SOME ASPECTS OF NONLINEAR STABILITY

by

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INTRODUCTION

Solutions to problems involving systems containing nonlinear elements are seldom clear-cut. All approaches to solutions of problems of this nature are laced with approximations and assumptions which, while simplifying the solutions, are, to the degree of the approximations and assumptions, inaccurate.

The observation (1) may be made that the coefficients in equations describing physical systems are never known to a high degree of accuracy. These coefficients must be found empirically as a result of experimental measurements which are always subject to error. Especially in nonlinear systems where the coefficients are functions of the operating conditions, it is difficult to determine values of the coefficients to a high degree of accuracy. Additionally, the physical parameters are often subject to change with ambient conditions such as time, temperature, and the like. Changes of this sort are probably not included in equations describing the system. As a result, the coefficients are subject to considerable uncertainty. Thus, depending on the nature of the nonlinearity and the method involved in the solution, the solution may or may not be of a high degree of accuracy. Consequently, there is reason to question whether or not the solution finally obtained actually applies to the physical system under study.

The obvious question which arises when a physical system subject to the above observations is involved is: despite the uncertainty inherent in the given data and the method of solution of the problems involved, is there an approach that will give enough insight into the problem that corrective action can be taken on the system to insure a reasonable interpretation, on the part of the system, of the desired performance?

The answer to this question lies in definition of the word "reasonable". In general the minimum requirement for a reasonable performance on the part of the system is the requirement that the system be stable. This in turn requires definitions of system stability.

Defining Stability

The following is an attempt to integrate the ideas of Hughes (2) and Cunningham (1) regarding definitions of stability. The definitions are for three types of stability: Asymptotic, orbital, and structural.

Perhaps the original essence of the stability idea asks: if a system is initially at rest or else operating in a steady state, either of which is to say that the system is operating at some equilibrium point, and a small perturbation is applied to the system, does the system return to its initial state, depart in a monotonic increase with time from the initial state, or achieve some ultimate state different from the initial state? At first glance this seems to be a fair evaluation of what can happen to a system and for a linear system it is quite adequate, the key words being "equilibrium point".

To see the significance of this, note that a physical system may be described by a set of simultaneous differential equations of the form:

$$
\begin{aligned}
dx_1/dt &= \dot{x}_1 = f_1(x_1, x_2, x_3, \ldots, x_n) \\
dx_2/dt &= \dot{x}_2 = f_2(x_1, x_2, x_3, \ldots, x_n) \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
dx_n/dt &= x_n = f_n(x_1, x_2, x_3, \ldots, x_n)
\end{aligned}
$$

with t the independent variable. In this case $x_1, x_2, \ldots, x_n$ are the dependent variables, and functions $f_1, f_2, \ldots, f_n$ are generally nonlinear functions of the dependent variables. The simplest equilibrium points are those points where $\dot{x}_1$, $\dot{x}_2$, $\ldots$, $\dot{x}_n$ are all zero simultaneously. The system is then at rest since all of the dependent variables are constant and unvarying with time.

For a linear system, functions $f_1, f_2, \ldots, f_n$ are linear functions only and when the derivatives are set equal to zero; the linear functions give the conditions for equilibrium as:

$$
\begin{aligned}
0 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots, a_{1n}x_n \\
0 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots, a_{2n}x_n \\
\vdots \quad & \quad \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \\
0 &= a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots, a_{nn}x_n
\end{aligned}
$$

and the a_{ij} coefficients are constants arising from the parameters of the physical system. Generally the determinant of the coefficients does not vanish; i.e.,

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

In this linear case, then, it is apparent that the variables x_i satisfying the system equations must all be zero. Thus a linear system has but a single equilibrium point at which all the dependent variables vanish.

For a linear system the previous definition for stability is rigorous since the necessity for considering equilibrium points other than zero is unnecessary. For a linear system this type of stability is the so-called "asymptotic stability".

For the nonlinear system, f_1, f_2, f_3, ..., f_n are nonlinear functions and lead to nonlinear equations in place of the linear set. These nonlinear equations may lead to solutions for the x_i which are not zero and additionally, more than a single set of solutions may exist, which is equivalent to saying that nonlinear equations may have many equilibrium points.

From the above, asymptotic stability which characterizes the system as stable if it returns to the initial state, unstable if it continues to diverge from the initial state, or "neutrally" or "conditionally" stable if it attains some new equilibrium state, does not describe adequately some conditions which may appear in nonlinear systems. For the nonlinear system it is necessary to specify that the initial disturbances of the x_i be small enough to keep them in the region of the equilibrium point in question. This definition may be shown graphically. If a system under study is operating about some equilibrium point and a small but finite perturbation is applied such as shown on the phase portrait in Fig. 1, then if P repre-

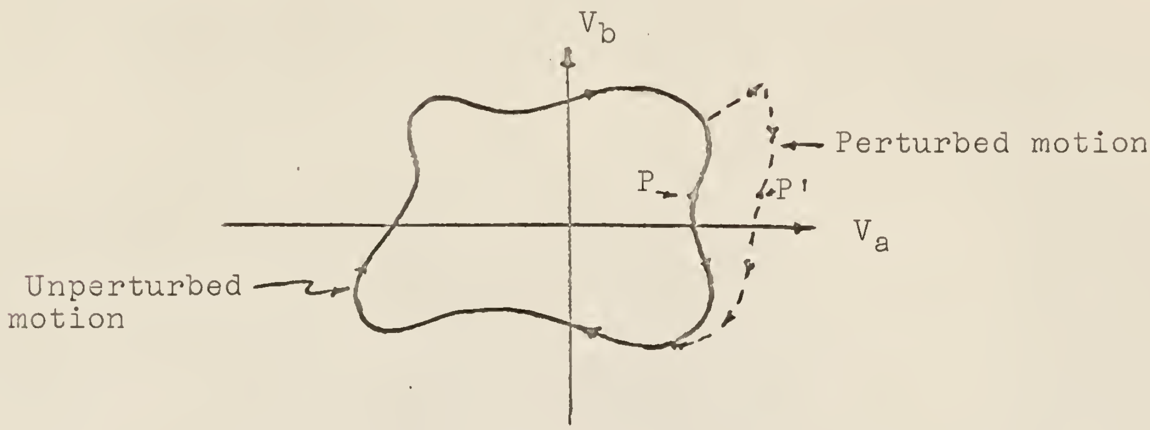


Fig. 1.

sents an unperturbed moving point on the phase portrait and P' represents the corresponding point on the perturbed portrait, and if

$$\lim_{t \to \infty} |P - P'| = 0$$

the system is said to be asymptotically stable. The motion shown is periodic but periodic motion is not necessary for the argument.

The significance of structural stability may be shown as follows (4): for nonlinear problems normally encountered, it is often necessary to express some nonlinear function of a variable and its derivatives with a finite number of terms of a power series. Consider, for example, a second order system whose phase portrait is described by

$$\frac{dV_b}{dV_a} = \frac{A(V_a, V_b)}{B(V_a, V_b)} \tag{1}$$

with A and B simply functions of V_a and V_b. Also consider a slightly different system in which the phase portrait is described by

$$\frac{dV_b}{dV_a} = \frac{A(V_a, V_b) + a(V_a, V_b)}{B(V_a, V_b) + b(V_a, V_b)} \qquad (2)$$

with $a(V_a, V_b)$ and $b(V_a, V_b)$ functions of V_a and V_b and very much smaller than $A(V_a, V_b)$ and $B(V_a, V_b)$, respectively. If the phase portraits of these two equations are plotted simultaneously with the same initial conditions, the plot might appear as shown in Fig. 2.

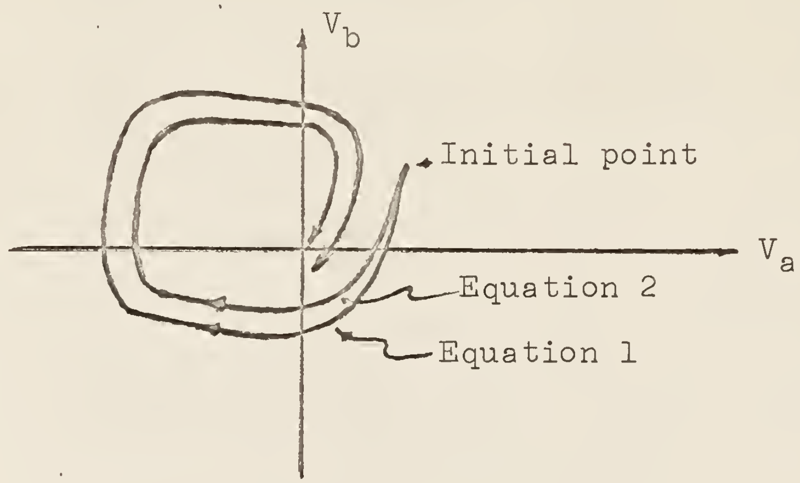


Fig. 2.

If these two phase portraits are qualitatively similar in nature, the original system may be defined as structurally stable. The above differential equations are really expressions of the nonlinear function as a finite polynomial in the regions of interest. The addition of the terms $a(V_a, V_b)$ and $b(V_a, V_b)$ means simply that the coefficients of the polynomials used to represent $A(V_a, V_b)$ and $B(V_a, V_b)$ may vary slightly. If these slight variations markedly change the system behavior as evidenced by the phase portrait, then the method used to approximate the nonlinear functions is of doubtful validity. Use of power series approximations and incremental solutions is thus complicated by structural instability. The indication is that, at best, incremental solutions must be much more precise in order to get a reasonable determination of system characteristics.

Finally, consider a system having a steady-state oscillatory motion represented as a closed curve in the phase plane as in Fig. 3. If the given motion is slightly perturbed and the new motion remains within the immediate vicinity of the old

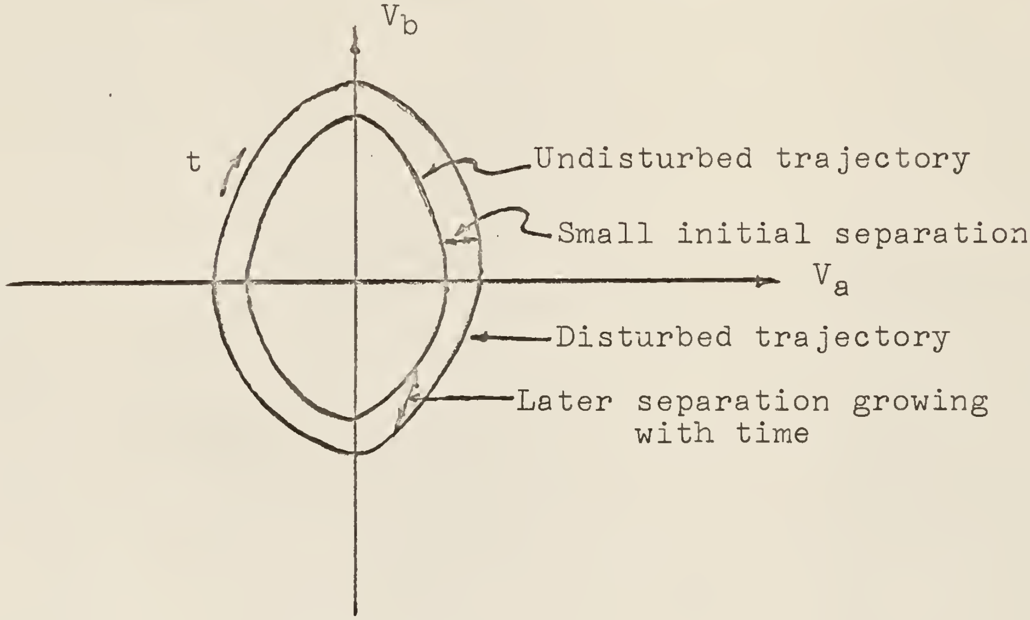


Fig. 3.

motion, then the system is said to be orbitally stable. More rigorously (2) in the case of periodic motion, a system is orbitally stable if for a given perturbation ϵ there is a number δ such that the distance between two corresponding points on these two phase portraits is never greater than δ. In view of this definition and the definition for asymptotic stability, a conservative nonlinear oscillating system is not orbitally unstable, although it is asymptotically unstable.

The purpose of this report is to review some of the methods of determining stability aspects of nonlinear systems.

THE DESCRIBING FUNCTION APPROACH

The so-called "describing function" is an application of the principle of harmonic balance to various nonlinear elements. The background for the principle of harmonic balance and the describing function is discussed in Appendix A. Describing functions for many types of nonlinearities have been derived, as in Kuo (3) and Gibson (5). As an example of the derivation of a describing function, which will be used in a stability study, take the case of a transmission element saturating abruptly (1).

Many physical transmission elements have the property of saturation. The output quantity is related linearly to the input quantity so long as the input magnitude is less than some critical value. If saturation is assumed to take place abruptly, the input-output relation can be described simply in terms of several linear algebraic equations. If the input instantaneous value is x and if the output instantaneous value is y, they may be related as in Fig. 4.

These relations are

$$-x_c \leq x \leq + x_c \quad - \quad y = kx$$

$$x \geq + x_c \quad - \quad y = +kx_c$$

$$x \leq - x_c \quad - \quad y = -kx_c$$

where k is a positive constant. Electronic amplifiers often saturate abruptly in this fashion.

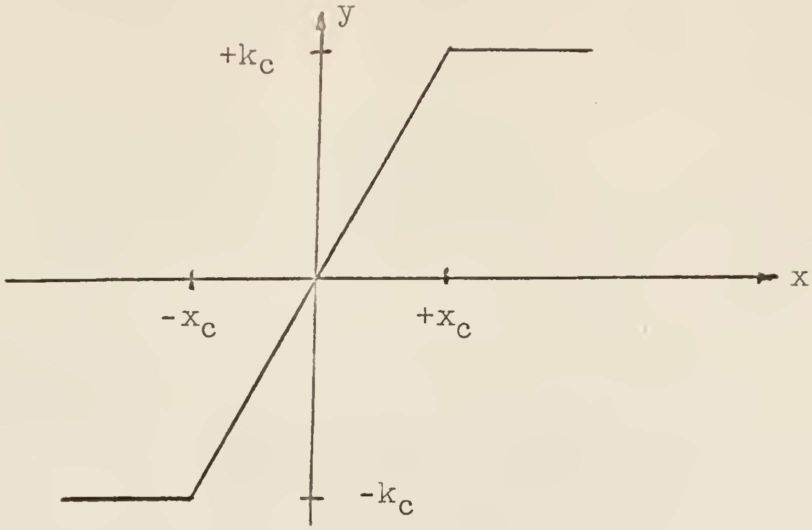


Fig. 4.

The describing function H may be found for this type of element by assuming that the input quantity varies in a simple harmonic fashion as $x = X \cos \omega t$, where X is the amplitude and ω is the angular frequency. The output quantity will also vary in a simple harmonic fashion if $X \leq x_c$, in which case $y = kx \cos \omega t$. If $X > x_c$, the output quantity is not simple harmonic but must be expressed as a Fourier series. The component of fundamental frequency is the one of interest and it is found in the usual manner. If the input and output quantities are plotted as functions of time, the result is shown in Fig. 5.

