MINIMUM-DRAG BODIES OF REVOLUTION
AT HYPERSONIC AIRSPEEDS

by

CHING-CHENG CHAO

Diploma, Air Technological School, Chinese Air Force
Kangshan, Taiwan, Republic of China, 1948

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

Approved by

Wilson Tripp

Major Professor

TABLE OF CONTENTS

NOMENCLATURE iv

SECTION 1. INTRODUCTION 1

SECTION 2. FUNDAMENTAL HYPERSONIC FLOW THEORY 3

2-1 Introductory Remarks 3

2-2 Normal Shock 4

2-3 Oblique Shock 5

2-4 Basic Hypersonic Relations 7

2-5 Newtonian Impact Theory 9

2-6 Centrifugal Force Effects 10

SECTION 3. BASIC PRINCIPLES OF CALCULUS OF VARIATIONS 15

3-1 Introductory Remarks 15

3-2 Maximum or Minimum of an Integral 15

3-3 Euler Equation 17

3-4 Typical Minimum-Drag Body Problem 19

3-5 Isoperimetric Rule of Calculus of Variations 22

SECTION 4. MINIMUM-DRAG BODIES OF REVOLUTION 23

4-1 Introductory Remarks 23

4-2 Fundamental Considerations 23

4-3 Discontinuous Solution for Minimum-Drag Body--Given Fineness Ratio 26

4-4 Parametric Equations for Minimizing Curve--Fineness Ratio Given 30

4-5 3/4 Power Body Approximation 32

4-6 Modified Theory--Given Fineness Ratio 34

4-7 2/3 Power Body Approximation 38

4-8 Minimum-Drag Body--Given Surface Area and Base Diameter 39

4-9 Modified Theory--Given Surface Area and Base Diameter 42

4-10 Continuous Solution for Minimum-Drag Body--Given Base Diameter and Volume 45

4-11 Discontinuous Solution for Minimum-Drag Body--Given Base Diameter and Volume 50

4-12 Parametric Equations for Minimizing Curve--Given Base Diameter and Volume 54

4-13 Modified Theory--Given Base Diameter and Volume 56

4-14 Minimum-Drag Body--Given Length and Volume 57

4-15 Modified Theory--Given Length and Volume 61

4-16 Minimum-Drag Body--Given Length and Surface Area 64

4-17 Modified Theory--Given Length and Surface Area 66

4-18 Results and Discussion 69

SECTION 5. CONCLUSION 71

ACKNOWLEDGEMENT 72

REFERENCES 73

NOMENCLATURE

All auantities in slug-foot-second units

Symbol	Meaning		Units
C_D	Drag coefficient	$\frac{4D}{q_\infty \pi D_B^2}$	
Cp	Pressure coefficient	$\frac{P - P_\infty}{q_\infty}$	
c	Constant of integration		
D	Pressure foredrag		lbf
D_B	Base diameter		ft
f, F	Integrand function		
F_R	Fineness ratio	$\frac{L}{D_B/2}$	
I_D	Drag parameter	$\frac{D}{2\pi q_\infty}$	ft^2
K	Ratio of specific heats		
L	Body length		ft
M	Mach number		
N	Distance measured normal to the body surface		ft
P	Static pressure		lbf/ft^2
q	Dynamic pressure		lbf/ft^2
R	Radius of curvature of streamline in plane containing symetric of body		ft
s	Length along surface		ft
S	Surface area		ft^2
V	Resultant velocity		ft/sec

Vol	Body volume	ft^3
x, y	Coordinates of point on meridian curve of body	ft
δ, θ	Turning angle, angle between free stream direction and tangent to body surface	rad
σ	Shock angle	rad
λ	Lagrange multiplier	
ρ	Density	$slug/ft^3$

Subscripts

0	Value at corner of meridian curve
1	Value at nose point of meridian curve or value before oblique shock
2	Value at base point of meridian curve or value after oblique shock
∞	Value of free stream
+	Right-hand limiting value of quantity at corner on minimizing curve
-	Left-hand limiting value of quantity at corner on minimizing curve
B	Value along meridian curve
C	Value due to surface curvature
s	Value behind shock
x	Value before normal shock
y	Value after normal shock

SECTION 1

INTRODUCTION

Since the advent of higher Mach number missiles, the problem of determining the shapes of nonlifting bodies of revolution having minimum pressure drag at supersonic and hypersonic speeds has received much attention from many investigators. Based on the small perturbation potential flow theory, von Karman and later on, Haack, Ferrari, Lighthill and Sears calculated the minimum-drag body shapes for various conditions. Their findings are representative of body shapes of practical fineness ratios at low supersonic Mach numbers. (Ref. 1)

Using a method similar to the present day calculus of variations, Newton was perhaps the first to calculate the body shape of minimum drag. The problem which Newton was concerned with was to determine the body shape of minimum drag at extremely high speeds, where the inertia forces are large compared to the elastic forces in the fluid. The drag law adopted by Newton, known as the impact theory, is a good approximation to that for hypersonic flow. It states: on impact, the approaching stream loses its normal component of momentum to the body surface at hypersonic speeds, while the tangential component remains unchanged. Since then, certain generalizations of Newton's work have been done by other investigators.

Taking into consideration the effects of centrifugal forces of flow over a curved surface, Eggers, Resnikoff and Denis made some modification to the simple impact theory and obtained the optimum body shapes for certain given geometric conditions (Ref. 2). By using the calculus of variations, solutions to the minimum-drag body problems are generally rather complicated parametric equations. The minimizing curve for given fineness ratio can be approximated by the 3/4 or 2/3 power body, depending on whether the centrifugal force effects are considered or not.

It is undertaken in this report to present methods for the determination of shapes of the nonlifting bodies of revolution having minimum pressure drag at hypersonic speeds by means of the calculus of variations and the Newtonian impact theory. Much of the material is taken from sources of references 1 and 2. In the latter part and some other respects, fuller development is given by the author of this report for the results from these references. The presentation will be carried out for the various combinations of conditions of given length, base diameter, surface area and volume. In each case, separate analysis will be made in accordance with simple and modified theory for comparison. In order to maintain the best possible continuity for the presentation of the subject, fundamentals of the hypersonic flow theory and the calculus of variations are given in the next two sections.

SECTION 2

FUNDAMENTAL HYPERSONIC FLOW THEORY

2-1 Introductory Remarks. The hypersonic speed range lies between Mach numbers from 5 to 40. Results obtained from the simple impact theory approximate satisfactorily for straight-side bodies if the free stream Mach number is of the order of 15 or higher. For curved surfaces, effects of the centrifugal forces should be taken into account especially at moderate hypersonic Mach numbers.

As the Mach number increases ($M_\infty > 6$), the extremely high temperatures behind the shock cause serious alterations in the thermodynamic properties of the air, and more complicated boundary layer and heat transfer problems are involed. In addition, the shock wave has a tendency to wrap itself around the body surface, forming the so called "hypersonic boundary layer" between the shock and the surface. The concept is of particular use in calculating the pressure coefficients over a surface when curvature effects are considered. Throughout this section, all analyses are based on the assumption that effects of viscosity, molecular vibration and dissociation are considered negligible.

Since the Newtonian simple impact theory is satisfactory only at extreme hypersonic speeds and the validity of the linearized theory is doubtful at $M_\infty > 3$, special considerations are given for flow at moderate hypersonic speeds, i.e., $3 \leq M_\infty \leq 15$. As certain shock relations are required in presenting materials on the hypersonic flow, a short summary of the pertinent relations is given in the present section.

2-2 Normal Shock.

Consider the air flow through a normal shock as shown in Fig. 2-1. The continuity, momentum and energy equations are given by

$$\rho_x V_x = \rho_y V_y \tag{2-1}$$

$$P_x - P_y = \rho_y V_y^2 - \rho_x V_x^2 \tag{2-2}$$

$$C_p (T_x - T_y) = \frac{V_y^2 - V_x^2}{2} \tag{2-3}$$

By using the perfect gas relations,

$$C_p T = \frac{KR}{K-1} \frac{P}{\rho R} = \frac{K}{K-1} \frac{P}{\rho}$$

Eq. (2-3) can be expressed as

$$\frac{K}{K-1} \left(\frac{P_x}{\rho_x} - \frac{P_y}{\rho_y} \right) = \frac{1}{2} \left(V_y^2 - V_x^2 \right) \tag{2-3a}$$

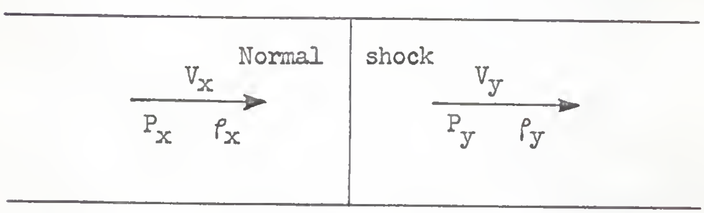

Fig. 2-1. Normal shock wave.

Eliminating V_x and V_y from Eqs. (2-1), (2-2) and (2-3a), the Rankine-Hugoniot relation is obtained, i.e.,

$$\frac{\rho_y}{\rho_x} = \frac{1 + \frac{K+1}{K-1} \frac{P_y}{P_x}}{\frac{K+1}{K-1} + \frac{P_y}{P_x}} \tag{2-4}$$

The ratio of pressures fore and aft of the shock can be written in terms of the initial Mach number M_x:

$$\frac{P_y}{P_x} = \frac{2K}{K+1} M_x^2 - \frac{K-1}{K+1} \qquad (2\text{-}5)$$

Substituting in Eq. (2-4), one obtains the density ration across the shock

$$\frac{\rho_y}{\rho_x} = \frac{K+1}{\frac{2}{M_x^2} + (K-1)} \qquad (2\text{-}6)$$

2-3 Oblique Shock.

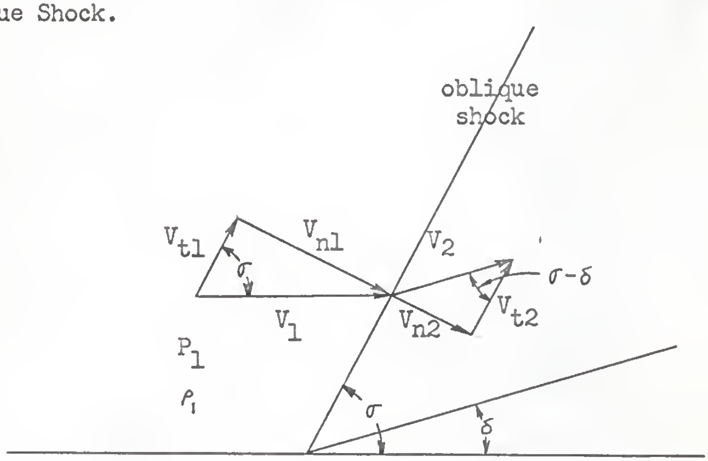

Fig. 2-2. Oblique shock wave

The oblique shock wave may be thought of as being formed by superimpoing a flow parallel to the normal shock wave such that the resultant velocities V_1 and V_2 are parallel to the walls fore and aft of the shock, respectively. The continuity, momentum and energy equations can be written as

Continuity:

$$\rho_1 V_{n1} = \rho_2 V_{n2} \qquad (2\text{-}7)$$

Momentum in t-direction:

$$(\rho_1 V_{n1}) V_{t1} = (\rho_2 V_{n2}) V_{t2} \tag{2-8}$$

$$V_{t1} = V_{t2} = V_t$$

Momentum in n-direction:

$$P_1 - P_2 = \rho_2 V_{n2}^2 - \rho_1 V_{n1}^2 \tag{2-9}$$

Energy:

$$c_p (T_1 - T_2) = \frac{1}{2}(V_2^2 - V_1^2)$$

or

$$\frac{K}{K-1}\left(\frac{P_1}{\rho_1} - \frac{P_2}{\rho_2}\right) = \frac{1}{2}\left(V_{n2}^2 - V_{n1}^2\right) \tag{2-10}$$

since

$$C_p T = \frac{K}{K-1}\frac{P}{\rho}$$

and

$$V_2^2 - V_1^2 = (V_{n2}^2 + V_t^2) - (V_{n1}^2 + V_t^2) = V_{n2}^2 - V_{n1}^2$$

Note the similarity between the continuity, momentum and energy equations for the normal shock and the oblique shock. If we put $V_{n1} = V_x$ and $V_{n2} = V_y$, all the relations derived for the normal shock are also applicable to the case of oblique shock. Substitution of M_1 sin for M_x in Eqs. (2-5) and (2-6) gives

$$\frac{P_2}{P_1} = \frac{2K}{K+1} M_1^2 \sin^2\sigma - \frac{K-1}{K+1} \tag{2-11}$$

$$\frac{\rho_2}{\rho_1} = \frac{K+1}{\frac{2}{M_1^2 \sin^2\sigma} + (K-1)} \tag{2-12}$$

Now, from the geometry of Fig. 2-2,

$$\frac{\tan\sigma}{\tan(\sigma-\delta)} = \frac{V_{n1}/V_t}{V_{n2}/V_t} = \frac{V_{n1}}{V_{n2}} = \frac{\rho_2}{\rho_1} \tag{2-13}$$

Combining Eqs. (2-12) and (2-13),

$$\frac{K+1}{\frac{2}{M_1^2\sin^2\sigma} + K-1} = \frac{\tan\sigma}{\tan(\sigma-\delta)}$$

$$\frac{2}{M_1^2\sin^2\sigma} = (K+1)\,\frac{\tan(\sigma-\delta)}{\tan\sigma} - (K-1)$$

$$= 2 + (K+1)\left[\frac{\sin(\sigma-\delta)\cos\sigma}{\cos(\sigma-\delta)\sin\sigma} - 1\right]$$

$$= 2 + (K+1)\,\frac{\sin(-\delta)}{\cos(\sigma-\delta)\sin\sigma}$$

$$\frac{1}{M_1^2\sin^2\sigma} = 1 - \frac{K+1}{2}\,\frac{\sin\delta}{\cos(\sigma-\delta)\sin\sigma} \tag{2-14}$$

2-4. Basic Hypersonic Relations.

The region between the body surface and the shock is known as the hypersonic boundary layer. In order to calculate pressures on bodies at hypersonic speeds, it is important to know the position and shape of the leading shock wave.

Referring to Eq. (2-12), when the free stream Mach number normal to the oblique shock, i.e., $M_1\sin\sigma$, is large, the density ratio across the shock becomes

$$\frac{\rho_2}{\rho_1} \cong \frac{K+1}{K-1} \tag{2-15}$$

The two limiting cases can be studied as follows

(i) Extreme hypersonic flow.

The free stream Mach number is very large. The tremendous temperatures behind the shock cause a large variation in the specific heat ratio K such that $K \rightarrow 1$ as $M_{\infty} \rightarrow \infty$. Thus for $K = 1$,

$$\frac{\rho_2}{\rho_1} \rightarrow \infty \tag{2-16}$$

$$\frac{V_{n2}}{V_{n1}} \rightarrow 0 \tag{2-17}$$

which indicates V_{n2} is negligible compared to V_{n1}. Furthermore, from Eq. 2-13

$$\frac{\rho_2}{\rho_1} = \frac{\tan\sigma}{\tan(\sigma - \delta)} \rightarrow \infty$$

where generally, $\sigma \neq 90°$, i.e., $\tan\sigma \neq \infty$

Therefore

$$\tan(\sigma - \delta) = 0$$

$$\tag{2-18}$$

or

$$\sigma = \delta$$

(ii) Moderate hypersonic flow.

The free stream Mach number is high but finite. Consider $K = 1.4$ as the limiting case behind the shock. For slender bodies, $\tan\sigma \simeq \sigma$, $\tan(\sigma - \delta) \simeq \sigma - \delta$, thus

$$\sigma = 6(\sigma - \delta)$$

or

$$\sigma = 1.2\,\delta \tag{2-19}$$

From Eqs. (2-18) and (2-19), it is seen that the shock wave closely follows the body surface, resulting in a hypersonic boundary layer of infinitesimal thickness. This is especially so for bodies of revolution in the three dimensional hypersonic flow.

2-5 Newtonian Impact Theory.

