APPLICATION OF FEED BACK CONTROL THEORY IN PRODUCTION AND INVENTORY CONTROL

by

GOKARAJU V. N. RAJU

B. Sc., Benaras Hindu University, 1960
B. I. E., Ohio State University, 1964

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Approved by:

John P. Clifton
Major Professor

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CHAPTER I

INTRODUCTION

In the past two decades, powerful and extremely general techniques have been developed for the analysis and control of mechanical and electrical systems. Today the field covers the mathematical areas of the theory of equations, differential equations, operational calculus, Laplace transform theory and functions of a complex variable. Two characteristic features of such systems are as follows[2]: first is the feed back loop by means of which (a) the actual response of the system is compared with the input (desired value) and (b) their difference is fed back into the system to reduce the difference to the desired value. The second important feature is that input and load (disturbance) affect the behavior of the system but are themselves unaffected by it. There are certain obvious analogies between such systems and production and inventory control systems. Even though at present one may not be ready to use the full range of techniques available to the servomechanism engineer to analyze and control management systems, one can subject to test the depth of the obvious analogies by analyzing specific production and inventory control systems using control theory. The purpose of this report is to explore such analogies.

In Chapter II a brief review of the techniques of control theory is presented, and in Chapter III application of these techniques on specific inventory and production control systems is discussed. Emphasis has been placed on formulating the problem in the language of control theory and determining criteria for evaluating the merit of a control system.

CHAPTER II

SELECTED THEORIES, PROCEDURES AND TECHNIQUES USEFUL IN THE ANALYSIS OF CONTROL SYSTEMS

To adequately describe control theory with all its facets and ramifications would be the work of a multivolume technical series. Therefore, control theory will be reviewed merely in terms of procedures and techniques useful in the synthesis and analysis of control systems.

I. Terminology

A. Open Loop and Closed Loop Systems

For the purpose of clarification, it will be convenient to classify control systems into two broad types. The distinction between the two systems is best shown by giving an example of each. Suppose that a heat treating furnace is to be carried through a prescribed temperature cycle. One method of accomplishing this task is to construct the system so that the rate of gas flow, and hence the furnace temperature, is controlled by adjusting the gas valve by means of a rotating cam. By proper design and calibration, the system will cause the furnace temperature to follow the desired time variation. A little reflection will, however, reveal some important limitations on the performance of such a system. Although the cam controls the position of the valve, the furnace temperature depends on several other things also. For instance, if the gas supply pressure should change from that assumed in design, a different gas flow rate would exist for a given valve position, resulting in a different temperature. In fact, any change from the design or calibration will cause a deviation from the desired temperature-time curve. A system which cannot take into account changes from calibration conditions, is called an open loop control system.

Consider the other type of control system. A good example of a closed loop system (or feedback control system) is the temperature control in a home heating system. In such a system there is a bimetallic element which measures temperature by means of the differential thermal expansion of two dissimilar metals. The element is designed to make and break the contact when the room temperature is low and high respectively. As the room heats up and when the desired temperature is reached, the bimetal opens the circuit and the burner will shut off. Should the room cool down, the contacts will close, starting the burner and returning the temperature to a desired level. This system will maintain the temperature near desired value irrespective of changes in fuel heating value, powerline voltage, environmental temperature or other conditions. Thus the closed loop control system actually measures the quantity to be controlled, compares the actual value to the desired value, and if they are not the same, institutes corrective action.

B. The Block Diagram

It is possible to describe the action of all types of feedback control systems in terms of a functional block diagram [3]. The blocks in the diagram should be interpreted as representing functions of components, and not isolated pieces of equipment. The lines and arrows on the diagram indicate the direction of the flow of information, energy or material.

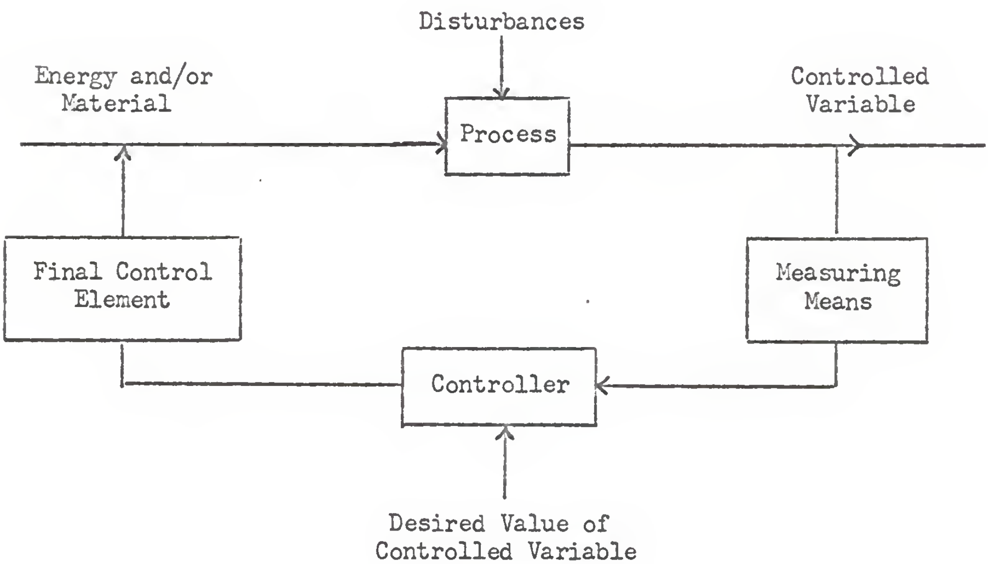

Figure 1. Functional block diagram of closed loop control systems.

The term process in this diagram is to be thought of in its most general sense. Thus some typical <u>process</u> might be heating a room, positioning the tool slide on a machine tool, or producing a certain number of an item. In general, there will be one or more conditions in a process which we wish to control in some way. One might wish to hold them constant at some predetermined value or vary them according to a given program or schedule. If it is desired to hold them constant, then the main reason for having a control at all is that there are disturbances acting on the system tending to change the process from its desired conditions.

In order to control the process, it is necessary to measure its actual condition as directly as possible. The controlled variable will be measured by the measuring means and the information will be sent back to the controller. Here the actual condition of the process is compared with the desired condition; and if they differ, the controller acts to correct the error; that is it signals the final control element to adjust the flow of material and/or energy to the process in such a direction as to decrease the error.

In discussing the nature of control system specifications, and also in our further analysis, standardized nomenclature will be used wherever it is desirable. This nomenclature is illustrated in Figure 2 which shows a generalized operational block diagram for a feedback control system [3]. An actual system may be more or less complicated than the diagram of Figure 2, but can in general be represented in terms of the symbols given there.

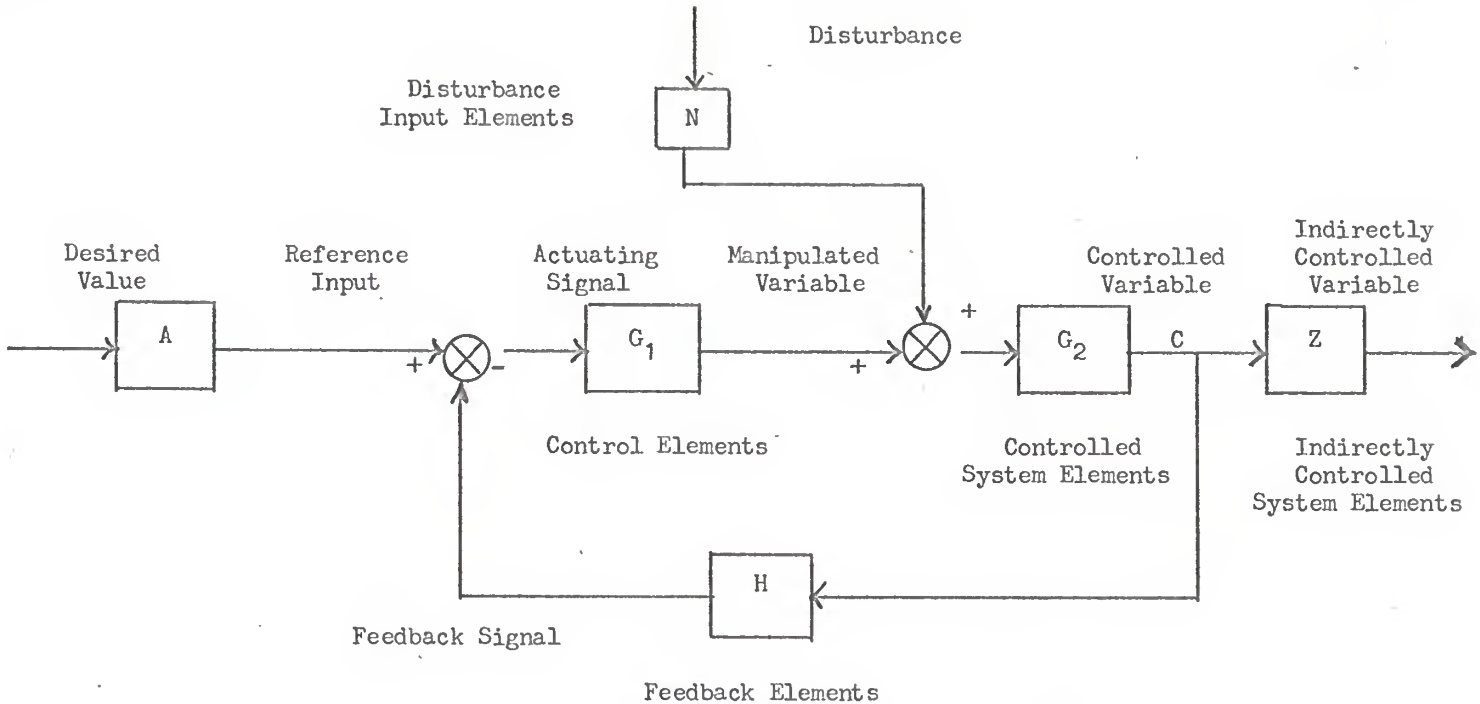

Figure 2. Standard symbology for feedback control systems. The symbol $\otimes$ represents differential device (or comparator).

