INVESTIGATING VISUAL ATTENTION WHILE SOLVING COLLEGE ALGEBRA PROBLEMS

by

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Abstract

This study utilizes eye-tracking technology as a tool to measure college algebra students' mathematical noticing as defined by Lobato and colleagues (2012). Research in many disciplines has used eye-tracking technology to investigate the differences in visual attention under the assumption that eye movements reflect a person's moment-to-moment cognitive processes. Motivated by the work done by Madsen and colleagues (2012) who found visual differences between those who correctly and incorrectly solve introductory college physics problems, we used eye-tracking to observe the visual attention difference between correct and incorrect solvers of college algebra problems. More specifically, we consider students' visual attention when presented tabular representations of linear functions. We found that in several of the problems analyzed, those who answered the problem correctly spend more time looking at relevant table values of the problem while those who answered the problem incorrectly spend more time looking at irrelevant table labels $x, y, y = f(x)$ of the problem in comparison to the correct solvers. More significantly, we found a noteworthy group of students, who did not move beyond table labels, using these labels solely to solve the problem. Future analyses need to be done to expand on the differences between eye patterns rather than just focusing on dwell time in the relevant and irrelevant areas of a table.

Keywords: Eye tracking, college algebra, students' mathematical noticing, tabular representations

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Chapter 1 - Introduction

Mathematical problems are often overloaded with information, both relevant and irrelevant. Students must sort through the abundance of information, as it is unreasonable to process everything at once, and decide on the relevant information that they will attend to and which they will use to solve the problem. Key research findings have constructed fundamental principles of experts' knowledge and how experts versus novices learn and apply their understanding in areas such as chess, physics, and mathematics. For instance, "experts notice features and meaningful patterns of information that are not noticed by novices" and "experts are able to flexibly retrieve important aspects of their knowledge" (NRC, 2000, p. 31).

Given a simple table of values, as shown below in Figure 1-1, a student may attend to multiple features, both particular features and meaningful patterns. We hope such details as will reveal students' conceptual understanding and misconceptions of linear functions in tabular form. As educators and educational researchers this is important as we move to base on instruction on students' understanding and misconceptions.

Figure 1-1 Table representing a linear function.

x	$y = f(x)$
0	4
1	7
2	10
3	13
4	16
5	19

One may begin observing the table by noticing the table labels denoting the independent and dependent variables and further that the table represents a function labeled by $y = f(x)$. It is important though that one can move quickly from focusing on the table labels to more relevant information of the table, namely the values in each column. Hence one may bring their focus down the column of x-values and see the values increasing by a constant 1 and the corresponding y-values increasing by a constant 3, which proves that the table of values yield a constant rate of change or slope. And one may conclude this table represents a particular function, a linear function. This reasoning puts an emphasis on the noticing of meaningful patterns and understanding of the patterns and what they represent. In this case, the student did attend to particular features such as the table labels but were able to quickly determine what was important to attend to as the values gave more meaningful information within the table.

Another observation may move one's eye patterns across the two columns and see the pattern in the additive differences between x-values and y-values, i.e. 4, 6, 8, 10, 12, 14. Or one may try to find a multiplicative relationship between the x-values and y-values, i.e. $1\times7 = 7$, $2\times5 = 10$, $4\times4 = 16$. These examples of reasoning may give light to a student's pattern recognition but the student may lack the conceptual understanding needed to determine which patterns are relevant in determining a function.

Some students may not notice any patterns in the table. A student may focus on the point (0, 4) located in the first row of values, as this point serves an importance as the y-intercept. Then the student may find the slope by using the formula $slope = \frac{y_2 - y_1}{x_2 - x_1}$ and the first two points given, (0, 4) = (x_1, y_1) and (1, 7) = (x_2, y_2). Therefore, the slope would be $\frac{7-4}{1-0} = \frac{3}{1} = 3$ and we have the linear equation $y = 3x + 4$. This reasoning is focusing on the particulars given in the table but is ignoring the rest of the information given, that is the subsequent rows of values. Although the student was able to correctly find the equation of the line, does the student show a deeper understanding of linear functions, i.e. that the slope is constant for any two ordered pairs presented in a table of values? These are the question that we may unfold when analyzing students' visual noticing patterns.

Through these examples, we overwhelmingly see how a student's level of understanding of linear functions will determine the information the student deems relevant as they make sense of the table. Attending to or noticing these certain features in mathematical problems is defined as (Lobato, J., Hohensee, C., & Rhodenhamel, B., 2014) "selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for one's attention" (p. 809). We can measure what students are noticing by analyzing their verbal responses, gestures, and eye movements. By examining students' mathematical noticing, we can determine whether or not students have even selected the relevant features presented in a problem statement, thus revealing their level of understanding of the mathematical concepts.

Both educators and educational researchers are commonly interested in understanding the students' thought processes during problem solving. The majority of these processes are not visible and furthermore, may be difficult to infer. We may listen to the students, observe as they write down their work, observe closely to their gestures, and even ask the students leading questions to gain a deeper understanding of their thinking. But there is a limit as to what we can accurately understand about a student's processes as the student gains meaning to the problem. In our study, we will focus on using students' eye movements as our primary source of data to measure visual attention and noticing. This will allow us to look much more closely into the mind of the student. In addition, we will use students' think aloud process to illuminate the students' processes and reasoning of what they are attending to in the problems and why they chose to attend to those features and patterns.

In this chapter, we will give a brief overview of eye movements and visual attention and how they relate to cognitive processing. We will then discuss our theoretical framework, students' mathematical noticing and APOS theory. Finally we will discuss the motivation for the study and our research questions.

Relevant Literature

Eye Movements and Visual Attention

Recording eye movements is a recent method used in cognitive psychology and educational research as a window into the cognitive processes (Ball, Lucas, Miles, & Gale, 2003; Deubel & Schneider, 1996; Merkley, & Ansari, 2010; Rayner, 2009; Stephen, Boncoddo, Magnuson, & Dixon, 2009; Thomas, & Lleras, 2007). Eye movements consist of a series of fixations (i.e. when eyes are stationary) and saccades (i.e. rapid eye movements between fixations). We track and analyze the locations, durations, and order of the saccades and fixations to assist us in perceiving the participants' cognitive processes. The theoretical justification of the connection between eye movements and cognitive processing is the "eye mind hypothesis" (Just & Carpenter, 1976, 1980; Rayer, 1998). The hypothesis assumes that there is a relationship between eye fixation and attention location. Furthermore, Just and Carpenter (1980) showed that in reading, where one fixates and the duration of fixation has a strong positive correlation to cognitive processing and cognitive processing demands, respectively. It has also been shown that more time is spent fixating on areas deemed as relevant than those believed to be irrelevant by the participant (Kaakinen, Hyönä, & Keenan, 2002; Liversedge, Paterson, & Pickering, 1998).

Although eye movements give a good indication to what students are attending to, the eye movements still do not give us information as to why they are attending to particular features or about their success or failure of their attempt to process the information. Therefore, along with the eye movements it is important to use think aloud interviews to help give light to the students' learning processes. The think aloud protocol (Kuusela & Paul, 2000) asks participants to "think aloud" by stating where they are looking and why they are looking there, and what they are thinking and feeling during the problem solving process.

Research on Visual Attention in Mathematics Education

Eye movements have been used extensively in context of educational research; the majority of which comes from the domain of reading (Rayner, 1998, 2009). More recently, eye-tracking has become an extensive instrument in multimedia learning research (Van Gog & Scheiter, 2010). In the domain of mathematics education, the use of eye tracking was used to compare expert and novice attentional differences in various realms of mathematics.

For instance, a study was done to compare expert and novice differences attending to university level multi-representational mathematics, which included both graphs and formulas (Andrà, et al., 2013). In this exploratory study, the students were given symbolic or graphical representations (along with text) of a set of functions and asked to determine, without paper and pencil, which one of the representations corresponds to the rule (e.g. $x < y$). The results show on average, in reading formulas students have a smaller number of longer fixations than those in graphs, which present a bigger number of shorter fixations but the overall dwell time was higher for graphs.

In a later study, Shvarts and Cumachenko (2013) studied expert vs. novice behavior when the students were given competing mathematical representations, i.e. both formulas and graphs, during a learning environment. Students then were asked how to represent the mathematical concept and eye-tracking was used to find which areas were more attended to and presumed most useful for the students. The authors found that during the learning stages, novices spend a longer time looking at graphs while experts spend more time on formulas and that there was a predominance of saccades between different representations in expert behavior. Mirroring Andrà and colleagues' results, the authors attributed the longer dwell time spent on graphs to the larger amount of information presented in the graphical representation. They observed that the experts were able to make decisions as to what was necessary information from the graph, while the novices were not, therefore reducing the total dwell time attending to the graph.

Furthermore, Iglis and Alcock (2012) compared proof validation behavior between beginning undergraduate students (novices) and research active mathematicians (experts). Overall the authors confirmed that experts validate proofs in a significantly different manner from novices. For instance, they found that novices spend more time focusing on "surface features" of arguments, such as algebraic manipulations, and spend less time attending to the logical structure of the proof. And mathematicians appear to spend more time inferring implicit between-line statements as their saccades showed them bouncing between lines more frequently.

It is important to remark that the extent of these prior studies have focused on comparing time spent attending to different representations (graphical, tabular, symbolic) and whether expert or novices attend to either/or representations. Our hope is to focus on tabular and graphical representations and understand students' visual patterns within a tabular or graphical representation.

Students' Mathematical Noticing

Defined by Lobato, Hohensee, and Rhodenhamel (2012), *mathematical noticing* refers to "selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for students' attention" (p. 438). The idea of noticing draws upon research in cognitive psychology. Cognitive science and neuroscience identifies three types of attention: alerting (i.e. preparing for a sensory signal), orientating (i.e. turning towards a sensory signal), and executive attention (Fan, McCandliss, Sommer, Raz, & Posner, 2002; Posner & Peterson, 1990). Noticing is similar to *executive attention* as it refers to the process of selecting pieces of information when competing sources of information are present.

Mathematical noticing further extends the notion of executive attention by basing it in the learning process of *reflective abstraction* (Campbell, 2001). Reflective abstraction is rooted in Piaget's work in cognitive development. As a learner attempts to understand new information, they go through the process of reflective abstraction as they construct meaning and form mathematical generalizations. Noticing therefore can be captured in these beginning stages as students begin to make sense of a mathematical problem by noticing features and patterns, which will lead the student to make meaning of the problem. Furthermore, by connecting noticing to reflective abstraction, Lobato and colleagues (2014) showed "that what one notices mathematically can serve as the rootstock upon which one constructs ways to reason in new situations" (p. 812). Therefore, we can connect mathematical noticing to learning.

APOS Theory

In efforts to characterize students' cognitive learning processes in mathematics, we draw upon APOS Theory, a theory of learning specific to university level mathematics education. APOS Theory states that students build mathematical concepts by constructing mental actions, processes, and objects, and organizing them into schemas to make sense of the situations and to solve problems (Asiala et al., 1996). APOS Theory is based on Piaget's idea of relative abstraction, as extended to advanced mathematics primarily by Dubinsky (1991a, 1991b).

In relation to understanding functions in college mathematics courses, the first level of understanding is that of action. A student in this level assumes a function is tied to specific rule or formula, which the answer depends on by manipulation of variables or replacing by numbers for calculations. Having a process conception of function, assumes a function is an input-output machine and is independent of the formula. An object is constructed from a process when the student becomes aware transformations, multiple representations, and properties, i.e. understanding the concept completely. Finally, a schema is developed from collection actions, processes, and objects across concepts, and thus building a framework, which will assist in problem solving in novel situations.

Multiple Representations of Linear Equations

The literature stemming from Dubinsky's APOS theory, suggests that students' experiences with multiple forms of representations (i.e. graphs, tables, and symbolic representations) help build the schema needed to develop the whole picture of mathematical relations (Eisenberg, 1992). The term mathematical representation refers to "any system of information or objects whose relationship with the mathematical domain or idea they reflect is established through shared mathematical conventions" (Adu-Gyamfi and Bossé, 2013). Particularly concerning instruction in the domain of functions, we represent and communicate the concept using words, symbols, graphs, and tables, although the representations may not be treated equally or the connections between them may not be made explicit.   Keller and Hirsch (1998) identified three factors that influence a student's preference of representation in solving mathematical problems: (a) the nature of students' experiences with each representation; (b) the students' perceptions of the acceptability of using a representation; and (c) the level of the task.

