ALGEBRAIC FUNCTIONS

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TABLE OF CONTENTS

INTRODUCTION

Algebraic functions, in many ways, are just a generalization of algebraic numbers. Algebraic numbers are the roots of a polynomial equation with rational coefficients while an algebraic function consists of the roots of a polynomial equation with rational functions as coefficients. In the first case, the equation has multiple roots if the discriminant is zero, while in the second, one finds for what complex values the equation has repeated roots by setting the discriminant equal to zero, and solving the resulting equation. Furthermore, the algebraic numbers of a given equation can be found by use of a procedure known as Newton's method. Likewise, an algebraic function, defined by a certain equation, can be found by a generalization of Newton's method.

This report is divided into three sections. In the first section we consider some of the properties of the resultant and discriminant for polynomials in one and two variables. These properties are used in sections two and three to determine the critical points and reducibility of a given equation.

In the second section we prove that an algebraic function is simply a certain type of analytic function. The proof involves finding the nature of the expansions at the different types of critical points. However, the actual determination of these expansions is left until the third section.

The third section shows how Newton's method can be generalized to actually find the algebraic function which corresponds to a given equation. The method is illustrated by several examples.

The following theorems will be used throughout this paper.

1. Theorem of Monodromy. If D is a simply connected domain, if a is in D, and if a power series f(z;a) can be continued analytically along every path in D, then the continuations define a holomorphic function in D.

2. Theorem of Rouche'. Let C be a simple closed rectifiable curve, and suppose f(z) is meromorphic in the interior of C, and on C, but has neither zeros or poles on C. If f(z) can be written as the sum of two functions g(z) and h(z) meromorphic in the interior of C, and on C, such that $g(z) \neq 0, \infty$ on C, and $|g(z)| > |h(z)|$ on C then the change in the argument of f(z) when z describes C is the same as the change in the argument of g(z), and the difference between the number of zeros and number of poles is the same for both functions.

3. Identity Theorem. Suppose that f(z) and g(z) are two functions holomorphic in the domains D_1 and D_2 respectively. Suppose that D_1 and D_2 intersect in a domain D ($D = D_1 \cap D_2$) and there exists an infinite sequence of distinct points z_n in D having at least one limit point in D such that $f(z_n) = g(z_n)$ $n = 1, 2, 3, \ldots$ Then f(z) = g(z) in D, and g(z) is the analytic continuation of f(z) in D_2, while f(z) is the analytic continuation of g(z) in D_1.

RESULTANTS AND DISCRIMINANTS

In our study of algebraic functions we shall need some of the properties of the resultant of two polynomials and of the discriminant of a single polynomial. With regard to this need, we make the following definition.

Definition 1.1 Let $f(z) = f_o z^n + f_1 z^{n-1} + \ldots + f_n$, and $g(z) = g_o z^m + g_1 z^{m-1} + \ldots + g_m$ be two polynomials in the complex variable z with coefficients from the complex field, and suppose $g_o \neq 0$, $f_o \neq 0$. Then the resultant, $R(f, g)$, is defined to be

$$R(f,g) = \begin{vmatrix} f_o & f_1 \ldots f_n & & 0 \ldots 0 \\ 0 & f_o \ldots & f_{n-1} & f_n \ldots 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 \ldots 0 & f_o & f_1 \ldots & f_n \\ g_o & g_1 \ldots g_m & & 0 \ldots 0 \\ 0 & g_o \ldots & g_{m-1} & g_m \ldots 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 \ldots 0 & g_o & g_1 \ldots & g_m \end{vmatrix} \begin{matrix} \text{m times} \\ \\ \\ \\ \text{n times} \\ \\ \\ \\ \end{matrix}$$

Theorem 1.1 A necessary and sufficient condition that the polynomials f and g have a root in common is $R(f, g) = 0$.

To prove the necessity, we consider the system of equations

$$\begin{aligned} f_o z^{n+m-1} + f_1 z^{n+m-2} + \ldots + f_n z^{m-1} &= 0 \\ f_o z^{n+m-2} + f_1 z^{n+m-3} + \ldots + f_n z^{m-2} &= 0 \\ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \end{aligned}$$

$$f_o z^n + f_1 z^{n-1} + \ldots + f_n = 0$$

$$g_o z^{n+m-1} + g_1 z^{n+m-2} + \ldots + g_m z^{n-1} = 0$$

$$g_o z^{n+m-2} + g_1 z^{n+m-3} + \ldots + g_m z^{n-2} = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$g_o z^m + g_1 z^{m-1} + \ldots + g_m = 0$$

In matrix form, this system of equations can be written,

$$\begin{bmatrix} f_o & f_1 \ldots f_n & & 0 \ldots 0 \\ 0 & f_o & f_{n-1} & f_n \ldots 0 \\ & \cdot & \cdot & \cdot \quad \cdot \\ 0 \ldots 0 & f_o\, f_1 & \ldots & f_n \\ g_o & g_1 \ldots g_m & & 0 \ldots 0 \\ 0 & g_o \ldots g_{m-1} & & g_m \ldots 0 \\ & \cdot & \cdot & \cdot \quad \cdot \\ 0 \ldots 0 & g_o\, g_1 & \ldots & g_m \end{bmatrix} \begin{bmatrix} z^{n+m-1} \\ z^{n+m-2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

If we make the substitution $x_{n+m-1} = z^{n+m-1}$, and regard the x's as independent variables, we have a system of n + m homogeneous equations in n + m unknowns. Now suppose z = a is a common root of f(z) and g(z). That is, f(a) = 0, g(a) = 0. Then $x_{n+m-1} = a^{n+m-1}$, and we see that each of the n + m homogeneous equations is satisfied. Therefore, the column vector $\begin{bmatrix} a^{n+m-1} \\ a^{n+m-2} \\ \cdot \\ \cdot \\ \cdot \\ a \\ 1 \end{bmatrix}$ is a colution of this system of equations. Since the solution is obviously nontrivial, we conclude that

$$R(f,g) = \begin{bmatrix} f_o & f_1 \cdots f_n & 0 \cdots 0 \\ 0 & f_o \cdots f_{n-1} & f_n \cdots 0 \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ 0 \cdots 0 & f_o \; f_1 \; \cdots & f_n \\ g_o \cdots\cdots g_m & & 0 \cdots 0 \\ 0 & g_o \quad g_{m-1} & g_m \cdots 0 \\ 0 \cdots 0 & g_o \quad \cdots & g_m \end{bmatrix} = 0$$

To prove the sufficiency, we note that the resultant can be written as

$$R(f,g) = f_o^{\,m} g_o^{\,n} \begin{vmatrix} 1 & \frac{f_1}{f_o} & \cdots & \frac{f_n}{f_o} & 0 \cdots 0 \\ 0 & 1 & \cdots & \frac{f_{n-1}}{f_o} \; \frac{f_n}{f_o} & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 \cdots 0 & 1 & \frac{f_1}{f_o} & \frac{f_2}{f_o} \cdots & \frac{f_n}{f_o} \\ 1 & \frac{g_1}{g_o} & \cdots & \frac{g_m}{g_o} & 0 \cdots 0 \\ 0 & 1 & \cdots & \frac{g_{m-1}}{g_o} \; \frac{g_m}{g_o} & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 \cdots 0 & 1 & \frac{g_1}{g_o} & \cdots & \frac{g_m}{g_o} \end{vmatrix}$$

since f_o and g_o are not equal to zero.

Now, let $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_m$ be the roots of f(z) and g(z) respectively. By the Factor Theorem,

$$f(z) = f_o(z-a_1)(z-a_2)\ldots(z-a_n),$$

$$g(z) = g_o(z-b_1)(z-b_2)\ldots(z-b_m).$$

We then have the following relationships.

$$-(a_1 + a_2 + \ldots + a_n) = \frac{f_1}{f_o} \qquad -(b_1 + b_2 + \ldots + b_m) = \frac{g_1}{g_o}$$

$$(-1)^2 \sum_{i<j} a_i a_j = \frac{f_2}{f_o} \qquad (-1)^2 \sum_{i<j} b_i b_j = \frac{g_2}{g_o}$$

$$(-1)^k \sum_{i_1<i_2<\ldots<i_k} a_{i_1} a_{i_2} \ldots a_{i_k} = \frac{f_k}{f_o} \qquad (-1)^k \sum_{t_1<t_2<\ldots<t_k} b_{t_1} b_{t_2} \ldots b_{t_k} = \frac{g_k}{g_o}$$

$$1 \leqq k \leqq n \qquad 1 \leqq k \leqq m$$

From these relationships, we see that R is a polynomial in the roots a_i and b_j which we denote by $R = R(a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_m)$. We shall show that R is homogeneous and of degree mn in the roots. That is, $R(pa_1, pa_2, \ldots pa_n, pb_1, \ldots pb_m) = p^{mn}R(a_1, \ldots, a_n, b_1, \ldots b_m)$.

The preceding formulas show that multiplying each root by a factor p multiplies $\frac{f_k}{f_o}$ and $\frac{g_i}{g_o}$ by p^k and p^i respectively.

Therefore,

$$R(pa_1, \ldots, pa_n, pb_1, \ldots pb_m) = f_o^m g_o^n \begin{vmatrix} 1 & p\frac{f_1}{f_o} \ldots p^n\frac{f_n}{f_o} & 0 \ldots 0 \\ 0 \quad 1 & \ldots p^{n-1}\frac{f_{n-1}}{f_o} & \ldots 0 \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ 0 \ldots 0 & p\frac{f_1}{f_o} \ldots & p^n\frac{f_n}{f_o} \\ 1 & p\frac{g_1}{g_o} \ldots \quad p^n\frac{g_m}{g_o} & 0 \ldots 0 \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ 0 \ldots 0 & p\frac{g_1}{g_o} \ldots & p^m\frac{g_m}{g_o} \end{vmatrix}$$