II. Representation of the System Under Study by a Time Differential Equation

After having captured the system in a block diagram, the analysis can be carried out by one of two ways: (1) the direct approach and (2) the indirect approach.

A. The Direct Approach

The direct approach which is common to almost all phases of deterministic engineering analysis is to represent the system under study by a time differential equation. (Many of the systems that are dealt with by mechanical and electrical engineers are described at least approximately by systems of linear differential equations with constant coefficients.) System stability (transient behavior) is determined from the homogeneous solution of the equation, and the steady state performance for any given input function (forcing function) from the particular solution obtained from substitution of the forcing input function in the basic equation. The usual solution method for linear homogeneous equations is to assume a solution of the form $\theta_0(t) = e^{rt}$ where θ_0 is the system output at time t, and "r" is a real or complex number to be evaluated.

This form reduces the homogeneous differential to an algebraic equation, the solution of which is a linear combination of $t^b e^{\gamma_i t} (i=1,2\ldots n)$ where "n" is the order of the system equation, γ_i is the i^{th} root of the resulting algebraic equation, and "b" is an integer between zero and one less than the multiplicity of the root. The constants are then evaluated from initial or other boundary conditions. Solutions of the particular equation depend on the form of the forcing input function, hence no general treatment is available [6].

B. The Indirect Methods

The differential equations representing many practical systems prove to be extremely difficult to solve by direct means. In such cases, one of two general approaches may be followed, either singly or in combination. The first of these is essentially analytic and involves the use of Laplace transforms. The second, an empirical approach, is to construct an experimental model and observe its performance under given test conditions. In further analysis the method used will employ Laplace transforms.

Laplace transforms are employed to provide system representation in terms of the complex frequency $S = \sigma + jw$ where $j = \sqrt{-1}$. After transformation to the S-plane, which for linear systems results in algebraic equations, the equation is rearranged to isolate either the error or the output in terms of the system parameters and forcing functions. The denominator of the error or output expression is factored and a partial fraction expansion effected which results in a linear series of functions of S. Ideally the inverse transformations of these are well known so that transformation back to the time domain is readily accomplished. The resulting expression represents the system error or output as a function of time, i. e. the solution to the original system differential equations. The defining equations and symbols of transform mathematics as applied to control theory are:

$f(t)$ = any function of time.

S = a complex variable having the form $\sigma + jw$.

$F(S)$ = the resulting equation in the transform variable, S, when $f(t)$ had been operated on by a Laplace integral.

$\mathcal{L}$ = an operational symbol indicating that the quantity which it prefixes is to be transformed by the Laplace integral.

Thus $F(S) \triangleq \mathcal{L}\left[f(t)\right]$ where the symbol $\triangleq$ means "equal to by definition." The Laplace integral, which has been represented by $\mathcal{L}$, is defined as: $\mathcal{L} \triangleq \int_0^{\infty} e^{-St}\, dt$

Therefore

$$\mathcal{L}\left[f(t)\right] = \int_0^{\infty} e^{-St}\, dt\left[f(t)\right] = \int_0^{\infty} f(t) e^{-St}\, dt.$$

Thus the Laplace transform of any equation or term in an equation may by obtained by multiplying by e^{-St} and then integrating the product from $t = 0$ to $t = \infty$. The following table(10) presents the Laplace transforms to a few common functions.

$f(t)$	$\mathcal{L}[f(t)]$
A (a constant)	$\frac{A}{S}$
$e^{-\alpha t}$	$\frac{1}{\alpha+S}$
$A\ e^{-\alpha t}$	$\frac{A}{\alpha+S}$
$Ae^{-\alpha t} + e^{-Bt}$	$A/(\alpha+S) + \frac{1}{(B+S)}$
$\mathrm{Sin}Bt$	$B/(S^2+B^2)$
$\mathrm{Cos}Bt$	$S/(S^2+B^2)$
t	$1/S^2$
t^2	$2/S^3$
t^n	$n!/S^{n+1}$

Not only can we go from the "t" domain to the complex "S" domain by use of Laplace transform; we can also perform inverse transformation which may be denoted symbolically as: $\mathcal{L}^{-1}\ F(S) = f(t)$

Thus 1.) If $F(S) = A/S$ then $\mathcal{L}^{-1}\ \frac{A}{S} = A$

2.) If $F(S) = 1/(\alpha+S)$ then $\mathcal{L}^{-1}\frac{1}{(\alpha+S)} = e^{-\alpha t}$

3.) If $F(S) = B/(S^2+B^2)$ then $\mathcal{L}^{-1}\ \frac{B}{S^2+B^2} = \mathrm{Sin}Bt$

In addition, the following theorems about Laplace transforms will also be useful in the later analysis. These theorems without attempting proofs are:

Real differentiation theorem.

$$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = SF(S) - f(0^+)$$

where $f(0^+)$ is the initial value of $f(t)$ evaluated at $t \rightarrow 0$ from positive values.

Real integration theorem.

$$\mathcal{L}\left[\int f(t)dt\right] = \frac{1}{S}\mathcal{L}\left[f(t)\right] + \frac{1}{S}\left[\int f(t)dt\right]_{t=0^+}$$

Final value theorem.

$$\lim_{t \rightarrow \infty} f(t) = \lim_{S \rightarrow 0} S\mathcal{L}\left[f(t)\right] \quad \text{if the limit exists.}$$

Initial value theorem.

$$\lim_{t \rightarrow 0} f(t) = \lim_{S \rightarrow \infty} S\mathcal{L}\left[f(t)\right]$$

These theorems will be of great use when studying the steady state and transient behavior of the system in Chapter III.

III. Performance Measures

A. Stability Consideration

Once the system equation is solved either by the direct method (classical method of solving differential equation) or by indirect method (Laplace transform method), the results can be compared with given performance requirements.

Chief among the performance parameters is that of stability or the convergence of the transient response. A system in which transients grow or fail to die out is immediately eliminated from further practical consideration since such a system cannot compensate for any error induced in the system. A procedure by which system stability can be determined directly from the system equation is the application of Routh-Hurwitz stability criterion. This criterion is based on the relationships between the form of the coefficients of the system output equation and location of singularities (poles in the case of real systems) of the equation.

B. Steady State Error

Next to stability the performance measure of interest is the steady state error. This is the difference between the desired and actual values when the system has attained equilibrium (steady state). The system response to step input, cyclic input such as sinusoidal excitation are factors to be considered.

With this brief review of control theory some of the applications to industrial processes will be demonstrated.

CHAPTER III

APPLICATIONS

I. Simple Inventory and Production Control Problems

In this chapter the control of the rate of production of a single item shall be considered. The item is supposed to be manufactured to standard specifications, placed in stock, and shipped out on order of customers. The item is manufactured continuously, and the control consists in issuing instructions that vary continually with the quantity to be manufactured per day (or other unit of time).

The aim of the control system is to minimize the cost of manufacture over a period of time. This cost, or the variable part of it, is assumed to depend on (1) the variations in the manufacturing rate (i.e. it costs more to make 1,000 items if the manufacturing rate fluctuates than if it is constant) and (2) the inventory of finished goods (i.e. increase in this inventory involves carrying costs: decrease in the inventory below a certain point involves delay in filling customers' orders). Hence the criterion by which we will judge the system will be some function of the magnitude of the fluctuations in manufacturing rate and the inventory of finished goods.

Let θ be the desired inventory level in number of units of product, I to be the actual inventory level. E will represent the deviation between actual and desired inventory level in units of product. Further symbols will be as follows:

D_I = inventory decision function

P = production order in units of product per unit of time

$I^{\bullet}$ = rate of change of inventory

I = actual inventory = $\int_0^t I^{\bullet}\, dt$

L = customer order rate in units of product per unit time

Such a system can be represented in the block diagram as in Figure 3.

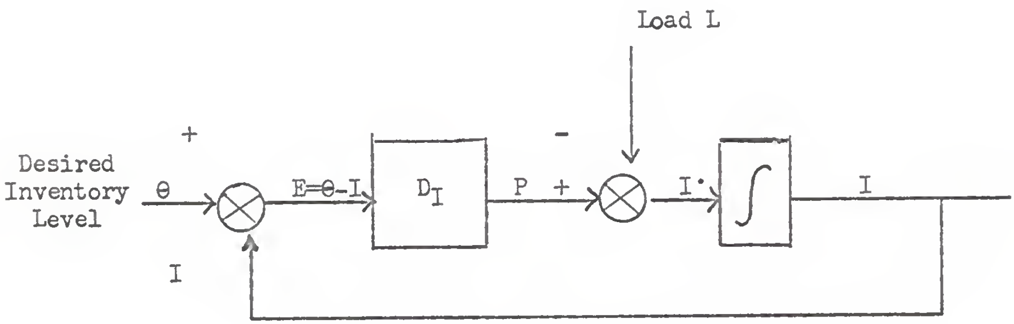

Figure 3. Simple production and inventory control system.