When examining high-school student actions, interpretations, and speech with respect to questions raised regarding tabular, graphical, and algebraic representations of functions, Adu-

Gyamfi and Bossé (2013) found that students notably struggled more on items that asked to identify linear functions given a tabular representation. Furthermore, when asked about the rate of change or slope, select students were able to correctly find a numeral value from the symbolic (algebraic equation), graphical and tabular representations but the graphical and tabular representations were never used as a basis for arguing linearity. The authors hypothesize that the reason may be from lack of instruction where rate of change is taught as a concept but rather instruction, which emphasizes rate of change as procedure without attaching meaning.

A study done by Lobato, Hohensee, and Rhodehamel (2013) also analyzed how students solved problems concerning linear functions in tabular form. The concept of noticing measured by students' speech and gestures was used to analyze the students' reasoning. The researchers found that students in one class first noticed two uncoordinated quantities (x and y columns) and then noticed coordinated additive number patterns in the table, which finally led to noticing coordinated quantities that persevered a multiplicative relationship. Students in another class had different noticing patterns, as they focused noticing additive growth of just a single quantity (either the y-values or x-values). The author's framework and theoretical basis provided explanations between these two noticing patterns. They were able to attribute these differences to the focusing interactions of the classroom, features of the mathematical tasks that were given, and the nature of the mathematical instruction; therefore students' mathematical noticing patterns are socially-situated. This reaffirms Adu-Gyamfi and Bossé's suggestion that instruction (or lack of instruction) develops students' conceptions of concepts and mathematical representations. Stemming from this research, we wish to focus on tabular and graphical representations of linear functions, which are focused on less in college algebra instruction, and develop a position of noticing with regard to the underlying concept of linear functions in those representations.

Motivation

This study was motivated by prior research in cognitive psychology and physics education research on visual attention and problem solving. Our goal was to extend the results of Madsen (2004) and Madsen, Larson, Loschky, and Rebello (2012) who found differences in visual attention of those who correctly and incorrectly solve introductory physics problems given in visual diagrams. Madsen, et al. (2012), found that those who answered correctly spent a higher percentage of total viewing time fixating on "thematically relevant areas" in the problem diagram.

We wished to use eye-tracking methods to show similar results in differences in solving introductory college algebra problems. Our focus was on introductory algebra problems given in tabular and graphical representations, which can be solved conceptually, i.e. visualizing the constant rate of change. This was greatly motivated by Adu-Gyamfi and Bossé's (2013) research, which showed differences in student reasoning and correctness when students were given graphical and tabular representations of a linear function.

This project was a part of our long-term goal to assist students in an online constructive learning environment of college mathematics courses. It is important though that we first understand which features college algebra solvers are observing and what elements they are noticing when problem solving. We will then be able to use this information to design visual cues that redirect students' attention to the relevant features and meaningful patterns of the problem either in the table or graph. Based on Lobato, Hohensee, and Rhodehamel (2013), the classroom now becomes the online learning environment where various mathematical tasks along with cues will be used to construct connections in the students' internal representations of the problem. We can hypothesize that this will have an effect on student's noticing patterns within the different representations.

Research Questions

Our overarching research goal was to explore the role of visual attention and mathematical noticing in college algebra conceptual problem solving. More specifically we want to answer the following research question:

1. Does visual attention differ between those who correctly and incorrectly solve introductory college algebra problems covering the topic of linear functions given in tabular and graphical representations?
    a. Do correct solvers spend more time noticing and attending to relevant features and patterns in a table of values?
    b. Do incorrect solvers spend more time noticing and attending to irrelevant features in the table of values?

Hypotheses

Our hope is to show a difference between incorrect and correct solvers' visual attention. Based on previous in mathematics and physics education, we suggest that there will be a difference and our study will investigate what that difference entails.

Furthermore, prior research shows there is a difference between experts and novices visual attention and that experts tend to notice and attend to meaningful patterns while novices do not notice meaningful patterns, and rather attend to particular features. We will assume that these particular features could be relevant or irrelevant features of the problems. Following Madsen, et. al. (2012), we hypothesize that those who answer correctly will spend a higher percentage of viewing time attending to relevant features and lower percentage of viewing time attending to irrelevant features. Conversely, those who answer incorrectly will spend a lower percentage of viewing time attending to irrelevant features and a higher percentage viewing time attending to irrelevant features.

Chapter 2 - Methodology

Participants

The participants were 23 students (14 females, 9 males) enrolled in Math 100 College Algebra during Summer 2014 semester at Kansas State University, Manhattan, Kansas. Due to the practical constraints, we used convenience sampling to recruit the college algebra students with a desired sample size of 20. Students participated in the study as an option for course credit. Participants were considered homogenous because they all were taking the course at the same time and the data was collected within a week. All the problems presented to the participants were previously covered in the course during the 1^{st} week of the semester. Data was collected in the 7^{th} week of the semester (out of a 8 week course).

Materials

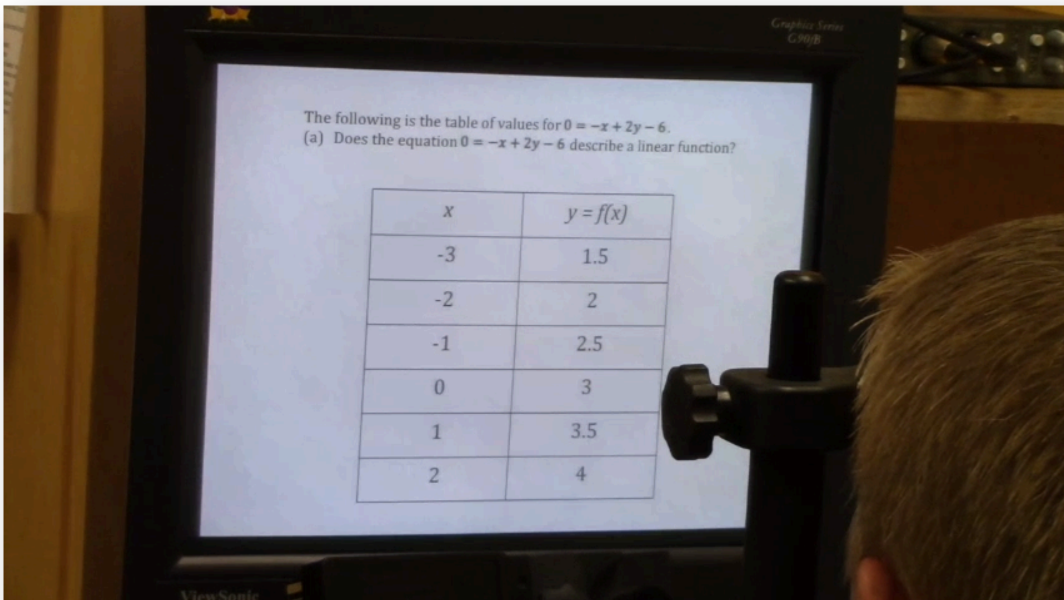
The materials consisted of twenty-one, visual-type algebra problems concerning functions and linear functions (See Appendix A for a list of problems). The problems were designed to be solved mentally, without paper or pencil, using the given table or graph. These problems were chosen based on prior experience and the literature on solving problems concerning linear functions given multi-representations.

Eye-Tracking Technology

For our study, eye movements were recorded with an EyeLink 1000 desktop mounted eye-tracking system (http://www.sr-research.com), which is accurate to less than 0.50° of visual angle. Participants were presented with algebra problems on a computer screen viewed at a distance of 24 inches using a chin and forehead rest to minimize participants' extraneous head movements. The eye tracker, forehead rest, and computer monitor are pictured in Figure 2-1. The resolution of the computer screen was set to 1024 x 768 pixels with a refresh rate of 85 Hz.

Each algebra problem subtended 33.3° x 25.5° of visual angle. An eye movement was classified as a saccade (i.e., in motion) if the eye's acceleration exceeded 8,500°/s^2 and the velocity exceeded 30°/s. Otherwise, the eye was considered to be in a fixation (i.e., stationary at a specific spatial location). A nine-point calibration and validation procedure was used at the beginning of the experiment.

Figure 2-1 Participant viewing computer screen with head in forehead rest and eye movements being recorded with Eye Link 1000 desktop eye tracker.



Procedure

The design of the study was observational casual-comparative research, as we are studying the relationship between visual attention (measured as the percentage of time spent on the problem) as the continuous dependent variable, and the correctness of answer as the categorical independent variable. A possible extraneous variable is the factor of learning that progresses as students are given similar types of problems with different graphs and tables. Viewing and solving similar problems may lead the students to look at the relevant features.

Therefore, to control for this sequencing effect, we randomized the order of the problems. Participants were randomly assigned to 3 different conditions in which they were given a different sequence of the randomized problems. We were then able to compare matching conditions controlling for the sequencing effect of learning.

Each participant took part in an individual interview session, which was between 20 and 40 minutes long. At the beginning of the session, participants were given a short explanation of the goal of the interview and eye tracking system and what to expect from the study. Furthermore, they were instructed to verbalize their thought process and explain their reasoning process as they answered each question. They were told they might be asked additional clarifying questions during their explanations. They were further told that they will not need a paper, pencil, or calculator to solve the problems and that they may give answers in a simplified form. We wanted students to focus on the concepts rather than the calculations. (See Appendix B for detailed study protocol.)

Participants were first given a vision test using the Freiburg Visual Acuity and Contrast Test (FrACT) (http://www.michaelbach.de/fract/index.html). Following the vision test, we calibrated and validated the eye tracking system to one eye (usually the right eye). If the validation's mean error was $\leq 0.50°$ of visual angle and maximum error was $\leq 1.00^{o}$ of visual angle, the experiment began, otherwise the calibration and validation was repeated until successful.

To illustrate the think aloud process, two training problems were given. The problems were basic algebra problems not similar to the problem set during the interview but gave students the familiarity of reading a table and graph (See Appendix A for the training problems). First, the experimenter went through the think aloud process solving the first training problem.

Participants were given time at this point to ask any questions and then they followed with the second training problem. Recording with a flip video camera started after finishing the training problems and student questions/concerns were addressed.

During the interview, participants were given one problem at a time. They were cued to continue to talk as they came to their conclusions. After stating their answer, each participant was asked to provide their reasoning. If a participant's explanation was unclear, they were asked follow-up questions. Participants were given unlimited time to answer the problems and provide their verbal reasoning. After stating their answer and reasoning, the experimenter recorded the response correct or incorrect. To be scored as correct, both the correct answer and correct reasoning must be given. Participants were not given any feedback from the experimenter as to if their answer was correct or incorrect. Between problems, a calibration drift correction procedure was done to ensure proper calibration throughout the experiment. This procedure required the participant to fixate on a small white dot in the middle of a gray screen and press a key. Pressing the key caused the screen to advance to the next problem when the participant's fixation was within a pre-defined area around the white dot. After the interview, participants were debriefed on the solutions to the problems and were thanked for their participation.

Chapter 3 - Analysis

Analysis of Interviews

Through the course of the interviews, we realized that a portion of the problems given were either too hard (i.e. the majority of the participants did not know how to approach the problem) or too easy. The purpose of the interviews was to determine what type of patterns and features of the problems where attended to by the incorrect and correct problem solvers. Therefore, to be able to compare the differences between incorrect and correct solvers, we decided to focus on a set of six problems that showed the greatest differences in the interviews. The problems we decided to focus on addressed linear functions and slope when given a table of values (Problems 4, 5, 6, 7, 8, 9ab) given in Figure 3-1.