We multiply the k row of R by p^{k-1} for $k = 1, \dots m$, and the (m+i)th row by p^{i-1} for $i = 1, 2, \dots n$. At the same time we take a factor $p^{-(1+\dots+n-1+1+\dots+m-1)}$ outside the determinant. The value of the determinant is unchanged, and we have,

$$R\,(pa_1, \dots, pa_n, pb_1, \dots pb_m) =$$

$$p^{-(1+\dots+n-1+1+\dots+m-1)} f_o^{\,m} g_o^{\,n} \begin{vmatrix} 1 & p\frac{f_1}{f_o} \dots & p^n\frac{f_n}{f_o} & & 0 \dots 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 \dots 0 & p^{m-1} \dots & p^{n+m-1} & & \frac{f_n}{f_o} \\ 1 & p\frac{g_1}{g_o} & \dots p^m\frac{g_m}{g_o} & & \dots\dots 0 \\ \cdot & \cdot & \cdot & & \cdot \\ 0 \dots 0 & p^{n-1} & p^{n+m-1} & & \frac{g_m}{g_o} \end{vmatrix}$$

$$= p^{(1+\dots+m+n-1) - (1+\dots+n-1+1+\dots+m-1)} f_o^{\,m} g_o^{\,n} \begin{vmatrix} 1 & \frac{f_1}{f_o} \dots & \frac{f_n}{f_o} & 0 \dots 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 \dots & 0 & 1 \dots & \frac{f_n}{f_o} \\ 1 & \frac{g_1}{g_o} \dots & \frac{g_m}{g_o} & 0 \dots 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 \dots & 0 & 1 \dots & \frac{g_m}{g_o} \end{vmatrix}$$

$$= p^{\frac{[m+(n-1)][m+n] - [(n-1)n+m(m-1)]}{2}} R(a_1, \dots a_n, b_1, \dots, b_m)$$

$$= p^{\frac{m^2+2mn-m+n^2-n-n^2+n-m^2+m}{2}} R(a_1, \dots, a_n, b_1, \dots, b_m)$$

$$= p^{mn} R(a_1, \dots, a_n, b_1, \dots, b_m).$$

Now suppose the b_i are held constant, and the a_k are allowed to vary, then R is a polynomial in the a_k of degree mn and by the necessity part of the proof $R = 0$ whenever $a_j = b_r$ $1 \le j \le n$, $1 \le r \le m$. Therefore $a_j - b_r$ is a factor of R. That is $R = f_o^m g_o^n \Pi(a_j - b_r) \quad M(a_1, a_2, \ldots, a_n)$

$= f_o^n g_o^m (a_1-b_1)(a_1-b_2)\ldots(a_1-b_m)(a_2-b_1)\ldots(a_2-b_m)\ldots(a_n-b_1)\ldots$

$(a_n-b_m) M(a_1, a_2, \ldots, a_n)$. Since R and $\underset{\substack{1\le j\le n \\ 1\le r\le m}}{(a_j-b_r)}$ are polynomials of degree mn in the a_k, $M(a_1, a_2, \ldots a_n)$ must be equal to a constant. The value of the constant can be found by considering the special case $f = z^n$, $g = z^m + 1$.

$$R(f,g) = \begin{vmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ . & . & . & . \\ 0 & \ldots & & 1 \end{vmatrix} = 1, \; f_o = 1, \; g_o = 1$$

$a_k = 0 (k=1, 2, \ldots n) \quad b_j = e^{\frac{(1+2j)\pi}{m}} \quad (j=1, 2, \ldots m)$

$$\underset{\substack{1\le j\le n \\ 1\le r\le m}}{\Pi}(a_j-b_r) = (-1)^{mn} \prod_{j=1}^{m} b_j^n = (-1)^{mn} \left[\exp\left(\sum_{j=1}^{m} \frac{(1+2j)\pi}{m}\right)\right]^n$$

$$= (-1)^{mn} \left[\exp\left(\frac{2m(m+1)\pi}{2m} + \pi\right)\right]^n = (-1)^{mn} \left[(-1)^{m+2}\right]^n = (-1)^{2mn} = 1.$$

Therefore, $M(a_1, a_2, \ldots a_n) = 1$.

If we now fix a_j, we have $R(f,g) = f_o^m g_o^n \quad (a_j - b_r)$.

Remembering the factored expressions for f and g, we can also write

$$R(f,g) = f_o^m \prod_{j=1}^{n} g(a_j) = (-1)^{mn} f_o^m g_o^n \; \Pi(b_r - a_j) = (-1)^{mn} g_o^n \prod_{r=1}^{m} f(b_r).$$

We conclude, if $R(f,g) = 0$, then, $f(z)$ and $g(z)$ have a root in common.

We next consider the discriminant of $f(z)$.

<u>Definition 1.2</u> The discriminant, D(f), of a polynomial f(z) is defined to be $(-1)^{\frac{n(n-1)}{2}} \frac{R(f, f')}{f_o}$ where $f' = \frac{df}{dz}$.

If f(z) has a multiple root, then this root must also appear in f'(z), and we have the following corollary to Theorem 1.

<u>Corollary 1.1</u> A necessary and sufficient condition that f(z) have multiple roots is D(f) = 0.

Now, let us extend these results to polynomials in two complex variables z and w.

<u>Definition 1.3</u> Let $f(z,w) = f_o(z)w^n + f_1(z)w^{n-1} + \ldots + f_n(z),\ f_o(z) \not\equiv 0$

$$g(z,w) = g_o(z)w^m + g_1(z)w^{m-1} + \ldots + g_m(z),\ g_o(z) \not\equiv 0$$

be polynomials in the complex variables w with coefficients that are polynomials in z, then R(z, f, g) =

$$\begin{array}{l} \\ \text{m times} \\ \\ \\ \\ \text{n times} \\ \\ \\ \end{array} \left| \begin{array}{cccccccc} f_o(z) & f_1(z) & \ldots & f_n(z) & 0 & \ldots & 0 \\ 0 & f_o(z) & \ldots & f_n(z) & & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0\ldots0 & f_o(z) & & \ldots & & & f_n(z) \\ g_o(z) & \ldots & & g_m(z) & 0\ldots0 & & \\ 0 & g_o(z) & & g_m(z) & & & \ldots0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0\ldots0 & g_o(z) & & \ldots & & & g_m(z) \end{array} \right|$$

<u>Definition 1.4</u> The discriminant of f(z, w) is the polynomial $D(z,f) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{f_o(z)} R(z,f,f')$ where $f' = \frac{\partial f}{\partial w}$, $f_o(z) \neq 0$. If $f_o(z) = 0$, $D(z,f) = (-1)^{\frac{n(n+1)}{2}} R(f, f', z)$.

Theorem 1.2 A necessary and sufficient condition that $f(z, w)$ have multiple roots is $D(z, f) = 0$.

If $f_o(z) \neq 0$, the result follows by Corollary 1.1. To handle the case where $f_o(z) = 0$, we expand the resultant by minors of the first column. After expanding each minor again, and setting $f_o(z) = 0$, we obtain the following expression for $D(z, f)$.

$$D(z, f) = (-1)^{\frac{n(n+1)}{2}} f_1(z) \begin{vmatrix} f_1(z) & \cdots & f_n(z) & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots 0 & f_1(z) & f_2(z) & \cdots & f_n(z) \\ (n-1)f_1(z) & \cdots & f_{n-1}(z) & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots 0 & (n-1)f_1(z) & \cdots & & f_{n-1}(z) \end{vmatrix}$$

If $f_o(z) = 0$, the above determinant is $R(f, f', z)$. Therefore, if $f_o(z) = 0$, $D(z, f) = (-1)^{\frac{n(n+1)}{2}} f_1(z)R(z, f, f')$. If $f_1(z) \neq 0$, then by Theorem 1.1 f and f' have equal roots if and only if $R(z, f, f') = 0$. By the above equality, $f(z)$ has multiple roots if and only if $D(z, f) = 0$. If $f_o(z) = 0$ and $f_1(z) = 0$ then we need to extend the definition of $f(z, w)$ to include the pair (z, ∞). We shall see later (Theorem 2.6) that $f(z, w)$ can be defined at the point (z, ∞) to make the theorem hold. That is, $w = \infty$ can be defined so that it is a multiple root of $f(z, w)$, if $f_o(z)$ and $f_1(z)$ equal 0, and if $w = \infty$ is a multiple root of $f(z, w)$ then $f_o(z) = 0$, $f_1(z) = 0$, and thus $D(z, f) = 0$.

Definition 1.5 The polynomial $f(z, w)$ is reducible if $f(z, w) = p(z, w)q(z, w)$ where $p(z, w)$ and $q(z, w)$ are polynomials in z and w, neither of which is equal to a constant. If the polynomial is not reducible, it is said to be irreducible

Theorem 1.3 For two irreducible polynomials $f(z, w)$ and $g(z, w)$ of positive degree in w, there are only a finite number of values of z for which f and g have equal roots unless f is a constant times g.

The proof is by contraposition. If there were an infinite number of values of z for which f and g have equal roots then $R(f, g, z) \equiv 0$. Otherwise, the resultant would be a polynomial of finite degree in z, and thus have a finite number of roots. By Theorem 1.2 we would then have a contradiction. Also, $f(z, w)$ and $g(z, w)$ are polynomials in w with coefficients (polynomials in z) from the field of Rational Functions, and therefore have a greatest common divisor, which we denote as $d(z, w)$. That is there exists polynomials in w, $h(z, w)$, $k(z, w)$, $p(z, w)$, $q(z, w)$, with coefficients that are rational functions in z such that

(a) $f(z, w) = h(z, w)d(z, w)$

(b) $g(z, w) = k(z, w)d(z, w)$

(c) $f(z, w)p(z, w) + g(z, w)q(z, w) = d(z, w)$

These equalities hold for all values of z and w for which the coefficients of the powers of w in d, h, k, p, and q exist. Since each coefficient is of the form $r(z)/t(z)$, where $r(z)$ and $t(z)$ are polynomials in z, and since p, g, h, k, d have finite degree in w, there are only a finite number of values for which the above equalities fail to hold. Suppose $z = z_0$ is a value of z for which the above equations hold. Now $R \equiv 0$, and therefore $f(z_0, w)$ and $g(z_0, w)$ have a root in common, say w_0.

From (c), $f(z_0, w)p(z_0, w) + g(z_0, w)q(z_0, w) = d(z_0, w)$, and $d(z_0 w_0) = 0$. Therefore, either $d(z_0, w) \equiv 0$ or $w-w_0$ is a factor of $d(z_0, w)$. From (a) we see that $d(z_o, w) \not\equiv 0$, since $f(z_o, w) = 0$ would imply that $z-z_o$ is a factor of each coefficient of $f(z, w)$. That is, $f(z, w)$ is reducible. Thus $d(z_o, w)$ is of positive degree in w. Letting z_o vary, we see that $d(z, w)$ is also of positive degree in w.

If $\ell(z)$ is a common denominator for the coefficient of the powers of z in h, d, and k, we may write

$$f(z, w) = \frac{1}{\ell(z)} h_1(z, w) d_1(z, w)$$

$$g(z, w) = \frac{1}{\ell(z)} k_1(z, w) d_1(z, w)$$

where h_1, d_1, k_1, are polynomials in z and w. The above equalities hold whenever $\ell(z) \neq 0$, and thus for all but a finite number of z's. For any z for which $\ell(z) \neq 0$ we can write $\ell(z)f(z) = h_1(z, w)d_1(z, w)$. Regarding z as fixed, the right and left sides of the equation are polynomials in w, and hence must be equal for all values of w. If we now let z vary, we see that these equalities must hold for all values of z and w. Therefore, the above equation is an identity. If $\ell(z)$ is a constant then $d_1(z, w)$ is a common polynomial factor of $f(z, w)$ and $g(z, w)$ of positive degree in w, and since f and g are irreducible h_1 and k_1 then must be constants. If not, let r be a root of $\ell(z)$ and $\ell(z) = \ell_1(z)(z-r)$. Consider h_1, k_1, d_1, as polynomials in w with coefficients which are polynomials in z. Substituting for $\ell(z)$,

$$(z-r)\ell_1(z)f(z, w) = h_1(z, w)d_1(z, w)$$

$$(z-r)\ell_1(z)g(z, w) = k_1(z, w)d_1(z, w)$$

We see that $h_1(r, w)d_1(r, w) = k_1(r, w)d_1(r, w) = 0$.