From the concluding remarks of the last paragraph, we know that the shock is nearly coincident with the body surface in a hypersonic flow. Furthermore, since V_{n2} is practical negligible compared with V_{n1}, it is seen that on impact the approaching stream loses all its normal component of momentum to the body surface while its tangential component of momentum remains unchanged. Eq.(2-9) can be written as

$$P - P_\infty = \rho_1 V_{n1}^2$$

Dividing by $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$, we obtain the local pressure coefficient for shock flow,

$$C_p = \frac{P-P_\infty}{q_\infty} = 2\left(\frac{V_{n1}}{V_\infty}\right)^2 = 2\sin^2\delta \tag{2-20}$$

where the local pressure is denoted by P and the free stream pressure by P_∞. δ is the angle between the tangent to the local surface and the free stream direction since $\delta \simeq \sigma$ according to Eqs. (2-18) and (2-19).

It should be noted that the Newtonian approximation, Eq. (2-20) does not specify the pressure coefficient on surfaces that do not "see" the flow, i.e. surfaces on which expansion flow would be predicted according to supersonic gasdynamics. In fact, Eq. (2-20) can be applied to problems of three-dimensional flow as well as two-diamensional flow. However, in the case of bodies with surfaces curved in the direction of the stream, the pressure relieving effects of the local centrifugal forces must be taken into consideration, in particular when the free stream Mach number is moderate. The resultant local pressure coefficient should be the sum of the pressure coefficient just behind the shock and the pressure coefficient due to the centrifugal forces. This will be treated in Art. 2-6.

Now, the pressure coefficient behind the shock wave can also be obtained as follows. From Eqs.(2-11) and (2-14),

$$C_{p_s} = \frac{2}{KM_1^2}\left(\frac{P_2}{P_1} - 1\right)$$

$$= \frac{2}{KM_1^2}\left[\frac{2K}{K+1} M_1^2 \sin^2\sigma - \frac{K-1}{K+1} - 1\right]$$

$$= \frac{4}{(K+1)M_1^2}\left[M_1^2 \sin^2\sigma - 1\right]$$

$$= \frac{2\sin\sigma \sin\delta}{\cos(\sigma-\delta)}$$

Since $\sigma = \delta$, the relation for hypersonic speeds becomes

$$C_{p_s} = 2\sin^2\delta \tag{2-21}$$

Thus it is shown by Eqs. (2-20) and (2-21) that a two dimensional body with no surface curvature in the stream direction has the same pressure on the surface as behind the leading shock. This is also true for straight-side three dimensional bodies.

2-6 Centrifugal Force Effects.

As stated before, the centrifugal force effects on the pressures at the body surfaces at high but finite Mach numbers must be taken into account. Consider the flow in the hypersonic boundary layer around the body of revolution as shown in Fig. 2-3.

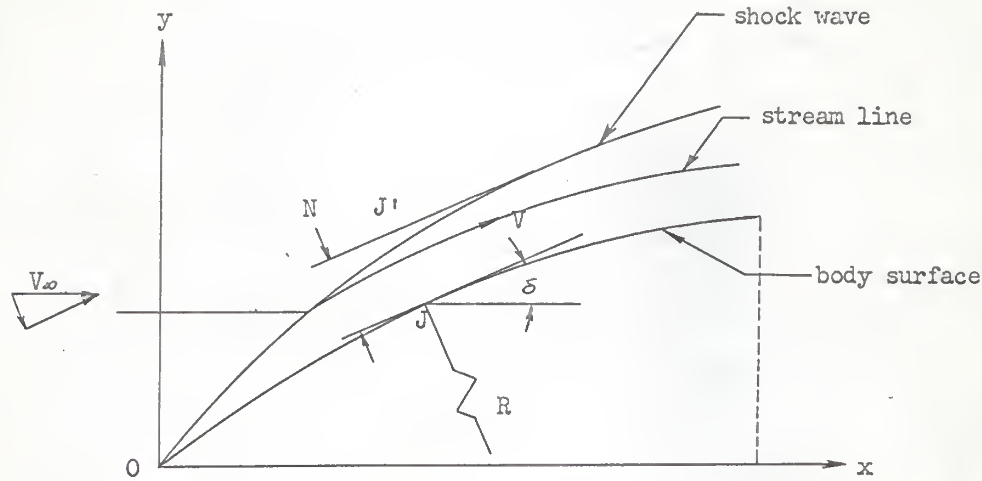

Fig. 2-3. Hypersonic boundary layer

The local pressure change from the surface to the shock due to centrifugal forces in the fluid is

$$\Delta P_c = \int_0^N \frac{dP}{dN}\, dN = \int_0^N \frac{\rho V^2}{R}\, dN \tag{2-22}$$

where $\rho V^2/R$ is the pressure gradient due to the centrifugal force of fluid of density ρ ; R is the local radius of curvature; and V is the local velocity in the hypersonic boundary layer. Assumption has been made that the streamlines in the hypersonic boundary layer are essentially parallel to the local surface. By using the mean values of the velocity and radius in the distance N, Eq. (2-22) can be written as

$$\Delta P_c = \frac{\bar{V}}{\bar{R}} \int_0^N \rho V dN \tag{2-23}$$

By continuity, the mass flow rate through the section ring JJ' is

$$\dot{m} \simeq 2\pi y \int_0^N \rho V dN = \pi y^2 \rho_\infty V_\infty^2$$

$$\int_0^N \rho V dN = \frac{y}{2} \rho_\infty V_\infty \qquad (2\text{-}24)$$

Substituting in Eq. (2-23),

$$\Delta P_c = \frac{\bar{V}}{\bar{R}} \frac{y}{2} \rho_\infty V_\infty \qquad (2\text{-}25)$$

Then, the pressure coefficient due to centrifugal force is obtained

$$C_{P_c} = \frac{\Delta P_c}{q_\infty} = \frac{y}{\bar{R}} \frac{\bar{V}}{V_\infty} \qquad (2\text{-}26)$$

From Eq. (2-26), the pressure coefficient due to centrifugal forces can be worked out for the two regions of hypersonic speeds as follows.

(i) Extreme hypersonic flow.

As shown in previous sections, at infinite Mach number, the hypersonic boundary layer is of infinitesimal thickness, and thus $\bar{R}$ equals the body surface radius of curvature,

$$\bar{R} = R_B \qquad (2\text{-}27)$$

$$\frac{1}{\bar{R}} = \frac{1}{R_B} = \frac{y''}{(1+y'^2)^{3/2}}$$

Since $\qquad y' = dy/dx = \tan$

$$y'' = \frac{d}{dx}\tan\delta = \sec^2\delta \frac{d\delta}{dx} = (1+y'^2)\frac{d\delta}{dx}$$

Therefore

$$\frac{1}{R_B} = \frac{1}{\sqrt{1+y'^2}} \frac{d\delta}{dx} = \frac{y'}{\sqrt{1+y'^2}} \frac{d\delta}{dy}$$

$$= \sin\delta \frac{d\delta}{dy} \qquad (2\text{-}28)$$

Assuming uniform thickness of the hypersonic boundary layer, i.e., constant velocity along streamlines downstream of the bow shock, the mean velocity up to a certain point J on the surface can be written as

$$\bar{V} = \frac{2V_\infty}{y_j^2} \int_0^{y_j} y \cos\delta \, dy \tag{2-29}$$

since
$$V = \frac{1}{\pi y_j^2} \int_0^{y_j} (2\pi y) V_\infty \cos\delta \, dy$$

Substitution of Eqs. (2-28) and (2-29) into Eq. (2-26), all in terms of the local coordinates, yields the pressure coefficient due to the centrifugal force at a point J on the surface

$$C_{p_c} = \frac{y_j}{\bar{R}} \frac{\bar{V}}{V_\infty} = y_j \sin\delta_j \frac{d\delta_j}{dy_j} \cdot \frac{2V_\infty}{y_j^2 V_\infty} \int_0^{y_j} y \cos\delta \, dy$$
$$= \frac{2\sin\delta_j}{y_j} \frac{d\delta_j}{dy_j} \int_0^{y_j} y \cos\delta \, dy \tag{2-30}$$

However, at infinite Mach numbers, $C_{p_s} >> C_{p_c}$, this modification is always neglected in the calculation of the pressure coefficients.

(ii) Moderate hypersonic flow.

At Mach numbers that are high but finite, the value of K is closer to 1.4 than 1. As shown in Eq. (2-19), $\sigma \cong 1.26$, that is, the hypersonic boundary layer is no longer of infinitesimal thickness. The evaluation of $\bar{R}$ and $\bar{V}$ from Eqs. (2-28) and (2-29) are in error. An improved schematic of hypersonic boundary layer is shown in Fig. 2-3 where the shock wave takes on a shape such that the lateral distance N between the surface and the shock increases downstream along streamlines in the hypersonic boundary layer. It should be noted that near the base of the body, $\bar{R} >> R_B$. The approximation of $\bar{R} = R_B$ is good only in the vicinity of the nose. According to Ref. 2, a better approximation is suggested as

$$\frac{\bar{R}}{R_B} = \frac{1}{1 - y/y_2} \tag{2-31}$$

where y_2 is the body ordinate of the base .

In the hypersonic boundary layer of nonuniform, finite thickness, pressure disturbances can be transmitted across the streamlines, and the velocity along streamlines downstream of the shock wave is not necessarily constant. A better approximation to the mean velocity $\overline{V}$ may be obtained from the impact theory, i.e.,

$$\overline{V} = V_\infty \cos\delta \tag{2-32}$$

Substitution of Eqs. (2-28), (2-31) and (2-32) in Eq. (2-26) gives the local pressure coefficient due to centrifugal force in the form of

$$\begin{aligned} C_{p_c} &= \frac{y}{\overline{R}}\frac{\overline{V}}{V_\infty} = \frac{y}{R_B}\left(1 - \frac{y}{y_2}\right)\cos\delta \\ &= y\left(1 - \frac{y}{y_2}\right)\sin\delta\cos\delta\,\frac{d\delta}{dy} \\ &= \frac{y}{2}\left(1 - \frac{y}{y_2}\right)\frac{d}{dy}\sin^2\delta \end{aligned} \tag{2-33}$$

The total pressure coefficient C_p is the sum of the pressure coefficient just behind the shock and the pressure coefficient due to the centrifugal force. Therefore, from Eqs. (2-20) and (2-33), one obtains

$$\begin{aligned} C_p &= C_{p_s} + C_{p_c} \\ &= 2\sin^2\delta + \frac{y}{2}\left(1 - \frac{y}{y_2}\right)\frac{d}{dy}\sin^2\delta \end{aligned} \tag{2-34}$$

This result will be used in Section 4 in the calculation of pressure drag for bodies of revolution in hypersonic flow of finite Mach numbers when the effects of centrifugal forces are to be taken into consideration.

SECTION 3

BASIC PRINCIPLES OF CALCULUS OF VARIATIONS

3-1 Introductory Remarks.

The simplest case of a typical variational problem is to determine the function y(x) for an integral

$$\int_{x_1}^{x_2} F\left(x,\ y(x),\ y'(x)\right)\, dx \tag{3-1}$$

takenalong a curve connecting two given points A and B, as shown in Fig. 3-1, in which the integral is to be a maximum or minimum. The determination of the maximum or minimum will be carried out in a way similar to that for ordinary functions. The purpose here is to present the basic principles of the calculus of variations that are needed in Section 4 in the treatment of the minimum-drag body profile problem at hypersonic speeds. No attempt is made to give a complete presentation on the subject in this report.

3-2 Maximum or Minimum of an Integral.

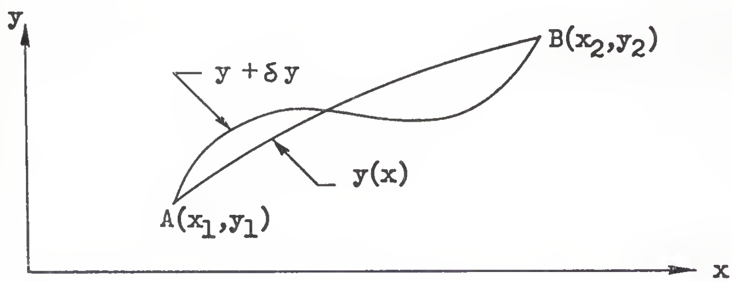

Fig. 3-1. Nomenclature for infinitesimal changes

The two end points A and B are joined by any curve y = y(x), therefore

$$y_1 = y(x_1), \quad y_2 = y(x_2)$$

Now, the integral is taken along curve C. When curve C is replaced by another curve joining A and B, the value of the integral is usually changed. As y is replaced by $y + \delta y$, the slope of the curve changes from y' to $y' + \delta y'$, and an infinitesimal change in I occurs. Expanding the integrand by Taylor series, one obtains

$$I + \Delta I = \int_{x_1}^{x_2} F(x,\ y+\delta y,\ y'\ \delta y')dx$$

$$\int_{x_1}^{x_2}\left[F(x,y,y') + \left(\delta y\frac{\partial}{\partial y} + \delta y'\frac{\partial}{\partial y'}\right)F + \frac{1}{2!}\left(\delta y\frac{\partial}{\partial y} + \delta y'\frac{\partial}{\partial y'}\right)^2 F + \cdots\right]dx$$

$$= \int_{x_1}^{x_2} F(x,y,y')dx + \int_{x_1}^{x_2}(\delta y F_y + \delta y' F_{y'})\,dx$$

$$+ \frac{1}{2}\int_{x_1}^{x_2}\left[(\delta y)^2 F_{yy} + 2\,\delta y\ \delta y'\ F_{yy'} + (\delta y')^2 F_{y'y'}\right]dx + \cdots \tag{3-2}$$

We shall call that part of the change in the integral which contains only the first or second order terms in δy and $\delta y'$ the first or second variation of the integral and denote it by δI or $\delta^2 I$ repectively, i.e.,

$$\delta I = \int_{x_1}^{x_2} (\delta y F_y + \delta y' F_{y'})\,dx \tag{3-3}$$

$$\delta^2 I = \frac{1}{2}\int_{x_1}^{x_2}\left[(\delta y)^2 F_{yy} + 2\,\delta y\ \delta y'\ F_{yy'} + (\delta y')^2 F_{y'y'}\right]dx \tag{3-4}$$

The preceding results are equivalent to differentiating under the integral sign in Eq. (3-1). All higher order variations of the integral are negligible compared with the first variation. A necessary condition for the integral to be stationary is that the first variation must vanish, i.e.,

$$\delta I = 0 \tag{3-5}$$

3-3 Euler Equation.

Let the varied curve be

$$Y = y + \delta y \tag{3-6}$$

where $y = y(x)$ is the original curve. By differentiation,

$$Y' = y' + \frac{d}{dx}(\delta y) \tag{3-7}$$

By definition of variation, one can also write

$$Y' = y' + \delta y' = y' + \delta\left(\frac{dy}{dx}\right) \tag{3-8}$$

Therefore

$$\frac{d}{dx}(\delta y) = \delta\left(\frac{dy}{dx}\right) \tag{3-9}$$

That is, the derivative of a variation is the variation of a derivative.

Now, integrate the second term of the first variation in Eq. (3-3) by parts, i.e., in the integral

$$\int_{x_1}^{x_2} \delta y' F_{y'}\, dx = \int_{x_1}^{x_2} F_{y'} \frac{d}{dx}(\delta y)\, dx$$

let $F_{y'} = u$ and $\frac{d}{dx}(\delta y)\, dx = dV$

then, $du = \frac{d}{dx} F_{y'}\, dx$ $\quad V = \int dV = \delta y$

Integrating by parts,

$$\int_{x_1}^{x_2} \delta y' F_{y'}\, dx = F_{y'} \delta y \Big|_{y_1}^{y_2} - \int_{x_1}^{x_2} \delta y \frac{d}{dx} F_{y'}\, dx$$

Substituting into Eq. (3-3),

$$\delta I = F_{y'} \delta y \Big|_{y_1}^{y_2} + \int_{x_1}^{x_2} \delta y \left(F_y - \frac{d}{dx} F_{y'}\right) dx \tag{3-10}$$

Now, A and B are fixed end points. The integrated part must vanish, thus

$$\delta I = \int_{x_1}^{x_2} \delta y \left(F_y - \frac{d}{dx} F_{y'} \right) dx \tag{3-11}$$

In order that the integral be stationary, the quantity in the parenthesis of the integrand in Eq. (3-11) must vanish, i.e.,

$$F_y - \frac{d}{dx} F_{y'} = 0 \tag{3-12}$$

since δy is an arbitrary function subject to only those general conditions such as δy must vanish at the end points, it should be a continuous function with first derivative or higher order derivatives, and either $|\delta y|$ or both $|\delta y|$ and $|\delta y'|$ should be small.

