magnetic field is applied.

Solution of the Equation

The method applied follows closely that of Hwang and Fan (3). Considering the mesh network of Fig. 4, a grid is imposed on the flow field. Equation (17) is then written in implicit finite difference form at any point (j, k) as follows:

$$U_{j,k} \frac{U_{j+1,k} - U_{j,k}}{\Delta X} + V_{j,k} \frac{U_{j,k+1} - U_{j,k-1}}{2\Delta Y}$$

$$= - \frac{P_{j+1} - P_j}{\Delta X} + \frac{1}{2}\left[\frac{U_{j+1,k+1} - 2U_{j+1,k} + U_{j+1,k-1}}{(\Delta Y)^2}\right.$$

$$\left. + \frac{U_{j,k+1} - 2U_{j,k} + U_{j,k-1}}{(\Delta Y)^2}\right] + M^2 \left(e - \frac{U_{j+1,k} + U_{j.k}}{2}\right) \quad (19)$$

Considering the U and P values in the j+1 column as unknowns, and those in the j column as known values, an equation for U and P is obtained.

$$(\alpha_k)\, U_{j+1,k-1} + (\beta_k)\, U_{j+1,k} + (\gamma_k)\, U_{j+1,k+1} + (\xi)\, P_{j+1} = (\phi_k) \quad (20)$$

where

$$\alpha_k = \left[\frac{1}{2(\Delta Y)^2}\right]$$

$$\beta_k = \left[- \frac{U_{j,k}}{\Delta X} - \frac{1}{(\Delta Y)^2} - \frac{M^2}{2}\right]$$

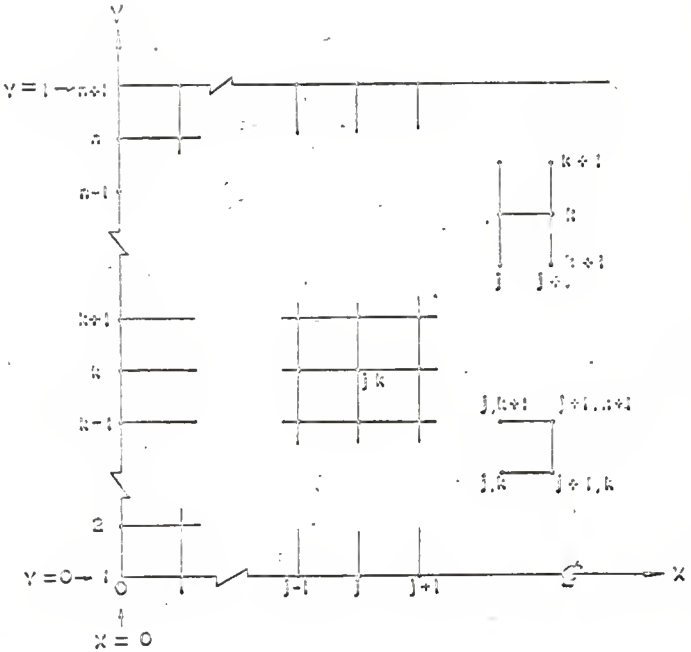

Fig.4. Mesh network for difference representation.

$$\gamma_k = \left[\frac{1}{2(\Delta Y)^2}\right]$$

$$\zeta = \left[-\frac{1}{\Delta X}\right]$$

$$\phi_k = \left[-\frac{U_{j,k}^2}{\Delta X} + V_{j,k}\left(\frac{U_{j,k+1} - U_{j,k-1}}{2\Delta Y}\right) - \frac{P_j}{\Delta X}\right.$$

$$\left. - \frac{1}{2(\Delta Y)^2}(U_{j,k-1} - 2U_{j,k} + U_{j,k-1}) - M^2\left(e - \frac{U_{j,k}}{2}\right)\right] \quad . \; .$$

Using Simpson's integration rule for equation (16), *we obtain*

$$\int_0^1 U dY = \frac{1}{3} \cdot \frac{1 - 0}{n}(U_1 + 4U_2 + 2U_3 + 4U_4 + 2U_5 + \; . \; . \; .) = 1$$

or

$$U_1 + 4\sum U_{2k} + 2\sum U_{2k+1} = 3n, \qquad k = 1, 2, 3, \; . \; . \; . \qquad (21)$$

Now, considering the points $k = 1, 2, \ldots, n$ in the $j+1$ column in equation (20), and combining them with equation (21), we obtain $(n + 1)$ equations and the same number of unknowns. Here we use the boundary conditions that $\frac{\partial U}{\partial Y} = 0$ for $x \geq 0$ and $Y = 0$ yields $U_{j+1,2} = U_{j+1,0}$, and that $U_{j+1,n+1} = 0$ for $j + 1 = 0, 1, 2, \ldots$.

The equations in matrix form are as follows.

$$
\begin{bmatrix}
1 & 4 & 2 & 4 & 2 & \cdots & 4 & 0 \\
\beta_1 & \alpha_1+\gamma_1 & 0 & 0 & 0 & \cdots & 0 & \xi \\
\alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & \cdots & 0 & \xi \\
0 & \alpha_3 & \beta_3 & \gamma_3 & 0 & \cdots & 0 & \xi \\
\vdots & & \ddots & \ddots & \ddots & & \vdots & \vdots \\
\vdots & 0 & & \alpha_{n-1} & \beta_{n-1} & & \gamma_{n-1} & \xi \\
\cdots & \cdots & \cdots & \cdots & 0 & \alpha_n & B_n & \xi
\end{bmatrix}
\begin{bmatrix}
U_{j+1,1} \\ U_{j+1,2} \\ U_{j+1,3} \\ U_{j+1,4} \\ \vdots \\ U_{j+1,n} \\ P_{j+1}
\end{bmatrix}
=
\begin{bmatrix}
3_n \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{n-1} \\ \phi_n
\end{bmatrix}
$$

The values of U, P, and V are known at the section $X = 0$ from boundary conditions. Thus let $j = 0$ at this section; then the values of U and P in the $j + 1 = 1$ column can be obtained by solving the $(n + 1)$ linear simultaneous equations. In order to solve V in this column, equation (15) is replaced by the following difference equation:

$$\frac{U_{j+1,k+1} + U_{j+1,k} - U_{j,k+1} - U_{j,k}}{2\Delta X} + \frac{V_{j+1,k+1} - V_{j+1,k}}{\Delta Y} = 0$$

Solving for $V_{j+1,k+1}$, one obtains

$$V_{j+1,k+1} = V_{j+1,k} - \frac{\Delta Y}{2\Delta X}\left(U_{j+1,k+1} + U_{j+1,k} - U_{j,k+1} - U_{j,k}\right) \tag{23}$$

Thus let $k = 1, 2, \ldots, n$ and substitute the values of U which have been calculated into equation (23), and remember $V_{j+1,1} = 0$ from boundary conditions; then values of $V_{1,k}$ $(k = 1, 2, \ldots, n)$ are obtained. After the values of U, V, and P are determined for the $j + 1 = 1$ column, the above procedure is applied to the $j + 1 = 2$ column. Thus it is possible to advance column by column along the channel until the flow becomes fully developed.

The $(n + 1)$ simultaneous equations were solved by Gaussian elimination method; the calculations were performed on an IBM 1620 digital computer. For Hartmann numbers equal to 2.5, 4, and 10, a 40-point mesh was used with $\Delta X = 0.00025$ from $X = 0$ to $X = 0.001$. Then a 20-point mesh with $\Delta X = 0.001$ was used until fully developed. Because of the relatively shorter entrance length and in an attempt to obtain better results for the case where Hartmann number equals 50, an 80-point mesh with $\Delta X = 0.000,05$ is used from $X = 0$ to $X = 0.0001$. Then a 40-point mesh with $\Delta X = 0.0003$ is used until $X = 0.001$. Thereafter ΔX is changed to 0.00025 with 40-point mesh until fully developed. Owing to the capacity of the digital computer, two programs are used successively in the case of 80-point mesh. In all cases, the ratio $\frac{\Delta X}{(\Delta Y)^2}$ was kept $\leq \frac{1}{2}$, and as close to $\frac{1}{2}$ as possible (3). Each of the above listed meshes (80, 40, 20 points) applied to one-half the channel height.

Results and Discussion

The velocity component, U, for Hartmann number, M, of 4, 10, and 50 in the entrance region is shown in Figs. 5a, 5b, and 5c for various values of Y. It is the same as in the case of developing velocity from a uniform velocity distribution at the entry, the dimensionless center line velocity approaches 1.284 for $M = 4$, 1.111 for $M = 10$, and 1.020 for $M = 50$. These values

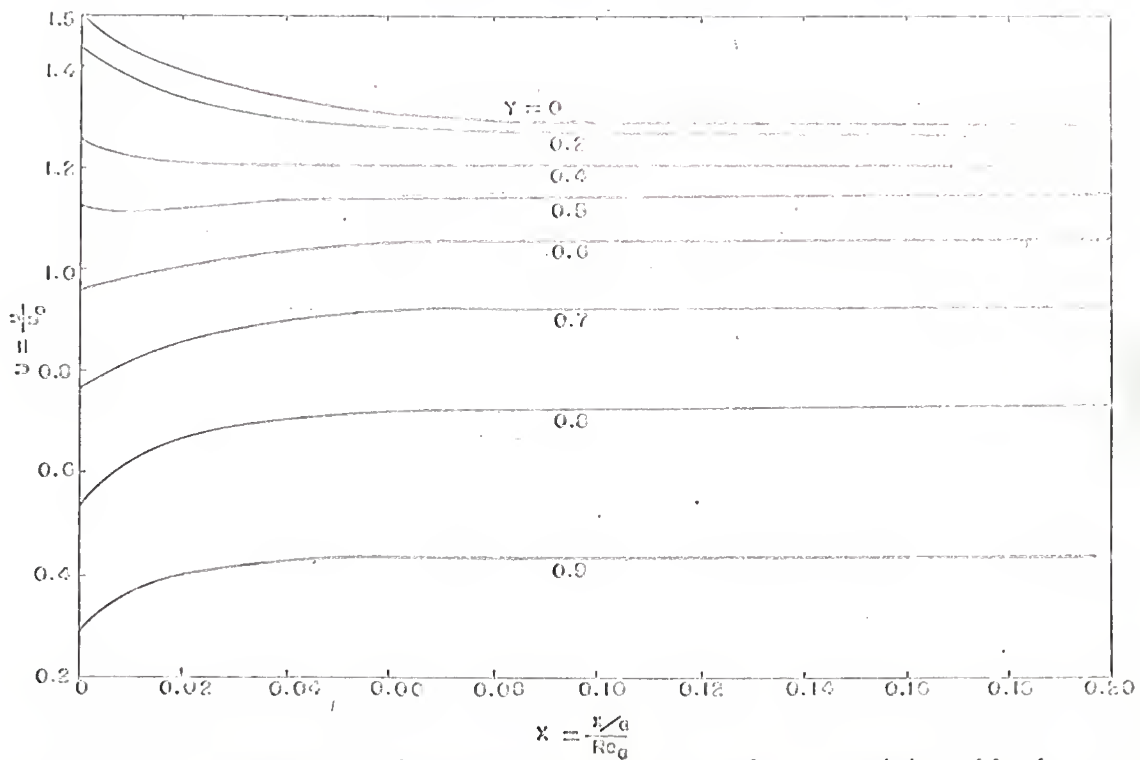

Fig.5a. Development of U in a channel of flat plates, M=4

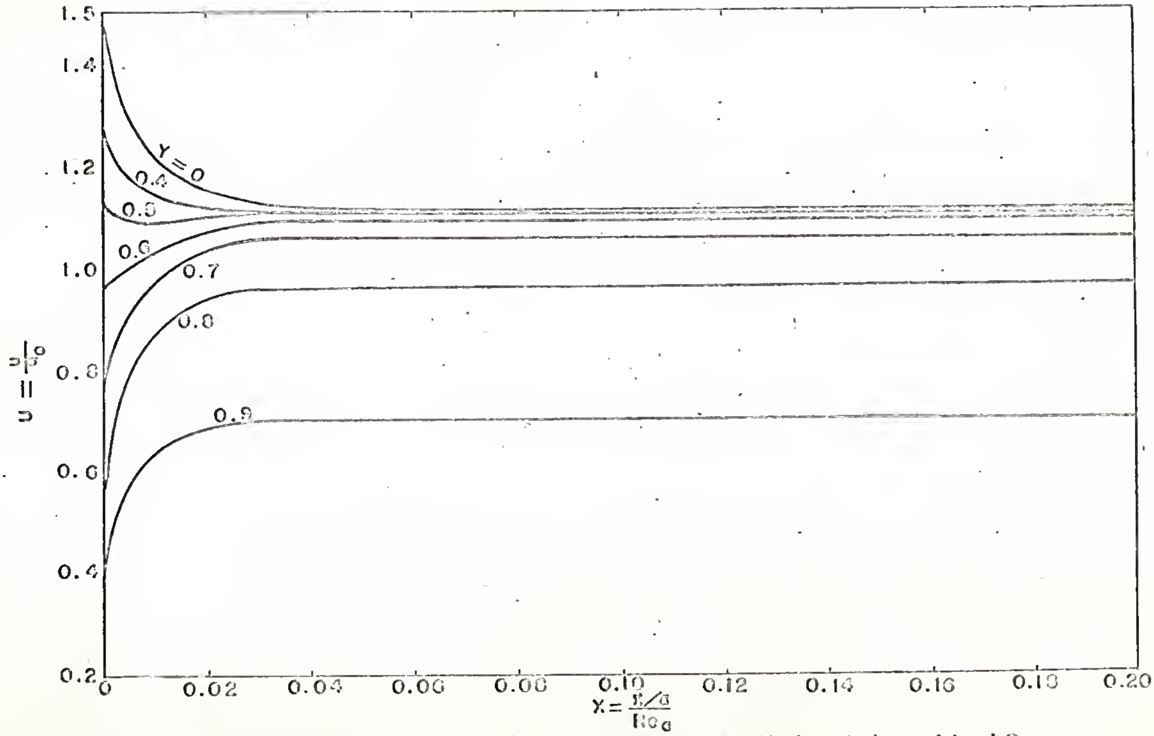

Fig. 5b Development of U in a channel of flat plats, M = 10.

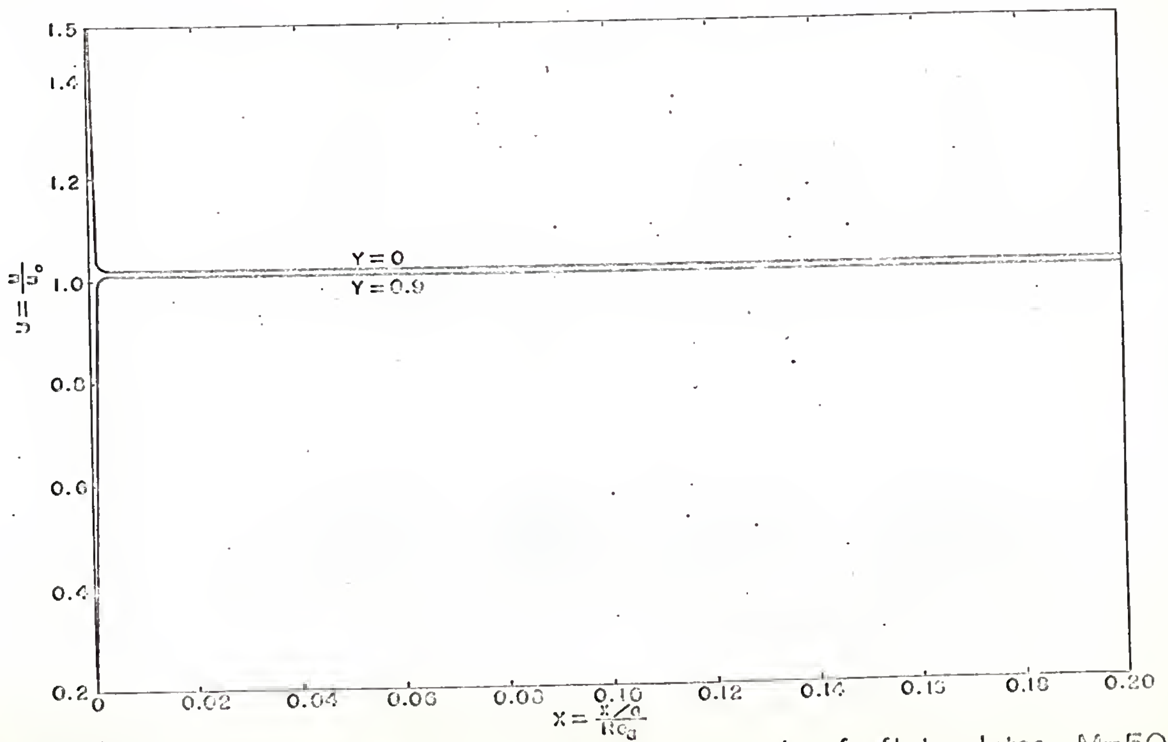

Fig 5c. Development of U in a channel of flat plates, M=50

can be obtained from the velocity equation of Hartmann (6), which is

$$U = M\left[\frac{\cosh M - \cosh (MY)}{M \cosh M - \sinh M}\right] . \tag{24}$$

The velocity profile for a different Hartmann number, M, in the entrance region is shown in Figs. 6a and 6b. It is well known that the effect of a magnetic field is to flatten the velocity profile. Therefore flows with velocity profiles nonmagnetically fully developed (parabolic profile) at the entry, in general require larger entrance length than flows with uniform velocity profile at the entry. Table 1 shows the entrance length for one per cent and five per cent deviation of center line velocity from fully developed one, and are compared with values from Hwang and Fan's (3) analysis of flows with a uniform entry profile.

Maciulaitis and Loeffler (5) defined the entrance length as the distance required for the friction factor to come within 10 per cent of the final developed value. In order to make comparisons with their results, the entrance length according to this definition for different M was also found. By definition, the friction factor f is

$$f = \frac{\tau_w}{\rho \bar{u}^2/2} = \frac{-(\partial u/\partial y)_{y=a}}{\rho \bar{u}^2/2} . \tag{25}$$

In dimensionless parameters, it becomes

$$f = -\frac{2}{Re_a}\left(\frac{\partial U}{\partial Y}\right)_{Y=1} . \tag{26}$$

Therefore, the ratio of the friction factor to that at the fully

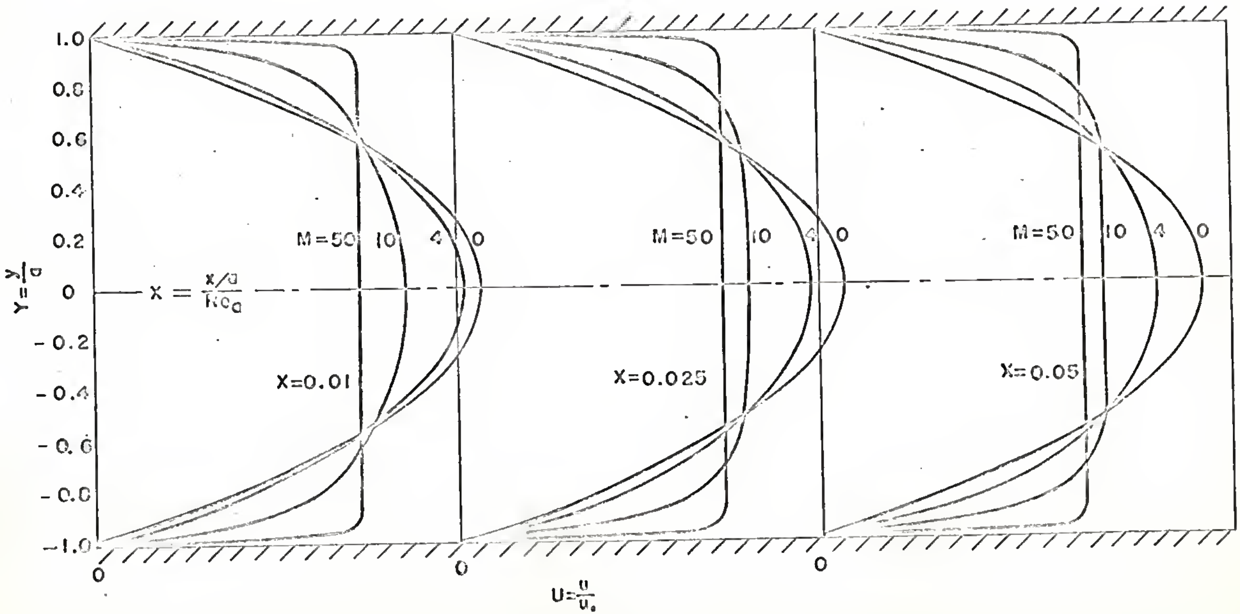

Fig. 6a. Development of U profile in a flat channel for M = 0, 4, 10, 50.

