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Scattering of electromagnetic waves by many small perfectly conducting or impedance bodies

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A theory of electromagnetic (EM) wave scattering by many small particles of an arbitrary shape is developed. The particles are perfectly conducting or impedance. For a small impedance particle of an arbitrary shape, an explicit analytical formula is derived for the scattering amplitude. The formula holds as $a \to 0$, where a is a characteristic size of the small particle and the wavelength is arbitrary but fixed. The scattering amplitude for a small impedance particle is shown to be proportional to $a^{2-\kappa}$, where $\kappa \in [0, 1)$ is a parameter which can be chosen by an experimenter as he/she wants. The boundary impedance of a small particle is assumed to be of the form $\zeta = ha^{-\kappa}$, where $h = \text{const}$, $\text{Re}h \geq 0$. The scattering amplitude for a small perfectly conducting particle is proportional to a^3, and it is much smaller than that for the small impedance particle. The many-body scattering problem is solved under the physical assumptions $a \ll d \ll \lambda$, where d is the minimal distance between neighboring particles and λ is the wavelength. The distribution law for the small impedance particles is $\mathcal{N}(\Delta) \sim 1/a^{2-\kappa} \int_\Delta N(x)dx$ as $a \to 0$. Here, $N(x) \geq 0$ is an arbitrary continuous function that can be chosen by the experimenter and $\mathcal{N}(\Delta)$ is the number of particles in an arbitrary sub-domain Δ. It is proved that the EM field in the medium where many small particles, impedance or perfectly conducting, are distributed, has a limit, as $a \to 0$ and a differential equation is derived for the limiting field. On this basis, a recipe is given for creating materials with a desired refraction coefficient by embedding many small impedance particles into a given material. [http://dx.doi.org/10.1063/1.4929965]

I. INTRODUCTION

Electromagnetic (EM) wave scattering is a classical area of research. Rayleigh stated in 1871, see Ref. 17, that the main part of the field, scattered by a small body, $ka \ll 1$, where k is the wave number and a is the characteristic size of the body, is the dipole radiation, but did not give formulas for calculating this radiation for bodies of arbitrary shapes. For spherical bodies, Mie, Ref. 4, gave a solution to EM wave scattering problem using separation of variables in the spherical coordinates. This method does not work for bodies of arbitrary shapes. Rayleigh and Mie concluded that EM field, scattered by a small body, is proportional to $O(a^3)$. We prove that the field scattered by a small impedance body (particle) of an arbitrary shape is proportional to $a^{2-\kappa}$, where $\kappa \in [0, 1)$ is a parameter which can be chosen by the experimenter as he/she wishes, see formula (1.3) below. Since $2 - \kappa < 3$, it follows, for $a \to 0$, that the scattering amplitude for small impedance particle is much larger than the scattering amplitude for perfectly conducting or dielectric small particle. This conclusion may be of practical importance.

There is a large literature on low-frequency wave scattering and multiple scattering, see Refs. 1, 3, 6, 7, and 19.

In this paper, a theory of EM wave scattering by perfectly conducting and by impedance small bodies of arbitrary shapes is developed. For one-body scattering problem, explicit formulas for

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the scattering amplitudes are derived for perfectly conducting and for impedance small bodies of arbitrary shapes. For many-body scattering problem, the solution is given as a sum of explicit terms with the coefficients that solve a linear algebraic system (LAS). If the size of the small bodies $a \to 0$ and their number $M = M(a) \to \infty$, a limiting integral equation is derived for the field in the limiting medium. This equation allows us to obtain a local differential equation for the field in the limiting medium and to give explicit analytic formulas for the refraction coefficient of the limiting medium.

As a result, we formulate a recipe for creating materials with a desired refraction coefficient by embedding many small impedance particles in a given material.

The methods developed in this paper were applied to acoustic problems in Refs. 9–12, to heat transfer in the medium where many small bodies are distributed in Ref. 13, and to wave scattering by many nano-wires in Ref. 14.

In Section II, the theory of EM wave scattering is developed for small perfectly conducting bodies (particles) of arbitrary shapes.

In Section III, the theory is developed for EM wave scattering by one impedance particle of an arbitrary shape.

In Section IV, the theory is developed for EM wave scattering by many small impedance particles of an arbitrary shape.

In Section V, a recipe for creating materials with a desired refraction coefficient is given. The problem of creating materials with a desired magnetic permeability is solved.

Physical assumptions in this paper can be described by the inequalities,

$$a \ll d \ll \lambda, \tag{1.1}$$

where λ is the wavelength in $\mathbb{R}^3 \setminus \Omega$, Ω is a bounded domain in which many small particles D_m are distributed, $1 \le m \le M = M(a)$, and d is the minimal distance between neighboring particles.

The boundary impedance is assumed to be

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \tag{1.2}$$

where $x_m \in D_m$ is an arbitrary point inside D_m, $h(x)$ is an arbitrary continuous function in Ω such that $\mathrm{Re}h \ge 0$, $\kappa \in [0,1)$ is a parameter.

One can choose $h = h(x)$*,* Re $h(x) \ge 0$*, and* $\kappa, \kappa \to [0,1)$*, as one wishes.*

The distribution of the small impedance particles in D is given by the formula

$$\mathcal{N}(\Delta) := \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx(1+o(1)), \quad a \to 0, \tag{1.3}$$

where $\Delta \subset \Omega$ is an arbitrary open set, $\mathcal{N}(\Delta)$ is the number of small particles in the set Δ, and $N(x) \ge 0$ is an arbitrary continuous function in Ω.

The experimenter can choose the function $N(x) \ge 0$ *as he/she wishes.*

One has

$$\mathcal{N}(\Delta) = \sum_{x_m \in \Delta} 1. \tag{1.4}$$

By ω, the frequency is denoted, $k = \frac{\omega}{c}$ is the wave number, and c is the velocity of light in the air.

II. SCATTERING BY PERFECTLY CONDUCTING PARTICLES

A. Scattering by one particle

The problem is to find the solution to Maxwell's equations

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\epsilon E, \quad \text{in } D' := \mathbb{R}^3 \setminus D, \tag{2.1}$$

where D is the small body, $ka \ll 1$, $a = 0.5\mathrm{diam}D$, ϵ and μ are dielectric and magnetic constants of the medium in D', $k = \omega\sqrt{\epsilon\mu}$, and the boundary condition is

$$[N,[E,N]] = 0 \quad \text{on } S := \partial D. \tag{2.2}$$

Here, and below, $N := N_s$ is the unit normal to S pointing into D', $[E, N] = E \times N$ is the vector product of two vectors, $E \cdot N = (E, N)$ is the scalar product, $|S|$ is the surface area.

The incident field E_0 is

$$E_0 = \mathcal{E} e^{ik\alpha\cdot x}, \qquad H_0 = \frac{\nabla \times E_0}{i\omega\mu}, \tag{2.3}$$

where $\alpha \in S^2$ is a unit vector, the direction of the incident plane wave, and it is assumed that $\mathcal{E} \cdot \alpha = 0$. This assumption implies that

$$\nabla \cdot E_0 = 0, \qquad \nabla \cdot H_0 = 0. \tag{2.4}$$

The field E to be found is

$$E = E_0 + v_E, \tag{2.5}$$

where the scattered field v_E satisfies the radiation condition

$$r\left(\frac{\partial v_E}{\partial r} - ikv_E\right) = o(1), \quad r := |x| \to \infty. \tag{2.6}$$

In Equation (2.6), $o(1)$ is uniform with respect to the direction $\beta := \frac{x}{r}$ of the scattered field as $r \to \infty$.

The scattering amplitude $A(\beta, \alpha, k)$ is defined as usual,

$$v_E = \frac{e^{ikr}}{r} A(\beta, \alpha, k) + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \beta = \frac{x}{r}. \tag{2.7}$$

The magnetic field $H = H_0 + v_H$,

$$H = \frac{\nabla \times E}{i\omega\mu}, \quad v_H = \frac{\nabla \times v_E}{i\omega\mu}. \tag{2.8}$$

Let us look for the solution to scattering problems (2.1)-(2.6) of the form

$$E = E_0 + \nabla \times \int_S g(x,t) J(t) dt, \qquad g(x,t) = \frac{e^{ik|x-t|}}{4\pi|x-t|}, \tag{2.9}$$

where J is a *tangential field to S*. We assume that $S \in C^2$, that is, S is twice continuously differentiable.

Equations (2.1) are satisfied if

$$\nabla \times \nabla \times E = k^2 E, \qquad H = \frac{\nabla \times E}{i\omega\mu}. \tag{2.10}$$

Since E_0 satisfies Equations (2.10), these equations are equivalent to

$$\nabla \times \nabla \times v_E = k^2 v_E, \qquad v_E = \frac{\nabla \times v_E}{i\omega\mu}. \tag{2.11}$$

Equation for v_E is equivalent to the following equations:

$$(\nabla^2 + k^2) v_E = 0, \qquad \nabla \cdot v_E = 0 \text{ in } D', \tag{2.12}$$

because $\nabla \times \nabla \times v_E = \nabla\nabla \cdot v_E - \nabla^2 v_E$ and $\nabla \cdot v_E = 0$. Conversely, Equations (2.12) are equivalent to (2.10) and to (2.1).

The radiation condition is satisfied by

$$v_E = \nabla \times \int_S g(x,t) J(t) dt$$

for any vector-function $J(t)$.

Boundary condition (2.2) yields

$$\frac{J}{2} + TJ := \frac{J}{2} + \int_S [N_s, [\nabla_s g(s,t), J(t)]] dt = -[N_s, E_0], \tag{2.13}$$

where the formula

$$\lim_{x\to s^-}[N, \nabla\times\int_S g(x,t)J(t)dt] = \frac{J(s)}{2} + TJ \tag{2.14}$$

was used, see Ref. 15. Let us prove that Equation (2.13) has a solution and this solution is unique in the space $C(S)$ of continuous on S functions. This proves that the scattering problem can be solved by formula (2.9) with J solving (2.13).

