THE COHOMOLOGY OF A FINITE MATRIX QUOTIENT GROUP

by

BRIAN PASKO

A.A.S., Milwaukee Area Technical College, 1996

B.S., Marquette University, 1998

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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the

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In this work, we find the module structure of the cohomology of the group of four by four upper triangular matrices (with ones on the diagonal) with entries from the field on three elements modulo its center. Some of the relations amongst the generators for the cohomology ring are also given. This cohomology is found by considering a certain split extension. We show that the associated Lyndon-Hochschild-Serre spectral sequence collapses at the second page by illustrating a set of generators for the cohomology ring from generating elements of the second page. We also consider two other extensions using more traditional techniques. In the first we introduce some new results giving degree four and five differentials in spectral sequences associated to extensions of a general class of groups and apply these to both the extensions.

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Notation

$GL_n(R)$	invertible $n \times n$ matrices with entries from the ring R
$UT_n(R)$	$n \times n$ upper triangular matrices with entries from the ring R
$H^*(G; M)$	Cohomology of the finite group G with coefficients in the G-module M
$R[G]$	group algebra of group G over ring R
$\beta(z)$	mod-p Bockstein of the class z
Inf_H^G	Inflation from the subgroup H to G
$< x, y, z >$	Massey product of classes x, y, z
$\mathcal{N}_H^G$	Evens norm from subgroup H to G
Res_H^G	Restriction from G to the subgroup H
Tr_H^G	Transfer or corestriction from subgroup H to G
$\mathbb{F}[\alpha_1, \alpha_2]$	Algebra over the field $\mathbb{F}$ generated by α_1 and α_2
$\mathbb{F}[\alpha_1, \alpha_2](x_1, x_2)$	$\mathbb{F}[\alpha_1, \alpha_2]$-module with basis $\{x_1, x_2\}$
$\mathrm{soc}_R(M)$	Socle of the R-module M
$\mathrm{rad}_R(M)$	Radical of the R-module M

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Chapter 1

Introduction

1.1 Background

The foundations of group cohomology were laid down in the early part of the twentieth century by topologists studying the cohomology of topological spaces. Later algebraists set group cohomology on a firm algebraic footing independent of the topology. The difference between the approaches is quite superficial however. Given a finite group G, one can construct a topological space, called the group's classifying space, whose first fundamental group is G and whose higher homotopy groups vanish. Such a space is an Eilenberg-Mac Lane space of G, denoted $K(G,1)$. The cohomology of a finite group is isomorphic to the cohomology of the classifying space of the group. So, a typical example of the ideas that follow familiar to the reader may be the case of a topological space X having a simplicial structure, say a CW-complex. It should now come as no suprise that group cohomology involves a rich interplay between Algebra and Topology. Often when one doesn't know how to solve a problem using one approach one translates it into the language of the other area and then (hopefully) decides the question using known results.

How the group structure is reflected in the cohomology of the group is still largely unknown. Some important information however is. For example, there is a 1-1 correspondence between splittings of the split extension (short exact sequence) $1 \to N \hookrightarrow G \overset{q}{\twoheadrightarrow} Q \to 1$ and elements in $H^1(G; M)$. When the normal subgroup N is abelian, we have a another 1-1 correspondence between the set of equivalent extensions of the above form and $H^2(G; M)$. That is, we can up to isomorphism, decide how many groups G fit into this extension once we fix N and Q. One may suppose that that cohomology is useful in distinguishing groups. Alas, different groups can have the same cohomology. The converse, however, is true. That is, groups with differing cohomology cannot be isomorphic. There are also connections between the mod-p cohomology ring of a group and the modular representations of the group. Indeed, the cohomology groups of $GL_4(\mathbb{F}_3)$ give universal characteristic classes for modular group representations.

The main tool in group cohomology is the spectral sequence. This device provides a computational means of finding the cohomology of a group. Generally, a spectral sequence is a sequence of objects called pages that "converges" to the cohomology of the group G. A page in a spectral sequence is a 1^{st} quadrant array of modules along with a multiplication making the page a ring. The rth page also has a differential, d_r, defined on it. That is, a map of bidegree $\{r, -r+1\}$ which is a derivation with respect to the multiplication. Derivation here means that $d_r(ab) = d_r(a)b + (-1)^{|a|} a d_r(b)$, where if a is in the $\{i, j\}$-position (written $a \in E_r^{i,j}$), then $|a| = i + j$; and $\{r, -r+1\}$-bigraded means that if $a \in E_r^{i,j}$, then $d_r(a) \in E_r^{i+r, j-r+1}$. Given the $E_r -$ page define the $\{i, j\}$ entry of the $E_{r+1} -$ page as $\mathrm{Ker}\, d_r|_{E_r^{i,j}}$ modulo $\mathrm{Im}\, d_r|_{E_r^{i-r, j+r-1}}$. That is, E_{r+1} is the homology of E_r.

Consider again the extension $1 \to N \hookrightarrow G \overset{q}{\twoheadrightarrow} Q \to 1$. A well known theorem says that the second page of the Lyndon-Hochschild-Serre spectral sequence is given as the cohomology of the quotient with coefficients being the cohomology of the normal subgroup. Symbolically,

$E_2^{i,j} \simeq H^i(Q; H^j(N; M))$. With this formula in hand, how to proceed is clear. Choose extensions with sub- and quotient groups having cohomologies that we understand well and from this knowledge determine the cohomology of G. We obtain in the words of McCleary: " $E_2^{i,j} \simeq$ "something computable" converging to H*, something desireable" ([8, Pg. 6]).

Of particular interest to algebraists is $H^*(GL_4(\mathbb{F}_3); \mathbb{F}_3)$, the group of four by four invertable matrices over the field with three elements. One way to find this cohomology is to first find $H^*(UT_4(\mathbb{F}_3); \mathbb{F}_3)$, the upper triangulars with ones on the diagonal, and then since $H^*(GL_4(\mathbb{F}_3); \mathbb{F}_3)$ injects into $H^*(UT_4(\mathbb{F}_3); \mathbb{F}_3)$ we may work backward to derive $H^*(GL_4(\mathbb{F}_3); \mathbb{F}_3)$. Hereafter, let H denote the quotient group $UT_4(\mathbb{F}_3)$ modulo its center. The center is a copy of $\mathbb{Z}/3\mathbb{Z}$ generated by $I_4 + a_{1,4}$. This dissertation will address this group of order 3^5.

The outline of this thesis is as follows: the current chapter gives background on the question of interest, and some standard definitions and results used in the later chapters including a more rigorous development of the ideas above. Chapter 2, our main result, concerns the spectral sequence associated to the extension $(\mathbb{Z}/3\mathbb{Z})^4 \longrightarrow H \longrightarrow \mathbb{Z}/3\mathbb{Z}$. We decompose $H^*((\mathbb{Z}/3\mathbb{Z})^4; \mathbb{F}_3)$ as a sum of indecomposable $\mathbb{F}_3[\mathbb{Z}/3\mathbb{Z}]$-modules to decide the $\mathbb{F}_3[Q]$-module structure of the coefficients, $H^*((\mathbb{Z}/3\mathbb{Z})^4; \mathbb{F}_3)$. From this data we may construct the E_2-page of the associated Lyndon-Hochschild-Serre spectral sequence. According to theorem 2.7, $E_2 = E_\infty$ which is proved by showing that a minimal set of ring generators of the E_2-page represents a minimal set of generators for E_∞. We define the ring generators of $H^*(H; \mathbb{F}_3)$ as Evens norms, Massey products and transfers and their Bocksteins. A complete set of multiplicative relations in $H^*(H; \mathbb{F}_3)$ is not given although, a partial list is presented in appendix A.