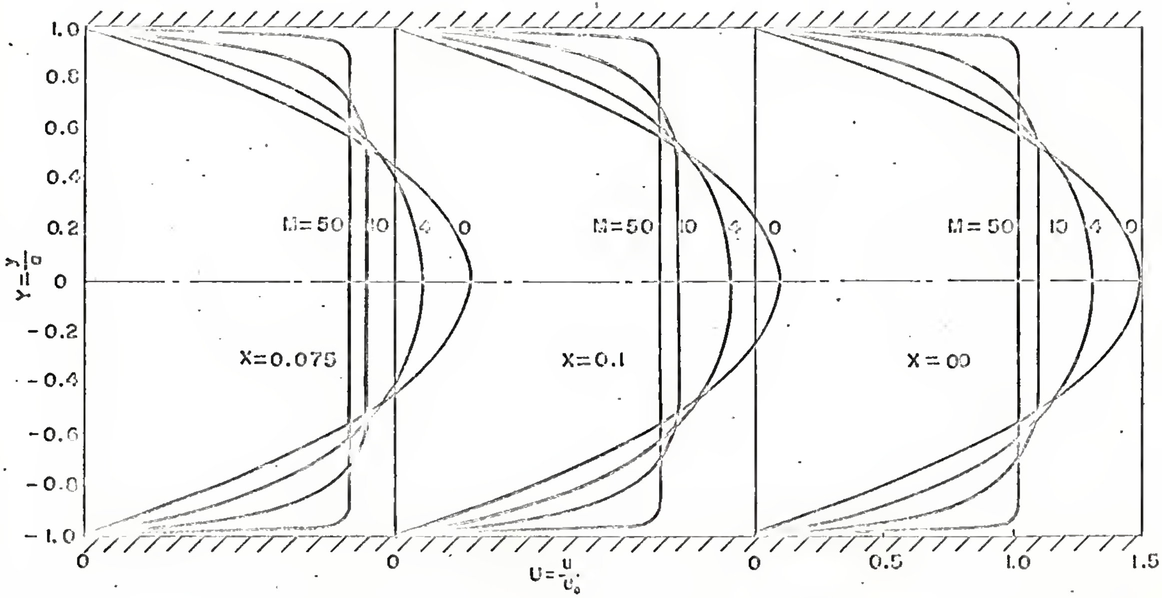

Fig. 6b. Development of U profile in a flat channel for M=0, 4, 10, 50.

Table 1. Comparison of the entrance length based on center line velocity.

		Parabolic entry (Present work)	Uniform entry (Ref. (3))
M = 4	$X_e'(1\%)$	0.019,3	0.018,8
	$X_e'(5\%)$	0.008,18	0.007,32
M = 10	$X_e'(1\%)$	0.007,31	0.003,04
	$X_e'(5\%)$	0.003,77	0.000,234
M = 50	$X_e'(1\%)$	0.000,275	
	$X_e'(5\%)$	0.000,162	

developed condition is

$$\frac{f}{f_{F.D.}} = \frac{\left(\frac{\partial U}{\partial Y}\right)_{Y=1}}{\left(\frac{\partial U}{\partial Y}\right)_{Y=1 \text{ at F.D.}}} \quad . \tag{27}$$

The value of $\left(\frac{\partial U}{\partial Y}\right)_{Y=1}$ at the fully developed condition is obtained from equation (24) as follows.

$$\left(\frac{\partial U}{\partial Y}\right)_{Y=1 \text{ at F.D.}} = - \frac{M^2 \sinh M}{M \cosh M - \sinh M} \tag{28}$$

Table 2 shows the comparison of the present work with that of Maciulaitis and Loeffler employing the entrance length in accordance with their definition for M = 4 and 10. Because of the drastic change of velocity near the wall for M = 50, the friction factor obtained by the finite difference technique is not accurate due to the capacity of the available computer. Therefore only M = 4 and 10 are shown in this table.

Table 2. Comparison of the entrance length based on friction fector.

	Present work (X_e')	Maciulaitis & Loeffler (X_e')
M = 2.5	0.0038	
M = 4	0.0045	0.0223
M = 10	0.00167	0.00496

The disagreement between the present work and Maciulaitis and Loeffler's is seen in Table 2. In their solutions, there is a minimum velue of δ/a for certain M, for example, for M = 4, $(\delta/a)_{min}$ is 0.612 ($=\sqrt{6}/4$), and for M = 10, $(\delta/a)_{min} = 0.24495$ ($=\sqrt{6}/10$). Beyond these minimum values of δ/a, the solutions ere not applicable. Thus the velocity profiles corresponding to these $(\delta/a)_{min}$ become the fully developed one which wes used to celculate the friction fector ratio. It is doubtful that the fully developed velocity profiles thus obteined epproximate the exact Hartmenn velocity profile. However, the Hartmann velocity profiles ere used in the present work, and the velocity distributions in the later part of the entrance region are very close to the fully developed Hertmann velocity profiles as can be seen from Fig. 6b.

The comparison of the entrence lengths in Tebles 1 end 2 shows that quite a large difference exists using different definitions. It is debateble that the definition by Reference (5) can be used.

The accuracy of present analysis can be seen in a reasonable agreement of the results with the known asymptotes, for instance, the center line velocities and $\frac{\partial U}{\partial Y}$ evaluated at the walls, as shown in Table 3.

Table 3. Comparison of the asymptotic values.

M	Asymptotic Center line velocity			Asymptotic $\left(\frac{\partial U}{\partial Y}\right)_{Y=1}$		
	Expected	Present work	Deviation (%)	Expected	Present work	Deviation (%)
4	1.2842	1.2862	0.156	5.3286	5.2563	1.356
10	1.1110	1.1123	0.117	11.1111	10.3810	6.571
50	1.0204	1.0218	0.137			

In general, as the Hartmann number increases the entrance length becomes shorter, as shown in Table 1. However, for very small Hartmann numbers, as the Hartmann number decreases the entrance length decreases also. In particular, when Hartmann number is zero, since there is no change in velocity profile throughout the flow field, the entrance length is zero. The same conclusion was also given in Reference (5). Table 4 shows the variation of entrance length defined by one per cent deviation of center line velocity, with M of 1.5, 2.5, 4, 10, and 50. The maximum entrance length will occur for the Hartmann number of approximately 2.5 ~ 4.

The dimensionless pressure drops over the entrance region are shown in Fig. 7 for M = 4, 10, and 50 for $e = \frac{1}{2}$. Figure 7 is not convenient for calculating the total pressure drop beyond

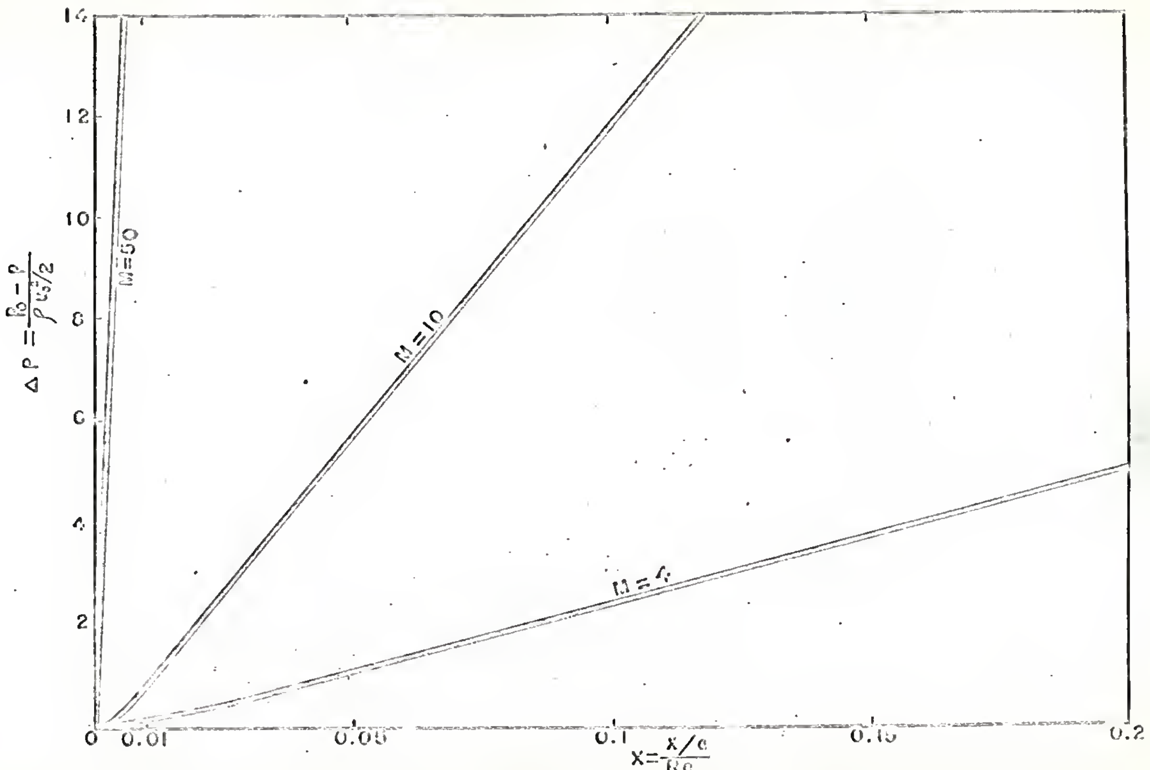

Fig. 7. Dimensionless entrance region pressure distribution for $e = 0.5$.

Table 4.

M	1.5	2.5	4	10	50
$X_e'(1\%)$	0.0145	0.0211	0.0205	0.0076	0.000,288

the entrence length. For this purpose, the following equation is used to express the pressure drop for X' greater then the entrence length.

$$\frac{p_0 - p}{\rho \bar{u}^2/2} = CX' + K \tag{29}$$

where C and K are constants to be determined.

Considering the fully developed region, equetion (17) becomes

$$\frac{dP}{dX} = \frac{d^2U}{dY^2} + M^2(e - U) \quad . \tag{30}$$

From equetion (24), we have

$$\frac{dU}{dY} = M^2\left(\frac{-\sinh MY}{M \cosh M - \sinh M}\right),$$

$$\frac{d^2U}{dY^2} = M^3\left(\frac{-\cosh MY}{M \cosh M - \sinh M}\right). \tag{31}$$

Substituting equetions (24) end (31) into (30) yields

$$\frac{dP}{dx} = M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right). \tag{32}$$

Integreting equetion (32) from X_e to X, one obtains

$$\int_{P_{X_e}}^{P} dP = M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right) \int_{X_e}^{X} dX ,$$

$$P - P_{X_e} = M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right)(X - X_e) ,$$

$$\frac{p - p_{X_e}}{\bar{u}^2/2} = 2M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right)(X - X_e) \quad .$$

The total pressure drop is the sum of the pressure drop in the entrance region and the pressure drop in the region after the velocity is fully developed. Therefore the total pressure drop is

$$\frac{p_0 - p}{\rho\bar{u}^2/2} = \frac{p_0 - p_{X_e}}{\rho\bar{u}^2/2} + \frac{p_{X_e} - p}{\rho\bar{u}^2/2}$$

$$= -2M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right)(X - X_e) + \frac{p_0 - p_{X_e}}{\rho\bar{u}^2/2} ,$$

$$\frac{p_0 - p}{\rho\bar{u}^2/2} = -2M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right) X + \frac{p_0 - p_{X_e}}{\rho\bar{u}^2/2}$$

$$+ 2M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right) X_e \tag{33}$$

Since $X = 4X'$, equation (33) becomes

$$\frac{p_0 - p}{\rho\bar{u}^2/2} = -8M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right) X' + \frac{p_0 - p_{X_e}}{\rho\bar{u}^2/2}$$

$$+ 2M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right) X_e \, . \tag{34}$$

Comparing equation (34) with (29), we have

$$C = -8M^2\left(e - \frac{M \cosh M}{M \cosh M - \sinh M}\right) \tag{35}$$

$$K = \frac{p_0 - p_{X_e}}{\rho \bar{u}^2/2} + 2M^2\left(a - \frac{M \cosh M}{M \cosh M - \sinh M}\right) X_e \qquad (36)$$

Table 5 shows the comparison of values of C calculated from equation (35) with those obtained from finite difference calculation, and also shows values of K obtained from this analysis. Their excellent agreement provides an additional measure of confidence in the results of the present analysis.

Table 5.

M	C: From equation (35)	C: Present work	C: Deviation (%)	K
4	106.628	106.596	0.03	-0.232
10	488.889	489.944	0.216	-0.369
50	10408.162	10436.896	0.276	-0.538

It is obvious from equation (36) that the correction factor K is a function of a. The values shown in Table 5 for $a = \frac{1}{2}$, and X_e is the entrance length listed in Table 1 (one per cent deviation). It is interesting to note that the values of K are negative. This means that contrary to the case of uniform velocity profile at the entry, the pressure drop in the entrance region is smaller than the pressure drop in the fully developed region of equal length. This is true because the free stream velocity is decreasing as it passes downstream. This causes the pressure to increase and compensates the pressure drop due to friction in the entrance region. Thus the overall pressure drop becomes smaller.

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ACKNOWLEDGMENT

The author wishes to express his sincere gratitude to his major adviser, Dr. C. L. Hwang, for his constant encouragement, advice, and valuable suggestions; Dr. L. T. Fan, Professor of Chemical Engineering, for his enthusiastic support and valuable suggestions; Dr. R. G. Nevins, Head of the Department of Mechanical Engineering, Dr. Wilson Tripp, Professor of Mechanical Engineering, and Dr. C. F. Koch, Assistant Professor of Mathematics, for being members of the author's advisory committee; the Air Force Office of Scientific Research Grant AF-AFOSR-463-66 for supporting this work.

The author also wishes to express his thanks to his parents and his wife for their encouragement.

APPENDICES

APPENDIX I. SCHILLER'S METHOD

The flow is in a horizontal circular tube with radius r_0. In Schiller's method (1, 2) the velocity profile is approximated by a curve comprised of two parabolas and a straight line (see Fig. 1); the vertices of the parabolas lie on the border of the boundary layer. The parabolic velocity profile in the boundary layer is assumed to be

$$\frac{u}{u_0} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad (1)$$

where u_0 is the velocity of the center core and is a function of x, the coordinate along the direction of the flow; δ is the thickness of the boundary layer, and $y = r_0 - r$.

The flow is considered to be steady, laminar, incompressible, and axisymmetrical. Furthermore, it is assumed that the fluid properties are constant, that the pressure is uniform at any given section and is a function of x only $\left(\frac{\partial P}{\partial y} = 0\right)$, and that the velocity at the inlet edge is uniform.

First of all, the momentum integral equation for flow in a circular tube is derived. Consider the flow region within the boundary layer between sections ab and cd, which are dx apart. (See Fig. 2.) Let the x-axis be in the same direction as the flow, and let the y-axis be measured from the wall. Then the mass flow rate into the section ab within the boundary layer is

$$M_{in} = \int_{r_1}^{r_0} \rho u(2\pi r)\,dr \ .$$

Since $r = r_0 - y$, $y = r_0 - r$, and $dy = -dr$, the above equation becomes

$$M_{in} = \int_{\delta}^{0} 2\pi\rho u(r_0 - y)(-dy)$$

$$= 2\pi\rho \int_0^{\delta} u(r_0 - y)\,dy \ .$$

The mass flow rate out of the region is

$$M_{out} = 2\pi\rho \int_0^{\delta} u(r_0 - y)\,dy + 2\pi\rho \frac{d}{dx}\left[\int_0^{\delta} u(r_0 - y)\,dy\right] dx \ .$$

The difference between M_{out} and M_{in} is

$$M_{out} - M_{in} = 2\pi\rho \frac{d}{dx}\left[\int_0^{\delta} u(r_0 - y)\,dy\right] dx$$

which is the mass flow rate into the region from the center core.

Therefore the momentum flux into the region due to the bulk flow (or convective flow) is

$$2\pi\rho\, u_0 \frac{d}{dx}\left[\int_0^{\delta} u(r_0 - y)\,dy\right] dx + 2\pi\rho \int_0^{\delta} u^2(r_0 - y)\,dy$$

The momentum flux out of the region due to the bulk flow is

$$2\pi\rho \int_0^{\delta} u^2(r_0 - y)\,dy + 2\pi\rho \frac{d}{dx}\left[\int_0^{\delta} u^2(r_0 - y)\,dy\right] dx \ .$$

Thus the net momentum flux out of the region is

$$2\pi\rho \frac{d}{dx}\left[\int_0^{\delta} u^2(r_0 - y)\,dy\right] dx - 2\pi\rho\, u_0 \frac{d}{dx}\left[\int_0^{\delta} u(r_0 - y)\,dy\right] dx. \quad (2)$$

For steady-state flow, the net momentum flux out of the region is equal to the net force acting on the region in the same direction. The forces exerted on the region are the pressure force end the shear force of the wall. Hence

$$\sum F_x = -\tau_w(2\pi r_0 dx) + p\left[\pi r_0^2 - \pi(r_0 - \delta)^2\right]$$

$$+ (p + \varepsilon \frac{dp}{dx} dx) \left[\pi r_1^2 - \pi(r_1 - d\delta)^2\right]$$

$$- (p + \frac{dp}{dx} dx) \left[\pi r_0^2 - \pi(r_0 - \delta - d\delta)^2\right]$$

where $0 < \varepsilon < 1$. The first term of the right-hand side represents the sheer force of the wall, the second term the force acting on the boundery leyer at section ab, the third term the force acting on the boundary from the centrel core, end the last term the force acting on the boundery layer at section cd.