The equations of the system are:

$$E = \theta - I \qquad 3.1$$

$$P = D_I\,(E) = D_I\,(\theta - I) \qquad 3.2$$

$$I^{\bullet} = (P - L) \qquad 3.3$$

Converting these into the "S" domain by taking Laplace transforms of each side of the equation:

$$\mathcal{L}(E) = \mathcal{L}(\theta) - \mathcal{L}(I) \qquad 3.4$$

$$\mathcal{L}(P) = D_I\,\mathcal{L}(E) \qquad 3.5$$

$$S\mathcal{L}(I) = \mathcal{L}(P) - \mathcal{L}(L) \qquad 3.6$$

Substituting the values from equations 3.5 and 3.4 into equation 3.6 and rearranging terms will give the following:

$$S\mathcal{L}(I) + D_I\mathcal{L}(I) = D_I\mathcal{L}(\theta) - \mathcal{L}(L) \qquad 3.7$$

from which is obtained

$$\mathcal{L}(I) = \frac{D_I\mathcal{L}(\theta)}{S+D_I} - \frac{\mathcal{L}(L)}{S+D_I} \qquad 3.8$$

Since we ideally want zero inventory to be kept, we let $\theta = 0$. Thus the following:

$$\mathcal{L}(I) = \left[-\frac{1}{S+D_I}\right]\mathcal{L}(L) \qquad 3.9$$

The term in the bracket $\left[-\frac{1}{S+D_I}\right]$ is called the inventory transfer function "Y", relating inventory to customer order pattern. To derive a production transfer function one goes back to equations 3.5 and 3.4 and gets the relation:

$$\mathcal{L}(P) = D_I\mathcal{L}(\theta) - D_I\mathcal{L}(I) \qquad 3.10$$

$$= D_I \quad (\theta) - D_I\frac{\mathcal{L}(P)}{S} + D_I\frac{\mathcal{L}(L)}{S}$$

$$\mathcal{L}(P) = \frac{D_I\mathcal{L}(\theta)}{1+\frac{D_I}{S}} + \frac{D_I\mathcal{L}(L)/S}{1+\frac{D_I}{S}} \qquad 3.11$$

Again letting $\theta = 0$ the following results:

$$\mathcal{L}(P) = \left[\frac{D_I}{(S+D_I)}\right]\mathcal{L}(L) \qquad 3.12$$

The quantity $\frac{D_I}{S+D_I}$ in the above equation 3.12 is the production transfer function, relating production to customer order pattern.

A. Steady State Behavior

One can now examine the inventory transfer function to discover for the particular system the decision rule D_I that will induce appropriate behavior in the actual inventory I. It is desired that I be as small as possible. Consider the steady state behavior of the system by using the final value theorem given in the previous section which is reproduced here.

$$\lim_{t \to \infty} f(t) = \lim_{S \to 0} S\mathcal{L}\left[f(t)\right] \text{ if the limit exists}$$

$$\text{Therefore I steady state} = \lim_{S \to 0} - \frac{S\mathcal{L}(L)}{S+D_I} \qquad 3.13$$

Case 1 - Suppose that up to time t = 0, customer orders have been zero, and after that time they are received at the rate of one order per unit of time.

$$\text{i.e.} \quad L = 1 \qquad 3.14$$

$$\therefore \mathcal{L}(L) = \frac{1}{S} \quad \text{(from table of Laplace transforms)} \qquad 3.15$$

$$\therefore I_{SS} = \lim_{S \to 0} - \frac{1}{S+D_I} \qquad 3.16$$

Since it is desired that I_{SS} be as close to zero as possible, this can be accomplished for example by choosing D_I as:

$$D_I = \frac{1}{S^k}(a+bS). \quad \text{Where } k > 1,\ a > 0,\ b > 0 \qquad 3.17$$

$$I_{SS} = \lim_{S \to 0} \frac{-1}{S+\frac{1}{S^k}(a+bS)}$$

the result is:

$$I_{SS} = \lim_{S \to 0} \frac{-S^k}{S^{k+1}+a+bS} = 0 \qquad 3.18$$

That is if the customer orders have been one order per unit of time, this kind of a decision function would result in zero steady state error.

Case 2 - Suppose that up to time t = 0 orders have been zero and that after that time they are received at the rate of t^n units per unit time.

i.e.

$$L = 0 \text{ for } t < 0 \qquad 3.19$$

$$= t^n \text{ for } t > 0$$

$$\mathcal{L}(L) = \frac{n!}{S^{n+1}} \qquad 3.20$$

$$I_{SS} = \lim_{S \to 0} \frac{-n/S^n}{S+D_I} = \frac{-n!}{S^{n+1}+S^n D_I} \qquad 3.21$$

In this case one can assure a zero steady state error with D_I of the same form as before (as in case 1, equation 3.17) but with $k \geqslant (n+1)$.

Case 3 - Suppose $L(t) = \cos(wt)$, for $t > 0$. For such an input one would like to study what the steady state output would be. It is known that if L (t) is sinusoidal, then the steady state output would also be a sinusoidal with the same frequency but altered amplitude and phase.

i.e.

$$I_{SS}(t) = B \cos(wt+\phi)$$

where

$$B = \sqrt{Y(jw) \cdot Y(-jw)}$$

where Y is the inventory transfer function

$$\frac{(-1)}{S+D_I}$$

and $j = \sqrt{-1}$

and $\phi = \tan^{-1} \dfrac{\text{Imaginary part of } Y(j)}{\text{Real part of } Y(j)}$

If for example the decision function D_I is given by $D_I = \frac{1}{S}$ $(a+bS)$, then,

$$Y(S) = \frac{-1}{S+a+\frac{b}{S}} = -\frac{S}{S^2+bS+a}$$

Let $S = jw$

then,

$$Y(jw) = \frac{-jw}{-w^2+bjw+a}$$

$$Y(-jw) = \frac{jw}{-w^2-bjw+a}$$

Therefore

$$Y(jw) \cdot Y(-jw) = \frac{w^2}{(a-w^2)^2+b^2w^2}$$

and

$$B = \frac{w}{\sqrt{(a-w)^2+b^2w^2}}$$

When w = 0, B is also zero and then as w increases, B increases to a maximum value; and then as w becomes greater and greater, B tends to get smaller and smaller.

$$\lim_{w \to \infty} B = \frac{w}{w^2} = 0$$

The effects of a and b on the transient response of the system can be studied. We have the inventory transfer function "Y" given by:

$$\mathcal{L}(I) = Y \cdot \mathcal{L}(L)$$

for L = cos(wt),

$$\mathcal{L}(I) = Y\, \mathcal{L}(\cos wt)$$

$$= \frac{-S}{S^2+bS+a} \cdot \frac{S}{S^2+w^2}$$

If k_1, k_2, k_3, and k_4 are constants and S_1 and S_2 the roots of the quadratic expression S^2+bS+a, then one can write the above equation using partial fraction expansion as:

$$L(I) = \frac{k_1}{S-S_1} + \frac{k_2}{S-S_2} + \frac{k_3S+k_4}{S^2+w^2}$$

Converting these to time domain, the following results:

$$I = k_1\, e^{-\left[\frac{-b+b^2-4a}{2}\right]t} + k_2\, e^{-\left[\frac{-b-b^2-4a}{2}\right]t} + k_5 \cos(wt+\emptyset)$$

The quantity given by the first two expressions of the above equation represents the transient response and the last term represents the steady state response. In order to have as small a transient as possible, a and b should be chosen such that the terms in the brackets in the above equation will be as large as possible.

It is now indicated that the properties that the decision rule (operator D_I) must possess to assure small or vanishing steady state inventory excesses and deficiencies for various customer orders or loads.

B. Stability of the System

A system will be stable if the roots of the characteristic equation of the system have negative real parts. If the roots have large negative real parts, the transient will be strongly damped. Examine now the aspect of stability for some decision functions.

Case 1

Let $D_I = \frac{a}{S}$, $a > 0$

then $Y = \frac{-1}{S+\frac{a}{S}} = -\frac{S}{S^2+a}$

Using partial fraction expansion of denominator gives

$$Y = \left\{ \frac{k_1}{S+j\sqrt{a}} + \frac{k_2}{S-j\sqrt{a}} \right\}$$

$$\therefore \mathcal{L}(I) = \left\{ \frac{k_1}{S+j\sqrt{a}} + \frac{k_2}{S-j\sqrt{a}} \right\} + \ldots$$

$$I = k_1 e^{-j\sqrt{at}} + k_2 e^{j\sqrt{at}} + \ldots$$

With this decision function termination will be with sustained oscillations because $e^{j\sqrt{at}}$ will never die out.

Case 2

Let $D_I = \frac{1}{S}(a+bS)$, a and b are real.

$$Y = -\frac{S}{S^2+bS+a}$$

The roots of the equation are $\frac{(-b\pm\sqrt{b^2-4a})}{2}$

The system will be stable if $b^2 > 4a$, $a > 0$, $b > 0$. Otherwise it is unstable.

Case 3

Let $D_I = \frac{1}{S}(a+bS+cS^2)$
with a, b, c real.

Then

$$Y = \frac{-S}{(1+c)S^2+bS+a}$$

Then the root of the denominator is

$$S = \frac{-b\pm\sqrt{b^2-4a(1+c)}}{2(1+c)}$$

The system is stable if a, b and (c+1) are all of the same sign; otherwise unstable. Looking at these decision rules, we realize that there is no hard and fast rule to find best values and the question of optimization is still a little farther off and not quite understood.

C. Interpretation of the Decision Operator

The operator D_I represents a rule of decision.

Since $\mathcal{L}(P) = \mathcal{L}(E)\, D_I$, this rule determines on the basis of information as to current deficit or excess of inventory E, at what rate P manufacture should be carried on. Among the operators that were previously found to possess satisfactory properties is $D_I = \frac{a}{S} + b$, with a and b large positive constants.

With this operator one can write:

$$\mathcal{L}(P) = \mathcal{L}(E) \quad (\tfrac{a}{S} + b)$$

i.e. $P = a \int E\, dt + bE$

Taking derivatives

$$\dot{P} = a\, E + b\, \dot{E}$$

which, interpreted, means: the rate of production should be increased or decreased by an amount proportional to the deficiency or excess of inventory plus an amount proportional to the rate at which the inventory is decreasing. The constants of proportionality a, and b, should be large if it is desired to keep the inventory within narrow bounds.