This is known as the Euler equation. If a maximum or a minimum value of I along a curve C exists, that curve must be a solution to this differential equation. Since $F = F(x, y, y')$, Eq. (3-12) can be written in the form of

$$F_y - F_{y'x} - y'F_{y'y} - y''F_{y'y'} = 0 \tag{3-13}$$

This is a second order differential equation. Since the curve passes through fixed end points A and B, the two arbitrary constants contained in its solution can be determined, if possible, from these two boundary conditions. However, there are only a few simple instances in which the Euler equations are integrable. In problems of finding the minimum-drag body, the integrand F depends on y and y' only,

$$F = F(y, y') \tag{3-14}$$

Then, $F_{y'x} = 0$. From Eq. (3-13), the Euler equation becomes

$$F_y - y'F_{y'y} - y''F_{y'y'} = 0 \tag{3-15}$$

Multiplying both sides by y', the left hand side of Eq. (3-15) turns into an exact derivative, i.e.,

$$y'(F_y - y'F_{y'y} - y''F_{y'y'}) = 0$$

$$y'F_y + y''F_{y'} - y''F_{y'} - y'^2F_{y'y} - y'y''F_{y'y'} = 0$$

$$\frac{d}{dx}(F - y'F_{y'}) = 0$$

Therefore

$$F - y'F_{y'} = c \tag{3-16}$$

The first order differential equation can be solved for y' either by separation of variables or by introducing a parameter.

For an ordinary function to be an extremum, the first necessary condition is that the first derivative of the function be zero and a maximum or minimum is determined by the sign of the second derivative. Likewise, in the calculus of variations, for a functional to be an extremum, the first condition is that the first variation of the functional be zero, i.e., solving the Euler equation the a maximum or minimum is determined by the sign of the second variation or the expression

$$(\delta y)^2 F_{yy} + 2(\delta y)(\delta y') F_{yy'} + (\delta y')^2 F_{y'y'}$$

under the integral in Eq. (3-4). The integral I is a maximum if the expression is negative or a minimum if the expression is positive.

3-4 Typical Minimum-Drag Body Problem.

The integral to be minimized in the typical minimum-drag body problem, in most cases, has its integrand in the form

$$F(y, y') = y\phi(y') \tag{3-17}$$

Thus, Eq. (3-16) turns to be

$$y\phi(y') - yy'\phi'(y') = c \tag{3-18}$$

Now, consider the second variation in Eq. (3-4),

$$F = y\phi(y'), \qquad F_y = \phi(y'), \qquad F_{yy} = 0$$

$$F_{y'} = y\phi'(y'), \qquad F_{yy'} = \phi'(y'), \qquad F_{y'y'} = y\phi''(y')$$

Substituting in Eq. (3-4) yields

$$\delta^2 I = \int_{x_1}^{x_2} \left[\delta y \ \delta y' \ \phi'(y') + \frac{1}{2} (\delta y')^2 \ y \ \phi''(y') \right] dx \tag{3-19}$$

Integrating $\int_{x_1}^{x_2} (\delta y)(\delta y')\phi'(y')dx$ by parts, let

$$\delta y \phi'(y') = u, \qquad (\delta y')dx = \frac{d}{dx}(\delta y)dx = dv$$

Then, $du = \frac{d}{dx}\left[\delta y \phi'(y')\right] dx \qquad v = \int dv = \delta y$

Therefore

$$\int_{x_1}^{x_2} \delta y \ \delta y' \ \phi'(y')\,dx = (\delta y)^2 \phi'(y') \Big|_{y_1}^{y_2} - \int_{x_1}^{x_2} \delta y \frac{d}{dx}\left[\delta y \phi'(y')\right] dx \tag{3-20}$$

or

$$\int_{x_1}^{x_2} \delta y \ \delta y' \ \phi'(y')\,dx = (\delta y)^2 \phi'(y') \Big|_{y_1}^{y_2} - \int_{x_1}^{x_2} (\delta y)^2 \frac{d}{dx}\phi'(y')\,dx - \int_{x_1}^{x_2} \delta y \ \delta y' \ \phi'(y')\,dx$$

$$\int_{x_1}^{x_2} \delta y \ \delta y' \ \phi'(y')\,dx = \frac{1}{2}(\delta y)^2 \phi'(y') \Big|_{y_1}^{y_2} - \frac{1}{2}\int_{x_1}^{x_2} (\delta y)^2 \frac{d}{dx}\phi'(y')\,dx \tag{3-21}$$

Note

$$\frac{d}{dx}\phi'(y') = \frac{d}{dy'}\phi'(y')\frac{dy'}{dx} = y''\phi''(y') \tag{3-22}$$

Substitution of Eqs. (3-21) and (3-22) in Eq. (3-19) gives

$$\delta^2 I = \frac{1}{2}\int_{x_1}^{x_2} \phi''(y')\left[y(\delta y')^2 - y''\ (\delta y)^2\right]dx + \frac{1}{2}(\delta y)^2\phi'(y')\Big|_{y_1}^{y_2} \tag{3-23}$$

Since A and B are fixed end points, the last term vanishes. Then

$$\delta^2 I = \frac{1}{2}\int_{x_1}^{x_2} \phi''(y')\left[y(\delta y')^2 - y''(\delta y)^2\right] dx \tag{3-24}$$

In general, I is a maximum if $\delta^2 I < 0$ and I is a minimum if $\delta^2 I > 0$. Further, from Eq. (3-18),

$$y = \frac{C}{\phi(y') - y'\phi'(y')} \tag{3-25}$$

$$y' = \frac{-C\frac{d}{dx}\left[\phi(y') - y'\phi'(y')\right]}{\left[\phi(y') - y'\phi'(y')\right]^2} = \frac{-C\left[y''\phi'(y') - y''\phi'(y') - y'y''\phi''(y')\right]}{\left[\phi(y') - y'\phi'(y')\right]^2}$$

$$= C\,\frac{y'y''\phi''(y')}{\left[\phi(y') - y'\phi'(y')\right]^2} \tag{3-26}$$

or

$$\phi''(y') = \frac{\left[\phi(y') - y'\phi'(y')\right]^2}{C\,y''}$$

It is seen that the sign of $\phi''(y')$ is invariable if the sign of y'' is invariable. For minimum-drag body problems, the ordinate y can be always taken as positive. If the curve is concave to the x-axis, $y'' < 0$. Therefore, the sign of the quantity inside the bracket in Eq. (3-24) is always positive, and the integrand will have the same sign as $\phi''(y')$. Then, it is obvious that for a curve concave to the x-axis, a maximum exists if $\phi'' < 0$ and a minimum exists if

$\phi'' > 0$.

3-5 Isoperimetric Rule of Calculus of Variations.

The integral $$I = \int F(x, y, y')dx \tag{3-28}$$

to be minimized sometimes is subject to a constraint in the form of

$$J = \int \psi(x, y, y')dx \tag{3-29}$$

From previous discussion, we must have

$$\delta I = 0 \tag{3-30}$$

If the constraint is to hold, only those variations δy and $\delta y'$ are allowed such that the variation

$$\delta J = 0 \tag{3-31}$$

Thus

$$\delta I + \lambda \delta J = 0 \tag{3-32}$$

where λ is a constant called the Lagrange multiplier. This means that the first variation of the integral

$$I + \lambda J = \int \left[F(x, y, y') + \lambda \psi(x, y, y')\right] \cdot dx \tag{3-33}$$

must vanish. Now, the problem can be handled as prescribed before and the Lagrange multiplier λ determined from the relation in Eq. (3-28). The techniques of variational calculus outlined in this section will be found very useful in the analysis of the minimum-drag bodies in Section 4.

SECTION 4

MINIMUM-DRAG BODIES OF REVOLUTION

4-1 Introductory Remarks.

The investigation undertaken in this section is concerned with the shapes of nonlifting bodies of revolution, having minimum pressure foredrag at high supersonic airspeeds in continuum flow. Methods of the calculus of variations will be employed, and it is desired to simplify the drag equation insofar as is practicable, consistent with retaining the salient features of the dependence of the pressure drag on the body shape and free stream conditions. The expression for pressure coefficient from the simple impact theory will be used to derive an expression for the pressure drag on a general nonlifting body of revolution. Once the condition for minimization are given, the calculus of variations is applied to the specific minimum-drag body problem.

4-2 Fundamental Considerations.

Consider the body of revolution in impact flow at zero angle of attack as shown in Fig. 4-1. The drag contribution of the base, where the surface does not "see" the flow is neglected according to Newtonian impact theory. Therefore the total pressure drag can be found by integrating the local dynamic pressure over the body length L,

$$D = 2\pi \int_0^L (P - P_\infty)\, y \,\frac{dy}{dx}\, dx \tag{4-1}$$

where P denotes the local static pressure and P_∞ the free stream static pressure. The pressure-drag coefficient is defined as

$$C_D = 4D/(q_\infty \pi D_B^{\,2})$$

in which q_∞ is the free stream dynamic pressure.

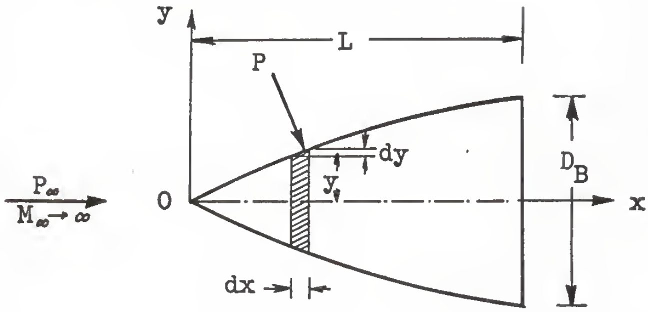

Fig. 4-1. Body of revolution in impact flow.

Dividing Eq. (4-1) by $4\pi q_\infty$, the integral to be minimized becomes

$$I = \frac{D}{4\pi q_\infty} = \frac{1}{2}\int_0^L \frac{P-P_\infty}{q_\infty} y y' dy = \frac{1}{2}\int_0^L C_p y y' dx \tag{4-2}$$

where C_p denotes the local pressure coefficient and $y' = dy/dx$.

According to Eq. (2-20),

$$C_p = 2\sin^2\delta = \frac{2y'^2}{1+y'^2} \tag{4-3}$$

Substituting Eq. (4-3) into Eq. (4-2) yields

$$I = \int_0^L \frac{y y'^3}{1+y'^2} dx \tag{4-4}$$

which is in the form of $\quad I = \int y\phi(y')dx$

where

$$\phi(y') = \frac{y'^3}{1+y'^2} \tag{4-5}$$

Differentiating,

$$\phi'(y') = \frac{3y'^2(1+y'^2)-2y'^4}{(1+y'^2)^2} = \frac{3y'^2+y'^4}{(1+y'^2)^2} \tag{4-6}$$

$$\phi''(y') = \frac{(1+y'^2)(6y'+4y'^3)-4y'(3y'^2+y'^4)}{(1+y'^2)^3} = \frac{2y'(3-y'^2)}{(1+y'^2)^3} \tag{4-7}$$

Substituting Eqs. (4-5) and (4-6) in Eq. (3-25) gives

$$y = \frac{c}{\phi(y') - y'\phi(y')} = \frac{c(1+y'^2)^2}{-2y'^3} = \frac{c_1(1+y'^2)^2}{y'^3} \tag{4-8}$$

where $c_1 = -c/2 > 0$ since $y > 0$ and $y' > 0$. Differentiating

$$y' = c_1 \frac{4y'^4(1+y'^2) - 3y'^2(1+y'^2)^2}{y'^6} \frac{dy'}{dx}$$

$$= \frac{c_1(1+y'^2)(y'^2-3)}{y'^4} y'' \tag{4-9}$$

It is seen from Eq. (4-9) that for $y' > \sqrt{3}$, $y'' > 0$, the meridian curve is convex to the x-axis; for $y' < \sqrt{3}$, $y'' < 0$, the meridian curve is concave to the x-axis. Furthermore, from Eq. (4-7), $\phi''(y') > 0$ when $y' < \overline{3}$. Therefore, according to the concluding remarks at the end of Art. 3-4, the integral and hence the forebody pressure drag has a minimum for a surface concave to the x-axis, for which $y' < \sqrt{3}$.

Now, let us return to Eq. (4-9). Note that $y'' = dy'/dx = y'(dy'/dx)$

$$dy = \frac{c_1(y'^2+1)(y'^2-3)}{y'^4} dy' \tag{4-10}$$

Since $y' = dy/dx$, the general expression for the x coordinate is

$$x = \int \frac{dy}{y'}$$

Substituting Eq. (4-10) in Eq. (4-11) and integrating,

$$x = c_1 \int \frac{(y'^2+1)(y'^2-3)}{y'^5} dy'$$

$$= c_1 \int \left(\frac{1}{y'} - \frac{2}{y'^3} - \frac{3}{y'^5} \right) dy'$$

or

$$X = C_1 \left[\ln y' + \frac{1}{y'^2} + \frac{3}{4y'^4} \right] + C_2 \tag{4-12}$$

$$y = \frac{C_1 (1 + y'^2)^2}{y'^3} \tag{4-8}$$

The generating curve for the minimum-drag body terminates at fixed points. Therefore the body surface is actually a zone of surface of revolution. Theoretically, elimination of the parameter y' between Eqs. (4-12) and (4-8) will result in an equation between x and y, and the constants c_1 and c_2 can be determined from the boundary conditions at the two fixed points.

The body profile thus obtained is not for least drag but for a minimum drag, i.e., the resistance experienced by the body profile of the solution will be a relative minimum rather than an absolute minimum. According to the concept of strong variations, the resistance can be made as small as desired by a zigzag line for the generating curve (Ref. 3). One can not pass from the body shape curve as obtained from the continuous solution to a zigzag line since the change in y' would not be infinitesimal. Actually, the zigzag body configuration is a violation against Newton's law of resistance since there will be an infinite number of places in which air is trapped.

4-3 Discontinuous Solution for Minimum-Drag Body--Given Fineness Ratio.

From the continuous solution obtained in Art. 4-2, it is seen that for the minimum-drag body of revolution, the shape must be concave to the x-axis, and $y' < \sqrt{3}$. In the case where the given fineness ratio requires that the straight line joining A and B is inclined to the x-axis at an angle greater than 60°, the continuous solution is not applicable. It is necessary that $y' > 0$ at all points at the surface. A blunt nosed body, as shown in Fig. 4-2, is considered as a discontinuous solution to this problem.

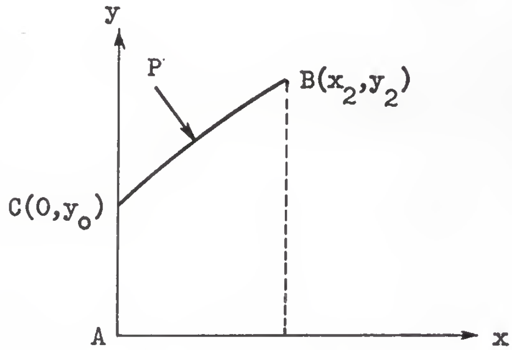

Fig. 4-2. Minimum-drag body for given fineness ratio.