Rearrenging the equetion yields

$$\sum F_x = -\tau_w(2\pi r_0 dx) + p\pi(2r_0\delta - \delta^2)$$

$$+ (p + \varepsilon \frac{dp}{dx} dx)\, \pi \left[2r_1 d\delta - (d\delta)^2\right]$$

$$- (p + \frac{dp}{dx} dx)\, \pi\left[(2r_0\delta - \delta^2) + 2r_0 d\delta - 2\delta d\delta - (d\delta)^2\right].$$

Neglecting the infinitesimel terms of higher orders yields

$$\sum F_x = -\tau_w(2\pi r_0 dx) + p\pi(2r_1 d\delta) - p\pi(2r_0 d\delta - 2\delta d\delta)$$

$$-\pi(2r_0\delta - \delta^2) \frac{dp}{dx} dx$$

or

$$\sum F_x = -\tau_w(2\pi r_0 dx) + p\pi(2r_1 d\delta) - p\pi\left[2(r_0 - \delta) d\delta\right]$$

$$- \pi(2r_0\delta - \delta^2) \frac{dp}{dx} dx . \tag{3}$$

Since

$$r_0 - \delta = r_1 ,$$

$$p\pi\left[2(r_0 - \delta)d\delta\right] = p\pi(2r_1 d\delta) \ . \tag{4}$$

From equations (3) and (4), we obtain

$$\Sigma F_x = -\tau_w(2\pi r_0 dx) - \pi(2r_0\delta - \delta^2)\frac{dp}{dx}dx \ .$$

From Euler's equation, we have

$$-\frac{dp}{dx} = \rho u_0 \frac{du_0}{dx} \quad .$$

From the definition we also have (note that τ_w is the stress exerted by the fluid on the wall):

$$\tau_w = -\mu\left(\frac{\partial u}{\partial r}\right)_{r=r_0} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} .$$

Therefore

$$\Sigma F_x = -2\pi r_0 \ \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} dx + \pi\rho u_0 \frac{du_0}{dx}(2r_0\delta - \delta^2)dx \ .$$

From equation (1), we obtain

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 2\frac{u_0}{\delta} \quad .$$

Substituting the above equation into the net force equation, we obtain

$$\begin{aligned}\Sigma F_x &= -2\pi r_0 \ \mu\left(2\frac{u_0}{\delta}\right)dx + \pi\rho(2r_0\delta - \delta^2)u_0\frac{du_0}{dx}dx \\ &= -4\pi\mu u_0\left(\frac{r_0}{\delta}\right)dx + \pi\rho r_0^2\left[2\left(\frac{\delta}{r_0}\right) - \left(\frac{\delta}{r_0}\right)^2\right]u_0\frac{du_0}{dx}dx\end{aligned} \tag{5}$$

According to the momentum theorem, equation (2) is set equal to the right-hand side of equation (5) to give

$$2\pi\rho \frac{d}{dx}\left[\int_0^\delta u^2(r_0 - y)\,dy\right]dx - 2\pi\rho u_0 \frac{d}{dx}\left[\int_0^\delta u(r_0 - y)\,dy\right]dx$$

$$= -4\pi\mu u_0(\frac{r_0}{\delta})\,dx + \pi\rho r_0^2\left[2(\frac{\delta}{r_0}) - (\frac{\delta}{r_0})^2\right]u_0 \frac{du_0}{dx}\,dx$$

or

$$\frac{d}{dx}\left[\int_0^\delta u^2(r_0 - y)\,dy\right] - u_0 \frac{d}{dx}\left[\int_0^\delta u(r_0 - y)\,dy\right]$$

$$= -2\nu u_0(\frac{r_0}{\delta}) + \frac{1}{2} r_0^2\left[2\frac{\delta}{r_0} - (\frac{\delta}{r_0})^2\right]u_0 \frac{du_0}{dx} \quad . \qquad (6)$$

Equation (6) is a momentum integral equation of the boundary layer for flow in a circular tube. Using the assumed velocity profile expressed in equation (1), we can perform the integration in the first and second terms on the left-hand side of equation (6), as follows:

$$\int_0^\delta u^2(r_0 - y)\,dy = \int_0^\delta u_0^2(4\frac{y^2}{\delta^2} - 4\frac{y^3}{\delta^3} + \frac{y^4}{\delta^4})\,(r_0 - y)\,dy$$

$$= u_0^2 r_0\left[\frac{4}{\delta^2}\frac{y^3}{3} - \frac{4}{\delta^3}\frac{y^4}{4} + \frac{1}{\delta^4}\frac{y^5}{5}\right]_0^\delta$$

$$- u_0^2\left[\frac{4}{\delta^2}\frac{y^4}{4} - \frac{4}{\delta^3}\frac{y^5}{5} + \frac{1}{\delta^4}\frac{y^6}{6}\right]_0^\delta$$

$$= \frac{8}{15} r_0\delta u_0^2 - \frac{11}{30}\delta^2 u_0^2$$

or

$$\int_0^\delta u^2(r_0 - y)\,dy = \frac{r_0^2}{30} u_0^2(16\frac{\delta}{r_0} - 11\frac{\delta^2}{r_0^2}) \quad . \qquad (7)$$

$$\int_0^\delta u(r_0 - y)dy = \int_0^\delta u_0(2\frac{y}{\delta} - \frac{y^2}{\delta^2})(r_0 - y)dy$$

$$= u_0 r_0 \left[\frac{2}{\delta}\frac{y^2}{2} - \frac{1}{\delta^2}\frac{y^3}{3}\right]_0^\delta - u_0\left[\frac{2}{\delta}\frac{y^3}{3} - \frac{1}{\delta^2}\frac{y^4}{4}\right]_0^\delta$$

$$= \frac{2}{3} r_0\delta u_0 - \frac{5}{12} u_0\delta^2$$

or

$$\int_0^\delta u(r_0 - y)dy = \frac{r_0{}^2}{12} u_0(8\frac{\delta}{r_0} - 5\frac{\delta^2}{r_0{}^2}) \quad . \tag{8}$$

Substituting equations (7) and (8) into equation (6), we obtain

$$\frac{r_0{}^2}{30}\frac{d}{dx}\left[u_0{}^2(16\frac{\delta}{r_0} - 11\frac{\delta^2}{r_0{}^2})\right] - \frac{r_0{}^2}{12}u_0\frac{d}{dx}\left[u_0(8\frac{\delta}{r_0} - 5\frac{\delta^2}{r_0{}^2})\right]$$

$$= -2\nu u_0(\frac{r_0}{\delta}) + \frac{1}{2}r_0{}^2\left[2\frac{\delta}{r_0} - \frac{\delta^2}{r_0{}^2}\right]u_0\frac{du_0}{dx}$$

or

$$\frac{r_0{}^2}{30}\left[u_0{}^2\frac{d}{dx}(16\frac{\delta}{r_0} - 11\frac{\delta^2}{r_0{}^2}) + 2(16\frac{\delta}{r_0} - 11\frac{\delta^2}{r_0{}^2})u_0\frac{du_0}{dx}\right]$$

$$- \frac{r_0{}^2}{12}u_0\left[u_0\frac{d}{dx}(8\frac{\delta}{r_0} - 5\frac{\delta^2}{r_0{}^2}) + (8\frac{\delta}{r_0} - 5\frac{\delta^2}{r_0{}^2})\frac{du_0}{dx}\right]$$

$$- \frac{1}{2}r_0{}^2(2\frac{\delta}{r_0} - \frac{\delta^2}{r_0{}^2})\ u_0\frac{du_0}{dx} = -2\nu u_0(\frac{r_0}{\delta}) \quad .$$

Rearranging the terms, we obtain

$$\frac{r_0{}^2}{60}u_0{}^2\frac{d}{dx}(32\frac{\delta}{r_0} - 22\frac{\delta^2}{r_0{}^2} - 40\frac{\delta}{r_0} + 25\frac{\delta^2}{r_0{}^2})$$

$$+ \frac{r_0^2}{60} u_0 \frac{du_0}{dx} (64 \frac{\delta}{r_0} - 44 \frac{\delta^2}{r_0^2} - 40 \frac{\delta}{r_0} + 25 \frac{\delta^2}{r_0^2} - 60 \frac{\delta}{r_0} + 30 \frac{\delta^2}{r_0^2})$$

$$= -2 \nu u_0 (\frac{r_0}{\delta})$$

or

$$\frac{r_0^2}{60} u_0 \frac{d}{dx} (3 \frac{\delta^2}{r_0^2} - 8 \frac{\delta}{r_0}) + \frac{r_0^2}{60} \frac{du_0}{dx} (11 \frac{\delta^2}{r_0^2} - 36 \frac{\delta}{r_0})$$

$$= -2 \nu (\frac{r_0}{\delta}) \; .$$

Simplifying the equation yields

$$u_0 \frac{d}{dx} (3 \frac{\delta^2}{r_0^2} - 8 \frac{\delta}{r_0}) + \frac{du_0}{dx} (11 \frac{\delta^2}{r_0^2} - 36 \frac{\delta}{r_0}) = - \frac{120}{r_0^2} \nu (\frac{r_0}{\delta}). \quad (9)$$

If $\bar{u}$ is the average velocity, the principle of conservation of mass gives

$$\pi r_0^2 \, \bar{u} = \int_{r_0-\delta}^{r_0} (2\pi r) u dr + u_0 \pi (r_0 - \delta)^2$$

$$= 2\pi \int_{\delta}^{0} (r_0 - y) \; u(-dy) + u_0 \pi (r_0 - \delta)^2$$

$$= 2\pi \int_{0}^{\delta} (r_0 - y) \; u dy + u_0 \pi (r_0 - \delta)^2$$

$$= 2\pi u_0 \int_{0}^{\delta} (r_0 - y)(2 \frac{y}{\delta} - \frac{y^2}{\delta^2}) dy + u_0 \pi (r_0 - \delta)^2$$

$$= 2\pi r_0 u_0 \left[\frac{2}{\delta} \frac{y^2}{2} - \frac{1}{\delta^2} \frac{y^3}{3} \right]_0^{\delta} - 2\pi u_0 \left[\frac{2}{\delta} \frac{y^3}{3} - \frac{1}{\delta^2} \frac{y^4}{4} \right]_0^{\delta}$$

$$+ u_0 \pi (r_0 - \delta)^2$$

or

$$\pi r_0^2 \, \bar{u} = \frac{4}{3} \pi u_0 r_0 \delta - \frac{5}{6} \pi u_0 \delta^2 + \pi u_0 (r_0 - \delta)^2.$$

Therefore

$$\frac{\bar{u}}{u_0} = \frac{4}{3} \left(\frac{\delta}{r_0}\right) - \frac{5}{6} \left(\frac{\delta}{r_0}\right)^2 + \left(1 - \frac{\delta}{r_0}\right)^2$$

or

$$\frac{\bar{u}}{u_0} = 1 - \frac{2}{3} \left(\frac{\delta}{r_0}\right) + \frac{1}{6} \left(\frac{\delta}{r_0}\right)^2.$$

Rearranging the equation, we obtain

$$\frac{1}{6} \left(\frac{\delta}{r_0}\right)^2 - \frac{2}{3} \left(\frac{\delta}{r_0}\right) + \left(1 - \frac{\bar{u}}{u_0}\right) = 0$$

or

$$\left(\frac{\delta}{r_0}\right)^2 - 4\left(\frac{\delta}{r_0}\right) + 6\left(1 - \frac{\bar{u}}{u_0}\right) = 0 .$$

This is a quadratic equation of $\left(\frac{\delta}{r_0}\right)$. The roots are

$$\frac{\delta}{r_0} = 2 \pm \sqrt{4 - 6\left(1 - \frac{\bar{u}}{u_0}\right)} .$$

But

$$\frac{\delta}{r_0} \leqq 1$$

Hence

$$\frac{\delta}{r_0} = 2 - \sqrt{4 - 6\left(1 - \frac{\bar{u}}{u_0}\right)} . \tag{10}$$

Equation (10) shows the relation between the boundary layer thickness and the velocity at the central core.

Let $\eta = \frac{u_0}{\bar{u}} - 1$

then

$$\frac{\bar{u}}{u_0} = \frac{1}{1+\eta} \quad . \tag{11}$$

Substituting equation (11) into equation (10), we obtain

$$\frac{\delta}{r_0} = 2 - \sqrt{4 - 6(1 - \frac{1}{1+\eta})}$$

or

$$\frac{\delta}{r_0} = 2 - \sqrt{\frac{4 - 2\eta}{1+\eta}} \quad . \tag{12}$$

From equation (11), we can obtain

$$u_0 = (1 + \eta)\bar{u} \tag{13}$$

and

$$\frac{du_0}{dx} = \bar{u}\frac{d\eta}{dx} \quad . \tag{14}$$

Substituting equations (12), (13), and (14) into equation (9) results in

$$\begin{aligned}(1+\eta)\bar{u}\frac{d}{dx}&\left[3(2 - \sqrt{\frac{4-2\eta}{1+\eta}})^2 - 8(2 - \sqrt{\frac{4-2\eta}{1+\eta}})\right] \\ &+ \bar{u}\frac{d\eta}{dx}\left[11(2 - \sqrt{\frac{4-2\eta}{1+\eta}})^2 - 36(2 - \sqrt{\frac{4-2\eta}{1+\eta}})\right] \\ &= -\frac{480}{D^2}\frac{\nu}{2 - \sqrt{\frac{4-2\eta}{1+\eta}}} \quad .\end{aligned} \tag{15}$$

where $D = 2\,r_0$, the diameter of the tube.

The left-hand side of this equation is simplified as follows.

$$(1+\eta)\bar{u}\,\frac{d}{dx}\left[3\left(4-4\sqrt{\frac{4-2\eta}{1+\eta}}+\frac{4-2\eta}{1+\eta}\right)-16+8\sqrt{\frac{4-2\eta}{1+\eta}}\right]$$

$$+\bar{u}\,\frac{d\eta}{dx}\,11\left(4-4\sqrt{\frac{4-2\eta}{1+\eta}}+\frac{4-2\eta}{1+\eta}\right)-72+36\sqrt{\frac{4-2\eta}{1+\eta}}$$

$$=(1+\eta)\bar{u}\,\frac{d}{dx}\left(\frac{12-6\eta}{1+\eta}-4\sqrt{\frac{4-2\eta}{1+\eta}}-4\right)+\bar{u}\,\frac{d\eta}{dx}\left(\frac{44-22\eta}{1+\eta}\right.$$

$$\left.-8\sqrt{\frac{4-2\eta}{1+\eta}}-28\right)$$

$$=(1+\eta)\bar{u}\left[\frac{(1+\eta)(-6)-(12-6\eta)}{(1+\eta)^2}-4\cdot\frac{1}{2}\,\frac{\dfrac{(1+\eta)(-2)-(4-2\eta)}{(1+\eta)^2}}{\sqrt{\dfrac{4-2\eta}{1+\eta}}}\right]\frac{d\eta}{dx}$$

$$+\bar{u}\left(\frac{d\eta}{dx}\right)\left(\frac{44-22\eta}{1+\eta}-8\sqrt{\frac{4-2\eta}{1+\eta}}-28\right)$$

$$=\bar{u}\,\frac{d\eta}{dx}\left[\frac{-18}{1+\eta}+\frac{12}{(1+\eta)\sqrt{\dfrac{4-2\eta}{1+\eta}}}+\frac{44-22\eta}{1+\eta}-8\sqrt{\frac{4-2\eta}{1+\eta}}-28\right]$$

$$=\bar{u}\,\frac{d\eta}{dx}\left[\frac{-2-50\eta}{1+\eta}+\frac{12}{(1+\eta)\sqrt{\dfrac{4-2\eta}{1+\eta}}}-8\sqrt{\frac{4-2\eta}{1+\eta}}\right].$$

Thus equation (15) becomes

$$\bar{u}\,\frac{d\eta}{dx}\left[\frac{-2-50\eta}{1+\eta}+\frac{12}{(1+\eta)\sqrt{\dfrac{4-2\eta}{1+\eta}}}-8\sqrt{\frac{4-2\eta}{1+\eta}}\right]$$

$$=-\frac{480}{D^2}\,\frac{\nu}{2-\sqrt{\dfrac{4-2\eta}{1+\eta}}}$$

or

$$\frac{d\eta}{dx}\left(2-\sqrt{\frac{4-2\eta}{1+\eta}}\right)\left[\frac{-2-50\eta}{1+\eta}+\frac{12}{(1+\eta)\sqrt{\frac{4-2\eta}{1+\eta}}}-8\sqrt{\frac{4-2\eta}{1+\eta}}\right]$$

$$=-\frac{480}{D}\frac{\nu}{D\bar{u}}$$

or

$$\frac{d\eta}{dx}\left[\frac{-4-100\eta}{1+\eta}+\frac{24}{(1+\eta)\sqrt{\frac{4-2\eta}{1+\eta}}}-16\sqrt{\frac{4-2\eta}{1+\eta}}+\frac{2\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta}\right.$$

$$\left.+\frac{50\eta\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta}-\frac{12}{1+\eta}+8\,\frac{4-2\eta}{1+\eta}\right]=-\frac{480}{D}\frac{1}{Re_D}$$

or

$$\left[\frac{-116\eta+16}{1+\eta}+\frac{24}{(1+\eta)\sqrt{\frac{4-2\eta}{1+\eta}}}-16\sqrt{\frac{4-2\eta}{1+\eta}}+\frac{2\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta}\right.$$

$$\left.+\frac{50\eta\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta}\right]d\eta=-\frac{480}{Re_D D}\,dx \quad . \tag{16}$$

Integrating this equation from $x = 0$ to x, and noting from equation (11) that when $x = 0$, $\eta = 0$, we obtain

$$\int_0^\eta \left[\frac{-116\eta + 16}{1+\eta} + \frac{24}{(1+\eta)\sqrt{\frac{4-2\eta}{1+\eta}}} - 16\sqrt{\frac{4-2\eta}{1+\eta}} + \frac{2\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta} + \frac{50\eta\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta}\right] d\eta = -\frac{480}{DRe_D}\int_0^x dx \quad . \tag{17}$$

The integrations on the left-hand side of the equation above are carried out term by term with the aid of integration tables as follows.

$$\int_0^\eta \frac{-116\eta + 16}{1+\eta}\, d\eta = \int_0^\eta \frac{-116(\eta+1) + 132}{1+\eta}\, d\eta$$

$$= \int_0^\eta \frac{132}{1+\eta}\, d\eta - \int_0^\eta 116 d\eta = -116\eta + 132 \ln(1+\eta) \quad . \tag{18}$$

$$\int_0^\eta \frac{24}{(1+\eta)\sqrt{\frac{4-2\eta}{1+\eta}}}\, d\eta = 24\int_0^\eta \frac{d\eta}{\sqrt{(1+\eta)(4-2\eta)}}$$

$$= 24\int_0^\eta \frac{d\eta}{\sqrt{-2\eta^2 + 2\eta + 4}} = 24\left[-\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{-4\eta+2}{6}\right)\right]_0^\eta$$

$$= -12\sqrt{2}\left[\sin^{-1}\left(\frac{-2\eta+1}{3}\right)\right]_0^\eta = 12\sqrt{2}\left[\sin^{-1}\left(\frac{2\eta-1}{3}\right)\right]_0^\eta$$

or

$$\int_0^\eta \frac{24 d\eta}{(1+\eta)\sqrt{\frac{4-2\eta}{1+\eta}}} = 12\sqrt{2}\sin^{-1}\left(\frac{2\eta-1}{3}\right) + 12\sqrt{2}\sin^{-1}\left(\frac{1}{3}\right) \quad . \tag{19}$$

$$\int_0^{\eta} -16\sqrt{\frac{4-2\eta}{1+\eta}}\, d\eta = -16\left\{\sqrt{(4-2\eta)(1+\eta)}\right.$$

$$\left. + 3\left[\frac{-1}{\sqrt{2}} \sin^{-1}\left(\frac{-4\eta+2}{6}\right)\right]\right\}_0^{\eta}$$

$$= -16\left[\sqrt{4+2\eta-2\eta^2} + \frac{3}{\sqrt{2}} \sin^{-1}\left(\frac{2\eta-1}{3}\right)\right]_0^{\eta}$$

$$= -16\left[\sqrt{4+2\eta-2\eta^2} + \frac{3}{\sqrt{2}} \sin^{-1}\left(\frac{2\eta-1}{3}\right) - 2\right.$$

$$\left. + \frac{3}{\sqrt{2}} \sin^{-1}\left(\frac{1}{3}\right)\right]$$

or

$$\int_0^{\eta} -16\sqrt{\frac{4-2\eta}{1+\eta}}\, d\eta = 32 - 24\sqrt{2} \sin^{-1}\left(\frac{1}{3}\right)$$

$$- 16\sqrt{4+2\eta-2\eta^2} - 24\sqrt{2} \sin^{-1}\left(\frac{2\eta-1}{3}\right) . \tag{20}$$

$$\int_0^{\eta} \frac{2\sqrt{\dfrac{4-2\eta}{1+\eta}}}{1+\eta}\, d\eta = 2\int_0^{\eta} \frac{\sqrt{4-2\eta}}{(1+\eta)^{3/2}}\, d\eta$$

$$= 4\left[-\frac{\sqrt{4-2\eta}}{\sqrt{1+\eta}} + \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{-4\eta+2}{6}\right)\right]_0^{\eta}$$

$$= 4\left[-\frac{\sqrt{4-2\eta}}{\sqrt{1+\eta}} - \frac{1}{\sqrt{2}} \sin^{-1} \frac{2\eta-1}{3}\right]_0^{\eta} .$$

$$= 4\left[-\frac{\sqrt{4-2\eta}}{\sqrt{1+\eta}} - \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{2\eta-1}{3}\right)\right.$$

$$\left. + 2 - \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{1}{3}\right)\right]$$

or

$$\int_0^{\eta} \frac{2\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta}\, d\eta = 8 - 2\sqrt{2}\sin^{-1}\left(\frac{1}{3}\right)$$

$$- 4\sqrt{\frac{4-2\eta}{1+\eta}} - 2\sqrt{2}\sin^{-1}\left(\frac{2\eta-1}{3}\right). \qquad (21)$$

$$\int_0^{\eta} \frac{50\eta\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta}\, d\eta = 50\int_0^{\eta} \frac{\eta\sqrt{4-2\eta}}{(1+\eta)^{3/2}}\, d\eta .$$

Let

$$1+\eta = v, \qquad dv = d\eta, \qquad \eta = v-1$$

then

$$50\int \frac{\eta\sqrt{4-2\eta}}{(1+\eta)^{3/2}}\, d\eta = 50\int \frac{(v-1)\sqrt{4-2v+2}}{v^{3/2}}\, dv$$

$$= 50\left(\int \frac{\sqrt{6-2v}}{\sqrt{v}}\, dv - \int \frac{\sqrt{6-2v}}{v^{3/2}}\, dv\right)$$

$$= 50\Bigg\{\sqrt{v(6-2v)} + 3\,\frac{-1}{\sqrt{2}}\sin^{-1}\left(\frac{-4v+6}{6}\right)$$

$$-2\left[\frac{-\sqrt{6-2v}}{\sqrt{v}} + \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{-4v+6}{6}\right)\right]\Bigg\}$$

$$= 50\Bigg[\sqrt{(1+\eta)(4-2\eta)} - \frac{3}{\sqrt{2}}\sin^{-1}\left(\frac{1-2\eta}{3}\right)$$

$$+ 2\sqrt{\frac{4-2\eta}{1+\eta}} - \frac{2}{\sqrt{2}}\sin^{-1}\left(\frac{1-2\eta}{3}\right)\Bigg].$$