Chapter 3 considers the central extension $(\mathbb{Z}/3\mathbb{Z})^2 \longrightarrow H \longrightarrow (\mathbb{Z}/3\mathbb{Z})^3$. Standard theorems give some differentials in the associated spectral sequence and some new results providing some d_4 and d_5 differentials are produced (on pages 42–43). These are of interest because of the general lack of theorems giving differentials in spectral sequences. The calculation of $H^*(H;\mathbb{F}_3)$ is not completed in this section although some comparison is made with the results of chapter 2.

Chapter 4 considers the central extension $\mathbb{Z}/3\mathbb{Z} \longrightarrow H \longrightarrow G_{27} \times \mathbb{Z}/3\mathbb{Z}$. Again, standard theorems as well as the results from chapter 3 give some differentials. We also introduce and apply Leary's circle method to this extension and observe some limitations of this technique.

The final chapter lists some immediate and distant desired results that this work points to. Some initial consideration of $H^*(UT_4(\mathbb{F}_3);\mathbb{F}_3)$ is made including a theorem with the d_2 and d_3 differentials in the spectral sequence associated to the central extension
$\mathbb{Z}/3\mathbb{Z} \longrightarrow UT_4(\mathbb{F}_3) \longrightarrow H$.

We recall the definition of the cohomology of a group. Let G be a finite group and M a $\mathbb{F}_3[G]$-module.

Choose a projective resolution of M over $\mathbb{F}_3[G]$, (X,∂):

$$\cdots \overset{\partial_{n+1}}{\to} X_n \overset{\partial_n}{\to} X_{n-1} \cdots \to X_1 \to X_0 \overset{\epsilon}{\to} M \to 0.$$

This is an exact sequence of $\mathbb{F}_3[G]$-modules and so the X_i are $\mathbb{F}_3[G]$-modules, the ∂_n are $\mathbb{F}_3[G]$-module maps, and $\partial_{n-1} \circ \partial_n = 0$. Apply the contravariant functor $\mathrm{Hom}_{\mathbb{F}_3[G]}(\_\_, M)$ dimension-wise to obtain a cochain complex, (X', δ).

$$\cdots \leftarrow X^{n+1} \overset{\delta_n}{\leftarrow} X^n \overset{\delta_{n-1}}{\leftarrow} X^{n-1} \cdots \leftarrow X^1 \leftarrow X^0.$$

We then define the cohomology of G with coefficients in M as follows:
$H^n(G;M) = \mathrm{Ker}(\delta_n) \,/\, \mathrm{Im}(\delta_{n-1})$. The reader may recognize this as $\mathrm{Ext}^n_{\mathbb{F}_3[G]}(\mathbb{F}_3, M)$, or as the simplicial cochain complex of the topological space $K(G,1)$.

As an example, let $G = \mathbb{Z}/p\mathbb{Z}$ and let M be the trivial $\mathbb{F}_p[G]$-module $\mathbb{F}_p$. Suppose $G =< a >$. A projective resolution, $X \to \mathbb{F}_p$ is $X_i = \mathbb{F}_p[G]e_i$ for $i \geq 0$, and

$$\partial(e_i) = \begin{cases} \sum_{j=1}^{p} a^j e_{i-1} & \text{for } i \text{ even,} \\ (a-1)e_{i-1} & \text{for } i \text{ odd.} \end{cases}$$

This resolution ends: $\cdots \xrightarrow{\sum_{j=1}^{p} a^j} \mathbb{F}_p[G] \xrightarrow{a-1} \mathbb{F}_p[G] \xrightarrow{\sum_{j=1}^{p} a^j} \mathbb{F}_p[G] \xrightarrow{a-1} \mathbb{F}_p[G] \xrightarrow{\epsilon} M \to 0.$

The map $\epsilon : X_0 \twoheadrightarrow M$ takes the basis element $e_0 \mapsto 1$ (the augmentation map). This resolution is minimal in the sense that $\mathrm{Ker}\epsilon = \mathrm{Im}\partial_1 \subseteq JX_0$, and for $n > 0$, $\mathrm{Ker}\partial_n = \mathrm{Im}\partial_{n+1} \subseteq JX_n$, where J denotes the Jacobson radical. Minimal resolutions are unique up to isomorphism of complexes (see [7] for additional details). These resolutions are particularly useful since upon applying the functor, $\mathrm{Hom}_{\mathbb{F}_3[G]}(\_\_, M)$ we obtain a cochain complex where all the coboundary maps are zero and in each degree is a copy of the coefficients.

$$\cdots \xleftarrow{0} \mathbb{F}_p \xleftarrow{0} \mathbb{F}_p \xleftarrow{0} \mathbb{F}_p.$$

Thus, $H^n(G; M) \simeq \mathbb{Z}/p\mathbb{Z}$ for all $n \geq 0$.

The cohomology of a group G admits a graded ring structure when the coefficients are a ring. Graded here means that $\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$, where $|z|$ denotes the degree of the class $z \in H^*(G; \mathbb{F}_p)$. Definitions and proofs of various properties of this *cup product* may be found in standard references (e.g. [5], [7]). What concerns us for the moment is the fact that we may use this product to write cohomologies as algebras. Thus, for example, we may write $H^*(\mathbb{Z}/p\mathbb{Z}; \mathbb{F}_p)$ as the tensor product of a polynomial algebra on one degree two generator α and an exterior algebra on one degree one generator x: $\mathbb{F}_p[\alpha] \otimes \wedge_{\mathbb{F}_p}(x)$.

We will often write cohomologies as modules over a polynomial subalgebra and refer to this description as an *additive description*. For example, $H^*(\mathbb{Z}/p\mathbb{Z}; \mathbb{F}_p) \simeq \mathbb{F}_p[\alpha] \otimes \wedge_{\mathbb{F}_p}(x)$ we write as $\mathbb{F}_p[\alpha](1, x)$; and, $H^*((\mathbb{Z}/p\mathbb{Z})^2; \mathbb{F}_p) \simeq \mathbb{F}_p[\alpha, \beta] \otimes \wedge_{\mathbb{F}_p}(x, y)$ as $\mathbb{F}_p[\alpha, \beta](1, x, y, xy)$, where x, y are degree one generators and α, β are degree two generators.

Continuing the example above, by the universal coefficient theorem, $H^*((\mathbb{Z}/p\mathbb{Z})^m; \mathbb{F}_p) \simeq \bigotimes_{j=1}^{m} H^*(\mathbb{Z}/p\mathbb{Z}; \mathbb{F}_p)$. We may obtain the same result by noticing that the tensor product of the usual minimal resolutions of each copy of $\mathbb{Z}/p\mathbb{Z}$ is a resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[(\mathbb{Z}/p\mathbb{Z})^m]$. Indeed it is a minimal resolution. In the associated cochain complex the coboundary maps are again all zero. Thus $H^n((\mathbb{Z}/p\mathbb{Z})^m; \mathbb{F}_p) \simeq \mathbb{F}_3[\alpha_1, \dots \alpha_m] \otimes \wedge_{\mathbb{F}_3}(x_1, \dots x_m)$ with x_i degree one and α_i degree two. Or additively,

$\mathbb{F}_3[\alpha_1, \dots \alpha_m](1, x_1, \dots x_m, x_i x_j, \dots, x_i x_j x_k, \dots x_1 x_2 \dots x_m)$ ♦

1.2 History

Let $UT_n(\mathbb{F}_p)$ denote the n by n upper triangular matrices (with ones on the diagonal) with entries from the field on p elements, $\mathbb{F}_p$. Let $GL_n(\mathbb{F}_p)$ denotes the invertible n by n matrices over $\mathbb{F}_p$.