The foregoing is obvious. What is perhaps not obvious is that derivative control (the term $b\dot{E}$ in the above equation) is essential to the stability of the system. Basing changes in production rate only on the size of the inventory ($b=0$) would introduce undamped fluctuations in the system.

II. <u>System with Production Lag</u>

The most important features missing from the previous system are a production lag and the availability of information about new orders. In actual cases a period of time will elapse from the moment when instructions are issued to increase the rate of production to the moment when increased flow of goods is actually produced. One is now ready to study a system that approximates more closely the problems expected to be encountered in actual situations.

In Figure 4 is shown a system with production lag.

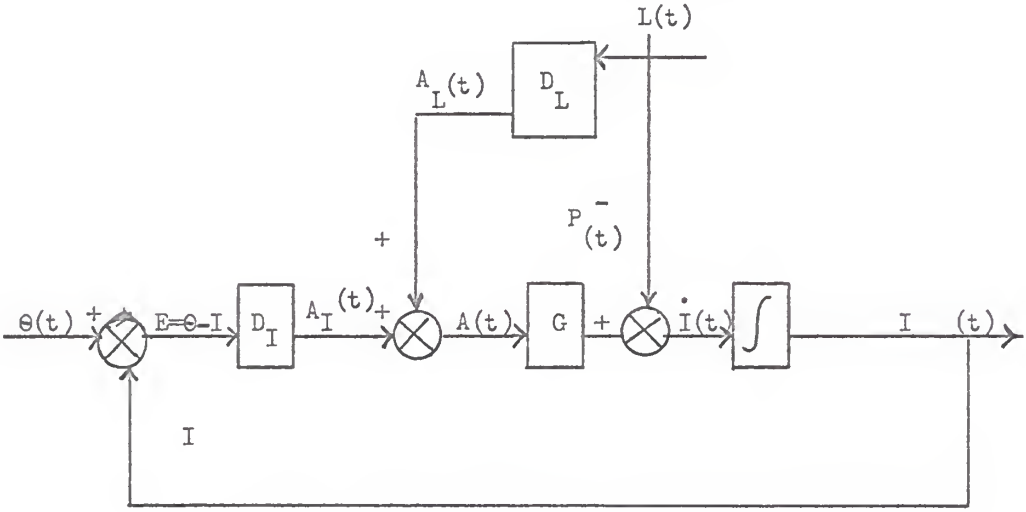

Figure 4. Inventory control system with production lag.

The equations of this system are:

$$E = \theta - I \qquad 4.1$$

$$A_I = D_I\,(E) \qquad 4.2$$

$$A_L = D_L\,(L) \qquad 4.3$$

$$A = A_L + A_I \qquad 4.4$$

$$P = (G)\,(A) \qquad 4.5$$

$$I^{\bullet} = P - L \qquad 4.6$$

The variable A represents the overall adjustment order at time t as to the production rate. Adjustment order A and the actual rate of production P are connected by factory G. The operators D_I and D_L correspond to the decision rule, which now depends both on inventory level and rate of new orders. Both operators are at one's disposal in seeking an optimal scheduling rule.

Assume that at the factory "G" there is a delay of T.

i.e. $P(t) = A(t-T)$ 4. 7

i.e. $\mathcal{L}(P) = \mathcal{L}(A)\, e^{-TS}$ 4. 8

Transforming all the equations from (4.1) to (4.6) into "S" domain and combining all the equations and setting 0 = 0, the inventory transfer function is obtained as:

$$\mathcal{L}(I) = \left[\frac{e^{-TS}D_L - 1}{S + e^{-TS}D_I}\right]\mathcal{L}(L) \qquad 4.\ 9$$

and the production transfer function as:

$$\mathcal{L}(P) = \left[\frac{D_I\ e^{-TS} + S\ D_L\ e^{-TS}}{S + D_I\ e^{-TS}}\right]\mathcal{L}(L) \qquad 4.10$$

A comparison of 4.9 to the corresponding equation with no production lags reveals that both numerator and denominator have been affected by the introduction of the production lag.

Consideration of the numerator of the inventory transfer function in equation 4.9 shows that control is not an easy problem. Setting $D_L = 1$ makes the numerator $(e^{-TS}-1)$, which approaches zero only as S approaches $\frac{2n\pi i}{T}$, where n is zero or any integer. Hence this procedure would stabilize the inventory only for a sinusoidal load whose frequency is an exact multiple of the frequency corresponding to the production lag. One can say with $D_L = 1$ the system performs better than when there is no information about orders.

A. Feed Forward of Information about New Orders

It shall now be seen what happens if the value of customers' orders L (Load) for T units of time can be predicted in advance of the actual receipt of orders.

Setting $D_L = e^{TS}$ 4.11

Then the numerator of 4.9 becomes

$$e^{-TS}\, e^{TS} - 1 = 0$$

Defining the variable ϕ so that

$$\phi\,(S) = e^{TS}\mathcal{L}(L) \qquad 4.12$$

Taking the inverse transformation of both sides:

$$\phi\,(t) = L\,(t+T) \qquad 4.13$$

Hence, setting $D_L = e^{TS}$ corresponds to predicting the value of L for T units of time in advance of the actual receipt of orders. If orders could be predicted over the time interval T, production could be scheduled in anticipation of the actual receipt of these orders thus avoiding any inventory fluctuation whatsoever. It will not be attempted to explore the problem of forecasting L (t+T), but optimal decision rules will be considered when future orders are not known with certainty.

B. Feed Back of Information about Inventories

Consider now the denominator of 4.9 which is $S+e^{-TS}D_I$. Because of the sinusoidal character of e^{-TS} this will behave roughly like $(S+D_I)$. Hence the system will behave in the same general manner as the system with no production lag.

For $D_L = (\frac{a}{S} + b)$ the denominator becomes $S + e^{-TS}(\frac{a}{S} + b)$. The roots of the characteristic equation,

$$S^2 + (a+bS)e^{-TS} = 0 \qquad 4.14$$

are not easily evaluated. So a method is suggested by which the fixed lag with operator e^{-TS} is replaced by a distributed lag which retains the algebraic character of the system transform but avoids the difficulties encountered in equation (4.14).

$P(t) = A(t-T)$ can be replaced by

$$P(t) = \int_0^t P_r(T)\, A(t-T)\, dT \qquad 4.15$$

$$\text{where} \int_0^\infty P_r(T)\, dt = 1$$

$Pr(T)$ may be regarded as the probability that the lag in producing a particular scheduled item will be of length T. For large values of T one would expect $Pr(T)$ to be zero.

$$\text{Suppose } Pr(T) = a^2 e^{-aT} \qquad 4.16$$

$$\text{then } L(Pr) = \frac{a^2}{a^2+S^2} \qquad 4.17$$

and

$$\mathcal{L}(I) = \frac{\frac{a^2}{a^2S^2} D_L - 1}{S + \frac{a^2}{a^2+S^2} D_I} \mathcal{L}(L) \qquad 4.18$$

The system transform defined by the above equation 4.18 can be analyzed by methods previously employed to determine suitable forms of D_L and D_I.

III. System with Production Lag Distributed with Probability Density Function Pr(T)

Now consider the period of time between the moment when instructions are issued to increase the rate of production to the moment when the increased flow of goods is actually produced to be a random variable with probability density function, Pr(T). It is represented by a system as in Figure 5 below.

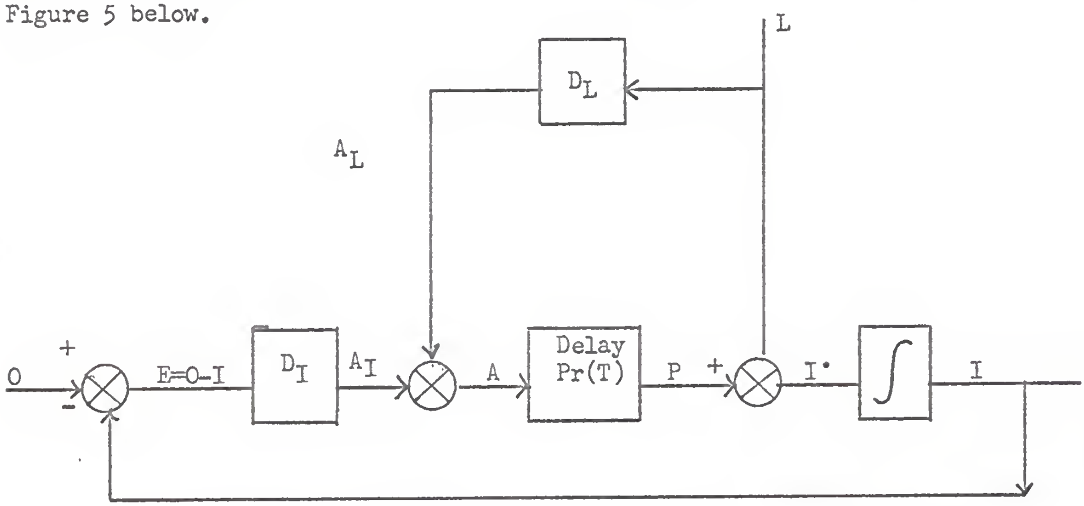

Figure 5. Inventory control system with varying production lag.

All the variables except Pr(T) are as defined previously

Pr(T)dt = probability that the lag in producing a particular scheduled item will be between length T and T+dt.