The flat portion AC is normal to the free stream direction, therefore its pressure drag is

$$D_{AC} = \pi y_o^2 (P - P_\infty) \tag{4-13}$$

$$\frac{D_{AC}}{2\pi q_\infty} = \frac{1}{2}\,\frac{P-P_\infty}{q_\infty}\,y_o^2 = \frac{1}{2}\,C_p\,y_o^2 = \frac{1}{2}(2\sin^2 90^\circ)\,y_o^2 = y_o^2 \tag{4-14}$$

For the curved portion CB, the pressure drag is

$$D_{CB} = 2\pi \int_0^{x_2} (P - P_\infty)\,y y'\,dx \tag{4-15}$$

$$\frac{D_{CB}}{2\pi q_\infty} = \int_0^{x_2} \frac{P-P_\infty}{q_\infty}\,y y' dx = \int_0^{x_2} C_p\,y y' dx \tag{4-16}$$

Adding Eqs. (4-14) and (4-16), the integral for the total pressure-drag to be minimized becomes

$$I = \frac{D_T}{2\pi q_\infty} = y_o^2 + \int_0^{x_2} C_p\,y y' dx \tag{4-17}$$

From Eq. (4-3),

$$C_p = 2y'^2/(1 + y'^2)$$

Substituting in Eq. (4-17),

$$I = y_0^2 + 2\int_0^{x_2} \frac{y\,y'^3}{1+y'^2}\,dx \tag{4-18}$$

The second term on the right side is of the same form as in Eq. (4-4) but twicw as large, i.e.,

$$F(y, y') = 2y\phi(y') \tag{4-19}$$

where

$$\phi(y') = \frac{y'^3}{1+y'^2} \tag{4-20}$$

Differentiating Eq. (4-19),

$$F_y = 2\phi(y') \qquad\qquad F_{y'} = 2y\phi'(y')$$

Sustituting in Eq. (3-10), and noting the additional first order term resulting from $(y_o + \delta y_o)^2$, one obtains

$$\delta I = 2y_0\delta y_0 + 2y\,\delta y\,\phi'(y')\Big|_{y_0}^{y_2} + 2\int_0^{x_2}\left\{\phi(y') - \frac{d}{dx}\left[y\,\phi'(y')\right]\right\}\delta y\,dx \tag{4-21}$$

For a minimum to exist, the Euler equation must be satisfied, i.e., that part of δI under the integral sign in Eq. (4-21) must vanish. Evaluation of the upper and lower limits on the second term on the right side of Eq. (4-21) gives

$$\delta I = 2y_o\Big[1 - \phi'(y')\Big]_{y_o}\delta y_o \tag{4-22}$$

since B is a fixed point. From Eq. (4-6),

$$\phi'(y') = \frac{3y'^2 + y'^4}{(1+y'^2)^2} \tag{4-23}$$

Substituting in Eq. (4-22),

$$\delta I = 2y_0\left[1 - \frac{3y_0'^2 + y_0'^4}{(1+y_0'^2)^2}\right]\delta y_0 \tag{4-24}$$

Again, for a minimum to exist, this remaining part of δI must also vanish. Since C is a point at the corner of the blunt nose, $y_o \neq 0$, $\delta y_o \neq 0$. Therefore,

$$1 - \frac{3y_o'^2 + y_o'^4}{(1+y_o'^2)^2} = 0 \tag{4-25}$$

$$y_o' = 1 \tag{4-26}$$

which means that the meridian curve must intersect the y-axis at an angle of 45°. Now, consider the second variation to ascertain that the solution found is a minimum. Referring to Eq. (3-23), and noting the additional second order term $(\delta y_o)^2$ resulting from $(y_o + \delta y_o)^2$, one obtains

$$\delta^2 I = (\delta y_o)^2 + \int_0^{X_2} \phi''(y')\left[y(\delta y')^2 - y''(\delta y)^2\right] dx + (\delta y)^2 \phi'(y')\Big|_{Y_o}^{Y_2}$$

$$= (\delta y_o)^2\left[1 - \phi'(y')\right]_{Y_o} + \int_0^{X_2} \phi''(y')\left[y(\delta y')^2 - y''(\delta y)^2\right] dx \tag{4-27}$$

From Eq. (4-25), we see

$$\left[1 - \phi'(y')\right]_{Y_o} = 0 \tag{4-28}$$

From Eq. (4-7),

$$\phi''(y') = \frac{2y'(3-y'^2)}{(1+y'^2)^3} \tag{4-29}$$

Substituting in Eq. (4-27),

$$\delta^2 I = \int_0^{X_2} \frac{2y'(3-y'^2)}{(1+y'^2)^3}\left[y(\delta y')^2 - y''(\delta y)^2\right] dx > 0 \tag{4-30}$$

since the body curve is concave to the x-axis and the initial slope at point C (the greatest on the curve CB) $y_o' = 1 < \sqrt{3}$. Therefore, the body shape thus obtained is a profile of minimum drag.

4-4 Parametric Equations for Minimizing Curve—Given Fineness Ratio.

The parametric equations for the meridian curve of a minimum-drag body of given fineness ratio have been obtained in Art. 4-2, namely

$$X = c_1\left[\ln y' + \frac{1}{y'^2} + \frac{3}{4y'^4}\right] + c_2 \tag{4-12}$$

$$Y = c_1 \frac{(1+y'^2)^2}{y'^3} \tag{4-8}$$

At point C, $x = o$, $y = y_o$ and $y' = 1$, thus

$y_o = 4c_1$, $\qquad c_1 = y_o/4$

$0 = (y_o/4)(1 + 3/4) + c_2$, $c_2 = -7y_o/16$

Substituting values of c_1 and c_2 in Eqs. (4-12) and (4-8),

$$X = \frac{y_o}{4}\left[\ln y' + \frac{1}{y'^2} + \frac{3}{4y'^4} - \frac{7}{4}\right] \tag{4-31}$$

$$Y = \frac{y_o}{4} \frac{(1+y'^2)^2}{y'^3} \tag{4-32}$$

At point B, $x = x_2 = L$, $y = y_2 = D_B/2$, thus

$$L = \frac{y_o}{4}\left[\ln y_2' + \frac{1}{y_2'^2} + \frac{3}{4y_2'^4} - \frac{7}{4}\right] \tag{4-33}$$

$$D_B = \frac{y_o}{2} \frac{(1+y_2'^2)^2}{y_2'^3} \tag{4-34}$$

Dividing Eq. (4-33) by Eq. (4-34) gives the fineness ratio

$$F_R = \frac{L}{D_B} = \frac{y_2'^3\left[\ln y_2' + \frac{1}{y_2'^2} + \frac{3}{4y_2'^4} - \frac{7}{4}\right]}{2(1+y_2'^2)^2} \tag{4-35}$$

as $y_2' = 1$, $F_R = 0$, which implies zero length;

$y_2' \to 0$, $F_R \to \infty$, which implies infinite length.

Therefore, there exists some real value of y_2' between 0 and 1 such that Eq. (4-35) is satisfied for given fineness ratio.Once the slope of the curve at the rear end point is known, the value of y_o can be obtained from Eq. (4-34).

For example, given the fineness ratio $F_R = 3.09$ and unit chord, i.e., $L = 1$, at point B: $y_2' = 0.12$, $\delta_2 = \arctan 0.12 = 6°51'$

at point C: $y_o = 0.0012$, $\delta_o = 45°$

The local tangent changes from 45° at the nose to 6°51' at the base.

It can be seen that over the major portion of the body surface, the local slope is very small in comparison with that at the nose. That is, for given fineness ratio, the pressure drag is minimized by accepting higher pressures on a relatively small area of large slope near the nose and achieving lower pressures on a large area of small slope near the base.

Since high-speed missiles will operate at both low and high supersonic speeds, it is of particular importance to compare the minimum-drag body shapes determined by the linearized theory with those determined by using the simple impact theory. Two minimizing curves of given fineness ratio determined by the respective methods are plotted in Fig. 4-3. The comparison shows that the shapes are similar, although the minimum-drag body for low supersonic speeds is generally the flatter of the two.

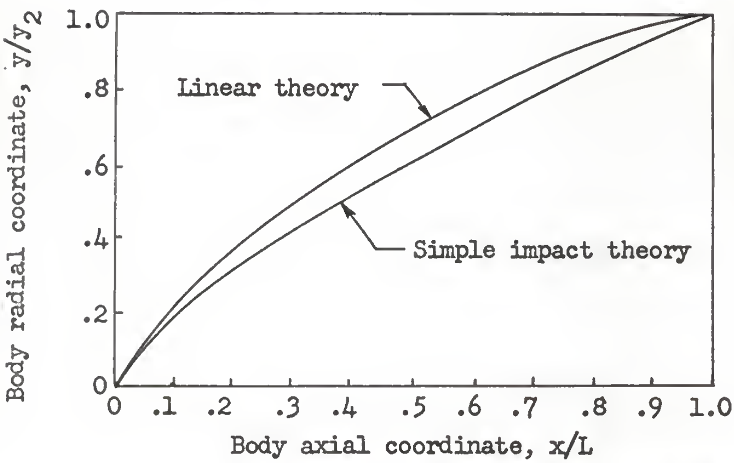

Fig. 4-3. Comparison of minimum-drag bodies of given fineness ratio.(After Eggers, et al., NACA)

4-5 3/4 Power Body Approximation.

In Ref. 2, the minimum-drag body shape given by the parametric equations Eqs. (4-31) and (4-32) was found to be approximated very closely by

$$\frac{Y}{D_B/2} = \left(\frac{X}{L}\right)^{3/4} \tag{4-36}$$

This is known as the 3/4 power body. When the effects of centrifugal forces are neglected, results obtained from (4-36) are in good agreement with those from Eqs. (4-31) and (4-32), particularly for large fineness ratio. Body curves obtained from the exact solution and the 3/4 power approximation are plotted in Fig. 4-4 for comparison.

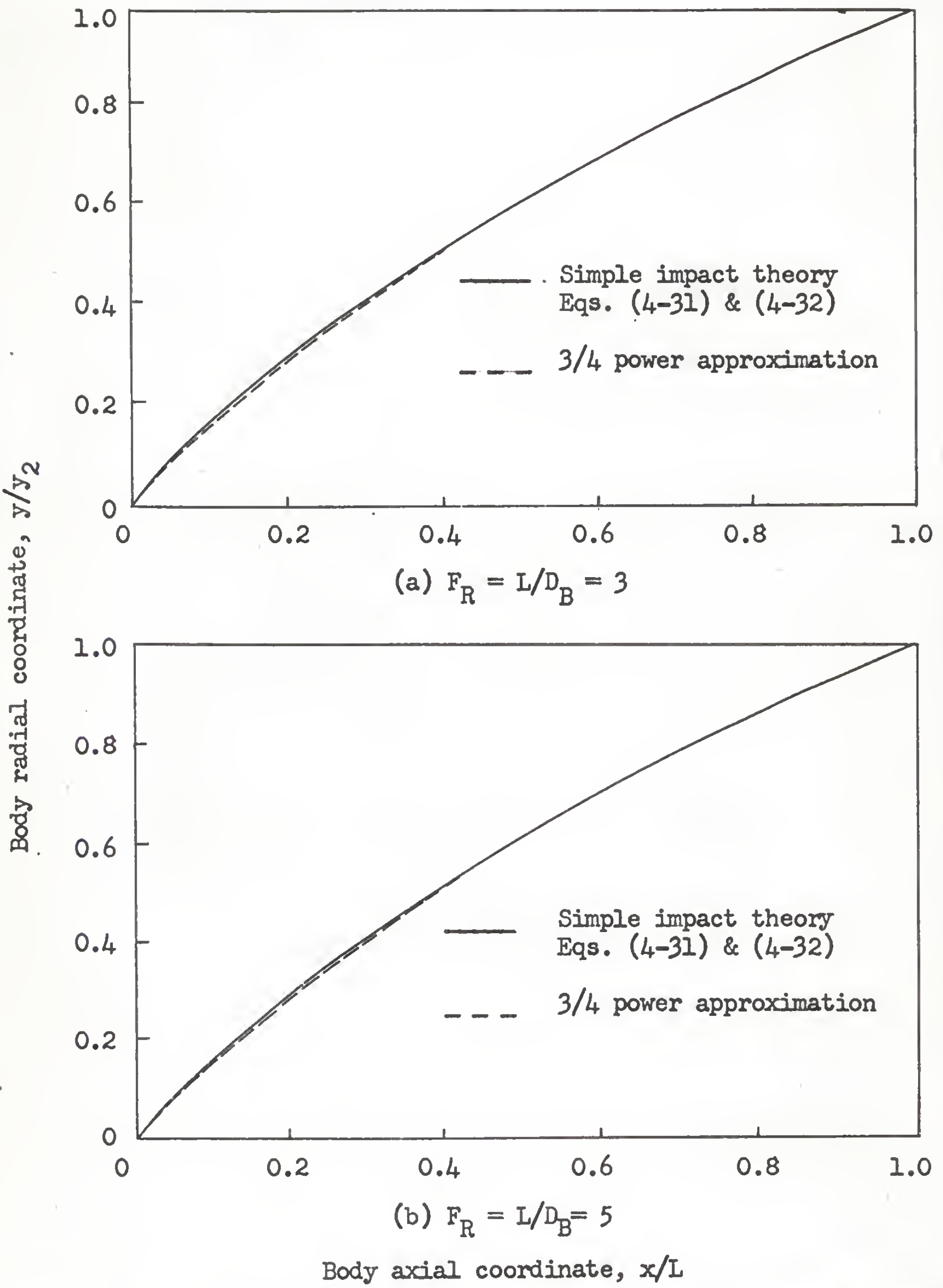

Fig. 4-4. Minimu-drag dodies of revolution for given fineness ratio. (After Eggers, et al., NACA)

4-6 Modified Theory—Given Fineness Ratio.

It was shown in Art. 2-5 that when the centrifugal force effects of flow over curved surface at hypersonic speeds are taken into consideration, the local pressure coefficient is the sum of the pressure coefficient behind the the shock and the pressure coefficient due to curvature. From Eq. (2-34),

$$C_p = 2\sin^2\delta + \frac{y}{2}\left(1-\frac{y}{y_2}\right)\frac{d}{dy}\sin^2\delta \tag{4-37}$$

Again, we take the case of a discontinuous solution for the minimum-drag body of given fineness ratio. The integral to be minimized is mainly the same as in Eq. (4-17), i.e.,

$$I = \frac{D_T}{2\pi q_\infty} = y_0^2 + \int_0^{x_2} C_p\, y y'\, dx \tag{4-38}$$

However, due to the centrifugal forces about the corner at $C(0, y_0)$, an additional term call the "leading edge thrust" should be included in Eq. (4-38). Considering forces over the ring element of infinitesimal thickness around the corner of the nose, the leading edge thrust is

$$T(y_0) = \lim_{\epsilon\to 0} q_\infty \int_{y_0-\epsilon}^{y_0+\epsilon} \frac{y}{2}\left(1-\frac{y}{y_2}\right)\left(\frac{d}{dy}\sin^2\delta\right)(2\pi y\, dy)$$

or

$$\phi(y_0) = \frac{T(y_0)}{2\pi q_\infty} = \lim_{\epsilon\to 0}\int_{y_0-\epsilon}^{y_0+\epsilon} \frac{y^2}{2}\left(1-\frac{y}{y_2}\right)\frac{d}{dy}\sin^2\delta\, dy \tag{4-39}$$

By using the mean value theorem,

$$\phi(y_0) = \frac{\bar{y}^2}{2}\left(1-\frac{\bar{y}}{y_2}\right)\lim_{\epsilon\to 0}\int_{y_0-\epsilon}^{y_0+\epsilon}\frac{d}{dy}\sin^2\delta\, dy$$

where $y_0 - \epsilon < \bar{y} < y + \epsilon$. Since $\epsilon \to 0$

$$\frac{\bar{y}^2}{2}\left(1-\frac{y}{Y_2}\right)=\frac{Y_0^2}{2}\left(1-\frac{Y_0}{Y_2}\right)$$

$$\phi(Y_0)=\frac{Y_0^2}{2}\left(1-\frac{Y_0}{Y_2}\right)\left[\sin^2\delta_0-\sin^2\delta_{0-}\right]$$

$$=\frac{Y_0^2}{2}\left(1-\frac{Y_0}{Y_2}\right)\left(\sin^2\delta_0-1\right)=-\frac{Y_0^2}{2}\left(1-\frac{Y_0}{Y_2}\right)\cos^2\delta_0 \tag{4-40}$$

since $\delta_{0-}=90°$. Letting

$$\phi(y)=-\frac{y^2}{2}\left(1-\frac{y}{Y_0}\right)\cos^2\delta \tag{4-41}$$

Eq. (4-40) can also be written as

$$\phi(Y_0)=-\int_0^{X_2}\frac{d}{dx}\phi(y)\,dx \tag{4-42}$$

since

$$-\int_0^{X_2}\frac{d}{dx}\phi(y)\,dx=-\left[\phi(Y_2)-\phi(Y_0)\right]=\phi(Y_0)$$

Now, adding Eq. (4-42) to Eq. (4-38),

$$I=Y_0^2+\int_0^{X_2}\left[C_p y y'-\frac{d}{dx}\phi(y)\right]dx \tag{4-43}$$

The integrand function to be minimized can be written as

$$F(y,\ y')=C_p\ yy'-\frac{d}{dx}\phi(y) \tag{4-44}$$

The necessary condition for a minimum to exist is that the Euler equation be satisfied. The integrand function F is independent of x, therefore, according to Eq. (3-16),

$$y'F_{y'} - F = c \tag{4-45}$$

Since

$$\sin^2\delta = \frac{y'^2}{1+y'^2} \tag{4-46}$$

one obtains

$$\cos^2\delta = \frac{1}{1+y'^2} \tag{4-47}$$

Substituting in Eq. (4-44) gives

$$C_p y y' = y y'\left[2\sin^2\delta + \frac{y}{2}\left(1-\frac{y}{y_2}\right)\frac{d}{dy}\sin^2\delta\right]$$

$$= \frac{2yy'^3}{1+y'^2} + \frac{y^2y'}{2}\left(1-\frac{y}{y_2}\right)\frac{d}{dy}\sin^2\delta \tag{4-48}$$