The cohomology of the two by two upper triangulars, $UT_2(\mathbb{F}_p)$ is well known. Indeed, this group is isomorphic to the cyclic group of order p. Quillen [16] found $H^*(UT_3(\mathbb{F}_2); \mathbb{F}_2)$. In this work he introduced many new and powerful techniques to approach the cohomology of a group. Indeed, this work largely marked the beginning of modern group cohomology. Lewis later [11] decided the ring structure of $H^*(UT_3(\mathbb{F}_3); \mathbb{Z})$. (Note: this group is isomorphic to the extra special group of order 27 and exponent 3.) In the 1980's and 90's, several authors gave arguments to find $H^*(UT_3(\mathbb{F}_3); \mathbb{F}_3)$. This cohomology appeared as a subalgebra of work by Mimh [15] who gave the ring structure, although, Milgram and Tezuka report that his answer was incorrect.

In [10], Leary gives the complete ring structure of $H^*(UT_3(\mathbb{F}_3); \mathbb{F}_3)$ including a full set of multiplicative relations. He introduced his 'circle method' in this, his PhD thesis. We shall address this method later. Leary revisited $H^*(UT_3(\mathbb{F}_3); \mathbb{F}_3)$ in [9]. This time he attacked the problem quite differently. He again considered a central extension but rather than applying the circle method, he used the known results from his previous work to find the d_2 and d_3 differentials. The Kudo and Serre transgression theorems gave some d_5 differentials and additionally, the author was able to produce new results giving the d_4 differential completely. We shall see generalizations of these results in Chapter 3. Benson and Carlson [4] published a summary of the work to date (1991) on $H^*(UT_3(\mathbb{F}_3); \mathbb{F}_3)$ and more generally, the extraspecial p groups including a algebraic determination of $H^*(UT_3(\mathbb{F}_3); \mathbb{F}_3)$.

Published the same year as [9], Milgram and Tezuka [14] obtain the ring $H^*(UT_3(\mathbb{F}_3); \mathbb{F}_3)$

by using Lewis' result. They used the fact that the short exact sequence of coefficients $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ induces a long exact sequence in cohomology:

$$\cdots \leftarrow H^{n+1}(G;\mathbb{Z}/p\mathbb{Z}) \xleftarrow{\delta_n} H^n(G;\mathbb{Z}) \leftarrow H^n(G;\mathbb{Z}) \leftarrow H^n(G;\mathbb{Z}/p\mathbb{Z}) \xleftarrow{\delta_{n-1}} \cdots,$$

from which they were able to reconstruct $H^*(UT_3(\mathbb{F}_3);\mathbb{F}_3)$. While the authors do not explicitly give a set of relations, they do give an additive description of the cohomology (i.e. wrote it as a sum of modules over polynomial subalgebras, algebraists refer to this as writting this algebra in 'normal form') while Leary did not. Of course, given a set of generators and relations, one can construct an additive description, however, the converse is not true.

Siegel [17] introduced means of constructing the E_2-page of a spectral sequence associated to a certain extension using representation theory, one which we shall make use of to obtain our principle result. He obtained the additive structure but stopped short of finding the relations or giving an additive description.

This discussion leads to the question: what is $H^*(UT_4(\mathbb{F}_3);\mathbb{F}_3)$? Some comments on the cohomology of $UT_4(\mathbb{F}_3)$ are given in chapter 5 but, this tract will mainly concern $H^*(H;\mathbb{F}_3)$ where H denotes $UT_4(\mathbb{F}_3)$ modulo its center. ♦

1.3 Preliminaries

We record some basic facts about matrix groups. Hereafter, we let $a_{i,j} = I_n + e_{i,j}$, where $e_{i,j}$ is the $n \times n$ matrix with 1 in the i,j-entry and zeroes elsewhere.

The Sylow-p subgroup of $GL_n(\mathbb{F}_p)$ is the $n \times n$ upper triangular matrices with 1's on the diagonal, $UT_n(\mathbb{F}_p)$. The center of $UT_n(\mathbb{F}_p)$ is a copy of $\mathbb{Z}/p\mathbb{Z}$ generated by $a_{1,n}$. The center of the quotient of $UT_n(\mathbb{F}_p)$ by its center is a copy of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ generated by $\overline{a_{1,n-1}}$, and $\overline{a_{2,n}}$.

Recall that multiplication in $UT_n(\mathbb{F}_p)$ is given as follows [18]:

$$se_{ij}te_{hk} = \begin{cases} (s+t)e_{ik} & \text{if } j = h, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we have a formula for the commutator of two elements: if $u < v < w$, then $[sa_{uv}, ta_{vw}] = sta_{uw}$.

In general, define $H_k = \{A \in UT_n(\mathbb{F}_p) | a_{ij} = 0 \text{ for } 0 < j - i < k\}$.

Proposition 1.1. *The collection* $\{sa_{i,i+1}\}_{i=1}^{n-1}$ *generates* $UT_n(\mathbb{F}_p)$.

Proof: We will only concern ourselves with the cases $n = 3, 4$ for which the result is easy to see by calculation. The more general proof is given in [18] which is just the obvious induction proof. □

Let H be $UT_4(\mathbb{F}_3)$ modulo its center, $H = UT_4(\mathbb{F}_3)/ < a_{1,4} >$. Make the following identifications: a_i denotes $a_{i,i+1}$, a_{2i+2} denotes $a_{i,i+2}$ and a_6 denotes $a_{1,4}$. We will use a few convenient abuses of notation. A 'subgroup generated by a_i' will mean the subgroup $< a_{ij} >$ as appropriate. Also, we will write elements in the group H as a_i rather than $\overline{a_i}$; and, elements of quotient groups of H as a_i instead of $\overline{\overline{a_i}}$, etc. It should be clear from the context which group and which elements we're referring to. The group H is generated by a_1, a_2, a_3. Indeed, $[a_1, a_2] = a_4$ and $[a_2, a_3] = a_5$, and $[a_4, a_5] = e$. H contains many copies of $UT_3(\mathbb{F}_3)$. The two copies $< a_4, a_2, a_1 >$ and $< a_5, a_3, a_2 >$ will be used regularly in the main body of this text. We will call these subgroups G_{27}^h and G_{27}^l, respectively. ♦

1.4 Maps from subgroups

This section contains material familiar to the area of study. It is included here to for the sake of the expository nature of this work. All the material in this section may be found in many references, among them [3], [5], [7]. We will make extensive use of these ideas throughout this text explicitly or otherwise. The transfer and Evens norm we present in the restricted case of finite groups and $\mathbb{F}_p$ coefficients since this is all that will be required in this work. The interested reader can find the more general statements and applications in the texts cited above.

In later chapters we will use 1.6 to show that certain elements in spectral sequences live to the $E_\infty-$page. We will be able to define non-zero classes in the cohomology of the group H as Evens norms or transfers of classes in a particular subgroup. In addition, in Appendix A, Proposition 1.3 will allow us to find relations amongst the ring generators of a group's cohomology.

We will not elaborate on the obvious maps $\mathrm{Res}^G_N\colon\ H^*(G;M) \to H^*(G;M)$ induced by the inclusion $N \hookrightarrow G$; and $\mathrm{Inf}^G_N\colon\ H^*(G/N;M^N) \to H^*(G;M)$ induced by the canonical quotient map $G \overset{q}{\twoheadrightarrow} G/N$ when $N \lhd G$. The Bockstein map is the connecting homomorphism $\beta\colon\ H^*(G;\mathbb{Z}/p\mathbb{Z}) \to H^{*+1}(G;\mathbb{Z}/p\mathbb{Z})$ in the long exact cohomology sequence induced by the sequence of coefficients $\mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$.