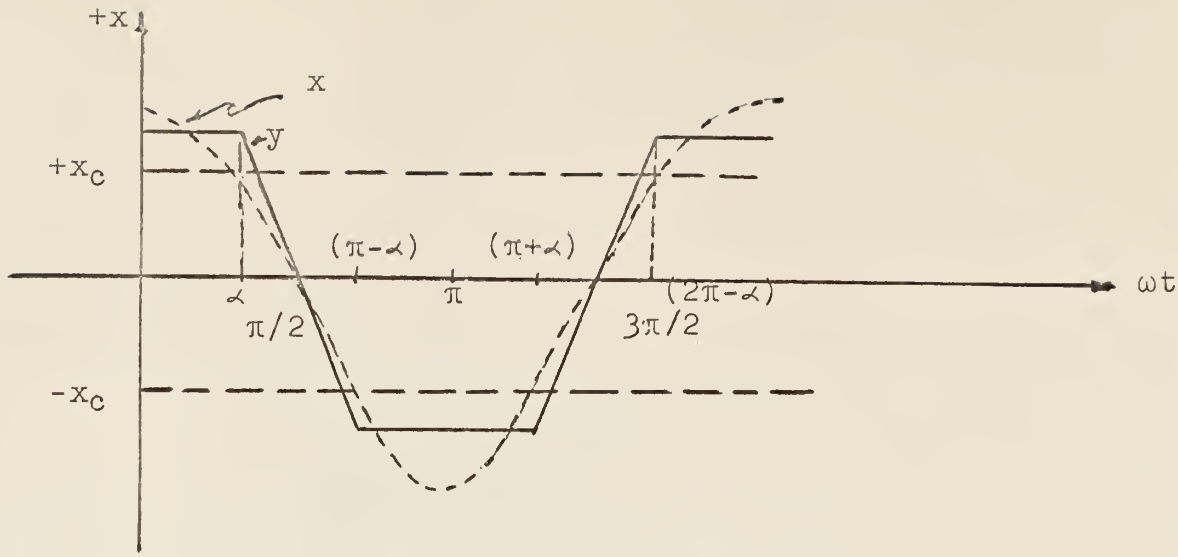


Fig. 5.

The relationships for the first half cycle are:

$0 \leq \omega t \leq \alpha$	$x \geq + x_c$	$y = +kx_c$
$\alpha \leq \omega t \leq \pi - \alpha$	$-x_c \leq x \leq + x_x$	$y = +kx \cos \omega t$
$\pi - \alpha \leq \omega t \leq \pi$	$x \leq -x_c$	$y = -kx_c$

where α is the angle for which $\cos \alpha = \dfrac{x_c}{x}$ and $X > x_c$. The second half cycle is symmetrical with the first, and thus it is unnecessary to consider it in the Fourier analysis. Because of the symmetry only cosine components appear in the output wave. The amplitude of the fundamental cosine component is:

$$A_1 = \frac{2}{\pi}\left[\int_0^{\alpha} kx_c \cos \omega t \, d(\omega t) + \int_{\alpha}^{\pi-\alpha} kx \cos^2 \omega t \, d(\omega t) + \int_{(\pi-\alpha)}^{\pi} (-kx_c) \cos \omega t \, d(\omega t)\right]$$

Carrying out the integration,

$$A_1 = \frac{2}{\pi}\left\{ kx_c \left[\int_0^{\alpha} \cos \omega t \, d(\omega t) - \int_{(\pi-\alpha)}^{\pi} \cos \omega t \, d(\omega t) \right] + kx \int_{\alpha}^{(\pi-\alpha)} \cos^2 \omega t \, d(\omega t) \right\}$$

$$A_1 = \frac{2}{\pi} kx_c \left[\sin \omega t \Big|_0^{\alpha} - \sin \omega t \Big|_{(\pi-\alpha)}^{\pi} \right] + \frac{2}{\pi} kx \left[\frac{1}{2} \sin \omega t \cos \omega t + \frac{1}{2} \omega t \Big|_{\alpha}^{(\pi-\alpha)} \right]$$

$$A_1 = kx \left[1 - \frac{2}{\pi} (\alpha - \sin \alpha \cos \alpha) \right]$$

If $X = x_c$, then $\alpha = 0$ and $A_1 = kx_c$. If $X >> x_c$, then $\alpha \to \frac{\pi}{2}$ and $A_1 \to \frac{4kx_c}{\pi}$.

The describing function is then the ratio $H = \frac{A_1}{x}$ and is

$$X \leq x_c \qquad H = k$$

$$X \geq x_c \qquad H = k \left[1 - \frac{2}{\pi} (\alpha - \sin \alpha \cos \alpha) \right]$$

where $\cos \alpha = \frac{x_c}{x}$. The describing function depends upon the input amplitude but not upon the frequency. It is a real number having magnitude but zero angle. It varies as a function of the ratio $\frac{X}{x_c}$ as shown in Fig. 6.

The describing function may be put to further use in a stability study after some preliminary remarks about Nyquist's criterion and the stability of feedback systems.

Assuming a knowledge of the Routh-Hurwitz criterion (7) for stability and the characteristic equation, this criterion may be

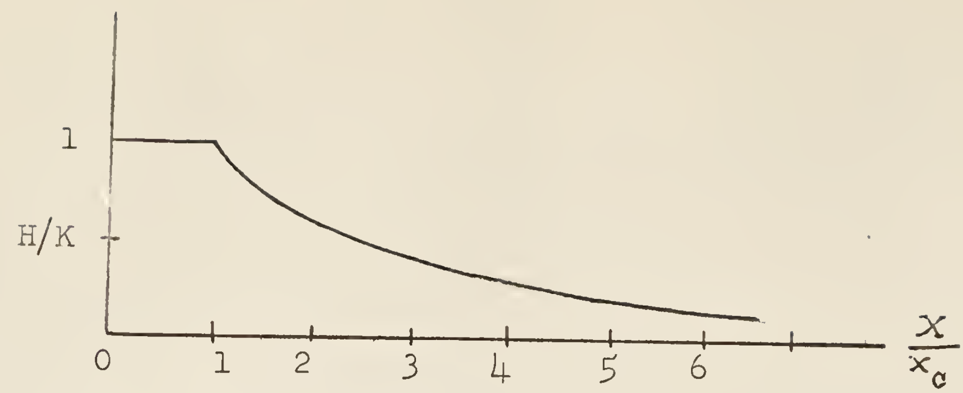


Fig. 6.

stated simply by saying that the requirement for stability is that the characteristic equation and its coefficients be positive, and further that the characteristic equation have no roots with positive real parts (or no pure imaginary roots, in a practical sense). The Routh-Hurwitz criterion, however, does not give any information concerning methods of improving the system.

The Nyquist criterion possesses the following features which make it particularly desirable for the stability analysis of feedback control systems (3).

1. It provides the same amount of information on the absolute stability of a feedback system as the Routh criterion.
2. In addition to the absolute system stability, the Nyquist criterion also indicates the degree of stability of a stable system and gives information about how the system stability may be improved if necessary.
3. The Nyquist locus gives information concerning the frequency response of the system.

The Nyquist method is based on conformal mapping of complex quantities and stripped of its rigorous background is quite simple to use.

Note (1) that the characteristic equation may be written

$$f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} \ . \ . \ . \ a_{n-1}\lambda + a_n = 0$$

The value of λ in the characteristic equation is generally a complex number and can be represented on the complex plane as in Fig. 7 with $\lambda = \delta + j\omega$. Any value of λ that is a root of the characteristic equation and leads to an unstable solution has a

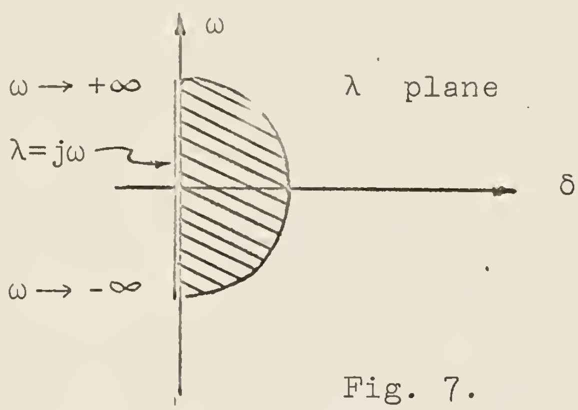


Fig. 7.

positive real part and would be located on the right half of the complex λ plane. This region is shown shaded in Fig. 7. Its boundaries can be traced out by starting at the lower end of the $j\omega$ axis where $\lambda = -j\omega \longrightarrow -j\infty$, moving up this axis until $\lambda = j\omega \longrightarrow +j\infty$, turning clockwise through a right angle, and returning to the starting point along a semicircle of very large radius. At the starting point, a second clockwise right angle turn is needed to begin retracing the original path. The shaded region is always to the right of this boundary as it is traced in the direction indicated.

The algebraic function $f(\lambda)$ is involved in the characteristic equation. Plot this function on the f plane as shown below as λ traces out the boundary of the figure just described for $f(\lambda) = \lambda^2 + g\lambda + h$, g and h constants. λ is allowed to be pure imaginary and takes on successive values between $-j\infty$ and $+j\infty$. Corresponding values of $f(j\omega)$ are calculated and plotted. This leads to the heavy curves shown on the λ plane (Fig. 7) and f plane (Fig. 8). The dotted curve of the f plane is found by tracing the solid curve to the point $\omega \longrightarrow +\infty$, turning clockwise through a right angle and returning to the point $\omega \longrightarrow -\infty$ along a circular path. As the boundaries in either the λ plane or f plane are traced in the directions indicated, the shaded areas correspond and are located to the right of the path.

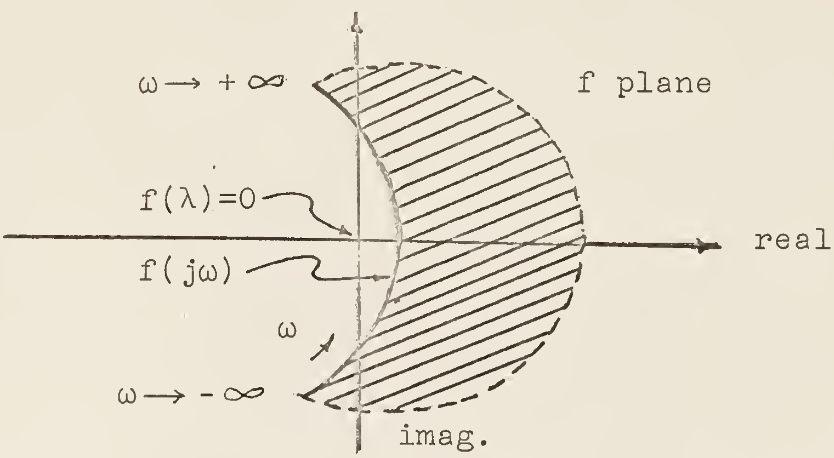


Fig. 8.

Since the polynomial $f(\lambda)$ is of degree n in λ, each point on the f plane has corresponding to it n points, generally all different, in the λ plane. By the same token, a given point on the plane has only one point corresponding to it on the f plane. Any point in the shaded region of the λ plane must have corresponding to it a point in the shaded region of the f plane.

These points represent values of λ which would lead to unstable solutions.

The characteristic equation is $f(\lambda) = 0$, which is represented by the origin of the f plane. Thus the roots of the characteristic equation, which determines properties of the solution for the original differential equation, are represented by those points in the λ plane which correspond to the origin of the f plane. Instability is indicated and at least one of the roots has a positive real part if the origin of the f plane is in the shaded area.

In the analysis of feedback systems the equation being studied often involves fractions of the form $f(\lambda) = 1 + \dfrac{P_1(\lambda)}{P_2(\lambda)}$ where $P_1(\lambda)$ and $P_2(\lambda)$ are polynomials in λ. Here $f(\lambda)$ may become infinite for certain values of $\lambda = j\omega$ such that $P_2(\lambda) = P_2(j\omega) = 0$. These are the poles of $f(\lambda)$. The procedure for mapping is to avoid the poles of $f(\lambda)$ by following a small semicircle around them. If $f(\lambda) = 1 + \dfrac{k}{\lambda(\lambda + \ell)}$, where k and ℓ are constants, a pole is located at $\lambda = 0$. This is analogous to the case already discussed. The path to be followed is shown in Fig. 9a, where a small detour is needed around the origin of the λ plane. The corresponding path in the f plane is shown in Fig. 9b, where a large semicircle appears corresponding to the small semicircle about the origin of the f plane. In each case as ω increases from negative to positive values, a clockwise right-angled turn is made as the semicircular path is entered. Figure 9 again indicates a stable system. Typically the Nyquist method involves considerable numerical calculation.

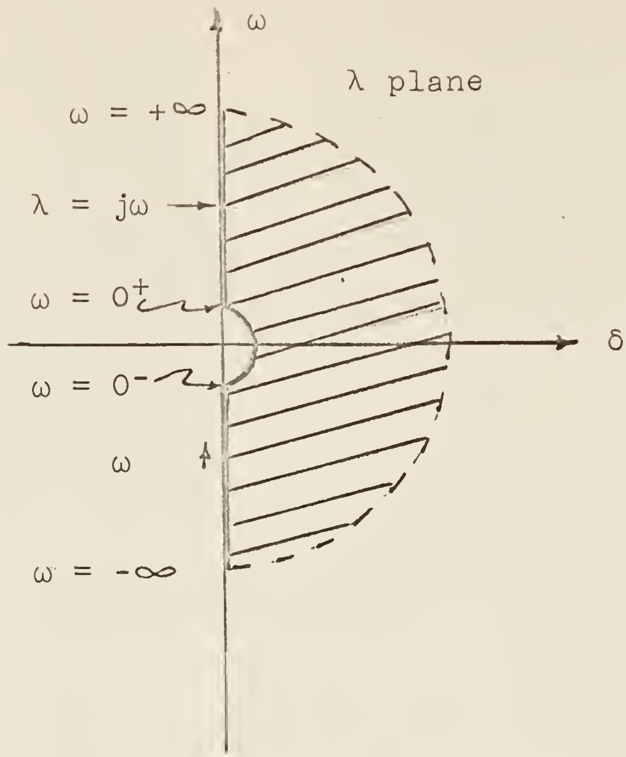
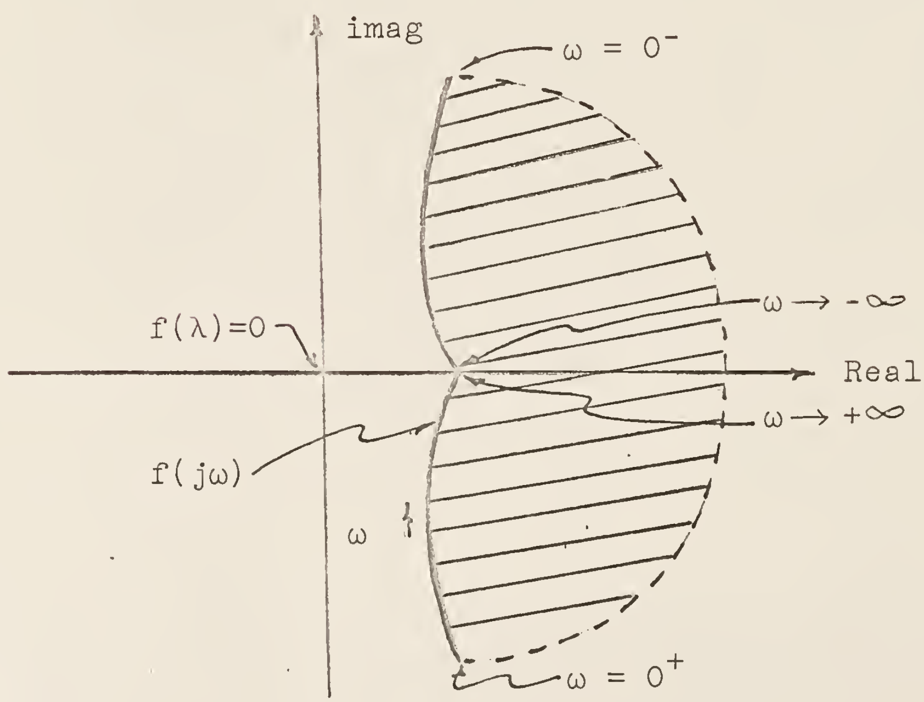


Fig. 9a.