Therefore, either z-r is a factor of every coefficient in d_1 or a factor of every coefficient in h_1 and k_1. In either case

$$(z-r)\ell_1(z)f(z,w) = (z-r)h_2(z,w)d_2(z,w)$$

$$(z-r)\ell_1(z)g(z,w) = (z-r)k_2(z,w)d_2(z,w)$$

where h_2, k_2, d_2, are polynomials in z and w, and d is of positive degree in w.

It follows, by cancelling z-r, that,

$$\ell_1(z)f(z,w) = h_2(z,w)d_2(z,w)$$

$$\ell_1(z)g(z,w) = k_2(z,w)d_2(z,w)$$

If $\ell_1(z)$ is a constant, we conclude, as before, f(z,w) and g(z,w) differ only by a constant. If not, we proceed as before. In any case, after a finite number of steps, we arrive at the desired result.

ALGEBRAIC FUNCTIONS

Let $f(z, w)$ be an irreducible polynomial of positive degree in w of the form $f(z, w) = f_o(z)w^n + \ldots + f_n(z)$ where each coefficient $f_k(z)$ is itself a polynomial in z with coefficients from the field of complex numbers, and $f_o(z) \not\equiv 0$. The largest exponent of z occurring in one of these coefficients will be denoted by m. We then make the following definition.

Definition 2.1 An algebraic function is a function, $w = F(z)$, defined for all values of z in the extended complex plane by an equation of the form $f(z, w) = 0$, if z is finite, and by $g(o, w) = 0$ where $g(t, w) = t^m f(\frac{1}{t}, w)$, $t = \frac{1}{z}$, if $z = \infty$.

We note that for each finite z the function $F(z)$ normally has n distinct values. Exceptions occur if

1. $f_o(z) = 0$, for the equation has at most degree $n-1$
2. $D(f, z) = 0$, for the equation has multiple roots (Theorem 1.2)

$F(\infty)$ may or may not have fewer than n distinct values depending on whether the determinant and/or the leading coefficient of $g(t, w)$ evaluated at $t = 0$ is equal or unequal to zero.

With regard to the above discussion we make the definition,

Definition 2.2 The algebraic critical points of $F(z)$ are all points z which satisfy one or more of the conditions: 1. $D(z) = 0$ 2. $f_o(z) = 0$ 3. $z = \infty$.

Since $f_o(z)$ is a nonzero polynomial of finite degree, it has a finite number of roots. (at most m) Also, by Theorem 1.3, there are only a finite number of z for which $D(z) = 0$. Therefore, the algebraic critical points of

$F(z)$ are finite in number. Now, let z_o be a finite critical point, and suppose $w_1(z_o) = w_2(z_o) = \dots = w_k(z_o)$. $(1 \le k \le n)$. That is, $f(z,w)$ has a root, $w_o = w_1(z_o)$ of multiplicity k. We shall show that $w_i(z)$ $1 \le i \le k$ is continuous at $z = z_o$.

Theorem 2.1 For sufficiently small $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|z-z_o| < \delta$ then $|w_i(z)-w_o| < \epsilon$ for $1 \le i \le k$. Moreover, if $z_1 \ne z_o$ and $|z_1 - z_o| < \delta$ then $w_r(z_1) \ne w_s(z_1)$ if $r \ne s$. $1 \le r \le k$, $1 \le s \le k$.

We first write $f(z,w)$ in the form,

$$f(z,w) = f_o(z) \left((w-w_o) + w_o\right)^n + f_1(z) \left((w-w_o) + w_o\right)^{n-1} + \dots + f_n(z).$$

After expanding the above equation, and arranging in descending powers of $w-w_o$ we obtain

$f(z,w) = \underline{f_o(z)}\,(w-w_o)^n + \underline{f_1(z)}\,(w-w_o)^{n-1} + \dots + \underline{f_i(z)}$ where $\underline{f_i(z)}$ $(1 \le i \le n)$ is a polynomial in z. Now we know that $\underline{f_n(z_o)} = \underline{f_{n-1}(z_o)} = \dots = \underline{f_{n-(k-1)}}(z_o) = 0$ and $\underline{f_{n-k}(z_o)} \ne 0$ since $\left.\frac{\partial^j f}{\partial w^j}\right|_{\substack{z = z_o \\ w = w_o}} = 0 \, (0 \le j \le k-1)$ and $\frac{\partial^k f}{\partial w^k} \ne 0$, ($w = w_o$ is a root of multiplicity k). Therefore, because $\underline{f_{n-k}(z_o)} \ne 0$, and is continuous, there exists a $\delta_1' > 0$ such that if $|z-z_o| \le \delta_1'$ then $|\underline{f_{n-k}(z)}| \ne 0$. Also, since the set of algebraic critical points of $F(z)$ is finite, there exists a $\delta_2' > 0$ such that whenever $z \ne z_o$ and $|z-z_o| \le \delta_2'$ then $D(z) \ne 0$. Let $\delta' = \min(\delta_1', \delta_2')$ and $\overline{C\delta'} = \{z;\ |z-z_o| \le \delta'\}$. We can write $f(z,w)$ as,

$$f(z,w) = f_{n-k}(z)\,(w-w_o)^k \left(\frac{\underline{f_o(z)}}{\underline{f_{n-k}(z)}} (w-w_o)^{n-k} + \frac{\underline{f_1(z)}}{\underline{f_{n-k}(z)}} (w-w_o)^{n-k-1} + \dots + \frac{\underline{f_{n-k-1}}(z)}{\underline{f_{n-k}(z)}} (w-w_o)^1 + \dots + \frac{\underline{f_n(z)}}{\underline{f_{n-k}(z)}} (w-w_o)^{-k} \right)$$

Let $A(z,w) = \frac{f_o(z)}{f_{n-k}(z)}(w-w_o)^{n-k} \quad \frac{f_1(z)}{f_{n-k}(z)} \quad (w-w_o)^{n-k-1} + \ldots + \frac{f_{n-k-1}(z)}{f_{n-k}(z)}(w-w_o)$

$$B(z,w) = \frac{f_{n-(k-1)}(z)}{f_{n-k}(z)} \ (w-w_o)^{-1} + \ldots + \frac{f_n(z)}{f_{n-k}(z)} \ (w-w_o)^{-k}$$

Then $f(z,w) = f_{n-k}(z)\ (w-w_o)^k\ (1+A+B)$.

Let c be the greatest lower bound of $|f_{n-k}(z)|$ for all z in $\overline{c_\delta{}'}$. We note that $c>0$, because for any z in $\overline{c_\delta{}'}$, $|f_{n-k}(z)| \neq 0$. Also, let m be an upper bound for $|f_{i(z)}|\ i = o, 1, \ldots n,\ z \varepsilon \overline{c_\delta{}'}$. Now suppose ε is any positive number such that $0 < \varepsilon < \min\left(\frac{c}{4m}, \frac{1}{2}\right)$. If $\bar{c}_\varepsilon = \{ w; |w-w_o| \leq \varepsilon \}$ then for any z in $\overline{c_\delta{}'}$ and w in $\bar{c}_\varepsilon$

$$|A(z,w)| \leq \frac{|f_o(z)|}{|f_{n-k}(z)|} \ |w-w_o|^{n-k} + \ldots + \frac{|f_{n-k-1}(z)|}{|f_{n-k}(z)|} \quad |w-w_o|$$

$$\leq \frac{m}{c}\varepsilon^{n-k} + \frac{m}{c}\varepsilon^{n-k-1} + \ldots + \frac{m}{c}$$

$$= \frac{m}{c}\varepsilon(1+\varepsilon^2+\ldots+\varepsilon^{n-k-1}) = \frac{m}{c}\varepsilon \ \frac{1-\varepsilon^{n-k-1}}{1-\varepsilon}$$

$$< \frac{m}{c}\ \left(\frac{1-\varepsilon^{n-k-1}}{1/2}\right) < \frac{2m}{c}\varepsilon < \frac{2m}{c}\ \frac{c}{4m} = \frac{1}{2} \ .$$

Therefore $|A(z,w)| < \frac{1}{2}$ for all z in $c_\delta{}'$ and w in c_ε .

Now, since $|f_n(z_o)| = |f_{n-1}(z_o)| = \ldots = |f_{n-(k-1)}(z_o)| = 0$

there exists a δ such that $0<\delta<\delta'$, and for any z in $c_\delta = \{z; |z-z_o| < \delta\}$,

$$|f_{n-i}(z)| < \frac{c}{2(\varepsilon^{-1}+\varepsilon^{-2}+\ldots+\varepsilon^{-k})} = a, \ i = o, 1, \ldots, k-1.$$

Then for all z and w for which $|z-z_o| < \delta$ and $|w-w_o| = \varepsilon$

$$|B(z,w)| \leq \frac{|f_{n-(k-1)}(z)|}{|f_{n-k}(z)|} \quad |w-w_o|^{-1} + \ldots + \frac{|f_n(z)|}{|f_{n-k}(z)|} \ |w-w_o|^{-k}$$

$< \frac{a}{c}(\mathcal{E}^{-1} + \ldots + \mathcal{E}^{-k})$. Substituting in for a, we have $|B(z,w)| < 1/2$. Let z be an arbitrary point in c_δ, then $f(z_1,w) = \underline{f_{n-k}}(z_1)(w-w_o)^k(1+A(z_1,w) + B(z_1,w))$. Furthermore, for any w such that $|w-w_o| = \mathcal{E}$, we have by previous results $|\underline{f_{n-k}}(z_1)(w-w_o)^k(A(z_1,w) + B(z_1,w))| \leq |\underline{f_{n-k}}(z_1)(w-w_o)^k| \left(|A(z_1,w)| + |B(z_1,w)|\right) \leq |\underline{f_{n-k}}(z_1)(w-w_o)^k|$

We now apply Rouche's Theorem to the functions:

$\underline{f_{n-k}(z_1)}(w-w_o)^k$, $\underline{f_{n-k}(z_1)}(w-w_o)^k(A+B)$ on the circle $|w-w_o| = \mathcal{E}$. Then $f(z_1,w)$ has the same number of zeros in the interior of $\overline{c_\mathcal{E}}$ as $\underline{f_{n-k}(z_1)}(w-w_o)^k$. Since $\underline{f_{n-k}(z_1)} \neq 0$ for any z in c_δ, we see that $f(z_1,w)$ has k zeros in the interior of $\overline{c_\mathcal{E}}$. That is, $|w_i(z_1) - w_o| < \mathcal{E}$ $i=1,2,\ldots k$. Furthermore, $w_i(z_1) \neq w_j(z_1)$ for $i \neq j$, since $D(z_1) \neq 0$.