And

$$\int Pr(T)dT = 1, \text{ since } \int Pr(T)dT = 0.0.$$

The system equations can be written as follows:

$$\mathcal{L}(E) = \mathcal{L}(\theta) - \mathcal{L}(I) \qquad 5.1$$

$$\mathcal{L}(A_I) = D_I \mathcal{L}(E) \qquad 5.2$$

$$\mathcal{L}(A_L) = D_I \mathcal{L}(L) \qquad 5.3$$

$$\mathcal{L}(A) = \mathcal{L}(A_I) + \mathcal{L}(A_L) \qquad 5.4$$

$$\mathcal{L}(P) = \mathcal{L}[Pr(T)] \cdot \mathcal{L}(L) \qquad 5.5$$

$$S\mathcal{L}(I) = \mathcal{L}(P) - \mathcal{L}(L) \qquad 5.6$$

From equations 5.5 and 5.6:

$$S\mathcal{L}(I) = \mathcal{L}[Pr(T)] \cdot \mathcal{L}(A) - \mathcal{L}(L)$$

$$= \mathcal{L}[Pr(T)][\mathcal{L}(A_I) + \mathcal{L}(A_L)] - \mathcal{L}(L) \qquad 5.7$$

$$= \mathcal{L}[Pr(T)]\{D_I[\mathcal{L}(\theta) - (I)] + D(L)\mathcal{L}(L)\} - \mathcal{L}(L)$$

Since $\theta = 0$ (desired inventory)

$$\mathcal{L}(\theta) = 0$$

Then:

$$\mathcal{L}(I)\{S + D_I\mathcal{L}[Pr(T)]\} = \mathcal{L}(L)\{\mathcal{L}[Pr(T)] \; D_L - 1\} \qquad 5.8$$

The inventory transfer function is given as

$$Y = \frac{\mathcal{L}(I)}{\mathcal{L}(L)} = \frac{\mathcal{L}[Pr(T)] D_L - 1}{S + D_I \; [Pr(T)]} \qquad 5.9$$

Similarly for obtaining the production transfer function,

$$\begin{aligned}\mathcal{L}(P) &= \mathcal{L}[Pr(T)] \cdot \mathcal{L}(A) \\ &= \mathcal{L}[Pr(T)] \left[\mathcal{L}(A_I) + \mathcal{L}(A_L)\right] \\ &= \mathcal{L}[Pr(T)] \left[D_I \mathcal{L}(I) + D_L \mathcal{L}(L)\right] \\ &= \mathcal{L}[Pr(T)] \left[-D_I \left\{\frac{1}{S}\mathcal{L}(P) - \frac{1}{S}\mathcal{L}(L)\right\} + D_L \mathcal{L}(L)\right] \\ &= \frac{\mathcal{L}[Pr(T)]}{S} \left[-D_I \mathcal{L}(P) + \mathcal{L}(L) \left\{D_I + SD_L\right\}\right]\end{aligned} \qquad 5.10$$

The resulting production transfer function is:

$$Z = \frac{\mathcal{L}(P)}{\mathcal{L}(L)} = \frac{\mathcal{L}[Pr(T)] \left[D_I + SD_L\right]}{S + \mathcal{L}[Pr(T)]} \qquad 5.11$$

A. Stability of the System

Assume the distribution of lag time to be exponential. Then

$$Pr(T) = \lambda e^{-\lambda t}$$

and,

$$\begin{aligned}\mathcal{L}[Pf(T)] &= \int_0^\infty e^{-\lambda t}\, e^{-St}\, dt \\ &= \lambda \int_0^\infty e^{-(\lambda + S)t}\, dt \\ &= \frac{\lambda}{\lambda + S}\end{aligned} \qquad 5.12$$

Therefore,

$$\mathcal{L}(I) = \frac{\frac{\lambda}{\lambda+S} D_L - 1}{S + D_I \frac{\lambda}{\lambda+S}} \mathcal{L}(L) = \frac{\lambda D_L - \lambda - S}{S^2 + S\lambda + D_I\lambda} \mathcal{L}(L)$$

$$= \frac{-S + (D_L - 1)\lambda}{S^2 + S\lambda + D_I\lambda} \mathcal{L}(L) \qquad 5.13$$

Given an inventory decision function such as

$$D_I = \frac{1}{S^k}(a+bS) \qquad 5.14$$

Then:

$$\mathcal{L}(I) = \frac{-S + (D_L - 1)\lambda\,\mathcal{L}(L)}{S^2+S\lambda + \frac{1}{S^k}(a+bS)\lambda}$$

$$= \frac{-S^{k+1} + S^k\lambda\,(D_L - 1)\,\mathcal{L}(L)}{S^{k+2}+S^{k+1}\lambda + bS\lambda + a\lambda} \qquad 5.15$$

The stability criterion dictates that all powers of S in the denominator expression from S to a should be present. Otherwise the system is unstable.

For the characteristic equation

$$S^{k+2} + S^{k+1}\lambda + bS\lambda + a\lambda = 0 \qquad 5.16$$

k may have values 0, 1, -1, -2, -3 which will permit stable system operation. All other values k might have do not lead to stable system operation.

IV. Cost Consideration in the Control of Inventories and Production Rate Fluctuation

The general criterion for the optimality of a production and inventory control system of the kind we are analyzing is minimizing the cost of production.

Large inventories involve warehousing costs, interest costs, possible costs through physical depreciation in storage. An inventory deficiency on the other hand involves a cost in the sense of delay in filling orders and consequent loss of customer good will, etc. Therefore in order to achieve optimality one should balance these two kinds of costs. It also appears reasonable to assume that the cost of producing a given quantity of output over a period of time is minimized if output remains the same during that period of time. If output is represented as a constant M, plus an oscillating function with zero mean, then it may be assumed that the rate at which cost is being incurred is a function of M and the frequency and amplitude of M(t).

From equation 4.6,

$$\frac{dI}{dt} = P(t) - L(t) \qquad 5.17$$

If it is succeeded in stabilizing I at I = 0, actual production rate P will not be constant but will follow L(t). Conversely if P is stabilized, I will not be constant but will follow the integral of L(t). A system cannot be devised that will simultaneously eliminate inventory and production fluctuations, but must instead, establish a balance between the two.

Consider the steady state of the system under sinusoidal inputs and outputs. If customer order pattern is sinusoidal then in steady state the actual rate of production P and the actual inventory I will also behave sinusoidally with the same period (but differing in amplitude). It is assumed that the cost associated with production rate P is proportional to the square of the amplitude of oscillation. Let this constant of proportionality be C_1. Similarly it is assumed that the cost of maintaining inventories I is proportional to the square of its amplitude of oscillation and with the constant of proportionality being represented by C_2.

And:

$$L(t) = a \cos(wt) \qquad 5.18$$

$$P(t) = b \cos(wt) + \beta \sin(wt) \qquad 5.19$$

$$I(t) = c \cos(wt) + \gamma \sin(wt) \qquad 5.20$$

with a, b, c, β, γ being real values.

From the relation $\frac{dI}{dt} = P(t) - L(t)$,

$$-C_w \text{Sin}(wt) + \gamma w \cos(wt) = b \cos(wt) + B \sin(wt) - a \cos(wt) \qquad 5.21$$

equating the coefficients of life terms, results

$$W\gamma = b - a, \; -wc = \beta \qquad 5.22$$

The criterion function to be minimized it

$$C_1(b^2 + \beta^2) + C_2 \; (C^2 + \gamma^2) = Z \qquad 5.23$$

Substituting for c and γ from (5.22) into the cost equation, taking derivatives of Z with respect to b and β, and setting these equal to zero, gives,

$$b = \frac{aC_2}{C_1 w^2 + C_2} \quad \text{and} \quad \beta = 0 \qquad 5.24$$

Therefore

$$C = 0, \; \gamma = \frac{aC_1 w}{(C_1 w^2 + C_2)} \qquad 5.25$$

for small w: $b \rightarrow a$ 5.26

and $\gamma \rightarrow 0$

for large W:

$$b \to 0$$

$$\gamma \to 0 \qquad 5.27$$

$$W\gamma \to -a$$

When interpreted, these results mean that the optimal decision-rule will adjust production rate and hold inventories down for long period fluctuations in customer order pattern, but for rapid fluctuations in orders it will stabilize production and permit inventories to fluctuate. The amplitude of manufacturing fluctuations b will vary inversely with w. The magnitude of inventory fluctuations γ will have a maximum for $w^2 = C_2/C_1$.

It is seen how feed back control theory can be used to analyze the behavior of a continuous inventory control system when it is presented with specific patterns of inputs (customer orders). The response to any other arbitrary input can be obtained by the same general approach.

V. Discrete Inventory and Production Control Systems

Although the assumption that customers' orders and production vary continuously in time may be a reasonable approximation to reality in some situations, in many inventory processes customers' orders are reviewed and production schedules are set at discrete intervals of time. It would be interesting to see how this change will modify the analysis. The block diagram, loop relations, etc., are as used before. A schematic representation of a discrete system is as shown in Figure 6.

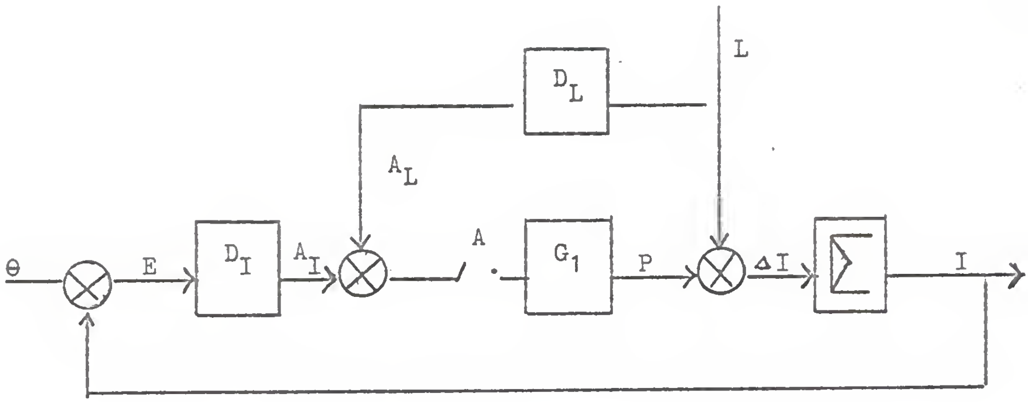

Figure 6. Schematic representation of a discrete inventory control system.