Theorem 2.1. *If D is sufficiently small, then Equation (2.13) is uniquely solvable in $C(S)$ and its solution J is tangential to S.*

Proof. Note that any solution to Equation (2.13) is a tangential to S field. To see this, just take the scalar product of N_s with both sides of Equation (2.13). This yields $N_s\cdot J(s) = 0$. In other words, J is a tangential to S field.

Let us check that the operator T is compact in $C(S)$. This follows from the formula

$$TJ = \int_S\left(\nabla_S g(s,t)N_s\cdot J(t) - J(t)\frac{\partial g(s,t)}{\partial N_s}\right)dt. \tag{2.15}$$

Indeed, if J is a tangential to S field, then

$$N_s\cdot J(s) = 0. \tag{2.16}$$

Since $S\in C^2$, relation (2.16) implies

$$|N_s\cdot J(t)| = O(|s-t|)|J(t)|, \quad |\nabla_S g(s,t)N_s\cdot J(t)| \le O\left(\frac{1}{|s-t|}\right)|J(t)|. \tag{2.17}$$

Thus, the first integral in (2.15) is a weakly singular compact operator in $C(S)$. The second integral in (2.15) is also a weakly singular compact operator in $C(S)$ because

$$\left|\frac{\partial g(s,t)}{\partial N_s}\right| = O\left(\frac{1}{|s-t|}\right), \tag{2.18}$$

if $S\in C^2$.

Consequently, Equation (2.13) is of Fredholm type in $C(S)$. The corresponding homogeneous equation has only the trivial solution if D is sufficiently small. This follows from the following argument. The homogeneous version of Equation (2.13) means that the function

$$v_E = \nabla\times\int_S g(x,t)J(t)dt$$

solves Equations (2.12), satisfies radiation condition (2.6), and

$$[N, v_E] = 0 \quad \text{on } S. \tag{2.19}$$

This implies that $v_E = 0$ in D'.

Lemma 2.1 (see below) implies that if $v_E = 0$ in D' then $J = 0$. This conclusion and the Fredholm alternative prove the existence and uniqueness of the solution to Equation (2.13). The smallness of the body D guarantees that k^2 is not a Dirichlet eigenvalue of the Laplacian in D. Theorem 2.1 is proved. □

Lemma 2.1. Assume that the following conditions hold:

a) $v_E = 0$ *in* D',
b) J *is tangential to* S, *and*
c) k^2 *is not a Dirichlet eigenvalue of the Laplacian in* D.
Then, $J = 0$.

Proof. Denote $A := \int_S g(x,t)J(t)dt$ and use the formula

$$\int_{D'}\nabla\times A\cdot B dx = \int_{D'} A\cdot\nabla\times B dx - \int_S N\cdot[A,B]ds = \int_{D'} A\cdot\nabla\times B dx, \tag{2.20}$$

valid for any $B \in C_0^\infty(D')$. If $\nabla \times A = 0$ in D', then formula (2.20) yields

$$\int_{D'} A \cdot \nabla \times B dx = 0, \quad \forall B \in C_0^\infty(D'). \tag{2.21}$$

Write this formula as

$$\int_S dt J(t) \cdot \int_{D'} g(x,t) F(x) dx = 0, \quad F := \nabla \times B. \tag{2.22}$$

The set of vector-fields F coincide with the set of divergence-free fields $\nabla \cdot F = 0$ in D', where $F \in C_0^\infty(D')$.

The set of vector-fields

$$G(t) = \int_{D'} g(x,t) F(x) dx, \quad \forall F \in C_0^\infty(D'),$$

where it is not assumed that the condition $\nabla \cdot F = 0$ holds, *is dense in the set $L^2(S)$ of vector fields*. Indeed, if there exists an $h \neq 0$ such that

$$\int_S h(t) \int_{D'} g(x,t) F(x) dx dt = 0, \quad \forall F \in C_0^\infty(D'), \tag{2.23}$$

and $w(x) := \int_S g(x,t) h(t) dt$, then

$$\int_{D'} w(x) F(x) dx = 0, \quad \forall F \in C_0^\infty(D').$$

Thus,

$$w(x) = \int_S g(x,t) h(t) dt = 0 \quad \text{in } D'. \tag{2.24}$$

Consequently,

$$(\nabla^2 + k^2) w = 0 \quad \text{in } D, \quad w = 0 \ \text{ on } \ S. \tag{2.25}$$

Since k^2 is not a Dirichlet eigenvalue of the Laplacian in D, Equation (2.25) implies $w = 0$ in D. Therefore, $w = 0$ in $D \cup D'$. This implies $h = \frac{\partial w}{\partial N_+} - \frac{\partial w}{\partial N_-} = 0$. Consequently, the set $G(t)$ is dense in the set $L^2(S)$ of vector fields on S.

We claim that if $\nabla \cdot F = 0$ in D', where $F \in C_0^\infty(D')$, then $\nabla \cdot G = 0$ on S.

Indeed,

$$\nabla_t \cdot \int_{D'} g(x,t) F(x) dx = -\int_{D'} \nabla_x g(x,t) \cdot F(x) dx = \int_{D'} g(x,t) \nabla \cdot F(x) dx = 0. \tag{2.26}$$

Conversely, if $\nabla \cdot G = 0$ on S, then Equation (2.26) show that

$$\int_{D'} g(x,t) \nabla \cdot F(x) dx = 0, \forall t \in S.$$

Let us use the local coordinate system with the axis x_3 directed along the outer normal N_s to S, and $x_1(s), x_2(s)$ are coordinates along two orthogonal axes tangential to S. Let us denote by $e_1(s)$ and $e_2(s)$ the unit vectors along these axes at a point $s \in S$.

Equation (2.22) can be written as

$$\int_S J(t) \cdot G(t) dt = 0 \tag{2.27}$$

for all smooth $G(t)$ such that $\nabla \cdot G = 0$ on S, $G = \int_{D'} g(x,t) F(x) dx, \nabla \cdot F = 0$.

Let $J(t) = J_1(t) e_1(t) + J_2(t) e_2(t)$ in the local coordinates. For an arbitrary small $\delta > 0$, one can choose $G_1(t)$ and $G_2(t)$ such that

$$\|\overline{J}_1 - G_1\|_{L^2(S)} + \|\overline{J}_2 - G_2\|_{L^2(S)} < \delta, \tag{2.28}$$

where the over-bar denotes the complex conjugate. With G_1 and G_2 so chosen, choose G_3 such that

$$\nabla \cdot G = 0 \quad \text{on} \quad S, \tag{2.29}$$

which is clearly possible. Then, Equation (2.27) yields

$$\int_S (|J_1|^2 + |J_2|^2)dt = O(\delta). \tag{2.30}$$

Since $\delta > 0$ is arbitrary small, relation (2.30) implies $J_1 = J_2 = 0$. Therefore, $J = 0$.
Lemma 2.1 is proved. □

As was stated above, it follows from Lemma 2.1 and from the Fredholm alternative that Equation (2.13) is uniquely solvable for any right-hand side if $k^2 \notin \sigma(\Delta_D)$, that is, if k^2 is not a Dirichlet eigenvalue of the Laplacian in D. If D is sufficiently small, which we assume since $a \to 0$, then a fixed number k^2 cannot be a Dirichlet eigenvalue of the Laplacian in D because the smallest Dirichlet eigenvalue of the Laplacian in D is $O(\frac{1}{a^2}) > k^2$ if $a \to 0$.

Remark 2.1. The assumption $k^2 \notin \sigma(\Delta_D)$ can be discarded if $g(x,t)$ is replaced by $g_\epsilon(x,t)$, the Green function of the Dirichlet Helmholtz operator in the exterior of a ball $B_\epsilon := \{x : |x| \le \epsilon\}$, where $\epsilon > 0$ is chosen so that $k^2 \notin \sigma(\Delta_{D\setminus B_\epsilon})$. This choice of $\epsilon > 0$ is always possible (see Ref. 8, p. 29).

Let us denote by V the operator that gives the tangential to S component $v_{E\tau}$ of the unique solution v_E to scattering problems (2.1)-(2.3) and (2.6),

$$E = E_0 + v_E, \qquad v_{E\tau} = V(-[N, E_0]). \tag{2.31}$$

If the tangential component $v_{E\tau}$ is known, then v_E is uniquely defined in D'. This is a known fact, see, for example, Ref. 15. The operator V is linear and bounded in $C(S)$. It maps $C(S)$ onto $C(S)$ and v_E has the same smoothness as the data $[N, E_0]$. For example, if $S \in C^\ell$, then $v_E \in C^\ell(D')$, where $\ell > 0$.

Define

$$Q := \int_S J(t)dt. \tag{2.32}$$

From formulas (2.7), (2.9), and (2.32), it follows that

$$A(\beta, \alpha, k) = \frac{ik}{4\pi}[\beta, Q]. \tag{2.33}$$

For body D, one has

$$\int_S [N, E_0]ds = \int_D \nabla \times E_0 dx = \nabla \times E_0\,|D| = \nabla \times E_0\, c_D a^3, \tag{2.34}$$

where $|D|$ is the volume of D and $c_D > 0$ is a constant depending on the shape of D. For example, if D is a ball of radius a, then $c_D = \frac{4\pi}{3}$.