Figure 3-1 Set of problems 4, 5, 6, 7, 8, 9ab

Problem 4:
Does the table of values represent a linear function?

x	0	1	2	3	4	5
$y = f(x)$	4	7	10	13	16	19

Problem 5:
Does the table of values represent a linear function?

x	-3	-2	-1	0	1	2
$y = f(x)$	9	4	1	0	1	4

Problem 6:
Does the table of values represent a linear function?

x	-5	-3	-2	0	3	4
$y = f(x)$	6	2	0	-4	-10	-12

Problem 7:
The table of values represents a linear function. What is the slope of the line?

x	0	2	4	6	8	10
$y = f(x)$	0	6	12	18	24	30

Problem 8:
The table of values represents a linear function. What is the slope of the line?

x	-4	-2	0	2	4	6
$y = f(x)$	12	12	12	12	12	12

Problem 9:
The table of values represents a linear function. (a) What is the slope of the line? (b) Find another point on the line?

x	-5	-2	-1	3	5	9
$y = f(x)$	16	10	8	0	-4	-12

During the interviews, the participants would think aloud, stating their process of reasoning as they worked through the problem toward their final response. Their final answer and reasoning for their answer was also recorded. Below we composed a table of the similar types of answers and final reasonings for the problems we are focusing on. A small percentage of students were unable to answer the questions giving an 'I don't know' (IDK) response. (For the complete reasoning on each problem for each student, see Appendix C.)

For each answer and reasoning, a process of inductive coding (Johnson & Christensen, 2014), for which the codes were generated by directly examining the data during the coding process, was used. Common themes in reasoning stated by students were characterized and labeled, using the terminology from the students' reasoning. My descriptive codes (i.e. constant slope, visualize the graph, etc.) are means of describing the reasoning used by those students in their final answers. This process was repeated to gain intracoder reliability, by which the codes were further refined.

Table 3-1 Number of participants providing each answer and reasoning on given problem.
*An asterisk denotes that students may have used the same type of reasoning but had errors either with their arithmetic or procedure therefore had a different final answer.

Problem Description and Question	Answer	Reasoning	Number of Participants (23 total)
4. Does the table of values represent a linear function? *(Table represents the line y = 3x + 4)*	Yes	Constant slope	8
	Yes	Visualize the graph	6
	*No, IDK	Focus on the equation y = f(x)	5
	Yes	Each input has exactly one output	3
	Yes	Multiplicative relationship between x and y values	1
5. Does the table of values represent a linear function? *(Table represents the*	No	Inconstant slope	1
	*No, Yes	Visualize the graph	9
	Yes, IDK	Focus on the equation y = f(x)	5

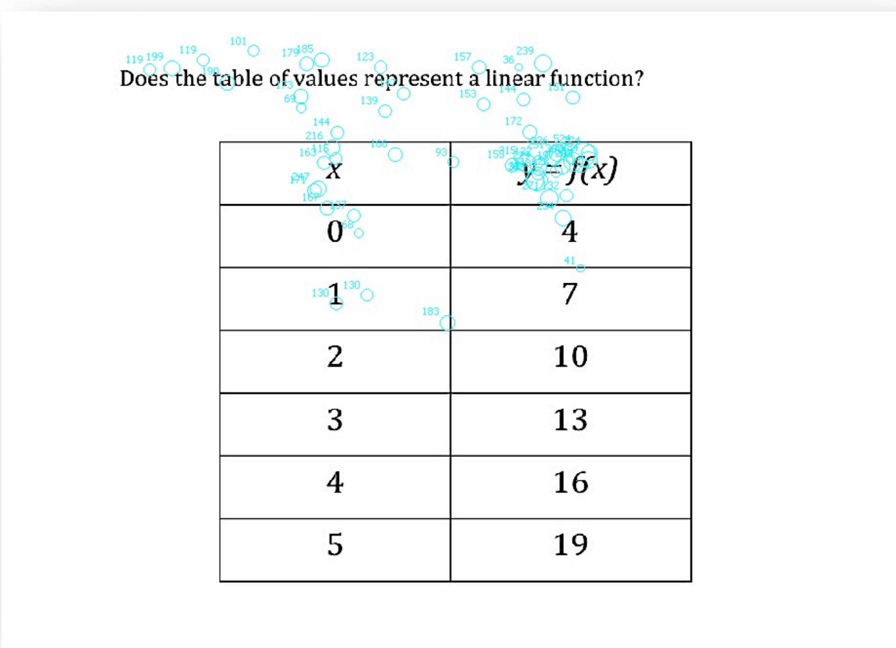
quadratic $y = x^2$)	No	Two different inputs give you the same output	3
	No	Relationship between x and y values ($y = x^2$)	3
	IDK	I don't know.	2
6. Does the table of values represent a linear function? *(Table represents the line $y = -2x - 4$)*	Yes	Constant slope	2
	No	Inconstant slope	1
	*Yes, No	Visualize the graph	7
	*Yes, No, IDK	Focus on the equation $y = f(x)$	5
	Yes	Each input has exactly one output	4
	Yes	Multiplicative relationship between x and y values	1
	Yes	All the y-values are even	1
	IDK	I don't know.	2
7. The table of values represents a linear function. What is the slope of the line? *(Table represents the line $y = 3x$)*	*3, 2	Constant slope	3
	*3, 1/3	Slope formula using 2 points	10
	3	Focus on the equation $y = f(x)$	2
	*3, 6	Multiplicative relationship between x and y values ($y = 3x$)	4
	2, None	Focus on the x-values	2
	6	Focus on the y-values	1
	None	Additive relationship between x and y values (changing)	1
8. The table of values represents a linear function. What is the slope of the line? *(Table represents the line $y = 12$)*	0	Visualize the graph	5
	*+/- 3, None	Multiplicative relationship between x and y values	3
	None	Not a linear function	4
	*+/- 3, 0, 16/14	Slope formula	4
	*-3, 14, 1, None	Focus on the equation	4
	*2, None	Focus on the x-values	2
	None	Focus on the y-values	1
9. What is the slope of the line? *(Table represents the line $y = -2x + 6$)*	*+/-2, -3	Slope formula using 2 points	13
	*16/-5, None	Multiplicative relationship between x and y values	2
	None	Focus on the equation	4
	*3, None	Focus on the x-values	3
	4	Focus on the y-values	1

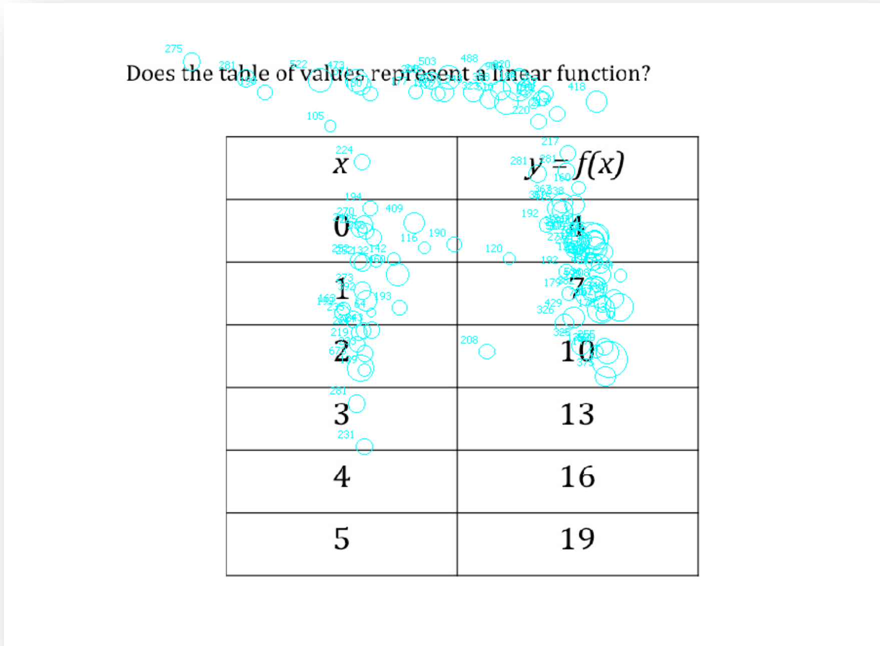
Area of Interest Analysis

An "area of interest" (AOI) analysis was used primarily in this study. In this analysis of eye movements, areas of the algebra problems were specified, for example, the area relevant or irrelevant to solving the problem. These areas were determined a posteriori based on the research questions guiding the study and the participant interviews. Then the amount of time (fixation dwell time) each participant spent in each AOI was determined from the eye movement records and transformed into a useful metric, percentage of total time viewing the diagram.

In figure 3-2, we see the fixations recorded for two different students. Each circle represents a fixation with larger circles indicating a longer fixation. Both students spend time reading the question but the first student has more fixations focused on the table labels *x* and $y = f(x)$, particularly the student was fixated on the $y = f(x)$. The second student briefly looked at the table labels and then moved on to focus on the table values. These fixations are recorded as the dwell time spent in the areas determined by the AOI. We already begin to see a difference between the students that spend more time in the table labels and more time in the table values.

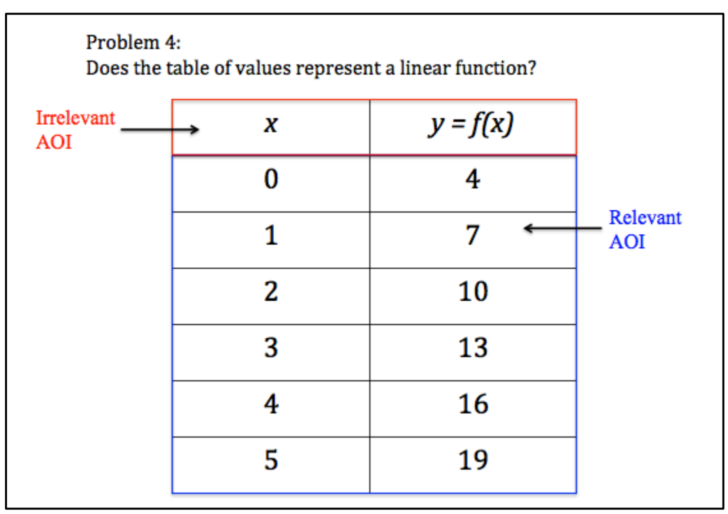
Figure 3-2 Fixations recorded by the eye tracker for two different students. Each circle represents a fixation with larger circles indicting a longer fixation.





During the interviews, there was a large difference between participants who knew where to look and those who did not (i.e. those who knew the relevant information in the table and those who focused on the table labels y = f(x)), as we can visually see in figure 3-2. Therefore we defined the relevant AOI as the table values and irrelevant AOI as the table labels.

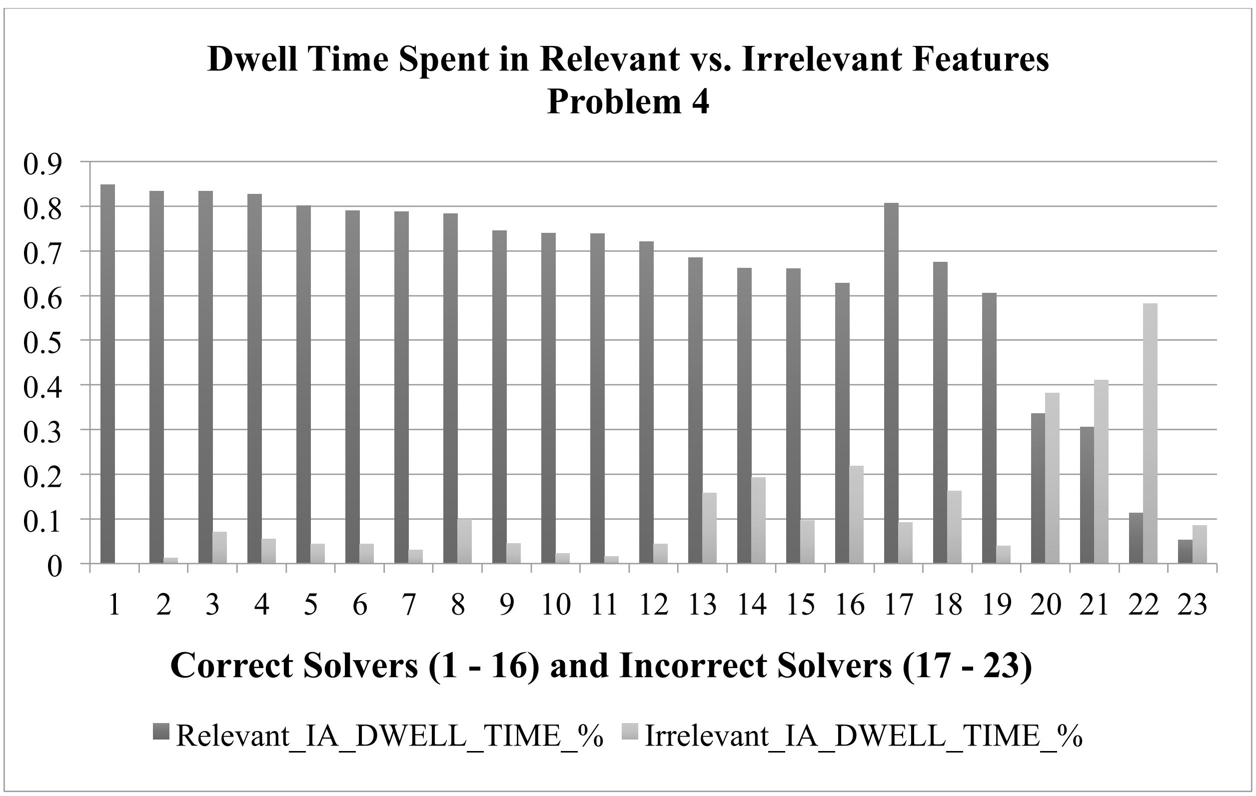
Figure 3-3 Area of Interest (AOIs) defined as irrelevant AOI in red and relevant AOI in blue.