Thus

$$50\int_0^{\eta} \frac{\eta\sqrt{4-2\eta}}{(1+\eta)^{3/2}}\, d\eta = \Big[50\sqrt{4+2\eta-2\eta^2} + 100\sqrt{\frac{4-2\eta}{1+\eta}}$$

$$+ 125\sqrt{2}\, \sin^{-1}\left(\frac{2\eta-1}{3}\right)\Big]_0^{\eta}$$

$$= 50\sqrt{4+2\eta-2\eta^2} + 100\sqrt{\frac{4-2\eta}{1+\eta}}$$

$$+ 125\sqrt{2}\, \sin^{-1}\left(\frac{2\eta-1}{3}\right)$$

$$- 100 - 200 + 125\sqrt{2}\, \sin^{-1}\left(\frac{1}{3}\right)$$

or

$$\int_0^{\eta} \frac{50\eta\sqrt{\frac{4-2\eta}{1+\eta}}}{1+\eta}\, d\eta = 50\sqrt{4+2\eta-2\eta^2} + 100\sqrt{\frac{4-2\eta}{1+\eta}}$$

$$+ 125\sqrt{2}\, \sin^{-1}\left(\frac{2\eta-1}{3}\right)$$

$$- 300 + 125\sqrt{2}\, \sin^{-1}\left(\frac{1}{3}\right) \quad . \qquad (22)$$

Now substituting equations (18) through (22) into equation (17), we obtain

$$-116\eta + 132 \ln(1+\eta) + (12\sqrt{2} - 24\sqrt{2} - 2\sqrt{2} + 125\sqrt{2})$$

$$\sin^{-1}\left(\frac{2\eta-1}{3}\right)$$

$$+ (12\sqrt{2} - 24\sqrt{2} - 2\sqrt{2} + 125\sqrt{2})\, \sin^{-1}\left(\frac{1}{3}\right) + (-16+50)\sqrt{4+2\eta-2\eta^2}$$

$$+ (-4+100)\sqrt{\frac{4-2\eta}{1+\eta}} + 32+8-300 = -\frac{480}{DRe_D}\, x$$

or

$$- \frac{480}{DRe_D} x = -116\eta + 132 \ln (1+\eta) + 111\sqrt{2} \sin^{-1} \left(\frac{2\eta-1}{3}\right)$$
$$+ 111\sqrt{2} \sin^{-1} \left(\frac{1}{3}\right) + 34\sqrt{4+2\eta-2\eta^2} + 96\sqrt{\frac{4-2\eta}{1+\eta}} - 260$$

or

$$\frac{x}{DRe_D} = - \frac{1}{480} \Big[-116\eta + 132 \ln (1+\eta) + 111\sqrt{2} \sin^{-1} \left(\frac{2\eta-1}{3}\right)$$
$$+ 111\sqrt{2} \sin^{-1} \left(\frac{1}{3}\right) + 34\sqrt{4+2\eta-2\eta^2} + 96\sqrt{\frac{4-2\eta}{1+\eta}} - 260 \Big]$$

or

$$\frac{x}{DRe_D} = \frac{1}{16}\Big[\frac{58}{15}\eta - \frac{22}{5} \ln (1+\eta) - \frac{17}{15}\sqrt{4+2\eta-2\eta^2} - \frac{16}{5}\sqrt{\frac{4-2\eta}{1+\eta}}$$
$$- \frac{37\sqrt{2}}{10} \sin^{-1} \left(\frac{2\eta-1}{3}\right) - \frac{37\sqrt{2}}{10} \sin^{-1} \left(\frac{1}{3}\right) + \frac{26}{3}\Big] . \quad (23)$$

Thus the velocity distribution in the entrance region is completely defined by equations (1), (12), and (23).

The Length of the Entrance Region

When the flow is fully developed, $\frac{u_0}{\bar{u}} = 2$, or $\eta = 1$. Therefore from equation (23) we obtain

$$f(\eta) = \frac{1}{16} (3.86667 - 3.04964 - 2.26667 - 3.2 - 1.77776$$
$$- 1.77776 + 8.66667) = \frac{0.46141}{16} = 0.0288 .$$

Thus the entrance length x_e is

$$\frac{x_e}{DRe_D} = 0.0288$$

or

$$x_e = 0.0288\ DRe_D \quad . \tag{24}$$

Pressure Drop

From Bernoulli equation, the difference in pressure between any two locations in the entrance region is

$$p_1 - p_2 = \frac{\rho}{2}(u_0)_2{}^2 - \frac{\rho}{2}(u_0)_1{}^2 \tag{25}$$

where p_1 and p_2 are the pressure at x_1 and x_2, respectively, and $(u_0)_1$ and $(u_0)_2$ are the velocities at the central core at x_1 and x_2, respectively.

From equation (11), we have

$$(u_0)_1 = \bar{u}(1+\eta_1) \quad , \qquad (u_0)_2 = \bar{u}(1+\eta_2) \quad .$$

Hence the equation (25) becomes

$$p_1 - p_2 = \frac{\rho}{2}\bar{u}^2\left[(1+\eta_2)^2 - (1+\eta_1)^2\right]$$

$$= \frac{\rho}{2}\bar{u}^2\left[(\eta_2{}^2 - \eta_1{}^2) + 2(\eta_2 - \eta_1)\right] . \tag{26}$$

The pressure drop at the entrance region and at the fully developed region can be obtained as follows.

For $x \leq 0.0288\ DRe_D$, the pressure difference between the inlet edge and any location in the entrance region is obtained from equation (26), as

$$p_0 - p = \frac{\rho\bar{u}^2}{2}\left[(\eta^2 - \eta_0{}^2) + 2(\eta - \eta_0)\right] .$$

Since $\eta_0 = 0$,

$$p_0 - p = \frac{\rho\bar{u}^2}{2}(\eta^2 + 2\eta) \quad .$$

If $x = x_e = 0.0288\ DRe_D$, $\eta = 1$. Therefore the totel pressure drop in the entrance region is

$$\Delta p = \frac{3}{2}\rho\bar{u}^2 \quad .$$

For $x > 0.0288\ DRe_D$, the pressure gredient in the fully developed region is constent and is given by

$$-\frac{dp}{dx} = \frac{32\mu\bar{u}}{D^2} = \frac{64}{Re_D}\left(\frac{\rho\bar{u}^2}{2}\right)\frac{1}{D}$$

The pressure difference between the inlet edge end e location in the fully developed region is the sum of the pressure drop in the entrance region end the pressure drop between $x = x_e$ and the location considered. Therefore

$$p_0 - p = \frac{3}{2}\rho\bar{u}^2 + \frac{64}{Re_D}\left(\frac{\rho\bar{u}^2}{2}\right)\frac{1}{D}\int_{x_e}^{x} dx$$

$$= \frac{3}{2}\rho\bar{u}^2 + \frac{64}{Re_D}\left(\frac{\rho\bar{u}^2}{2}\right)\frac{1}{D}(x - 0.0288\ Re_D D)$$

or

$$\frac{p_0 - p}{\frac{\rho\bar{u}^2}{2}} = 3 + \frac{64}{Re_D D}x - 64(0.0288)$$

or

$$\frac{p_0 - p}{\frac{\rho\bar{u}^2}{2}} = \frac{64}{Re_D D}x + 1.16 \quad . \tag{27}$$

If the fluid enters the pipe from a region where the pressure is p_∞ and the velocity is negligible, there is a pressure drop of $\frac{\rho \bar{u}^2}{2}$ between that region and the inlet edge. Then equation (27) becomes

$$\frac{p_\infty - p}{\frac{\rho \bar{u}^2}{2}} = \frac{64\ x}{D Re_D} + 2.16 \quad . \tag{28}$$

References

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APPENDIX II. LANGHAAR'S LINEARIZING APPROXIMATION METHOD

The Family of Velocity Profiles in a Circular Tube

The axial direction (x-direction, see Fig. 1) component of the Navier-Stokes equation in rectangular coordinates is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

where g_x is the x component of the acceleration of gravity. (Reference 1.)

It is assumed that the flow is steady, incompressible and laminar, the fluid properties are constant; the transverse velocities v and w are small compared to u; hence the pressure gradient $-\frac{\partial P}{\partial x}$ is a function of x only. Moreover, the velocity at the entrance is assumed to be uniform and equal to the average velocity $\bar{u}$, and the term $\frac{\partial^2 u}{\partial x^2}$ may be dropped in equation (1) in comparison with $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 u}{\partial y^2}$.

With the above assumptions, equation (1) reduces to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) . \tag{2}$$

In order to linearize the differential equation the convective terms on the left-hand side of equation (2) are replaced by the following term:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nu \beta^2 u \qquad (3)$$

where β is a function of x only. Equation (2) then becomes a linear differential equation,

$$\nu \beta^2 u = g_x - \frac{1}{\rho} \frac{dp}{dx} + \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) .$$

Dividing both sides by ν yields

$$\beta^2 u = \frac{1}{\nu} \left(g_x - \frac{1}{\rho} \frac{dp}{dx}\right) + \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

or

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \beta^2 u = \alpha \qquad (4)$$

where

$$\alpha = -\frac{1}{\nu} \left(g_x - \frac{1}{\rho} \frac{dp}{dx}\right)$$

a function of x alone.

The assumption expressed by equation (3) can be justified partially on the following theoretical grounds:

1. Equation (3) is satisfied by β = 0 in the fully developed region.
2. Equation (3) is satisfied at all points on the wall irrespective of the value of β because u and the left-hand side of equation (3) both vanish there.
3. If it is assumed that the laminar state is attained by the development of a boundary layer in the entrance region, then equation (3) is satisfied at all points in the central core of the fluid. This conclusion is

reached from the fact that $\frac{\partial u}{\partial z}$ and $\frac{\partial u}{\partial y}$ in equation (3) vanish over the flat cantral portions of the velocity profilas. Accordingly, equation (3) becomes

$$u \frac{\partial u}{\partial x} = \nu \beta^2 u$$

or

$$\frac{\partial u}{\partial x} = \nu \beta^2 \ . \tag{5}$$

Since u and $\frac{\partial u}{\partial x}$ sre functions of x alone for the potential flow in the central core region, it follows that aquation (3) is satisfied by equation (5) in that ragion.

4. Equation (3) is satisfied at all points of tha inlet section $x = 0$, since the thicknass of the boundary layer is zaro.
5. Equation (3) is justifiad except for thosa points which lia within the boundary layer. In the latter part of the antranca region, a distinct boundary layer does not exist since the central parts of the velocity profilas ara appraciably rounded. However, in that region the flow approachas the fully davalopad laminar stata for which aquation (3) is valid, as axplainad in 1.

In cylindrical coordinatas, we have

$$y^2 + z^2 = r^2$$

$$u = u(r)$$

$$\frac{\partial u}{\partial y} = \frac{du}{dr}\frac{\partial r}{\partial y} = \frac{du}{dr}\frac{y}{r}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{du}{dr}\frac{y}{r}\right) = \frac{y}{r}\frac{\partial}{\partial r}\left(\frac{du}{dr}\right)\frac{\partial r}{\partial y} + \frac{du}{dr}\frac{\partial}{\partial y}\left(\frac{y}{r}\right)$$

$$= \frac{y^2}{r^2}\frac{d^2u}{dr^2} + \frac{du}{dr}\,\frac{r - y\dfrac{\partial r}{\partial y}}{r^2}$$

$$= \frac{y^2}{r^2}\frac{d^2u}{dr^2} + \frac{du}{dr}\,\frac{r - \dfrac{y^2}{r}}{r^2} \quad .$$

Similarly,

$$\frac{\partial^2 u}{\partial z^2} = \frac{z^2}{r^2}\frac{d^2u}{dr^2} + \frac{du}{dr}\,\frac{r - \dfrac{z^2}{r}}{r^2} \quad .$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{d^2u}{dr^2}\left(\frac{y^2 + z^2}{r^2}\right) + \frac{du}{dr}\frac{1}{r^2}\left(r - \frac{y^2}{r} + r - \frac{z^2}{r}\right)$$

$$= \frac{d^2u}{dr^2} + \frac{du}{dr}\frac{1}{r^2}\left(2r - \frac{y^2 + z^2}{r}\right)$$

$$= \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} \; .$$

Equation (4) then becomes

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \beta^2 u = \alpha \; . \qquad (6)$$

The particular solution, by inspection, is

$$u = -\frac{\alpha}{\beta^2} \; . \qquad (7)$$

The complementary equation is

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \beta^2 u = 0 \tag{8}$$

which is a modified Bessel's equation of zero order. The solution of equation (8) is (Ref. 3)

$$u = C_1 \ I_0(\beta r) + C_2 \ K_0(\beta r)$$

where I_0 is a modified Bessel function of first kind and of zero order, and K_0 is a modified Bessel function of second kind and of zero order.

Since $K_0(\beta r)$ approaches to infinity while u remains finite as $r \longrightarrow 0$, therefore C_2 must be zero. Thus

$$u = C_1 \ I_0(\beta r). \tag{9}$$

The solution of equation (6) is the sum of the particular solution (7) and the complementary solution (9). Hence

$$u = C_1 \ I_0(\beta r) - \frac{\alpha}{\beta^2} \ . \tag{10}$$

From the boundary condition that $u = 0$ at $r = r_0$, we have

$$0 = C_1 \ I_0(\beta r_0) - \frac{\alpha}{\beta^2}$$

or

$$\frac{\alpha}{\beta^2} = C_1 \ I_0(\gamma) \tag{11}$$

where

$$\gamma = \beta r_0 \ .$$

The continuity equation is used to eliminate

$\alpha \left[= -\frac{1}{\nu}\left(g_x - \frac{1}{\rho}\frac{dp}{dx}\right) \right]$. The continuity equation is

$$\pi r_0^{\,2}\,\bar{u} = \int_0^{r_0} 2\pi r u dr$$

or

$$\frac{1}{2}\, r_0^{\,2}\,\bar{u} = \int_0^{r_0} r u dr \quad .$$

Substituting equation (10) into this equation yields

$$\frac{1}{2}\, r_0^{\,2}\,\bar{u} = \int_0^{r_0} r\left[C_1\ I_0(\beta r) - \frac{\alpha}{\beta^2}\right] dr$$

$$= C_1 \int_0^{r_0} r I_0(\beta r)\, dr - \frac{\alpha}{\beta^2} \int_0^{r_0} r dr$$

or

$$\frac{1}{2}\, r_0^{\,2}\,\bar{u} = \frac{C_1}{\beta^2} \int_0^{\gamma} (\beta r)\ I_0(\beta r)\, d(\beta r) - \frac{\alpha}{\beta^2} \int_0^{r_0} r dr$$

$$= \frac{C_1}{\beta^2}\Big[(\beta r)\ I_1(\beta r)\Big]_0^{\gamma} - \frac{\alpha}{\beta^2}\cdot\frac{1}{2}\, r_0^{\,2}$$

$$= \frac{C_1}{\beta^2}\Big[(\gamma)\ I_1(\gamma)\Big] - \frac{\alpha}{\beta^2}\,\frac{r_0^{\,2}}{2}$$

or

$$\frac{1}{2}\,\beta^2 r_0^{\,2}\,\bar{u} = C_1\Big[(\gamma)\ I_1(\gamma)\Big] - \frac{\alpha}{\beta^2}\cdot\frac{\beta^2 r_0^{\,2}}{2}$$

or

$$\frac{1}{2}\,\gamma^2\,\bar{u} = C_1 \gamma\, I_1(\gamma) - \frac{\alpha}{\beta^2}\cdot\frac{\gamma^2}{2} \quad . \tag{12}$$

Substituting equation (11) into equation (12), we obtain

$$\frac{1}{2}\,\gamma^2\,\bar{u} = C_1 \gamma\, I_1(\gamma) - \frac{\gamma^2}{2}\, C_1\ I_0(\gamma)$$

or

$$C_1 \left[I_1(\gamma) - \frac{\gamma}{2} I_0(\gamma) \right] = \frac{\gamma}{2} \bar{u} \quad .$$

Dividing both sides of this equation by $\gamma/2$ yields

$$C_1 \left[\frac{2}{\gamma} I_1(\gamma) - I_0(\gamma) \right] = \bar{u} \; .$$

Using the relationship (Ref. 4)

$$I_2(\gamma) = I_0(\gamma) - \frac{2}{\gamma} I_1(\gamma)$$

or

$$\frac{2}{\gamma} I_1(\gamma) - I_0(\gamma) = - I_2(\gamma)$$

we obtain

$$- C_1 I_2(\gamma) = \bar{u}$$

or

$$C_1 = - \frac{\bar{u}}{I_2(\gamma)} \quad . \tag{13}$$

Substituting equation (13) into equation (11) yields

$$\frac{\alpha}{\beta^2} = - \bar{u} \frac{I_0(\gamma)}{I_2(\gamma)} \quad . \tag{14}$$

Substituting equation (13) and (14) into equation (10), we obtain the equation for the velocity profile given by

$$u = \frac{I_0(\gamma)}{I_2(\gamma)} \bar{u} - \frac{I_0(\beta r)}{I_2(\gamma)} \bar{u} \; .$$

Dividing both sides by the average velocity $\bar{u}$ yields

$$\frac{u}{\bar{u}} = \frac{I_0(\gamma) - I_0(\beta r)}{I_2(\gamma)}$$

or

$$\frac{u}{\bar{u}} = \frac{I_0(\gamma) - I_0(\gamma \frac{r}{r_0})}{I_2(\gamma)} \quad . \tag{15}$$

Thus if the relation between γ (= βr_0) and x is known, equation (15) completely defines the velocity profiles in the entrance region. The next task is to find the function $\gamma(x)$.

The Function $\gamma(x)$

The momentum equation can be obtained by integrating the Navier-Stokes equation over the cross section.

In the cylindrical coordinate system, the simplified Navier-Stokes equation, equation (2) becomes (Ref. 2)

$$v_r \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial x} = g_x - \frac{1}{\rho} \frac{dp}{dx} + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{\gamma} \frac{\partial u}{\partial r}\right) \tag{16}$$

and the continuity equation becomes

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial u}{\partial x} = 0$$

or

$$\frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial u}{\partial x} = 0 \tag{17}$$

where v_r is the velocity in the radial direction.