One has the formula (see Ref. 15, p. 8)

$$-\int_S \frac{\partial g(s,t)}{\partial N_s} ds = \frac{1}{2} + o(1), \quad a \to 0. \tag{2.35}$$

Since $N_s \cdot J(s) = 0$ and S is C^2 – smooth, it follows that $|N_s \cdot J(t)| \le c|s - t||J(t)|$. Therefore,

$$I := \left| \int_S ds \int_S dt \nabla_s g(s,t) N_s \cdot J(t) \right| \le c \int_S ds \int_S dt \frac{1}{|s-t|} |J(t)| \tag{2.36}$$

and $I \le O(a) \int_S |J(t)|dt$. If I would satisfy the estimate $I = o(Q)$, as $a \to 0$, then the theory would simplify considerably and one would have $Q = -\nabla \times E|D| = -\nabla \times E c_D a^3$. Unfortunately, estimate $I = o(Q)$ is not valid, and one has to give a new estimate for the integral $I_1 := \int_S ds \int_S dt \nabla_s g(s,t) N_s \cdot J(t)$. To do this, integrate Equation (2.13) over S, use Equations (2.15) and (2.35), and get

$$Q + I_1 = -c_D a^3 \nabla \times E_0. \tag{2.37}$$

Let us write I_1 as

$$I_1 = e_p \int_S \Gamma_{pq}(t) J_q(t) dt, \tag{2.38}$$

where $\{e_p\}_{p=1}^3$ is an orthonormal basis of $\mathbb{R}^3$,

$$\Gamma_{pq}(t) := \int_S \frac{\partial g(s,t)}{\partial s_p} N_q(s) ds, \tag{2.39}$$

and the integral in formula (2.39) is understood as a singular integral. Thus, Equation (2.37) takes the form

$$(I+\Gamma)Q = -c_D a^3 \nabla \times E_0. \tag{2.40}$$

Here, the constant matrix Γ is determined from the relation

$$\Gamma Q = e_p \int_S \Gamma_{pq}(t) J_q(t) dt, \tag{2.41}$$

and the summation is understood over the repeated indices p,q, so Γ is the matrix which sends a constant vector Q onto the constant vector I_1 defined by Equation (2.38).

One can prove that the constant matrix Γ exists and can be determined by Equation (2.41), and the matrix $I+\Gamma$ is non-singular.

To prove that a constant matrix Γ exists, assume that for every $p = 1,2,3$, the set of functions $\{\Gamma_{pq}(t)\}_{q=1}^3$ is linearly independent in $L^2(S)$, $\int_S \Gamma_{pq}^2(t)dt \neq 0$, and $Q = \int_S J(t)dt \neq 0$. Here, $J(t) = \sum_{q=1}^3 e_q J_q(t)$. For a fixed p, let M_p be the set in $L^2(S)$ orthogonal to the linear span of $\Gamma_{pq}(t)$. Then, every function $J_q(t)$ can be represented as $J_q(t) = J_q^0(t) + \sum_{j=1}^3 c_{qj}\Gamma_{pj}(t)$, where $J_q^0 \in M_p$ and c_{qj} are constants. One has

$$\sum_{q=1}^3 \int_S \Gamma_{pq}(t) J_q(t) = \sum_{q,j=1}^3 c_{qj} \int_S \Gamma_{pq}(t)\Gamma_{pj}(t) dt := \sum_{q,j=1}^3 c_{qj}\gamma_{p;qj},$$

where $\gamma_{p;qj}$ is a constant non-singular matrix for each p because the set $\{\Gamma_{pq}(t)\}_{q=1}^3$ is assumed linearly independent. To satisfy Equation (2.41), one has to satisfy the following equation:

$$\sum_{q,j=1}^3 c_{qj}\gamma_{p;qj} = \sum_{q=1}^3 \Gamma_{pq} Q_q.$$

Since we assumed that $Q \neq 0$, at least one of the numbers $Q_q \neq 0$. If there is just one such number, say, $Q_{q_1} \neq 0$ and $Q_q = 0$ for $q \neq q_1$, then we set $\Gamma_{pq_1} = Q_{q_1}^{-1} \sum_{q,j=1}^3 c_{qj}\gamma_{p;qj}$, $Q = e_{q_1} Q_{q_1}$ where there is no summation over q_1, and $\Gamma_{pq} = 0$ for $q \neq q_1$. If, for example, $Q_{q_b} \neq 0$, $b = 1,2$, then we may set $\Gamma_{pq_b} = \frac{1}{2} Q_{q_b}^{-1} \sum_{q,j=1}^3 c_{qj}\gamma_{p;qj}$ and $\Gamma_{pq} = 0$ for $q \neq q_b$, $b = 1,2$. If $Q_q \neq 0$ for $q = 1,2,3$, then we may set $\Gamma_{pq} = \frac{1}{3} Q_q^{-1} \sum_{q,j=1}^3 c_{qj}\gamma_{p;qj}$.

A more physical choice of Γ_{pq} is the following one:

$$\Gamma_{pq} := \frac{\overline{Q_q}}{\sum_{m=1}^3 |Q_m|^2} \sum_{b,j=1}^3 c_{bj}\gamma_{p;bj}, \qquad \sum_{m=1}^3 |Q_m|^2 > 0.$$

Corresponding to this choice weights are $\frac{\overline{Q_q}}{\sum_{m=1}^3 |Q_m|^2}$, so that $\sum_{q=1}^3 \Gamma_{pq} Q_q = \sum_{b,j=1}^3 c_{bj}\gamma_{p;bj}$.

A simpler approach to finding $\Gamma = (\Gamma_{pq})$, which automatically leads to a diagonal matrix $\Gamma = \gamma I$ with a number γ and the identity matrix I, is to find γ from the condition $|e_p \int_S \Gamma_{pq}(t) J_q(t) dt - c e_p \int_S J_p(t) dt| = min$, where the minimization is taken over the number c and $|\cdot|$ is the length of a vector. The solution of this minimization problem is $c_{min} := \gamma = \frac{\sum_{p=1}^3 \overline{Q_p} X_p}{\sum_{p=1}^3 |Q_p|^2}$, where $X_p := \int_S \Gamma_{pq}(t) J_q(t) dt$. For this choice of Γ, one has $(I+\Gamma)^{-1} = (1+\gamma)^{-1} I$.

From the computational point of view, it is simpler to use the formula with the diagonal Γ to calculate the number $c_\gamma := (1+\gamma)^{-1}$, and to calculate the Q by the formula

$$Q = -c_\gamma c_D a^3 \nabla \times E_0, \qquad c_\gamma := (1+\gamma)^{-1}.$$

The existence of the constant matrix Γ_{pq} in Equation (2.41) is proved.

To prove the second claim, namely, that the matrix $I + \Gamma$ is non-singular, it is sufficient to prove that $\dim R(I + \Gamma) = 3$, where $R(B)$ is the range of the matrix B. The range of the matrix $I + \Gamma$ consists of the vectors $-c_D a^3 \nabla \times E_0$. Let us check that the range of the set of vectors $\{\nabla \times E_0\}$ equals to 3, $\dim\{\nabla \times E_0\} = 3$, where $E_0 = \mathcal{E} e^{ik\alpha\cdot x}$, $\alpha \cdot \mathcal{E} = 0$, $\alpha \in S^2$, and $\mathcal{E}$ runs through the set of arbitrary constant vectors. Since $\nabla \times E_0 = ik[\alpha, \mathcal{E}]e^{ik\alpha\cdot x}$ and one can obviously choose three pairs of vectors $\mathcal{E}, \alpha$ such that the three vectors $[\alpha, \mathcal{E}]$ are linearly independent and $\alpha \cdot \mathcal{E} = 0$, the second claim is proved.

Since the matrix $I + \Gamma$ is non-singular, Equation (2.40) yields a formula for Q,

$$Q = -c_D a^3 (I + \Gamma)^{-1} \nabla \times E_0. \tag{2.42}$$

Let us formulate the result using the simplified diagonal form of the matrix Γ.

Theorem 2.2. *One has*

$$Q = -c_D a^3 c_\gamma \nabla \times E_0, \quad a \to 0, \qquad c_\gamma := (1 + \gamma)^{-1}. \tag{2.43}$$

To use this result practically one has to solve numerically integral equation (2.13) for J, calculate $Q := \int_S J(t)dt := \sum_{p=1}^3 e_p Q_p$, then calculate $\gamma = \frac{\sum_{p=1}^3 \overline{Q_p} X_p}{\sum_{p=1}^3 |Q_p|^2}$, where $X_p := \int_S \Gamma_{pq}(t) J_q(t)dt$, and then use formula (2.43).

From formulas (2.3), (2.33), and (2.43), one calculates $A(\beta, \alpha, k)$.

B. Many-body scattering problem

Let D_m, $1 \le m \le M = M(a)$ be small perfectly conducting bodies of the characteristic size a, $x_m \in D_m$, $D := \bigcup_{m=1}^M D_m$, $D_m \subset \Omega, D' = \mathbb{R}^3 \setminus D$. Assume that D_m are distributed in a bounded domain Ω according to the formula

$$\mathcal{N}(\Delta) = \frac{1}{a^3} \int_\Delta N(x)dx(1 + o(1)), \quad a \to 0, \tag{2.44}$$

where $\Delta \subset \Omega$ is an arbitrary open subset of Ω, $N(x) \ge 0$ is a continuous in Ω function which can be chosen by the experimenter as he/she wishes. Let us assume that relation (1.1) holds. If Ω is a cube with the size L, then

$$\left(\frac{L}{d}\right)^3 = O(M) = O(\frac{1}{a^3}),$$

so $d = O(La)$. Therefore, condition $d \gg a$ can hold if L is sufficiently large. If L is fixed, then the condition $d \gg a$ can hold if $N \ll 1$, because under this assumption about N one has

$$d = O\left(\frac{a}{(\int_\Omega N(x)dx)^{1/3}}\right) \gg a.$$

The many-body scattering problem consists of solving Equations (2.1) with $D = \bigcup_{m=1}^M D_m$, with boundary conditions (2.2), where $S = \bigcup_{m=1}^M S_m$, and with radiation condition (2.6). The solution to this problem is unique.