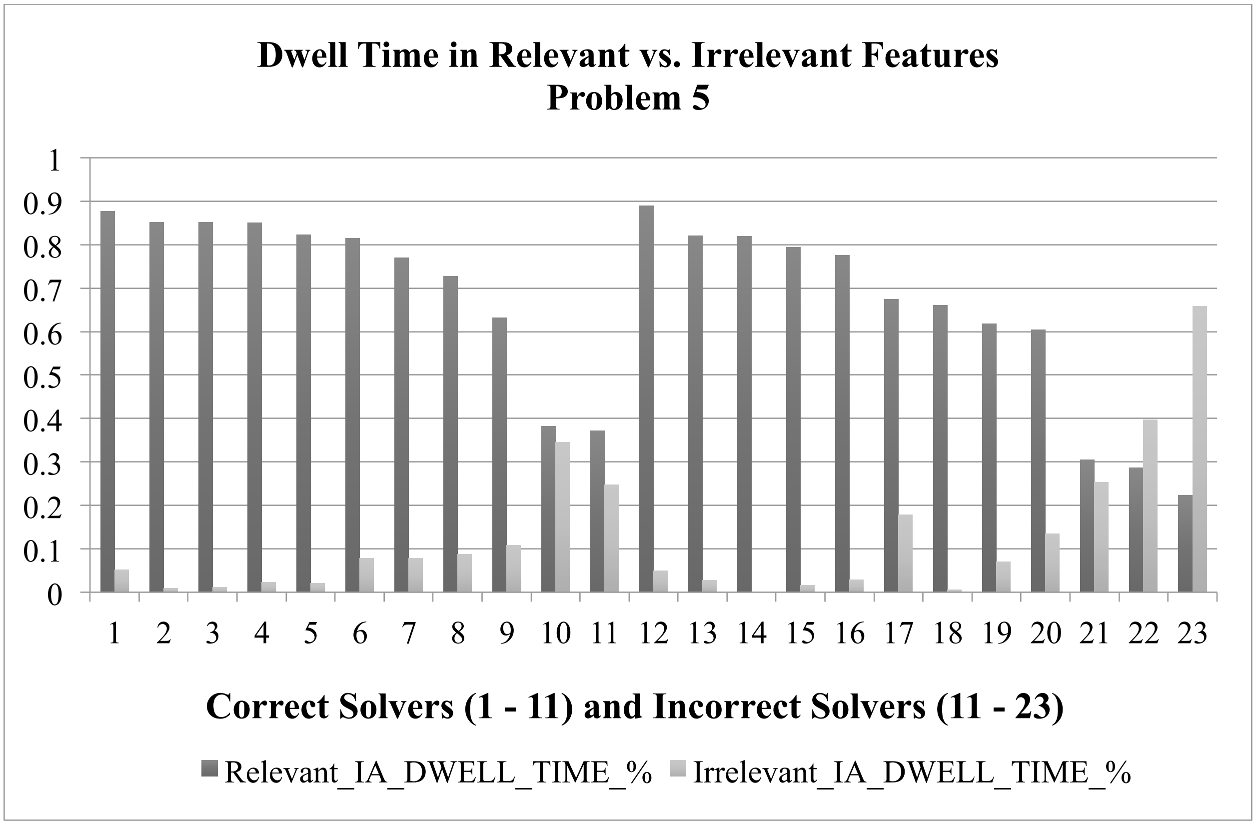


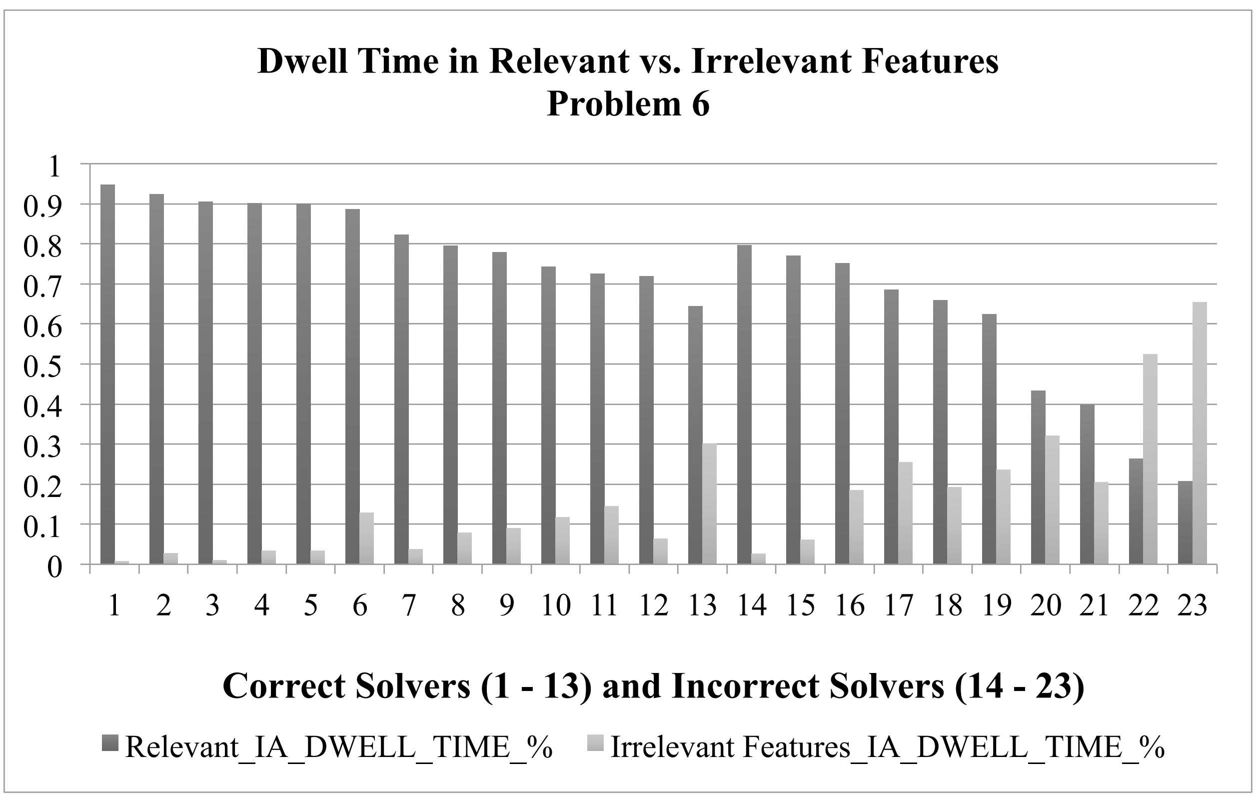
Statistical Analysis

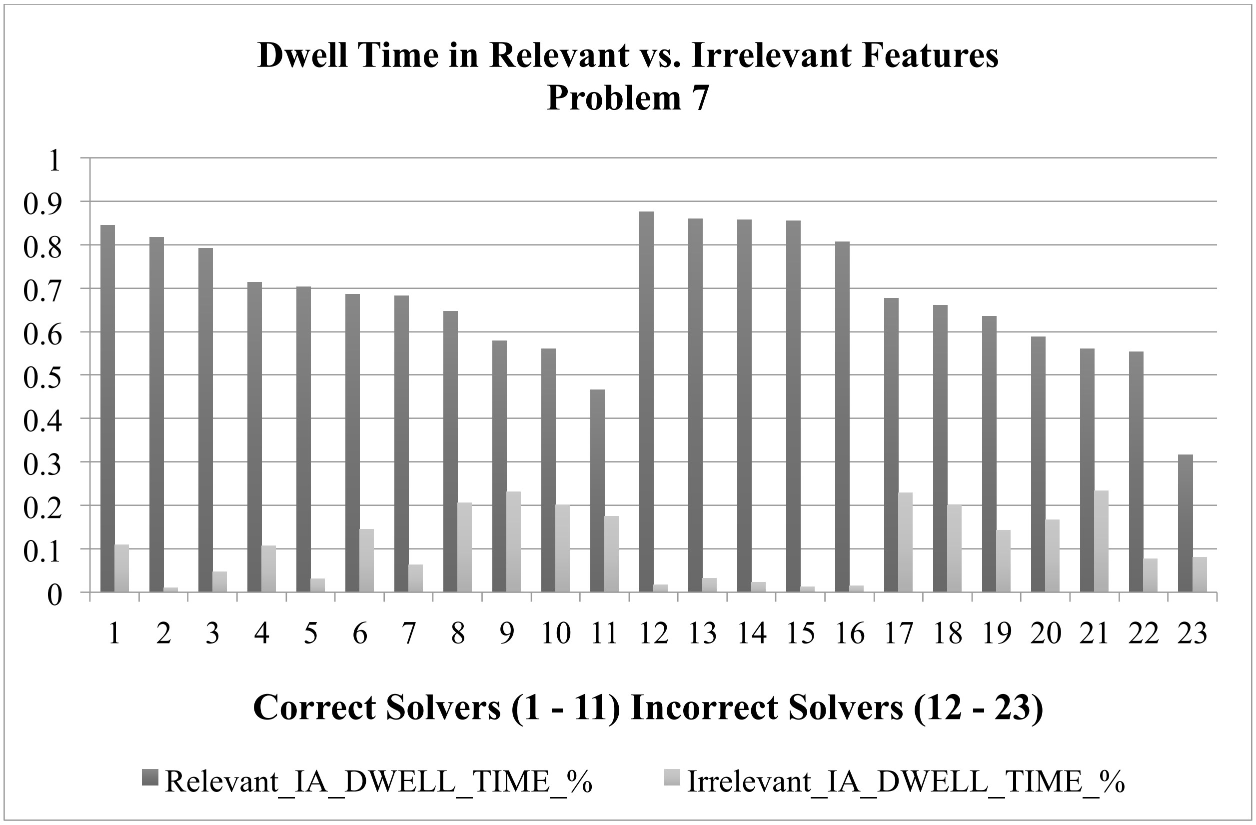
We began sorting through the AOI data by graphing and comparing participants dwell times in both the relevant and irrelevant features, given in Figure 3-3. A bar graph was generated for each student, on each problem. The following figures exhibit the dwell time per each student and separate the correct and incorrect solvers (left to right). We begin to see a trend in each problem where incorrect solvers were spending more of their total viewing time attending to irrelevant features (light gray) than relevant features (dark gray). Furthermore, this trend is more apparent in some problems than others, which leads us to question what is different between these problems which led students to spend between ten and sixty percent of their time attending to the table labels instead of the table values.