We note that if $D(z_o) \neq 0$, then k=1, and thus for every $z=z_1$ in c_δ there is one and only one root of $f(z_1,w) = 0$ in c_δ. Therefore, by the above theorem, this root, say $w_1(z)$, is a continuous single valued function of z in c_δ. Using these remarks we can prove the following theorem.

<u>Theorem 2.2</u> If k = 1, and $w_1(z_o) = w_o$ in the preceding theorem, then $w_1(z)$ is holomorphic in c_δ.

Let z_1 and $z_1 + h$ be arbitrary points in c_δ, and suppose $w_1(z_1) = w_1^*$, $w_1(z_1+h) = w_1^* + \ell$. Then $f(z_1, w_1^*) = 0$ and $f(z_1+h, w_1^*+1) = 0$. The theorem will be proved if we show that $\lim_{h\to 0} \frac{w_1(z_1+h) - w_1(z_1)}{h} = \lim_{h\to 0} \frac{\ell}{h}$ exists. To prove this we expand $f(z_1+h,\ w_1^*+\ell)$ in a Taylor series about (z_1, w_1^*). Therefore, $f(z_1+h,\ w_1^*+\ell) = f(z_1,w_1^*) + h\frac{\partial f}{\partial z}(z_1,w_1^*) + \frac{\partial f}{\partial w}(z_1,w_1^*) + \left\{\text{terms which contain at least the factors } h^2,\ \ell^2 \text{ or } \ell h\right\}$

Since $f(z_1, w_1^*) = f(z_1 + h,\ w_1^* + \ell) = 0$, we can write $0 = h\left(\frac{\partial f}{\partial z}\Big|_{\substack{z = z_1 \\ w = w_1^*}} + P(z,w)h + Q(z,w)\ell\right) + \ell\left(\frac{\partial f}{\partial w}\Big|_{\substack{z = z_1 \\ w = w_1^*}} + R(z,w)\ell\right)$ where P, Q, and R are polynomials in z and w. Now $\frac{\partial f}{\partial w}\Big|_{\substack{z=z_1 \\ w=w_1^*}} \neq 0$ because $D(z\ ) \neq 0$, and therefore we can choose ℓ small enough so that for all z and w in c_δ and $\overline{c_\varepsilon}$ respectively $|R\ell| < \left|\frac{\partial f}{\partial w}\begin{pmatrix} z=z_1 \\ w=w_1^* \end{pmatrix}\right|$. Then $\frac{\partial f}{\partial w}\Big|_{\substack{z=z_1 \\ w=w_1^*}} + R(z,w)\ell \neq 0$, and thus

$$\frac{\ell}{h} = \frac{-\frac{\partial f}{\partial z}(z_1, w_1^*) + P(z,w)h + Q(z,w)\ell}{\frac{\partial f}{\partial w}(z_1, w_1^*) + R(z,w)\ell}$$

. If we pass to the limit as $h \to 0$, we have $\lim_{h\to 0} = \frac{-\frac{\partial f}{\partial z}(z_1, w_1^*)}{\frac{\partial f}{\partial w}(z_1, w_1^*)}$. Obviously this limit exists.

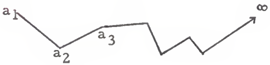
Since we have shown $w_1(z)$ is holomorphic in c_δ, it can be expanded in a power series with center z_o and whose radius of convergence is at least δ . We shall denote this series by $w_1(z;z_o)$. Now for each non critical point z_o there are n distinct roots $w_i(z_o)$ $i = 1, 2, \ldots n$ of $f(z,w) = 0$. Then, from the previous two theorems, each root is holomorphic in some neighborhood c_{δ_i} of z_o, and can be expanded in a power series $w_i(z;z_o)$ whose radius of convergence is at least δ_i.

If the critical points are excluded, we can easily show that each series $w_i(z;z_o)$ can be continued throughout the punctured plane.

<u>Theorem 2.3</u> Each series $w_i(z;z_o)$ can be continued along any path in the punctured plane.

Obviously we only need to prove the theorem for $w_1(z;z_o)$. Let c_δ be any circle in which $w_1(z;z_o)$ converges, and suppose z_1 is an arbitrary noncritical point on the boundary of c_δ. Then, because z_1 is a noncritical point, there exist $\delta_i > 0$ $(i = 1, 2, \ldots n)$ such that $w_1(z;z_1)$ converges in $c\delta_i$. Also, if z_2 is in $c_\delta \cap (\bigcap_{i=1}^{n} c_{\delta_i})$ then for some j $(1 \leq j \leq n)$ $w_j(z_2;z_1) = w_1(z_2;z_o)$. Since $w_j(z_2;z_1) \neq w_i(z_2;z_1)$ for $i \neq j$, we can choose a sufficiently small neighborhood c_δ', such that z_2 is in c_δ', and $c_\delta' \subset c_\delta \cap (\bigcap_{i=1}^{n} c_{\delta_i})$, and $w_j(z;z_1) = w_1(z;z_o)$ for all z in c_δ'. Then, by the Identity Theorem $w_j(z;z_1) = w_1(z;z_o)$ for all $z \in c_\delta \cap c_{\delta_j}$. Therefore, $w_j(z;z_1)$ is a continuation of $w_1(z;z_o)$.

We shall let $a_1, a_2, \ldots a_r, \infty$, denote the critical points of $f(z, w)$, and suppose line segments are drawn connecting a_i to $a_{i-1} (i=2, 3, \ldots r)$ and a_r to ∞. From the preceding theorem, we can easily show that each of the functions w_i is holomorphic in the cut plane. That is, the complex plane with this line excluded.



<u>Corollary 2.1</u> Each function $w_i(z)$, $i = 1, 2, \ldots n$ is holomorphic in the cut plane.

From the preceding theorem, $w_i(z;z_o)$ can be continued along any path in the cut plane. Since the cut plane is simply connected, we conclude, by the Theorem of Monodromy, that $w_i(z)$ is holomorphic in the cut plane.

Now, we wish to show that the $w_i(z)$ are all branches of an n-valued analytic function. Equivalently, we wish to show that $w_i(z)$ can be continued into $w_j(z)$ $(1 \leq j \leq n)$ by a suitable crossing of the polygonal line connecting the critical points. To this end we investigate the behavior of the $w_i(z)$ at the critical points. We shall prove,

Theorem 2.4 At a finite critical point, $z = a$, the roots of $f(z, w) = 0$ can be expressed as Laurent series in z' where $z - a = z'^{\ell}$ (ℓ a positive integer).

Let a be one of the finite critical points of $f(z, w)$, k the circumference of a circle about a, which neither encloses nor contains any other critical points; z_o, a point of k. Suppose k has positive orientation, then by previous results each of the $w_i(z;z_o)$ can be continued along k. Let $w_1^*(z;z_o)$ be the continuation of $w_1(z;z_o)$ around k. Then for some j $(1 \leq j \leq n)$, $w_j(z_o;z_o) = w_1^*(z_o;z_o)$. Since $w_j(z_o;z_o) \neq w_i(z_o;z_o)$ for $i \neq j$, there exists a neighborhood c_δ containing z_o such that $w_1^*(z;z_o) = w_j(z;z_o)$ for all z in c_δ. Therefore, by the Identity Theorem, $w_1^*(z;z_o) = w_j(z;z_o)$. Again, we can continue $w_j(z;z_o)$ along k, obtaining $w_j^*(z;z_o)$. By the same reasoning as before, for some s $(1 \leq s = n)$ $w_s(z;z_o) \equiv w_j^*(z;z_o)$. Now, two distinct roots cannot be continued into the same root, since otherwise the inverse continuation would transform this root into the two distinct roots. So, continuing in the manner described, we see that after h_1 times $(1 \leq h_1 \leq n)$ we must have for some $p(1 \leq p \leq n)$ $w_p(z;z_o) = w_1(z;z_o)$.

Suppose the $w_i(z)$ are numbered so that $w_1(z;z_o)$ goes into $w_2(z;z_o)$, $w_2(z;z_o)$ into $w_3(z;z_o)\ldots$, $w_{h_1-1}(z;z_o)$ into $w_{h_1}(z;z_o)$ and $w_{h_1}(z;z_o)$ into $w_1(z;z_o)$. (Note that h_1 may equal one). If $h_1 = n$, then all the $w_i(z)$ are in the same cycle. If not $w_{h_1+1}(z;z\ )$ can be continued along k, obtaining $w_{h_1*+1}(z;z\ )$. We see that $w_{h_1*+1} \neq w_t$ for $1 \leq t \leq h_1$, since the inverse continuation would give

w_{t-1}, w_{h_1+1} if > 1

w_{h_1}, w_{h_1+1} if t = 1

Therefore $w^*_{h_1+1}(z;z_o) = w_q(z;z_o)$ $h_1+1 \leq q \leq n$. Continuing in this manner, after h_2 times, we must have $w_r(z;z_o) \equiv w_{h_1+1}(z;z_o)$, $h_1+1 \leq r \leq n$. Suppose the w_i are numbered so that w_{h_1+1} goes into w_{h_1+2}, w_{h_1+2} into $w_{h_1+3}, \ldots,$ $w_{h_1+h_2}$ into w_{h_1+1}. After applying this process j times $w_{h_1+h_2+\ldots+h_{j-1}+1}$ goes into $w_{h_1+\ldots,+h_{j-1}+2}, \ldots,$ $w_n(z)$ goes into $w_{h_1+h_2+\ldots+h_{j-1}+1}$. We see that $\sum_{i=1}^{j} h_i = n$, where j equals the number of permutations.

The proof will be completed if we show that the roots which form the k^{th} permutation $1 \leq k \leq j$ with h_k elements can be expressed as a Laurent series $f_k(z') = \sum_{n=-\infty}^{\infty} c_n z'^n$ in the complex z' plane where $(z-a) = (z')^{h_k}$. Obviously, we need show the above only for the permutation $w_1, w_2 \ldots w_{h_1}$. Making the substitution in $w_1(z)$, we have $w_1(z) = w_1((z')^{h_1}+a) = f_1(z')$. Let k' be a circle with radius R, and origin as center in the z' plane, chosen small enough, so that the origin is the only point which has for its image a critical point. Then, as z' goes around k' once, $(z')^{h_1}$ encircles the origin, and hence z the point a precisely h_1 times. Therefore, the argument of z-a is increased by $2\pi h_1$.