Integrating equation (16) over the cross section, we have

$$\int_0^{r_0} \left(v_r \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial x}\right) 2\pi r dr$$

$$= \int_0^{r_0} g_x 2\pi r dr - \int_0^{r_0} \frac{1}{\rho}\frac{dp}{dx} 2\pi r dr + \int_0^{r_0} \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) 2\pi r dr$$

$$2\int_0^{r_0} \left(v_r \frac{\partial u}{\partial r} + u\frac{\partial u}{\partial x}\right) r dr = g_x \int_0^{r_0} d(r^2) - \frac{1}{\rho}\frac{dp}{dx}\int_0^{r_0} d(r^2)$$
$$+ 2\nu \int_0^{r_0} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) r dr$$

or

$$2\int_0^{r_0} \left(v_r \frac{\partial u}{\partial r} + u\frac{\partial u}{\partial x}\right) r dr = g_x {r_0}^2 - \frac{1}{\rho}\frac{dp}{dx} {r_0}^2$$
$$+ 2\nu \int_0^{r_0} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) r dr \quad . \qquad (18)$$

The integration of the left-hand side of equation (18) can be simplified as follows:

$$2\int_0^{r_0} \left(v_r \frac{\partial u}{\partial r} + u\frac{\partial u}{\partial x}\right) r dr = 2\left[\int_0^{r_0} v_r \frac{\partial u}{\partial r} r dr + \int_0^{r_0} u\frac{\partial u}{\partial x} r dr\right]$$
$$= 2\left[\int_0^{r_0} v_r \frac{\partial u}{\partial r} r dr + \frac{1}{2}\int_0^{r_0} \frac{\partial (u^2)}{\partial x} r dr\right]. \qquad (19)$$

Let

$$r v_r = u' \quad \text{and} \quad \frac{\partial u}{\partial r} dr = dv'$$

then

$$du' = d(r v_r) \text{ and } v' = u \; .$$

Thus integrating by parts yields

$$\int_0^{r_0} v_r \frac{\partial u}{\partial r} r dr = \Big[r v_r u \Big]_0^{r_0} - \int_0^{r_0} u\, d(r v_r) = - \int_0^{r_0} u\, d(r v_r)$$

or

$$\int_0^{r_0} v_r \frac{\partial u}{\partial r} rdr = - \int_0^{r_0} u \frac{1}{r} \frac{\partial (rv_r)}{\partial r} rdr \quad . \qquad (20)$$

From equation (17), we obtain

$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) = - \frac{\partial u}{\partial x} \quad .$$

Hence equation (20) becomes

$$\int_0^{r_0} v_r \frac{\partial u}{\partial r} rdr = \int_0^{r_0} u \frac{\partial u}{\partial x} rdr = \frac{1}{2} \int_0^{r_0} \frac{\partial (u^2)}{\partial x} rdr \quad . \qquad (21)$$

Substituting equation (21) into equation (19) yields

$$2 \int_0^{r_0} (v_r \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial x}) rdr = 2 \left[\frac{1}{2} \int_0^{r_0} \frac{\partial (u^2)}{\partial x} rdr + \frac{1}{2} \int_0^{r_0} \frac{\partial (u^2)}{\partial x} rdr \right]$$

$$= 2 \int_0^{r_0} \frac{\partial (u^2)}{\partial x} rdr$$

or

$$2 \int_0^{r_0} (v_r \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial x}) rdr = 2 \frac{d}{dx} \int_0^{r_0} u^2 rdr \quad . \qquad (22)$$

Thus equation (18) becomes

$$2 \frac{d}{dx} \int_0^{r_0} u^2 rdr = g_x r_0{}^2 - \frac{1}{\rho} \frac{dp}{dx} r_0{}^2 + 2\nu \int_0^{r_0} (\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}) rdr$$

From equation (15), we obtain

$$\frac{\partial u}{\partial r} = - \frac{\bar{u}}{I_2(\gamma)} \frac{\partial}{\partial r} I_0 (\frac{\gamma}{r_0} r)$$

$$= - \frac{\bar{u}}{I_2(\gamma)} (\frac{\gamma}{r_0}) \; I_1 (\frac{\gamma}{r_0} r)$$

or

$$\frac{\partial u}{\partial r} = - \frac{\bar{u}\beta}{I_2(\gamma)} I_1(\beta r) \tag{23}$$

and

$$\frac{\partial^2 u}{\partial r^2} = - \frac{\bar{u}\beta^2}{I_2(\gamma)} I_1{}'(\beta r) \ . \tag{24}$$

Therefore the last term of equation (18) becomes

$$2\nu\int_0^{r_0} (\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}) r dr = -2\nu\int_0^{r_0} \left[\frac{\bar{u}\beta^2}{I_2(\gamma)} I_1{}'(\beta r)\right] r dr$$

$$-2\nu\int_0^{r_0} \frac{\bar{u}\beta}{I_2(\gamma)} I_1(\beta r) dr$$

$$= -2\nu \frac{\bar{u}}{I_2(\gamma)} \int_0^{\gamma} (\beta r) I_1{}'(\beta r) d(\beta r) - 2\nu \frac{\bar{u}}{I_2(\gamma)} \int_0^{\gamma} I_1(\beta r) d(\beta r)$$

$$= - \frac{2\nu\bar{u}}{I_2(\gamma)} \left\{\left[(\beta r) I_1(\beta r)\right]_0^{\gamma} - \int_0^{\gamma} I_1(\beta r) d(\beta r) + \int_0^{\gamma} I_1(\beta r) d(\beta r)\right\}$$

or

$$2\nu \int_0^{r_0} (\frac{\partial^2 u}{\partial r^2}) + \frac{1}{r}\frac{\partial u}{\partial r}) r dr = - \frac{2\nu\bar{u}}{I_2(\gamma)} \left[\gamma I_1(\gamma)\right] \ . \tag{25}$$

Substituting equations (22) and (25) into equation (18), we obtain the momentum integral equation

$$- \frac{2\nu\bar{u}\gamma I_1(\gamma)}{I_2(\gamma)} + r_0{}^2 \ (g_x - \frac{1}{\rho}\frac{dp}{dx}) = 2 \frac{d}{dx} \int_0^{r_0} u^2 r dr \ . \tag{26}$$

Since the radial velocity is zero in the central core, equation (16) becomes

$$g_x - \frac{1}{\rho}\frac{dp}{dx} = u_0 \frac{du_0}{dx} - \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)_{r=0} \ . \tag{27}$$

From equations (23) and (24), we have

$$\frac{1}{r}\frac{\partial u}{\partial r} = -\frac{\bar{u}\beta}{I_2(\gamma)}\frac{I_1(\beta r)}{r} ,$$

$$\lim_{r\to 0}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) = -\frac{\bar{u}\beta}{I_2(\gamma)}\lim_{r\to 0}\frac{I_1(\beta r)}{r} = -\frac{\bar{u}\beta^2}{I_2(\gamma)}\lim_{r\to 0} I_1{}'(\beta r)$$

$$= -\frac{\bar{u}\beta^2}{I_2(\gamma)} I_1{}'(0) = -\frac{\bar{u}\beta^2}{I_2(\gamma)}\frac{I_0(0) - I_2(0)}{2}$$

$$= -\frac{1}{2}\frac{\bar{u}\beta^2}{I_2(\gamma)} \tag{28}$$

and

$$\left(\frac{\partial^2 u}{\partial r^2}\right)_{r=0} = -\frac{\bar{u}\beta^2}{I_2(\gamma)} I_1{}'(0) = -\frac{\bar{u}\beta^2}{I_2(\gamma)}\frac{I_0(0) - I_2(0)}{2}$$

$$= -\frac{1}{2}\frac{\bar{u}\beta^2}{I_2(\gamma)} . \tag{29}$$

Because of equations (28) and (29), equation (27) can be written as

$$g_x - \frac{1}{\rho}\frac{dp}{dx} = u_0\frac{du_0}{dx} + \frac{\nu\bar{u}\beta^2}{I_2(\gamma)}$$

or

$$g_x - \frac{1}{\rho}\frac{dp}{dx} = u_0\frac{du_0}{dx} + \frac{\nu\,\bar{u}\gamma^2}{r_0{}^2 I_2(\gamma)} \tag{30}$$

Substituting this equation into equation (26), we obtain

$$-\frac{2\nu\,\bar{u}\gamma I_1(\gamma)}{I_2(\gamma)} + r_0{}^2 u_0\frac{du_0}{dx} + \frac{\nu\,\bar{u}\gamma^2}{I_2(\gamma)} = 2\frac{d}{dx}\int_0^{r_0} u^2 r\,dr$$

$$\frac{\nu \bar{u}\gamma^2 - 2\nu\bar{u}\gamma I_1(\gamma)}{I_2(\gamma)} = 2\frac{d}{dx}\int_0^{r_0} u^2 r dr - r_0^2 \frac{1}{2}\frac{d}{dx}(u_0^2)$$

$$\doteq 2\frac{d}{dx}\int_0^{r_0} u^2 r dr - \frac{d}{dx}\int_0^{r_0} u_0^2 r dr$$

$$= \frac{d}{dx}\int_0^{r_0}(2u^2 - u_0^2) r dr ,$$

or

$$\frac{\gamma^2 - 2\gamma I_1(\gamma)}{I_2(\gamma)} = \frac{1}{\nu\bar{u}}\frac{d}{dx}\int_0^{r_0}(2u^2 - u_0^2) r dr$$

$$= \frac{\bar{u}r_0^2}{\nu}\frac{d}{dx}\int_0^{1}\left(2\frac{u^2}{\bar{u}^2} - \frac{u_0^2}{\bar{u}^2}\right)\left(\frac{r}{r_0}\right)d\left(\frac{r}{r_0}\right) . \tag{31}$$

Let

$$U = \frac{u}{\bar{u}} , \; U_0 = \frac{u_0}{\bar{u}} , \; R = \frac{r}{r_0} , \; Re_r = \frac{\bar{u}r_0}{\nu} \text{ and } X = \frac{x}{r_0 Re_r} .$$

Then equation (31) becomes

$$\frac{\gamma^2 - 2\gamma I_1(\gamma)}{I_2(\gamma)} = r_0 \, Re_r \frac{d}{dX}\left[\int_0^{1}(2U^2 - U_0^2) R dR\right]\frac{dX}{dx}$$

$$= \frac{d}{dX}\int_0^{1}(2U^2 - U_0^2) R dR$$

or

$$\frac{2\gamma I_1(\gamma) - \gamma^2}{I_2(\gamma)} = \frac{d}{dX}\int_0^{1}(U_0^2 - 2U^2) R dR . \tag{32}$$

Noting that equation (15) can be written as

$$U = \frac{I_0(\gamma) - I_0(\gamma R)}{I_2(\gamma)} , \tag{33}$$

the integration on the right-hand side of equation (32) can be performed as follows:

$$\int_0^1 (U_0^2 - 2U^2)RdR = \int_0^1 \left\{ \left[\frac{I_0(\gamma) - I_0(0)}{I_2(\gamma)} \right]^2 - 2\left[\frac{I_0(\gamma) - I_0(\gamma R)}{I_2(\gamma)} \right]^2 \right\} RdR$$

$$= \frac{[I_0(\gamma) - 1]^2}{I_2^2(\gamma)} \cdot \frac{1}{2}\left[R^2\right]_0^1 - \frac{2}{I_2^2(\gamma)} \int_0^1 \left[I_0^2(\gamma) - 2I_0(\gamma) I_0(\gamma R) + I_0^2(\gamma R) \right] RdR$$

$$= \frac{[I_0(\gamma) - 1]^2}{2I_2^2(\gamma)} - \frac{2}{I_2^2(\gamma)} \left[\frac{I_0^2(\gamma)}{2} - 2I_0(\gamma) \frac{1}{\gamma^2} \int_0^\gamma (\gamma R) I_0(\gamma R) d(\gamma R) + \int_0^1 I_0^2(\gamma R) RdR \right]$$

$$= \frac{[I_0(\gamma) - 1]^2}{2I_2^2(\gamma)} - \frac{I_0^2(\gamma)}{I_2^2(\gamma)} + \frac{4I_0(\gamma)}{\gamma^2 I_2^2(\gamma)} \left[(\gamma R) I_1(\gamma R) \right]_0^\gamma - \frac{2}{I_2^2(\gamma)} \int_0^1 I_0^2(\gamma R) RdR$$

$$= \frac{[I_0(\gamma) - 1]^2}{2I_2^2(\gamma)} - \frac{I_0^2(\gamma)}{I_2^2(\gamma)} + \frac{4I_0(\gamma)\, I_1(\gamma)}{\gamma I_2^2(\gamma)} - \frac{2}{I_2^2(\gamma)} \int_0^1 I_0^2(\gamma R) RdR \quad . \tag{34}$$

For the integration of the last term of equation (34), the following formula is used (Ref. 3).

$$\int_0^x x I_n^2(\lambda x) dx = -\frac{x^2}{2} \left[I_n'^2(\lambda x) - \left(1 + \frac{n^2}{\lambda^2 x^2}\right) I_n^2(\lambda x) \right] . \tag{35}$$

Thus

$$\int_0^1 I_0^2(\gamma R)RdR = -\left[\frac{R^2}{2}\left\{I_0'^2(\gamma R) - (1 + \frac{0}{\gamma^2 R^2})\ I_0^2(\gamma R)\right\}\right]_0^1$$

$$= -\frac{1}{2}\left[I_0'^2(\gamma) - I_0^2(\gamma)\right]$$

or

$$\int_0^1 I_0^2(\gamma R)RdR = \frac{1}{2}\left[I_0^2(\gamma) - I_0'^2(\gamma)\right] = \frac{1}{2}\left[I_0^2(\gamma) - I_1^2(\gamma)\right]\ . \tag{36}$$

Equation (34) now becomes

$$\int_0^1 (U_0^2 - 2U^2)RdR = \frac{[I_0-1]^2}{2I_2^2} - \frac{I_0^2}{I_2^2} + \frac{4I_0I_1}{\gamma I_2^2} - \frac{2}{I_2^2}\cdot\frac{1}{2}(I_0^2-I_1^2)$$

$$= \frac{(I_0-1)^2}{2I_2^2} - \frac{I_0^2}{I_2^2} + \frac{2I_0}{I_2^2}(I_0-I_2) - \frac{I_0^2-I_1^2}{I_2^2}$$

$$= \frac{(I_0-1)^2}{2I_2^2} + \frac{I_1^2 - 2I_0I_2}{I_2^2}$$

or

$$\int_0^1 (U_0^2 - 2U^2)RdR = \frac{(I_0-1)^2 - 4I_0I_2 + 2I_1^2}{2I_2^2}, \tag{37}$$

in which the argument of I's is γ.

Substituting equation (37) into equation (32) gives

$$\frac{2\gamma I_1 - \gamma^2}{I_2} = \frac{d}{dX}\ \frac{(I_0-1)^2 - 4I_0I_2 + 2I_1^2}{2I_2^2}\ . \tag{38}$$

Let

$$f(\gamma) = \frac{4I_0I_2 - (I_0-1)^2 - 2I_1^2}{2I_2^2} \tag{39a}$$

$$g(\gamma) = \frac{I_2}{2\gamma I_1 - \gamma^2} \tag{39b}$$

then equation (38) becomes

$$\frac{1}{g(\gamma)} = - \frac{d}{dX}\left[f(\gamma)\right]$$

or

$$\frac{1}{g(\gamma)} = - \frac{d}{d\gamma}\left[f(\gamma)\right]\frac{d\gamma}{dX}$$

or

$$\frac{1}{g(\gamma)} = - f'(\gamma)\frac{d\gamma}{dX} \quad . \tag{39}$$

This is a differential equation which expresses γ as a function of X.

For a large value of γ, $I_n(\gamma)$ can be represented by an asymptotic series, as (Ref. 4)

$$I_n(\gamma) \approx \frac{e^{\gamma}}{(2\pi\gamma)^{1/2}}\left[1 - \frac{4n^2-1^2}{8\gamma \cdot 1!} + \frac{(4n^2-1)(4n^2-3^2)}{(8\gamma)^2\ 2!} - \ldots\right]$$

$$\approx \frac{e^{\gamma}}{(2\pi\gamma)^{1/2}}\left[1 + \varepsilon_n(\gamma)\right], \tag{40}$$

where $\varepsilon_n(\gamma) = 0$ as $\gamma \longrightarrow \infty$.

Accordingly, from equation (33), we have

$$\lim_{\gamma\to\infty} U = \lim_{\gamma\to\infty}\left[\frac{I_0(\gamma) - I_0(\gamma R)}{I_2(\gamma)}\right]$$

$$= \lim_{\gamma\to\infty}\frac{I_0(\gamma)}{I_2(\gamma)} - \lim_{\gamma\to\infty}\frac{I_0(\gamma R)}{I_2(\gamma)}$$

$$= 1 - \lim_{\gamma\to\infty}\frac{\dfrac{e^{\gamma R}}{(2\pi\gamma R)^{1/2}}}{\dfrac{e^{\gamma}}{(2\pi\gamma)^{1/2}}}$$

$$= 1 - \lim_{\gamma \to \infty} \frac{1}{\sqrt{R}} \quad e^{\gamma(R-1)}$$

$$= 1 - \frac{1}{\sqrt{R}} \lim_{\gamma \to \infty} \frac{1}{e^{\gamma(1-R)}} \quad .$$

Since

$$0 \leq R \leq 1 \text{ or } 1 - R \geq 0$$

$$\lim_{\gamma \to \infty} U = 1 - \frac{1}{\sqrt{R}} \, 0 = 1 \, .$$

But

$$U = \frac{u}{\bar{u}} = 1$$

at the inlet section, $x = 0$. Therefore,

$$\lim_{\gamma \to \infty} X(\gamma) = 0 \, .$$

Thus integrating equation (39) from $X = 0$ to X leads to

$$\int_0^X dX = - \int_\infty^\gamma g(\gamma) \; f'(\gamma) d\gamma$$

or

$$X = \int_\gamma^\infty g(\gamma) \; df(\gamma) \quad . \tag{41}$$

For the numerical integration of equation (41), the following change of variables is made.

$$Z = f(\gamma) \quad \text{and} \quad Y = g(\gamma) \tag{42}$$

The values of Z and Y as $\gamma \longrightarrow \infty$ can be found as follows:

From equation (40), we know that as $\gamma \to \infty$

$$I_0 = I_1 = I_2 = I_n = \frac{e^\gamma}{(2\pi\gamma)^{1/2}} \longrightarrow \infty .$$

These relations will be used in the evaluation of the values of Z and Y as $\gamma \to \infty$ as follows:

$$\lim_{\gamma\to\infty} Z = \lim_{\gamma\to\infty} f(\gamma) = \lim_{\gamma\to\infty} \frac{4I_0I_2 - (I_0-1)^2 - 2I_1^2}{2I_2^2}$$

$$= 2 - \frac{1}{2} - 1 + \lim_{\gamma\to\infty} \frac{2I_0-1}{2I_2^2} = \frac{1}{2} \; ,$$

because

$$\lim_{\gamma\to\infty} \frac{2I_0-1}{2I_2^2} = \lim_{\gamma\to\infty} \frac{2 - \dfrac{1}{I_0}}{2\,\dfrac{I_2^2}{I_0}} = \lim_{\gamma\to\infty} \frac{2 - 0}{2I_2} = 0 \quad .$$

$$\lim_{\gamma\to\infty} Y = \lim_{\gamma\to\infty} g(\gamma) = \lim_{\gamma\to\infty} \frac{I_2}{2\gamma I_1 - \gamma^2}$$

$$= \lim_{\gamma\to\infty} \frac{1}{2\gamma\dfrac{I_1}{I_2} - \dfrac{\gamma^2}{I_2}} = \lim_{\gamma\to\infty} \frac{1}{2\gamma - \dfrac{\gamma^2}{I_2}} \quad .$$

Since

$$\lim_{\gamma\to\infty} \frac{\gamma^2}{I_2} = \lim_{\gamma\to\infty} \frac{\gamma^2}{\dfrac{e^\gamma}{(2\pi\gamma)^{1/2}}} = (2\pi)^{1/2} \lim_{\gamma\to\infty} \frac{\gamma^{5/2}}{e^\gamma} = 0$$

as L'Hospital's rule is applied repeatedly, therefore

$$\lim_{\gamma\to\infty} Y = \lim_{\gamma\to\infty} \frac{1}{2\gamma} = 0 \quad .$$

Next, we shall find the limit of Z and Y as $\gamma \to 0$. I functions may be expressed in power series as (Ref. 4)

$$I_n(\gamma) = i^{-n} J_n(i\gamma)$$

$$= \frac{(\frac{1}{2}\gamma)^n}{n!}\left[1 + \frac{(\frac{1}{2}\gamma)^2}{1(n+1)} + \frac{(\frac{1}{2}\gamma)^4}{1\cdot 2(n+1)(n+2)} + \frac{(\frac{1}{2}\gamma)^6}{1\cdot 2\cdot 3(n+1)(n+2)(n+3)} + \cdots\right].$$

Let

$$\frac{1}{2}\gamma = a,$$

then

$$I_0(\gamma) = 1 + a^2 + \frac{a^4}{4} + \frac{a^6}{36} + \cdots,$$

$$I_1(\gamma) = a\left[1 + \frac{a^2}{2} + \frac{a^4}{12} + \frac{a^6}{144} + \cdots\right],$$

$$I_2(\gamma) = \frac{a^2}{2}\left[1 + \frac{a^2}{3} + \frac{a^4}{24} + \frac{a^6}{360} + \cdots\right].$$

Thus

$$4I_0I_2 = 2a^2\left(1 + \frac{4}{3}a^2 + \frac{5}{8}a^4 + \cdots\right),$$

$$(I_0 - 1)^2 = a^4 + \frac{a^6}{2} + \cdots,$$

$$2I_1^2 = 2a^2\left[1 + a^2 + \frac{5}{12}a^4 + \cdots\right],$$

$$2I_2^2 = \frac{a^4}{2}\left[1 + \frac{2}{3}a^2 + \cdots\right].$$

Now Z can be written as

$$Z = \frac{4I_0I_2 - (I_0-1)^2 - 2I_1^2}{2I_2^2}$$

$$= \frac{\left(2a^2 + \frac{8a^4}{3} + \frac{5a^6}{4} + \dots\right) - \left(a^4 + \frac{a^6}{2} + \dots\right) - \left(2a^2 + 2a^4 + \frac{5}{6}a^6 \dots\right)}{\frac{a^4}{2} + \frac{a^6}{3} + \dots}$$

$$= \frac{-\frac{a^4}{3} - \frac{a^6}{12} + \dots}{\frac{a^4}{2} + \frac{a^6}{3} + \dots} .$$

Since $\gamma \longrightarrow 0$, as $a \longrightarrow 0$, we have

$$\lim_{\gamma \to 0} Z = \lim_{a \to 0} \left[\frac{-\frac{a^4}{3} - \frac{a^6}{12} + \dots}{\frac{a^4}{2} + \frac{a^6}{3} + \dots} \right]$$

$$= \lim_{a \to 0} \left[\frac{-\frac{1}{3} - \frac{a^2}{12} + \dots}{\frac{1}{2} + \frac{a^2}{3} + \dots} \right] = -\frac{2}{3} .$$

Similarly we have for Y

$$Y = \frac{I_2}{2\gamma I_1 - \gamma^2} = \frac{\frac{a^2}{2}\left[1 + \frac{a^2}{3} + \frac{a^4}{24} + \dots\right]}{4a\left[a + \frac{a^3}{2} + \frac{5}{12}a^5 + \dots\right] - 4a^2}$$

$$= \frac{\frac{a^2}{2} + \frac{a^4}{6} + \frac{a^6}{48} + \dots}{2a^4 + \frac{5}{6}a^6 + \dots} ,$$

and

$$\lim_{\gamma\to 0} Y = \lim_{a\to 0} \left(\frac{\frac{a^2}{2} + \frac{a^4}{6} + \frac{a^6}{48} + \ldots}{2a^4 + \frac{5}{6} a^6 \ldots} \right) = \infty \quad .$$

Summarizing, we have the following limits for new variables Z and Y:

$$Z \longrightarrow \frac{1}{2} , \quad Y \longrightarrow 0 \quad \text{as} \quad \gamma \longrightarrow \infty$$

$$Z \longrightarrow -\frac{2}{3} , \quad Y \longrightarrow \infty \quad \text{as} \quad \gamma \longrightarrow 0 \quad .$$

Thus, equation (41) may be expressed as

$$X = \int_Z^{1/2} Y dZ \ . \tag{43}$$

Equation (42) may be regarded as a parametric representation of a curve, and equation (43) shows that $X(\gamma)$ is the area under this curve in Y-Z coordinates in the interval $(Z, \frac{1}{2})$. Therefore, if we assign a set of values to γ , a table of points on this curve may be calculated. Then by numerically integrating equation (43), we obtain the corresponding value of X.