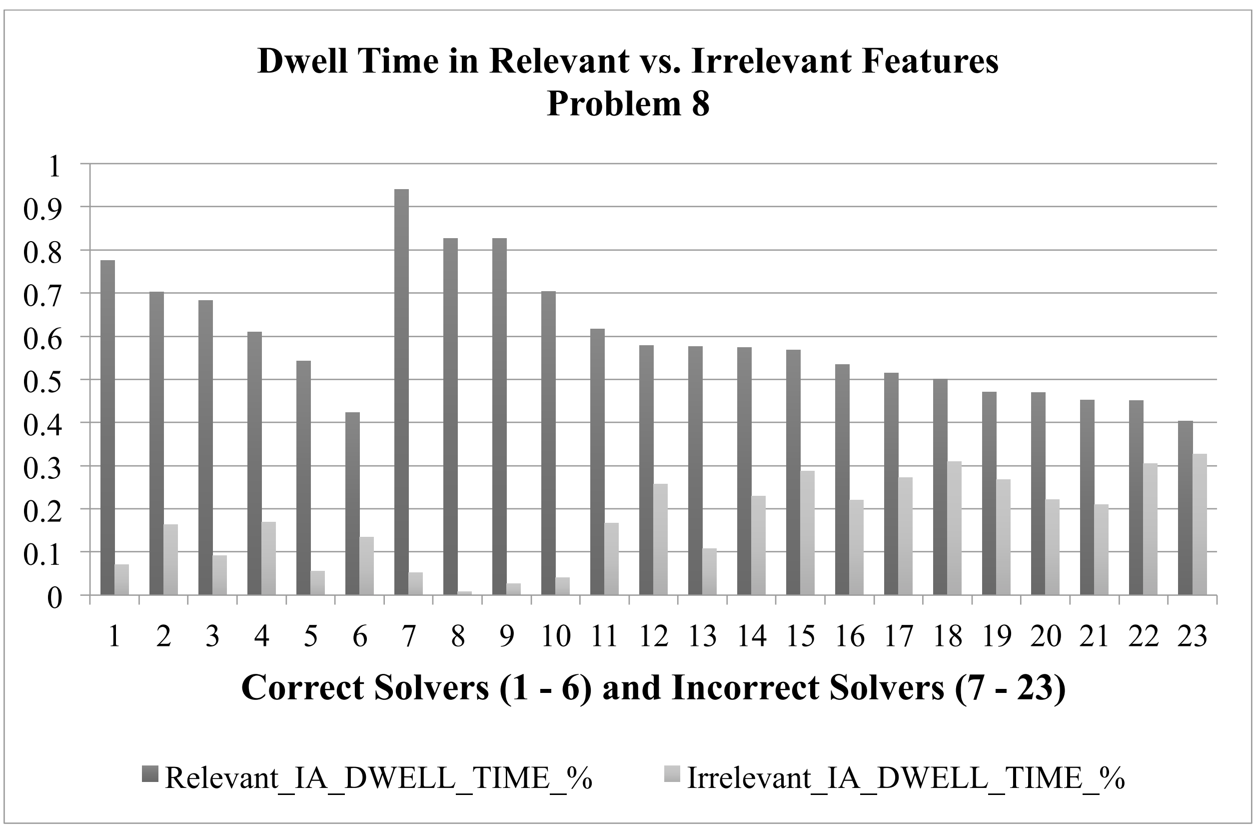
Figure 3-4 Bar graphs exhibiting the percent time spent (dwell time) attending to relevant vs. irrelevant features for each participant and each problem.

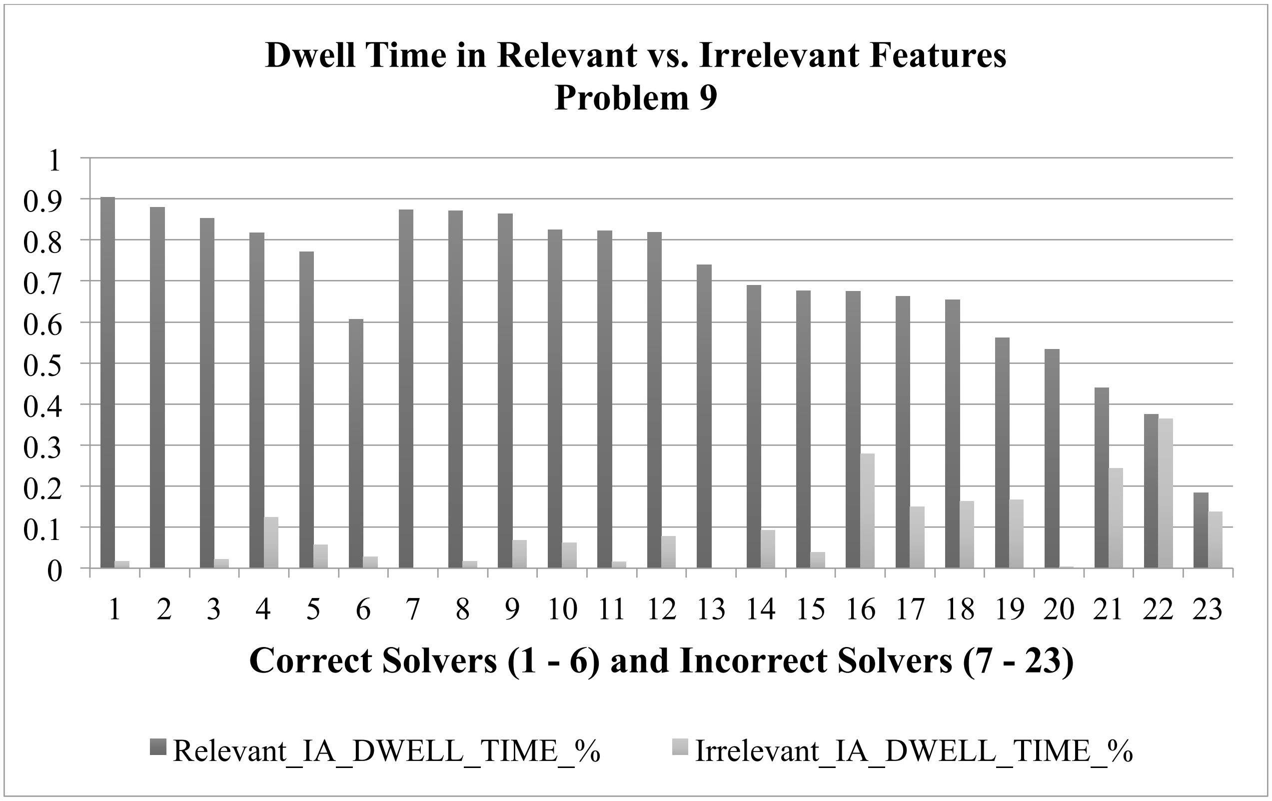



Dwell Time in Relevant vs. Irrelevant Features
Problem 5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
Correct Solvers (1 - 11) and Incorrect Solvers (11 - 23)
Relevant_IA_DWELL_TIME_%
Irrelevant_IA_DWELL_TIME_%


Dwell Time in Relevant vs. Irrelevant Features
Problem 6
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
Correct Solvers (1 - 13) and Incorrect Solvers (14 - 23)
Relevant_IA_DWELL_TIME_%
Irrelevant Features_IA_DWELL_TIME_%


Dwell Time in Relevant vs. Irrelevant Features
Problem 7
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
Correct Solvers (1 - 11) Incorrect Solvers (12 - 23)
Relevant_IA_DWELL_TIME_%
Irrelevant_IA_DWELL_TIME_%


Dwell Time in Relevant vs. Irrelevant Features
Problem 8
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
Correct Solvers (1 - 6) and Incorrect Solvers (7 - 23)
Relevant_IA_DWELL_TIME_%
Irrelevant_IA_DWELL_TIME_%



Guided by our preliminary results found in the graphs and our goal to compare how the correct solvers and incorrect solvers spend time attending to relevant and irrelevant features, we next found the mean percentage time spent during the entire problem period for relevant and irrelevant AOIs. First, we found the *sample mean*, which in our case is the average amount of time participants (separating correct and incorrect solvers) spent attending to relevant and irrelevant features. Then we found the *sample variance* and *sample standard deviation*, which are measures of the spread of the data. We can evaluate the variance of a set of data from the mean, that is, how far the observations deviate from the mean.

The following table 3-3 gives the mean percentage time spent, variance, and standard deviation for the relevant and irrelevant AOIs for the correct and incorrect solvers.

Table 3-1 Mean percentage time spent, variance, and standard deviation during entire problem period for relevant and irrelevant AOIs for participants who answered the question correctly versus incorrectly.

AOI Type	Problem Number	Answered Correctly	Answered Incorrectly
Relevant	4	75.53, 48.06, 6.93 n=16	41.36, 829.03, 28.79 n=7
	5	72.29, 341.17, 18.47 n=11	62.24, 527.08, 22.96 n=12
	6	76.90, 371.11, 19.26 n=13	62.83, 406.68, 20.17 n=10
	7	68.71, 131.31, 11.46 n=12	68.10, 292.94, 17.12 n=11
	8	62.31, 160.02, 12.65 n=6	58.88, 229.54, 15.15 n=17
	9a	80.51, 115.94, 10.77 n=6	66.26, 375.16, 19.37 n=17
Irrelevant	4	7.19, 42.71, 6.54 n=16	25.05, 429.86, 20.73 n=7
	5	9.65, 113.46, 10.65 n=11	15.16, 398.87, 19.97 n=12
	6	13.05, 308.57, 17.57 n=13	20.39, 223.93, 14.96 n=10
	7	10.24, 58.63, 7.66 n=12	12.06, 76.08, 8.72 n=11
	8	11.44, 23.32, 4.83 n=6	19.48, 115.58, 10.75 n=17
	9a	4.13, 20.19, 4.49 n=6	11.06, 114.29, 10.69 n=17

Our goal is to be able to compare the means between the correct and incorrect solvers, so we will use a statistical analysis, *t – test*, to determine whether the means between the two groups are *statistically different*. Since the variances of the two groups (given in Table 3-3) are unequal, we can use the *Welch's t-test* to compare the means. Welch's t-test is a two-sample test to check the hypothesis that two populations have equal means. Furthermore, the Welch's t-test is an adaptation of the more commonly used *Student's t-test* but is used, as previously stated, when two samples have unequal variances.

Welch's t-test defines the *t-statistic* by the formula:

Equation 3-1 Welch's t-statistic

$t = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$ where $\overline{x_1}$ and $\overline{x_2}$ are the two sample means, s_1 and s_2 are the two sample variances, and n_1 and n_2 are the two sample sizes.

The degrees of freedom ν is approximated by:

Equation 3-2 Degrees of freedom

$\nu \approx \frac{(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2})^2}{\frac{s_1^4}{n_1^2 v_1} + \frac{s_1^4}{n_2^2 v_2}}$ where s_1 and s_2 are the two sample variances, n_1 and n_2 are the two sample sizes, and v_1 and v_2 are the degrees of freedom associated with the respective variance estimate.

Once *t* and *v* have been computed we can use the *t-test* to test the null hypothesis that the two population means are equal (H_0: $\mu_1 = \mu_2$). The sampling distribution used to determine the probability value (*p – value)* is the *t-distribution,* which describes samples drawn from a population. Under the assumption that the null hypothesis is true, we can reject the null hypothesis when the value of *t* is large (i.e. when it falls in one of the two tails of the t-distribution). When the *t-value* falls in one of the two tails of the t-distribution it is considered an unlikely event under the assumption that the null hypothesis is true and therefore we claim to accept the alternative hypothesis that the population means are unequal (H_1: $\mu_1 \neq \mu_2$).

It is important to remark that the Welch's *t*-test is stronger than Student's *t*-test and maintains type 1 error rates (i.e. incorrect rejection of the null hypothesis). Furthermore, the power of Welch's *t*-test comes close to that of Student's *t*-test, even when the population variances are equal and sample sizes are balanced.

In the next chapter the results from the Welch's t-test are reported along with a summary of the results and what the results mean in our study.