The main differences between the continuous and discrete system are as follows. The discrete system is a sample data system because only at discrete intervals is information about the stock levels available. When the relay in the discrete system is closed information is transmitted about stock level. When the relay is open there is no flow of information. Also, summation replaces the integral operator.

A considerable body of information on discrete systems has been developed for use in control processes. The approach is similar to that for continuous systems except that it operates on difference equations and "Z-transforms" instead of differential equations and Laplace transforms.

The underlying basis for Z-transforms is quite simple. The Z-transform of a variable f_n which takes on values at the points n = 0, 1, 2 . . . . . is defined as:

$$F(Z) = \sum_{n=0}^{\infty} f_n Z^n \qquad 6.1$$

For example, suppose $f_n = a^n$, then,

$$F(Z) = \sum_{n=0}^{\infty} f_n Z^n = \sum_{n=0}^{\infty} (aZ)^n = \frac{1}{1-aZ} \qquad 6.2$$

If the input distribution (distribution of customer order) is a pulse then

$$\begin{aligned} f_n &= 1, \; n = 0 \\ &= 0 \text{ when } n \neq 0 \end{aligned} \qquad 6.3$$

$$\begin{aligned} F(Z) &= \sum_{n=0}^{\infty} f_n Z^n \\ &= (1)Z^0 = (0)Z^1 + \dots \\ &= 1 \end{aligned} \qquad 6.4$$

For a unit step input,

$$\begin{aligned} f_n &= 1 \qquad n \geqslant 0 \\ &= 0 \qquad n < 0 \end{aligned} \qquad 6.5$$

$$\begin{aligned} F(Z) &= (1)Z^0 + (1)Z^1 + (1)Z^2 + \dots \\ &= \frac{1}{1-Z} \end{aligned} \qquad 6.6$$

For a unit ramp (trend input),

$$\begin{aligned} f_n &= n; \qquad n \geqslant 0 \\ &= 0 \qquad n < 0 \end{aligned} \qquad 6.7$$

$$\begin{aligned} F(Z) &= (0)Z^0 + (1)Z^1 + 2Z^2 + \dots \\ &= Z \left[(0)Z^{-1} + (1)Z^0 + (2)Z^1 + 3Z^2 + \dots\right] \end{aligned}$$

The series within the bracket on the right hand side is the derivative of the step-input series. Taking the derivative

$$\frac{d}{dZ}\left[\frac{1}{1-Z}\right] = \left[\frac{1}{(1-Z)^2}\right] \qquad 6.8$$

Therefore $F(Z) = \dfrac{Z}{(1-Z)^2}$ 6.9

An abbreviated table of Z-transforms is given below.

Table of Z-Transforms

Original Function	Z-Transform
unit impulse, $t = 0$	1
unit step, a constant	$\frac{1}{1-Z}$
unit ramp, trend	$\frac{Z}{(1-Z)^2}$
P_t (function of t)	$P(Z)$
P_{t+1}	$\frac{P(Z)}{Z}$
P_{t-k}	$\frac{P(Z)}{(Z)^{-k}}$
$\sum_{0}^{t} P_t$	$\frac{P(Z)}{(1-Z)}$

Consider a discrete inventory system in which we make the extra production in the j+1st period a fraction of the inventory shortage at the end of the j period.

Defining:

L_j = incremental sales (Load) during period j

P_j = incremental production during period j

I_j = inventory shortage at end of period j

The system equations are

$$I_j = \sum_{n=0}^{j} (L_n - P_n) \qquad 6.10$$

$$P_{j+1} = aI_j \qquad 6.11$$

$$P_{j+1} = a\sum_{n=0}^{j} (L_n - P_n) \qquad 6.12$$

$$P_{j+1} + a\sum_{n=0}^{j} P_n = a\sum_{n=0}^{j} L_n \qquad 6.13$$

Transforming

$$\sum_{j=0} Z^j P_{j+1} + a\sum_{j=0} Z^j \sum_{n=0}^{j} P_n = a\sum_{j=0} Z^j \sum_{n=0}^{j} L_n \qquad 6.14$$

Using the table of Z-transforms,

$$\frac{P(Z)}{Z} + \frac{a}{1-Z} P(Z) = \frac{a}{1-Z} L(Z) \qquad 6.15$$

$$P(Z) \left[1-Z+aZ\right] = aZS(Z) \qquad 6.16$$

So the production transfer function is

$$\frac{P(Z)}{L(Z)} = \frac{aZ}{1-(1-a)Z} \qquad 6.17$$

and the inventory transfer function is

$$\frac{I(Z)}{L(Z)} = \frac{1}{1-(1-a)Z} \qquad 6.18$$

A. <u>System Response to Specific Inputs</u>

<u>Impulse Input</u>

Suppose up until time t = 0 customer orders remain the same and then there is sudden increase in demand followed by a return to the previous stage existing at t = 0.

For unit impulse

$$L(Z) = 1 \qquad 7.\ 1$$

From equation 6.17,

$$P(Z) = \frac{aZ}{1-(1-a)Z} \qquad 7.\ 2$$

Similarly from 6.18,

$$I(Z) = \frac{1}{1-(1-a)Z} \qquad 7.\ 3$$

Performing inverse transformation on 7.2 and 7.3,

$$P_j = a(1-a)^{j-1} \qquad 7.\ 4$$

and

$$I_j = (1-a)^j \qquad 7.\ 5$$

Equations 7.4 and 7.5 are plotted in Figure 7.

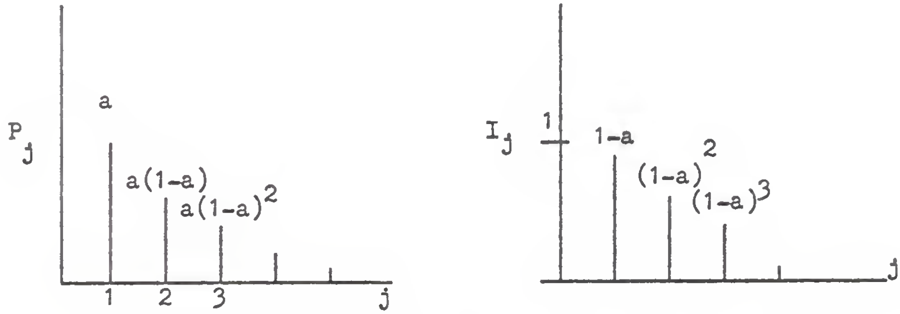

Figure 7. Production and inventory responses for impulse input.

Production comes back to the steady state and inventory shortages die out and hence the system response is satisfactory for impulse input.

Step Input

Suppose that up to time unit t = 0 customers' orders have been zero and after that they are received at the rate of one order per unit of time.

For unit step, from the table of Z-transforms,

$$L(Z) = \frac{1}{1-Z} \qquad 8.1$$

Substituting L(Z) in production and inventory transfer functions (6.17) and (6.18) yields

$$P(Z) = \frac{aZ}{(1-Z)[1-(1-a)Z]} = \frac{1}{1-Z} + \frac{-1}{1-(1-a)Z} \qquad 8.2$$

and

$$I(Z) = \frac{1}{(1-Z)\ [1-(1-a)Z]} = \frac{\frac{1}{a}}{1-Z} + \frac{\frac{1-a}{a}}{1-(1-a)Z} \qquad 8.\ 3$$

Taking inverse transforms,

$$P_j = 1-(1-a)^j \qquad 8.\ 4$$

$$I_j = \frac{1}{a} - \frac{1-a}{a}(1-a)^j = \frac{1}{a}\left[1-(1-a)^{j+1}\right] \qquad 8.\ 5$$

Figure 8 represents the production and inventory responses to the step input.

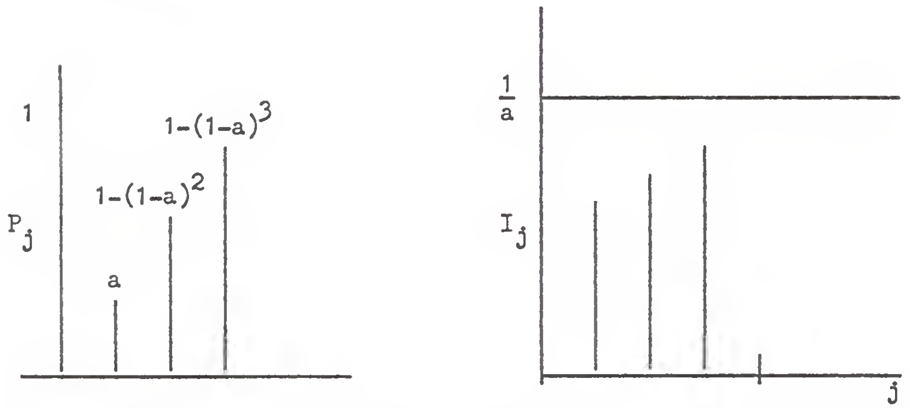

Figure 8. Production and inventory responses to step input.

The production catches up to unity fairly rapidly and therefore is a satisfactory response. However the inventory shortage becomes larger and larger with each period reaching a maximum of $\frac{1}{a}$. So inventory response is unsatisfactory.

It can be noticed that if a=1 an impulse of sales will be controlled after one period, a very fast response. Nevertheless, a step in sales will create a constant unit inventory discrepancy. Naturally a ramp increase in customer orders will cause a linear increase in inventory shortage.

An important feature of the decision function

$$P_j+1 = a\sum_{n=0}^{j} (L_n-P_n)$$

is that when a = 1 the equation represents the familiar P-model regulator when the output is being measured from the X_0 level as a base. Under the P- system of inventory control there is a fixed order period and varying order size. The procedure is that at periodic intervals--the period being analytically determined--the amount of inventory is reviewed and an order is placed. The P-model regulator can be represented by the following decision function.

$$P_{j+1} = M-E(I) + \sum_{n=0}^{j} (L_n-P_n) \qquad 8.\ 6$$

Where M is the maximum stock level defined for the P-model and E(I) is the expected inventory level.