In this manner, values of X corresponding to various values of γ are calculated and listed in Table 1 (Ref. 1). Thus, by giving a value of γ , X can be found from this table. X being given, the velocity profile of this corresponding section is then defined by equation (15) or (33). Hence the velocity field in the entrance region is completely defined.

The Length of Entrance Region

As usual, the length of the entrance region is defined as the distance from the entry to a section where the center line velocity is 99 per cent of the fully developed one; that is,

$$u_0 = 0.99\ (u_0)_{F.D.}$$

$$\frac{u_0}{\bar{u}} = 0.99\ \frac{(u_0)_{F.D.}}{\bar{u}}\ .$$

However,

$$\frac{(u_0)_{F.D.}}{\bar{u}} = 2$$

hence

$$U_0 = \frac{u_0}{\bar{u}} = 0.99 \times 2 = 1.98.$$

This is equivalent to saying that the entrance length is the distance from the entry to a section where the dimensionless center line velocity, U_0, is equal to 1.98. The values of U_0 at various sections can be obtained from equation (33) with $R = 0$; that is,

$$U_0 = \frac{I_0(\gamma) - 1}{I_2(\gamma)}\ .$$

Values of U_0 are listed in Table 1. From this table

$$X_e = 0.227 \text{ when } U_0 = 1.98,$$

or

$$X_e = \frac{x_e}{R_0\ Re_r} = 0.227\ ,$$

$$\frac{x_e}{r_0 \frac{\bar{u}r_0}{\nu}} = 0.227 ,$$

$$\frac{4x_e}{D \frac{\bar{u}D}{\nu}} = 0.227 .$$

Thus we can conclude that

$$\frac{x_e}{D\ Re_D} = 0.0568 .$$

The Pressure Drop

In the entrance region, the pressure gradient $\frac{dp}{dx}$ is higher than in the fully developed flow, not only because of the increased friction loss but also because of the increase in kinetic energy of the fluid as it passes downstream.

The method which is employed to calculate the pressure drop involves an energy balance in the entrance region. The power which is dissipated by the fluid friction in the interval x = 0 to x = x is given by the dissipation function, as follows (Ref. 2):

$$E = \mu \iiint \left[2 \left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right] dV$$

$$= \mu \int_0^x dx \int_A \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right] dA$$

where A denotes the cross-sectional area. It is assumed that w and v may be neglected in this equation. Then, by the continuity equation, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, we conclude that $\frac{\partial u}{\partial x} = 0$.

Thus the preceding equation reduces to

$$E = \mu \int_0^x dx \int_A \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dA \ . \tag{44}$$

During the time interval dt, the net work of pressure on the body of fluid in the interval x = 0 to x = x is $\bar{u}A(p_0 - p)dt$. The work of friction on this body is -Edt, where E is given by equation (44). It is assumed that the tube is horizontal, so that the work of body force is zero. Since the work of all forces is equal to the change in kinetic energy, we have the following relation:

$$\bar{u}A(p_0 - p)dt - Edt = \left[\frac{1}{2} \rho \int_A (u^3 - \bar{u}^3)\, dA \right] dt$$

or

$$\bar{u}A(p_0 - p) = \mu \int_0^x dx \int_A \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dA + \frac{1}{2} \rho \int_A (u^3 - \bar{u}^3)\, dA \ . \tag{45}$$

In cylindrical coordinates, we have

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cos \theta, \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \sin \theta \ .$$

Hence

$$\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 \ .$$

From equations (23) and (35), we thus obtain

$$\int_A \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] dA = \frac{\bar{u}^2 \ \beta^2}{I_2{}^2(\gamma)} \int_0^{r_0} I_1{}^2(\beta r) 2\pi r dr$$

or

$$\int_A \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] dA = 2\pi \frac{\bar{u}^2 \ \beta^2}{I_2{}^2} \int_0^{r_0} I_1{}^2(\beta r) r dr$$

$$= 2\pi \frac{\bar{u}^2 \ \beta^2}{I_2{}^2(\gamma)} \left[\frac{r^2}{2}\left\{- I_1{}'^2(\beta r) + \left(1 + \frac{1}{\beta^2 r^2}\right) I_1{}^2(\beta r)\right\}\right]_0^{r_0}$$

$$= \frac{\pi \bar{u}^2 \beta^2 r_0{}^2}{I_2{}^2(\gamma)} \left[-I_1{}'^2(\beta r_0) + \frac{1 + \beta^2 \gamma_0{}^2}{\beta^2 r_0{}^2} I_1{}^2(\beta r_0)\right]$$

$$= \frac{\pi \bar{u}^2}{I_2{}^2(\gamma)} \left[-\left\{\gamma I_1(\gamma)\right\}^2 + (1+\gamma^2) \ I_1{}^2(\gamma)\right]$$

$$= \frac{\pi \bar{u}^2}{I_2{}^2(\gamma)} \left[-\left\{-I_1(\gamma) + \gamma I_0(\gamma)\right\}^2 + (1+\gamma^2) I_1{}^2(\gamma)\right]$$

$$= \frac{\pi \bar{u}^2}{I_2{}^2(\gamma)} \left[\gamma^2 I_1{}^2(\gamma) + 2\gamma I_1(\gamma) I_0(\gamma) - \gamma^2 I_0{}^2(\gamma)\right]$$

$$= \frac{\pi \bar{u}^2 \gamma^2}{I_2{}^2(\gamma)} \left[I_1{}^2(\gamma) + I_0(\gamma) \frac{2 I_1(\gamma)}{\gamma} - I_0{}^2(\gamma)\right]$$

$$= \frac{\pi \bar{u}^2 \gamma^2}{I_2{}^2(\gamma)} \left[I_1{}^2(\gamma) + I_0(\gamma)\left\{I_0(\gamma) - I_2(\gamma)\right\} - I_0{}^2(\gamma)\right]$$

$$= \frac{\pi \bar{u}^2 \gamma^2}{I_2^2(\gamma)} \left[I_1^2(\gamma) - I_0(\gamma)\, I_2(\gamma) \right] .$$

Thus, the first term of the right-hand side of equation (45) becomes

$$\mu \int_0^x dx \int_A \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dA$$

$$= \mu \pi \bar{u}^2 \int_0^x \frac{\gamma^2}{I_2^2} \left[I_1^2 - I_0 I_2 \right] dx$$

$$= \mu \pi \bar{u}^2 \int_0^X \frac{\gamma^2}{I_2^2} \left[I_1^2 - I_0 I_2 \right] d(r_0 \frac{\bar{u} r_0}{\nu} X)$$

$$= \rho \pi r_0^2 \, \bar{u}^3 \int_0^X \frac{\gamma^2}{I_2^2} \left[I_1^2 - I_0 I_2 \right] dX \quad . \qquad (46)$$

The second term of the right-hand side of equation (45) becomes

$$\frac{1}{2} \rho \int_A (u^3 - \bar{u}^3)\, dA = \frac{\rho}{2} \bar{u}^3 \int_0^{r_0} (U^3 - 1)\, 2\pi r dr$$

$$= \pi \rho \bar{u}^3 \int_0^1 (U^3 - 1)(r_0 R)\, d(r_0 R)$$

$$= \pi \rho \bar{u}^3 r_0^2 \int_0^1 (U^3 - 1) R dR \quad . \qquad (47)$$

Substituting equations (46) and (47) into equation (45) yields

$$\bar{u} \pi r_0^2 (p_0 - p) = \rho \pi r_0^2 \, \bar{u}^3 \int_0^X \frac{\gamma^2}{I_2^2} \left[I_1^2 - I_0 I_2 \right] dX + \rho \pi r_0^2 \, \bar{u}^3 \int_0^1 (U^3 - 1) R dR \quad .$$

Thus

$$p_0 - p = \rho \bar{u}^2 \int_0^X \frac{\gamma^2}{I_2^2} \left[I_1^2 - I_0 I_2 \right] dX + \rho \bar{u}^2 \int_0^1 (U^3 - 1) R dR$$

or

$$\frac{p_0 - p}{\frac{1}{2}\rho \bar{u}^2} = 2\int_0^X \frac{\gamma^2}{I_2^2}\left[I_1^2 - I_0 I_2\right] dX + 2\int_0^1 (U^3 - 1) R dR \ .$$

According to the Bernoulli equation, the pressure drop between a reservoir where the velocity is small and the inlet section is $\frac{1}{2}\rho \bar{u}^2$. Therefore, the total pressure drop between the supply reservoir and section x is $\frac{1}{2}\rho \bar{u}^2 + (p_0 - p)$, or

$$\frac{p_\infty - p}{\frac{1}{2}\rho \bar{u}^2} = 1 + 2\int_0^X \frac{\gamma^2}{I_2^2}\left[I_1^2 - I_0 I_2\right] dX + 2\int_0^1 (U^3 - 1) R dR \ . \qquad (48)$$

Let

$$\phi(\gamma) = \frac{\gamma^2}{I_2^2}\left[I_1^2 - I_0 I_2\right],$$

then we have

$$\int_0^X \frac{\gamma^2}{I_2^2}\left[I_1^2 - I_0 I_2\right] dX = \int_0^X \phi(\gamma)\, dX \ . \qquad (49)$$

The evaluation of this integral is difficult due to the fact that $\phi(\gamma) \longrightarrow \infty$ as $X \longrightarrow 0$ $(\gamma \longrightarrow \infty)$. Therefore we make use of a limiting relation. Since

$$f(\gamma) = \frac{4 I_0 I_2 - (I_0 - 1)^2 - 2 I_1^2}{2 I_2^2}$$

$$f'(\gamma) = 2 \cdot \frac{I_2^2 (I_0 I_2)' - (I_0 I_2)(I_2^2)'}{I_2^4}$$

$$- \frac{1}{2}\, \frac{I_2^2 \cdot 2(I_0 - 1) I_0' - (I_0 - 1)^2 (I_2^2)'}{I_2^4} - \frac{I_2^2 (I_1^2)' - I_1^2 (I_2^2)'}{I_2^4}$$

$$= 2\,\frac{I_2^2(I_0I_2'+I_2I_0')-2I_0I_2^2I_2'}{I_2^4} - \frac{I_2^2(I_0-1)I_0'-(I_0-1)^2I_2I_2'}{I_2^4}$$

$$- 2\,\frac{I_2^2I_1I_1' - I_1^2I_2I_2'}{I_2^4} \quad .$$

Substituting the relations $I_1' = I_0 - \frac{I_1}{\gamma}$, $I_2' = I_1 - \frac{2}{\gamma} I_2$,

and $I_0' = I_1$, we have

$$I_2^3 f'(\gamma) = 2\left\{I_2\left[I_0(I_1 - \frac{2}{\gamma} I_2) + I_1I_2\right] - 2I_0I_2\left[I_1 - \frac{2}{\gamma} I_2\right]\right\}$$

$$- I_2(I_0-1)I_1 + (I_0^2 - 2I_0 + 1)(I_1 - \frac{2}{\gamma} I_2)$$

$$- 2I_1I_2(I_0 - \frac{I_1}{\gamma}) + 2I_1^2(I_1 - \frac{2}{\gamma} I_2).$$

Simplifying and using the relation $\frac{2I_1}{\gamma} = I_0 - I_2$, we obtain

$$I_2^3 f'(\gamma) = - 6I_0I_1I_2 + 3I_1I_2^2 + I_0^2I_1 + 2I_1^3 + \frac{4}{\gamma} I_0I_2^2$$

$$- \frac{2}{\gamma} I_0^2I_2 + I_1I_2 - 2I_0I_1 + \frac{4}{\gamma} I_0I_2 + I_1 - \frac{2}{\gamma} I_2 \ . \tag{50}$$

For very large values of γ, the functions of I's are expressed in asymptotic series as

$$I_0(\gamma) \sim \frac{e^\gamma}{\gamma^2\sqrt{2\pi\gamma}}\left[\gamma^2 + \frac{\gamma}{8} + \frac{9}{128} + \gamma^2\varepsilon_0(\gamma)\right],$$

$$I_1(\gamma) \sim \frac{e^\gamma}{\gamma^2\sqrt{2\pi\gamma}}\left[\gamma^2 - \frac{3}{8}\gamma - \frac{15}{128} + \gamma^2\varepsilon_1(\gamma)\right], \tag{51}$$

$$I_2(\gamma) \sim \frac{e^\gamma}{\gamma^2\sqrt{2\pi\gamma}}\left[\gamma^2 - \frac{15}{8}\gamma + \frac{105}{128} + \gamma^2\varepsilon_2(\gamma)\right],$$

where $\lim_{\gamma\to\infty} \gamma^2\varepsilon_n(\gamma) = 0, \quad n = 0, 1, 2.$

Substituting equation (51) into equation (50) yields

$$\left[\gamma^2 - \frac{15}{8}\gamma + \frac{105}{128}\right]^3 f'(\gamma)$$

$$= -6\left[\gamma^2 + \frac{\gamma}{8} + \frac{9}{128}\right]\left[\gamma^2 - \frac{3}{8}\gamma - \frac{15}{128}\right]\left[\gamma^2 - \frac{15}{8}\gamma + \frac{105}{128}\right]$$

$$+ 3\left[\gamma^2 - \frac{3}{8}\gamma - \frac{15}{128}\right]\left[\gamma^2 - \frac{15}{8}\gamma + \frac{105}{128}\right]^2$$

$$+ \left[\gamma^2 + \frac{\gamma}{8} + \frac{9}{128}\right]^2\left[\gamma^2 - \frac{3}{8}\gamma - \frac{15}{128}\right] + 2\left[\gamma^2 - \frac{3}{8}\gamma - \frac{15}{128}\right]^3$$

$$+ \frac{4}{\gamma}\left[\gamma^2 + \frac{\gamma}{8} + \frac{9}{128}\right]\left[\gamma^2 - \frac{15}{8}\gamma + \frac{105}{128}\right]^2$$

$$- \frac{2}{\gamma}\left[\gamma^2 + \frac{\gamma}{8} + \frac{9}{128}\right]^2\left[\gamma^2 - \frac{15}{8}\gamma + \frac{105}{128}\right] + \frac{\gamma^2\sqrt{2\pi\gamma}}{e^{\gamma}} F(\gamma), \tag{52}$$

where $F(\gamma)$ is a polynomial function of γ resulting from the substitution of equation (51) into the expression

$I_1I_2 - 2I_0I_1 + \frac{4}{\gamma} I_0I_2 + I_1 - \frac{2}{\gamma} I_2$ in equation (50).

Since $\lim\limits_{\gamma\to\infty} \frac{\gamma^2\sqrt{2\pi\gamma}}{e^{\gamma}} F(\gamma) = 0$, equation (52) yields

$$\left(\gamma^2 - \frac{15}{8}\gamma + \frac{105}{128}\right)^3 f'(\gamma) = \gamma^4 + k'\gamma^3 + \ldots ,$$

after simplification, where k' is a numerical constant. From this we obtain

$$f'(\gamma) = \frac{1}{\gamma^2} + k\frac{1}{\gamma^3} + \ldots , \tag{53}$$

where k is a numerical constant. Moreover, substituting equation (51) into

$$g(\gamma) = \frac{I_2}{2\gamma I_1 - \gamma^2}$$

yields

$$g(\gamma) = \frac{1}{2\gamma} + \ldots \quad . \tag{54}$$

Therefore, from equations (41), (53), and (54), we obtain

$$X = \int_{\gamma}^{\infty} g(\gamma) f'(\gamma) d\gamma$$

$$= \frac{1}{2} \int_{\gamma}^{\infty} (\gamma^{-3} + \ldots) d\gamma = \frac{1}{4\gamma^2} + \ldots \quad .$$

It follows that

$$\lim_{\gamma \to \infty} \gamma^2 X = \frac{1}{4} \, .$$

It is pointed out by Langhaar (1) that setting $\gamma^2 X = \frac{1}{4}$ in the range $\gamma \geqq 100$ is a good approximation. With this approximation the integration of equation (49) becomes easy and is carried out as follows:

$$\int_0^X \phi(\gamma) dX = \int_0^{0.000026} \phi(\gamma) dX + \int_{0.000026}^X \phi(\gamma) dX$$

where X = 0.000026 corresponds to γ = 100. From equation (51), we have

$$\frac{I_0}{I_2} \sim 1 + \frac{2}{\gamma} + \frac{3}{\gamma^2} + \ldots$$

$$\frac{I_1}{I_2} \sim 1 + \frac{3}{2\gamma} + \frac{15}{8\gamma^2} + \ldots$$

$$\frac{I_1^{\,2}}{I_2^{\,2}} \sim 1 + \frac{3}{\gamma} + \frac{6}{\gamma^2} + \ldots .$$

Hence

$$\frac{I_1^{\,2}}{I_2^{\,2}} - \frac{I_0}{I_2} \sim \frac{1}{\gamma} + \frac{3}{\gamma^2} + \frac{5}{\gamma^3} + \ldots .$$

Making the approximation that $\gamma^2 X = \frac{1}{4}$ for $\gamma \geq 100$, we obtain

$$\begin{aligned}\phi(\gamma) &= \gamma^2 \left(\frac{I_1^{\,2}}{I_2^{\,2}} - \frac{I_0}{I_2}\right) \\ &\sim \gamma + 3 + \frac{5}{\gamma} \\ &\sim \frac{1}{2} X^{-1/2} + 3 + 10X^{1/2}\end{aligned}$$

and

$$\int_0^{0.000026} \phi(\gamma)\,dX = \int_0^{0.000026} \left(\frac{1}{2} X^{-1/2} + 3 + 10X^{+1/2}\right) dX$$

$$= 0.00518.$$

Therefore,

$$\int_0^{X} \phi(\gamma)\,dX = 0.00518 + \int_{0.000026}^{X} \phi(\gamma)\,dX .$$

The last term in this equation can be evaluated by a numerical method because it is a proper integral.