Chapter 4 - Results and Discussion

Results from Eye Movements

Visual attention was analyzed for each of the different problems using a Welch's t-test with *percentage of time attending to* both AOI types as the dependent variable and *correctness of answer* as the independent variable. We hypothesize that there is difference in visual attention in attending to relevant and irrelevant features between those who correctly versus incorrectly solve the problem. Furthermore, our hypothesis predicts a difference in the population mean percentage dwell time attending to relevant and irrelevant areas between the correct solvers (μ_C) and incorrect solvers (μ_I), i.e. null H_0: $\mu_C = \mu_I$ and alternative H_1: $\mu_C \neq \mu_I$.

More specifically, we hypothesize that those with adequate concept knowledge to correctly answer a problem will spend more time fixating on and attending to relevant areas and patterns within the table than on irrelevant areas of the table. Conversely, we predict that those who answer incorrectly will spend more time fixating on irrelevant areas.

Results of the Welch's t-test are reported in Table 4-2. Mean percentage of fixation dwell time, standard error for the correct and incorrect solvers, and the number of observations for the correct and incorrect solvers are also shown in Table 4-2. An asterisk indicates a statistically significant difference at the $\alpha = 0.05$ level.

Table 4-1 Mean percentage time spent (± standard error) and results of Welch's t-test during entire problem period for relevant and irrelevant AOIs for participants who answered the question correctly versus incorrectly.

AOI Type	Problem Number	Answered Correctly	Answered Incorrectly	t(df)	p
Relevant	4*	75.53 (± 1.73), n=16	41.36 (± 10.88), n=7	t(6.307) = -3.1008	0.0198
	5	72.29 (± 5.57), n=11	62.24 (± 6.63), n=12	t(20.678) = -1.1606	0.259
	6	76.90 (± 5.34), n=13	62.83 (± 6.38), n=10	t(19.035) = -1.6916	0.107

	7	68.71 (± 4.94), n=12	68.10 (± 3.46), n=11	t(19.31) = 0.1011	0.9205
	8	62.31 (± 5.16), n=6	58.88 (± 3.67), n=17	t(10.503) = -0.5419	0.5992
	9a*	80.51 (± 4.40), n=6	66.26 (± 4.70), n=17	t(16.299) = -2.2146	0.0414
Irrelevant	4	7.19 (± 1.63), n=16	25.05 (± 7.84), n=7	t(6.528) = 2.2307	0.0636
	5	9.65 (± 3.21), n=11	15.16 (± 5.77), n=12	t(17.077) = 0.8357	0.4149
	6	13.05 (± 4.87), n=13	20.39 (± 4.73), n=10	t(20.726) = 1.0816	0.2919
	7	10.24 (± 2.52), n=12	12.06 (± 2.31), n=11	t(20.968) = -0.5333	0.5994
	8*	11.44 (± 1.97), n=6	19.48 (± 2.61), n=17	t(19.32) = 2.4583	0.0236
	9a*	4.13 (± 1.83), n=6	11.06 (± 2.60), n=17	t(19.994) = 2.1811	0.0413

We found that on all seven problems we reported on, those who answered the problem correctly spent a higher percentage of total viewing time dwelling in the relevant area with two of the problems showing statistically significant differences between correct and incorrect solvers (Table 4-1). Furthermore, we found that on all seven problems reported, those that answered the problem incorrectly spent a higher percentage of total viewing time dwelling in the irrelevant area with two of the problems showing statistically significant difference between incorrect and correct solvers (Table 4-1).

Concentrating on the two problems that showed a statistical difference (Problem 4 and 9a), we first concluded problem 4 was one of the conceptually easiest problems to solve while problem 9a was one of the hardest problems to solve, which the numbers of answered correctly and incorrectly concur. Therefore, it may appear that these problems weeded out the students that knew what features were relevant but did not have the knowledge to solve the problem correctly and we were then able to distinguish between the students that were unable to determine the relevant features and those that were able to determine the relevant features.

Grouping by problem type, there was a statistically significant difference of dwell time in the relevant area between those who answered correctly and incorrectly on determining if the table represented a linear function and determining the slope of the line from a table (Table 4-2). There was also a statistically significant difference of dwell time in the irrelevant area between those who answered correctly and incorrectly on determining if the table represented a linear function (Table 4-2).

Table 4-2 Mean percentage time spent (± standard error) per problem set and results of Welch's t-test for relevant and irrelevant AOIs for participants who answered the question correctly versus incorrectly.

AOI Type	Problem Set	Answered Correctly	Answered Incorrectly	t	p
Relevant	4, 5, 6 Linear Function*	75.08 (± 2.36), n=40	57.40 (± 4.55), n=29	t(42.912) = -3.4482	0.0013
	7, 8, 9 Linear Slope*	69.82 (± 2.72), n=23	63.26 (± 2.57), n=46	t(56.507) = -2.1285	0.0377
Irrelevant	4, 5, 6 Linear Function*	9.77 (± 1.92), n=40	19.35 (± 3.41), n=29	t(45.244) = 2.4509	0.0182
	7, 8, 9 Linear Slope	9.83 (± 1.45), n=23	13.96 (± 1.61), n=46	t(62.775) = 1.9042	0.0615

As the interviews also suggest, incorrect solvers were largely concerned with the irrelevant information to solving the problem (i.e. the table labels). Particularly, incorrect solvers were looking for a symbolic representation of the function such as an equation so they could plug in values or find the slope as the coefficient of the *x*. Therefore, in Table 4-3 below, we see incorrect solvers spent 14.16% and 8.93% of their total viewing time looking specifically at the table label $y = f(x)$.

Table 4-3 Mean percentage time spent (± standard error) per problem set and results of Welch's t-test for "y = f(x)" AOI for participants who answered the question correctly versus incorrectly.

AOI Type	Problem Set	Answered Correctly	Answered Incorrectly	t	p
"y = f(x)"	4, 5, 6 Linear Function*	6.03 (± 1.56), n=40	14.16 (± 2.79), n=29	t(45.04) = 2.5444	0.0144
	7, 8, 9 Linear Slope*	5.15 (± 1.14), n=23	8.93 (± 1.35), n=46	t(62.438) = 2.1244	0.0376

In addition, there is a statistically significant difference between correct and incorrect solvers in dwell time attending to the "y = f(x)" label when asked whether the table of values represents a linear function and finding the slope of the line. This mirrors the interview results where 20/69 = 29% of students' reasonings focused on using the equation to determine whether the table of values represents a linear equation. Also, 8/69 = 11.5% of students' reasonings focused on using the "equation", y = f(x) label, to determine the slope of the line.

Chapter 5 - Conclusion

Overview of Research

The purpose of our work was to begin investigating visual attention of College Algebra students when giving a table of values representing a linear function. Our hopes were to look to compare the differences between the correct and incorrect solvers using a eye-tracker and use APOS theory to conceptualize their understanding of functions. Upon revisiting our goals of this study, we find some statistical evidence that there are differences between correct and incorrect solvers' visual attention, which confirms our hypotheses, which are restated below.

1. Visual attention differs between those who correctly and incorrectly solve introductory college algebra problems.
    a. Correct solvers spend more time attending to relevant features of the problem and less time attending to irrelevant features.
    b. Incorrect solvers spend more time attending to irrelevant features of the problem and less time attending to relevant features.

Research Question 1a

We hypothesized that that those with adequate conceptual knowledge to correctly answer a problem would be able to determine particular features that were irrelevant in providing a solution to the problem and therefore spend more time fixating on relevant areas of a problem. In our study, we determined the irrelevant features of the table being the labels x and $y = f(x)$. We assumed that the correct solvers would initially attend to the table labels but would be able to then move down the table and look for patterns among the table values (relevant features). We found that when given a tabular representation of a linear function, the trend showed the correct solvers spent more time attending to the relevant features of the problem. Furthermore, in participant interviews, correct solvers stated that they were using the values in either a procedure, looking for relationships and/or patterns, or in visualizing the graph of the function. There is a strong focus on looking for either particular features of the values (e.g. y-intercept) and also patterns within the values (e.g. multiplicative relationship between x and y values, or constant change in y values). The interviews and dwell time suggest that correct solvers were much less focused on the table labels.

Research Question 1b

We also hypothesized that incorrect solvers in comparison to the correct solvers would spend more time attending to the irrelevant features of the table. We found that when given a tabular representation of a linear function, the trend showed the incorrect solvers spent more time than the correct solvers in attending to the irrelevant features of the problem. Furthermore, there is a statistically significant difference between incorrect and correct solvers in their time spent attending to the label $y = f(x)$. More significantly, there is a group of incorrect solvers for which their interviews focused on finding the equation. In this case the label $y = f(x)$ served as the equation because of the presence of variables. Either the students were looking for the $y = mx + b$ form in the variables, or used the label $y = f(x)$ to plug in the values. Our assumption based on the APOS theory assumes that these students are at the Action level of APOS learning theory and have not developed yet the notion of a function as a process. The student's focus on the equation as determining whether the table represents a function or not brings us to this conclusion.

Overall, we did find a difference between incorrect and correct solvers in what they are noticing and spending time attending to when given a table of values. But what became interesting was this difference between types of incorrect solvers: incorrect solvers that spent more time attending to the table values and those who spent more time attending to the table labels. This leads us to wonder what misconceptions of the table lead students to spend between ten and sixty percent of their time attending to the labels *x* and $y = f(x)$? Can we distinguish between this group of incorrect solvers and the group of incorrect solvers that did move past the table labels and attended to the values in efforts to solve the problem?

Future Work

In future work, we would like to expand on these preliminary results and begin by observing where the differences between incorrect and correct solvers lie. Our results suggest that there are students that do not know how to use the table values and therefore do not know where to look. But there is also a difference between the set of students that know where the relevant features lie in the table but solve the problem either incorrectly or correctly. Our future work would attempt to place students based on their eye movements into one of the three categories. Furthermore, further analyses would need to be done to expand on the differences between eye patterns rather than just focusing on dwell time in AOIs. Our hypothesis is that different relevant and irrelevant patterns will become apparent to determine the difference between the incorrect and correct solvers who attended to the table values.

The anticipated broader impact of our work is to move beyond understanding College Algebra students' visual patterns in multi-representational problems to developing an effective instructional environment based on these findings. Specifically, in an online learning environment where an instructor is not present to cue the student, having visual cues built in may hint students toward the relevant features of the graph, as would an instructor if he/she was present. In conclusion, there is still much work to be done to understand how students attend to features of a table or graph before we can begin constructing effective cues. This study offers hope that with eye-tracking we can begin to understand students inner conceptions of what they see on paper or on the screen.

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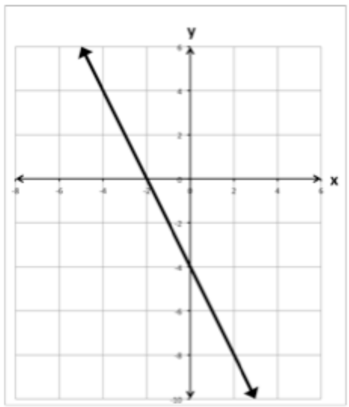
Appendix A – List of Problems

Training Problems

The following 'training problems' were given to students to help assist students in the think-aloud process.