We look for the solution of the form

$$E = E_0 + \sum_{m=1}^M \nabla \times \int_{S_m} g(x,t) J_m(t)dt. \tag{2.45}$$

This formula can be written as

$$E = E_0 + \sum_{m=1}^M [\nabla g(x, x_m), Q_m] + f, \quad Q_m := \int_{S_m} J_m(t)dt, \tag{2.46}$$

where

$$f := \sum_{m=1}^{M} \nabla \times \int_{S_m} (g(x,t) - g(x,x_m))J_m(t)dt := \sum_{m=1}^{M} f_m. \tag{2.47}$$

Let us show that for all m one has

$$|f_m| \ll |I_m| := |[\nabla g(x,x_m), Q_m]|, \quad a \to 0. \tag{2.48}$$

If (2.48) holds, then the asymptotically exact solution of the many-body scattering problem is of the form

$$E = E_0 + \sum_{m=1}^{M} [\nabla g(x,x_m), Q_m], \qquad a \to 0. \tag{2.49}$$

This is a basic result: it reduces the solution to the many-body scattering problem to finding quantities Q_m rather than to finding the vector-functions $J_m(t)$. Such a reduction makes it possible to solve the many-body scattering problem for so many particles that it was not possible to do earlier.

Our assumption is $a \ll d \ll \lambda$. Since $k = \frac{2\pi}{\lambda}$, it follows that $a \ll d \ll k^{-1}$.

To check inequality (2.48), note that

$$|\nabla g(x,x_m)| \le O\left((k + d^{-1})\frac{1}{d}\right) = O(\frac{1}{d^2}), \qquad |x - x_m| = d, \tag{2.50}$$

$$|\nabla g(x,t) - \nabla g(x,x_m)| \le O\left(a(k + d^{-1})\frac{1}{d^2}\right), \quad |t - x_m| \le a. \tag{2.51}$$

Thus, $|I_m| = O\left(|Q_m|\frac{1}{d^2}\right)$, $|f_m| \le O\left(|Q_m|a(k + d^{-1})\frac{1}{d^2}\right)$, and $Q_m \neq 0$. Consequently,

$$\left|\frac{f_m}{I_m}\right| \le O\left(ka + ad^{-1}\right) \ll 1. \tag{2.52}$$

Note that our basic physical assumption $a \ll d \ll \lambda$ implies $ka \ll ad^{-1}$ because $k = \frac{2\pi}{\lambda}$, so $k \ll \frac{1}{d}$ and $ka \ll \frac{a}{d}$.

Let us define the notion of the *effective field* E_e acting on the j-th particle

$$E_e := E_0(x) + \sum_{m\neq j}^{M} \nabla \times \int_{S_m} g(x,t)J_m(t)dt. \tag{2.53}$$

As $a \to 0$, the effective field is asymptotically equal to the full field because the radiation from one particle is proportional to $O(a^3)$, see Theorem 2.3 in Section II A.

If

$$ka + \frac{a}{d} \ll 1, \tag{2.54}$$

then, with the error negligible as $a \to 0$, one has

$$E = E_0 + \sum_{m=1}^{M} [\nabla g(x,x_m), Q_m], \tag{2.55}$$

where

$$Q_m = -a^3 c_{D_m} (I + \Gamma)^{-1} \nabla \times E_e(x_m). \tag{2.56}$$

If the quantities $A_m := (I + \Gamma)^{-1}(\nabla \times E_e)(x_m), 1 \le m \le M$ are found, then the solution of the many-body scattering problem for perfectly conducting small bodies of an arbitrary shape can be found by formulas (2.55) and (2.56).

The shape of the small bodies enters only through the constants c_{D_m}, since $|D_m| = c_{D_m} a^3$.

In order to solve many-body scattering problem, one needs to find the quantities A_m. Let us reduce the problem of finding A_m and E_m to solving linear algebraic systems (LASs).

Put $x = x_j, E_0(x_j) := E_{0j}$ in (2.55), assume for simplicity that $c_{D_m} = c_D$ for all m, that is, the small bodies are identical, let $m \neq j$ and get

$$E_j = E_{0j} - c_D \sum_{m \neq j}^{M} [\nabla g(x_j, x_m), A_m]\, a^3, \quad 1 \leq j \leq M. \tag{2.57}$$

There are $2M$ vector unknowns A_m and E_m in this LAS and M equations. One needs another set of M linear equations for finding these unknowns.

To derive these equations, apply the operator $(I+\Gamma)^{-1}\nabla\times$ to Equation (2.55), denote $A_{0j} := (I+\Gamma)^{-1}(\nabla \times E_0)(x_j)$, and set $x = x_j, j \neq m$ in the resulting equation. This yields a LAS,

$$A_j = A_{0j} - c_D a^3 \sum_{m \neq j}^{M} \Big((I+\Gamma)^{-1}\nabla_x \times [\nabla g(x, x_m), A_m]\Big)|_{x=x_j}, \quad 1 \leq j \leq M. \tag{2.58}$$

Formula

$$\nabla_x \times [F(x), A] = (A, \nabla)F(x) - A(\nabla, F(x)),$$

valid if the vector A is independent of x, can be useful.

If M is very large, then the order of LAS (2.57) and (2.58) can be *drastically reduced* by the following method.

Let $\bigcup_{p=1}^{P} \Delta_p$ be a partition of Ω into a union of cubes Δ_p with the side $b = b(a)$, $\lim_{a\to 0} b(a) = 0$. Assume that

$$b \gg d \gg a, \quad \lim_{a \to 0} \frac{d}{b} = 0. \tag{2.59}$$

At the points $x_m \in \Delta_p$, the values of A_m and of $\nabla g(x, x_m)$, where $x \notin \Delta_p$, are asymptotically equal as $a \to 0$. Therefore, Equation (2.58) can be rewritten as

$$A_q = A_{0q} - c_D \sum_{p \neq q} \Big((I+\Gamma)^{-1}\nabla_x \times [\nabla g(x, x_p), A_p]\Big)|_{x=x_q} a^3 \sum_{x_m \in \Delta_p} 1, \tag{2.60}$$

and Equation (2.57) can be transformed similarly. Here, $x_p \in \Delta_p$ is an arbitrary point, $D_m \subset \Delta_p$, $x_m \in \Delta_p$, D_m are small bodies in Δ_p. Since Δ_p is small the quantities A_m, E_m, and $g(x_j, x_m)$ for x_m in Δ_p and $x_j \in \Delta_q$, $p \neq q$, are equal to A_p, E_p, and $g(x_q, x_p)$, respectively, up to the quantities of higher order of smallness as $a \to 0$.

By (2.44), one has

$$a^3 \sum_{x_m \in \Delta_p} 1 = a^3 \mathcal{N}(\Delta_p) = N(x_p)|\Delta_p|, \quad a \to 0, \tag{2.61}$$

where $|\Delta_p|$ is the volume of Δ_p. Thus,

$$E_q = E_{0q} - c_D \sum_{p \neq q} [\nabla g(x_q, x_p), A_p] N(x_p)|\Delta_p|, \quad 1 \leq q \leq P, \quad a \to 0, \tag{2.62}$$

$$A_q = A_{0q} - c_D \sum_{p \neq q}^{P} \Big((I+\Gamma)^{-1}\nabla_x \times [\nabla g(x, x_p), A_p] N(x_p)|\Delta_p|\Big)|_{x=x_q}, \quad 1 \leq q \leq P, \quad a \to 0. \tag{2.63}$$

Equations (2.62) and (2.63) are a LAS for $2P$ unknowns A_q, E_q, $P \ll M$. Computational work can be considerably reduced if one solves first system (2.63) for P unknown vectors A_p and then calculate P unknowns E_p by formula (2.62).

Since $P \ll M$, the order of LAS (2.62) and (2.63) is *much smaller than the order of LAS (2.57) and (2.58).*

A similar argument allows one to replace Equation (2.55) by the following equation:

$$E_{eq} = E_{0q} - c_D\Big(\nabla \times \sum_{p \neq q}^{P} g(x, x_p)((I+\Gamma)^{-1}\nabla \times E_e)(x_p) N(x_p)|\Delta_p|\Big)|_{x=x_q}, \tag{2.64}$$

where the formula $[\nabla g(x), A] = \nabla \times (gA)$ was used. This formula is valid for a scalar function g of x and a vector A, independent of x.

Formula (2.64) is a Riemannian sum for the following *limiting integral equation*:

$$E(x) = E_0(x) - c_D \nabla \times \int_\Omega g(x,y)(I+\Gamma)^{-1}\nabla \times E(y)N(y)dy. \tag{2.65}$$

The method used for the derivation of equation (2.65) in contrast to the usual assumptions of the homogenization theory does not use periodicity assumption and the operator of our problem does not have a discrete spectrum.

Let us state our result.

Theorem 2.3. *If assumptions (2.54) hold, then the unique solution to the many-body scattering problem can be calculated by formula (2.49), where Q_m are given in (2.56) and $(I+\Gamma)^{-1}(\nabla \times E_e)(x_m) := A_m$ and $E_e(x_m) := E_m$ are found from the LAS (2.57) and (2.58). The order of LAS (2.57) and (2.58) can be drastically reduced if assumptions (2.59) hold, and one obtains LAS (2.62) and (2.63) of the order $P << M$. As $a \to 0$, the electric field in the medium tends uniformly to the limit $E(x)$ which satisfies equation (2.65).*

Apply the operator $\nabla \times \nabla\times$ to Equation (2.65) and use the formulas

$$\nabla \times \nabla \times = \nabla\nabla \cdot - \nabla^2, \quad \nabla \cdot E = 0, \quad -\nabla^2 g = k^2 g + \delta(x-y).$$

Assume for simplicity that Γ is a diagonal matrix, $\Gamma := \gamma I$, and let $C_D := \frac{c_D}{1+\gamma}$. Then,

$$\nabla \times \nabla \times E = k^2 E - C_D \nabla \times (N(x)\nabla \times E) = k^2 E - C_D N(x) \nabla \times \nabla \times E - C_D[\nabla N, \nabla \times E]. \tag{2.66}$$

Consequently,

$$\nabla \times \nabla \times E = \frac{k^2 E}{1 + C_D N(x)} - \frac{C_D[\nabla N, \nabla \times E]}{1 + C_D N(x)}. \tag{2.67}$$

It is clear from (2.67) that the refraction coefficient in the medium where many small perfectly conducting particles are distributed is changed: the new refraction coefficient is proportional to $\big(1 + C_D N(x)\big)^{-1}$. The second term on the right-hand side of Equation (2.67) can be interpreted as coming from the new magnetic permeability. Indeed, if $\mu = \mu(x)$ in Maxwell equations, then taking $\nabla\times$ of the first equation and using the second equation, one gets:

$$\nabla \times \nabla \times E = k^2 E + [\frac{\nabla \mu(x)}{\mu(x)}, \nabla \times E].$$

Compare this formula with Equation (2.67) and conclude that $\mu(x) = \big(1 + C_D N(x)\big)^{-1}$.