From equation (33), we see that U is a function of R for any cross section in the entrance region. If we let $(U^3 - 1)R = F(R)$, then

$$\int_0^1 (U^3-1)R\,dR = \int_0^1 F(R)\,dR .$$

This integral can be evaluated by Simpson's rule. Thus, the total pressure drop defined in equation (46) is obtained. Table 2 (Ref. 1) shows the values of the integrals

$$\int_0^X \phi(\gamma)\,dX \quad \text{and} \quad \int_0^1 (U^3-1)\,R\,dR$$

and the results of the pressure drop at different locations from the entry.

From this table, we may plot a curve of $\dfrac{p_\infty - p}{\frac{1}{2}\rho \bar{u}^2}$ against X. This curve determines the pressure drop, $p_\infty - p$, as a function of x. The slope of the curve is

$$\frac{d}{dX}\left(\frac{p_\infty - p}{\frac{1}{2}\rho \bar{u}^2}\right) = -\frac{1}{\frac{1}{2}\rho \bar{u}^2}\frac{dp}{dX} = -\frac{r_0{}^2}{(\frac{1}{2}\rho\nu\bar{u})}\frac{dp}{dx} .$$

In the fully developed laminar flow region (Hagan-Poiseville flow), $\dfrac{dp}{dx} = -\dfrac{8\rho\nu\bar{u}}{r_0{}^2}$. Thus the slope of this curve at the end of the entrance region is

$$-\frac{r_0{}^2}{(\frac{1}{2}\rho\nu\bar{u})}\left(-\frac{8\rho\nu\bar{u}}{r_0{}^2}\right) = 16.$$

Consequently the curve is asymptotic to a line with slope 16.

For a value X_1, somewhat greater than the entrance length X_e, and if $X > X_e$, equation (48) becomes

$$\frac{p_\infty - p}{\frac{1}{2}\rho\bar{u}^2} = 1 + 2\int_0^1 (U^3-1)R\,dR + 2\int_0^{X_1} \phi(\gamma)\,dX + 2\int_{X_1}^{X} \phi(\gamma)\,dX\ . \qquad (55)$$

In this case, the flow is fully developed; therefore, $U = 2(1-R^2)$, and

$$\int_0^1 (U^3-1)R\,dR = 8\int_0^1 (1-R^2)^3 R\,dR - \int_0^1 R\,dR$$

$$= 8\int_0^1 (R-3R^3+3R^5-R^7)\,dR - \frac{1}{2}$$

$$= 8\left[\frac{1}{2} - 3\cdot\frac{1}{4} + 3\cdot\frac{1}{6} - \frac{1}{8}\right] - \frac{1}{2} = \frac{1}{2}\,.$$

From the definition of $\phi(\gamma)$ and the power series for I functions, we have

$$\phi(\gamma) = \frac{\gamma^2}{I_2^2}(I_1^2 - I_0 I_2) = \frac{\gamma^2}{I_2}\left(\frac{I_1^2}{I_2} - I_0\right)$$

or

$$\phi(\gamma) = \frac{\gamma^2}{\frac{1}{8}\gamma^2 + \frac{1}{96}\gamma^4 + \ldots}\left[\frac{\frac{1}{4}\gamma^2 + \frac{1}{16}\gamma^4 + \ldots}{\frac{1}{8}\gamma^2 + \frac{1}{96}\gamma^4 + \ldots} - I_0\right]\ .$$

Thus

$$\lim_{\gamma\to 0}\phi(\gamma) = \lim_{\gamma\to 0}\frac{\gamma^2}{\frac{1}{8}\gamma^2 + \frac{1}{96}\gamma^4 + \ldots}$$

$$\left[\lim_{\gamma \to 0} \frac{\frac{1}{4}\gamma^2 + \frac{1}{16}\gamma^4 + \ldots}{\frac{1}{8}\gamma^2 + \frac{1}{96}\gamma^4 + \ldots} - \lim_{\gamma \to 0} I_0\right]$$

$$= 8\left[2 - 1\right] = 8 .$$

Since $x \longrightarrow \infty$ as $\gamma \longrightarrow 0$, it follows that in the fully developed region $\phi(\gamma) = 8 =$ constant. Therefore,

$$\int_{X_1}^{X} \phi(\gamma)\, dX = 8 \int_{X_1}^{X} dX = 8(X - X_1) .$$

By setting $\int_0^{X_1} \phi(\gamma) = K_1$, equation (55) gives

$$\frac{p_\infty - p}{\frac{1}{2}\rho \bar{u}^2} = 1 + 1 + 2K_1 + 16(X - X_1)$$

$$= 16X + 2K_1 + 2 - 16X_1 .$$

Let

$$2K_1 + 2 - 16X_1 = m,$$

then

$$\frac{p_\infty - p}{\frac{1}{2}\rho \bar{u}^2} = 16X + m .$$

We see from Table 2 that if $X_1 = 0.304$ ($\gamma = 0.4$), $K_1 = 2.5739$. Thus

$$m = 2 \times 2.5739 + 2 - 16 \times 0.304 = 2.28 .$$

The total pressure drop becomes

$$\frac{p_\infty - p}{\frac{1}{2}\rho \bar{u}^2} = 16X + 2.28$$

$$= \frac{64\ x}{DRe_D} + 2.28.$$

References

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APPENDIX III. SCHLICHTING'S MATCHING METHOD

Consider that the flow in a straight channel comprised of two flat parallel plates has uniform velocity distribution at the entry. Furthermore assume that the flow is incompressible and two-dimensional. In this method (1) the flow field is divided into two sections, the upstream section and the downstream section. In the upstream section, the boundary layers develop in the same way as they would along a flat plate at zero incidence. The integration is then performed in the downstream direction so that the boundary layers are calculated by finding the acceleration of the central stream. In the downstream section, the velocity distribution is assumed to be the sum of the fully developed profile and a deviation velocity which approaches zero as $x \longrightarrow \infty$. The integration is then performed in the upstream direction in order to find this deviation velocity. Having obtained both solutions in the form of series expansions, they are joined at a point where both solutions are valid. In this way an approximate description of the flow field in the entire entrance region is obtained.

Upstream Solution

Here the x-axis is fixed along one of the walls of the channel in the flow direction, and the ordinate is denoted by y' as shown in Fig. 8. In the downstream solution, however, the x-axis is taken as the center line of the channel and the

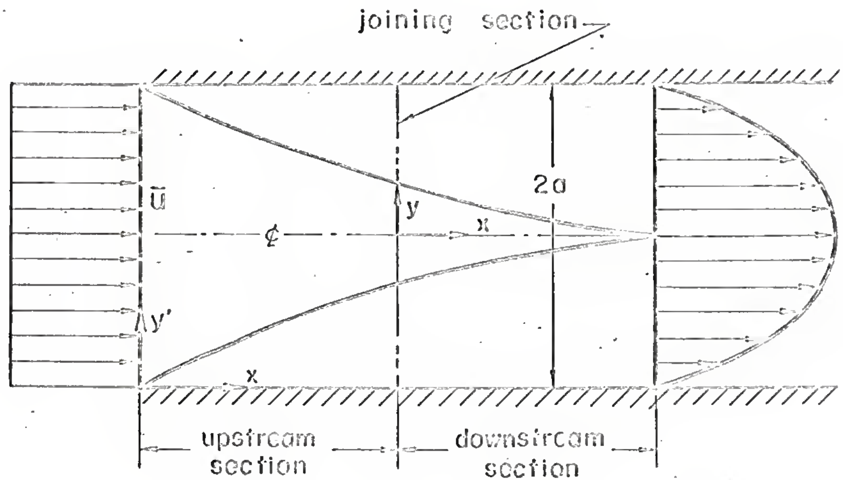

Fig. 8. Configeration of a channel with parallel plates for matching method.

ordinate y is measured from the center line. Following the usual assumptions, the boundary layer equation becomes (see Chapter 1, equation (11))

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad . \tag{1}$$

Let

$$\varepsilon = \sqrt{\frac{\nu x}{a^2 \bar{u}}}$$

and assume the center line velocity, u_0, to be in the following form

$$u_0(x) = \bar{u}(1 + k_1\varepsilon + k_2\varepsilon^2 + \ . \ . \ . \ ) \tag{2}$$

which is in a power series of $\sqrt{x}$. The coefficients k_1, k_2, ... are unknown.

According to Bernoulli's equation, the pressure gradient in equation (1) can be written as

$$- \frac{1}{\rho} \frac{dp}{dx} = u_0 \frac{du_0}{dx}$$

$$= \bar{u}(1 + k_1\varepsilon + k_2\varepsilon^2 + \ . \ . \ .)$$

$$\left[\bar{u}(k_1 + 2\varepsilon k_2 + 3\varepsilon^2 k_3 + \ . \ . \ .) \frac{d\varepsilon}{dx}\right]$$

where

$$\frac{d\varepsilon}{dx} = \frac{1}{2}\sqrt{\frac{\nu}{a^2 \bar{u} x}} = \frac{1}{2}\sqrt{\frac{\nu x}{a^2 \bar{u}}}\Big/ x = \frac{1}{2}\frac{\varepsilon}{x} \ .$$

Therefore, we have

$$- \frac{1}{\rho} \frac{dp}{dx} = \frac{\bar{u}^2 \varepsilon}{2x} (1 + k_1\varepsilon + k_2\varepsilon^2 + \ldots)(k_1 + 2\varepsilon k_2 + 3\varepsilon^2{}_3 + \ldots)$$

$$= \frac{\bar{u}^2 \varepsilon}{2x} \left[k_1 + (K_1{}^2 + 2K_2)\varepsilon + 3(K_1K_2+K_3)\varepsilon^2 + \ldots \right] . \quad (3)$$

Introducing a new variable

$$\eta = \sqrt{\frac{\bar{u}}{\nu x}}\, y'$$

and assuming that the stream function, $\Psi(x, y')$, can be expanded in a series in ε in a form similar to $u_0(x)$ as shown below

$$\Psi(x, y') = \bar{u}a\left[\varepsilon f_0(\eta) + \varepsilon^2 f_1(\eta) + \ldots\right]$$

we have

$$u(x, y') = \frac{\partial \Psi}{\partial y'} = \bar{u}a \left[\varepsilon f_0{}'(\eta) + \varepsilon^2 f_0{}'(\eta) + \ldots\right] \frac{\partial \eta}{\partial y'}$$

where

$$\frac{\partial \eta}{\partial y'} = \sqrt{\frac{\bar{u}}{\nu x}} = \sqrt{\frac{\bar{u}a^2}{\nu x}}\Big/ a = \frac{1}{a\,\varepsilon} .$$

Thus

$$u(x, y') = \bar{u}\left[f_0{}'(\eta) + \varepsilon f_1{}'(\eta) + \ldots\right]. \quad (4)$$

Similarly,

$$v(x, y') = -\frac{\partial \Psi}{\partial x}$$

$$= -\bar{u}a\left[f_0 + 2\varepsilon f_1 + 3\varepsilon^2 f_2 + \ldots\right]\frac{d\varepsilon}{dx}$$

$$-\bar{u}a\left[\varepsilon f_0{}' + \varepsilon^2 f_1{}' + \varepsilon^2 f_2{}' + \ldots\right]\frac{\partial \eta}{\partial x} .$$

Since

$$\eta = \sqrt{\frac{\bar{u}}{\nu x}}\, y' , \quad \frac{\partial \eta}{\partial x} = \sqrt{\frac{\bar{u}}{\nu}}\, y'(-\frac{1}{2}\, x^{-3/2}) = -\frac{1}{2x}\,\eta$$

and

$$\frac{d\varepsilon}{dx} = \frac{\varepsilon}{2x}$$

velocity v(x, y') becomes

$$v(x,y') = -\frac{\bar{u}a}{2x}\varepsilon\left[f_0 + 2\varepsilon f_1 + 3\varepsilon^2 f_2 + \ldots\right] + \frac{\bar{u}a\varepsilon\eta}{2x}$$

$$\left[f_0' + \varepsilon f_1' + \varepsilon^2 f_2' + \ldots\right] . \qquad (5)$$

From equation (4), we have

$$\frac{\partial u}{\partial x} = \bar{u}\left[f_1' + 2\varepsilon f_2' + 3\varepsilon^2 f_3' + \ldots\right]\frac{d\varepsilon}{dx}$$

$$+ \bar{u}\left[f_0'' + \varepsilon f_1'' + \varepsilon^2 f_2'' + \varepsilon^3 f_3'' + \ldots\right]\frac{\partial\eta}{\partial x}$$

$$= \frac{\varepsilon\bar{u}}{2x}\left[f_1' + 2\varepsilon f_2' + 3\varepsilon^2 f_3' + \ldots\right] - \frac{\bar{u}}{2x}\eta\left[f_0'' + \varepsilon f_1''\right.$$

$$\left. + \varepsilon^2 f_2'' + \varepsilon^3 f_3'' + \ldots\right] . \qquad (6)$$

$$\frac{\partial u}{\partial y'} = \bar{u}\left[f_0'' + \varepsilon f_1'' + \varepsilon^2 f_2'' + \ldots\right]\frac{\partial\eta}{\partial y'}$$

$$= \frac{\bar{u}}{a\varepsilon}\left[f_0'' + \varepsilon f_1'' + \varepsilon^2 f_2'' + \ldots\right] . \qquad (7)$$

$$\frac{\partial^2 u}{\partial y'^2} = \frac{\bar{u}}{\varepsilon a}\left[f_0''' + \varepsilon f_1''' + \varepsilon^2 f_2''' + \ldots\right]\frac{\partial\eta}{\partial y'}$$

$$= \frac{\bar{u}}{a^2\varepsilon^2}\left[f_0''' + \varepsilon f_1''' + \varepsilon^2 f_2''' + \ldots\right] . \qquad (8)$$

Substituting equations (3) through (8) into equation (1), we have

$$\frac{\varepsilon\bar{u}^2}{2x}\left[f_0' + \varepsilon f_1' + \varepsilon^2 f_2' + \ldots\right]\left[f_1' + 2\varepsilon f_2' + 3\varepsilon^2 f_3' + \ldots\right]$$

$$- \frac{\bar{u}^2}{2x} \eta \left[f_0' + \varepsilon f_1' + \varepsilon^2 f_2' + \ldots\right]\left[f_0'' + \varepsilon f_1'' + \varepsilon^2 f_2'' + \varepsilon^3 f_3'' + \ldots\right]$$

$$- \frac{\bar{u}^2}{2x} \left[f_0'' + \varepsilon f_1'' + \varepsilon^2 f_2'' + \ldots\right]\left[f_0 + 2\varepsilon f_1 + 3\varepsilon^2 f_2 + \ldots\right]$$

$$+ \frac{\bar{u}^2}{2x} \eta \left[f_0'' + \varepsilon f_1'' + \varepsilon^2 f_2'' + \ldots\right]\left[f_0' + \varepsilon f_1' + \varepsilon^2 f_2' + \ldots\right]$$

$$= \frac{\nu \bar{u}}{a^2 \varepsilon^2} \left[f_0''' + \varepsilon f_1''' + \varepsilon^2 f_2''' + \ldots\right] + \frac{\bar{u}^2 \varepsilon}{2x} \left[K_1 + (K_1^2 + 2K_2)\varepsilon + 3(K_1 K_2 + K_3)\varepsilon^2 + \ldots\right]$$

where

$$\frac{\nu \bar{u}}{a^2 \varepsilon^2} = \frac{\bar{u}^2}{x} .$$

Simplifying, we obtain

$$\varepsilon \left[f_0' + \varepsilon f_1' + \varepsilon^2 f_2' + \ldots\right]\left[f_1' + 2\varepsilon f_2' + 3\varepsilon^2 f_3' + \ldots\right]$$

$$- \left[f_0'' + \varepsilon f_1'' + \varepsilon^2 f_2'' + \ldots\right]\left[f_0 + 2\varepsilon f_1 + 3\varepsilon^2 f_2 + \ldots\right]$$

$$= 2 \left[f_0''' + \varepsilon f_1''' + \varepsilon^2 f_2''' + \ldots\right] + \varepsilon \left[K_1 + (K_1^2 + 2K_2)\varepsilon + 3(K_1 K_2 + K_3)\varepsilon^2 + \ldots\right]$$

Collecting terms in powers of ε gives a series of differential equations for f_0, f_1, f_2, ... The equation of the first approximation for the zero power of ε is

$$f_0 f_0'' + 2 f_0''' = 0 \tag{9}$$

with boundary conditions $f_0 = f_0' = 0$ at $\eta = 0$ and $f_0' = 1$ at $\eta \longrightarrow \infty$ as obtained from equations (4) and (5).

This is identical to the differential equation of flow over a flat plate given and solved by Blasius (1). The values of f_0, f_0', f_0'' can be found in the table (Ref. 1).

The second approximation (for the first power of ε) is determined by the equation

$$2f_1''' + f_0 f_1'' - f_0' f_1' + 2f_0'' f_1 = -K_1 \tag{10}$$

which is a linear differential equation of third order. The boundary conditions are $f_1 = f_1' = 0$ at $\eta = 0$ and $f_1' = K_1$ at $\eta = \infty$. The first two boundary conditions are obtained from equations (4) and (5); the last one is obtained from the right-hand sides of equations (2) and (4), since $u(x, y') = u_0(x)$ as $\eta \longrightarrow \infty$.

For the higher approximation, we can obtain equations of imilar type. From these equations, values of f and K can be found. Thus, the velocity distribution in this region is completely defined. However, a great number of terms would be needed in the series given by (4) if the entrance region problem is to be solved merely by this upstream solution. Each added term would make the solution more and more difficult and complex. Therefore, another mathematical analysis is applied in the downstream region where the deviation velocity from fully-developed flow is found.

Downstream Solution

Assume that

$$u(x, y) = u_f(y) - u^*(x, y) \tag{11}$$

where

$$u_f(y) = \frac{3}{2}\,\bar{u}(1 - \frac{y^2}{a^2}) \tag{12}$$

the fully-developed velocity distribution, and $u*(x, y)$ is a deviation of velocity from the fully developed one. Here, y is measured from the center line of the channel.