Transfer

Let D be a set of coset representatives for a subgroup of finite index, H, in G. Let $X \to \mathbb{F}_p$ be a resolution of $\mathbb{F}_p$ over $\mathbb{F}_p[G]$. Note that by restriction it is also a $\mathbb{F}_p[H]$-resolution. Suppose the cochain f represents $\alpha \in H^*(H;\mathbb{F}_p)$. At the cochain level, the transfer is given as the map $T : \mathrm{Hom}_{\mathbb{F}_p[H]}(X, \mathbb{F}_p) \to \mathrm{Hom}_{\mathbb{F}_p[G]}(X, \mathbb{F}_p)$ by

$$T_{G/H}(f) = \sum_{x \in D} x \cdot f.$$

It is obvious that T is well defined and commutes with the differentials in both cochain complexes.

Definition 1.2. The *transfer* is the map $\mathrm{Tr}_H^G : H^*(H; \mathbb{F}_p) \to H^*(G; \mathbb{F}_p)$ induced by T.

Proposition 1.3. *Let $H, K < G$ with $[G : H] < \infty$ and let D be a set of coset representatives for the double cosets HgK. For $\alpha \in H^*(G; M)$,*
$\mathrm{Res}_K^G(Tr_H^G(\alpha)) = \sum\limits_{g \in D} Tr_{K \cap gKg^{-1}}^K(\mathrm{Res}_{H \cap gHg^{-1}}^{gHg^{-1}}(c_{g^{-1}}(\alpha)))$. *When $H \lhd G$ this becomes*
$\mathrm{Res}_K^G(Tr_H^G(\alpha)) = \sum\limits_{g \in D} c_{g^{-1}}(\alpha)$.

Evens Norm

First defined in [6], the Evens norm, denoted $\mathcal{N}$, will be an important idea in this text. The reader may reference that text for the more general statements.

Let X denote G/H and Y denote H. Let $S = \{s \in S(X \times Y) | s(x, y) = (s(x), y) \; \forall \; (x, y) \in X \times Y\}$. The product of $|X|$ copies of H, H^X, is in 1:1-correspondence with the set $\{s \in S(X \times Y) | s(x, y) = (x, h_x(y)) \; \forall \; (x, y) \in X \times Y \text{ for some } h_x \in H\}$. It is easy to see that $S \cap H^X$ consists only of the identity and that S is in the normalizer of H^X in $S(X \times Y)$. We then define the *wreath product* of X and Y, $\mathcal{W} = H^X \rtimes S$ and denote it by $\mathcal{W} = S \int H$.

We now define the map $1 \int \alpha : H^*(H; \mathbb{F}_p) \to H^*(S(G/H) \int H; \mathbb{F}_p)$. To do so we require Nakaoka's theorem:

Theorem 1.4. $H^*(S(G/H) \int H; \mathbb{F}_p) \simeq H^*(S; H^*(H; \mathbb{F}_p)^{\otimes G/H})$.

Note that $H^0(S; H^*(H;\mathbb{F}_p)^{\otimes G/H})$ consists of those classes in $H^*(H;\mathbb{F}_p)^{\otimes G/H}$ invariant under the action of S. For $\alpha \in H^{even}(H;\mathbb{F}_p)$, the class $\alpha^{\otimes [G:H]} = \underbrace{\alpha \otimes \cdots \otimes \alpha}_{[G:H] \text{ times}} \in H^*(H;\mathbb{F}_p)^{\otimes G/H}$ is such an invariant. Using Nakaoka's result we define $1 \int \alpha$ as the corresponding class in $H^*(S(G/H) \int H;\mathbb{F}_p)$.

We construct the map $\Psi : G \to S(G/H) \int H$, called the *monomial representation*, as follows. Let T be a set of coset representatives for the cosets G/H and let M by an H-module. Using the usual construction of the *induced module* we make $\mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} M \simeq \bigoplus_{t \in T} t\mathbb{F}_p[H] \otimes_{\mathbb{F}_p[H]} M$ a G-module by letting $gt = \overline{gt} h_{g,t}$ where $h_{g,t} \in H$ and $\overline{s}$ denotes the coset representative in T for s. Note G acts on T by left multiplication $gt = \overline{gt}$. That is, we have $\pi(g) \in S(T) \simeq S(G/H)$. We finally define $\Psi(g) = \pi(g) \prod_{t \in T} h_{g,t}$.

Definition 1.5. If $\alpha \in H^*(H;\mathbb{F}_p)$ is of even degree, define $\mathcal{N}_H^G(\alpha) = \Psi^*(1 \int \alpha)$.

Proposition 1.6. *[7, Theorem 6.1.1 (N3), (N4)] Suppose H is a subgroup of G of finite index. If $G = \bigcup_{x \in D} KxH$ is a double coset decomposition of G, then for all $\alpha \in H^*(H;\mathbb{F}_p)$ of even degree,* $\mathrm{Res}_K^G(\mathcal{N}_H^G(\alpha)) = \prod_{x \in D} \mathcal{N}_{K \cap xHx^{-1}}^K(\mathrm{Res}_{K \cap xHx^{-1}}^{xHx^{-1}}(c_{x^{-1}}(\alpha)))$. *When $H \lhd G$, this becomes* $\mathrm{Res}_K^G(\mathcal{N}_H^G(\alpha)) = \prod_{x \in D} c_{x^{-1}(\alpha)}$.

♦

1.5 Representation Theory

In this section we gather some basic definitions and facts from modular representation theory that will be used in this work. The reader is advised to see [2] for additional details. In this section, A is a finite-dimensional algebra with unit element over $\mathbb{F}_p$, and U is a left A-module that is finite-dimensional over $\mathbb{F}_p$.

Definition 1.7. The socle, $soc_A(U)$, is the largest semisimple (i.e. direct sum of simples) submodule of U.

Definition 1.8. The radical, $rad_A(U)$, is the smallest submodule of U with semisimple quotient.

Alperin [2] states the following theorem. The proof is omitted here since, in Alperin's words, "...[it] presents no techniques we shall have any further use for" ([2, Pg. 17]). We are interested in a corollary of this result which places important conditions on the structure of $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$-modules.

Theorem 1.9. *[2, Theorem 2] The number of simple $\mathbb{F}_p[G]$-modules equals the number of conjugacy classes of G of elements whose order is not divisible p.*

Corollary 1.10. *[2, Corollary 3] The only simple $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$-module is the 1-dimensional trivial module $\mathbb{F}_p$.*

Proof: The trivial module is simple and by 1.9 the identity is the only conjugacy class of order not divisible by p. $\square$

In chapter 2 we will use the usual minimal resolution for the group $Q \simeq \mathbb{Z}/p\mathbb{Z} =< a >$. Let S be a $\mathbb{F}_3[Q]$-module.

Proposition 1.11. *The socle and radical of S are given by:*

i .) $\mathrm{soc}_{\mathbb{F}_3[Q]}(S) = Ker(a-1)$

ii .) $\mathrm{rad}_{\mathbb{F}_3[Q]}(S) = Im(a-1)$.