Differentiating Eq. (4-41)

$$\frac{d}{dx}\phi(y) = \cos^2\delta\,\frac{d}{dx}\left(-\frac{y^2}{2}+\frac{y^3}{2y_2}\right) - \frac{y^2}{2}\left(1-\frac{y}{y_2}\right)\frac{d}{dx}\cos^2\delta$$

$$= \frac{yy'}{1+y'^2}\left(-1+\frac{3}{2}\frac{y}{y_2}\right) - \frac{y^2}{2}\left(1-\frac{y}{y_2}\right)\frac{d}{dx}\cos^2\delta \tag{4-49}$$

Subtracting Eq. (4-49) from (4-48) gives

$$F(y,y') = \frac{yy'}{1+y'^2}\left(2y'^2+1-\frac{3}{2}\frac{y}{y_2}\right) = yy'\left(2-\frac{1+\frac{3}{2}\frac{y}{y_2}}{1+y'^2}\right) \tag{4-50}$$

$$F_{y'} = yy'\,\frac{2y'\left(1+\frac{3}{2}\frac{y}{y_2}\right)}{(1+y'^2)^2} + y\left(2-\frac{1+\frac{3}{2}\frac{y}{y_2}}{1+y'^2}\right)$$

$$= \frac{2yy'^2}{(1+y'^2)^2}\left(1+\frac{3}{2}\frac{y}{y_2}\right) + \frac{F}{y'} \tag{4-51}$$

Substitution of Eqs. (4-50) and (4-51) in Eq. (4-45) yields

$$y' F_{y'} - F = \frac{2yy'^2}{(1+y'^2)^2}\left(1+\frac{3}{2}\frac{y}{y_2}\right) = C = 2C_1 \tag{4-52}$$

or

$$\frac{3}{2y_2}y^2 + y - \frac{C_1(1+y'^2)^2}{y'^3} = 0$$

$$y = \frac{y_2}{3}\left[-1+\sqrt{1+\frac{6}{y_2}\frac{C_1}{y'^3}(1+y'^2)^2}\right] \tag{4-53}$$

At end point B, $y = y_2$, $y' = y_2'$. From Eq. (4-52), the constant c_1 is

$$C_1 = \frac{5}{2}\frac{y_2 y_2'^3}{(1+y_2'^2)^2}$$

Substituting in Eq. (4-53),

$$y = \frac{y_2}{3}\left[\sqrt{1+\frac{15y_2'^3}{(1+y_2'^2)^2}\frac{(1+y'^2)^2}{y'^3}} - 1\right] \tag{4-55}$$

$$x = \int \frac{dy}{y'} \tag{4-56}$$

From Eq. (4-43), $\quad I = y_0^{\,2} + \int_0^{x_2} F(y, y')dx$

For a minimum to exist, the remaining part besides the Euler equation must vanish. Referring to Eq. (3-10), it can be written as

$$2y_0\,\delta y_0 + F_{y'}\,\delta y\Big|_{y_0}^{y_2} = \left(2y_0 - F_{y'}\Big|_{y_0}\right)\delta y_0 = 0 \tag{4-57}$$

where the upper limit vanishes since $\delta y = 0$ at B. From Eq. (4-51)

$$F_{y'}\Big|_{y_o} = \frac{2y_o y_o'^2}{(1+y_o'^2)^2}\left(1+\frac{3}{2}\frac{y_o}{y_2}\right)+2y_o - \frac{y_o}{1+y'^2}\left(1+\frac{3}{2}\frac{y_o}{y_2}\right)$$

Substituting in Eq. (4-57),

$$y_o\left(1+\frac{3}{2}\frac{y_o}{y_2}\right)\left[\frac{1}{1+y_o'^2} - \frac{2y_o'^2}{(1+y_o'^2)^2}\right] = 0$$

Therefore

$$y_o' = 1 \tag{4-58}$$

That is, the meridian curve intersects the y-axis at an angle of 45°. By substituting $y_o' = 1$ in Eq. (4-55), it is seen that

$$y_o > 0 \tag{4-59}$$

Fig. 4-5 shows the minimum-drag body of fineness ratio $F_R = 6.18$ with results obtained from both the simple impact theory(Eqs.4-31 and 4-32) and the modified impact theory (Eqs. 4-55 and 4-56). It is seen that the body curve determined by taking centrifugal force effects into consideration is more blunt and has more curvature in the nose section.

4-7 2/3 Power Body Approximation.

For the simple impact theory, it was noted that the minimum-drag body of large fineness ratio was closely approximated by the 3/4 power body shape. In the present case of modified theory where centrifugal forces are taken into account, it is further noted that the 2/3 power body

$$\frac{y}{D_B/2} = \left(\frac{x}{L}\right)^{2/3} \tag{4-60}$$

provides a good approximation to the exact solution given by Eqs. (4-55) and (4-56). In fact, the plot of Eq. (4-60) for $F_R = 6.18$ falls on top of the curve determined by the modified theory in Fig. 4-5.

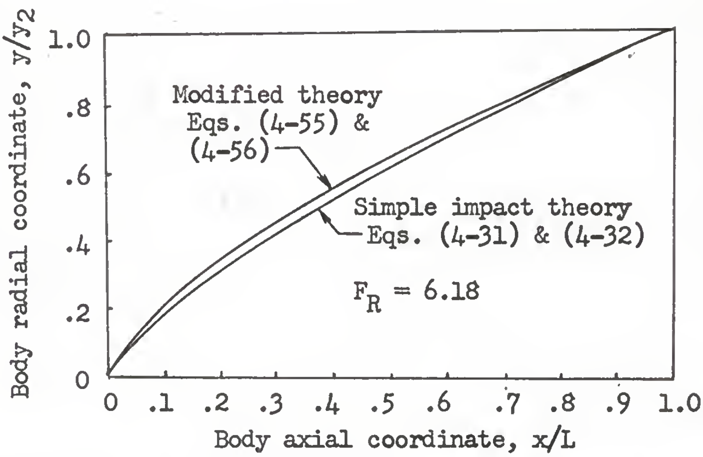

Fig. 4-5. Comparison of minimum-drag body for simple and modified theory. (After Eggers et al., NACA)

4-8 Minimum-Drag Body—Given Surface Area and Base Diameter.

Following the same procedure in Art. 4-3, i.e., assuming a blunt nose for a general form, the pressure drag parameter to be minimized is

$$I_D = y_0^2 + \int_0^{x_2} \frac{2yy'^3}{1+y'^2}\, dx \tag{4-61}$$

However, the solution is subject to a constraint

$$S = \pi y_0^2 + 2\pi \int_0^{x_2} y\sqrt{1+y'^2}\, dx = \text{constant} \tag{4-62}$$

Since length is not given, we can take $y_0 = 0$. By the isoperimetric rule in Eq. (3-33),

$$I = \int_0^{x_2} \left[\frac{2yy'^3}{1+y'^2} + \lambda y\sqrt{1+y'^2} \right] dx \tag{4-63}$$

where λ is the Lagrange multiplier. The integrand function takes on the form

$$F(y, y') = y\phi(y')$$

in which

$$\phi(y') = \frac{2y'^3}{1 + y'^2} + \lambda\sqrt{1 + y'^2} \tag{4-64}$$

According to Eq. (3-18),

$$y\phi(y') - yy'\phi'(y') = c \tag{4-65}$$

Differentiating Eq. (4-64),

$$\phi'(y') = \frac{6y'^2(1 + y'^2) - 4y'^4}{(1 + y'^2)^2} + \frac{\lambda y'}{\sqrt{1 + y'^2}}$$

$$= \frac{2(3y'^2 + y'^4)}{(1 + y'^2)^2} + \frac{\lambda y'}{\sqrt{1 + y'^2}} \tag{4-66}$$

Substituting Eqs. (4-64) and (4-66) in Eq. (4-65),

$$y\left[\frac{2y'^3}{1 + y'^2} + \lambda\sqrt{1 + y'^2} - \frac{6y'^3 + 2y'^5}{(1 + y'^2)^2} - \frac{\lambda y'^2}{\sqrt{1 + y'^2}}\right] = c$$

$$y\left[\frac{\lambda}{\sqrt{1 + y'^2}} - \frac{4y'^3}{(1 + y'^2)^2}\right] = c \tag{4-67}$$

Since the curve must meet the x-axis, c = 0. Thus

$$\lambda = \frac{4y'^3}{(1 + y'^2)^{3/2}} = 4\left(\frac{y'^2}{1 + y'^2}\right)^{3/2} \tag{4-68a}$$

or

$$\frac{1 + y'^2}{y'^2} = \left(\frac{4}{\lambda}\right)^{2/3}$$

$$y' = \frac{dy}{dx} = \frac{1}{\sqrt{(4/\lambda)^{2/3} - 1}} \tag{4-68b}$$

Integrating

$$y = \frac{x}{\sqrt{(4/\lambda)^{2/3} - 1}} \tag{4-69}$$

where the constant of integration is seen to be zero, since x = 0, y = 0. The generating curve is a straight line, therefore, the required minimum-drag body of given surface and base diameter is a right circular cone, with the **slope** $y' = y/x = y_2/x_2$ = constant, the surface area is given by

$$\begin{aligned} S &= 2\pi \int_0^{x_2} y\sqrt{1 + y'^2}\, dx \\ &= 2\pi \sqrt{\frac{1 + y'^2}{y'^2}} \int_0^{y_2} y\, dy = \pi y_2^2 \sqrt{\frac{1 + y'^2}{y'^2}} \end{aligned} \tag{4-70}$$

or

$$\sqrt{\frac{y'^2}{1 + y'^2}} = \frac{\pi y_2^2}{S}$$

Substituting in Eq. (4-68a),

$$\lambda = 4\left(\frac{\pi y_2^2}{S}\right)^3 \tag{4-71}$$

To ascertain that the solution found in Eq. (4-69) is a minimum, let us consider the second variation of the integral in Eq. (4-63). Referring to Eq. (3-24), since $y'' = 0$ for the cone, therefore

$$\delta^2 I = \frac{1}{2}\int_0^{x_2} y(\delta y')^2 \phi''(y')\, dx \tag{4-72}$$

the sign of which depends solely on $\phi''(y')$. Differentiating Eq. (4-66),

$$\phi''(y') = \frac{2\left[(1 + y'^2)(6y' + 4y'^3) - 4y'(3y'^2 + y'^4)\right]}{(1 + y'^2)^3} + \lambda\left[\frac{-y'^2}{(1 + y'^2)^{3/2}} + \frac{1}{(1 + y'^2)^{1/2}}\right]$$

$$\phi''(y') = \frac{2(6y' + 10y'^3 + 4y'^5 - 12y'^3 - 4y'^5)}{(1+y'^2)^3} + 4\left(\frac{y'^2}{1+y'^2}\right)^{3/2} \frac{1}{(1+y'^2)^{3/2}}$$

$$= \frac{12y'}{(1+y'^2)^3} > 0 \tag{4-73}$$

since $y' > 0$. Therefore the body profile thus found gives minimum pressure drag.

4-9 Modified Theory—Given Surface Area and Base Diameter.

By taking centrifugal force effects into account, we can treat the problem in the same way as for given fineness ratio in Art. 4-6, except for the addition of the constraint

$$S = \pi Y_0^2 + 2\pi \int_0^{X_2} Y\sqrt{1+Y'^2}\, dx = \text{const.} \tag{4-74}$$

With the aid of Eqs. (4-43) and (4-44), the new integrand function to be minimized here is

$$f = F(y, y') + \lambda y\sqrt{1+y'^2} \tag{4-75}$$

where, as in Eq. (4-50),

$$F = yy'\left(2 - \frac{1 + \frac{3}{2}\frac{Y}{Y_2}}{1+y'^2}\right)$$

Hence

$$f_{y'} = F_{y'} + \frac{\lambda y y'}{\sqrt{1+y'^2}} \tag{4-76}$$

Solution of the Euler equation gives

$$y' f_{y'} - f = c$$

or

$$y' F_{y'} - F + \lambda y \sqrt{1 + y'^2} \left(\frac{y'^2}{1 + y'^2} - 1 \right) = c \tag{4-77}$$

Substitution of Eq. (4-52) in Eq. (4-77) yields

$$\frac{2yy'^3}{(1 + y'^2)^2} \left(1 + \frac{3}{2} \frac{y}{y_2}\right) - \frac{\lambda y}{\sqrt{1 + y'^2}} = c \tag{4-78}$$

Since $y_o = 0$, $c = 0$. Therefore

$$\lambda = \frac{2y'^3}{(1 + y'^2)^{3/2}} \left(1 + \frac{3}{2} \frac{y}{y_2}\right)$$

$$y = \frac{y_2}{3} \left[\frac{\lambda (1 + y'^2)^{3/2}}{y'^3} - 2 \right] \tag{4-79}$$

Differentiating Eq. (4-79),

$$dy = \frac{\lambda y_2}{3} \, \frac{3y'^2 (1 + y'^2)^{\frac{1}{2}} - 3(1 + y'^2)^{3/2}}{y'^4} \, dy'$$

$$= \lambda y_2 \left[\frac{(1 + y'^2)^{\frac{1}{2}}}{y'^2} - \frac{(1 + y'^2)^{3/2}}{y'^4} \right] dy'$$

$$x = \int_0^y \frac{dy}{y'} = \lambda y_2 \left[\int_0^{y'} \frac{(1 + y'^2)^{1/2}}{y'^3} \, dy' - \int_0^{y'} \frac{(1 + y'^2)^{3/2}}{y'^5} \, dy' \right]$$

By using table of integrals,

$$x = \frac{\lambda y_2}{8} \left[\frac{2(1 + y'^2)^{3/2}}{y'^4} + \frac{(1 + y'^2)^{1/2}}{y'^2} - \ln \frac{1 - \sqrt{1 + y'^2}}{y'} - c \right] \tag{4-80}$$

With $y_0 = 0$, the value of λ is determined as follows:

$$S = 2\pi\int_0^{x_2} y\sqrt{1+y'^2}\,dx = 2\pi\int_0^{y_2} y\sqrt{\frac{1+y'^2}{y'^2}}\,dy \tag{4-81}$$

From Eq. (4-79),

$$\sqrt{\frac{1+y'^2}{y'^2}} = \frac{1}{\lambda^{1/3}}\left(\frac{3y}{y_2}+2\right)^{\frac{1}{3}}$$

hence

$$S = \frac{2\pi}{\lambda^{1/3}}\int_0^{y_2} y\left(\frac{3y}{y_2}+2\right)^{\frac{1}{3}} dy \tag{4-82}$$

Transforming variable, let

$$\frac{3y}{y_2} + 2 = z^3$$

$$y = \frac{y_2}{3}(z^3-2), \qquad dy = y_2 z^2 dz$$

$$y=0, \quad z = \sqrt[3]{2}; \qquad y = y_2, \quad z = \sqrt[3]{5}$$

Thus

$$S = \frac{2\pi}{\lambda^{1/3}}\int_{\sqrt[3]{2}}^{\sqrt[3]{5}} \frac{y_2}{3}(z^3-2)\cdot z\cdot y_2 z^2 dz$$

or

$$\frac{\lambda^{\frac{1}{3}} S}{y_2^2} = \frac{2\pi}{3}\int_{\sqrt[3]{2}}^{\sqrt[3]{5}} (z^6 - 2z^3)\,dz = \frac{2\pi}{3}\left[\frac{z^7}{7} - \frac{z^4}{2}\right]_{\sqrt[3]{2}}^{\sqrt[3]{5}}$$

$$= \frac{2\pi}{3}\left[\frac{15}{14}\sqrt[3]{5} + \frac{3}{7}\sqrt[3]{2}\right] = 4.95$$

$$\lambda = 121.6\left(\frac{y_2^2}{S}\right)^3 \tag{4-83}$$

Minimizing curves obtained from both the simple and modified impact theory are plotted in Fig. 4-6 for comparison. In this case, both bodies have pointed noses since the length is not fixed. The body shape determined by the modified theory has curvature in areas behind the nose while that determined by the simple impact theory has straight sides, i.e., a right circular cone.