Fib. 9b.

Another remark is in order concerning the application of the Nyquist criterion. Block diagrams as shown in Fig. 10 are frequently used in describing feedback and other systems.

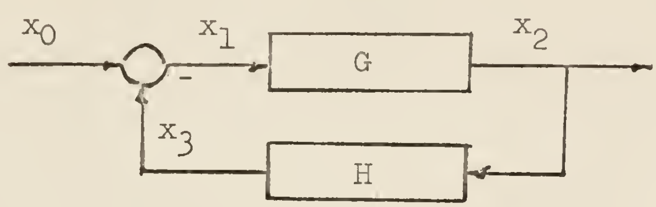


Fig. 10.

The system above may be represented by the equations

$$x_1 = x_0 - x_3 = \frac{x_2}{G(D)} = x_0 - H(D)x_2$$

or $$x_2 = \frac{G(D)x_0}{1 + G(D)H(D)}$$

where G and H are generally functions of the derivative operator $D = \frac{d}{dt}$.

Asymptotic stability of the system is determined by what happens in the absence of any input signal to the system, so that $x_0 = 0$. With $x_0 = 0$, $x_2(1 + G(D)H(D)) = 0$. Typically, a solution for this differential equation is assumed as $x_2 = Xe^{\lambda t}$, where X is an arbitrary constant and λ is the characteristic exponent. Substituting the solution into $(1 + G(D)H(D))x_2 = 0$ yields $f(\lambda) = 1 + G(\lambda)H(\lambda) = 0$. Thus the system stability may be tested by the Nyquist method. In applying the Nyquist plot it is slightly simpler to plot not $f(j\omega)$ but rather the curve representing just the product $G(j\omega)H(j\omega)$. Stability is governed by the relation of this curve to the point at $-1 + j0$. In addition, any physical system is a low-pass device so that the product $G(j\omega)H(j\omega)$ approaches zero as ω approaches infinity. Therefore the curve representing the product closes on itself. The system is stable if the curve for $G(j\omega)H(j\omega)$ does not enclose the point $-1 + j0$. The system is unstable if the point is enclosed.

In applying the Nyquist criterion as described, functions G and H for the parts of the system become merely the transfer functions defined with simple harmonic variations. If some part of the system is slightly nonlinear, its transfer function becomes the describing function as previously noted. Provided that the nonlinearity is not too great and that the wave forms in the system are essentially sinusoidal in shape, a prediction of this kind may be essentially correct. Testing the stability in this manner is open to question if the system is such that the wave forms depart considerably from a sinusoidal shape.

In view of the preceding discussions, an analysis may be made of a system containing a nonlinear element utilizing the techniques described (1). The circuit shown in the block diagram in Fig. 11 consists of an electronic amplifier with a phase-shifting network connected between its output and input terminals. Since any practical amplifier saturates if the magnitude of the signal voltage applied to it becomes too large, the amplifier becomes nonlinear in the large signal node.

Assume at first that the amplifier is linear without saturation effects, that is, operating in the small signal mode. The limiter is thus not considered, and therefore e_2 and e_3 are

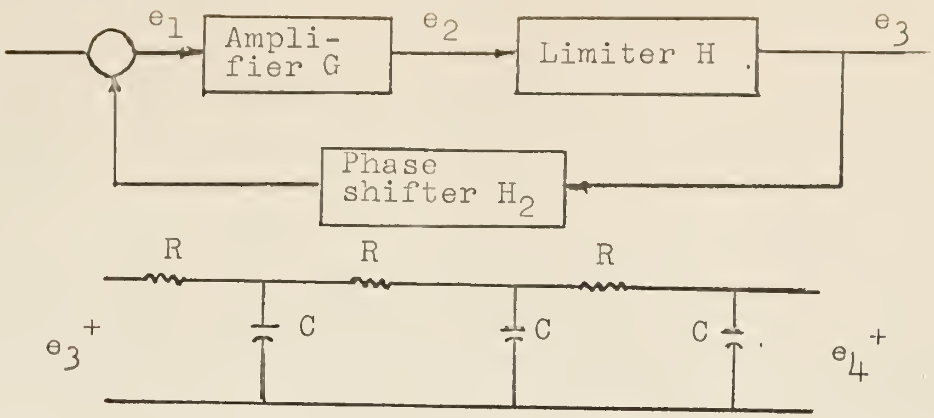


Fig. 11.

identical. A number of phase-shifting networks may be employed with oscillators of this type. The one shown is sometimes used. The transfer function for this phase-shifting network is (4):

$$H_2(D) = \frac{e_4}{e_3} = \frac{1}{(RCD)^3 + 5(RCD)^2 + 6(RCD) + 1}$$

Attenuation and phase shift both occur on a signal passing through this network and both increase as frequency increases.

Specifically, take the amplifier as a single-stage vacuum-tube circuit with voltage amplification $G - \frac{e_2}{e_1} = 35$. Typically, this amplifier would reverse the polarity of the signal, justifying the negative sign on e_4 and the algebraic signs of the equations. In this case, then, the stability of the system is given by

$$1 + GH_2(\lambda) = 0$$

and the test for stability consists of, in part, plotting

$$GH_2(j\omega) = \frac{35}{\left[-5(RC\omega)^2 + 1\right] + j\left[-(RC\omega)^3 + 6(RC\omega)\right]}$$

as ω varies from $-\infty$ to $+\infty$. This curve is shown as Fig. 12.

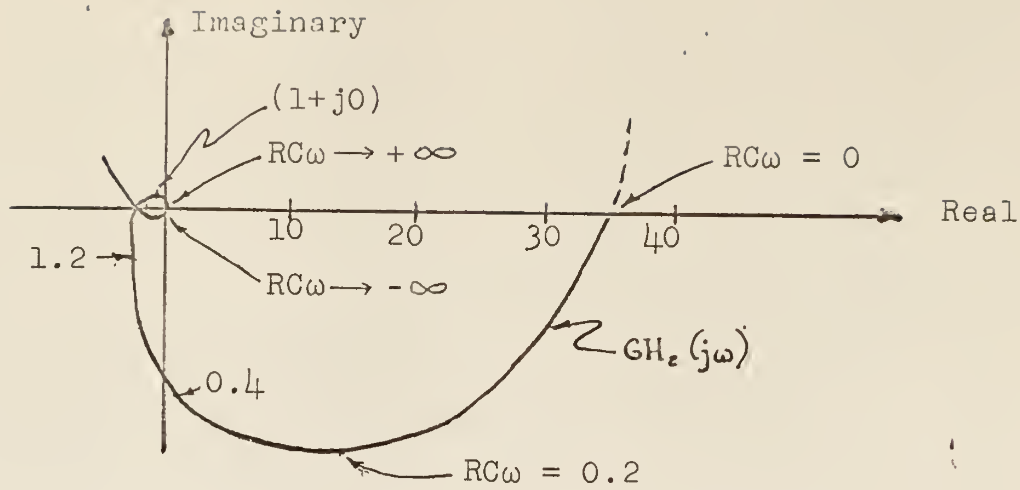


Fig. 12.

As ω increases in the positive direction, the point -1 + j0 is always to the right of the curve and is completely circled by the curve. The system is asymptotically unstable, and oscillation will occur with increasing amplitude.

A practical amplifier always saturates if the applied signal becomes too large. The saturation effect can be considered by including a limiter following the ideal amplifier as was shown in Fig. 11. A simple limiter was the object of one of the preceding discussions and the results will be used here. The limiter is assumed to saturate abruptly when the magnitude of the input voltage e_2 becomes too large; that is, exceeds a critical value e_c, so that the following relationships apply.

$$-e_c \leq e_2 \leq +e_c: \qquad e_3 = e_2$$

$$e_2 \geq e_c: \qquad e_3 = +e_c$$

$$e_2 \leq -e_c: \qquad e_3 = -e_c$$

These are the relationships of the describing function example with k replaced by unity. With a sinusoidal input voltage of amplitude E_2 applied to the limiter, the describing function for it has been shown to be

$E_2 \leq e_c$: $\qquad H = 1$

$E_2 \geq e_c$: $\qquad H = \left[1 - \frac{2}{\pi}(\alpha - \sin\alpha\cos\alpha)\right]$

where $\alpha = \frac{e_c}{E_2}$ and $k = 1$.

The stability of the system including the limiter is accordingly governed by the relation

$$1 + GH(E_2)H_2(\lambda) = 0 \qquad (1)$$

This is the equation

$$f(\lambda) = 1 + H_1(\lambda)H_2(\lambda) \quad \text{with } H_1 = GH(E_2)$$

where $H(E_2)$ is the describing function for the limiter. The equation can be put into more convenient form for study by writing

$$\frac{1}{GH_2(\lambda)} = -H(E_2) \qquad (2)$$

which is the equation for steady-state oscillation. For this example $G = 35$, $H_2(\lambda)$ was previously given with D replaced by λ, and $H(E_2)$ is given by the final figure (Fig. 3) with $k = 1$ and $X = E_2$. In Fig. 13 are plotted two curves representing the two sides of equation (2) with λ replaced by $j\omega$. The left side of the equation is a curve with certain values of the quantity $RC\omega$ indicated along it. The right side of the equation is simply that portion of the negative real axis lying between the point $-1 + j0$ and the origin. Certain values of the ratio $\frac{E_2}{e_c}$ are indicated along this line. Equation (2) is satisfied when the two curves intersect, and conditions determined by the intersection point correspond to steady-state conditions in the nonlinear circuit. Essentially the effect is this: because of the asymptotic instability, the amplitude of oscillation increases from an initial small value; however, the limiter reduces the effective amplification until at steady state the effective amplification plus limiter is just sufficient to overcome attenuation in the phase-shift network.

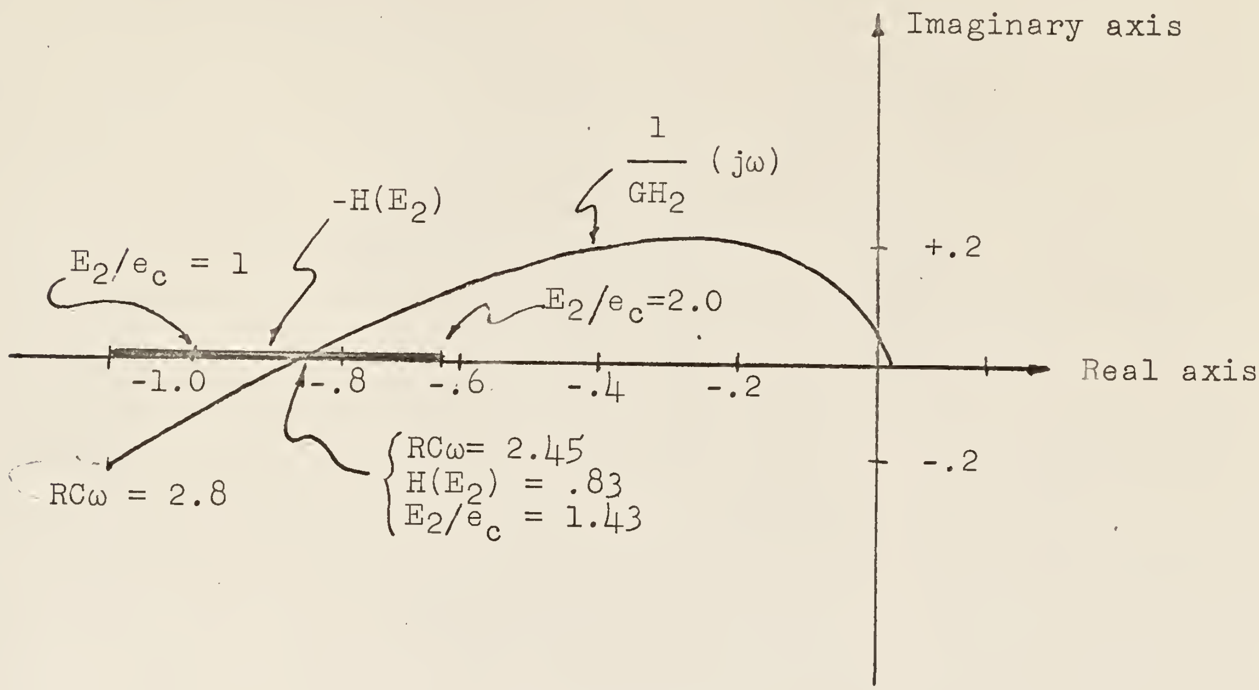


Fig. 13.

The intersection of the curves above with the circuit parameters used in the example indicates that steady-state oscillation should occur with a frequency of RCω = 2.45 and an amplitude such that $\frac{E_2}{e_c} = 1.43$. The flat-topped wave of the output as shown in the section on the describing function should be somewhat similar to the limiter output wave form. The limiter output is a sinusoid with a peak 1.43 times the limiting level. Since the phase-shifting network is a low-pass filter, the wave form at the input of the amplifier should be nearly sinusoidal. If this is the case, the prediction based on the describing function for the nonlinear element should be fairly accurate. If amplification G were increased, a larger quantitative error could be expected. As can be seen from the output in the section on describing functions, the greater the gain, G (X in that discussion), the more nearly the output approaches a square wave and the less applicable the describing function method becomes.

ANALYSIS BY MEANS OF SINGULAR POINTS

The tunnel diode is an example of a two-terminal resistance in which the instantaneous current i_r and instantaneous voltage e_r are related by a curve similar to that of Fig. 14 (1, 9). Other means are possible to obtain this curve. Utilizing the ideas in Appendix B concerning singular points and considering the nonlinear element as being composed of several linear elements, a stability analysis can often be made.