Since $\nabla \cdot E = 0$, one has $\nabla \times \nabla \times E = \nabla\nabla \cdot E - \nabla^2 E = -\nabla^2 E$, and since $N(x) \geq 0$ is compactly supported, Equation (2.67) is a Schrödinger-type equation with compactly supported potential and the terms with the first derivatives, the coefficients in front of which are compactly supported. The solution of this equation satisfies the radiation condition at infinity.

III. SCATTERING BY ONE IMPEDANCE PARTICLE OF AN ARBITRARY SHAPE

The problem consists of finding the solution to system (2.1), assuming that $E = E_0 + v_E$, E_0 is given in (2.3), the scattered field v_E satisfies radiation condition (2.6), and E satisfies the impedance boundary condition

$$[N,[E,N]] = \zeta[N,H] \quad \text{on } S, \quad Re\zeta \geq 0, \tag{3.1}$$

where ζ is a number, the boundary impedance. We will use condition (3.1) in the form

$$[N,[v_E,N]] - \frac{\zeta}{i\omega\mu}[N, \nabla \times v_E] = -f, \tag{3.2}$$

where

$$f := [N, [E_0, N]] - \frac{\zeta}{i\omega\mu}[N, \nabla \times E_0]. \tag{3.3}$$

Let us look for the solution of the scattering problem with the impedance boundary condition in form (2.9) where $J(t)$ is a tangential field to S.

It was not known if this solution could be represented in form (2.9). We prove in this section that one can find the solution in form (2.9) and the scattered field can be found in form (3.4), see below.

The uniqueness of the solution to the EM wave scattering problem by an impedance body is known (see a proof in Ref. 15). The existence of the solution in the form involving a sum of four boundary integrals was known (see Ref. 2), but such a representation of the solution is not useful for our purposes. We want to give an explicit closed-form formula for the field scattered by a small impedance particle of an arbitrary shape.

The integral equation for J, which one gets by substituting

$$v_E = \nabla \times \int_S g(x,t)J(t)dt \tag{3.4}$$

into boundary condition (3.2), is not of a Fredholm class: it is a singular integral equation.

Our approach to solving the scattering problem for a small impedance particle can be described as follows. We prove that its solution exists and can be represented in form (3.4) by using the general theory of elliptic systems (see Ref. 18) and checking that *the complementing or covering condition, also known as Lopatinsky-Shapiro (LS) condition,* is satisfied (see Ref. 16).

Note that if the solution exists, it can be found in form (3.4). Indeed, one can calculate $[N, e]$ on S, and solve the problem for perfectly conducting particle with the boundary condition $[N, e]$ on S. If $[N, e]$ on S is known, then e is uniquely determined, so the corresponding scattering problem is uniquely solvable and its solution, as follows from Theorem 2.1, can be found in form (3.4).

Next, we prove that asymptotically, as $a \to 0$, the main term in the scattered field is given by the formula

$$v_E = [\nabla g(x, x_1), Q], \qquad a \to 0, \tag{3.5}$$

where $x_1 \in D$ is an arbitrary point inside the small particle D, and

$$Q := \int_S J(t)dt. \tag{3.6}$$

This is an important point: not the function $J(t)$ but just the quantity Q defines main term of the scattered field if the body D is small, $ka \ll 1$. From the physical point of view, solving the scattering problem is reduced to finding vector Q rather than the vector-function $J(t)$. From the numerical point of view, such a reduction makes it possible to solve scattering problems with so many small particles that it was impossible to solve such problems earlier.

Finally, we give, as $a \to 0$, a formula for Q,

$$Q = -\frac{\zeta|S|}{i\omega\mu}\tau_1 \nabla \times E_0, \tag{3.7}$$

see formula (3.40) below, where $\tau_1 := (I + \Gamma)^{-1}\tau$, and τ is defined in formula (3.8).

In formula (3.7), $|S|$ is the surface area of $S := \partial D$, ζ is the boundary impedance (see condition (3.2)), and the tensor τ is defined as follows:

$$\tau_{jp} := \delta_{jp} - b_{jp}, \quad b_{jp} := \frac{1}{|S|}\int_S N_j(t)N_p(t)dt. \tag{3.8}$$

Formulas (3.5), (3.7), and (3.8) solve the EM wave scattering problem for a small impedance body of an arbitrary shape. It follows from formula (3.7) that $Q = O(a^{2-\kappa})$ because $|S| = O(a^2)$ and $\zeta = O(a^{-\kappa})$ as $a \to 0$.

Let us prove these statements. We start with the uniqueness and existence of the solution of the scattering problem with the impedance boundary condition.

Uniqueness of this solution is known (see Ref. 15, p. 81). Let us reduce solving Maxwell's system (2.1) to an equivalent elliptic system for E. If E is found then H is given by the formula

$$H = \frac{\nabla \times E}{i\omega\mu}. \tag{3.9}$$

Assume that $\mu = \text{const}$ in D'. Apply the operator $\nabla\times$ to first equation (2.1) and use second equation (2.1) to get

$$\nabla \times \nabla \times E = k^2 E, \quad \nabla \cdot E = 0, \quad \text{in } D', \tag{3.10}$$

where $k^2 = \omega^2 \epsilon\mu$. Equations (3.10) imply

$$\left(\nabla^2 + k^2\right) E = 0, \qquad \nabla \cdot E = 0 \quad \text{in } D'. \tag{3.11}$$

Since E_0 solves Equations (3.11) in $\mathbb{R}^3$, one concludes that

$$\left(\nabla^2 + k^2\right) v_E = 0, \qquad \nabla \cdot v_E = 0 \quad \text{in } D'. \tag{3.12}$$

Equations (3.12) could be replaced by one elliptic system

$$\left(-\nabla^2 - k^2\right) v_E = 0 \qquad \text{in } D' \tag{3.13}$$

and the boundary condition

$$\nabla \cdot v_E = 0 \qquad \text{on } S. \tag{3.14}$$

Indeed, the function $\psi(x) := \nabla \cdot v_E$ solves the problem

$$\left(\nabla^2 + k^2\right)\psi = 0 \quad \text{in } D', \quad \psi|_S = 0, \tag{3.15}$$

and ψ satisfies radiation condition (2.6). This implies (see Ref. 8, p. 28) that $\psi = 0$ in D'.

Therefore, our scattering problem is reduced to solving elliptic system (3.13) with boundary conditions (3.14) and (3.2) and radiation condition (2.6).

Let $w(x) := (1 + |x|^2)^{-\gamma}$, where $\gamma > \frac{1}{2}$, be a weight function. This weight is chosen so that the functions v_E, that are $O(\frac{1}{|x|})$ as $|x| \to \infty$, belong to $L^2(D', w)$. By $H^2(D', w)$, the weighted Sobolev space is denoted.

Theorem 3.1. *The solution v_E to elliptic system (3.13) with boundary conditions (3.14) and (3.2) and radiation condition (2.6) exists in $H^2(D', w)$, is unique, $v_E = O(\frac{1}{|x|})$ as $|x| \to \infty$, and v_E can be found of form (3.4).*

Proof. Clearly, system (3.13) is elliptic. Let us check that the LS (complementary) condition is satisfied. The principal symbol of operator (3.13) is $\xi^2 \delta_{pq}$, where ξ is the parameter of the Fourier transform

$$u(x) = \int_{\mathbb{R}^3} \tilde{u}(\xi) e^{i\xi \cdot x} d\xi. \tag{3.16}$$

If $D_j := -i\frac{\partial}{\partial x_j}$, then Equation (3.13) can be rewritten as follows:

$$\sum_{j=1}^{3} D_j^2 v_E - k^2 v_E = 0 \quad \text{in } D'. \tag{3.17}$$

Boundary conditions (3.2) and (3.14) can be written in the form

$$B(D) v_E = F, \qquad F := \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{3.18}$$

where $D := (D_1, D_2, D_3)$, f is a two-dimensional vector in the tangential to S plane in the local coordinates, and the zero component in the vector F in formula (3.18) comes from the condition $\nabla \cdot v_E = 0$ on S. The matrix $B(D)$ is defined by one vector boundary condition (3.2) and one scalar boundary condition (3.14). In the local coordinates on S, in which the exterior unit normal N to S

is directed along the z-axis, one has $N = (0,0,1)$, and the principal symbol of the boundary matrix differential operator $B(D)$ is

$$B(\xi) := \frac{\zeta}{i\omega\mu}\begin{pmatrix} -i\xi_3 & 0 & i\xi_1 \\ 0 & -i\xi_3 & i\xi_2 \\ i\xi_1 & i\xi_2 & i\xi_3 \end{pmatrix}. \tag{3.19}$$

The operator D_j is mapped by Fourier transform (3.16) onto ξ_j.