Let z_1' be an arbitrary point on k', and suppose we continue $f_1(z';z_1')$ along k', obtaining $f_1^*(z';z_1')$. Then z goes around a h_1 times, so that the corresponding continuation in the z plane $w_1^*(z; a+(z_1')^{h_1})$ is identically equal to $w_1(z;a + (z_1')^{h_1})$. Hence $f_1(z;z_1') \equiv f_1^*(z';z_1')$. Since $f_1(z')$ is single valued for $0 < z' < R$, it is holomorphic in this annulus. Consequently, $f_1(z')$ can be developed in a Laurent series,

$$f_1(z') = w_1(a + (z')^{h_1}) = \sum_{n=-\infty}^{\infty} c_n z'^n = \sum_{n=-\infty}^{\infty} c_n (z-a)^{\frac{n}{h_1}} .$$

From this theorem, we can easily prove the following theorem.

Theorem 2.5 If $f(z_o, w) = 0$ has a root of multiplicity ℓ $(1 < \ell \leq n)$ and $w_1(z_o) = w_2(z_o) = \ldots = w_\ell(z_o)$ then the $w_t(z)$ $(1 \leq t \leq \ell)$ form k permutations $(k \leq \ell)$ each of which has a Laurent series in z' with no negative terms.

From our previous discussion we see that $w_1(z)$, $w_2(z), \ldots w_\ell(z)$ form themselves into k permutations $(k \leq \ell)$ where the i^{th} permutation $(1 \leq i \leq k)$ has h_i elements and $\sum_{i=1}^{k} h_i = \ell$. By Theorem 2.1 the $w_t(z)$ are continuous as $z = a$, so that the Laurent series for the i^{th} permutation is $f_i(z) = \sum_{n=0}^{\infty} c_{in}(z-a)^{\frac{n}{h_i}}$.

We shall now consider the case $f_o(a) = 0$.

Theorem 2.6 If $f_o(a) = f_1(a) = \ldots = f_{p-1}(a) = 0$, and $w_{p+1}(a)$, $w_{p+2}(a), \ldots, w_n(a)$ are the roots of $f(a, w) = 0$ then $w_1(z)$, $w_2(z), \ldots w_p(z)$ form j permutations $(j \leq p)$ each of which has a Laurent series in z', which has a positive number of negative powers.

Let $w = \frac{1}{u}$ and $g(z, u) = u^n f(z, \frac{1}{u}) = f_n(z)u^n + \ldots + f_o(z)$. Then $g(a, u) = 0$ has a root, $u = 0$, of multiplicity p at $z = a$, since $\left.\frac{\partial^i g}{\partial u^i}\right|_{z=a} = i!\, f_i(a) = 0$ for $i = 0, 1, 2, \ldots p-1$. We note that $p < n$, because $f(z, w)$ is irreducible. Using the results obtained from the previous discussion, we see that $u_1, u_2, \ldots u_p$

give rise to j permutations each of which has a Laurent series in z' which has no negative terms. Then, the i^{th} permutation with h_i elements can be expressed as $G_i(z') = u_i((z')^p + a) = z'^{\ell} \sum_{n=1}^{\infty} c_n z'^n = \sum_{n=1}^{\infty} c_n (z-a)^{\frac{n}{h_i}}$ ($c_o = 0$, because $g(a, 0) = 0$). Suppose $c_o = c_1 = c_2 \ldots c_{\ell-1} = 0$, then $G_i(z') = z'^{\ell} \sum_{n=0}^{\infty} c'_n z'^n$ where $c'_o = c_{\ell}$, $c'_n = c_{n+\ell}$. We assume w can be expressed in a Laurent series $\sum_{n=-\ell}^{\infty} a_n z'^n$. Substituting the power series for u and w into the equation $w = \frac{1}{u}$, we have $(a_{-\ell} + a_{-\ell+1} z' + a_{-\ell+2} z'^2 + \ldots) = \dfrac{1}{c'_o + c'_1 z' \ldots}$.

Multiplying term by term, $a_{-\ell} = \dfrac{1}{c'_o}$ (since $c'_o \neq 0$), $a_{-\ell+1} = \dfrac{-c_1' a_{-\ell}}{c'_o}$.

From this recursion process the a_i are determined. Therefore,

$w_i((z')^{h_i} + a) = \sum_{n=-\ell}^{\infty} a_n (z-a)^{\frac{n}{h_i}}$.

The following theorem is a result of the previous two.

Theorem 2.7 The n valued function defined by $f(z, w) = 0$ can be expressed at a finite critical point, a, by some combination of,

1. Power series in z-a.

2. Laurent series in $(z-a)^{1/p}$ ($p \geq 1$) with no negative powers.

3. Laurent series in $(z-a)^{1/p}$ with a finite number of negative powers.

If $z = \infty$, by definition, the roots are solutions of the equation $g(o, w) = 0$ where $g(t, w) = t^m f(\frac{1}{t}, w)$, $t = \frac{1}{z}$. Our previous results apply to the point $t = 0$, so we have the following theorem.

Theorem 2.8 The n valued function defined by $f(z, w) = 0$ can be expressed at $z = \infty$ by some combination of,

1. Power series in $\frac{1}{z}$.

2. Laurent series in $(\frac{1}{z})^{1/p}$ $(p \geq 1)$ with no negative powers.

3. Laurent series in $(\frac{1}{z})^{1/p}$ with a finite number of negative powers.

Before we prove the next theorem, several definitions are in order.

Definition 2.3 Two power series are said to be equivalent relative to a set E if one can be obtained from the other by continuation along a path in E.

Definition 2.4 An analytic function, G(z), is an equivalence class of power series defined on a set E, together with expansions, if they exist, of the form:

1. $(z-a)^{-p}$ times power series in $z-a$, $p \geq 1$.

2. $(z-a)^{\frac{p}{n}}$ times power series in $(z-a)^{\frac{1}{n}}$, $n > 1$.

3. $z^{\frac{p}{n}}$ times power series in $z^{\frac{-1}{n}}$ $n \geq 1$.

Theorem 2.9 The n valued function defined by $f(z,w) = 0$ is analytic.

If the $w_i(z)$ were not branches of the same analytic function, then continuation of one of the roots, for example $w_1(z)$ yields only a proper subset of the roots, say $w_1(z)$, $w_2(z), \ldots, w_k(z)$, where $k < n$. To eliminate this possibility, we examine the function defined by $g(z,w) = w^k - (w_1 w_2 \ldots w_k) w^{k-1} + (-1)^2 \sum_{i<j} w_i w_j w^{k-2} + \ldots + (-1)^k \; w_1 w_2 \ldots w_k = 0$.

Let $E_1(z) = w_1$

$$E_2(z) = \sum_{i<j} w_i w_j$$

$$\vdots$$

$$E_k(z) = w_1 w_2 \ldots w_k$$

Then $g(z,w) = w^k - E_1(z)w^{k-1} + E_2(z)w^{k-2} + \ldots + (-1)^k E_k(z)$. Now each of the $E_i(z)$ $(1 \leq i \leq k)$ represent a single valued analytic function, since continuation merely interchanges the k roots, and thus leaves each $E_i(z)$ unchanged. Furthermore, the $E_i(z)$ have only a finite number of singular points, and these must be located at singular points of the roots. Also, a given singular point must be either a pole or a removable singularity, since each $w_i(z)$ has at most a pole at this point. Similarly, we see that $z = \infty$ is at most a pole of $E_i(z)$. Thus the $E_i(z)$ are holomorphic in the extended plane, except for poles, so they must be rational functions. If we multiply $g(z,w)$ by the common denominator of the $E_i(z)$, we obtain an equation of the form, $h(z,w) = h_o(z)w^k + h_i(z)w^{k-1} + \ldots + h_k(z)$ which is satisfied by the functions $w_1, w_2, \ldots, w_k$, where $h_i(z)$ $0 \leq i \leq k$ is a polynomial in z. Also, $w_1, w_2, \ldots, w_k$ are roots of $f(z,w) = 0$, so that $R(z,f,h) = 0$. By Theorem 1.3 we have reached a contradiction, since f and h are irreducible, and h is of degree less than f. Therefore, $w_1, w_2, \ldots, w_n$ are all branches of the same analytic function.

The converse of Theorem 2.9 holds.

Theorem 2.10 If $F(z)$ is an n-valued analytic function, with a finite number of critical points, defined throughout the extended complex plane, and if $F(z)$ is expressible at a critical point, b, by Laurent series in $(z-b)^{1/n}$, if b is finite, and by Laurent series in $1/z^{1/n}$, if $b = \infty$, each with a finite number of negative powers, then there exists one and only one irreducible polynomial equation of the n^{th} degree in z and w which $F(z)$ satisfies. Furthermore, if this equation is expressed in powers of w, then its coefficients are polynomials in z, and the leading coefficient can only have zeros at critical points of F(z).

Let $w_1(z{:}z_o)$, $w_2(z{:}z_o), \ldots, w_n(z{:}z_o)$ be elements of F(z) with center z_o. If the critical points of F(z) are joined together in some manner, then in the resulting cut plane each of the $w_i(z{:}z_o)$ can be extended along any path in it, so by the Theorem of Monedromy the continuations of $w_i(z{:}z_o)$ define a function $w_i(z)$ which is holomorphic in the cut plane.

Let $E_1(z) = w_1(z) + w_2(z) + \ldots + w_n(z)$

$$E_2(z) = \sum_{i<j} w_1(z)w_j(z) \qquad i = 1, 2, \ldots n-1 \qquad j = 1, 2, \ldots n$$

$$\vdots$$

$$E_n(z) = w_1(z)w_2(z) \ldots w_n(z)$$

These expressions represent a single valued analytic function in the extended complex plane, since analytic continuation can merely interchange the $w_i(z)$, and leaves their symmetric functions unchanged. Furthermore, the $E_i(z)$ have only a finite number of singularities, and these must be located at critical points of F(z). Therefore, the singularities of the $E_i(z)$ are isolated, and are consequently either poles or essential singularities. The latter possibility is easily excluded. If a is a finite critical point of F(z), then F(z) is expressible at z = a by Laurent series of the form $\sum_{k=-n}^{\infty} a_k(z-a)^{-\frac{k}{m}}$ $(m \geq 1)$. Therefore, either $w_i(z)$ becomes infinite as $z \to a$ or $w_i(z)$ remains finite. Since the $E_j(z)$ are merely combinations of the $w_i(z)$, $E_j(z)$ either becomes infinite as $z \to a$ or remains finite. The same type of argument can be given for $z = \infty$. Therefore, the $E_j(z)$ have at most poles in the extended complex plane, and so must be rational functions.