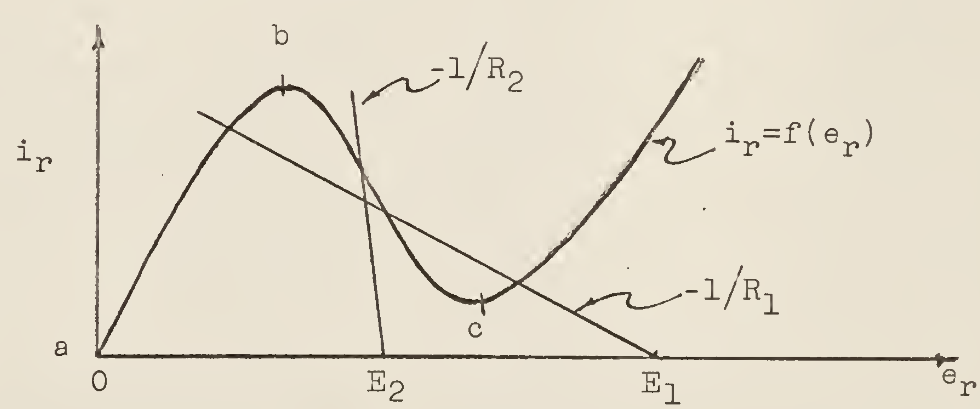


Fig. 14.

The functional relationship is $i_r = f(e_r)$ and may possibly be found by experiment or expressed in analytic form. In the central region the variational resistance is negative, implying a source of power. For the tunnel diode, the region between points b and c on the curve represents a decrease in the tunneling effect with subsequent decrease in current as the valence band in one region of the diode is raised to a position opposite a forbidden band in the other region by the increasing voltage (9). The variational resistance at any point on the curve is defined by $r = \dfrac{1}{\dfrac{di_r}{de_r}}$.

Consider the circuit in Fig. 15 for an analysis by means of singular points where the components other than the box are linear and a constant voltage E is applied to the circuit.

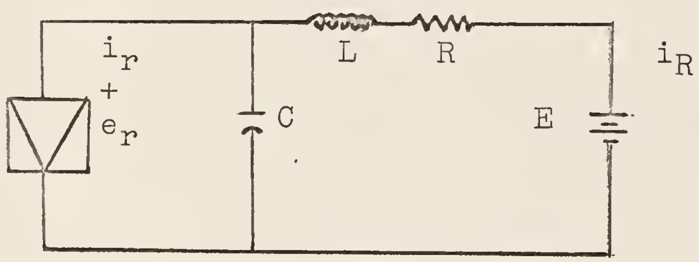


Fig. 15.

Let i_R be the battery current and i_r the current through the nonlinear element. The current through the capacitor is thus $(i_R - i_r)$. The loop equations are

$$-e_r = \frac{1}{C}\int \left[f(e_r) - i_r\right]dt$$

$$\frac{de_r}{dt} = \frac{1}{C}\left[i_r - f(e_r)\right] \tag{1}$$

and

$$E = Ri_R + \frac{Ldi_R}{dt} + \frac{1}{C}\int (i_R - i_r)\,dt$$

$$\frac{di_R}{dt} = \frac{1}{L}(E - Ri_R - e_r) \tag{2}$$

The singular or equilibrium points exist where $\frac{di_R}{dt} = 0$ and $\frac{de_r}{dt} = 0$ simultaneously. Thus at the singular points:

$$\left.\begin{aligned} E &= Ri_{RS} + e_{rS} \\ i_{RS} &= f(e_r) = i_{rS} \end{aligned}\right\}$$

The fact that at a singularity $i_{RS} = i_{rS}$ indicates that the capacitor current is zero.

A conventional technique can be used to determine the singularities. From a point on the e_r axis corresponding to steady voltage E, a load line may be erected with slope determined by resistance R. The slope of this line is $-\frac{1}{R}$. The intersection of $f(e_r)$ with this load line determines the singular points. The singularities are thus dependent not only upon the nonlinear characteristic but upon the applied voltage and loading resistor as well. Cases of special interest have an intersection in the negative resistance region. In Fig. 1 two cases are shown; the three intersection and single intersection cases. There is a basic difference in operation in the two cases, the first case being a switching operation and the second case leading to an oscillation.

Following the route described in Appendix B, near a singularity the substitutions can be made:

$$i_R = i_{RS} + i \quad \text{and} \quad e_R = e_{rS} + e$$

where i and e are small changes. Equations (1) and (2) thus become:

$$\frac{de}{dt} = \left(\frac{1}{C}\right)\cdot\left[i - \left(\frac{1}{r}\right)e\right] \tag{3}$$

$$\frac{di}{dt} = \left(\frac{1}{L}\right)(-Ri - e) \tag{4}$$

or

$$\frac{di}{de} = \frac{\left(\frac{1}{L}\right)(-e - Ri)}{\left(\frac{1}{C}\right)\left[-\left(\frac{1}{r}\right)e + 1\right]} \tag{5}$$

From (5) coefficients may be written:

$$a = -\left(\frac{1}{r}\right)C,\ b = \frac{1}{C},\ C = -\frac{1}{L},\ d = -\frac{R}{L}$$

and the characteristic roots are:

$$\lambda_1, \lambda_2 = \frac{1}{2}\left\{-\left(\frac{1}{rC} + \frac{R}{L}\right) \pm \left[\left(\frac{1}{RC} + \frac{R}{L}\right)^2 + \left(\frac{4}{LC}\right)\left(-1 - \frac{R}{r}\right)\right]^{1/2}\right\} \tag{6}$$

An analysis can now be made for various conditions in terms of the characteristic roots λ_1 and λ_2.

Now consider the straight-line approximation to the true characteristic shown in Fig. 16. The figure may be divided into three regions. Taking as a first case the situation where E and R are large enough so that three intersections occur, the variational resistances are $-r_1$, $+r_2$, and $+r_3$ in regions 1, 2, and 3, respectively. The slope of the load line is $-1/R$ in all regions. Now the types of singularities in each region may be examined. In region 1 the slope of the load line, $-1/R$, is less in magnitude than the slope of the negative resistance characteristic, $-1/r_1$. Thus $R > r_1$. Substitution into equation (6) indicates

$$\lambda_1, \lambda_2 = \frac{1}{2}\left\{-\left(\frac{1}{rC} + \frac{R}{L}\right) \pm \left[\left(\frac{1}{rC} + \frac{R}{L}\right)^2 + \frac{4}{LC}\ (\text{positive number})\right]^{1/2}\right\}$$

and the characteristic roots are real and of opposite sign. The singularity in this region is a saddle and unstable.

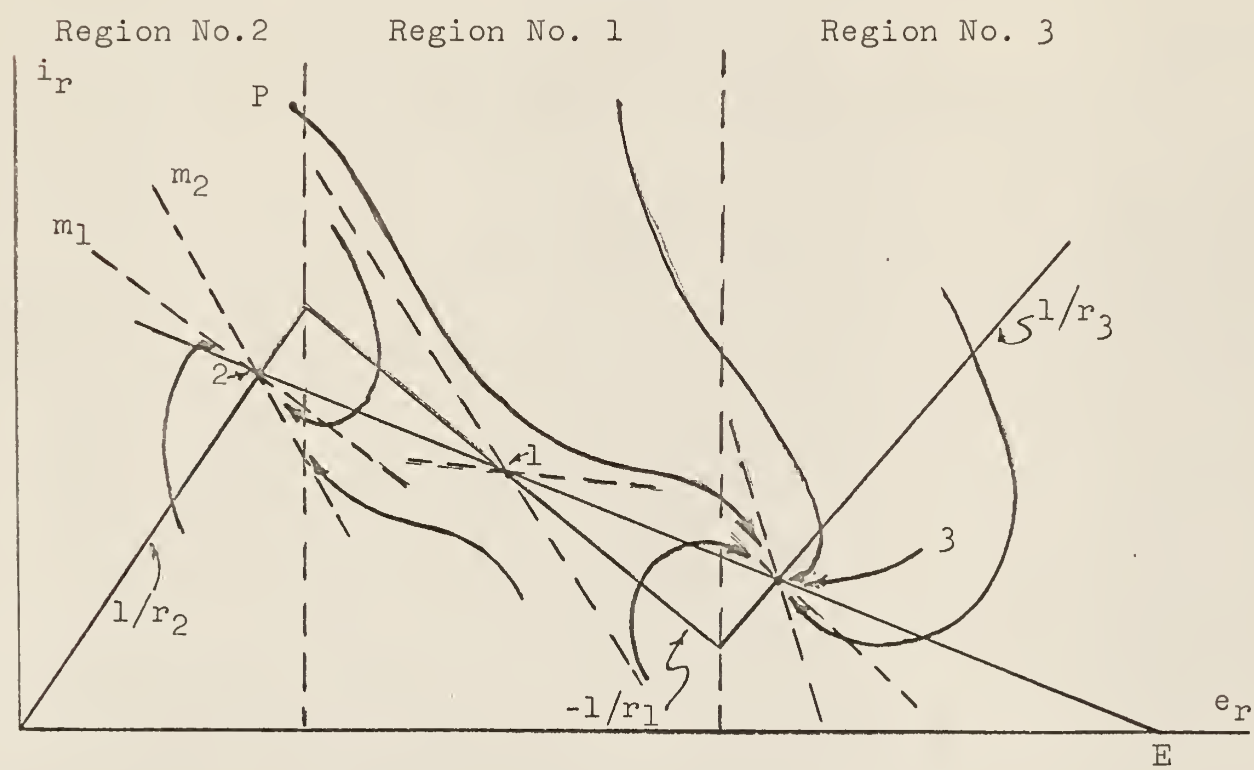


Fig. 16.

The variational resistances are $+r_2$ and $+r_3$ in regions 2 and 3, respectively. Substitution into equation (6) indicates that the singularities in these regions are stable and are either nodes or foci depending upon the relations among the circuit element values. The circuit used under these conditions can be the basis for certain types of triggering circuits characterized as having two stable states separated by an unstable state, the triggering action coming from suitable influences applied from outside.

Several solution curves for small variations in current and voltage are shown sketched in Fig. 16. Since the types of singularities are known, the only additional knowledge necessary is that $\frac{di}{de} = 0$ along $i = -e/r$, the resistance load line, and $\frac{di}{de} = \infty$ along $i = e/r$, the characteristic of the nonlinear resistance element, plus the slopes of m_1 and m_2 near each singularity. As noted, in regions 2 and 3 the singularities are stable and are now assumed to be nodes. The curves represent the way small variations i in the current i_R and e in the voltage e_r change with time following some small initial disturbance of the system. Within any one region the system is assumed linear so i and e are not necessarily limited to small values. If the system starts at some initial point it does not necessarily come to rest at the equilibrium point nearest the initial point, as may be seen by observing point P in Fig. 16; but the system will come to rest at one of the two stable points.

If it is specifically assumed that the circuit is in the stable state characterized by the singularity in region 2 and voltage E is then raised by an amount ΔE sufficient to cause the load line to shift, a possible situation is shown in Fig. 17. ΔE is actually a large voltage for the condition shown. The

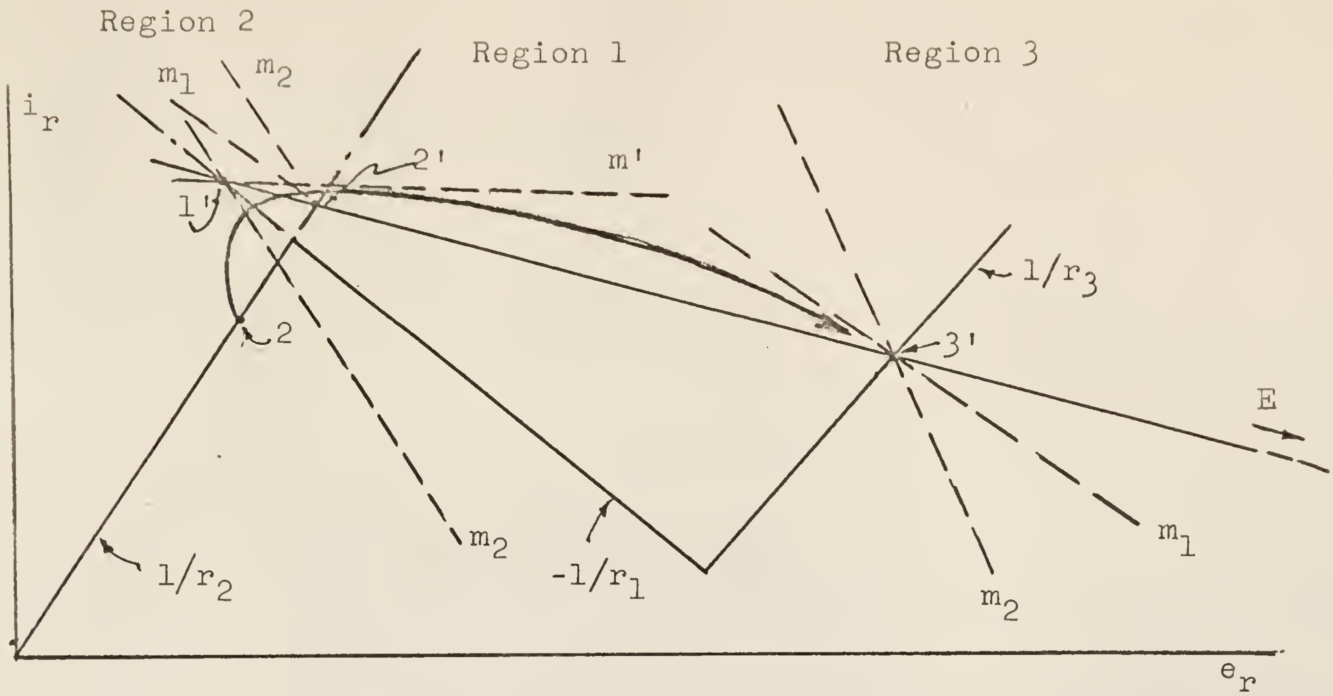


Fig. 17.

singularity in region 3 has moved to position 3'. However, singularities associated with regions 1 and 2 have moved outside these regions. If the straight-line characteristics of the non-linear resistance are extended, however, "virtual" singularities can be located as shown in Fig. 17. Virtual singularity 1' is associated with region 1 even though it is actually located in region 2. By the same token, 2' is associated with region 2 even though it is located in region 1. The singularities are of the same type as before the change in E and solution curves are sketched as before.

In the case where only one intersection occurs, $R < r_1$. Figure 18 indicates this situation.