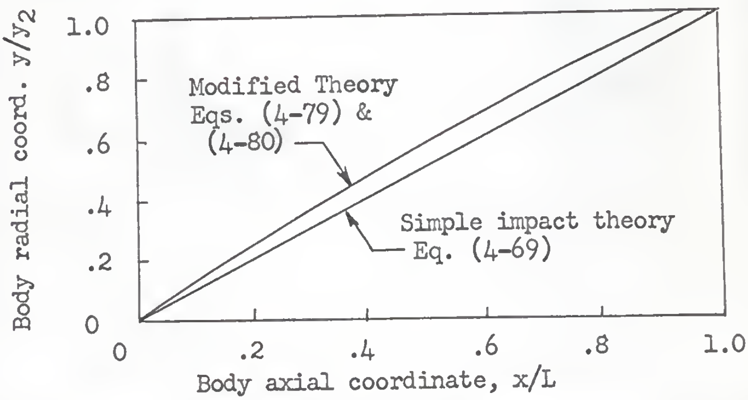

Fig. 4-6. Minimum-drag body for given surface area and base diameter determined by simple and modified impact theory. (After Eggers, et al., NACA)

4-10 Continuous Solution for Minimum-Drag Body--Given Base Diameter and Volume.

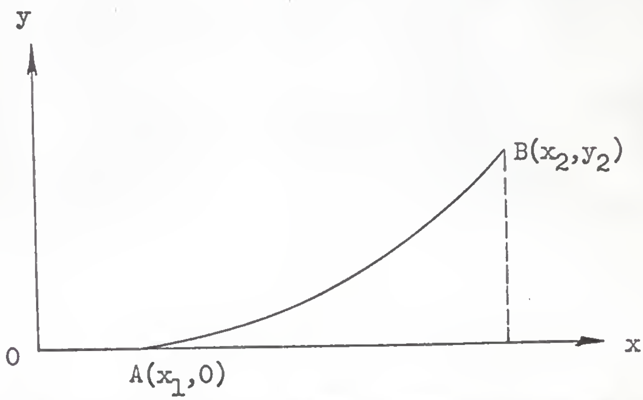

Fig. 4-7. Minimum-drag body for given base diameter and volume, continuous solution; schemation.

The expression for the pressure drag on the body surface described by the generating curve AB in Fig. 4-7 is given by

$$I_D = \int_{x_1}^{x_2} \frac{2yy'^3}{1+y'^2}\,dx \tag{4-84}$$

where the coordinate x_1 is not fixed yet. Being subject to the auxiliary condition

$$Vol. = \pi\int_{x_1}^{x_2} y^2 dx = constant \tag{4-85}$$

the integral to be minimized becomes

$$I = \int_{x_1}^{x_2} \left(\frac{2yy'^3}{1+y'^2} + \lambda y^2\right) dx \tag{4-86}$$

with the integrand

$$F = \frac{2yy'^3}{1+y'^2} + \lambda y^2 \tag{4-87}$$

Since F is independent of x, the solution of Euler equation is

$$y'F_{y'} - F = c \tag{4-88}$$

From Eq. (4-87),

$$F_{y'} = \frac{6yy'(1+y'^2) - 4yy'^4}{(1+y'^2)^2} = \frac{2y(3y'^2+y'^4)}{(1+y'^2)^2} \tag{4-89}$$

Substituting Eqs. (4-87) and (4-89) in Eq. (4-88),

$$\frac{2yy'(3y'^2+y'^4)}{(1+y'^2)^2} - \frac{2yy'^3}{1+y'^2} - \lambda y^2 = c \tag{4-90}$$

At point A, y = 0, c = 0. Therefore

$$\lambda y = \frac{2y'(3y'^2+y'^4)}{(1+y'^2)^2} - \frac{2y'^3}{1+y'^2} = \frac{4y'^3}{(1+y'^2)^2} \tag{4-91}$$

$$y = \frac{4}{\lambda}\frac{y'^3}{(1+y'^2)^2} = \frac{cy'^3}{(1+y'^2)^2} \tag{4-92}$$

where $c = 4/\lambda$ The minimizing body curve can be proved to be a hypocycloid. From the relation $y' = \tan\theta$, Equation (4-92) becomes

$$y = c\tan^3\theta/\sec^4\theta = c\cos\theta\sin^3\theta \tag{4-93}$$

then

$$dy = c(3\sin^2\theta\cos^2\theta - \sin^4\theta)\,d\theta$$

$$= c\sin\theta\left[3\sin\theta(1-\sin^2\theta) - \sin^3\theta\right]d\theta$$

$$= c\sin\theta\sin 3\theta\,d\theta \tag{4-94}$$

$$dx = dy/y' = c(\sin\theta/\tan\theta)\sin 3\theta\,d\theta$$

$$= c\cos\theta\sin 3\theta\,d\theta \tag{4-95}$$

Hence

$$ds = \sqrt{(dx)^2+(dy)^2} = c\sin 3\theta\,d\theta \tag{4-96}$$

From Eq. (4-93), note that when $y = 0$, $\theta = 0$, or 90°. Take the point where $y = 0$, $\theta = 0$ as a starting point for the measurement of the arc length. Integration of Eq. (4-96) gives

$$\int_0^s ds = c\int_0^\theta \sin 3\theta\,d\theta$$

or

$$s = -\frac{c}{3}\cos 3\theta\Big|_0^\theta = \frac{c}{3}(1-\cos 3\theta) \tag{4-97}$$

To find the volume of the body, substitute Eqs. (4-93) and (4-95) in Eq. (4-85),

$$\text{Vol.} = \pi c^3\int_0^{\theta_2}\cos^3\theta\sin 3\theta\sin^6\theta\,d\theta \tag{4-98}$$

where θ_2 is the value of θ at point B (x_2, y_2). From Eq. (4-93),

$$y_2 = c\cos\theta_2\sin^3\theta_2$$

or

$$c = y_2/\cos\theta_2 \sin^3\theta_2 \tag{4-99}$$

Integrating Eq. (4-98) and substituting for c from Eq. (4-99),

$$\begin{aligned} Vol. &= \pi c^3 \int_0^{\theta_2} \cos\theta(1-\sin^2\theta)(3-4\sin^2\theta)\sin^7\theta \, d\theta \\ &= \frac{\pi y_2^3}{\sin\theta_2 \cos^3\theta_2}\left(\frac{3}{8} - \frac{7}{10}\sin^2\theta_2 + \frac{1}{3}\sin^4\theta_2\right) \end{aligned} \tag{4-100}$$

$$\begin{aligned} \text{or} \quad Vol. &= \frac{\pi y_2^3}{\sin\theta_2 \cos^3\theta_2}\left[\frac{3}{8}\left(\cos^4\theta_2 + 2\cos^2\theta_2\sin^2\theta_2 + \sin^4\theta_2\right) - \frac{7}{10}\sin^2\theta_2\left(\cos^2\theta_2 + \sin^2\theta_2\right) + \frac{1}{3}\sin^4\theta_2\right] \\ &= \pi y_2^3\left(\frac{\tan^3\theta_2}{120} + \frac{\tan\theta_2}{20} + \frac{3}{8\tan\theta_2}\right) \end{aligned} \tag{4-101}$$

It is clear that the value of Vol. can be made as large as desired but it has a lower limit below which the continuous solution is no longer applicable. The minimum permissible volume can be determined as follows:

Differentiating Eq. (4-101) with respect to θ_2,

$$\begin{aligned} \frac{d\,Vol.}{d\theta_2} &= \pi y_2^3\left(\frac{1}{40}\tan^2\theta_2 \sec^2\theta_2 + \frac{1}{20}\sec^2\theta_2 - \frac{3}{8}\,\frac{\sec^2\theta_2}{\tan^2\theta_2}\right) \\ &= \frac{\pi}{40} y_2^3 \csc^2\theta_2\left(\tan^4\theta_2 + 2\tan^2\theta_2 - 15\right) \end{aligned} \tag{4-102}$$

Setting $dVol/d\theta_2 = 0$, one obtains

$$\tan\theta_2 = \sqrt{3}, \quad \text{or} \quad \theta = 60° \tag{4-103}$$

since $y_2 \neq 0$, $1 \leq \csc^2\theta_2 < \infty$, and $\tan^2\theta_2 = -5$ is unreasonable. Therefore

$$Vol_{min} = \pi y_2^3\left(\frac{1}{40} + \frac{1}{20} + \frac{1}{8}\right)\sqrt{3} = \frac{\sqrt{3}}{5}\pi y_2^3 \simeq 1.089\, y_2^3 \tag{4-104}$$

Given the base diameter $D_B = 2y_2$, if the required volume is less than the value specified above, the problem should be attacked in a different way (see discontinuous solution).

Now, to ascertain that the solution thus found is a minimum, let us determine the sign of the second variation. According to Eq. (3-4),

$$\delta^2 I = \frac{1}{2}\int_{x_1}^{x_2}\left[(\delta y)^2 F_{yy} + 2\,\delta y\,\delta y' F_{yy'} + (\delta y')^2 F_{y'y'}\right] dx \tag{4-105}$$

From Eq. (4-87)

$$F = \frac{2yy'}{1+y'^2} + \lambda y^2 = 2y\,\phi(y') + \lambda y^2 \tag{4-106}$$

where $\quad \emptyset(y') = y'^3/(1 + y'^2)$, the same form as in Eq.(4-5). (4-107)

Differentiating,

$$F_{yy} = 2\lambda, \qquad F_{yy'} = 2\emptyset'(y'), \qquad F_{y'y'} = 2y\emptyset''(y') \tag{4-108}$$

where, from Eqs. (4-6) and (4-7),

$$\phi'(y') = \frac{3y'^2 + y'^4}{(1+y'^2)^2} \tag{4-109}$$

$$\phi''(y') = \frac{2y'(3-y'^2)}{(1+y'^2)^3} \tag{4-110}$$

Substituting Eq. (4-108) in Eq. (4-105),

$$\delta^2 I = \int_{x_1}^{x_2}\left[2\,\delta y\,\delta y'\,\phi'(y') + (\delta y')^2 y\,\phi''(y') + \lambda(\delta y)^2\right] dx \tag{4-111}$$

From Eq. (4-92),

$$y = \frac{4}{\lambda}\,\frac{y'^3}{(1+y'^2)^2}$$

Hence

$$y' = \frac{4}{\lambda}\,\frac{(1+y'^2)(3y'^2) - y'^3(4y')}{(1+y'^2)^3}\,y'' = \frac{4}{\lambda}\,\frac{3y'^2 - y'^4}{(1+y'^2)^3}\,y'' \tag{4-112}$$

Comparing with Eq. (4-110) and simplying,

$$\lambda = 2\phi''(y')\, y'' \tag{4-113}$$

Substituting in Eq. (4-111) and noting that the first two terms of the integrand function in Eq. (4-111) is just the same as that in Eq. (3-19), except for a factor of 2, one can follow Eq. (3-23) and write

$$\delta^2 I = \int_{x_1}^{x_2} \phi''(y') \left[y(\delta y')^2 + y''(\delta y)^2 \right] dx + (\delta y)^2 \phi'(y') \Big|_0^{y_0} \tag{4-114}$$

$$\delta^2 I = \int_{x_1}^{x_2} \phi''(y') \left[y(\delta y')^2 + y''(\delta y)^2 \right] dx \tag{4-115}$$

in which the integrated part has vanished and the sign of the second variation depends solely on $\phi''(y')$, since the minimizing curve, a hypocycloid, is convex to the x-axis, and hence $y'' > 0$. Eq. (4-110) shows $\phi''(y') > 0$ when $y' < \sqrt{3}$, i.e., $\theta < 60°$. Therefore, for the continuous solution found for given base diameter and volume to give a minimum drag, the meridian curve must intercept the base at an angle no less than 30°, which corresponds to the condition of the minimum permissible volume. In other words, for given volume not less than $\sqrt{3}\, y_2^{\,3}/5$, the solution found is a minimum.

4-11 Discontinuous Solution for Minimum Drag Body--Given Base Diameter and Volume. Similar to that for the body of given fineness ratio, the body under consideration is made up of a curved portion AC and a normal portion CB as shown in Fig. 4-8. The integral to be minimized for the pressure drag on the portion AC is

$$I_{AC} = \int_{x_1}^{x_2} \left(\frac{2yy'^3}{1+y'^2} + \lambda y^2 \right) dx \tag{4-116}$$

where λ is the Lagrange multiplier for the constraint condition of given volume. On the portion CB, where the flow direction is normal to the surface, i.e., $\delta = 90°$, $C_p = 2$, (see Eq. 4-14), the integral to be minimized is

$$I_{CB} = y_2^2 - y_o^2 \tag{4-117}$$

where the ordinate y_o is to be determined. Combining Eqs. (4-116) and (4-117),

$$I = y_2^2 - y_o^2 + \int_{x_1}^{x_2} \left(\frac{2yy'^3}{1+y'^2} + \lambda y^2 \right) dx \tag{4-118}$$

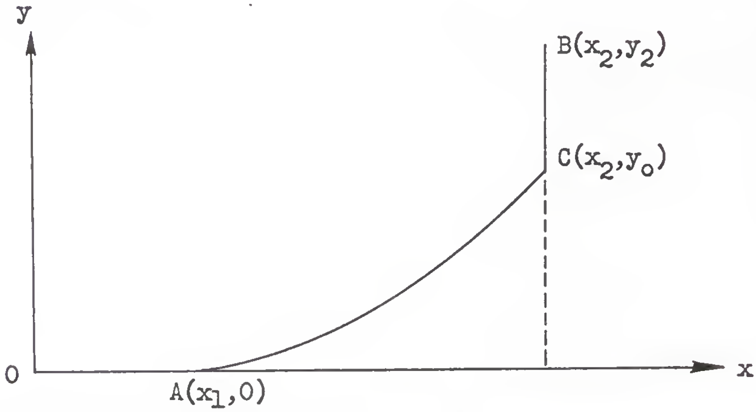

Fig. 4-8. Body curve for given base diameter and volume, discontinuous solution; schematic

Note that the integral is the same as in Eq. (4-86). Referring to Eq. (4-92), the solution of Euler's equation yields

$$y = \frac{4}{\lambda} \frac{y'^3}{(1+y'^2)^2} = \frac{cy'^3}{(1+y'^2)^2} \tag{119}$$

Furthermore, for a minimum to exist, the remaining part of δI must also vanish. According to Eq. (3-10),

$$F_{y'} \delta y \Big|_0^{y_o} - 2y_o \delta y_o = 0 \tag{4-120}$$

in which the second term is the first order term resulting from $-(y_o + \delta y_o)^2$.

Following Eq. (4-106), the integrand of Eq. (4-118) is

$$F = 2y\phi(y') + \lambda y^2 \tag{4-121}$$

where

$$\phi(y') = \frac{y'^3}{1+y'^2} \tag{4-122}$$

$$\phi'(y') = \frac{3y'^2 + y'^4}{(1+y'^2)^2} \tag{4-123}$$

$$\phi''(y') = \frac{2y'(3-y'^2)}{(1+y'^2)^3} \tag{4-124}$$

At point $A(x_1, 0)$, $\delta y = 0$. Thus, with the lower limit of the integrated part vanished, Eq. (4-120) becomes

$$2y_o\phi(y_o')\,\delta y_o - 2y_o\,\delta y_o = 0 \tag{4-125}$$

or

$$\phi(y_o') = \frac{3y_o'^2 + y_o'^4}{(1+y_o'^2)^2} = 1 \tag{4-126}$$

Therefore,

$$y_o' = 1 \tag{4-127}$$

Now, to ascertain that the solution found is a minimum, let us consider the sign of the second variation. Referring to Eq. (4-114),

$$\delta^2 I = \int_{x_1}^{x_2} \phi'(y')\left[y(\delta y')^2 + y''(\delta y)^2\right]dx + (\delta y)^2\phi'(y')\Big|_0^{y_o} - (\delta y_o)^2 \tag{4-128}$$

where $-(\delta y_o)^2$ is the second order term arising from the variation of $-(y_o+\delta y_o)^2$. Again, note that the lower limit of the integrated part vanishes. From Eq. (4-126), the last two terms cancel each other, since

$$(\delta y_o)^2\phi'(y_o') - (\delta y_o)^2 = (\delta y_o)^2 - (\delta y_o)^2 = 0 \tag{4-129}$$

Hence, Eq. (4-128) becomes

$$\delta^2 I = \int_{x_1}^{x_2} \phi''(y') \left[y(\delta y')^2 + y''(\delta y)^2 \right] dx \qquad (4\text{-}130)$$

which is of the same form as in Eq. (4-115). Therefore, the concluding remarks for the continuous solution in Art. 4-10 also hold for the discontinuous solution here. Since at point C (x_2, y_0), where the curve AC has the steepest slope with $y' = 1 < \sqrt{3}$, the solution found is a minimum.