Suppose we now form $g(z,w) = \left(w-w_1(z)\right)\left(w-w_2(z)\right)\dots\left(w-w_n(z)\right) = w^n - E_1(z)w^{n-1} + \dots + (-1)^n E_n(z)$. Multiplying by the common denominator of the coefficients, we have $f(z,w) = f_o(z)w^n + f_1(z)w^{n-1} + \dots + f_o(z) = 0$ where $f_i(z)$ is a polynomial in z. It remains to show that the polynomial on the left side of the equation is the only one satisfied by F(z). To this end, let us assume f(z, w) is reducible. Then F(z) satisfies an equation of degree n in w, which is expressible as a product of irreducible polynomials, each of degree less than n in w. F(z) would then satisfy at least one of the equations defined by setting each irreducible polynomial equal to zero. If the degree of this equation is $m < n$, then, by Theorem 2.9, it defines an m-valued analytic function. Obviously, we have reached a contradiction, because F(z) is n-valued. By Theorem 1.3 it follows that F(z) can satisy only one irreducible polynomial equation.

From this theorem, we can state an equivalent definition of an algebraic function, and its critical points.

Definition 2.5 An algebraic function, F(z) is an n-valued analytic function, with a finite number of critical points, defined throughout the extended complex plane, and is expressable at a critical point, b, by Laurent series in $(z)^{-1/n}$, if $b = \infty$, and by Laurent series in $(z-b)^{1/n}$ if b is finite, each with a finite number of negative powers.

Using this definition, we can easily prove the following theorem.

Theorem 2.11 If there exists a sequence of distinct points $\langle z_n \rangle_{n=1}^{\infty}$ at which an algebraic function, $F(z)$, assumes one of its values an infinite number of times, and if the sequence $\langle z_n \rangle_{n=1}^{\infty}$ converges to a limit c, which is a noncritical point, then the range of $F(z)$ is finite. Furthermore, the equation which $F(z)$ satisfies is of zero degree in z.

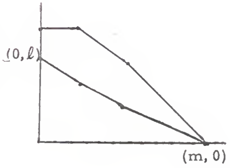
Let $w_1(z)$, $w_2(z), \ldots w_n(z)$ represent the n branches of $F(z)$. If $F(z) = F(w_1, w_2, \ldots, w_n)$ assumes one of its values an infinite number of times, then each $w_i(z)$ has to assume a value an infinite number of times. Therefore, we can find a subsequence $\langle z_{i_n} \rangle_{n=1}^{\infty}$ of $\langle z_n \rangle_{n=1}^{\infty}$ which converges to c, at which $w_i(z_{i_k}) = w_i(z_{i_m}) = di$ for arbitrary positive integers k and m. Also, a given singular point must be either a pole or a removable singularity.

Now, $g_i(z) = w_i(z) - d_i$ is holomorphic in any domain not containing a critical point, so, by the Identity Theorem, $g_i(z) \equiv 0$ in the complex plane without the critical points of $F(z)$. Also, since $F(z)$ has only a finite number of critical points, it can assume only a finite number of values at these points. Therefore, the range of $F(z)$ is finite. Since the symmetric functions formed by the $w_i(z)$ are constants, we conclude that the equation satisfied by $F(z)$ is of zero degree in z.

ALGEBRAIC EXPANSIONS

In this section we shall show that the expansion of an algebraic function, $F(z)$, at a critical point can actually be determined by use of a general procedure known as Newton's method.

A. Expansions of Multiple Roots. Suppose $w = b$ is a finite root of multiplicity ℓ, corresponding to the critical point $z = a$. Then, $f(z, w) = a_{ij}(z-a)^i(w-b)^j$, $a_{00} \neq 0$. In the x-y plane we plot the points (i, j) such that $a_{ij} \neq 0$. We then form the least convex polygon which contains these points. The polygon always has at least one point on each of the axes



since $f(z, w)$ is irreducible. Also, since $w = b$, is of multiplicity ℓ, the point on the y-axis nearest the origin must be $L = (0, \ell)$. This follows because $\frac{\partial^\ell f}{\partial w^\ell}(a, b) \neq 0$, $\frac{\partial^j f}{\partial w^j}(a, b) = 0$ $(j \leq \ell)$.

Let $M = (m, 0)$ be the point on the x-axis nearest the origin. Then, the polygon has a broken line $(L, \ldots, M)$ joining L with M, and this line consists of at most ℓ segments, since for each j $(0 \leq j \leq \ell)$, there exists at most one k $(1 \leq k \leq m)$ such that (k, j) lies on the broken line. Furthermore, all points (i, j) lie on $(L, \ldots, M)$ or above it. (See figure)

To determine the expansions, we start with the segment whose left endpoint is L. Suppose the equation of the line on this segment is $p_1 x + q_1 y = s_1$. Here p_1 and q_1 can be chosen as positive integers, and we may assume p_1 and q_1 are relatively prime. The slope of this line is $\frac{-p_1}{q_1}$.

Suppose the lower endpoint of this line segment is (m, n), where m and n are integers. Then, $m/\ell - n = q_1/p_1$, and because q_1 and p_1 are relatively prime. $m = r_1 q_1$, $n = \ell - r_1 p_1$, where r_1 is a positive integer. We shall show there $r_1 p_1$ branches of $F(z)$ associated with this segment.

To show this we substitute $z-a = t^{p_1}$, $w-b = t^{q_1}u$ into the Taylor series for $f(z,w)$. Then $f(z,w) = \sum a_{ij} t^{ip_1 + jq_1} u^j = 0$. For every point (i,j) on this line, we have $ip_1 + jq_1 = s_1$, for all other points, $ip_1 + jq_1 > s_1$. Thus the equation must be divisible by t^{s_1}, and after division we have an expression of the form $g(t,u) = \sum'' a_{ij} t^{ip_1 + jq_1 - s_1} u^j + \sum' a_{ij} u^j$. The second sum contains at least the two terms corresponding to the endpoints of the line segment. Therefore, the second sum can be written as, $u^{\ell - r_1 p_1} p_{r_1}(u^{p_1})$ where $p_{r_1}(v)$ is a polynomial of degree r_1. Also t can be factored out of the first sum, so $g(t,u)$ can be written as $g(t,u) = u^{\ell - r_1 p_1} p_{r_1}(u^{p_1}) + tg_1(t,u) = 0$ where $g_1(t,u)$ is a polynomial in (t,u).

Suppose $p_{r_1}(v) = 0$ has the roots $a_1^{p_1}, a_2^{p_1}, \ldots, a_{r_1}^{p_1}$, which, for the moment, we assume to be distinct. If $a\lambda$ is some p_1th root of a^{p_1} $(1 \le \lambda \le r_1)$ then $g(t,u)$ vanishes for $t = 0$, $u = a$. Since $a\lambda$ is a distinct root of $g(0,u)$, by Theorem 2.2, it can be expanded in a power series $u(t) = a\lambda + \sum_{k=1}^{\infty} c\lambda k t^k$. Substituting, $(z-a)^{1/p_1} = t$, $w-b = (z-a)^{q_1/p_1} u$, we have, $w = (z-a)^{q_1/p_1} (a\lambda + \sum_{k=1}^{\infty} c\lambda k (z-a)^{k/p_1}$. Now, $a\lambda e^{2\pi n i/p_1} (z-a)^{1/p_1} =$ $a\lambda \left((z-a)e^{2\pi n i}\right)^{1/p_1} = a\lambda (z-a)^{1/p_1}$, so from each root of $a\lambda^{p_1}$, we obtain the same expansion. However, because $a_1^{p_1}, a_2^{p_1}, \ldots, a_{r_1}^{p_1}$, are distinct, we have r_1 cycles each of p_1 branches corresponding to the first segment of Newton's polygon.

To obtain the above result, we assumed the roots $a_1, a_2, \ldots a_{r_1}$ were distinct. If they are not, say $a_1{}^{p_1} = a_2{}^{p_1} = \ldots = a_k{}^{p_1}$ $(k \leq r_1)$, then the entire process must be applied again to the polynomial $g(t, u)$ at the point $(0, a_1{}^{p_1})$. Bliss[1] shows that after a finite number of applications, one obtains an expression similar to $g(t, u)$ with a polynomial which has only simple roots. Furthermore, he shows, the corresponding expansions of w in terms of z have $r_1 p_1$ branches.

We handle the other sides in the same manner. Suppose that the difference of the x-coordinate of the endpoints of the second line segment is $r_2 q_2$ and that of the y-coordinates is $r_2 p_2$, so that the slope of the line is $\frac{-p_2}{q_2}$. Here the slope of this line is necessarily different from the first. Proceeding as before, we arrive at an expression, $h(t, u) =$ $u^{\ell - r_1 p_1 - r_2 p_2} p_{r_2}(u^{p_2}) + t\, h_1(t, u)$ where $pr_2(v)$ is a polynomial of degree r_2. If the roots of $pr_2(v) = 0$ are distinct, we obtain r_2 different cycles with p_2 branches in each cycle of the form, $w - b = (z-a)^{q_2/p_2}(a\lambda + \sum_{k=1}^{\infty} c\lambda_k (z-a)^{k/p_2})$ $\lambda = 1, 2, \ldots r_2$. If the roots are not distinct, as above, we must go through the entire process again.

In this manner, we find the expansions corresponding to the different

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[1] Bliss, Gilbert Ames. Algebraic Functions. New York: Colloquium Publications, American Mathematical Society, Vol. XII, 1933, pp. 37-39.

sides. We see that if the polygon has h_1 sides then $\sum_{i=1}^{h_1} r_i p_i = \ell$. This relation asserts that the total number of branches obtainable by this method is ℓ, that is, we have accounted for all branches which tend to b as z approaches a.

B. Expansions where a root is infinite.

The expansions which become infinite at a finite critical point z = a can also be found by Newton's method. That is, suppose $f_0(a) = f_1(a) = \ldots = f_{\ell-1}(a) = 0$, $f_\ell(a) \neq 0$. To find the expansions set $1/v = w$, and $h(z,v) = v^n f(z, 1/v)$. Then $h(z,v)$ has a root $v = 0$ of multiplicity ℓ, at $z = a$. (See Theorem 2.6) Applying the previous method to this root, we find the expansions in v which correspond to $v = 0$. The reciprocals of these expansions for v are the expansions of w which have poles at $z = a$.

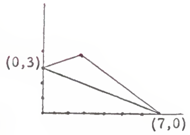
C. Expansions at $z = \infty$ To find the expansions at $z = \infty$, set $s = 1/z$, $k(s,w) = s^m f(1/s, w)$, and apply the process described in A and B to $k(s,w)$ at $s = 0$. Substituting $1/z = s$ in the resulting expansions will give the expansions at $z = \infty$.