The singularity is located in the negative resistance region. Extending the portions of the characteristic with

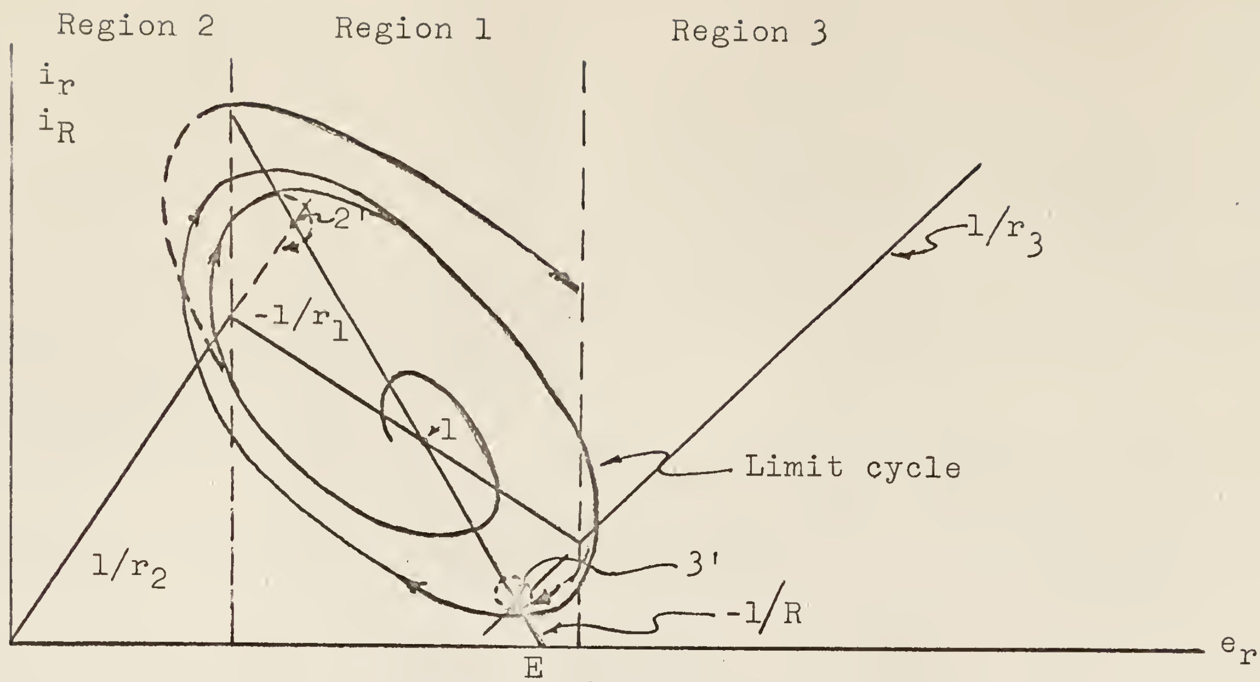


Fig. 18.

positive slope two virtual singularities may again be found, this time in region 1. If it is possible for the components to be chosen such that $r_1 < L/RC$, the magnitude of the resistance in the region of negative slope is less than the impedance of the parallel LCR circuit at its resonant frequency. Again substituting in equation (6) indicates that λ_1 and λ_2 must have positive real parts in region 1. Since $R < r_1$, the singularity cannot be a saddle and must be a node or focus. If it is a focus,

$$\left(\frac{4}{LC}\right)\left(1 + \frac{R}{r_1}\right) > \left(\frac{1}{r_1 C} + \frac{R}{L}\right)^2$$

and oscillations are implied. For the virtual singularities the argument is: $r > 0$, λ_1 and λ_2 have negative real parts from (6), and the operation is stable.

In Fig. 18 it is assumed that an unstable focus exists with singularity 1 while stable foci exist for both 2' and 3'. The solution curves relating instantaneous values of i_R and e_r are spiral curves, spiraling outward from singularity 1. When the curves cross into regions 2 or 3, they become spiral curves going inward toward virtual singularities 2' and 3'. An important result of these actions is that eventually a closed solution is reached, as indicated by the elliptical path in Fig. 18. This closed curve is the "limit cycle" and appears only in systems having nonlinear negative resistance. Quantitatively, the existence of the limit cycle may be justified. If initially the amplitude is very small, the solution curve is entirely in region 1 and the amplitude must grow. Alternatively, if an extremely large amplitude were to exist initially, the solution curve would be mostly in regions 2 and 3, where the amplitude decays. Such a small portion of the curve would be in region 1, with increasing amplitude, that the net effect would be decay. Therefore initial large amplitude decays, initial small amplitude grows, and there must be some intermediate amplitude which neither grows nor decays.

Experiments with a circuit similar in nature to that of Fig. 15 containing a tunnel diode, voltage source, magnetic coil, and resistance in a configuration such as Fig. 19, indicate that at least the inductance or capacitance of Fig. 15 must be considered for the switching or oscillation operations.

By means of Fig. 17, an attempt can be made to show why these actual components or else parasitic reactances must be

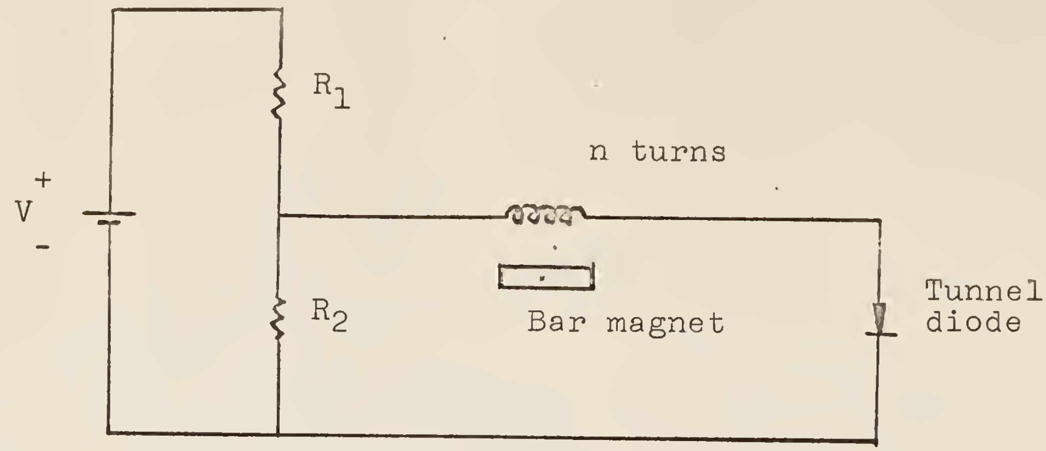


Fig. 19.

utilized. Figure 17 indicates that at least one reactive element is essential for an explanation of what takes place during the interval of triggering. During this interval the path followed must be the point relating instantaneous values of i_r, and e_r must be the characteristic of the nonlinear resistance element. If no reactances were present, the point relating instantaneous values of i_R and e_r would have to move along the straight-line path of the resistance load line. Without reactance, i_R must equal i_r and points corresponding to these two currents must coincide. As previously noted, this is not the case except at singularities and triggering between singularities could not occur. Thus a difference between paths could not occur except through a component of voltage drop across the series inductor or because of a current through capacitor C.

Possibly only one of the reactive elements is actually necessary, in which case a single first-order equation is sufficient to describe the circuit. The equation describing the circuit if L = 0 is

$$C \frac{de}{dt} + r + \frac{Re}{eR} = 0$$

where r is $-r_1$, $+r_2$, or $+r_3$, depending upon the region of operation. There is a single characteristic root:

$$\lambda = -\left(\frac{1}{C}\right)\left[\frac{(r + R)}{rR}\right]$$

In region 1, $r < 0$, and $R > |r|$ so that $\lambda > 0$ and the system is not stable. In regions 2 and 3, $r > 0$ so that $\lambda < 0$ and the system is stable. Assuming the inductance present but $C = 0$, yields

$$L\left(\frac{di}{dt}\right) + (R + r)i = 0$$

In this case, $\lambda = -\dfrac{(R + r)}{L}$. In region 1, $r < 0$, and $R > |r|$, so that again $\lambda < 0$ and the system is stable. In regions 2 and 3, $r > 0$ so that $\lambda < 0$ and the system is stable. This result, however, is a contradiction of the previous analyses which were experimental in nature,and the conclusion seems to be that a capacitance in parallel with the voltage-controlled resistance is necessary in obtaining the operating characteristics of the circuit.

Finally, it seems apparent from the solution curves that whatever reactances are present in the circuit play a part in the time to trigger from one stable state to the other. Generally it is desired to keep this time at a minimum and it is possible that parasitic or stray reactances are sufficient to serve the purpose without the insertion of actual reactances of larger magnitude.

DIRECT METHOD OF LYAPUNOV AND VARIABLE GRADIENT METHOD FOR STABILITY ANALYSIS

Stability information about a system may at times be obtained without actually solving the differential equations describing the system (5, 6, 8, 10). For linear systems the Routh-Hurwitz criteria provide an approach of this nature. The core idea of the direct, or second, method of Lyapunov is that it is sometimes possible to form functions of the system equation(s) and time which possess certain properties useful in the analysis of the system. Chief items of interest are several theorems leading to a "Lyapunov" function or "V" function, the generation of this function, and the analysis of the function.

The equations exhibited in the introduction form what is sometimes known as a "canonic form" and are again presented here.

$$dx_1/dt = X_1(x_1, x_2, \ldots, x_n)$$
$$dx_2/dt = X_2(x_1, x_2, \ldots, x_u)$$
$$\vdots$$
$$dx_n/dt = X_n(x_1, x_2, \ldots, x_u)$$

The first theorem attributable to Lyapunov regarding these equations may be stated.

For a set of first-order differential equations of the form,

$$\dot{x}_i = X_i(x_1, x_2, \ldots, x_u), \quad i = 1, 2, \ldots, n$$

and such that,

1. $\dot{x}_i = X_i(0, 0, \ldots, 0) = 0 \quad i = 1, 2, \ldots, n$
2. The functions X_i are continuous with respect to all variables x_i in the entire state space.

Then there exists a real-valued scalar function $V(x_1, x_2, \ldots, x_u)$ with the properties:

a. $V(x_1, x_2, \ldots, x_n)$ is continuous and has continuous first partial derivatives.

b. $V(x_1, x_2, \ldots, x_n) > 0$ except when $x_i = 0$ for $i = 1, \ldots, n$; (i.e., V is positive definite).

c. $V(0, 0, \ldots, 0) = 0$.

d. $V(x_1, x_2, \ldots, x_n) \longrightarrow \infty$ for $\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \longrightarrow \infty$

e. $\dot{V} = \dfrac{dv}{dt} = \sum_{i=1}^{n}\left(\dfrac{\partial v}{\partial x_i} \cdot \dfrac{dx_i}{dt}\right) < -\epsilon \sum_{i=1}^{n} x_i^2$

(i.e., $\dot{V}$ is negative definite).

Then the system is asymptotically stable in the large. Asymptotic stability in the large assures that for any real finite initial conditions on the system the output of the system will approach the equilibrium state of $x = 0$, $\dot{x} = 0$, as $t \longrightarrow \infty$.

Some further definitions are necessary here. The function V is called <u>positive definite</u> or <u>negative definite</u> in a given region about the origin if at all points in this region it has the same size (positive or negative), and, except at the origin, is nowhere zero. The function V is called <u>semidefinite</u> if it has the same sign throughout the region except at certain points at which it is zero; it must be zero at the origin as well. The function V is called <u>indefinite</u> if in the given region about the origin it takes on varying signs.

As a simple example of the use of the theorem, consider the V function (without considering for the moment how it was generated):

$$V = \frac{1}{3}\left(4x_1^2 + x_1x_2 + x_2^2\right)$$

It may be readily verified that conditions (a), (c), and (d) of the theorem are satisfied. The question arises then as to whether the quadratic form above is positive definite. For the general quadratic form

$$F(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$$

and a necessary and sufficient condition for it to be positive definite is that $a > 0$ and $ac - b^2 > 0$. For the form above, $a = 1.33$, $b = .33$, and $c = .33$, and condition (b) of the theorem is satisfied. In order to determine whether condition (e) is satisfied:

$$\dot{V} = \frac{8}{3} x_1\dot{x}_1 + \frac{x_1\dot{x}_2}{3} + \frac{x_2\dot{x}_1}{3} + \frac{2}{3} x_2\dot{x}_2$$

In general, there will be a relationship between $\dot{x}_1$ and $\dot{x}_2$. Let $\dot{x}_2 = (-3x_2 - 3x_1)$, $x_2 = \dot{x}_1$. Thus

$$\dot{V} = -\left(x_1^2 + \frac{x_1x_2}{3} + \frac{5}{3} x_2^2\right)$$

The last line above is also a quadratic form and, in addition, is negative definite. Since all the conditions of the theorem are satisfied, it is asymptotically stable in the large by

Lyapunov's direct method.

An analogy may be drawn between the potential energy of a system and the Lyapunov function, V. The positive V and negative $\dot{V}$ correspond to a system that dissipates energy. The response to any initial condition will cause the system to dissipate energy until the energy is zero. Conditions (1) and (2) of the theorem are satisfied by this condition. Herein lies a possible key to the generation of V functions but there are a number of analytical methods of generating these functions which appear more promising.

A geometric interpretation may be placed on the theorem and this interpretation can be most easily shown in two-dimensional space. Assume a V function of two variables that satisfies conditions (a), (b), (c), and (d) of the theorem. If $V(x_1, x_2)$ is set equal to a constant C, then the resulting equation describes a closed bounded curve in the x_1, x_2 (state) plane. If $V(x_1, x_2)$ is a quadratic form, then the curves are ellipses. If C is allowed to have different values, i.e., $0 < C_1 < C_2 < C_3, \ldots,$ $V(x_1, x_2)$ would appear as in Fig. 20.