Proof: i .) The socle $\mathrm{soc}_{\mathbb{F}_3[Q]}(S)$ is a sum of simple $\mathbb{F}_3[Q]$-modules. By 1.10 on the preceding page, these simples are copies of the trivial module $\mathbb{F}_p$ thus these simples are contained in $\mathrm{Ker}(a-1)$. Now, $\mathrm{Ker}(a-1)$ is a trivial module and is therefore semisimple. Since $\mathrm{soc}_{\mathbb{F}_3[Q]}(S)$ is the largest semisimple submodule, $\mathrm{Ker}(a-1) \subset \mathrm{soc}_{\mathbb{F}_3[Q]}(S)$.

ii .) It is well known that $\mathrm{rad}_{\mathbb{F}_3[Q]}(S) = \mathrm{rad}(\mathbb{F}_3[Q])S$. The subalgebra of $\mathbb{F}_3[Q]$, $\mathrm{rad}(\mathbb{F}_3[Q])$ is the collection of elements which annihilate every simple $\mathbb{F}_3[Q]$-module. Thus $\mathrm{rad}(\mathbb{F}_3[Q]) = \{q-1 | q \in Q\}$. □

Corollary 1.12. *In addition,*

i .) $\mathrm{soc}^2_{\mathbb{F}_3[Q]}(S) = Ker(a-1)^2$

ii .) $\mathrm{rad}^2_{\mathbb{F}_3[Q]}(S) = Im(a-1)^2$.

Proof: i .) We define $\mathrm{soc}^2_{\mathbb{F}_3[Q]}(S) = q^{-1}(\mathrm{soc}(S/\mathrm{soc}_{\mathbb{F}_3[Q]}(S))$ where $q : S \to S/\mathrm{soc}_{\mathbb{F}_3[Q]}(S)$ is the canonical quotient map. Using the alternative description of $\mathrm{soc}_{\mathbb{F}_3[Q]}(S)$ as the collection of all $u \in S$ such that $\mathrm{rad}(\mathbb{F}_3[Q])u = 0$ and part *ii.* of 1.11, the result is clear.

ii .) Clear: $\mathrm{rad}^2_{\mathbb{F}_3[Q]}(S) = \mathrm{rad}(\mathrm{rad}_{\mathbb{F}_3[Q]}(S))$. ♦

1.6 Massey products

The Massey triple product will be used throughout this text. Let P_* be the bar resolution for $\mathbb{F}_p$ over $\mathbb{F}_p[G]$. Suppose $x, y, z \in H^*(G; \mathbb{F}_p)$ with representative cocycles x, y, z in P^* satisfy xy and yz are both zero in $H^*(G; \mathbb{F}_p)$. Then, for some $\alpha \in P^{deg\, x + deg\, y - 1}$ and $\beta \in P^{deg\, y + deg\, z - 1}$, $\delta(\alpha) = xy$ and $\delta(\beta) = yz$.

For x, y and z as above, we call the collection of cochains $A_{ij} = (a_{ij})$ with $a_{11} = x,\ a_{22} = y,\ a_{33} = z,\ a_{12} = \alpha$ and $a_{23} = \beta$ a *defining system* for the Massey product of x, y and z. The cocycle $(-1)^{|\alpha|}\alpha z - x\beta$ is a *related cocycle* for the Massey product of x, y and z

Definition 1.13. The *Massey product* $< x, y, z >$ is defined to be the class represented by all such related cocycles. This operation is only well defined modulo $xH^{|y||z|-1}(G; \mathbb{F}_p) + H^{|x||y|-1}(G; \mathbb{F}_p)z$ since any cocycle representing a class in $H^*(G; \mathbb{F}_p)$ will have coboundary zero. We call this set the *indeterminacy* of the product $< x, y, z >$.

Definition 1.14. Suppose the Massey product $< x, x, x >$ is defined in $H^*(G; \mathbb{F}_p)$. If we restrict the defining system for this product A_{ij} to $a_{ii} = x$ and $a_{12} = a_{23}$, then, we call the resulting class a *restricted* Massey product and denote it $< x >^3$.

It is not hard to see that $< x >^3$ is defined without indeterminacy. This definition will be useful in defining generating classes in cohomologies. The proposition below will be used implicitly in later chapters.

Proposition 1.15. *[8, Theorem 1.4] If $u \in H^{2m+1}(G; \mathbb{F}_p)$, then $< u >^p$ is defined as a single class in $H^{2mp+2}(G; \mathbb{F}_p)$ and $< u >^p = -\beta(P^m u)$.*

Massey products satisfy the following well-known properties proofs may be found in [8] or [10]: given $u, v, w, x, y \in G$ such that all of the products are defined, (a). Massey products are additive in each argument,

(b). $< u, v, w > x + (-1)^u u < v, w, x > \equiv 0 \bmod uH^* x$,

(c). $(-1)^{wu} < u, v, w > +(-1)^{uv} < v, w, u > +(-1)^{vw} < w, u, v > \equiv 0$

$\bmod uH^* + vH^* + wH^*$,

(d). $< u, v, w > +(-1)^{|u||v|+|v||w|+|u||w|} < w, v, u > \bmod uH^* + H^* w$.

Proposition 1.16. *Suppose* $x \in H^1(G; \mathbb{F}_3)$, *then* $- < x >^3 = \beta(x) = < x, x, x >$.

Proof: The first equality is from Proposition 1.15 and the second from [10]. □

Quadruple and higher Massey products are defined in various references [12], [8]. McCleary [13] gives some interesting applications of Massey products to problems in Topology. In [12] May introduced further generalizations to matric Massey products. A corollary of his work there is the following.

Proposition 1.17. *Every class in* $H^*(G; \mathbb{F}_p)$ *can be described as a matric Massey product.*

We will not have use of these generalizations in this work and so will not pursue these ideas further. ♦

1.7 $\mathbf{G_{27}}$

Let G_{27} be the extra-special group of order 27 and exponent 3. $G_{27} \simeq UT_3(\mathbb{F}_3)$. We may present G_{27} as $< a, b, c | a^3 = b^3 = c^3 = [a, c] = [b, c] = 1, [a, b] = c >$. As a ring, $H^*(G_{27}; \mathbb{F}_3)$ consists of nine generators which we label as follows:

$$\underbrace{x_1, x_2,}_{deg\,1} \quad \underbrace{\alpha_1, \alpha_2, \sigma, \tau,}_{deg\,2} \quad \underbrace{\mu, \nu}_{deg\,3} \text{ and } \underbrace{\gamma.}_{deg\,6}$$

Since $H^1(G_{27}; \mathbb{F}_3) \simeq \mathrm{Hom}_{\mathbb{F}_3[G_{27}]}(G_{27}, \mathbb{F}_3)$, we may define $x_i \in H^1(G_{27}; \mathbb{F}_3)$ as follows.

$$x_1(g) = \begin{cases} 1 & \text{if } g = a, \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } x_2(g) = \begin{cases} 1 & \text{if } g = b, \\ 0 & \text{otherwise.} \end{cases} \quad ;$$

$\alpha_i = \beta(x_i) =< x_i, x_i, x_i >=< x_i >^3$ according to 1.16. Take σ as $< x_1, x_1, x_2 >$ and τ as $< x_1, x_2, x_2 >$ (see 1.18). Let $\mu = \beta(\sigma)$ and $\nu = \beta(\tau)$. The class γ we define as an Evens norm, $\gamma = \mathcal{N}(\delta)$, where δ is a degree two generator in $H^*(< a, c >; \mathbb{F}_3)$ which is the inflation of the degree two generator in $H^*(< c >; \mathbb{F}_3)$.

For $g \in G_{27}$, let $g = c^t b^s a^u$. In the bar resolution for G_{27} consider the cochains: $u[g] = u \quad s[g] = s \quad t[g] = t$. Note that u represents x_1 and s represents x_2.