With all information available from the continuous and the discontinuous solutions, the range of volume can be determined as follows.

At point C (x_2, y_0), $y_0' = 1$. Substituting in Eq. (4-119),

$$y_0 = 1/\lambda \qquad (4\text{-}131)$$

Substituting $\theta_c = 45°$ for θ_2 in Eq. (4-101), the given volume becomes

$$\text{Vol.} = \pi y_0^3 \left(\frac{1}{120} + \frac{1}{20} + \frac{3}{8} \right) = \frac{13}{30} \pi y_0^3 \qquad (4\text{-}132)$$

Since $y_0 \leq y_2$, the greatest admissible volume is

$$\text{Vol.} = \frac{13}{30} \pi y_2^3 \cong 1.361\, y_2^3 \qquad (4\text{-}133)$$

From the findings in Articles 4-10 and 4-11, the choice between the continuous and discontinuous solutions for least resistance* body of revolution of given volume and base diameter can be made as follows:

(1) If the given volume is less than $\sqrt{3}\,\pi y_2^3/5$, the discontinuous solution must be taken. The continuous solution is not applicable.

(2) If the given volume lies between $\sqrt{3}\,\pi y_2^3/5$ and $13\pi y_2^3/30$, discontinous

---

* The body curve found for minimum drag is also for least drag if the restriction that the slope be of the same sign at all points is imposed. A zigzag line can not be taken as a minimizing curve in this case.

solution should be taken as it gives the least resistance (Ref. 4).

(3) If the given volume is greater than $13\pi y_2^3/30$, the continuous solution must be taken. The discontinuous solution is not applicable.

These rules can be summarized as

Volume Range	$\longleftarrow \frac{\sqrt{3}}{5}\pi y_2^3$	$\longleftrightarrow$	$\frac{13}{30}\pi y_2^3 \longrightarrow$
Type of Solution	Discontinuous	Discontinuous	Continuous

4-12 Parametric Equations for Minimizing Curve--Given Base Diameter and Volume.

The equation for y has been found in Eq. (4-119),

$$y = \frac{4}{\lambda}\frac{y'^3}{(1+y'^2)^2} \tag{4-134}$$

Differentiating,

$$dy = \frac{4}{\lambda}\frac{3y'^2(1+y'^2)-4y'^4}{(1+y'^2)^3}\,dy'$$

$$= \frac{4}{\lambda}\frac{3y'^2-y'^4}{(1+y'^2)^3}\,dy' \tag{4-135}$$

Substituting in the general form of the parametric equation for x,

$$x = \int_0^y \frac{dy}{y'} = \frac{4}{\lambda}\int_0^{y'}\frac{3y'-y'^3}{(1+y'^2)^3}\,dy' \tag{4-136}$$

Integrating,

$$x = \frac{2}{\lambda}\frac{y'^4+3y'^2}{(1\ y'^2)^2} \tag{4-137}$$

Given the base diameter and volume, the value of the tangent to the body curve at the base, y_2' can be calculated by trial and error from Eq. (4-101). Thus, the Lagrange multiplier can be calculated from Eq. (4-134),

$$\lambda = \frac{4}{y_2} \frac{y_2'^3}{(1+y_2'^2)^2} \tag{4-138}$$

Substituting in Eqs. (4-134) and (4-137),

$$X = \frac{y_2}{2} \frac{(1+y_2'^2)^2}{y_2'^3} \frac{y_1'^4+3y_1'^2}{(1+y_1'^2)^2} \tag{4-139}$$

$$y_1 = y_2 \frac{(1+y_2'^2)^2}{y_2'^3} \frac{y_1'^3}{(1+y_1'^2)^2} \tag{4-140}$$

The above result is plotted in Fig. 4-9 together with four other curves for different given conditions where the fineness ratio is 5 for all bodies with the ordinate to an expanded scale to better indicate the individual profiles.

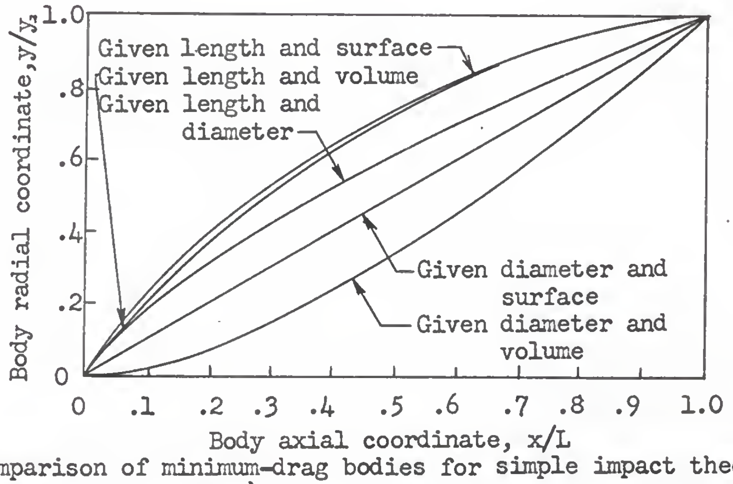

Fig. 4-9. Comparison of minimum-drag bodies for simple impact theory (After Eggers, et al., NACA)

4-13 Modified Theory--Given Base Diameter and Volume.

By taking the centrifugal force effects into consideration, the problem can be solved in the same way as in Art. 4-9 except for a new constraint of given volume

$$\text{Vol.} = \pi\int_{x_1}^{x_2} y^2 dx = \text{constant} \tag{4-141}$$

The integrand function to be minimized here is

$$f = F(y, y') + \lambda y^2 \tag{4-142}$$

in which according to Eq. (4-50),

$$F = y y'\left(2 - \frac{1+\frac{3}{2}\frac{y}{y_2}}{1+y'^2}\right)$$

$$f_{y'} = F_{y'} \tag{4-143}$$

The solution of Euler's equation gives

$$y'f_{y'} - f = c$$

i.e.,

$$y'F_{y'} - F - \lambda y^2 = c \tag{4-144}$$

Substituting Eq. (4-52) in Eq. (4-144),

$$\frac{2yy'^3}{(1+y'^2)^2}\left(1+\frac{3}{2}\frac{y}{y_2}\right) - \lambda y^2 = c \tag{4-145}$$

Since the curve intercepts the x-axis, i.e., $y_1 = 0$, $c = 0$.

$$\frac{2y'^3}{(1+y'^2)^2}\left(1+\frac{3}{2}\frac{y}{y_2}\right) = \lambda y \tag{4-146}$$

Rearranging

$$y = \frac{2y_2}{\frac{\lambda y_2(1+y'^2)^2}{y'^3} - 3} \tag{4-147}$$

from which

$$x = \int_{y_1}^{y} \frac{dy}{y'} \tag{4-148}$$

At point B, $y = y_2$, $y' = y_2'$. Substituting in Eq. (4-146),

$$\lambda y_2 = \frac{5 y_2'^3}{(1+y_2'^2)^2} \tag{4-149}$$

For fineness ratio $F_R > 1/2$, $y_1 = 0$, $y_1' = 0$. From previous experience in Art. 4-10 (see Eq. 4-103), the practical range of y_2' is

$$0 \leq y_2' \leq \sqrt{3}$$

Hence from Eq. (4-149),

$$0 \leq \lambda y_2 \leq 15\ 3/16^{*}$$

4-14 Minimum-Drag Body—Given Length and Volume.

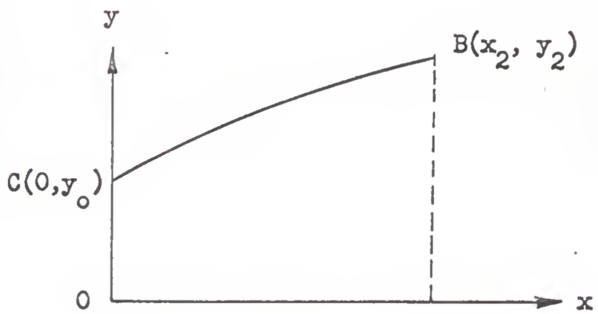

Fig. 4-10. Minimum-drag body for given length and volume.

Since the length is specified, it can be expected that the body shape

---

* The value found by the author of this report is different from the result given on P. 145 of Ref. 1.

assumes a blunt nose as shown in Fig. 4-10. From previous knowledge, the pressure-drag parameter to be minimized is

$$I = y_o^2 + \int_0^{x_2} \left(\frac{2yy'^3}{1+y'^2} + \lambda y^2 \right) dx \tag{4-150}$$

Note that the integrand function

$$F = \frac{2yy'^3}{1+y^2} + \lambda y^2 \tag{4-151}$$

is the same as that in Eq. (4-86), leading to the solution of the Euler's equation in Eq. (4-90),

$$\frac{2yy'(3y'^2+y'^4)}{(1+y'^2)^2} - \frac{2yy'}{1+y'^2} - \lambda y^2 = c$$

which reduces to

$$\lambda y^2 - \frac{4yy'^3}{(1+y'^2)^2} + c = 0 \tag{4-152}$$

The constant c can be expressed in terms of the corner coordinate y_o in the following way. Note

$$F = F(y, y') \tag{4-153}$$

Referring to Eq. (3-10), for a minimum to exist, the integrated part of δI plus the additional first order term must also vanish, i.e.,

$$2y_o \delta y_o + F_{y'} \delta y \Big|_{y_o}^{y_2} = 0 \tag{4-154}$$

Further, in order to meet the boundary conditions, the upper and lower limits should vanish independently. Since $\delta y_o \neq 0$, $\delta y_2 \neq 0$, it follows that

$$F_{y'}\Big|_{y_2} = 0 \tag{4-155}$$

$$2y_o - F_{y'}\Big|_{y_o} = 0 \tag{4-156}$$

From Eq. (4-89),

$$F_{y'} = \frac{2y(3y'^2 + y'^4)}{(1+y'^2)^2} \tag{4-157}$$

substituting in Eq. (4-155),

$$y_2' = 0 \tag{4-158}$$

Substituting Eq. (4-157) in Eq. (4-156) gives

$$2y_o\left[1 - \frac{3y_o'^2 + y_o'^4}{(1+y_o'^2)^2}\right] = 0 \tag{4-159}$$

$$y_o' = 1 \tag{4-160}$$

Substituting in Eq. (4-152),

$$c = y_o - \lambda y_o^2 \tag{4-161}$$

Hence, Eq. (4-152) becomes

$$\lambda y^2 - \frac{4y'^3}{(1+y'^2)^2}\, y + y_o - \lambda y_o^2 = 0 \tag{4-162}$$

Solving for y,

$$y = \frac{2y'^3}{\lambda(1+y'^2)^2} + \sqrt{\left[\frac{2y'^3}{\lambda(1+y'^2)^2}\right]^2 - \frac{y_o - \lambda y_o^2}{\lambda}} \tag{4-163}$$

$$x = \int_{y_o}^{y} \frac{dy}{y'} \tag{4-164}$$

An inspection of Eqs. (4-163) and (4-164) reveals that the values of λ and y_o can be determined from the conditions of given length and volume. According to Ref. 1, the range of λ is

$$\lambda \to -\infty \qquad \text{as} \qquad L \to 0 \tag{4-165}$$

$$\lambda \to 0 \qquad \text{as} \qquad L \to \infty\text{, i. e., } F_R \to \infty \tag{4-166}$$

$$-\infty < \lambda < 0 \qquad\qquad 0 < L < \infty \tag{4-167}$$

At point $C(0,y_o)$, $y_o{}' = 1$. Substituting y_o and $y_o{}'$ in Eq. (4-163),

$$y_o = \frac{1}{2\lambda} + \sqrt{\frac{1}{4\lambda^2} - \frac{y_o - \lambda y_o^2}{\lambda}}$$

$$= \frac{1}{2\lambda} \pm \frac{1}{2\lambda}(1 - 2\lambda y_o)$$

or

$$2\lambda y_o - 1 = \pm(1 - 2\lambda y_o) \tag{4-168}$$

The "-" sign on the right side is a trivial solution. This is the reason why only the "+" sign was used in front of the square root in Eq. (4-163). Hence,

$$2\lambda y_o = 1 \tag{4-169}$$

For given volume, it can be reasonably assumed that*

$$\text{as} \quad L \to \infty\ (F_R \to \infty), \qquad y_o \to 0 \qquad \text{and hence} \quad \lambda \to \infty \tag{4-170}$$

$$L \to 0, \qquad y_o \to \infty \qquad \lambda \to 0 \tag{4-171}$$

$$0 < L < \infty, \qquad \infty > y_o > 0 \qquad 0 < \lambda < \infty \tag{4-172}$$

Further more, substitution of Eq. (4-169) in Eq. (4-163) yields

$$y = \frac{4y_o y'^3}{(1+y'^2)^2} + \sqrt{\left[\frac{4y_o y'^3}{(1+y'^2)^2}\right]^2 - 2y_o^2 + y_o^2}$$

$$= \frac{4y'^3 y_o}{(1+y'^2)^2} + y_o\sqrt{\left[\frac{4y'^3}{(1+y'^2)^2}\right]^2 - 1} \tag{4-173}$$

---

* The result found by the author of this report is different from Ref. 1, P. 143.

4-15 Modified Theory—Given Length and Volume.

Figure 4-10 shows a schematic configuration of the body shape for this case. Centrifugal force effects are considered here. Referring to Eq. (4-50), the pressure drag expression to be minimized can be written as

$$I = y_o^2 + \int_0^{x_2} (F + \lambda y^2)dx \tag{4-174}$$

where $F = yy'\left(2 - \dfrac{1 + \frac{3}{2}\frac{y}{y_o}}{1+y'^2}\right)$

The new integrand function here is

$$f = F + \lambda y^2 \tag{4-175}$$

$$f_{y'} = F_{y'} \tag{4-176}$$

Solution of the Euler's equation yields

$$y'f_{y'} - f = c$$

or

$$y'F_{y'} - F - \lambda y^2 = c \tag{4-177}$$

Substituting Eq. (4-52) for $y'F_{y'} - F$ in Eq. (4-177),

$$\frac{2yy'^3}{(1+y'^2)^2}\left(1 + \frac{3}{2}\frac{y}{y_2}\right) - \lambda y^2 = c$$

Rearranging,

$$\left[\frac{3y'^3}{y_2} - \lambda(1+y'^2)^2\right]y^2 + 2y'^3 y - c(1+y'^2)^2 = 0 \tag{4-178}$$

which results in

$$y = \frac{-y'^3 + \sqrt{y'^6 + c(1+y'^2)^2\left[(3y'^3/y_2) - \lambda(1+y'^2)^2\right]}}{3y'^3/y_2 - \lambda(1+y'^2)^2} \tag{4-179}$$

$$x = \int_{y_0}^{y} \frac{dy}{y'} \tag{4-180}$$

Since the base diameter is not given, from Eq. (4-155),

$$f_{y'}\Big|_{y_2} = 0 \tag{4-181}$$

With the aid of Eqs.(4-176) and (4-51), one obtains

$$\frac{5\,y_2 y_2'^2}{(1+y_2'^2)^2} + y_2\left(2 - \frac{5}{2}\,\frac{1}{1+y_2'^2}\right) = 0$$

Simplifying,

$$4y_2'^4 + 13y_2'^2 - 1 = 0 \tag{4-182}$$

$$y_2'^2 = \frac{-13+\sqrt{185}}{8} = 0.075$$

$$y_2' = 0.274 \tag{4-183}$$

From Eq. (4-156),

$$2y_o - F_{y'}\Big|_{y_o} = 0 \tag{4-184}$$

Substituting Eq. (4-51) for $F_{y'}$ in Eq. (4-184) gives

$$\frac{1}{1+y_o'^2} - \frac{2\,y_o'^2}{(1+y_o'^2)^2} = 0$$

from which

$$y_o' = 1 \tag{4-185}$$

i.e., the body curve intercepts the y-axis at 45°.