If the time derivative of $V(x_1, x_2)$ is taken, the result is:

$$\frac{dv}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt}$$

Consider a phase trajectory crossing the curve $V(x_1, x_2) = C_3$ at point P. The partial derivatives may be written:

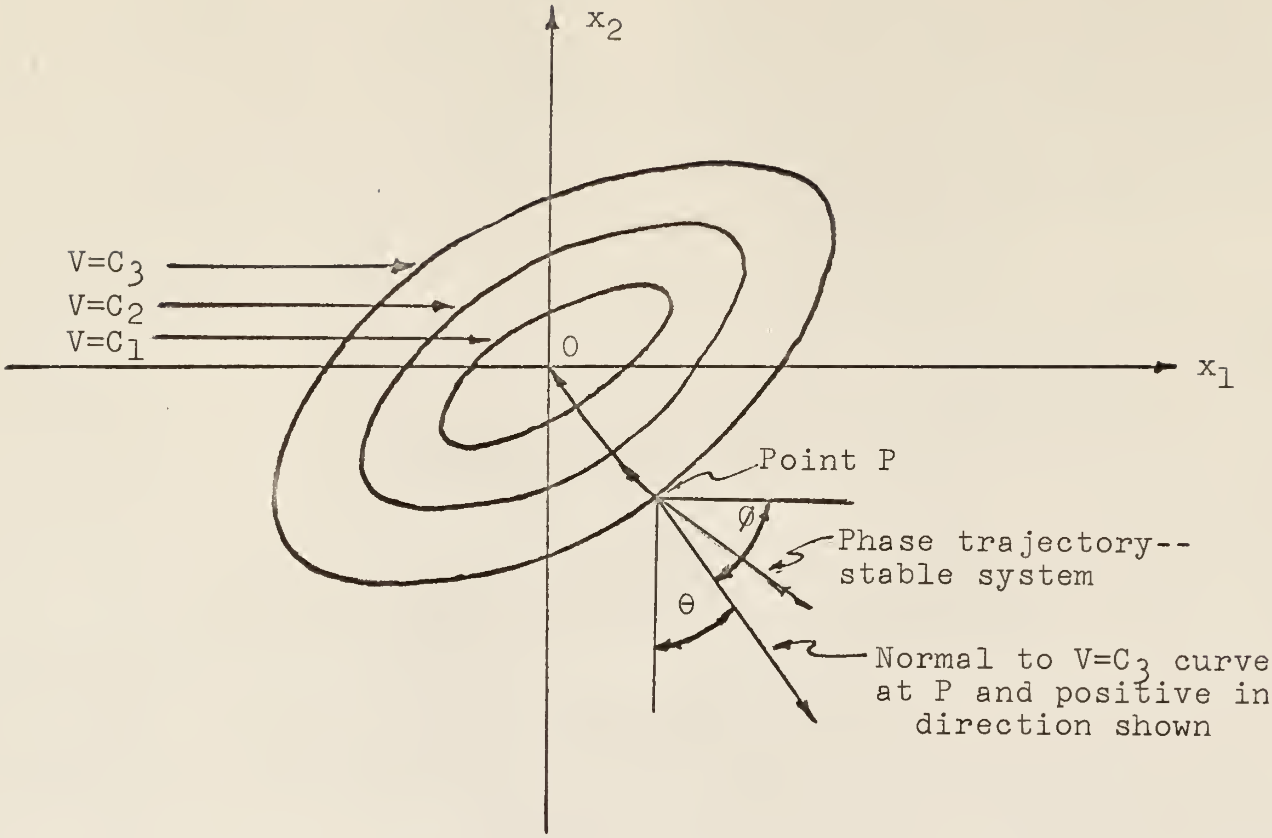


Fig. 20.

$$\frac{\partial V}{\partial x_1} = \left[\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2\right]^{1/2} \cos \phi$$

$$\frac{\partial V}{\partial x_2} = \left[\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2\right]^{1/2} \cos \theta$$

Thus

$$\frac{dv}{dt} = \left[\left(\frac{\partial V}{\partial x_1}\right)^2 + \left(\frac{\partial V}{\partial x_2}\right)^2\right]^{1/2} \left[\frac{dx_1}{dt} \cos \phi + \frac{dx_2}{dt} \cos \theta\right]$$

The second factor on the right represents a projection of the tangential velocity along the trajectory at point P onto the normal of the $V(x_1, x_2) = C_3$ curve at point P. It is obvious that if $\dot{V}$ is negative, then

$$\left[\frac{dx_1}{dt} \cos \phi + \frac{dx_2}{dt} \cos \theta\right]$$

must be negative and that the point P crosses the $V(x_1, x_2)$ curves from outside to inside. This is sufficient to assure that as $t \longrightarrow \infty$, the state variables x_1 and x_2 will approach zero and the discussion may be considered a rough proof of the theorem for the two-dimensional case.

If a system is unstable, Lyapunov's direct method may be used to establish this also. A theorem to do this is identical to the theorem previously stated with the exception that in condition (e), $\dot{V}$ is required to be positive definite (that is, of the same sign as V). This theorem may be stated in a somewhat different manner also. For a system described by the canonic set previously shown, if there exists a real valued function $V(x_1, x_2, \ldots, x_n)$ with the following properties, (1) $V(x_1, x_2, \ldots, x_n)$ is continuous and (2) the time derivative $\frac{dV}{dt}$ is negative definite, then:

1. The system is unstable in the finite region for which V is not positive semidefinite.
2. The response of the system is unbounded as $t \longrightarrow \infty$ if V is not globally positive semidefinite.

A graphical interpretation could be made of this also as for the stability theorem with the net effect of showing the point P of the phase trajectory crossing the C_i curves from the inside out.

A definite point should be made concerning the stability criteria obtained by Lyapunov's direct method which is that they are sufficient to establish stability; they are not necessary criteria. This is much the same as saying that if a function cannot be discovered which will either satisfy the conditions for stability or instability does not mean that the function does not exist; further, a Lyapunov function which establishes a stability or instability for a system is not unique.

It is apparent from the stability and instability theorems that the major problem is the actual generation of V functions in order to test for stability or instability. A promising approach to the generation of V functions is called the variable gradient method. The method is summarized below.

If a system is stable in a given state space, then a V function exists for the system. This statement follows the stability theorem and discussion following it. Provided only that V exists, its gradient ∇V exists. The last statement is discussed in reference (6) and a discussion of the proof given. Thus given the gradient, both V and $\dot{V}$ may be calculated. Actually, the approach is round about as would be suspected from the use of ∇V rather than V. Without, as yet, specific knowledge of ∇V except that it will be some function of the state variables, we proceed to determine $\dot{V}$ from ∇V. Thus

$$\dot{V} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + \ldots \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} \qquad (1)$$

The system canonic equations may now be inserted for the $\dot{x}_i$ and $\dot{V}$ put into vector form:

$$\dot{V} = \nabla \underline{V} \cdot \underline{\dot{x}} \tag{2}$$

Now V may be obtained by the line integration of equation (2).

$$V = \int_0^{(x_1, x_2, \ldots, x_u)} \nabla \underline{V} \cdot \underline{dx} \tag{3}$$

This line integration is independent of the path. The simplest path of integration and perhaps the easiest for actual integration is:

$$V \doteq \int_0^{x_1} V_1(\gamma_1, 0, 0, \ldots 0)\, d\gamma_1 + \int_0^{x_2} V_2(x_1, \gamma_2, 0, 0, \ldots 0)\, d\gamma_2$$

$$+ \ldots + \int_0^{x_n} V_n(x_1, x_2, \ldots, x_{n-1}, \gamma_n)\, d\gamma_n \tag{4}$$

where the subscripts on the V_i under the integral sign refer to the rows in the matrix representation of ∇V. Reference (10) discusses a uniqueness theorem which states that for a unique scalar function V to be obtained by a line integration of a vector function ∇V, the generalized curl equations

$$\frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i} \quad \text{or} \quad \nabla \times \nabla V = 0 \tag{5}$$

must hold. This is the n dimensional representation of Stokes' theorem. Proceeding, an arbitrary column vector ∇V is constructed with coefficients which are themselves functions of the state variables.

$$\nabla V = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{pmatrix} \tag{6}$$

Reference (6) discusses the coefficients a_{ii} which are restricted to functions of x_i only while a_{nn} is set equal to 2 to simplify the process of closedness. The a_{ij} are composed of a constant part a_{ijk} and variable part $a_{ijv}(x_1, x_2, \ldots, x_{n-1})$.

In summary, the outline for formal application of the method is:

1. Assume a gradient of the form of equation (6).
2. From the variable gradient form: $\dot{V} = \nabla \underline{V} \cdot \dot{\underline{x}}$ (equation 2).
3. In conjunction with and subject to the requirements of the generalized curl equations, equation (5), constrain dV/dt to be at least negative semidefinite.
4. From the known gradient, determine V and the region of closedness of V.
5. Invoke the necessary theorem to establish stability.

As an example, following the steps indicated above, consider the system shown in Fig. 21. Let the nonlinear

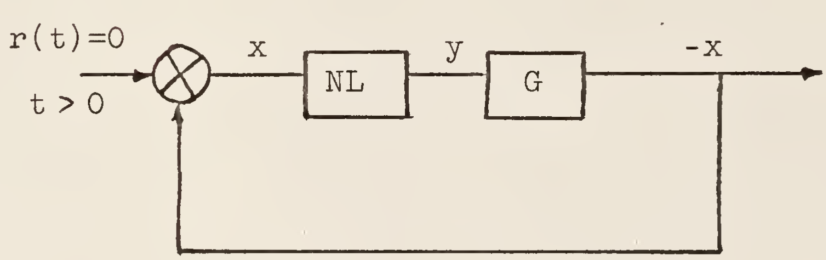


Fig. 21.

element be $y = x^3$ and $G = \dfrac{1}{S(S+1)}$. If it is assumed that the equations of motion written in state variable form are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - x_1^3$$

then:

Step 1

$$\nabla V = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} \nabla V_1 \\ \\ \nabla V_2 \end{bmatrix}$$

Step 2

$$\dot{V} = (a_{11}x_1 + a_{12}x_2)\dot{x}_1 + (a_{21}x_1 + 2x_2)\dot{x}_2$$

$$\dot{V} = x_1x_2(a_{11} - 2x_1^2 - a_{21}) + x_2^2(a_{12} - 2) - a_{21}x_1^4$$

Step 3

To constrain $\dot{V}$ to be at least negative semidefinite, the coefficient of x_1x_2 in the last equation in Step 2 can be set equal to zero and also $0 \leq a_{12} \leq 2$. a_{21} can be any positive number. Thus

$$\dot{V} = -x_2^2(2 - a_{12}) - a_{12}x_1^4 \ ; \quad a_{11} = a_{21} + 2x_1^2$$

and

$$\nabla V \quad \begin{bmatrix} a_{21}x_1 + 2x_1^3 + a_{12}x_2 \\ \\ a_{21}x_1 + 2x_2 \end{bmatrix} \qquad 0 \leq a_{12} \leq 2$$

For simplicity let $a_{12} = 1$.

Then

$$\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$$

implies

$$1 = \frac{\partial(a_{21}x_1)}{x_1} = a_{21x} + a_{21v} + x_1 \frac{\partial\{a_{21v}\}}{x_1}$$

and if $a_{21v} = 0$ and $a_{21x} = 1$, the above relationship is satisfied. Finally,

$$\nabla V = \begin{bmatrix} 2x_1^3 + x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$$

Step 4

$$V = \int_0^{x_1} (2x_1^3 + x_1)\,dx_1 + \int_0^{x_2} (x_1 + 2x_2)\,dx_2$$

$$V = \frac{x_1^4}{2} + \frac{x_1^3}{2} + x_1x_2 + x_2^2$$

Step 5

Since V is positive definite and $\dot{V}$ is negative definite, the conclusion is that the system is globally asymptotically stable. Other valid choices of a_{12} yield other valid V functions.

In summary, autonomous systems seem to be the largest field of application of the variable gradient technique. The method is applicable to single valued continuous nonlinearities where the nonlinearity is known as a polynomial, a specific function of x or a curve determined from experimental results. The method generates V functions to suit the problem at hand.

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APPENDICES

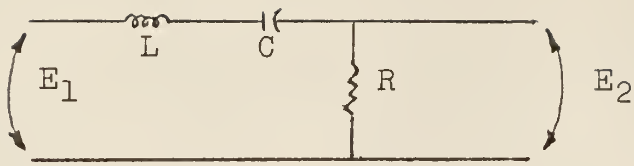
APPENDIX A

The Method of Krylov and Bogoliubov

1. General. The method of Krylov and Bogoliubov (1, 2, 5) is a series approximation technique for determining the free periodic oscillations of second order (and with a more general approach, higher order) systems. It is of some interest not only as an introduction to their principle of harmonic balance and the describing function technique which is based upon it, but also for itself.

Since the method in its more general application is an equivalent linearization process leading in some cases to a kind of "transfer function" for a nonlinear system, it will be of some merit to briefly review the linear transfer function concept.

The transfer function is defined when simple harmonic variations with respect to time exist. It is the complex ratio of the resulting response to a driving force. Depending upon the particular situation, the driving force and response may be either currents or voltages in any combination (or other physical analogs to these quantities). The transfer function may therefore have dimensions of impedance, admittance or pure numeric and consists of real and imaginary parts, or magnitude and angle. Both parts are generally functions of frequency but not of amplitude for a linear system. Transfer functions are usually not difficult to derive; for example:



$$\begin{pmatrix} 1 & mL \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{mC} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{R} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{mC} + mL \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{R} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{1}{R}\left(\frac{1}{mC} + mL\right) & \frac{1}{mC} + mL \\ \frac{1}{R} & 1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} ;$$

$$= \frac{E_2}{E_1} = \frac{1}{A_{11}} = \frac{R}{R + j(\omega L - \frac{1}{\omega C})}$$

where E_1 = input voltage

E_2 = output voltage

ω = frequency of input

m = $j\omega$

In the course of analyzing a nonlinear system, it is often desirable to separate the linear and nonlinear elements so far as possible. The transfer function may be found in the usual way for linear elements. A kind of equivalent transfer function is found for the nonlinear elements considering fundamental components only. This equivalent transfer function is called the <u>describing function</u> for the element. Generally it is a function dependent upon amplitude which may or may not depend upon frequency. It leads to results of useful accuracy and has meaning only in the event that the nonlinearity is such that the response to a simple harmonic driving force does not itself differ too much from being simple harmonic.

2. The Krylov and Bogoliubov Method for Second-order Systems. Consider the following equation:

$$\ddot{x} + F(x, \dot{x}) = 0 \tag{1}$$

x = the control (state variable

$F(x, \dot{x})$ = a nonlinear function in which linear terms are contained if they arise.

Put $F(x, \dot{x})$ in the following form:

$$F(x, \dot{x}) = \omega_0{}^2 x + \mu f(x, \dot{x}) \tag{2}$$

Equation (1) may now be written:

$$\ddot{x} + \omega_0{}^2 x + \mu f(x, \dot{x}) = 0 \tag{3}$$

The approximations below require that μ be small. A solution is desired of the form:

$$x = a \sin(\omega_0 t + \phi) \tag{4}$$

$$\dot{x} = a\, \omega_0 \cos(\omega_0 t + \phi) \tag{5}$$

with

$$a = a(t) \tag{6}$$

$$\phi = \phi(t) \tag{7}$$

unknown

For equation (5) to hold, special conditions must be imposed. For $\mu = 0$ in (3), the solution is simple harmonic motion of the form given in (4) with a and ϕ constant. For $\mu \neq 0$, the solution is no longer simple harmonic motion. Functions (6) and (7) permit correction of nominal frequency ω_0 and if they can be found, equation (4) can be rigorously correct. This is no mean task and an approximation to $a(t)$ and $\phi(t)$ is usually sought.