Proposition 1.18. *Let $\sigma = u_1^2 s_2 - u_1 v_2$ and $\tau = u_1 s_2^2 - v_1 s_2$. The cohomology classes represented by these cocycles $\sigma =< x_1, x_1, x_2 >$ and $\tau =< x_1, x_2, x_2 >$.*

Proof: The proof for τ is similar so we only prove the claim for σ. We need show both that the class represented by the cocycle is an element of the Massey product and that this product is defined as a unique class in $H^2(G_{27}; \mathbb{F}_3)$ (i.e. is defined without indeterminacy) so that the equality makes sense.

By computing the coboundary of u^2 in the bar resolution for G_{27}: $\delta(u^2)[c^t b^s a^u | c^{t'} b^{s'} a^{u'}] = u^2\{[c^{t'} b^{s'} a^{u'}] - [c^{t+t'+us'} b^{s+s'} a^{u+u'}] + [c^t b^s a^u]\} = u'^2 - (u + u')^2 + u^2 = -uu'$, we see that

$\delta(u^2) = u_1u_2$ (u_1u_2 represents the cohomology class $x_1 \cup x_1$). Similarly, $\delta(s_1^2) = s_1s_2 = x_2 \cup x_2$ and $\delta(t_1) = -u_1s_2 = -x_1 \cup x_2$. The system A with

$a_{12} = u^2, a_{23} = t, a_{11} = u, a_{22} = u$ and $a_{33} = s$ is a defining system for $< x_1, x_2, x_2 >$. A related cocycle is $u_1^2s_2 - u_1v_2$.

The indeterminacy of σ is $x_1H^1{+}H^1x_2$. As a set, this indeterminacy is $\{x_1x_1, x_1x_2, x_2x_2\}$. But, $x_i^2 = 0$ since these are both degree one classes and $x_1x_2 = 0$ also. Thus σ is defined without indeterminacy. Setwise, the indeterminacy of σ is the same as τ and so is zero as well. Alternatively, we could quote a suitably modified version of a result of Kraines, 1.19. □

Proposition 1.19. *[8, Lemma 20] For an odd prime p, if $x, y, z \in H^{odd}(G; \mathbb{F}_p)$ such that $< x, y, z >$ is defined, then $< x, y, z >$ has no indeterminacy.*

□

The relations amongst the ring generators for $H^*(G_{27}; \mathbb{F}_3)$ are:

$$x_i^2 = x_1x_2 = \mu^2 = \nu^2 = \mu\nu = 0$$

$$x_1\tau = x_2\sigma \qquad x_1\sigma = x_2\tau = x_2\alpha_1 = x_1\alpha_2$$

$$\sigma\tau = \alpha_1\alpha_2 \qquad \sigma^2 = \alpha_1\tau = x_2\mu + \alpha_2\sigma \qquad \tau^2 = \alpha_2\sigma$$

$$x_1\mu = \alpha_1\sigma - \alpha_1\alpha_2 \qquad x_1\nu = -x_2\mu \qquad x_2\nu = \alpha_2\tau - \alpha_1\alpha_2$$

$$\tau\mu = -\sigma\nu \qquad \sigma\mu = -\alpha_1\nu = -\tau\nu = \alpha_2\mu \qquad x_1\alpha_1\alpha_2 = x_2\alpha_2\sigma$$

$$\alpha_1^2\alpha_2 = \alpha_2^2\sigma - x_2\alpha_2\mu \qquad \alpha_1^2\tau = -\alpha_1\alpha_2\sigma - \alpha_1\alpha_2^2$$

$$-\alpha_1\alpha_2\mu = \alpha_2\sigma\nu$$

$$\alpha_1^2\alpha_2\sigma = \alpha_2^3\sigma.$$

Additively, $H^*(G_{27};\mathbb{F}_3)$ is isomorphic to

$$\mathbb{F}_3[\gamma] \otimes \begin{cases} \mathbb{F}_3[\alpha_1](1, x_1, \sigma, \mu)\oplus \\ \mathbb{F}_3[\alpha_2](x_2, \alpha_2, \tau, \nu, x_1\alpha_2, x_2\sigma, \alpha_1\alpha_2, \alpha_2\sigma, x_2\mu, \alpha_2\mu, \sigma\nu, \alpha_1\alpha_2\sigma). \end{cases}$$

Consider the four copies of $(\mathbb{Z}/3\mathbb{Z})^2$ in G_{27}: H_{11}, H_{12}, H_{01} and H_{10} defined by $H_{ij} =< b^i a^j, c >$.

Proposition 1.20. *The subgroups H_{ij} form a collection of detecting subgroups for the cohomology of G_{27}.*

Proof: We will write $H^*(< b^i a^j, c >;\mathbb{F}_3) = \mathbb{F}_3[\alpha, \delta](1, x, d, xd)$, where the d is the inflation of the one-dimensional generator of $H^*(< c >;\mathbb{F}_3)$, δ its Bockstein and x, α analogously. The claim follows from Table 1.1 which lists the restrictions of the generators of $H^*(G_{27};\mathbb{F}_3)$ to the H_{ij}. Restrict each class, $z \in H^*(G_{27};\mathbb{F}_3)$ to see that $\mathrm{Res}^{G_{27}}_{H_{ij}}(z) \neq 0$ for at least one H_{ij}. $\square$

Table 1.1: Restrictions in G_{27}:

$\mathrm{Res}^{G_{27}}_{H}$	$\overset{<ba,c>}{H_{11}}$	$\overset{<ba^2,c>}{H_{12}}$	$\overset{<b,c>}{H_{10}}$	$\overset{<a,c>}{H_{01}}$
x_1	x	$-x$	0	x
x_2	x	x	x	0
α_1	α	$-\alpha$	0	α
α_2	α	α	α	0
σ	$\alpha - xd$	$\alpha + xd$	0	$-xd$
τ	$\alpha + xd$	$-\alpha + xd$	xd	0
μ	$-\alpha d + x\delta$	$\alpha d - x\delta$	0	$-\alpha d + x\delta$
ν	$\alpha d - x\delta$	$\alpha d - x\delta$	$\alpha d - x\delta$	0
γ	$\delta^3 - \alpha^2\delta$	$\delta^3 - \alpha^2\delta$	$\delta^3 - \alpha^2\delta$	$\delta^3 - \alpha^2\delta$

Proposition 1.21. *G_{27} has no non-trivial essential cohomology. That is, $H^*(G_{27};\mathbb{F}_3)$ contains no non-trivial classes whose restriction to every subgroup is zero.*

Proof: Directly from table 1.1. Or, by [14], $H^*(G_{27};\mathbb{F}_3)$ is free and finitely generated over a polynomial subalgebra (i.e. is Cohen-Macaulay) and so, by [1] G_{27} does not contain non-trivial essential cohomology. It should be noted that the additive description given on the preceding page does is not describe this cohomology as Cohen-Macaulay. $\square$

The following table of transfers is included in this section for completeness.