The results of A, B, and C are summed up by the following theorem.

Theorem 3.1 The programs described in A, B, and C gives all expansions of $F(z)$ at a critical point $z = a$ or $z = \infty$.

Example 1

Let $f(z,w) = z^3 w^4 + w^3 - 8z^7 = 0$. We shall find the expansions of $F(z)$ at $z = 0$ and $z = \infty$. At $z = 0$ the equation has a root $w = 0$ of multiplicity 3. To find the expansions corresponding to this root, we plot the exponents of z and w in the x-y plane, with the z exponent as abcissa, and w exponent as ordinate. Next, we draw the least convex ploygon which contains these points. The slope of the line segment nearest the origin is -3/7. Therefore, $q_1 = 7$, $p_1 = 3$, and we substitute $z = t^3$, $w = t^7 u$ into $f(z,w)$. We obtain, $f(z,w) = g(t,u) = (t^3)^3(t^7u)^3 - 8(t^3)^7 = 0$. Division by t^{21} yields,



$t^{16}u^4 + u^3 - 8 = 0$. When $t=0$, $u^3=8$, $u=8^{1/3}$. We note that $8^{1/3}$ represents the three roots 2, $2e^{\frac{2\pi i}{3}}$, $2e^{\frac{4\pi i}{3}}$. Since the three roots are distinct at $t = 0$, the algebraic function defined by $t^{16}u^4 + u^3 - 8 = 0$, by Theorem 2.2, must have three distinct power series expansions at this point.

Let $u = \sum_{n=0}^{\infty} c_n t^n$. Substituting into the equation,

$$t^{16}\left(\sum_{n=0}^{\infty} c_n t^n\right)^4 + \left(\sum_{n=0}^{\infty} c_n t^n\right)^3 - 8 = 0.$$

Following the usual procedure, we obtain

$$c_0^3 - 8 = 0, \quad c_0 = 8^{1/3}.$$

$$c_i = 0 \ (i = 1, 2, \ldots 5).$$

$$c_0^4 + 3c_{16}c_0^2 = 0, \quad c_{16} = \frac{-c_0^2}{3} = \frac{-8^{2/3}}{3}. \quad \text{For } c_0 = 2, \ c_{16} = -4/3$$

$$c_j = 0 \ (j = 17, 18, \ldots, 31).$$

$$4c_0^3 c_{16} + 3c_{32}c_0^2 + 3c_{16}^2 c_0 = 0.$$

$$c_{32} = \frac{-4c_0^3 c_{16} - 3c_{16}^2 c_0}{3c_0^2} = \frac{8^{5/3}}{(3)(8^{2/3})} = \frac{8}{3}$$

One of the power series in u can be written

$$u = 2 - \frac{4}{3}t^{16} + \frac{8}{3}t^{32} \ \ldots$$

Substituting $t = z^{1/3}$, $w = z^{7/3}u$, we have

$$w = z^{7/3}(2 - \frac{4}{3} z^{16/3} + \frac{8}{3} z^{32/3} + \ldots).$$

The other two values of c_0 also yield the same result since

$$e^{\frac{2\pi i}{3}} z^{\frac{1}{3}} = (e^{2\pi i} z)^{\frac{1}{3}} = z^{\frac{1}{3}}.$$

$$e^{\frac{4\pi i}{3}} z^{\frac{1}{3}} = (e^{4\pi i} z)^{\frac{1}{3}} = z^{\frac{1}{3}}$$

This series is the expansion corresponding to the root $w = 0$, of multiplicity three at $z = 0$. From the equation and Theorem 2.6, we see that the fourth root must be infinite at $z = 0$. To find its expansion, we let $w = \frac{1}{u}$, $h(z, u) = u^4 f(z, \frac{1}{u}) = z^3 + u - 8z^7u^4 = 0$. When $z = 0$, this equation has a root $u = 0$ of multiplicity one. By Theorem 2.2, this root is holomorphic in a neighborhood of $z = 0$. Therefore, $u = \sum_{n=0}^{\infty} c_n z^n$.

Substituting in the equation,

$$z^3 + \sum_{n=0}^{\infty} c_n z^n - 8z^7 (\sum_{n=0}^{\infty} c_n z^n)^4 = 0.$$

From this equation we obtain the relationships,

$$c_0 = c_1 = c_2 = 0.$$

$$c_3 + 1 = 0, \; c_3 = -1.$$

$$c_i = 0 \quad (i = 4, 5, \ldots, 18).$$

$$c_{19} - 8c_3^4 = 0.$$

$$c_{19} = 8$$

$$c_j = 0, \; (j = 20, 21, \ldots, 34).$$

$$c_{35} - 32c_3^3 c_{19} = 0, \; c_{35} = -256.$$

So, $u = z^3(-1 + 8z^{16} - 256z^{32} \ldots)$.

Then, $w = 1/u = z^{-3} \dfrac{1}{-1 + 8z^{16} - 256z^{32} \ldots}$.

$$\frac{1}{-1 + 8z^{16} - 256z^{32} \ldots} = b_o + b_1 z + \ldots$$

$b_o = -1.$

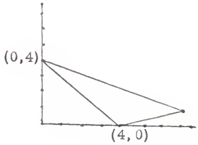
$8b_o - b_{16} = 0,\ b_{16} = -8.$

$-b_{32} + 8b_{16} - 256b_o = 0.$

$b_{32} = 256 - 64 = 192.$

So, $w = z^{-3}(-1 - 8z^{16} + 192z^{32} + \ldots)$.

At $z = \infty$, $g(s,w) = s^7 f(1/s, w) = s^4w^4 + s^7w^3 - 8 = 0$, where $z = 1/s$. By Theorem 2.6, we see that all the roots are infinite at $s = 0$. Therefore, let $w = 1/u$, $h(s,u) = u^4 g(s, 1/u) = s^4 + s^7u - 8u^4 = 0$. When $s = 0$, $u = 0$ is a root of multiplicity 4. The Newton's diagram of the equation is shown in the figure. The slope of the line segment nearest the origin is -1. So, $q_1 = 1$,



$p_1 = 1$, and we substitute $s = t$, $u = tv$ into $h(s,u)$. Therefore, $h(s,u) = k(t,v) = t^4 + t^8v - 8t^4v^4 = 0$. Dividing by t^4, $1 + t^4v - 8v^4 = 0$. When $t = 0$, $v^4 = 1/8$, $v = 8^{-1/4}$. Since the roots are distinct, the expansions for v at $s = 0$ are four distinct power series.

$v_i = \sum_{n=0}^{\infty} c_{in} t^n$ (i = 1, 2, 3, 4). Substituting into the equation,

$$1 + t^4\left(\sum_{n=0}^{\infty} c_{in}t^n\right) - 8\left(\sum_{n=0}^{\infty} c_{in}t^n\right)^4 = 0.$$

From this equation, we have the following relationships.

$$1 - 8c_{i0}^{\ 4} = 0, \ c_{i0} = 8^{-1/4}.$$

$$c_{ij} = 0, \ j = 1, 2, 3.$$

$$c_{i0} - 8(4c_{i0}^{\ 3}c_{i4}) = 0.$$

$$c_{i4} = \frac{1}{32c_{i0}^{\ 2}} = \frac{8^{1/2}}{32} .$$

$$c_{ik} = 0, \ k = 5, 6, 7.$$

$$c_{i4} - 8(6c_{i0}^{\ 2}c_{i4}^{\ 2} + 4c_{i8}c_{i0}^{\ 3}) = 0.$$

$$c_{i8} = \frac{c_{i4} - 48c_{i0}^{\ 2}c_{i4}^{\ 2}}{32c_{i0}^{\ 3}} = \frac{-8^{1/4}}{256}.$$

Then, $v_i = 8^{-1/4} + \frac{8^{1/2}}{32}t^4 - \frac{8^{1/4}}{256}t^8 + \ldots$

Since $s = t$, $u = tv$,

$$u_i = s(8^{-1/4} + \frac{8^{1/2}}{32}s^4 - \frac{8^{1/4}}{256}s^8 + \ldots)$$

Because $w_i = \frac{1}{u_i}$, $w_i = s^{-1}\dfrac{1}{8^{-1/4} + \frac{8^{1/2}}{32}s^4 - \frac{8^{1/4}}{256}s^8 + \ldots}$.

$$\frac{1}{8^{-1/4} + \frac{8^{1/2}}{32}s^4 - \frac{8^{1/4}}{256}s^8 + \ldots} = b_0 + b_1 s + \ldots$$

$$b_0 = 8^{1/4}.$$

$$b_4 8^{-1/4} + \frac{8^{1/2}}{32}b_0 = 0, \ b_4 = -\frac{1}{4}.$$

$$b_8(8^{-1/4}) + b_4\frac{8^{1/2}}{32} - \frac{8^{1/4}}{256}b_0 = 0,$$

$$b_8 = \frac{-b_4 8^{1/2}}{32} + \frac{8^{1/4}}{256} b_0 \; 8^{1/4} = \frac{(3)(8)^{3/4}}{256}.$$

Therefore, $w_i = s^{-1}(8^{1/4} - 1/4\, s^4 + \frac{(3)(8)^{3/4}}{256} s^8 + \ldots)$.

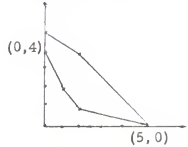
Substituting, $s = z^{-1}$,

$$w_i = z(8^{1/4} - 1/4\, z^{-4} + \frac{(3)(8)^{3/4}}{256}\, z^{-8} + \ldots) \; (i = 1, 2, 3, 4).$$

<u>Example 2</u>

$f(z, w) = z^2w^4 + w^5 + 2w^4 + 2z^2w^2 - zw^2 + z^2w + z^3w - z^5 = 0.$

We shall find the expansions of $f(z, w)$ only for $z = 0$. At $z = 0$, the equation has roots $w = 0$, of multiplicity 4, and $w = -2$. As before, to find the expansions corresponding to $w = 0$, we plot the powers of z and w in the x-y plane, and draw the least convex polygon which contains these points. The slope of the line connecting $(0, 4)$ and $(1, 2)$ is -2. Therefore, $q_1 = 1$.