Now, let us rewrite Eq. (4-179) in the form of

$$y = \frac{-y'^3 + \sqrt{y'^6 + (c/y_2)(1+y'^2)^2\left[3y'^3 - \lambda y_2(1+y'^2)^2\right]}}{3y'^3 - \lambda y_2(1+y'^2)^2}\,y_2 \tag{4-186}$$

or

$$y = \frac{-1 + \sqrt{1 + \frac{c}{y_2}\,\frac{(1+y'^2)^2}{y'^3}\left[3 - \lambda y_2\,\frac{(1+y'^2)^2}{y'^3}\right]}}{3 - \lambda y_2\,\frac{(1+y'^2)^2}{y'^3}}\,y_2 \tag{4-187}$$

At point B (x_2, y_2), $y_2' = 0.274$,

$$\frac{(1+y_2'^2)^2}{y_2'^3} = 56.188$$

Substituting in Eq. (4-187),

$$4 - 56.188\lambda y_2 = \sqrt{1 + 56.188(c/y_2)(3 - 56.188\lambda y_2)}$$

$$\frac{c}{y_2} = \frac{(4 - 56.188\lambda y_2)^2 - 1}{56.188(3 - 56.188\lambda y_2)} \qquad (4\text{-}188)$$

Substitution of Eq. (4-188) in Eq. (4-187) yields

$$y = y_2 \phi(y', \lambda y_2) \qquad (4\text{-}189)$$

By using the relation in Eq. (4-189), the length becomes

$$L = \int_{y_0}^{y} \frac{dy}{y'} = y_2 \int_{1}^{.274} \frac{1}{y'} \frac{d\phi}{dy'} dy' = y_2 \zeta(\lambda y_2) \qquad (4\text{-}190)$$

and the volume is given by

$$\text{Vol.} = \pi \int_{0}^{x_2} y^2 dx = \pi y_2^3 \int_{1}^{.274} \frac{\phi^2}{y'} \frac{d\phi}{dy'} dy'$$

$$= \pi y_2^3 \Gamma(\lambda y_2) \qquad (4\text{-}191)$$

Combination of Eqs. (4-190) and (4-191) yields

$$\frac{\Gamma}{\zeta^3} = \frac{\text{Vol}}{\pi L^3} \qquad (4\text{-}192)$$

Given Vol. and L, the value of λy_2 can be determined from the Γ and ζ functions. Thereby, we can calculate the values of c/y_2 and y_2 from Eqs. (4-188) and

(4-190) or (4-191), respectively. Values of y_2, λy_2 and c/y_2 thus obtained serve the parametric equations Eqs. (4-180) and (4-186) for the calculation of the minimum-drag body shape of given volume and length. Eggers, et al.,(Ref.2) gives numerical values of the ζ and Γ functions which are obtained for the various values of λy_2 that makes the relation of Eq. (4-192) hold for given volume and length for interpolation.

4-16 Minimum-Drag Body--Given Length and Surface Area.

As in Art. 4-8, the pressure-drag parameter to be minimized can be written as

$$I_D = y_0^2 + \int_0^{x_2} \frac{2yy'^3}{1+y'^2}\, dx \qquad (4\text{-}193)$$

which is subject to an auxiliary condition

$$S = \pi y_0^2 + 2\pi \int_0^{x_2} y\sqrt{1+y'^2}\, dx \qquad (4\text{-}194)$$

According to the isoperimetric rule of the calculus of variations, the integral to be minimized becomes

$$\begin{aligned} I &= y_0^2 + \int_0^{x_2} \left(\frac{2yy'^3}{1+y'^2} + \lambda y\sqrt{1+y'^2} \right) dx + \frac{\lambda}{2} y_0^2 \\ &= \left(1+\frac{\lambda}{2}\right) y_0^2 + \int_0^{x_2} \left(\frac{2yy'^3}{1+y'^2} + \lambda y\sqrt{1+y'^2} \right) dx \end{aligned} \qquad (4\text{-}195)$$

where the integrand function F is exactly the same as in Eq. (4-63), which has led to the solution (see Eq. 4-67),

$$y\left[\frac{\lambda}{\sqrt{1+y'^2}} - \frac{4y'^3}{(1+y'^2)^2} \right] = C_1 \qquad (4\text{-}196)$$

Therefore

$$y = \frac{C_1 (1+y'^2)^2}{4y'^3 - \lambda(1+y'^2)^{3/2}} \tag{4-197}$$

and

$$x = \int_{y_0}^{y} \frac{dy}{y'} \tag{4-198}$$

Since the base diameter is not given, it is required that

$$F_{y'}\Big|_{y_2} = 0 \tag{4-199}$$

Hence, from Eq. (4-66),

$$F_{y'}\Big|_{y_2} = y_2\left[\frac{2(3y_2'^2 + y_2'^4)}{(1+y_2'^2)^2} + \frac{\lambda y_2'}{\sqrt{1+y_2'^2}}\right] = 0 \tag{4-200}$$

$$y_2' = 0 \; * \tag{4-201a}$$

or

$$6y_2' + 2y_2'^3 + \lambda(1+y_2'^2)^{3/2} = 0 \tag{4-201b}$$

from which y_2' can be determined uniquely in terms of λ.

Also, from Eqs. (4-156) and (4-195),

$$(2+\lambda)y_0 - F_{y'}\Big|_{y_0} = 0 \tag{4-202}$$

With the aid of Eq. (4-66),

$$(2+\lambda) - \left[\frac{2(3y_0'^2 + y_0'^4)}{(1+y_0'^2)^2} + \frac{\lambda y_0'}{\sqrt{1+y_0'^2}}\right] = 0 \tag{4-203}$$

Now, let us determine the range of λ. For given surface area, when the length approaches zero, or when L is fixed at a very small value, the body shape assumes a blunt nose. From previous knowledge, the meridian curve

---

* The value found by the author is different from Ref. 1, P. 145 but agrees with the slope of the curve of given length and surface area in Fig 4-9.

intercepts the y-axis at an angle of 45°, i.e., $y_o' = 1$. Substituting in Eq. (4-203),

$$\lambda = 0$$

On the other hand, when the length approaches infinity, $y_o = 0$ and $y_o' = 0$ for fixed surface area. From Eq. (4-203),

$$\lambda = -2$$

To sum up, $\lambda \to 0$ as $L \to 0$ (4-204)

$\lambda \to -2$ as $L \to \infty$ (4-205)

For bodies of ordinary lengths, the practical range of is

$$-2 < \lambda < 0 \tag{4-206}$$

and

$$0 = y_2' < y_o' < 1 \tag{4-207}$$

where y_o' depends on λ .

4-17 Modified Theory—Given Length and Surface Area.

Referring to Art. 4-9, we see the problem can be attacked in the same way except for different terminal conditions. From Eq. (4-78), we obtain

$$\frac{3}{Y_2}\frac{y'^3}{(1+y'^2)^2}y^2 + \left[\frac{2y'^3}{(1+y'^2)^2} - \frac{\lambda}{\sqrt{1+y'^2}}\right]y - C = 0 \tag{4-208}$$

$$3y'^3y^2 + \left[2y'^3 - \lambda(1+y'^2)^{3/2}\right]Y_2 y - CY_2(1+y'^2)^2 = 0$$

Therefore,

$$y = \frac{\left[\lambda(1+y'^2)^{3/2} - 2y'^3\right]Y_2 \pm \sqrt{\left[\lambda(1+y'^2)^{3/2} - 2y'^3\right]^2 Y_2^2 + 12CY_2y'^3(1+y'^2)^2}}{6y'^3} \tag{4-209}$$

or

$$y = \frac{Y_2}{6}\left\{\lambda\left(\frac{1+y'^2}{y'^2}\right)^{3/2} - 2 \pm \sqrt{\left[\lambda\left(\frac{1+y'^2}{y'^2}\right)^{3/2} - 2\right]^2 + \frac{C}{Y_2}\frac{(1+y'^2)^2}{y'^3}}\right\} \tag{4-210}$$

$$x = \int_{y_0}^{y} \frac{dy}{y'} \tag{4-211}$$

From Eqs. (4-76) and (4-51),

$$f_{y'} = \frac{2y'^2 y}{(1+y'^2)^2}\left(1+\frac{3}{2}\frac{y}{y_2}\right) + 2y - \frac{y}{1+y'^2}\left(1+\frac{3}{2}\frac{y}{y_2}\right) + \frac{\lambda y' y}{\sqrt{1+y'^2}}$$

$$= \frac{y'^2-1}{(1+y'^2)^2}\left(1+\frac{3}{2}\frac{y}{y_2}\right)y + 2y + \frac{\lambda y' y}{\sqrt{1+y'^2}} \tag{4-212}$$

According to Eq. (4-199), the terminal condition is

$$f_{y'}\Big|_{y_2} = 0$$

i.e.,

$$\frac{5}{2}\,\frac{y_2'^2-1}{(1+y_2'^2)^2} + 2 + \frac{\lambda y_2'}{\sqrt{1+y_2'^2}} = 0 \tag{4-213}$$

From Eq. (4-202), another terminal condition can be found as

$$f_{y'}\Big|_{y_0} - (2+\lambda)y_0 = 0$$

i.e.,

$$\frac{y_0'^2-1}{(1+y_0'^2)^2}\left(1+\frac{3}{2}\frac{y_0}{y_2}\right) + \lambda\left(\frac{y_0'}{\sqrt{1+y_0'^2}} - 1\right) = 0 \tag{4-214}$$

From Eq. (4-210), one can determine the value of c/y_2 in terms of λ and y_2' by letting $y = y_2$. Also, from Eq. (4-213) it is seen that y_2' can be expressed in terms of λ . Therefore, the value of c/y_2 depends on λ only., and Eq. (4-210) can be written as

$$y = y_2\phi(y', \lambda) \tag{4-215}$$

At the same time, it is seen that y_0' also depends on λ. Then the given length and surface area can be expressed as

$$L = \int_{y_0}^{y_2} \frac{dy}{y'} = y_2 \int_{y_0'(\lambda)}^{y_2'(\lambda)} \frac{1}{y'} \frac{d\phi}{dy'} dy' = y_2 \varsigma(\lambda) \tag{4-216}$$

$$S = 2\pi \int_0^{x_2} y\sqrt{1+y'^2}\, dx = \int_{y_0}^{y_2} \frac{y\sqrt{1+y'^2}}{y'} dy$$

$$= 2\pi y_2^2 \int_{y_0'(\lambda)}^{y_2'(\lambda)} \frac{\sqrt{1+y'^2}}{y'} \phi \frac{d\phi}{dy'} dy' = 2\pi y_2^2 \beta(\lambda) \tag{4-217}$$

Combination of Eqs. (4-216) and (4-217) gives

$$\frac{\beta}{\varsigma^2} = \frac{S}{2\pi L^2} \tag{4-218}$$

As in Art. 4-15, the values of ς and β functions can be obtained by numerical integration for various values of λ to allow interpolation for the value of λ that makes $\beta/\varsigma^2 = S/(2\pi L^2)$. The corresponding values thus found for λ, ς, β satisfy the given length and surface area requirements and yield values of y_2 and c/y_2 for Eqs. (4-210) and (4-211) to plot the minimizing curve.

4-18 Results and Discussion.

Analysis carried out in this section reveals two characteristics which are common in minimum-drag body shapes. When the body length is fixed, the body is found with a blunt nose; whereas, when the body length is not fixed, (i.e., no restriction) the body is found with a sharp nose. This may be attributed to the fact that with the restriction on the length, reduced drag is achieved by accepting higher pressures on a comparatively small area of large slope near the nose, and thus maintaining lower pressures over a large area of small slope near the base. However, if the restriction on length is relaxed, the body can be made more slender with a sharp nose, and the pressure drag is reduced.

For the convenience of a quantitative comparison, typical meridian curves calculated with the simple impact theory have been plotted in Fig. 4-9 on the basis of the same fineness ratio $F_R = 5$ with the ordinates to an expanded scale to better indicate the individual profiles. It is seen that for minimum-drag, the body shape of given length and surface area assumes the maximum bluntness, while the body shape of given base diameter and volume has the maximum sharpness (i.e., a cusp nose). From Fig. 4-9, it is noted that the flat-nosed portions of the three meridian curves for the given length bodies are in all cases very small. As expected, the degree of bluntness increases when the fineness ratio decreases.

It is also of interest to compare minimum-drag body shapes determined by linear theory with those determined by the simple impact theory, i.e., bodies particularly designed for flight at low and high supersonic speeds respectively. Such meridian curves are plotted in Fig. 4-3 for given fineness ratio. It is seen that the minimum-drag body for supersonic speeds is generally flatter than the minimum-drag body for hypersonic speeds but the shapes are similar in spite of the marked difference in laws governing the surface pressures.

All the analysis in this section has been based on the assumption that the air flow at hypersonic speeds can be approximated by the Newtonian-type flow. Several calculated body shapes of fineness ratio 3 and 5, including those minimum-drag bodies for given length and base diameter and for given base diameter and surface area, were tested at Mach numbers from 2.73 to 6.28 at NACA Ames Aeronautical Laboratory by Eggers et al., in 1955. Test results showed that these body shapes are good approximations to correct profiles for minimum-drag (Ref. 2).

SECTION 5

CONCLUSION

It has been undertaken in this report to present a method for the determination of the shapes of nonlifting bodies of revolution having minimum pressure drag at hypersonic speeds. Problems for the various combinations of conditions of given length, base diameter, surface area and volume have been solved by means of Newton's impact theory and the calculus of variations. It is noted that the minimum-drag body found generally has a blunt nose if the length is fixed, as in Newton's classical problem, and a sharp nose if the length is not fixed.

At moderate hypersonic Mach numbers, effects of curvature in the stream direction have been investigated by a simple modification of the Newtonian impact theory. Calculation of body shapes have also been carried out for the same conditions as stated above. Comparison of results indicates that shapes thus found are blunter at the nose section and have more curvature in the region downstream of the nose. Also, a slight reduction in the pressure drag is noticed.

Several bodies of revolution of fineness ratios 3 and 5, calculated according to the simple impact theory for given length and base diameter and for given base diameter and surface area, were tested at Mach numbers from 2.73 to 6.28 at NACA Ames Aeronautical Laboratory by Eggers et al., in 1955. Test results showed that the calculated body shapes were good approximations to correct profiles of the corresponding conditions for minimum-drag.

ACKNOWLEDGEMENT

The author wishes to express his sincere gratitude to his major adviser, Dr. Wilson Tripp, Professor of Mechanical Engineering, for his constant encouragement, advice and valuable suggestions; Dr. R. G. Nevins, Head of the Department of Mechanical Engineering, Dr. Chi-Lung Huang, Professor of Applied Mechanics, and Dr. J. M. Marr, Professor of Mathematics, for their enthusiastic support and guidance as members of the author's advisory committee.

REFERENCES

1. R. W. Truitt, "Hypersonic Aerodynamics," Ronald Press Co., New York, 1959, PP. 105-145.

2. A. J. Eggers, Jr., M. M. Resnikoff and D. H. Dennis, "Bodies of Revolution Having Minimum-Drag at High Supersonic Airspeeds," February, 1956.

3. A. R. Forsyth, "Calculus of Variations," Cambridge University Press, London, 1927, PP. 320-324.

4. Issac Todhunter, "Researches in the Calculus of Variations," Macmillian and Co., London and Cambridge, 1871.

5. A. H. Shapiro, "The Dynamics and Thermodynamics of Compressible Fluid Flow," Volume I and II, Ronald Press Co., New York, 1954.

6. R. N. Cox and L. F. Crabtree, "Elements of Hypersonic Aerodynamics," Academic Press, New York and London.

7. W. D. Hayes and R. F. Probstein, "Hypersonic Flow Theory, " Academic Press, New York and London, 1959.

8. L. E. Elsgolc, "Calculus of Variations, " Addison-Wesley Publishing Co. Inc., Reading, Mass., 1962.

MINIMUM-DRAG BODIES OF REVOLUTION AT HYPERSONIC AIRSPEEDS

by

CHING-CHENG CHAO

Diploma, Air Technological School, Chinese Air Force
Kangshan, Taiwan, Republic of China, 1948

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

Shapes of minimum-drag bodies at zero angle of attack in the hypersonic flow are determined by means of Newton's impact theory and the calculus of variations. The investigation is carried out for various combinations of conditions of given body length, base diameter, surface and volume. Usually, the optimum body shape assumes a blunt nose when the length is fixed; whereas, a sharp nose when the length is not fixed. Due to curvature of flow over the surface, the bluntness of the meridian curve is increased in the nose section of the body. According to the theoretical investigation, these modifications show only a slight reduction in the pressure drag.

Several bodies of revolution of fineness 3 and 5 were tested at Mach numbers from 2.73 to 6.28 at Ames Aeronautical Laboratory by Eggers et al., in 1955. A comparison of the theorectical and experimental findings showed that the calculations were reasonable approximations to the correct shapes for minimum drag at hypersonic airspeeds.