Table 1.2: Transfers from $< c >\simeq \mathbb{Z}/3\mathbb{Z}$ in G_{27}:

	$Tr^{G_{27}}_{\mathbb{Z}/3\mathbb{Z}}(\quad)$
x_4	0
α_4	0
$x_4\alpha_4$	$-x_1\alpha_1 + x_2\sigma$
α_4^2	$-\alpha_1^2 + \alpha_2\sigma - x_2\mu$
$x_4\alpha_4^2$	$\alpha_1\mu + \sigma\nu$

Chapter 2

A Non-central Split Extension

In this part, we consider the split extension

$$N \longrightarrow H = N \rtimes Q \longrightarrow Q, \tag{2.1}$$

with $N =< a_5, a_4, a_3, a_1 >\simeq (\mathbb{Z}/3\mathbb{Z})^4$ and $Q =< \overline{a_2} >\simeq \mathbb{Z}/3\mathbb{Z}$. Recall that $H^*(N;\mathbb{F}_3) = \mathbb{F}_3[\alpha_1, \alpha_3, \alpha_4, \alpha_5] \otimes \wedge_{\mathbb{F}_3}(x_1, x_3, x_4, x_5)$. We define $x_i \in H^1(N;\mathbb{F}_3) \simeq \mathrm{Hom}_{\mathbb{F}_3}(N, \mathbb{F}_3)$ by $x_i(a_5^{j_5}a_4^{j_4}a_3^{j_3}a_1^{j_1}) = j_i$; and, α_i as the Bockstein of x_i. Similarly, we write $H^*(Q;\mathbb{F}_3) = \mathbb{F}_3[\alpha_2] \otimes \wedge_{\mathbb{F}_3}(x_2)$ and define x_2 and α_2 analogously. The action of Q on $H^*(N;\mathbb{F}_3)$ is given by $a_2(z_5) = z_5,\ a_2(z_4) = z_4,\ a_2(z_3) = z_5 + z_3,\ a_2(z_1) = z_4 - z_1$ for $z_i = x_i$ or α_i.

We follow the approach used by Siegel in [17]. In this article Siegel gives a method for constructing the E_2- page for a non-central extension where the quotient is cyclic. In outline, he describes the E_2- page $\simeq H^*(Q; H^*(N;\mathbb{F}_3))$ by decomposing $H^*(H;\mathbb{F}_3)$ as a direct sum of indecomposable $\mathbb{F}_3[Q]$-modules.

As in Siegel, for a $\mathbb{F}_3[Q]$-module M we have

$$H^r(Q;M) \simeq \begin{cases} \mathrm{soc}_{\mathbb{F}_3[Q]}(M) & \text{for } r = 0, \\ \mathrm{soc}^2_{\mathbb{F}_3[Q]}(M)/\mathrm{rad}_{\mathbb{F}_3[Q]}(M) & \text{for } r \text{ odd}, \\ \mathrm{soc}_{\mathbb{F}_3[Q]}(M)/\mathrm{rad}^2_{\mathbb{F}_3[Q]}(M) & \text{for } r > 0 \text{ even}. \end{cases} \tag{2.2}$$

Equation 2.2 is clear when we consider the usual minimal resolution for $\mathbb{F}_3$ over $\mathbb{F}_3[Q]$, $X \to \mathbb{F}_3$:

$$X_i = \mathbb{F}_3[Q]e_i \text{ for } i \geq 0, \text{ and } \partial(e_i) = \begin{cases} \sum_{j=1}^{3} a_2^j e_{i-1} = (a_2-1)^2 e_{i-1} & \text{for } i \text{ even,} \\ (a_2-1)e_{i-1} & \text{for } i \text{ odd.} \end{cases}$$

We must determine the $\mathbb{F}_3[Q]$-module structure of $H^*(N;\mathbb{F}_3)$. There are, up to isomorphism, 3 indecomposable $\mathbb{F}_3[Q]$-modules (see [2, Pg. 24 ff.]). We denote these by J_i, for $1 \leq i \leq 3$, where J_i has dimension i. Note that $J_1 = \mathbb{F}_3$, $J_2 = \mathbb{F}_3^2$ and $J_3 = \mathbb{F}_3^3$. J_3 is projective, indeed $J_3 \simeq \mathbb{F}_3[Q]$. The notation J_i is used because the action of the generator $a_2 \in Q$ is given by a single Jordan block of size i. We will write $J_2 =< x, y >$ and $J_3 =< x, y, z >$ if these are $\mathbb{F}_3$-bases cyclic with respect to $a_2 - 1$, that is $(a_2-1)z = y$ and $(a_2-1)y = x$.

Let $W = \mathbb{F}_3[\alpha_1, \alpha_3](1, \alpha_4, \alpha_4^2)(1, \alpha_5, \alpha_5^2)$ and $R = \wedge_{\mathbb{F}_3}(x_1, x_3, x_4, x_5)$.

Proposition 2.1. *$R = 5J_1 + 4J_2 + J_3$ and $W = 2J_1 + 2J_2 + 3J_3\mathbb{F}_3[\alpha_3] + 3J_3\mathbb{F}_3[\alpha_1, \alpha_3]$.*

Proof: We show that R and W are as claimed by exhibiting bases for these modules. Let σ_1 denote x_1x_4, $\tau_2 = x_3x_5$ and $\pi = x_1x_5 - x_3x_4$.

$5J_1:\ (1),\ (\sigma_1),\ (\tau_2),\ (\pi),\ (\sigma_1\tau_2)$

$4J_2: < -x_1,\ x_4 > (1,\ \tau_2),\ < x_3,\ x_5 > (1,\ \sigma_1)$

$J_3: < x_1x_3,\ -x_3x_4 - x_1x_5 - x_1x_3,\ x_4x_5 >$

For W, let ω denote $\alpha_1\alpha_5 + \alpha_3\alpha_4$.

$2J_1:\ (1),\ (\omega)$

$2J_2: < -\alpha_1,\ \alpha_4 >,\ < \alpha_3,\ \alpha_5 >$

$3J_3: < -\alpha_1^2,\ \alpha_1\alpha_4 + \alpha_1^2,\ \alpha_4^2 > \mathbb{F}_3[\alpha_3],\ < -\alpha_3^2,\ -\alpha_3\alpha_5 + \alpha_3^2,\ \alpha_5^2 > \mathbb{F}_3[\alpha_3],$

$< -\alpha_3\omega,\ (\alpha_3 - \alpha_5)\omega - \alpha_1\alpha_3^2,\ \alpha_4\alpha_5^2 > \mathbb{F}_3[\alpha_3]$

$3J_3: < \alpha_1\alpha_3,\ \alpha_3\alpha_4 - \alpha_1\alpha_5 - \alpha_1\alpha_3,\ \alpha_4\alpha_5 > \mathbb{F}_3[\alpha_1, \alpha_3],$

$< -\alpha_1\omega,\ (\alpha_1+\alpha_4)\omega+\alpha_1^2\alpha_3,\ \alpha_4^2\alpha_5 > \mathbb{F}_3[\alpha_1,\alpha_3],$

$< -\omega^2-\alpha_1^2\alpha_3^2,\ \omega^2+(\alpha_1\alpha_3-\alpha_4\alpha_5)(\alpha_3\alpha_4-\alpha_1\alpha_5)+\alpha_1^2\alpha_3^2,\ \alpha_4^2\alpha_5^2 > \mathbb{F}_3[\alpha_1,\alpha_3].$ □

Let S denote

$H^*((\mathbb{Z}/3\mathbb{Z})^4;\mathbb{F}_3) = R\otimes W = (5J_1+4J_2+J_3)\otimes(2J_1+2J_2+3J_3\mathbb{F}_3[\alpha_3]+3J_3\mathbb{F}_3[\alpha_1,\alpha_3])$.