Let $D_t := D_3$. The LS condition holds if the following problem

$$\left(-\frac{d^2}{dt^2} + \rho^2\right) u(\xi_1,\xi_2,t) = 0, \quad t > 0, \quad \rho^2 := \xi_1^2 + \xi_2^2, \tag{3.20}$$

$$B(\xi_1,\xi_2,D_t)u(\xi_1,\xi_2,t)|_{t=0} = 0 \tag{3.21}$$

has only the zero solution, provided that one uses exponentially decreasing, as $t \to \infty$, solution of Equation (3.20), that is, $u = e^{-t\rho}v$, $v = v(\xi_1,\xi_2) = (v_1,v_2,v_3)$.

Therefore, the LS condition holds if and only if the matrix

$$\begin{pmatrix} -i\rho & 0 & i\xi_1 \\ 0 & -i\rho & i\xi_2 \\ i\xi_1 & i\xi_2 & i\rho \end{pmatrix} \tag{3.22}$$

is non-degenerate for $\rho > 0$. The determinant of this matrix equals to

$$-i\rho(\rho^2 + \xi_1^2 + \xi_2^2) \neq 0 \qquad \text{if } \rho > 0. \tag{3.23}$$

Thus, the LS condition holds. This implies the Fredholm property of the corresponding problem in the spaces $H^m(D',w)$ where $w = (1+|x|^2)^{-\gamma}$, $\gamma > \frac{1}{2}$, that is, in the weighted Sobolev spaces with the norm $\|v\|_m^2 := \int_{D'} \sum_{l=0}^m |D^l v|^2 w(x)dx$. The weight w is chosen so that the functions decaying as $O(|x|^{-1})$ at infinity belong to $H^m(D',w)$. Since the LS condition holds, the elliptic estimate holds for the solution to problems (3.13), (3.14), (3.2), and (2.6):

$$\|v_E\|_{m+2} \leq c\left(\|(\nabla^2 + k^2)v_E\|_m + |B(D)v_E|_{m+\frac{1}{2}} + \|\eta v_E\|_0\right) \\ \leq c\left(|f|_{m+\frac{1}{2}} + \|\eta v_E\|_0\right), \tag{3.24}$$

where η is a smooth non-negative cut-off function vanishing near infinity, $\|v\|_m$ is the norm in $H^m(D',w)$, and $|v|_m$ is the norm in the Sobolev space $H^m(S)$ on the boundary S, see Ref. 18.

Due to the uniqueness of the solution to the scattering problem, one can reduce estimate (3.24) to the following estimate:

$$\|v_E\|_{m+2} \leq c|f|_{m+\frac{1}{2}}, \tag{3.25}$$

where $c > 0$ here and below denotes various estimation constants. To prove estimate (3.25), assume that it is false and derive a contradiction. If estimate (3.25) is false, then there is a sequence v_{En}, $\|v_{En}\|_{m+2} = 1$, such that

$$\|v_{En}\|_{m+2} \geq n|f_n|_{m+\frac{1}{2}}. \tag{3.26}$$

Therefore, in any compact subdomain D'' of D', one can select a convergent in $H^l(D'')$, $l < m$, subsequence which we denote again v_{En}. Assume for concreteness that $m = 0$. Then, by the Sobolev embedding theorem, v_{En} converges strongly in $H^l(D'')$ for $l < 2$. Estimate (3.24) implies that

$$\|v_{Ej} - v_{Em}\|_{H^2(D'')} \leq c(|f_j - f_m|_{1/2} + \|\eta(v_{Ej} - v_{Em})\|_{H^0(D'')} \to 0 \text{ as } j,m \to \infty.$$

Thus, v_{En} converges in $H^2(D'')$ to some element v, $\|v\|_2 = 1$. It follows from estimate (3.26) and from the relation $\|v_{En}\|_2 = 1$ that $|f_n|_{\frac{1}{2}} \to 0$ as $n \to \infty$. Let us check that v satisfies the radiation condition. This is done as follows. Denote $v_{En} := v_n$ and write the Green's formula,

$$v_n(x) = \int_S \Big(\frac{\partial g(x,s)}{\partial N} v_n(s) - g(x,s)\frac{\partial v_n}{\partial N}\Big) ds. \tag{3.27}$$

Pass to the limit $n \to \infty$ in this formula. This is possible since, by the Sobolev embedding theorem, the embedding of $H^l(D'')$ into $H^1(S)$ is compact if $l > \frac{3}{2}$ provided that $D'' \subset \mathbb{R}^3$, see Ref. 5. Due to the local convergence in $H^l(D'')$, $\frac{3}{2} < l < 2$, one can pass to the limit $n \to \infty$ in Equation (3.27) and get

$$v(x) = \int_S \Big(\frac{\partial g(x,s)}{\partial N} v(s) - g(x,s)\frac{\partial v}{\partial N}\Big) ds. \tag{3.28}$$

This implies that v satisfies the radiation condition.

Therefore, v solves the homogeneous scattering problem and, by the uniqueness of the solution to this problem, $v = 0$. This contradicts the normalization $\|v\|_2 = 1$, and the contradiction proves estimate (3.25).

The index of our problem is zero.

This follows from the uniqueness of the solution to the homogeneous version of scattering problems (3.13), (3.14), (3.2), and (2.6), see also Lemma 2.1.

Equation (3.2) can be written as

$$v_{E\tau} = \frac{\zeta}{i\omega\mu} V([N, \nabla \times v_{E\tau}]) - V(f), \tag{3.29}$$

where the operator V was introduced in formula (2.31), and $v_{E\tau}$ is the tangential component of v_E. Let us assume that $f \in H^m(S)$. If $v_{E\tau} \in H^m(S)$, then $\nabla \times v_{E\tau} \in H^{m-1}(S)$. Therefore, it follows from Equation (3.29) that $V([N, \nabla \times v_{E\tau}]) \in H^m(S)$. This means that V acts from $H^{m-1}(S)$ into $H^m(S)$. Since the embedding from $H^m(S)$ into $H^{m-1}(S)$ is compact, V is compact in $H^m(S)$.

We have proved the existence of the unique solution to problems (3.13), (3.14), (3.2), and (2.6). This problem is equivalent to scattering problems (2.1), (3.1), (2.5), and (2.6).

Let us prove that if a solution to this scattering problem exists, then the scattered field v_E can be represented in form (3.4).

Let E solve problems (2.1), (3.1), (2.5), and (2.6). The tangential component $[N,[E,N]]$ on S determines uniquely E in D'. There is a one-to-one correspondence between E and v_E, where $v_E = E - E_0$, and v_E satisfies boundary condition (3.2) with f defined in (3.3). The v_E of form (3.4) can be found from equation of type (2.13). Theorem 2.1 guarantees that this equation is solvable for J and the solution is unique. The corresponding v_E, defined by formula (3.4), is the scattered field, and $E = E_0 + v_E$ is the unique solution to scattering problems (2.1), (3.1), (2.5), and (2.6).

Theorem 3.1 is proved. □

Corollary 3.2. The smoothness of v_E is $\frac{3}{2}$ derivatives more than the smoothness of the data f, as follows from estimate (3.24).

Lemma 3.3. Formula (3.5) is asymptotically exact.

Proof. The proof is similar to the proof of formula (2.49). Namely, one has

$$E = E_0 + [\nabla g(x,x_1), Q] + \nabla \times \int_S (g(x,x_1) - g(x,t))\, J(t)dt, \tag{3.30}$$

where $x_1 \in D$ is an arbitrary point and

$$Q := \int_S J(t)dt. \tag{3.31}$$

Note that $g(x,x_1) = O(\frac{1}{d})$, where $d := |x - x_1|$. When one differentiates g, one gets

$$|\nabla g(x,x_1)| = O\Big(\frac{1}{d}(k + \frac{1}{d})\Big), \qquad d := |x - x_1|, \tag{3.32}$$

$$|\nabla (g(x,x_1) - g(x,t))| = O\left((\frac{k}{d} + \frac{1}{d^2})a(k + \frac{1}{d})\right), \quad a = |x_1 - t| \ll d \ll k^{-1}. \tag{3.33}$$

The quantity Q does not vanish. Thus, the ratio of the third to the second term on the right-hand side of Equation (3.30) is of the order

$$O\left(ka + \frac{a}{d}\right) \ll 1. \tag{3.34}$$

Lemma 3.3 is proved. □

Corollary 3.4. Formula (3.5) shows that solving the scattering problem by a small body ($ka \ll 1$) amounts to finding one quantity Q rather than the function $J(t)$ on S.

This is crucial for the solution of the many-body scattering problem that we present in Section IV.

Lemma 3.4. Formula (3.7) holds as $a \to 0$.

Proof. Proof of Lemma 3.4 is based on the following idea: we take the vector product of N_s with Equation (3.2), then integrate the resulting equation over S, and keep the main term as $a \to 0$. If

$$\zeta = \frac{h}{a^\kappa}, \quad \kappa \in [0,1), \quad Re\ h \geq 0, \tag{3.35}$$

then one obtains

$$Q = O\left(a^{2-\kappa}\right), \quad a \to 0. \tag{3.36}$$

Theorem 3.1 gives a mathematical justification of the smoothness of v_E and, therefore, of $J(t)$ provided that the data are smooth, see Corollary 3.2. This result is important for mathematical justification of the boundedness of the second derivatives of the function $J(t)$, which is assumed but not justified on p. 91 in Ref. 15. The estimates, necessary for a justification of formula (3.7), are given on pp. 88-93 in Ref. 15. The term $\int_S ds \int_S dt \nabla_s g(s,t) N(s) \cdot J(t)$ was neglected in Ref. 15. This term depends on a vector whose components are $\int_S \Gamma_{pq}(t) J_q(t) dt$. Here, and below, over the repeated indices, summation is understood and $\Gamma_{pq}(t) := \int_S \frac{\partial g(s,t)}{\partial s_p} N_q(s) ds$, where the integral is understood as a singular integral.