Apparently the actual frequency at any time is:

$$\omega = \omega_0 + \frac{d\phi}{dt} \tag{8}$$

What conditions are necessary if (5) is to hold? First differentiate equation (4) with respect to time.

$$\dot{x} = \frac{da}{dt} \sin(\omega_0 t + \phi) + a\left\{\cos(\omega_0 t + \phi)(\omega_0 + \frac{d\phi}{dt})\right\}$$

$$\dot{x} = \frac{da}{dt} \sin(\omega_0 t + \phi) + a\frac{d\phi}{dt} \cos(\omega_0 t + \phi) + a\omega_0 \cos(\omega_0 t + \phi) \tag{9}$$

If (5) is to hold:

$$\frac{da}{dt} \sin(\omega_0 t + \phi) + a\frac{d\phi}{dt} \cos(\omega_0 t + \phi) = 0 \tag{10}$$

Also differentiate (5) with respect to time.

$$\ddot{x} = \frac{da}{dt}\omega_0 \cos(\omega_0 t + \phi) - a\,\omega_0 \left\{\sin(\omega_0 t + \phi)(\omega_0 + \frac{d\phi}{dt})\right\}$$

$$\ddot{x} = \frac{da}{dt}\omega_0 \cos(\omega_0 t + \phi) - a\,\omega_0^2 \sin(\omega_0 t + \phi) - a\,\omega_0 \frac{d\phi}{dt} \sin(\omega_0 t + \phi) \tag{11}$$

Insert (10) and (11) into (3) with $z = (\omega_0 t + \phi)$, yielding:

$$\frac{da}{dt}\omega_0 \cos z - a\omega_0^2 \sin z - a\omega_0 \frac{d\phi}{dt} \sin z + \omega_0^2\, a \sin z + \mu f(a \sin z, a\omega_0 \cos z) = 0$$

$$\frac{da}{dt}\omega_0 \cos z - a\omega_0 \frac{d\phi}{dt} \sin z = -\mu f(a \sin z, a\omega_0 \cos z) \tag{11a}$$

$$\frac{da}{dt} \sin z - a \frac{d\phi}{dt} \cos z = 0 \tag{10}$$

Multiply (10) by $\omega_0 \cos z$ and (11a) by $\sin z$ and add, yielding:

$$\frac{d\phi}{dt} = \frac{1}{a\omega_0} \mu f(a \sin z,\ a\omega_0 \cos z) \sin z \tag{12}$$

Multiply (10) by $\omega_0 \sin z$ and (11a) by $\cos z$ and add, yielding:

$$\frac{da}{dt} = -\frac{1}{\omega_0} \mu f(a \sin z,\ a\omega_0 \cos z) \cos z \tag{13}$$

$a(t)$ and $\phi(t)$ could not be found by integration, but an approximate solution for the set must usually suffice. Equations (12) and (13) indicate that if μ is small, $\frac{da}{dt}$ and $\frac{d\phi}{dt}$ are small, and that $a(t)$ and $\phi(t)$ are slowly varying functions which may be considered to be constant at their average values over a single cycle. Thus:

$$\frac{da}{dt} \cong \frac{1}{2\pi\omega_0} \int_0^{2\pi} \mu f(a \sin z,\ a\omega_0 \cos z) \cos z \, dz \tag{14}$$

$$\frac{d\phi}{dt} \cong \frac{1}{2\pi a\omega_0} \int_0^{2\pi} \mu f(a \sin z,\ a\omega_0 \cos z) \sin z \, dz \tag{15}$$

Under the integral, a is assumed constant so the integration is not overly difficult. If (2) is integrated:

$$F(x, \dot{x}) = \omega_0^2 x + \mu f(x, \dot{x})$$

$$F(x, \dot{x}) - \omega_0^2 x = \mu f(x, \dot{x})$$

$$F(a \sin z,\ a\omega_0 \cos z) - \omega_0^2 (a \sin z) = \mu f(a \sin z,\ a\omega_0 \cos z)$$

$$\frac{da}{dt} = - \frac{1}{2\pi\omega_0} \int_0^{2\pi} F(a \sin z, a\omega_0 \cos z) \cos z \, dz$$

$$+ \frac{\omega_0{}^2}{\omega_0 2\pi} \int_0^{2\pi} a \sin z \cos z \, dz$$

$$\frac{da}{dt} \cong - \frac{1}{2\pi\omega_0} \int_0^{2\pi} F(a \sin z, a\omega_0 \cos z) \cos z \, dz \qquad (16)$$

Employing the square of (8) with (15) yields:

$$\omega^2 = \omega_0{}^2 + 2\omega_0 \frac{d\phi}{dt} + \left(\frac{d\phi}{dt}\right)^2$$

$$\omega^2 = \omega_0{}^2 + 2\omega_0 \frac{1}{2\pi a\omega_0} \int_0^{2\pi} F(a \sin z, a\omega_0 \cos z) dz$$

$$- \frac{\omega_0{}^2}{\pi a} \int_0^{2\pi} a \sin^2 z \, dz$$

$$\omega^2 = \frac{1}{\pi a} \int_0^{2\pi} F(a \sin z, a\omega_0 \cos z) \cos z \, dz \qquad (17)$$

With (16) and (17) the response of a system of the form of (1) may be found in the form of

$$x = a(t) \sin z(t)$$

where $a(t)$ and $z(t)$ are given by

$$\frac{da}{dt} = - \frac{a}{2\omega_0} b(a, \omega_0) \qquad (18)$$

$$\left(\frac{dz}{dt}\right)^2 = \omega^2 = g(a, \omega_0) \qquad (19)$$

and where, approximately,

$$b(a, \omega_0) = \frac{1}{\pi a} \int_0^{2\pi} F(a \sin z, a\omega_0 \cos z) \cos z \, dz \qquad (20)$$

$$g(a, \omega_0) = \frac{1}{\pi a}\int_0^{2\pi} F(a \sin z, a\omega_0 \cos z)\sin z \, dz \tag{21}$$

3. Generalization of the Method. In the above form, solution may be obtained for nonlinear equations in which the nonlinearity was of the form

$$F(x, \dot{x}) = g(a, \omega)x + b(a, \omega)\dot{x}$$

and as developed applied to second-order systems. It is apparently possible to obtain solutions of a similar nature to higher order systems with a more general approach; however, for demonstration of the method the second-order system is illustrative.

Now consider the nonlinear function $y = F(x, \dot{x})$ and let $x = a \sin \omega t$. Then:

$$y = F(a \sin \omega t, a\omega \cos \omega t)$$

Let $u = \omega t$ and expand in a Fourier series.

$$y = \frac{1}{2\pi}\int_0^{2\pi} F(a \sin u, a\omega \cos u)du$$

$$+ \left\{\frac{1}{\pi}\int_0^{2\pi} F(a \sin u, a\omega \cos u)\sin u \, du\right\} \sin \omega t$$

$$+ \left\{\frac{1}{\pi}\int_0^{2\pi} F(a \sin u, a\omega \cos u)\cos u \, du\right\} \cos \omega t$$

$$+ \text{higher harmonics}$$

If the nonlinearity is symmetrical, the first integral in the series is zero. This is not always so. For a symmetrical nonlinearity, neglecting higher harmonics, a function is achieved of the form:

$$y = g(a, \omega)x + \frac{b(a, \omega)\dot{x}}{\omega} \tag{22}$$

where $x = a \sin \omega t$, $\dot{x} = a\omega \cos \omega t$ from the last section; $g(a, \omega)$ and $b(a, \omega)$ are given by equations (21) and (20), respectively.

Equation (22) is linear even though the original $y = F(x,\dot{x})$ is nonlinear. In connection with this, note that what has been found thus far by the method of Krylov and Bogoliubox are approximation functions which replace a nonlinear function by an equivalent linear function, and thus the appellation "equivalent linearization". The approximations are useful if the nonlinearity μ is small so that the amplitude and phase characteristics of the nonlinear element may be considered constant for some period of time. In other words, a nonlinear element may be approximated by an equivalent linear one and the approximation is useful if the original nonlinearity is not too large. The approximation may be accomplished by means of a truncated Fourier series.

4. Example. The object of equivalent linearization in conjunction with the principle of harmonic balance, then, is to choose the linear replacement element such that the fundamental sinz and cosz components are the same for both linear and nonlinear elements under a simple harmonic motion. The equivalent linear element may be denoted by a function which is a "describing function" for the nonlinear element subject to the restrictions of the approximations.

As an example, consider a combination of electrical dissipative and reactive nonlinear elements (or possibly just a term in a differential equation) for which the voltage is some function of both the current and its first derivative. Thus:

$$v = f(i, \frac{di}{dt})$$

For an equivalent linearization of this function, the linear function is:

$$v = gi + b\frac{di}{dt}$$

If it is assumed that the current through the element is sinusoidal and described by

$$i = I\cos(\omega t + 0) = I\cos z$$

the voltage across the nonlinear element or term is:

$$v = f\left[I\cos z, -\omega I\sin z\right]$$

The fundamental sine and cosine terms are respectively,

$$V_{s1} = \frac{1}{\pi}\int_0^{2\pi} f(I\cos z, -\omega I\sin z)\sin z\,dz$$

$$V_{c1} = \frac{1}{\pi}\int_0^{2\pi} f(I\cos z, -\omega I\sin z)\cos z\,dz$$

The voltage across the equivalent linear element is:

$$v = (g\,I\cos z) - (b\omega\,I\sin z)$$

By the principle of harmonic balance the values of g and b may be found.

$$g = \frac{1}{\pi}\int_0^{2\pi} f(I\cos z, -\omega\,I\sin z)\cos z\,dz$$

$$b = \frac{1}{\pi}\int_0^{2\pi} f(I\cos z, -\omega\,I\sin z)\sin z\,dz$$

The function $v = gi + b\frac{di}{dt}$ is thus a function describing the relationship between current and voltage for this combination of elements, or a "sort of" transfer function.

APPENDIX B

Analysis of Singular Points

The analysis of the singular points of a differential equation is an extension of phase plane analysis and can be useful in determining the properties of the solution. Qualitative as well as some quantitative aspects of the solution can be had through a study of the locations and types of solution curves existing near singular points. It is usually desirable to have appropriate equations relating the variables of the system although in some cases relations available only in graphical form may be used (1, 2).

A graphical representation of solution curves on a plane surface with two dimensions is conveniently used in a study of singularities; thus such a study is limited to the case of two variables. If a differential equation of the form $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$ is investigated, where $P(x, y)$ and $Q(x, y)$ may be nonlinear functions of x and y, the equation is equivalent to the two equations:

$$\frac{dx}{dt} = P(x, y) \qquad \frac{dy}{dt} = Q(x, y)$$

and is obtained by eliminating the independent variable t. Elimination of t makes $\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}$ an autonomous equation and limits it to situations where any forcing function is either entirely absent or extremely simple.

Singularities of $\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}$ are those values of x and y for which both P and Q become simultaneously zero. At the singular point $x = x_s$ and $y = y_s$ and $P(x_s, y_s) = 0$, $Q(x_s, y_s) = 0$, and $\frac{dy}{dx}$ becomes indeterminate. There may be a number of singularities if P and Q are nonlinear. Since $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ at a singularity, the singularity is always a point of equilibrium. As was noted in the introduction, the nature of solutions near a singularity may be explored by substituting $x = x_s + u$ and $y = y_s + v$, where u and v are small variations. With these substitutions,

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}$$

becomes:

$$\frac{dy}{dx} = \frac{dv}{au} = \frac{Q(x_s,y_s) + Cu + Dv + C_2u^2 + D_2v^2 + F_2uv + \ldots\ldots}{P(x_s,u_s) + Au + Bv + A_2u^2 + B_2v^2 + E_2uv + \ldots\ldots}$$

A, B, C, D, ... real constants. A Taylor's series expansion may be necessary. The most important terms in determining the solution near a singularity are the linear terms in u and v. The kind of singularity depends only upon the linear terms, provided these terms are present if nonlinear terms are present. In other words, if $C_2 \neq 0$ in the numerator so that a term u^2 appears, the linear term Cu must be present with $C \neq 0$ for the singularity to be simple. The same condition applies in both numerator and denominator for both u and v. Under this condition with a simple singularity, the properties of the solution near a singularity depend upon the equation:

$$\frac{dv}{au} = \frac{Cu + Dv}{Au + Bv} \qquad (1)$$

Only linear terms appear. If there are higher power terms in u and v present, a study cannot be made from this equation alone. Equation 1 is equivalent to the pair of simultaneous first-order equations:

$$\frac{du}{dt} = Au + Bv \qquad \frac{dv}{dt} = Cu + Dv$$

This pair of equations is a simple case of the more general set of n simultaneous first-order equations.

A slight digression here for a discussion of a technique which simplifies matters later is in order. The set of n first-order equations may be written

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n
\end{aligned} \qquad (2)$$

where $x_1 \ldots x_n$ are the n dependent variables. $\dot{x} = \dfrac{dx}{dt}$.

Moreover:

$$\begin{aligned}
\{x\} &= \{x_n\} \\
\{\dot{x}\} &= [A]\{x\}
\end{aligned} \qquad (3)$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Solutions for equations of the form (2) are well known and involve exponential functions which retain their form upon differentiation. Thus

$$\{x\} = \{c_x\} \exp(\lambda t)$$

λ is a constant determined by the coefficients in the differential equations. Differentiating the solution and substituting into (3) yields

$$\lambda \{x\} = [A]\{x\}$$

which may be written

$$\left[[A] - \lambda [I] \right] = \{0\}$$

where I is the identity matrix. The equation may be satisfied, except for the trivial case, only if

$$\left| [A] - \lambda [I] \right| = 0$$

where the λ's are the characteristic roots or eigenvalues.

Physical systems can often be described by differential equations of the form of (1) and usually the right sides of the equations involve several of the dependent variables. A system in which there is no coupling would have on the right side of each equation only the single variable which appears on the left. Such an equation is said to be in normal form. A system with coupling may be converted to one having no coupling by the mathematical process of changing the variables through an appropriate linear transformation. Thus

$$\begin{aligned} x_1 &= p_{11}y_1 + p_{12}y_2 + \cdots p_{1n}y_n \\ x_2 &= p_{21}y_1 + p_{22}y_2 + \cdots p_{2n}y_n \\ &\vdots \\ x_n &= p_{n1}y_1 + p_{n2}y_2 + \cdots p_{nn}y_n \end{aligned}$$

In matrix form, $\{x\} = [B]\{y\}$, P_{ij} are constant quantities.

Rewritten:

$$\{y\} = [P]^{-1}\{x\}, \quad |P| \neq 0$$

Rewritten:

$$[P]^{-1}\{\dot{x}\} = \{\dot{y}\} = [P]^{-1}[A]\{x\} = [P]^{-1}[A][P]\{y\}$$

Thus:

$$\{\dot{y}\} = [B]\{y\}, \quad [B] = [P]^{-1}[A][P] \qquad (4)$$

A solution for (4) has the form $\{y\} = \{C_y\}\exp(\lambda t)$ and the characteristic roots of (4) are given by: $\left|[B] - \lambda_B[I]\right| = 0$.

Thus:

$$[B] - \lambda_B[I] = [P]^{-1}[A][P] - \lambda_B[P]^{-1}[I][P]$$

$$= [P]^{-1}\left[[A] - \lambda_B[I]\right][P]$$

The determinant of both sides must vanish. Since $|P| \neq 0$ and $|P^{-1}| \neq 0$, it must be that $\left|[A] - \lambda_B[I]\right| = 0$. Therefore the transformation $[P]$ changing $[A]$ to $[B]$ does not change the set of roots for the set of equations.