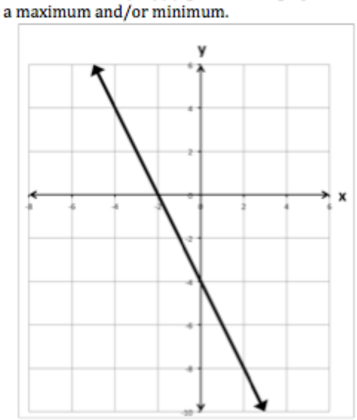
Training Problem 1:
(a) Fill in the table of values for the function *y = f(x)* given in the graph below.

x	*y = f(x)*
-4	4
-3	
	0
0	
	-6
2	-8
3	



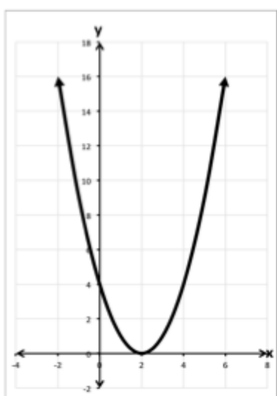
Training Problem 1:
(a) Fill in the table of values for the function *y = f(x)* given in the graph below.
(b) Does this function have a maximum and/or minimum.

x	*y = f(x)*
-4	4
-3	
	0
0	
	-6
2	-8
3	



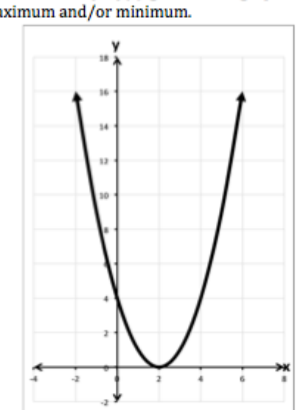
Training Problem 2:
(a) Fill in the table of values for the function *y = f(x)* given in the graph below.

x	*y = f(x)*
-2	
-1	9
0	
1	1
	0
3	1
	4



Training Problem 2:
(a) Fill in the table of values for the function *y = f(x)* given in the graph below.
(b) Does this function have a maximum and/or minimum.

x	*y = f(x)*
-2	
-1	9
0	
1	1
	0
3	1
	4



Interview Problems

The following fourteen problems (twenty-one problems including multiple parts) were given to each participant in a randomized order during the interview. They were presented on a non-numbered slide such as in Figure A-1. Problems with two parts (a and b) were given to participants sequentially and on separate slides.

Figure A-5-1 Example of non-numbered problem slide given to participants during the interview.

The table of values represents a linear function.
What is the slope of the line?

x	$y = f(x)$
-4	12
-2	12
0	12
2	12
4	12
6	12

Problem 1:

(a) Does the table below describe y as a function of x?

(b) Does the table below describe x as a function of y?

x	-3	-2	-1	0	1	2	3
y	3	0	-1	0	3	8	15

Problem 2:

(a) Does the table below describe y as a function of x?

(b) Does the table below describe x as a function of y?

x	-4	-2	0	2	4	6
y	12	12	12	12	12	12

Problem 3:

(a) Does the table below describe y as a function of x?

(b) Does the table below describe x as a function of y?

x	-4	-1	0	1	3	7	12
y	5	7	3	15	8	9	10

Problem 4:

Does the table of values represent a linear function?

x	0	1	2	3	4	5
y = f(x)	4	7	10	13	16	19

Problem 5:

Does the table of values represent a linear function?

x	-3	-2	-1	0	1	2
y = f(x)	9	4	1	0	1	4

Problem 6:

Does the table of values represent a linear function?

x	-5	-3	-2	0	3	4
y = f(x)	6	2	0	-4	-10	-12

Problem 7:

The table of values represents a linear function. What is the slope of the line?

x	0	2	4	6	8	10
y = f(x)	0	6	12	18	24	30

Problem 8:

The table of values represents a linear function. What is the slope of the line?

x	-4	-2	0	2	4	6
y = f(x)	12	12	12	12	12	12

Problem 9:

The table of values represents a linear function.

(a) What is the slope of the line?

(b) Find another point on the line?

x	-5	-2	-1	3	5	9
$y = f(x)$	16	10	8	0	-4	-12

Problem 10:

The following is the table of values for $y = \frac{-x+6}{2}$.

(a) Does the equation $y = \frac{-x+6}{2}$ describe a linear function?

(b) If it is a linear equation, what is the slope of the line?

x	-4	-1	2	5	8	11
$y = f(x)$	5	3.5	2	0.5	-1	-2.5

Problem 11:

The following is a table of values for $0 = -x + 2y - 6$.

(a) Does the equation $0 = -x + 2y - 6$ describe a linear function?

(b) If it is a linear equation, what is the slope of the line?

x	-3	-2	-1	0	1	2
$y = f(x)$	1.5	2	2.5	3	3.5	4

Problem 12:

The following is a table of values for $y = xy + 4$.

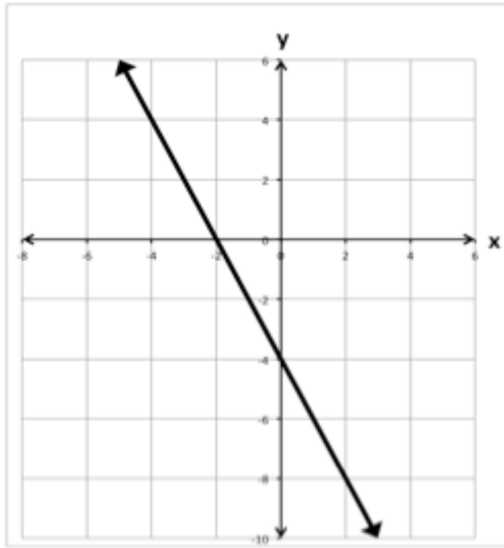
(a) Does the equation $y = xy + 4$ describe a linear function?

(b) If it is a linear equation, what is the slope of the line?

x	-3	-1	0	2	3	5
$y = f(x)$	1	2	4	-4	-2	-1

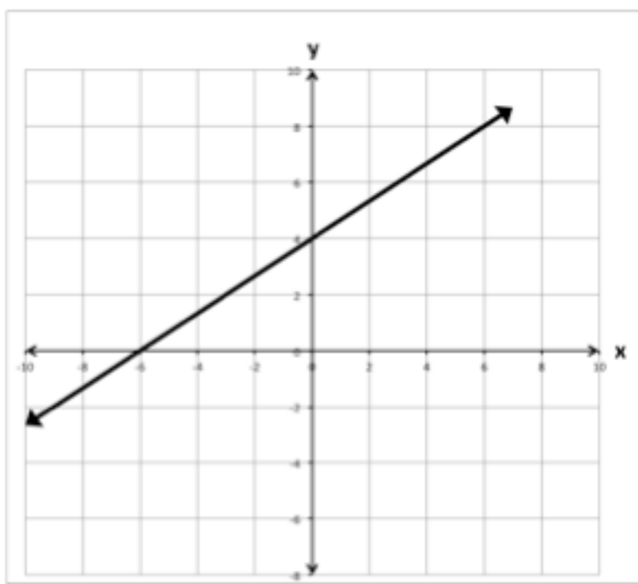
Problem 13:

Find the equation of the linear function graphed below.



Problem 14:

Find the equation of the linear function graphed below.



Appendix B – Interview Protocol

The following interview protocol was used for each participants interview.

Before the Interview:

1. Prepare for the interview at least 5 minutes before the scheduled time. (SEE EXPIREMENTER CHECKLIST)
2. Check the waiting room for the participant. Introduce yourself and welcome the participant by name. Close the door with the "Experiment in Progress. Do not disturb" sign on the door. Let the participant have a seat where they will be seated for the vision test.
3. Explain the purpose of the interview and eye tracking.

We are interviewing students in College Algebra to better understand how students look at and understand visual – type college algebra problems with diagrams, in hopes to improve college algebra instruction.

Today you will be working through a few sets of college algebra problems. The problems may look similar to one another but I will assure you that they are random. You will not need a calculator and do not need to write anything down. Answers may be given in un-simplified form such as "18 times 5 divided by 4". It is okay if you are not sure how to solve the problems; just try your best. There is no penalty for wrong answers.

In each question, you will be asked to think aloud. It is critical in our study that we know exactly what you are thinking as you are thinking it, so you will be asked to verbalize your thought process thoroughly. I would like it if you would just say anything that comes to mind while solving the problem. Remember, there is no penalty for wrong answers. Sometimes I may prompt you to keep talking or keep thinking aloud or may ask you follow-up questions for clarification purposes. As you are working and talking through the problems, we will use an infrared eye tracker to record your eye movements.

This interview should take approximately 40 minutes. Remember that your participation is completely voluntary and you may terminate the interview at any time. If you decide to participate and finish the interview, you will receive extra credit equal to one homework assignment in Math 100. You will earn the extra credit points as long as you make an honest *attempt to answer the questions, regardless of whether or not you answer the questions correctly. In the event we include any of your interview in any discussion or publication, your privacy will be maintained by the use of a pseudonym.*

In order to participate, I need you to sign a consent form. We have two copies of an Informed Consent form for you to sign: one for our records and one for you to keep. Please look over the form and let me know if you have any questions. Print your name, sign, and date the form.

4. Have the participant read and sign the Informed Consent form. If they decline to sign the form, thank them for their time and terminate the interview. Otherwise sign and date the form as witness and then proceed to the interview.
5. Fill out the data log with the relevant information. Ask the student for their age and native language.
6. Begin the Vision Test: On the desktop Choose "Acuity C" in FrACT.

We will start by a quick vision test for purposes of standardizing our data. You will see a series of semi-circles on the screen. Use the arrow keys on the keyboard to indicate where the opening is.

7. Record the V.A. decimal and Snellen fraction from the vision test in the data log.
8. Let the participant take a seat where they will be seated for the eye tracking.
9. Calibrate the eye-tracker: Help the participant adjust the chin/forehead rest. Click on "Camera Setup." Adjust the camera and focus on one eye (usually the right eye). Click on "Calibrate" and a dot will appear on the participant's screen.

We will now calibrate the eye-tracker to your eyes. Focus on the center of the dot. The dot will move in a random sequence to different parts of the screen. Do not move your eyes from the center of the dot until the dot disappears. Press [spacebar] when you are ready to begin.

10. Validate the calibration: Repeat the calibration sequence to validate.

We now must validate your calibration. Press [spacebar] when you are ready to begin.

11. Training Problems: Click on "Output/Record" and an instruction for training problem screen will be presented. Press [spacebar] after the instruction screen in presented.

We are going to begin by a set of training problems. It might seem unnatural to keep talking while solving a problem, so you will be given time to practice using the think-aloud process before we begin the actual interview. First, I will solve a problem using the think aloud process. Then, you will practice using the think aloud. We can then discuss what you could have done for me to better know what you are thinking and you can ask any questions you may have before we begin. For every problem, we will start by reading the question then begin thinking through the problem.

12. Go through the training problems and discuss with the participant. The training problems are not recorded. Press [0] to advance to the next screen. Present an instruction screen.

We will now begin the eye tracking interview. Remember to think aloud through the problems and to be clear about your final answer. Just a reminder: During the experiment I won't be able to give any hints or tell you if you are incorrect or correct. Between each problem you will see a break screen in which you can take a break, relax, and ask any questions you may have. Then there will be a drift correct to make sure that the eye-tracker is still tracking your eye. Focus on the dot on the center on the screen until it passes. Press [spacebar] whenever you are ready to begin.

13. Start recording.
14. Problems are presented. Participant answers verbally and interviewer may need to ask for clarification during the think aloud.
15. **Experimenter presses [0/1] to indicate if the answer is incorrect/correct**.
16. **Experimenter presses [c/p/n] to indicate if the reasoning was conceptual/ procedural/ neither.** Use this button press to advance to the next problem (or part).
17. At the end of the experiment, a thank-you screen is presented. Press [Spacebar] to begin saving the results. Debrief the student.

Thank you so much for participating. Do you have any questions or comments concerning the study? You are always welcome to email me later if you have any additional comments or questions. We ask that you don't discuss the specific details of the problems with anyone you know who may sign up to participate.

The extra credit will be added to KSOL at the end of the summer session. If for some reason your points don't show up, please email your instructor or me.

18. Stop the recorder.
19. Prepare for the next interview or shut everything down for the day. If there is a break between participants or if the end of the day, back up the results and videos.

Experimenter Checklist

BEFORE THE PARTICIPANT COMES IN:

________ 1) Unlock BH 486 Visual Cognition Lab and prop the door open for the participant to come in.

________ 2) Turn on the camera and all three computers (check the power strip under the desk to make sure it is on).

________ 3) While computers are turning on, get out the paperwork you will need:

________ Data Log

________ Experimenter Checklist/Protocol

________ 2 Blank IRB Forms

________ 4) Check refresh rate of participant monitor is set to **85 Hz**. Double click on the desktop short cut labeled "Display - Shortcut". On the left side of the panel, click on the "Adjust Resolution" button. Click on the "Advanced Settings" button on the right side of the menu. A new dialog box should open. Click on the "Monitor" tab. One of the settings is called the "Screen refresh rate" and there is a drop down menu that you can use to change the refresh rate.