If one takes into account the term

$$\int_S ds \int_S dt \nabla_s g(s,t) N(s) \cdot J(t) = e_p \int_S \Gamma_{pq}(t) J_q(t) dt, \tag{3.37}$$

where $\{e_p\}_{p=1}^3$ is an orthonormal basis of $\mathbb{R}^3$, then in place of Equation (3.7) one obtains the following equations:

$$\int_S J_p(t) dt + \int_S \Gamma_{pq}(t) J_q(t) dt = -\frac{\zeta |S|}{i\omega\mu} (\tau \nabla \times E_0, e_p), \qquad 1 \leq p \leq 3. \tag{3.38}$$

There exists a constant matrix $\Gamma := (\Gamma_{pq})$ such that

$$e_p \int_S \Gamma_{pq}(t) J_q(t) dt = \Gamma Q, \tag{3.39}$$

provided that $Q \neq 0$, which is our case. Equation (3.38) in this case takes the form $(I + \Gamma) Q = -\frac{\zeta |S|}{i\omega\mu} \tau \nabla \times E_0$, and the matrix $I + \Gamma$ is non-singular since $Q \neq 0$. Therefore,

$$Q = -\frac{\zeta |S|}{i\omega\mu} (I + \Gamma)^{-1} \tau \nabla \times E_0. \tag{3.40}$$

Lemma 3.4 is proved. □

The many-body scattering problem is discussed in Sec. IV on the basis of formula (3.7). This is done for simplicity of notations, since formula (3.40) can be identified with formula (3.7) if one replaces τ by $\tau_1 := (I + \Gamma)^{-1} \tau$.

IV. MANY-BODY SCATTERING PROBLEM

This problem consists of finding E and $H = \frac{\nabla\times E}{i\omega\mu}$, which satisfy Equations (2.1) with $D = \cup_{m=1}^{M} D_m \subset \Omega$, E is of form (2.5) and satisfies the impedance boundary conditions on $S_m = \partial D_m$,

$$[N,[E,N]] = \frac{\zeta_m}{i\omega\mu}[N, \nabla\times E] \quad \text{on } S_m; \quad Re\ \zeta_m \geq 0, \tag{4.1}$$

and radiation condition (2.6) for the scattered field v_E. We look for v_E of the form

$$v_E = \sum_{m=1}^{M} \nabla\times \int_{S_m} g(x,t)J_m(t)dt, \quad E = E_0 + v_E, \tag{4.2}$$

where J_m is a tangential to S_m field.

The basic physical (and mathematical) assumptions are (1.1) and (1.3).

The basic results of this section can be described as follows:

1. The above EM wave scattering problem has a solution; this solution is unique and can be found in form (4.2).
2. As $a \to 0$, the main term of the solution to the EM wave scattering problem is

$$E = E_0 + \sum_{m=1}^{M} [\nabla g(x,x_m), Q_m], \quad a\to 0; \quad Q_m := \int_{S_m} J_m(t)dt, \tag{4.3}$$

   where $x_m \in D_m$ are arbitrary points.
3. An explicit, asymptotically exact as $a \to 0$, formula for Q_m is derived,

$$Q_m = -\frac{\zeta_m|S_m|}{i\omega\mu}\tau_m(\nabla\times E_e)(x_m), \quad 1\leq m\leq M, \tag{4.4}$$

   where $|S_m|$ is the surface area of S_m, $\zeta_m = \frac{h(x_m)}{a^\kappa}$, $Re\ h \geq 0$, where $h \in C(\Omega)$ is a function the experimenter may choose as desired as well as the parameter κ, $\kappa \in [0,1)$, τ_m is the tensor defined by formula (3.8) with $S = S_m$, and $E_e(x)$ is *the effective field acting on the particle* D_m,

$$E_e(x) := E_0(x) + \sum_{p\neq m}^{M} \nabla\times \int_{S_p} g(x,t)J_p(t)dt. \tag{4.5}$$

   Equation (4.5) is valid not only in a neighborhood of x_m. The field scattered by m-th particle is proportional to $a^{2-\kappa}$ and is negligible compared with $E_e(x)$ at any point x.
4. Derivation of a LAS for calculating Q_m.
5. Proof of the existence of the limit $E(x)$ of the effective field $E_e(x)$ as $a \to 0$ and the derivation of the equation for the limiting field $E(x)$.
6. Physical interpretation of the equation for the limiting field $E(x)$. Explicit formulas for the new refraction coefficient and magnetic permeability.

The uniqueness and existence of the solution are proved similarly to the proof given in the case of the scattering problem for one body. Formulas (4.3) and (4.4) are established as in our theory of EM wave scattering by one body. An important point is the following one:

Each of the M small bodies can be considered under our basic assumption (1.1) as a single scatterer on which the incident field $E_e(x)$ is scattered. Therefore, formula (3.7) remains valid after replacing E_0 by E_e, and this yields formula (4.4).

Formula (4.3) is derived along the same lines as formula (2.55). If

$$(\nabla\times E_e)(x_m) := A_m, \qquad (\nabla\times E_0)(x_m) := A_{0m}, \qquad \text{and } |S_m| = c_m a^2,$$

then Equations (4.3)-(4.5) imply

$$A_j = A_{0j} - \Big(\nabla\times \sum_{j\neq m}^{M}\Big[\nabla g(x,x_m), \frac{h(x_m)c_m a^{2-\kappa}}{i\omega\mu}\tau_m A_m\Big]\Big)\Big|_{x=x_j}, \quad 1\leq j\leq M. \tag{4.6}$$

This is a LAS for finding A_m. If A_m are found, then

$$Q_m = -\frac{h(x_m)c_m a^{2-\kappa}}{i\omega\mu}\tau_m A_m. \tag{4.7}$$

For simplicity, one may assume in what follows that $c_m = c_0$ and $\tau_m = \tau$ do not depend on m. One can write Equation (4.3) as

$$E_e(x_j) = E_0(x_j) - \frac{c_0 a^{2-\kappa}}{i\omega\mu}\Big(\sum_{j\neq m}^{M}[\nabla g(x,x_m), \tau(\nabla\times E_e)(x_m)]\, h(x_m)\Big)|_{x=x_j}, \quad 1\le j\le M. \tag{4.8}$$

The order of LAS (4.6) and (4.8) can be drastically reduced.

Namely, consider a partition of Ω into a union of small cubes Δ_p, $\cup_{p=1}^{P}\Delta_p = \Omega$. Assume that the side $b = b(a)$ of Δ_p is much larger than d, $b >> d$, so that there are many small bodies D_m in every cube Δ_p, and

$$\lim_{a\to 0} b(a) = 0. \tag{4.9}$$

Recall that $x_m \in D_m$ is a point inside D_m. Let $x_p \in \Delta_p$ be an arbitrary point. For all $x_m \in \Delta_p$, the values $h(x_m) = h(x_p)$ up to the error that tends to zero as $a \to 0$, because h is a continuous function and $b(a) \to 0$ as $a \to 0$. The same is true for $\nabla g(x_j, x_m)$ and for $\tau(\nabla\times E_e)(x_m)$. Consequently, (4.8) implies

$$\begin{aligned} E_e(x_q) &= E_0(x_q) - \frac{c_0}{i\omega\mu}\sum_{q\neq p}^{P}\big[\nabla g(x_q,x_p), \tau(\nabla\times E_e)(x_p)\big]\, h(x_p) a^{2-\kappa}\sum_{x_m\in\Delta_p} 1 \\ &= E_0(x_q) - \frac{c_0}{i\omega\mu}\sum_{q\neq p}^{P}\big[\nabla g(x_q,x_p), \tau(\nabla\times E_e)(x_p)\big]\, h(x_p)N(x_p)|\Delta_p|. \end{aligned} \tag{4.10}$$

Here, we have used assumption (1.3) in the form

$$a^{2-\kappa}\sum_{x_m\in\Delta_p} 1 = \int_{\Delta_p} N(x)dx\big(1+o(1)\big) \approx N(x_p)|\Delta_p|, \tag{4.11}$$

where $|\Delta_p|$ is the volume of Δ_p, diam $\Delta_p \to 0$ as $a \to 0$.

Equation (4.10) is the Riemannian sum corresponding to the integral equation,

$$E(x) = E_0(x) - \frac{c_0}{i\omega\mu}\nabla\times\int_\Omega g(x,y)h(y)N(y)\tau\nabla\times E(y)dy. \tag{4.12}$$

Thus, the effective field E_e has a limit E, as $a \to 0$, and this limit satisfies Equation (4.12). We have proved the following theorem.

Theorem 4.1. *The effective field $E_e(x)$ in Ω tends to the limit $E(x)$ in $C(\Omega)$ and the limiting field $E(x)$ solves Equation (4.12).*

Let us *interpret physically Equation (4.12)*. Let us apply the operator $\nabla\times\nabla\times$ to Equation (4.12). This yields, after using the formulas $\nabla\times\nabla\times = \nabla\nabla\cdot - \nabla^2$ and $\nabla\cdot\nabla\times = 0$, the following equation:

$$\nabla\times\nabla\times E = \nabla\times\nabla\times E_0 - \frac{c_0}{i\omega\mu}\nabla\times\int_\Omega\big(-\nabla^2 g(x,y)\big)\, h(y)N(y)\tau\nabla\times E(y)dy. \tag{4.13}$$

Since $\nabla\times\nabla\times E_0 = k^2 E_0$ and $-\nabla^2 g(x,y) = k^2 g(x,y) + \delta(x-y)$, Equation (4.13) can be written as follows:

$$\nabla\times\nabla\times E = k^2 E - \frac{c_0}{i\omega\mu}\nabla\times\big(h(x)N(x)\tau\nabla\times E(x)\big). \tag{4.14}$$

Assume that τ is a *diagonal* tensor. For example, if D_m are balls, then $\tau_{pq} = \frac{2}{3}\delta_{pq}$, so $\tau = \frac{2}{3}I$, where I is the unit tensor. In this case,

$$\nabla \times \left(hN\tau\nabla \times E\right) = \frac{2}{3}hN\nabla \times \nabla \times E + \frac{2}{3}\left[\nabla(hN), \nabla \times E\right]. \tag{4.15}$$

Therefore, in this case, Equation (4.14) can be rewritten as follows:

$$\nabla \times \nabla \times E = \frac{k^2}{1 + \frac{2c_0}{3i\omega\mu}h(x)N(x)} - \frac{2c_0}{3i\omega\mu}\frac{[\nabla(hN), \nabla \times E]}{1 + \frac{2c_0}{3i\omega\mu}h(x)N(x)}. \tag{4.16}$$

The physical meaning of this equation becomes clear if one applies the operator $\nabla\times$ to first equation (2.1) assuming that $\mu = \mu(x)$, that is, assuming that μ is a function of x.