$p_1 = 2$, and we substitute $z = t^2$, $w = tu$ into the equation. Then, $(t^2)^2(tu)^4 + (tu)^5 + 2(tu)^4 + 2(t^2)^2(tu)^2 - t^2(tu)^2 + (t^2)^2(tu) + (t^2)^3 (tu) - (t^2)^5 = 0$. Dividing out a factor of t^4, $t^4u^4+tu^5+2u^4+2t^2u^2-u^2+tu+t^3u-t^6 = 0$. $(2u^2-1)u^2+t(u^5+t^3u^4+2tu^2+u+t^2u-t^5) = 0$. When $t = 0$, $u = 0 \; \pm\frac{\sqrt{2}}{2}$. Since the roots $\pm\frac{\sqrt{2}}{2}$ are distinct, by Theorem 2.2, these roots are holomorphic in a neighborhood of $t = 0$. Therefore,

$u = \sum_{n=0}^{\infty} c_n t^n$. Substituting into the equation, and equating coefficients in like powers of t, $t^4(\sum_{n=0}^{\infty} c_n t^n)^4 + t(\sum_{n=0}^{\infty} c_n t^n)^5 + 2(\sum_{n=0}^{\infty} c_n t^n)^4 + 2t^2(\sum_{n=0}^{\infty} c_n t^n)^2 - (\sum_{n=0}^{\infty} c_n t^n)^2 + t(\sum_{n=0}^{\infty} c_n t^n) + t^3(\sum_{n=0}^{\infty} c_n t^n) - t^6 = 0.$

$$2\,c_0^{\,4} - c_0^{\,2} = 0,\ c_0 = \pm \frac{\sqrt{2}}{2}.$$

$$c_0^{\,5} + 8c_0^{\,3}c_1 - 2c_0c_1 + c_0 = 0.$$

$$c_0^{\,4} + 8c_0^{\,2}c_1 - 2c_1 + 1 = 0.$$

$$c_1 = \frac{-c_0^{\,4}-1}{8c_0^{\,2}-2} = \frac{-5/4}{4-2} = \frac{-5}{8}.$$

$$5c_0^{\,4}c_1 + 2(4c_0^{\,3}c_2 + 6c_1^{\,2}c_0^{\,2}) + 2c_0^{\,2} - c_1^{\,2} - 2c_2c_0 + c_1 = 0.$$

$$c_2 = \frac{-12c_1^{\,2}c_0^{\,2} + c_1^{\,2} - 5c_0^{\,4}c_1 - 2c_0^{\,2} - c_1}{8c_0^{\,3} - 2c_0}.$$

$$c_2 = \frac{-12(25/64)(1/2) + 25/64 - 4(1/4)(-5/8) - 2(1/2) + 5/8}{\pm \frac{2\sqrt{2}}{2}} = \pm \frac{99}{64\sqrt{2}}.$$

Therefore, $u_i = \pm \frac{2}{\sqrt{2}}\ -5/8t \pm \frac{99}{64\sqrt{2}}t^2 + \ldots \quad (i = 1,\ 2).$

Since $t = z^{1/2}$, $w = z^{1/2}u$,

$$w = z^{1/2}\left(\pm \frac{\sqrt{2}}{2}\ \frac{-5z^{1/2}}{8} \pm \frac{99}{64\sqrt{2}}\,z + \ldots\right).$$

Now, $e^{i\pi}z^{1/2} = (e^{2i\pi}z)^{1/2} = z^{1/2}$, so that the series can be written simply as

$$w = z^{1/2}\left(\frac{\sqrt{2}}{2} - 5/8z^{1/2} + \frac{99}{64\sqrt{2}}\,z + \ldots\right).$$

The slope of the line connecting (1, 2) and (2, 1) is -1, so $w = tu$, $z = t$.

Substituting in the equation,

$$t^2(tu)^4+(tu)^5+2(tu)^4+2t^2(tu)^2-t(tu)^2+t^2(tu)+t^3(tu)-t^5 = 0.$$

Dividing by the factor t^3,

$t^3u^4+t^2u^5+2tu^4+2tu^2-u^2+u+tu-t^2 = 0.$ When $t = 0$, $u = 0,\ 1$. Since $u = 1$ is a simple root, $u = \sum_{n=0}^{\infty} d_n t^n$. Substituting $u = \sum_{n=0}^{\infty} d_n t^n$ into the equation,

$$t^3\left(\sum_{n=0}^{\infty} d_n t^n\right)^4 + t^2\left(\sum_{n=0}^{\infty} d_n t^n\right)^5 + 2t\left(\sum_{n=0}^{\infty} d_n t^n\right)^4 + 2t\left(\sum_{n=0}^{\infty} d_n t^n\right)^2 -$$
$$\left(\sum_{n=0}^{\infty} d_n t^n\right)^2 + \sum_{n=0}^{\infty} d_n t^n + t\sum_{n=0}^{\infty} d_n t^n - t^2 = 0.$$

$$-d_0^2 + d_0 = 0, \; d_0 = 1.$$

$$2d_0^4 + 2d_0^2 - d_1 d_0 + d_1 + d_0 = 0, \; d_1 = \frac{-2d_0^4 - 2d_0^2 - d_0}{1 - 2d_0} = 5.$$

$$d_0^5 + 8d_0^3 d_1 + 4d_1 d_0 - 2d_0 d_2 - d_1^2 + d_2 + d_1 - 1 = 0.$$

$$1 + 40 + 20 - 2d_2 - 25 + d_2 + 5 - 1 = 0, \; d_2 = 40.$$

Therefore, $u = 1 + 5z + 40z^2 + \ldots$

Since $w = tu$, $z = t$

$$w = z(1 + 5z + 40z^2 + \ldots).$$

The slope of the line connecting (2, 1) and (5, 0) is $-\frac{1}{3}$, therefore, $z = t$, $w = t^3 u$. Substituting in, $t^2(t^3u)^4 + (t^3u)^5 + 2(t^3u)^4 + 2t^2(t^3u)^2 - t(t^3u)^2 + t^2(t^3u) + t^3(t^3u) - t^5 = 0$. Dividing by t^5,

$t^9u^4 + t^{10}u^5 + 2t^7u^4 + 2t^3u^2 - t^2u^2 + u + tu - 1 = 0$. When $t = 0$, $u = 1$, so by Theorem 2.2, $u = \sum_{n=0}^{\infty} h_n t^n$.

Substituting in, $t^9\left(\sum_{n=0}^{\infty} h_n t^n\right)^4 + t^{10}\left(\sum_{n=0}^{\infty} h_n t^n\right)^5 + 2t^7\left(\sum_{n=0}^{\infty} h_n t^n\right)^4 +$

$$2t^3\left(\sum_{n=0}^{\infty} h_n t^n\right)^2 - t^2\left(\sum_{n=0}^{\infty} h_n t^n\right)^2 + \sum_{n=0}^{\infty} h_n t^n + t\sum_{n=0}^{\infty} h_n t^n - 1 = 0.$$

$$h_0 = 1.$$

$$h_1 + h_0 = 0, \; h_1 = -1.$$

$$-h_0^2 + h_2 + h_1 = 0, \; h_2 = 2$$

Therefore, $u = 1 - t + 2t^2 + \ldots$

Since $z = t$, $w = t^3 u$,

$$w = z^3(1 - z + 2z^2 + \ldots).$$

The expansions corresponding to $w = 0$ are:

$$w_1 = z^{1/2}\left(\frac{\sqrt{2}}{2} - 5/8z^{1/2} + \frac{99}{64\sqrt{2}} z + \ldots\right).$$

$$w_2 = z(1 + 5z + 40z^2 + \ldots).$$

$$w_3 = z^3(1 - z + 2z^2 + \ldots).$$

The expansion corresponding to $w = -2$ can be found by substituting $w = \sum_{n=0}^{\infty} b_n z^n$ into the equation.

$$z^2\left(\sum_{n=0}^{\infty} b_n z^n\right)^4 + \left(\sum_{n=0}^{\infty} b_n z^n\right)^5 + 2\left(\sum_{n=0}^{\infty} b_n z^n\right)^4 + 2z^2\left(\sum_{n=0}^{\infty} b_n z^n\right)^2 -$$

$$z\left(\sum_{n=0}^{\infty} b_n z^n\right)^2 + z^2\left(\sum_{n=0}^{\infty} b_n z^n\right) + z^3\left(\sum_{n=0}^{\infty} b_n z^n\right) - z^5 = 0.$$

$$b_0^5 + 2b_0^4 = 0, \; b_0 = -2.$$

$$5b_1 b_0^4 + 8b_1 b_0^3 - b_0^2 = 0, \; b_1 = \frac{b_0^2}{5b_0^4 + 8b_0^3} = 1/4.$$

$$b_0^4 + 5b_2 b_0^4 + 10b_1^2 b_0^3 + 8b_2 b_0^3 + 12b_1^2 b_0^2 + 2b_0^2 -$$

$$2b_1 b_0 + b_0 = 0.$$

$$16 + 80b_2 - 5 - 64b_2 + 8 + 1 - 2 + 3 = 0.$$

$$16b_2 = -21, \; b_2 = -21/16.$$

Therefore, $w_4 = -2 + 1/4\, z - 21/16\, z^2 + \ldots$

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ALGEBRAIC FUNCTIONS

by

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ABSTRACT

An algebraic function, $w = F(z)$, is defined by an irreducible equation of the form

$f(z, w) = f_o(z)w^n + f_1(z)w^{n-1} + \ldots + f_n(z) = 0$ where $f(z)$ is a polynomial in z, and it is understood for each z in the extended complex plane the equation is to be solved for w. For each z, $F(z) = w$ normally has n distinct values, but exceptions can occur when the discriminana of f is zero, $f_o(z) = 0$, or $z = \infty$. The set of z which satisfies one of more of these conditions is finite, and a member of the set is called a critical point. At a finite critical point, a, $F(z)$ can be expressed as Laurent Series in $(z-a)^{1/k}$ $(k \geq 1)$ with at most a finite number of negative powers. At $z = \infty$, $F(z)$ has expansions which are Laurent Series in $1/z^{1/k}$, $k \geq 1$, with at most a finite number of negative powers. Moreover, $F(z)$ is an analytic function. That is, if c and d are noncritical points then any one of the n distinct power series in $z-c$ can be continued into any one of the n distinct power series in $z-d$.

The expansion of $F(z)$ at a noncritical point, a, can be found by direct substitution of a power series $w_i = \sum_{k=0}^{\infty} c_{ik}(z-a)^k$ $(i = 1, 2, \ldots n)$ into $f(z, w) = 0$. By equating coefficients in like powers of z, the c_{ik} are determined. For a critical point a generalization of Newton's method must be used to find the expansions of $F(z)$. The proof that this method gives all expansions of $F(z)$ is quite difficult, since a great number of subcases have to be considered. However, for a given algebraic function, the method is straightforward, even though it may require numerous calculations. Essentially, the

procedure is to make substitutions of the form $w = b + t^{q_i}$, $z = a + t^{p_i}$, into $f(z, w) = 0$ until the resulting equation in t and u has noncritical point at $t = 0$. The exponents p_i and q_i are determined by plotting the exponents (i, j) of $f(z, w) = \sum a_{ij}(z-a)^i(w-b)^j = 0$ in the x-y plane. The least convex polygon which contains these points has a face nearest the origin whose sides have slope $\frac{-p_i}{q_i}$.