If the transformation matrix $[P]$ is chosen properly, $[B]$ can be made a diagonal matrix with all the elements not on the main diagonal zero. The set of equations described by $[B]$ are then said to be in <u>normal form</u>. The eigenvalues of $[B]$ are the same as those of $[A]$. The elements on the main diagonal of $[B]$ must be the eigenvalues. Thus the set of equations in normal form is:

$$\dot{y}_1 = \lambda_1 y_1$$

$$\dot{y}_2 = \lambda_2 y_2$$

$$\vdots$$

$$\dot{y}_n = \lambda_n y_n$$

$$[B] = \begin{bmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & & \lambda_n \end{bmatrix}$$

If two roots happen to form a complex pair as $\lambda_1 = \delta + j\omega$, $\lambda_2 = \delta - j\omega$, these two elements of $[B]$ would be complex quantities. Often it is desirable to have only real quantities appear. An equivalent form for the equation when λ_1 and λ_2 are complex quantities is

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -(\delta^2 + \omega^2)y_1 + 2\,\delta y_2$$

$$\vdots$$

$$\dot{y}_n = \lambda_n y_n$$

$$[B] = \begin{bmatrix} 0 & 1 & & 0 \\ -(\delta^2 + \omega^2) & 2\delta & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$$

The alternate form is usually preferable when complex roots occur. The original set of equations $\{\dot{x}\} = [A]\{x\}$, has solutions $\{x\} = \{Cx\}\exp(\lambda t)$. In normal form $\{y\} = \{Cy\}\exp(\lambda t)$. The only feature of the solutions which can be determined from the differential equations alone are the eigenvalues which are the same in each case. The coefficients $\{Cx\}$ and $\{Cy\}$ can be found only from initial conditions. Solution in terms of x and y can then be said to be equivalent, at least so far as qualitative properties are concerned.

Now the second-order system is described by just two equations. The matrix $[P]$ needed to reduce this kind of system to normal form is quickly found. Consider the use of the following symbols.

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad [A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad [P] = \begin{bmatrix} \alpha & \beta \\ & \delta \end{bmatrix}$$

$$\{y\} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} \qquad [B] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Originally, only $\{x\}$ and $[A]$ are known, and $[B]$, $[P]$, and $\{y\}$ must be found. The characteristic equation is

$$\left| [A] - \lambda [I] \right| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

or $\lambda^2 - (a + d)\lambda - (bc - ad) = 0$. The two characteristic roots are:

$$(\lambda_1, \lambda_2) = \frac{1}{2}\left\{(a + d) \pm \left[(a + d)^2 + 4(bc - ad)\right]^{1/2}\right\}$$

where, in all that follows, λ_1 is the root found with the positive sign. If the roots are real, λ_1 is always the more positive root. These values of λ_1 and λ_2 serve to determine matrix $[B]$.

Matrix $[P]$ must satisfy $[B] = [P]^{-1} [A] [P]$. Therefore

$[P][B] = [A][P]$. When these matrix products are found, the result is:

$$\begin{bmatrix} \alpha\lambda_1 & \beta\lambda_2 \\ \gamma\lambda_1 & \delta\lambda_2 \end{bmatrix} = \begin{bmatrix} \alpha a + \gamma b & \beta a + \delta b \\ \alpha c + \gamma d & \beta c + \delta d \end{bmatrix}$$

Since corresponding elements of equal matrices must be identical, the following simultaneous equations exist: $\alpha\lambda_1 = \alpha a + \gamma b$; $\gamma\lambda_1 = \alpha c + \gamma d$. The ratio $\frac{\gamma}{\alpha}$ defined as m_1 is:

$$m_1 = \frac{\gamma}{\alpha} = \frac{\lambda_1 - a}{b} = \frac{c}{\lambda_1 - d}$$

In a similar manner, the ratio $\frac{\delta}{\beta}$ is defined as m_2 and is:

$$m_2 = \frac{\delta}{\beta} = \frac{\lambda_2 - a}{b} = \frac{c}{\lambda_2 - d}$$

Both forms are necessary on the right-hand side of the above equations since sometimes one form is indeterminate. The ratios $\frac{\gamma}{\alpha}$ and $\frac{\delta}{\beta}$ are fixed by these equations and thereby fix the elements of matrix P within constant factors. The coordinate transformation is $\{x\} = [P]\{y\}$, or

$$x_1 = \alpha y_1 + \beta y_2$$

$$x_2 = \gamma y_1 + \delta y_2 = m_1\alpha y_1 + m_2\beta y_2$$

The types of singularities of a second-order system can now be investigated and classified. The simplest cases are those in which the two characteristic roots are real. Equations for the system in normal form are:

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

and

$$\frac{\dot{y}_2}{\dot{y}_1} = \frac{\lambda_2 y_2}{\lambda_1 y_1} = \frac{dy_2}{dy_1}$$

Possibilities may be listed as follows where λ_1 designates the more positive root.

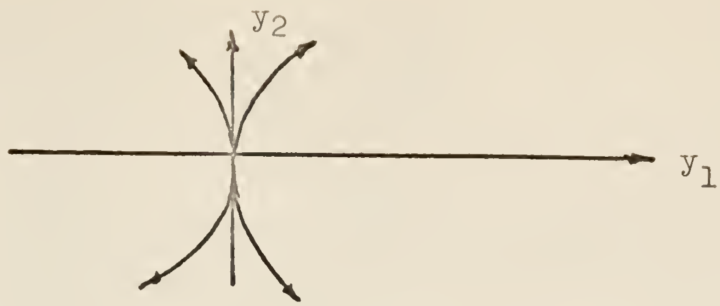
1. Both Roots Real and Positive.

$$0 < \lambda_2 < \lambda_1 \; ; \qquad\qquad 0 < \frac{\lambda_2}{\lambda_1} < +1$$

The equation for a curve representing a solution for the differential equation on the $y_1 y_2$ plane can be found directly by integration:

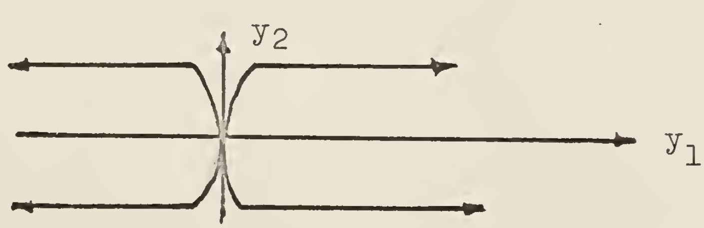
$$y_2 = C y_1^{(\lambda_2/\lambda_1)}$$

C is an arbitrary constant dependent upon initial conditions. These curves are generally parabolic in shape, with the exact shape determined by $\frac{\lambda_2}{\lambda_1}$ and constant C. The slope of the curves may be found from $\frac{dy_2}{dy_1} = C \frac{\lambda_2}{\lambda_1} y_1^{((\lambda_2 \lambda_1)-1)}$ and near the origin, $\frac{dy_2}{dy_1} \longrightarrow \infty$ as $y_1 \longrightarrow 0$, since $\frac{\lambda_2}{\lambda_1} < +1$. Thus all solution curves have a definite direction near the origin, being parallel to the y_2 axis. Shown below is the case where values of λ_1 and λ_2 are not far different, i.e., $0 << \frac{\lambda_2}{\lambda_1} < 1$. The curves represent the locus of points determined by corresponding values of y_1 and y_2. As independent variable t increases, the point relating instantaneous values of y_1 and y_2 moves along the curve in the direction of the arrowheads. Initial conditions determine the value of constant C, and thus the quadrant



within which a particular solution lies. Since the roots are positive, both y_1 and y_2 increase without bound as t increases and this type of singularity is said to be unstable. On the other hand, if y_1 and y_2 were both to vanish as t increased, the singularity would be called stable. One of the important features of a singular point is the question of its stability. Node is the name given to this type of singularity, referring to the fact that the solution curves have a definite direction near the singularity. Either the y_1 or y_2 axis can be a solution if the initial conditions are such that one of the variables y_1 or y_2 is exactly zero. The axes then represent special cases of solution curves to correspond to special initial conditions.

If the values of λ_1 and λ_2 are such that $0 < \frac{\lambda_2}{\lambda_1} << 1$, the solution curves take the shape of the figure below. These curves show a similarity to the curves above, but as they change direction the breaks are much sharper.



Consider the following solutions of the form:

$$y_1 = C_1 \exp(\lambda_1 t) \; ; \qquad y_2 = C_2 \exp(\lambda_2 t)$$

$$\frac{dy_2}{dy_1} = \frac{\lambda_2 C_2 \exp(\lambda_2 t)}{\lambda_1 C_1 \exp(\lambda_1 t)} \qquad 0 < \lambda_2 << \lambda_1$$

If t is large and positive:

$$e^{\lambda_1 t} >> e^{\lambda_2 t} \text{ and } \frac{dy_2}{dy_1} \approx 0$$

If t is large and negative:

$$e^{\lambda_1 t} << e^{\lambda_2 t} \text{ and } \frac{dy_2}{dy_1} \approx \infty$$

Thus if λ_1 and λ_2 differ sufficiently in magnitude, transition between these two occurs suddenly and solution curves are essentially two straight lines with a sharp break joining them.

2. Both Roots Real and Negative.

$$\lambda_2 < \lambda_1 < 0 \quad ; \qquad \frac{\lambda_2}{\lambda_1} > +1 \quad .$$

Solution curves for the normal form are again parabolic, but near the origin $\frac{dy_2}{dy_1} = 0$ and the curves are parallel to the y_1 axis. The singularity is again a node, but since negative eigenvalues lead to ultimate disappearance of y_1 and y_2 with increasing t, the node is stable. For increasing t the point representing corresponding values of y_1 and y_2 moves closer to the singularity at the origin but reaches it only at the infinite value of t. The y_1 and y_2 axes are again special solution curves.

3. Both Roots Real and Opposite in Sign.

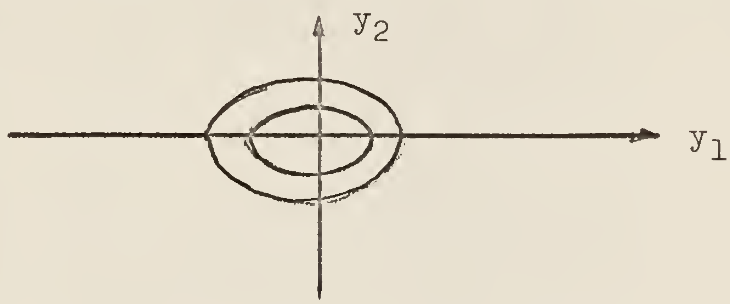
$$\lambda_2 < 0 < \lambda_1 \qquad \frac{\lambda_2}{\lambda_1} < 0$$

In this case $\frac{dy_2}{dy_1} = \left|\frac{\lambda_2}{\lambda_1}\right| (\frac{y_2}{y_1})$ and $y_2 y_1^{|\lambda_2/\lambda_1|} = C$. Solution curves in the y_1y_2 plane are hyperbolic in shape and generally pass by the singularity at the origin. Now λ_1 is positive and y_1 ultimately increases without bound and the solution is unstable, even though y_2 ultimately vanishes. This singularity is the saddle.

4. <u>Roots Pure Imaginaries</u>.

$$\delta = 0 \begin{cases} \lambda_1 = +j\omega \\ \lambda_2 = -j\omega \end{cases}$$

The equation for a solution curve can be found from $\frac{dy_2}{dy_1} = -\omega^2 \frac{y_1}{y_2}$ as $\omega^2 y_1{}^2 + y_2{}^2 = C$; the equation is that of an ellipse about the singularity at the origin. The figure below shows a general case. This type of singularity is known as a vortex. The solution is a periodic oscillation in time with no change in amplitude.



There is neither growth nor decay; the solution has a "neutral" stability and the amplitude and size of the ellipse are determined by the initial conditions.

5. <u>Roots Complex Conjugates</u>.

$$\lambda_1 = \delta + j\omega \; ; \qquad \lambda_2 = \delta - j\omega$$

In this case the normal form for the equations can be written

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -(\delta^2 + \omega^2) & & 2\delta \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$$\frac{\dot{y}_2}{\dot{y}_1} = \frac{dy_2}{dy_1} = \frac{-(\delta^2 + \omega^2)y_1 + 2\,\delta y_2}{y^2}$$

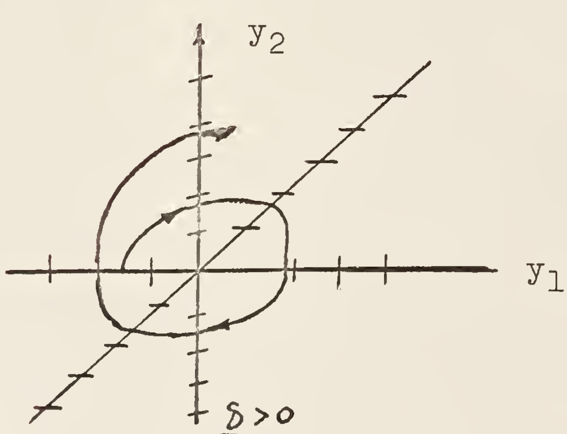
and solution curves depend upon this equation. Integration is possible following an appropriate change of variable. The qualitative nature of its solution curves, however, may be obtained more easily from an observation of lines of equal slope. The following seem evident from the equation:

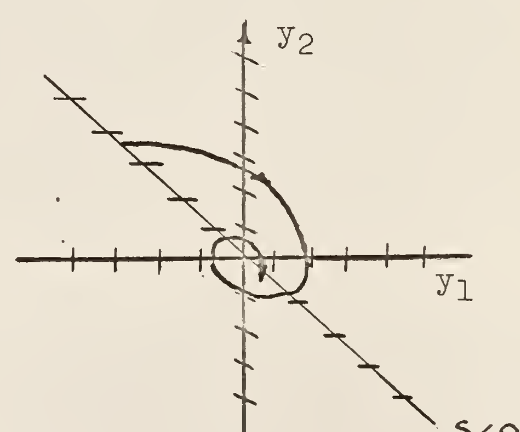
$$\frac{dy_2}{dy_1} = 0 \quad \text{along} \quad y_2 = \frac{(\delta^2 + \omega^2)y_1}{2\delta}$$

$$\frac{dy_2}{dy_1} = \infty \quad \text{along} \quad y_2 = 0$$

$$\frac{dy_2}{dy_1} = 2\delta \quad \text{along} \quad y_1 = 0.$$

These isoclines carrying directed line segments of appropriate slope are shown below and typical solution curves sketched.





The curves form spirals about the singularity at the origin which is designated as a focus. If $\delta \quad 0$, the solution ultimately grows without bound and is unstable. If $\delta \quad 0$, the solution ultimately vanishes and is stable.

For simple singularities, then, of two first-order linear equations, there are only four possibilities: node, saddle, vortex, and focus. The node and focus may be either stable or unstable, the saddle is always unstable, and the vortex is "neutrally" stable. Only the types of solution curves associated with these four singularities can exist.

SOME ASPECTS OF NONLINEAR STABILITY

by

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Three approaches to investigations of nonlinear stability are described in this report. The three approaches are: <u>The Describing Function Method</u>, <u>Analysis by Means of Singular Points</u>, and <u>The Variable Gradient Method of Generating "V" Functions</u> used in conjunction with certain theorems attributable to Lyapunov.

In Appendices A and B some derivations and descriptions are contained which provide background material applicable to the describing function and singularity point analysis approaches, respectively. The main body of the report was then utilized to exhibit some results of the various approaches applied to nonlinear stability problems.