________ 5) Check the data log to see which participant number and condition to run in the session.

________ 6) Open the shortcut to the experiment located on the desktop called "MATH_AOI_STUDY"

________ Name the .edf using the following convention: CA1_PP_#.edf where CA1 stands for College Algebra Study 1; PP is the two digit participant number; # is the condition number 1, 2, or 3. EXAMPLE: If participant 10 runs condition 1, the name would be CAI_10_1.edf

________ 7) Set up the recording equipment. (DON'T TURN ON UNTIL AFTER CALIBRATION.)

________Check the batteries and amount of memory left to make sure the devices will last for the entire session.

________Point the video camera at the participant's monitor. Prop the video camera on the corner of the table behind the participant.

________Make sure that the microphone is connected to the video camera. Place the microphone between the experimenter and the participant.

________ 8) Set up the vision test. Open the Freinburg Visual Acuity and Contrast Test located on the desktop called "FrACT3.8.0d.exe."

________ 9) Wipe down the chin/forehead rest and keyboard for the upcoming participant.

________ 10) Look up the upcoming participant's name and check the waiting room.

________ 12) Direct participant to the eye tracking room and place the "Experiment in Progress. Do not disturb" sign on the door. Close the door.

RUNNING THE SESSION:

________ 1) Introduce yourself and describe the purpose of the session.

________ 2) Have the participant sign the IRB form.

________ 3) Fill out the data log with the relevant information. Ask for age and native language.

________ 4) Perform the vision test. Record the V.A. decimal and Snellen fraction in data log.

________ 5) Calibrate the eye tracker (instructions below).

________ 6) Talk through the training problems.

________ 7) Start the recorder.

________ 8) Begin the experiment. PRESS [1] FOR CORRECT ANSWERS, AND [0] FOR INCORRECT ANSWERS.

PRESS [c] FOR CONCEPTUAL REASONING, [p] FOR PROCEDURAL REASONING AND [n] FOR NEITHER.

________ 9) Take note in the study log about any logistical problems.

________ 10) Debrief the student.

________ 11) Stop the recorder.

SETTING UP/CALIBRATE THE EYE TRACKER:

________ 1) With your help, have participant adjust the chair and chin/forehead rest to get comfortable.

________ 2) Click on "Camera Setup" on the right hand side of the screen to get the screen to adjust the camera.

________ 3) Adjust the camera and focus on one eye (usually the right eye). Make sure the text under the close-up video of the eye is green and says "PUPIL OK" and "CR OK". Also, make sure the left eye is highlighted on the bottom of the camera set up.

________ 4) Click on "Calibrate" located on the right hand side of the screen. If the white dot doesn't appear on screen, press the [c] on the keyboard.

________ 5) Instruct participant to look at the center of the dot and to press the space bar when they are ready to follow the dot with their eyes.

________ 6) While calibration is going on, a set of green crosses will appear on the experimenter screen. They should look like a grid. If not, readjust the camera and try tracking other eye, just make sure to click the appropriate "Eye Tracked" button at the bottom of the camera set up.

________ 7) After calibration, click on "Validate" to make sure the calibration is good. The computer will let you know if the calibration is good. If not, redo the calibration. We want $\leq 0.50^{o}$ average error and $\leq 1.00^{o}$ maximum error.

________ 8) When done calibrating, click on "Output/Record" to start the experiment.

AFTER THE PARTICIPANT LEAVES:

________ 1) Put data sheets in appropriate folders.

________ 2) If there is less than one hour free on the camera, transfer videos to computer. RENAME THE FILE WITH THE PARTICIPANT NUMBER.

________ 3) If there is another session, set up for the next session.

If there is no one coming in after you, turn off the computers, shut off the lights, and lock the door behind you.

________ 5) At the end of the day, back up the results and videos by transferring them to a flash drive and saving under "Math_AOI_Data" folder on the computer in BH 484.

CALIBRATION TROUBLESHOOTING/TIPS

- **If you have trouble getting green "PUPIL OK" and "CR OK" messages when adjusting the camera**, try switching to the other eye. Just make sure to click on the appropriate "Eye Tracked" button at the bottom of the camera setup screen.
- **You can recalibrate in between the different problem sets (trials).** When the white dot appears on the screen for the mini calibration (drift correct), press the "c" button on your keyboard OR click on "Calibrate". After validation, click on the "Output/Record" button to return to the experiment.

Appendix C - Complete List of Student Reasonings

For the interviews, we recorded the think-aloud process and final answer, which included their final reasoning. We based whether the participants were incorrect or correct on their final answer and reasoning. Below is a table with the participants' final answers and reasonings for each problem along with the coded reasoning used in the interview analysis.

Problem Description and Question	Participant	Answer	Student Reasoning	Coded Reasoning
4. Does the table of values represent a linear function? *(Table represents the line $y = 3x + 4$)*	1	Yes	Yes. Because the y's each increase by 3.	Constant Slope
	2	No	No. Because the equation doesn't look like y = mx + b	Focus on Equation
	3	Yes	Yes, y-intercept is 4 and the y's are going up 3, …slope is 3. Y-intercept is 4 and slope is 3.	Constant Slope
	4	Yes	Yes it does. All the y-values are going up by 3.	Constant Slope
	5	Yes	Yes, because each input has one output. I look at the inputs and outputs.	Input/Outputs
	6	Yes	Yes it does. I was actually looking this time to see if there are repeating inputs and outputs this time.	Input/Outputs
	7	No	No. Because when I put in 4 for y does not equal f(0) when I put 0 for x.	Focus on Equation
	8	Yes	Yeah it would be linear. I try to graph it in my head and it would be going positive.	Visualize Graph
	9	Yes	Yes it does. Because the values are in a line and the line is going up.	Visualize Graph
	10	Yes	Yes it does. Yes it would be because when you plot it, it continually goes up it has to be a linear function.	Visualize Graph
	11	Yes	Looking at the values, yes it is linear function. As x increases, y is increasing.	Visualize Graph
	12	No	No because I'm trying to find the equation y = mx + b but I can't get it to work out. It doesn't represent a linear function.	Focus on Equation
	13	Yes	Yes because, they are positive so they are all going to go up to infinity so they	Visualize Graph

			are going to make a straight line.	
	14	Yes	Same slope for all points makes it linear. So I find the slope between the points. I get the same slope…it's constant so it is linear function.	Constant Slope
	15	Yes	I can just look at the change in y. It changes by 3 each time so yes it is linear.	Constant Slope
	16	Yes	Yes. Because y is a product of x.	Multiplicative relationship
	17	Yes	Yes. I graphed it in my head and it is a linear line.	Visualize Graph
	18	Yes	Yes. Because every value of y increases by 3. For every value of x moving up by 1 the y increases by 3. This continues.	Constant Slope
	19	Yes	It is a linear function. None of the x's repeat and go up by 1 and the y's don't repeat and all go up by 3.	Constant Slope
	20	No	I'm guessing no because there is no equation that gives y = mx + b.	Focus on equation
	21	Yes	Each input has only ones output. For this one each input only has one output so I would say it is a linear function.	Input/Outputs
	22	Yes	Yes it does because it is going up by 3 evenly.	Constant Slope
	23	IDK	I don't know. I need the equation.	Focus on Equation

5. Does the table of values represent a linear function? *(Table represents the quadratic $y = x^2$)*	1	No	No because you have two different inputs for the same output so it's not a function.	Input/Outputs
	2	No	No it's not linear. I'm visualizing the graph and it makes a parabola…it goes down then back up.	Visualize Graph
	3	No	No and the function would by $y = x^2$. It's a parabola. I look at the x-values and corresponding y – values.	Multiplicative relationship
	4	No	To me it looks like a quadratic function. The points are going up to –infinity and +infinity.	Visualize Graph
	5	Yes	Yes it is linear function. I can plug in the values and get y-values.	Focus on Equation
	6	Yes	Yes, it does. I'm looking for an exponent and I don't see any. Since there are not exponents, it must be	Focus on Equation

			linear.	
	7	IDK	I don't know.	I don't know.
	8	Yes	Yeah. I think it would have a positive slope. I can visualize it going up.	Visualize Graph
	9	No	I think it is parabola so it's not a linear function.	Visualize Graph
	10	No	No it is not linear. You can see that it goes from being positive then 0 and goes back positive.	Visualize Graph
	11	Yes	Um I believe it is a linear function. If I were to graph it, I can make a line.	Visualize Graph
	12	No	Yeah this is not a linear function because you get the same input for different outputs.	Input/Outputs
	13	No	It's not linear. I would say it does because if you square root each y. That's a parabola. When you square the x values you get the y values.	Multiplicative relationship
	14	IDK	I don't know	I don't know
	15	No	The graph goes from a higher value in y to a lower value and then a higher value again so it would be a parabola. Not linear.	Visualize Graph
	16	Yes	Yes. I was looking to see if x is in the formula for y. If you take -3…if x is -3 and put -3 in the place of x you get y.	Focus on Equation
	17	No	No, the function graphs a parabola.	Visualize Graph
	18	No	No, the line doesn't have the same difference between the y-values.	Inconstant Slope
	19	No	No. Because there are two 1's in the y.	Inputs/Outputs
	20	Yes	Yes, I can take any of these numbers and put them into the function to be linear. I plug in 4 = f(-2) that makes it linear.	Focus on Equation
	21	Yes	Yes this represents a linear function. I feel like it is because every one of these answers multiplied by itself give me y = f(x).	Multiplicative relationship
	22	No	No because they [y-values] all stay positive while this [x-values] are moving from negative to positive. The graph would not be linear then.	Visualize Graph
	23	IDK	I need the equation to be able to know if it is y = mx + b	Focus on Equation

	1	Yes	Yes it's linear, I can visualize the graph	Visualize Graph

6. Does the table of values represent a linear function? *(Table represents the line y = -2x – 4)*			and it seems to be going down in a straight line.	
	2	No	No, I'm visualizing the graph and it doesn't make a straight line.	Visualize Graph
	3	Yes	Yes, it's linear. The x changes and I can calculate the slope between points, which is -2.	Constant Slope
	4	No	No it does not. Because as we change x from -5 to 6 we don't have the same value change in y for the increment levels. It's not the same amount.	Inconstant Slope
	5	Yes	Yes, because each output has only one input.	Inputs/Outputs
	6	Yes	Yes because y = f(x)…this does represent a linear function because again there are no exponents.	Focus on Equation
	7	Yes	Yes. Because the points all lie on the line. I plotted them in my head.	Visualize Graph
	8	Yes	Yes, I think it does. I can't remember what a linear function is but I think it is y=mx + b which the equation is.	Focus on Equation
	9	Yes	Yes it does. Because I think the y is positive then goes 0 then negative. So it would be a line.	Visualize Graph
	10	Yes	Yes, since the x is going from negative to positive and the y is going from positive to negative and it doesn't show any signs of going back to positive therefore the line would be sloped but not curved.	Visualize Graph
	11	Yes	I think it is a linear function just because I don't see anything wrong with it. If I would plug the values into the equation y = f(x) I would get y = 6 and x = -5 so 6 = f(5)…I just don't see anything wrong with the values.	Focus on Equation
	12	Yes	Um yes. Because each input is getting a different output and it would be one to one.	Inputs/Outputs
	13	Yes	I think it does. Because it is not a parabola. When I plot them in my head there are different lines but they aren't crossing and they are all straight lines. So that is why it is a linear function.	Visualize Graph
	14	IDK	I don't know.	I don't know
	15	Yes	Yes, I checked the slope between all two	Constant Slope

			points and it's all the same slope so it's linear.	
	16	Yes	Yes, because y is a product of x.	Relationship between x and y
	17	Yes	Yeah, but I just guessed because it looks like it. It would be a line.	Visualize Graph
	18	Yes	Yes. Every input has a different output.	Inputs/Outputs
	19	Yes	Going through the x's. No repeated x's. Going through the y's. No repeating y's. So yes.	Inputs/Outputs
	20	IDK	Yes, I don't know why though.	I don't know
	21	No	If we plugged in -5 where x is we would have f(-5) = 6 so no it does not.	Focus on Equation
	22	Yes	Yes, because all the numbers on the y's are even.	Different reasoning: even values
	23	IDK	I don't know because I need the equation.	Focus on Equation