Let x_0 = expected order quantity

i.e. $x_0 = M-E(I)$ 8. 7

$$P_{j+1} = x_0 + \sum_{n=0}^{j} (L_n-P_n) \qquad 8.\ 8$$

but if one considers the P_{j+1} as being measured from x_0 level as a base

$$P_{j+1} = \sum_{n=0}^{j} (L_n-P_n) \qquad 8.\ 9$$

So the results obtained when a = 1 will be the response of the P-model for inventory control.

B. Time Lag in the Implementation of Production Decisions.

Consider the effect of time lag on the system representing the P-model. Assume that the demand distribution is stationary and let k be the period of delay in the implementation of production decisions. Let the period of delay be k.

Thus the model becomes

$$P_j+1 = \sum_{n=0}^{j-k} I_n-k \sum_{n=0}^{j-k} P_{n-k} \qquad 9.1$$

Using the table of Z-transforms 9.1 can be written as

$$\frac{P(Z)}{Z} = \frac{I(Z)}{Z^{-k}(1-Z)} - \frac{P(Z)}{Z^{-k}(1-Z)} \qquad 9.2$$

Reducing we get the production transfer function

$$\frac{P(Z)}{I(Z)} = \frac{I(Z)Z^{k+1}}{I-Z+Z^{k+1}} \qquad 9.3$$

and for I_j,

$$P_{j+1} = I_{j-k} \qquad 9.4$$

$$\frac{P(Z)}{Z} = \frac{I(Z)}{Z^{-k}} \text{ or } P(Z) = Z^{k+1}I(Z) \qquad 9.5$$

So combining the two P(Z) equations and simplifying yields the inventory transfer function as

$$\frac{I(Z)}{L(Z)} = \frac{1}{1-Z+Z^{k+1}} \qquad 9.6$$

When k = 0 the equation 8.14 and 8.15 reduce to equation 8.16 and 8.17 when there is no delay. Now study the effect of such impulse in demand on the system.

Let k = 1, then:

$$P(Z) = \frac{Z^2}{1-Z+Z^2} \qquad 9.7$$

$$S(Z) = \frac{1}{1-Z+Z^2} \qquad 9.8$$

These transformations are not in the table of Z-transforms presented but they are obtained by using long division to obtain a power series, and these power series are

$$P(Z)=(0)Z^0+(0)Z^1+(1)Z^2+(1)Z^3+(0)^4+(-1)Z^5+(-1)Z^6+ \ldots \qquad 9.9$$

and

$$I(Z)=(1)Z^0+(1)Z^1+(0)Z^2+(-1)Z^3+(-1)Z^4+(0)Z^5+(1)Z^6+ \ldots \qquad 9.10$$

What these equations show is that inventory shortage oscillates. Figure 9 illustrates endless oscillations which the system experiences.

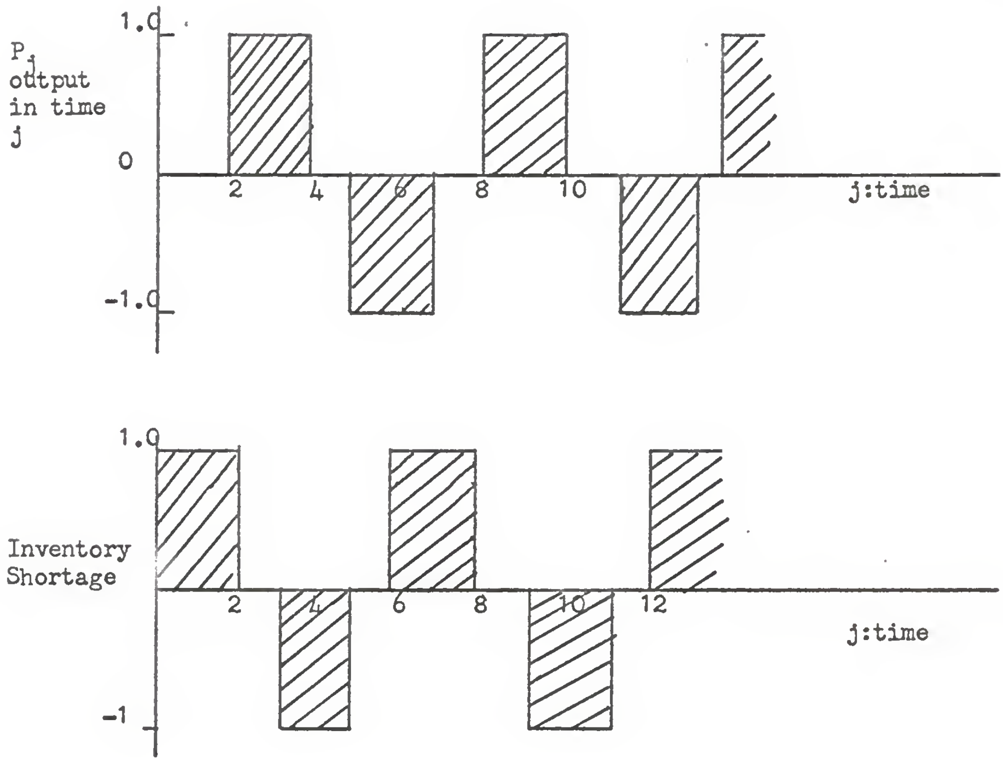

Figure 9. Production and inventory response for the system with time lag.

The cost of this oscillatory behavior can be very high indeed. This is the weakness of the P-model. It is extremely vulnerable to transmission and process delays.

C. <u>A Further Consideration of the Decision Rule</u>.

It will now be seen if a decision rule can be used in which cumulative shortage days as well as actual shortage levels are used to see if it will have superior step and ramp responses.

Consider the model

$$P_{j+1} = A\sum_{n=0}^{j}(L_n-P_n) + B\sum_{n=0}^{j}\sum_{n=0}^{j}(L_n-P_n) \qquad 10.\ 1$$

where

$$\sum_{n=0}^{j}(L_n-P_n) = \text{units short during period 0 to } j$$

and

$$\sum_{n=0}^{j}\sum_{n=0}^{j}(L_n-P_n) = \text{number of unit-days short.}$$

Obtaining Z-transforms of the equation 10.1,

$$\frac{P(Z)}{Z} = \frac{AL(Z)}{1-Z} + \frac{BL(Z)}{(1-Z)^2} - \frac{AP(Z)}{1-Z} - \frac{BP(Z)}{(1-Z)^2} \qquad 10.\ 2$$

which yields the production transfer function as

$$\frac{P(Z)}{L(Z)} = \left[\frac{(A+B)Z-AZ^2}{(A+B-2)Z+(1-A)Z^2+1}\right] \qquad 10.\ 3$$

and similarly the inventory transfer function is

$$\frac{I(Z)}{L(Z)} = \frac{1-Z}{(A+B-2)Z + (1-A)Z^2+1} \qquad 10.\ 4$$

An IBM 1620 computer program is written to simulate the responses of the system with this decision rule to an impulse input when A and B take on different values. Figures 10a, b, and c and d present the responses graphically.

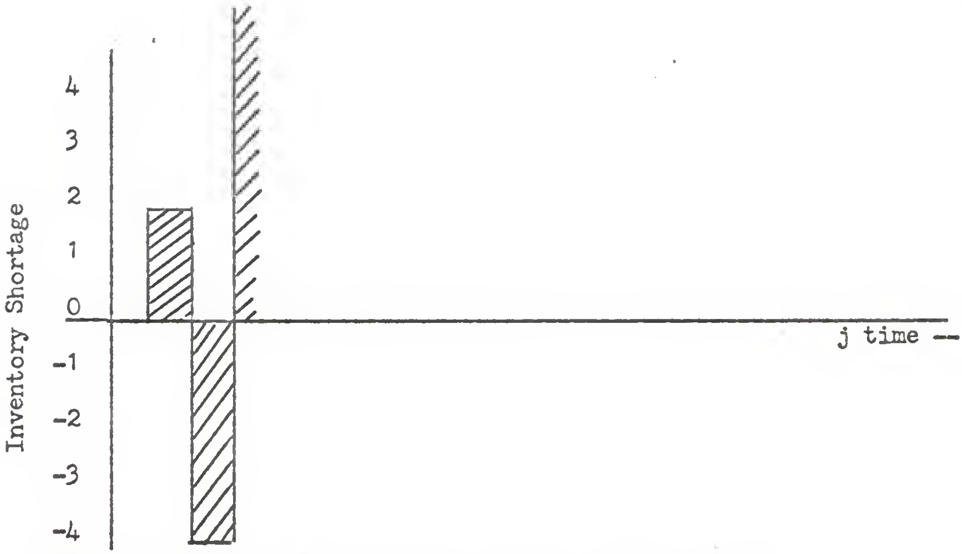

Figure 10a. For A = 2, B = 1, divergent oscillations.