Substituting equation (11) into (1), we have

$$(u_f - u*)(-\frac{\partial u*}{\partial x}) + v(\frac{du_f}{dy} - \frac{\partial u*}{\partial y})$$
$$= \nu(\frac{d^2u_f}{dy^2} - \frac{\partial^2 u*}{\partial y^2}) - \frac{1}{\rho}\frac{dp}{dx} \quad . \tag{13}$$

Differentiating equation (13) with respect to y, the pressure term is eliminated because it is a function of x alone, as follows.

$$-\frac{du_f}{dy}\frac{\partial u*}{\partial x} - u_f\frac{\partial^2 u*}{\partial x \partial y} + \frac{\partial u*}{\partial y}\frac{\partial u*}{\partial x} + u*\frac{\partial^2 u*}{\partial x \partial y} + \frac{\partial v}{\partial y}(\frac{du_f}{dy} - \frac{\partial u*}{\partial y})$$
$$+ v(\frac{d^2u_f}{\partial y^2} - \frac{\partial^2 u*}{\partial y^2}) = \nu(\frac{d^3u_f}{\partial y^3} - \frac{\partial^3 u*}{\partial y^3}) \quad . \tag{14}$$

From equation (11), we have

$$\frac{\partial u}{\partial x} = -\frac{\partial u*}{\partial x} \quad . \tag{15}$$

Using the continuity equation, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, equation (15) becomes

$$\frac{\partial v}{\partial y} = \frac{\partial u*}{\partial x} \quad . \tag{16}$$

Substituting equation (16) into equation (14) and simplifying, we obtain

$$- u_f \frac{\partial^2 u*}{\partial x \partial y} + u* \frac{\partial^2 u*}{\partial x \partial y} + v \left(\frac{d^2 u_f}{dy^2} - \frac{\partial^2 u*}{\partial y^2}\right) = \nu \left(\frac{d^3 u_f}{dy^3} - \frac{\partial^3 u*}{\partial y^3}\right). \quad (17)$$

For the deviation velocity $u*(x, y)$, Schlichting assumed a series of the form

$$\frac{u*}{\bar{u}} = C_1 e^{-\lambda_1 \varepsilon^2} \varphi_1'\left(\frac{y}{a}\right) + C_2 e^{-\lambda_2 \varepsilon^2} \varphi_2'\left(\frac{y}{a}\right) + \ldots \quad .$$

In this series, only the first term is used; the higher order terms are neglected in the first approximation. Thus

$$\frac{u*}{\bar{u}} = C_1 e^{-\lambda_1 \varepsilon^2} \varphi_1'\left(\frac{y}{a}\right) \quad . \quad (18)$$

From equation (18), we have

$$\frac{\partial u*}{\partial y} = \frac{C_1}{a} \bar{u}\, e^{-\lambda_1 \varepsilon^2} \varphi_1''\left(\frac{y}{a}\right)$$

$$\frac{\partial u*}{\partial x} = \bar{u}\, C_1 \varphi_1'\left(\frac{y}{a}\right) \left[e^{-\lambda_1 \varepsilon^2} (-2\lambda_1 \varepsilon) \frac{d\varepsilon}{dx} \right]$$

$$= \bar{u}\, C_1 \varphi_1'\left(\frac{y}{a}\right) e^{-\lambda_1 \varepsilon^2} (-2\lambda_1 \varepsilon) \frac{\varepsilon}{2x}$$

$$= - \frac{\bar{u} \lambda_1 C_1 \varepsilon^2}{x} e^{-\lambda_1 \varepsilon^2} \varphi_1'\left(\frac{y}{a}\right) \quad (19)$$

$$\frac{\partial^2 u*}{\partial x \partial y} = - \frac{\bar{u} \lambda_1 C_1 \varepsilon^2}{ax} e^{-\lambda_1 \varepsilon^2} \varphi_1''\left(\frac{y}{a}\right) \quad (20)$$

$$\frac{\partial^2 u*}{\partial y^2} = \frac{\bar{u} C_1}{a^2} e^{-\lambda_1 \varepsilon^2} \varphi_1'''\left(\frac{y}{a}\right) \quad (21)$$

$$\frac{\partial^3 u*}{\partial y^3} = \frac{\bar{u} C_1}{a^3} e^{-\lambda_1 \varepsilon^2} \varphi_1^{iv}\left(\frac{y}{a}\right) \quad (22)$$

From equation (12), we have

$$\frac{du_f}{dy} = -\frac{3}{a^2}\bar{u}y$$

$$\frac{d^2u_f}{dy^2} = -\frac{3}{a^2}\bar{u} \qquad (23)$$

$$\frac{d^3u_f}{dy^3} = 0. \qquad (24)$$

From equations (16) and (19), we have

$$v = \int \frac{\partial u^*}{\partial x}\, dy$$

$$= -\frac{\bar{u}\lambda_1 C_1 \varepsilon^2}{x} \quad e^{-\lambda_1\varepsilon^2} \int \psi_1'(\frac{y}{a})\, dy$$

$$= -\frac{a\bar{u}\lambda_1 C_1 \varepsilon^2}{x} \quad e^{-\lambda_1\varepsilon^2}\, \psi_1(\frac{y}{a}) \ . \qquad (25)$$

Since

$$\varepsilon = \sqrt{\frac{\nu x}{a^2\bar{u}}}$$

$$\nu = \frac{\varepsilon^2 a^2 \bar{u}}{x} \ . \qquad (26)$$

Substituting equations (12) and (18) and equations (20) through (26) into equation (17) yields

$$\left[\frac{3}{2}\,\bar{u}(1 - \frac{y^2}{a^2})\right] \frac{\bar{u}\lambda_1 C_1 \varepsilon^2}{ax} \quad e^{-\lambda_1\varepsilon^2}\, \psi_1''$$

$$- \bar{u}C_1\; e^{-\lambda_1\varepsilon^2}\, \psi_1' \; \frac{\bar{u}\lambda_1 C_1 \varepsilon^2}{ax} \; e^{-\lambda_1\varepsilon^2}\, \psi_1''$$

$$- \frac{a\bar{u}\lambda_1 C_1 \varepsilon^2}{x} e^{-\lambda_1\varepsilon^2} \psi_1 (- \frac{3}{a^2} \bar{u} - \frac{\bar{u}C_1}{a^2} e^{-\lambda_1\varepsilon^2} \psi_1''')$$

$$= \frac{\varepsilon^2 a^2 \bar{u}}{x} (0 - \frac{\bar{u}C_1}{a^3} e^{-\lambda_1\varepsilon^2} \psi_1^{iv}) .$$

Simplifying the above equation, we obtain

$$\psi_1^{iv} + 3\lambda_1 \left[\psi_1 + \frac{1}{2} (1 - \frac{y^2}{a^2}) \psi_1''\right]$$

$$= C_1\lambda_1 e^{-\lambda_1\varepsilon^2} (\psi_1' \psi_1'' - \psi_1 \psi_1''') .$$

The right-hand side of the above equation is zero as higher order terms of ψ_1 can be neglected. Then we obtain the following equation.

$$\psi_1^{iv} + 3\lambda_1 \left[\psi_1 + \frac{1}{2} (1 - \frac{y^2}{a^2}) \psi_1''\right] = 0 \qquad (27)$$

with λ_1 as its eigenvalue. The boundary conditions

$$v = 0 \text{ and } \frac{\partial u}{\partial y} = 0 \text{ at } y = 0$$

$$v = 0 \text{ and } u = 0 \text{ at } y = a$$

are equivalent to

$$\psi_1 = \psi_1'' = 0 \quad \text{at} \quad y = 0 \qquad (28)$$

$$\psi_1 = \psi_1' = 0 \text{ at } y = \pm a . \qquad (29)$$

The other boundary condition is chosen as

$$\psi_1' = 1 \quad \text{at} \quad y = 0 . \qquad (30)$$

This is possible because C_1 is still free. Having these boundary conditions, the equation (27) can be solved by a power series method.

Let

$$\psi_1\left(\frac{y}{a}\right) = \sum_{n=0}^{\infty} A_n\left(\frac{y}{a}\right)^n$$

$$= A_0 + A_1\left(\frac{y}{a}\right) + A_2\left(\frac{y}{a}\right)^2 + \ldots .$$

Then we have

$$\psi_1' = \sum_{n=1}^{\infty} nA_n\left(\frac{y}{a}\right)^{n-1}$$

$$= A_1 + 2A_2\left(\frac{y}{a}\right) + 3A_3\left(\frac{y}{a}\right)^2 + \ldots ,$$

$$\psi_1'' = \sum_{n=2}^{\infty} n(n-1)A_n\left(\frac{y}{a}\right)^{n-2}$$

$$= 2A_2 + 6A_3\left(\frac{y}{a}\right) + 12A_4\left(\frac{y}{a}\right)^2 + \ldots ,$$

$$\psi_1''' = \sum_{n=3}^{\infty} n(n-1)(n-2)A_n\left(\frac{y}{a}\right)^{n-3}$$

$$= 6A_3 + 24A_4\left(\frac{y}{a}\right) + 60A_5\left(\frac{y}{a}\right)^2 + \ldots ,$$

$$\psi_1^{iv} = \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)A_n\left(\frac{y}{a}\right)^{n-4}$$

$$= 24A_4 + 120A_5\left(\frac{y}{a}\right) + 360A_6\left(\frac{y}{a}\right)^2 + \ldots .$$

Substituting the above series for ψ_1, ψ_1'', and ψ^{iv} into equation (27), we obtain

$$\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)A_n\left(\frac{y}{a}\right)^{n-4}$$

$$+ 3\lambda_1 \left[\sum_{n=0}^{\infty} A_n \left(\frac{y}{a}\right)^n + \frac{1}{2} \left(1 - \frac{y^2}{a^2}\right) \sum_{n=2}^{\infty} n(n-1) A_n \left(\frac{y}{a}\right)^{n-2} \right] = 0$$

or

$$\sum_{n=0}^{\infty} (n+4)(n+3)(n+2)(n+1) A_{n+4} \left(\frac{y}{a}\right)^n$$

$$+ 3\lambda_1 \left[\sum_{n=0}^{\infty} A_n \left(\frac{y}{a}\right)^n + \frac{1}{2} \sum_{n=0}^{\infty} (n+2)(n+1) A_{n+2} \left(\frac{y}{a}\right)^n \right.$$

$$\left. - \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) A_n \left(\frac{y}{a}\right)^n \right] = 0 .$$

Therefore, we can write

$$4! A_4 + 3\lambda_1 \left[A_0 + \frac{1}{2} (2 \cdot 1) A_2 \right] = 0 ,$$

$$5 \cdot 4 \cdot 3 \cdot 2 A_5 + 3\lambda_1 \left[A_1 + \frac{1}{2} (3 \cdot 2) A_3 \right] = 0,$$

and

$$(n+4)(n+3)(n+2)(n+1) A_{n+4}$$

$$+ 3\lambda_1 \left\{ \left[1 - \frac{1}{2} n(n-1) \right] A_n + \frac{1}{2} (n+2)(n+1) A_{n+2} \right\} = 0 .$$

The general recursion formula for the coefficient then is

$$A_{n+4} = \frac{3\lambda_1 \left[(n-2)(n+1) A_n - (n+1)(n+2) A_{n+2} \right]}{2(n+1)(n+2)(n+3)(n+4)} . \tag{31}$$

From the boundary conditions shown in equation (28), we obtain $A_0 = A_2 = 0$. Therefore A_4 and all other constants of even subscripts are zero as seen from recursion equation (31). All constants with odd subscripts from five to infinity can be represented in terms of A_1 and A_3, therefore, A_1 and A_3 are left as arbitrary constants. From the boundary condition given in equation (30), A_1 is found to be 1. The remaining two boundary conditions $\psi_1(\pm 1) = 0$ and $\psi_1'(\pm 1) = 0$ as shown in equation (29) are used for determining A_3 and the eigenvalue λ_1. Since these boundary conditions are in the form of an infinite series, A_3 and λ_1 are found by a trial and error method. First, a value of λ_1 is chosen, then different values of A_n are found in terms of A_3 by equation (31). Using one of the remaining boundary conditions, A_3 is found. Then the remaining boundary condition is used to check the values of A_3 and λ_1. If these values do not satisfy the last boundary condition, a new value of λ_1 is chosen and the process is repeated until the correct value of λ_1 is found. Schlichting gave the value as $\lambda_1 = 18.75$. In this manner, all the constant terms can be found. Substituting these constants into the assumed series for $\psi_1(\frac{y}{a})$, $\psi_1'(\frac{y}{a})$ is defined for various values of $\frac{y}{a}$.

For the complete solution of the velocity profiles, only C_1 and the values of ϵ at the joining point remain to be found. These are found by matching the upstream and downstream solutions.

Upstream Pressure Distribution

Near the entrance of the channel, the boundary layers are very thin. Most of the flow is composed of the accelerating potential flow in the central core of the channel. Therefore, the pressure distribution is

$$- \frac{1}{\rho} \frac{dp}{dx} = u_0 \frac{du_0}{dx} .$$

Integrating from $x = x_0$, where $p = p_0$ and $u_0 = \bar{u}$, to any section in the upstream region gives

$$-\frac{1}{\rho}\int_{p_0}^{p} dp = \frac{1}{2}\int_{\bar{u}}^{u_0} d(u_0^2)$$

or

$$p - p_0 = \frac{1}{2}\rho(\bar{u}^2 - u_0^2)$$

or

$$\frac{p - p_0}{\frac{1}{2}\rho\,\bar{u}^2} = 1 - \left(\frac{u_0}{\bar{u}}\right)^2 . \tag{32}$$

Downstream Pressure Distribution

From equation (13), we have

$$\frac{1}{\rho}\frac{dp}{dx} = \nu\left(\frac{d^2u_f}{dy^2} - \frac{\partial^2 u^*}{\partial y^2}\right) + (u_f - u^*)\frac{\partial u^*}{\partial x} - v\left(\frac{du_f}{dy} - \frac{\partial u^*}{\partial y}\right)$$

Since the pressure varies with x only, $\frac{1}{\rho}\frac{dp}{dx}$ can be evaluated at $y = \pm a$. Therefore all terms except the first two on the right-hand side of the above equation vanish. The equation then becomes

$$\frac{1}{\rho}\frac{dp}{dx} = \nu\left(\frac{d^2u_f}{dy^2} - \frac{\partial^2 u^*}{\partial y^2}\right) \tag{33}$$

Substituting equations (21) and (23) into equation (33), we have

$$\frac{1}{\rho}\frac{dp}{dx} = \nu\left[-\frac{3}{a^2}\bar{u} - \frac{\bar{u}C_1}{a^2}e^{-\lambda_1\epsilon^2}\psi_1'''(1)\right] \tag{34}$$

It is to be noted that $\psi_1'''(\frac{y}{a})$ is defined for all values of $\frac{y}{a}$, since $\psi_1(\frac{y}{a})$ has been found. By definition we have $\varepsilon = \sqrt{\frac{\nu x}{a^2\bar{u}}}$. Therefore

$$dx = 2aRe_a \varepsilon d\varepsilon \ .$$

Let the value of ε at joining point be denoted by ε^*. Integrating equation (34) from $\varepsilon = \varepsilon^*$, where $p = p^*$, to ε, one obtains

$$\frac{1}{\rho}\int_{p^*}^{p} dp = -\frac{3\nu\bar{u}}{a^2} \cdot 2aRe_a \int_{\varepsilon^*}^{\varepsilon} \varepsilon d\varepsilon - \frac{\nu\bar{u}C_1}{a^2}\psi_1'''(1)\cdot 2aRe_a \int_{\varepsilon^*}^{\varepsilon} e^{-\lambda_1\varepsilon^2} d\varepsilon$$

$$\frac{p - p^*}{\frac{1}{2}\rho\,\bar{u}^2} = -\frac{6Re_a}{\frac{\bar{u}a}{\nu}}(\varepsilon^2 - \varepsilon^{*2}) + \frac{2C_1}{\lambda_1}\,\frac{Re_a}{\frac{\bar{u}a}{\nu}}\,\psi_1'''(1)(e^{-\lambda_1\varepsilon^2} - e^{-\lambda_1\varepsilon^{*2}})$$

or

$$\frac{p - p^*}{\frac{1}{2}\rho\,\bar{u}^2} = -6(\varepsilon^2 - \varepsilon^{*2}) + \frac{2C_1}{\lambda_1}\psi_1'''(1)(e^{-\lambda_1\varepsilon^2} - e^{-\lambda_1\varepsilon^{*2}}) \ . \quad (35)$$

For the complete solution of the pressure distribution, only C_1 and ε^* remain to be evaluated. These values are determined in the process of matching the upstream and downstream solutions.

Matching of Upstream and Downstream Solutions

The upstream and downstream solutions of velocity and pressure distribution are to be matched at a point where both solutions are valid. Because there are two undetermined quantities,

C_1 and ε^*, in the equations to be joined, two conditions are needed to solve for them. One condition is that the center line velocities of both solutions are equal at the joining point. Another condition used is that the slopes of pressure distribution $\frac{dp}{dx}$, for the two solutions match at the joining point. These two conditions are used to develop two simultaneous equations from which C_1 and ε^* are determined.

The downstream center line velocity, from equations (11) and (18), is

$$u_0 = u_f(0) - \bar{u}C_1 e^{-\lambda_1 \varepsilon^2} \psi_1{}'(0) \ .$$

Because of equation (12) and the boundary condition, equation (30), the above equation can be written as

$$u_0 = \frac{3}{2}\bar{u} - \bar{u}C_1 e^{-\lambda_1 \varepsilon^2} \ . \qquad (36)$$

Equating the center line velocity represented by equations (2) and (36) gives

$$1 + K_1\varepsilon + K_2\varepsilon^2 + K_3\varepsilon^3 + K_4\varepsilon^4 = \frac{3}{2} - C_1 e^{-\lambda_1 \varepsilon^2} \ . \qquad (37)$$

It is to be noted that the infinite series of equation (2) is terminated at the fifth term because only K_1, K_2, K_3, and K_4 have been found (Ref. 2).

The upstream pressure slope equation is

$$\frac{dp}{dx} = -\rho u_0 \frac{du_0}{dx} \ .$$

Substituting the center line velocity in the upstream section gives

$$\frac{dp}{dx} = -\rho\bar{u}^2(1 + K_1\varepsilon + K_2\varepsilon^2 + K_3\varepsilon^3 + K_4\varepsilon^4)(K_1 + 2K_2\varepsilon + 3K_3\varepsilon^2 + 4K_4\varepsilon^3)\frac{d\varepsilon}{dx}$$

$$= \frac{-\bar{u}^2\rho}{2\varepsilon a Re_a}(1 + K_1\varepsilon + K_2\varepsilon^2 + K_3\varepsilon^3 + K_4\varepsilon^4)(K_1 + 2K_2\varepsilon + 3K_3\varepsilon^2 + 4K_4\varepsilon^3) \quad (38)$$

Equating equation (34), the pressure slope equation in the downstream section, and equation (38), we have

$$\frac{3\nu\bar{u}}{a^2} + \frac{\nu\bar{u}C_1}{a^2}e^{-\lambda_1\varepsilon^2}\psi_1''' = \frac{\bar{u}^2}{2\varepsilon a\, Re_a}(1 + K_1\varepsilon + K_2\varepsilon^2 + K_3\varepsilon^3 + K_4\varepsilon^4)(K_1 + 2K_2\varepsilon + 3K_3\varepsilon^2 + 4K_4\varepsilon^3)$$

or

$$3 + C_1e^{-\lambda_1\varepsilon^2}\psi_1'''(1) = \frac{1}{2\varepsilon}(1 + K_1\varepsilon + K_2\varepsilon^2 + K_3\varepsilon^3 + K_4\varepsilon^4)(K_1 + 2K_2\varepsilon + 3K_3\varepsilon^2 + 4K_4\varepsilon^3) \quad (39)$$

Eliminating $C_1e^{-\lambda_1\varepsilon^2}$ from equations (37) and (39) gives a seventh order equation in ε as follows (Ref. 2).

$$K_1 + \varepsilon\left[K_1^2 + 2K_2 - 6 - \psi_1'''(1)\right] + \varepsilon^2\left[3K_1K_2 + 3K_3 + 2K_1\psi_1'''(1)\right] + \varepsilon^3\left[4K_1K_3 + 2K_2^2 + 4K_4 + 2K_2\psi_1'''(1)\right] + \varepsilon^4\left[5K_1K_4 + 5K_2K_3 + 2K_3\psi_1'''(1)\right] + \varepsilon^5\left[6K_2K_4 + 3K_3^2 + 2K_4\psi_1'''(1)\right] + 7K_3K_4\varepsilon^6 + 4K_4^2\varepsilon^7 = 0 \quad (40)$$

The values of K's were not given in Schlichting's paper. In Ref. (3) they are $K_1 = 1.7207$, $K_2 = -2.238$, $K_3 = 19.81$, and

K_4 = -208.896 (2). Schlichting gave the value ε at the matching point, that is, $\varepsilon = \varepsilon^*$, as 0.1264, and substituting this value into equation (37) yields $C_1 = 0.3485$.

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ANALYTICAL AND NUMERICAL SOLUTIONS OF SOME ENTRANCE REGION FLOW PROBLEMS

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B. S., National Taiwan University, 1958

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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Manhattan, Kansas

1966

For many years, the study of the developing velocity profiles and pressure drops in the hydrodynamic entrance region of tubes and ducts has attracted the interests of many investigators. Although the exact analytical solution is not available at the present time, many approximate solutions have been presented. The approximation methods of solution can be classified into four different categories: the momentum integral method, linearizing method, matching method, and finite difference method.

The purpose of this report is to present a detailed study of these four different approximation methods and to present the results obtained by applying a finite difference method in solving an MHD entrance region problem in which the velocity profiles are developed from a nonmagnetically fully developed one (parabolic) at the entry to a Hartmann velocity profile.

A brief introduction of the Navier-Stokes equation and the boundary layer equations is given in Chapter 1. In Chapter 2 Schiller's solution by the momentum integral method for a flow in a circular tube is presented in detail. A momentum integral equation is derived first. Then by applying the assumed velocity profile, which involves the boundary layer thickness, to the momentum integral equation, a relation between the boundary layer thickness and the axial coordinate is obtained. The solution is completed after carrying out the integration.

Chapter 3 contains a detailed account of Langhaar's linearizing method for the flow in a circular tube. The nonlinear partial differential equation of motion for the axial direction is linearized by the assumption that the convective terms are a function of axial direction only. The solution is presented in closed form in terms of Bessel functions.

Chapter 4 concerns Schlichting's matching method in solving the entrance region problem of a flow in a parallel plate channel. The flow is divided into two sections, the upstream and downstream sections. The solution for the upstream section corresponds to the Blasius series solution over a flat plate, and the solution representing the progressive deviation of the profile from its asymptotic parabolic distribution is obtained in a series form for the downstream section. Solutions for these two separate sections are then matched at a point where both solutions are applicable.

In Chapter 5, Hwang and Fan's finite difference method is applied in solving the MHD entrance region problem with the new boundary conditions mentioned above. The continuity equation and boundary layer equation are reduced to finite difference equations and solved on a digital computer. The results obtained are compared with those obtained by a momentum integral method.