Then, one gets

$$\nabla \times \nabla \times E = i\omega\mu(x)\nabla \times H + i\omega[\nabla\mu(x), H]. \tag{4.17}$$

Using second equation (2.1), one reduces (4.17) to the following equation:

$$\nabla \times \nabla \times E = k^2 n^2(x)E + \left[\frac{\nabla\mu}{\mu}, \nabla \times E\right], \qquad k^2 := \omega^2\epsilon\mu(x). \tag{4.18}$$

Comparing (4.18) with (4.16), one concludes that the following theorem is proved.

Theorem 4.2. *The refraction coefficient in the new limiting medium is given by the formula*

$$n(x) = \frac{1}{\sqrt{1 + \frac{2c_0}{3i\omega\mu}h(x)N(x)}}, \tag{4.19}$$

and the magnetic permeability in this medium is given by the formula

$$\mu(x) = \frac{\mu}{1 + \frac{2c_0}{3i\omega\mu}h(x)N(x)}, \tag{4.20}$$

where $\mu = const$ is the magnetic permeability in the original medium.

Note that according to formulas (4.16) and (4.18), one has

$$\frac{\nabla\mu(x)}{\mu(x)} = -\frac{2c_0}{3i\omega\mu}\frac{\nabla\left(h(x)N(x)\right)}{1 + \frac{2c_0}{3i\omega\mu}h(x)N(x)}. \tag{4.21}$$

V. CREATING MATERIALS WITH A DESIRED REFRACTION COEFFICIENT AND A DESIRED MAGNETIC PERMEABILITY

Formulas (4.19) and (4.20) allow one to give recipes for creating materials with a desired refraction coefficient or a desired magnetic permeability.

Suppose that one wants to create a material with a desired refraction coefficient $n(x)$ by embedding in a given material many small impedance particles. One has to choose a bounded domain Ω, where the small particles should be distributed, and give a distribution law (1.3) for these particles in Ω. The function $N(x) \geq 0$ in (1.3) can be chosen by the experimenter. Next, one has to give boundary impedances, defined by formula (1.2), where $h(x)$, $Re\ h \geq 0$, is a continuous in Ω function, which can also be chosen by the experimenter as he/she wishes, as well as the parameter $\kappa \in [0, 1)$.

Let us prove the following theorem:

Theorem 5.1. *Any refraction coefficient $n(x)$ can be obtained by choosing a suitable $h(x)$.*

Proof. Suppose that

$$h = h_1(x) + ih_2(x), \quad h_1(x) := Re\, h \geq 0, \quad N(x) = N = const, \quad \frac{2c_0 N}{3\omega\mu} := c_1 > 0.$$

Then, formula (4.19) yields

$$n(x) = \frac{1}{\sqrt{1 - ic_1 h_1(x) + c_1 h_2(x)}}. \tag{5.1}$$

Let us define

$$\sqrt{z} = |z|^{1/2} e^{i\frac{\varphi}{2}}, \quad \varphi = \arg z, \quad 0 \le \varphi \le 2\pi.$$

Since $h_1 \ge 0$ and h_2 are arbitrary real-valued functions, let us denote

$$u(x) := 1 + c_1 h_2(x), \qquad v(x) := c_1 h_1(x),$$

and write

$$\frac{1}{\sqrt{1 + c_1 h_2(x) - ic_1 h_1(x)}} = \frac{1}{\sqrt{u^2 + v^2(x)}} e^{-\frac{i}{2} \arg\left(1 + c_1 h_2(x) - ic_1 h_1(x)\right)}. \tag{5.2}$$

If $|u|$ and $|v|$ are arbitrary, so is $\frac{1}{\sqrt{u^2+v^2}}$. The argument φ of $1 + c_1 h_2(x) - ic_1 h_1 \in (\pi, 2\pi)$ if $h_1 \ge 0$, so $-\frac{\varphi}{2} \in \left(-\frac{\pi}{2}, -\pi\right)$. Choosing u and v suitably one can get a desirable amplitude $\frac{1}{\sqrt{u^2+v^2}}$ of the refraction coefficient and a desirable phase of it.

Theorem 5.1 is proved. □

Example. If $-\frac{\varphi}{2} \approx -\pi$, *then Re* $n(x) < 0$ *and Im* $n(x) < 0$ *can be made as small as one wishes, so it will be negligible. Thus, the obtained material has negative refraction: the phase velocity is directed opposite to the group velocity in this material. Recall that the phase velocity is* $v_p = \frac{\omega}{|k|} \frac{k}{|k|}$, *while the group velocity is* $v_g = \nabla_k \omega(k)$.

Similar reasoning leads to a conclusion that *a desired magnetic permeability can also be created.*

To do this, one uses formula (4.19). Indeed,

$$\mu(x) = \frac{\mu}{u(x) - iv(x)} = \frac{\mu}{\sqrt{u^2(x) + v^2(x)}} e^{-i\varphi}, \quad \varphi \in (\pi, 2\pi). \tag{5.3}$$

The quantity $\frac{1}{\sqrt{u^2(x)+v^2(x)}}$ can be made arbitrary if $h_1(x) \ge 0$ and $h_2(x)$ can be chosen arbitrarily. The argument $\varphi \in (\pi, 2\pi)$ can be chosen arbitrarily.

Remark 5.1. Principal differences of our results and the results of other authors on wave scattering by small bodies are:

1. *For wave scattering by one body: we derive a closed-form explicit formula for the scattering amplitude for small bodies of an arbitrary shape for four types of the boundary conditions (the Dirichlet, the Neumann, the impedance, and the interface (transmission)), see Refs. 11 and 15.*
2. *For many-body wave scattering problems for small bodies of arbitrary shapes our condition* $ka + ad^{-1} \ll 1$ *allows one to have* $kd \ll 1$, *that is, it allows to have many small particles on the wavelength. This means that the effective field in the medium in which many small particles are distributed and the above conditions hold the effective field, acting on each small particle, may differ very much from the incident field. That is, the multiple scattering effects are essential and cannot be neglected.*
3. *For solving problems of many-body wave scattering by small bodies an efficient numerical method is developed.*
4. *For many-body wave scattering problems the limiting equation for the effective field is derived in the limit when the size of small impedance particles tends to zero while the number of these particles tends to infinity.*
5. *A recipe is given for creating materials with a desired refraction coefficient by embedding many small particles with prescribed boundary impedances into a given material.*

[1] H. Ammari and H. Kang, "Reconstruction of small inhomogeneities from boundary measurements," in *Lecture Notes in Mathematics* (Springer-Verlag, Berlin, 2004), Vol. 1846.
[2] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory* (Springer-Verlag, Berlin, 1992).
[3] G. Dassios and R. Kleinman, *Low Frequency Scattering* (Oxford University Press, New York, 2000).
[4] G. Mie, "Beiträge zur Optik trüber Medien, speziell kolloidaler Metallösungen," Ann. Phys. **25**, 377-445 (1908).
[5] L. Kantorovich and G. Akilov, *Functional Analysis* (Pergamon Press, New York, 1982).
[6] M. Lax, "Multiple scattering of waves," Rev. Mod. Phys. **23**, 287-310 (1951).
[7] P. Martin, *Multiple Scattering. Interaction of Time-Harmonic Waves with N Obstacles*, Encyclopedia of Mathematics and Its Applications (Cambridge University Press, Cambridge, 2006), Vol. 107.
[8] A. G. Ramm, *Scattering by Obstacles* (D. Reidel, Dordrecht, 1986).
[9] A. G. Ramm, *Wave Scattering by Small Bodies of Arbitrary Shapes* (World Scientific Publishers, Singapore, 2005).
[10] A. G. Ramm, "Wave scattering by many small bodies and creating materials with a desired refraction coefficient," Afrika Mat. **22**(N1), 33-55 (2011).
[11] A. G. Ramm, "Many-body wave scattering problems in the case of small scatterers," J. Appl. Math. Comput. **41**(N1), 473-500 (2013).
[12] A. G. Ramm, "Wave scattering by many small bodies: Transmission boundary conditions," Rep. Math. Phys. **71**(N3), 279-290 (2013).
[13] A. G. Ramm, "Heat transfer in a medium in which many small particles are embedded," Math. Modell. Nat. Phenom. **8**(N1), 193-199 (2013).
[14] A. G. Ramm, "Scattering of electromagnetic waves by many nano-wires," Mathematics **1**, 89–99 (2013), Open access Journal: http://www.mdpi.com/journal/mathematics.
[15] A. G. Ramm, *Scattering of Acoustic and Electromagnetic Waves by Small Bodies of Arbitrary Shapes. Applications to Creating New Engineered Materials* (Momentum Press, New York, 2013).
[16] A. G. Ramm and M. Schechter, "Existence of the solution to electromagnetic wave scattering problem for an impedance body of an arbitrary shape," Appl. Math. Lett. **41**, 52-55 (2015).
[17] J. W. Strutt (Lord Rayleigh), *Scientific Papers* (Dover, New York, 1964).
[18] M. Schechter, *Modern Methods in Partial Differential Equations* (McGraw-Hill, New York, 1977).
[19] V. Twersky, "Multiple scattering by arbitrary configurations in three dimensions," J. Math. Phys. **3**, 83-91 (1962).