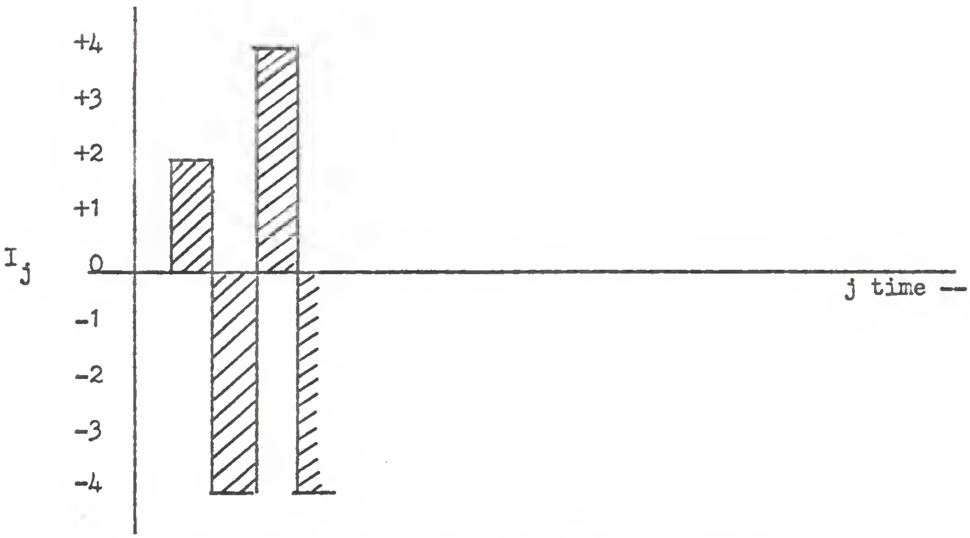

Figure 10b. For A = 1, B = 2, undamped oscillations.

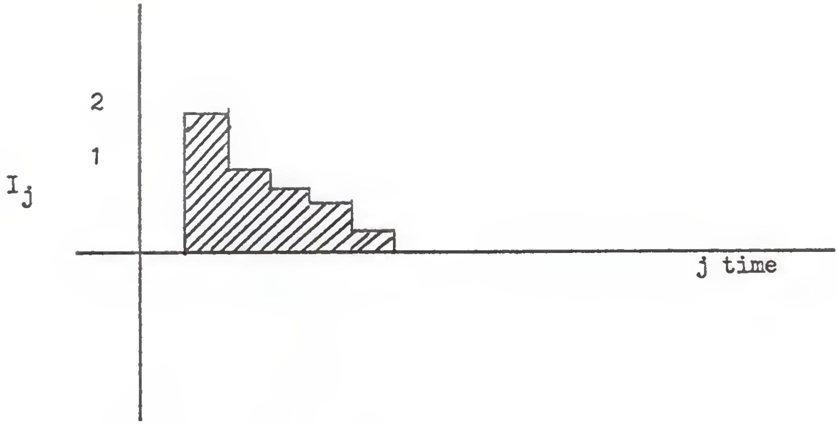

Figure 10c. For $A = \frac{1}{2}$, $B = 0$, overdamping.

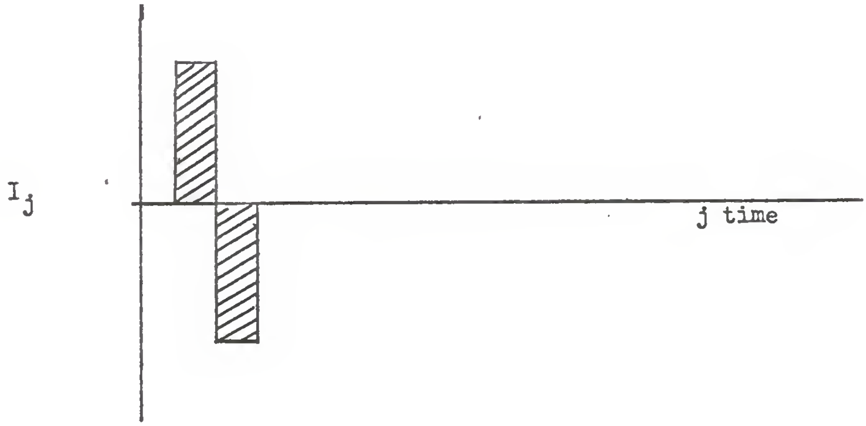

Figure 10d. For $A = 1$, $B = 1$, critical damping.

Both over damping and critical damping seem to be permissible. However when $A = 1$, $B = 1$, one has the case of critical damping restoring the stock level to its original value.

So "critical damping" for this system corresponds to A=B=1. One should use these values if production facilities permit their use. Thus with A = 1; B = 1; from 10.3 and 10.4:

$$\frac{P(Z)}{L(Z)} = \frac{2Z-Z^2}{1} \qquad 11.\ 1$$

and

$$\frac{I(Z)}{L(Z)} = \frac{1-Z}{1} \qquad 11.\ 2$$

Figure 11 represents the production and inventory responses of this system to impulse, step and ramp inputs.

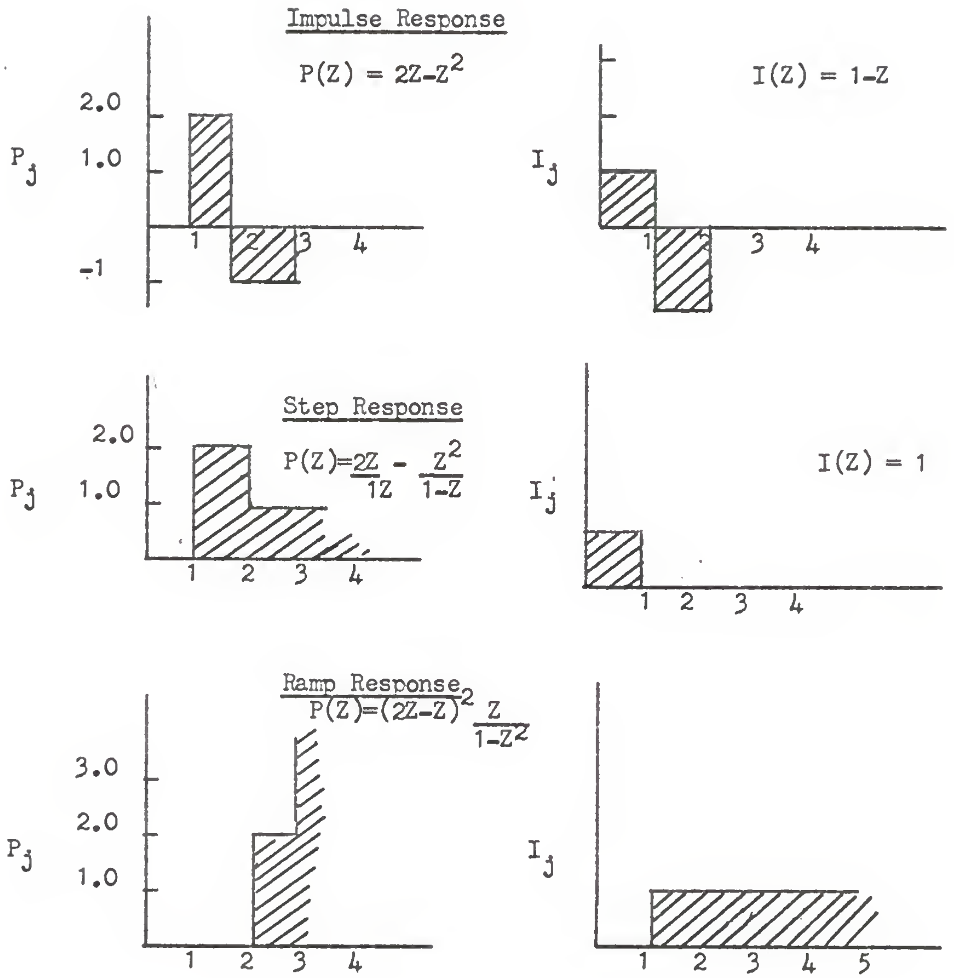

Figure 11. Production and inventory responses to impulse, step and ramp inputs.

It is observed from Figure 11 that impulse and step responses are quite satisfactory. For ramp input, although production response is satisfactory, a constant inventory shortage of one unit will occur. The control exerted by this system is generally satisfactory.

Note the simplification in the system that has been produced. Instead of a system which depends upon inventory shortage and shortage days a more easily implemented result has been obtained. The impulse response shows that the production in the next month is made equal to twice the present month's sales less previous month's sales. Such simplification can often by obtained using the systems analysis.

CHAPTER IV

CONCLUSIONS

In this report only a few decision functions and very simplified versions of the real life production and inventory problems were studied. However many other similar problems with more complications can be studied with the same general approach. Which of these is to be preferred in a given case depends upon the circumstances of specific problems.

The general conclusion to be drawn from this study is that the basic approach and fundamental techniques of control theory can be profitably applied in the analysis and design of production and inventory control systems. The conclusions that have been reached about the inventory control problem studied might in a qualitative sense by reached intuitively, but intuition has been aided by the frame of reference that control theory provides. Even at this early stage, the theory permits actual numbers to be inserted for the construction of specific decision rules that would apply to actual situations. The method could also provide a means of optimizing work processing and communications by means of block diagrams. The application of feed back control techniques to industrial functions represents an area in which investigations and studies will improve the performance of industrial enterprises.

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APPLICATION OF FEED BACK CONTROL THEORY
IN PRODUCTION AND INVENTORY CONTROL

by

GOKARAJU V. N. RAJU

B. Sc., Benaras Hindu University, 1960
B. I. E., Ohio State University, 1964

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Determining the quantity of product to produce in any given period of time is one of the major problems encountered in connection with the operation of a manufacturing organization. In this report the problem of controlling the rate of production is stated in terms of feed back control theory and the well developed methods of that theory are employed to study the behavior of a control system. This is illustrated for both the continuous and discrete systems. Continuous systems are those in which it is assumed that information about customer orders and inventory levels are available continually, whereas discrete systems are those in which such information is not continually available. The production and inventory control system under study is represented by means of time differential equations for continuous system and difference equations in the case of discrete system. Laplace transform and Z-transform methods are introduced and some of their elementary uses for studying the stability and steady state behavior of production and inventory control systems are illustrated.

Included in the production systems studied are systems with no production lag, with fixed production lag, and with random production lag. Response of the system to specific customer order patterns and alternative decision functions are studied. A cost criterion based on costs due to fluctuating production, carrying costs, and shortage costs is constructed to evaluate alternative decision rules. The general conclusion from this study is that the basic approach and fundamental techniques of control theory can profitably be applied in the analysis and design of production and inventory control systems.