Proposition 2.2. *$S = 10J_1\oplus 10J_2\oplus 15J_3\mathbb{F}_3[\alpha_3]\oplus 15J_3\mathbb{F}_3[\alpha_1,\alpha_3]\oplus$*

$8J_2\oplus 8(J_1\oplus J_3)\oplus 12(J_3\oplus J_3)\mathbb{F}_3[\alpha_3]\oplus 12(J_3\oplus J_3)\mathbb{F}_3[\alpha_1,\alpha_3]\oplus$

$2J_3\oplus 2(J_3\oplus J_3)\oplus 3(J_3\oplus J_3\oplus J_3)\mathbb{F}_3[\alpha_3]\oplus 3(J_3\oplus J_3\oplus J_3)\mathbb{F}_3[\alpha_1,\alpha_3]$.

After collecting terms, $S = 18J_1\oplus 18J_2\oplus 12J_3\oplus 48J_3\mathbb{F}_3[\alpha_3]\oplus 48J_3\mathbb{F}_3[\alpha_1,\alpha_3]$.

Proof: By [2, Lemma 5, pg. 50], $J_2\otimes J_2\simeq J_1\oplus J_3$. Of course, $J_1\otimes J\simeq J$. Since J_3 is projective, $J_2\otimes J_3\simeq J_3\oplus J_3$ and $J_3\otimes J_3\simeq J_3\oplus J_3\oplus J_3$. □

We give bases for these modules. When constructing a basis for $J_2 = < a,b > \otimes < x,y > = J_2$, the resulting $J_1\oplus J_3$ can be given as:

$< ay-bx > \oplus < (a_2-1)^2by, (a_2-1)by, by >$. Similarly,

$$\begin{array}{ccc} J_2\otimes J_3 & & J_3\oplus J_3 \\ < a,b > \otimes < x,y,z > & \to & < (a_2-1)^2az, (a_2-1)az, az > \oplus < (a_2-1)^2bz, (a_2-1)bz, bz > \end{array}$$

$$\begin{array}{ccc} J_3\otimes J_3 & & J_3\oplus J_3\oplus J_3 \\ < a,b,c > \otimes < x,y,z > & \to & < (a_2-1)^2az, (a_2-1)az, az > \oplus < (a_2-1)^2bz, (a_2-1)bz, bz > \oplus \\ & & < (a_2-1)^2cz, (a_2-1)cz, cz > . \end{array}$$

In terms of our above labels, we may write:

$$\begin{array}{lll} J_2 \otimes J_2 & & J_1 \oplus J_3 \\ <a,b> \otimes <x,y> & \to & <ay-bx> \oplus <ax, ax+bx+ay, by> \\ J_2 \otimes J_3 & & J_3 \oplus J_3 \\ <a,b> \otimes <x,y,z> & \to & <ax,ay,az> \oplus <-ax-ay+bx, ay+by+az, bz> \\ J_3 \otimes J_3 & & J_3 \oplus J_3 \oplus J_3 \\ <a,b,c> \otimes <x,y,z> & \to & <ax,ay,az> \oplus <-ax-ay+bx, ay+by+az, bz> \oplus \\ & & <ax-ay+az-bx-by+cz, by+bz+cy, cz> . \end{array}$$

Let $\mu_1 = x_1\alpha_4 - x_4\alpha_1,\ \nu_2 = x_3\alpha_5 - x_5\alpha_3,\ \kappa = x_1\alpha_5 + x_4\alpha_3,\ \lambda = x_3\alpha_4 + x_5\alpha_1,$ $\zeta_1 = (1 + a_2 + a_2^2)(x_4x_5\alpha_4^2\alpha_5),\ \zeta_2 = (1 + a_2 + a_2^2)(x_4x_5\alpha_4\alpha_5^2),\ \theta_1 = (1 + a_2 + a_2^2)(x_4\alpha_4^2\alpha_5^2),$ $\theta_2 = (1 + a_2 + a_2^2)(x_5\alpha_4^2\alpha_5^2)$, and $\eta = (1 + a_2 + a_2^2)(x_4x_5\alpha_4^2\alpha_5^2)$. To ease the notation and to make the proof of later statements clear (e.g. 2.3 on the next page), in the list below, we denote the copies of J_2 and J_3 only by the basis element for their socles. For example, $\{\alpha_3\}$ denotes J_2 with basis $<\alpha_3,\ \alpha_5>$ and $\{x_1x_3\}$ denotes $<x_1x_3,\ -x_3x_4 - x_1x_5,\ x_4x_5>$.

$10J_1:\ (1,\omega)(1,\ \sigma_1,\ \tau_2,\ \pi,\ \sigma_1\tau_2)$

$10J_2:\ (1,\ \sigma_1, \tau_2,\ \pi,\ \sigma_1\tau_2)\{\alpha_1, \alpha_3\}$

$15J_3:\ (1,\ \sigma_1,\ \tau_2,\ \pi,\ \sigma_1\tau_2)\{\alpha_1\alpha_3,\ \alpha_3^2,\ \alpha_3\omega\}\mathbb{F}_3[\alpha_3]$

$15J_3:\ (1,\ \sigma_1, \tau_2,\ \pi,\ \sigma_1\tau_2)\{\alpha_1^2,\ \alpha_1\omega,\ \omega^2\}\mathbb{F}_3[\alpha_1,\ \alpha_3]$

$8J_2:\ (1,\omega)\{x_1,\ x_1\tau_2,\ x_3,\ x_3\sigma_1\}$

$8(J_1 \oplus J_3):\ (1,\tau_2)(\mu_1,\ \kappa)$ $\qquad (1,\tau_2)\{x_1\alpha_1,\ x_1\alpha_3,\}$

$(1,\sigma_1)(\nu_2,\ \lambda)$ $\qquad (1,\sigma_1)\{x_3\alpha_1,\ x_3\alpha_3\}$

$12(J_3 \oplus J_3):\ (1,\sigma_1)\{x_3\alpha_1\alpha_3,\ x_3\alpha_3^2,\ \alpha_3\nu_2,\ \alpha_1\nu_2 - x_3\omega,\ x_3\alpha_3\omega,\ \nu_2\omega\}\mathbb{F}_3[\alpha_3]$

$(1,\tau_2)\{x_1\alpha_1\alpha_3,\ x_1\alpha_3^2,\ \alpha_3\kappa,\ \alpha_3\mu_1 + x_1\omega,\ x_1\alpha_3\omega,\ \kappa\omega\}\mathbb{F}_3[\alpha_3]$

$12(J_3 \oplus J_3):\ (1,\sigma_1)\{x_3\alpha_1^2,\ \alpha_1\lambda,\ \lambda\omega,\ x_3\alpha_1\omega,\ x_3\omega^2,\ \theta_2\}\mathbb{F}_3[\alpha_1,\ \alpha_3]$

$(1,\tau_2)\{x_1\alpha_1^2,\ \alpha_1\mu_1,\ \mu_1\omega,\ x_1\alpha_1\omega,\ x_1\omega^2,\ \theta_1\}\mathbb{F}_3[\alpha_1,\ \alpha_3]$

$2J_3:\ (1,\ \omega)\{x_1x_3\}$

$2(J_3\oplus J_3):\ \{x_1x_3\alpha_1,\ x_1x_3\alpha_3,\ x_3\mu_1+\alpha_1\pi,\ x_1\nu_2-\alpha_3\pi\}$

$3(J_3\oplus J_3\oplus J_3):\ \{x_1x_3\alpha_1\alpha_3,\ x_1x_3\alpha_3^2,\ \nu_2\kappa,\ \kappa\lambda+\pi\omega,\ \alpha_3(x_1\nu_2-\alpha_3\pi),$

$\quad -\alpha_1\alpha_3\pi+x_1x_3\omega-\alpha_3x_3\mu_1x_1x_3\alpha_3\omega,\ (x_1\nu_2-\alpha_3\pi)\omega,\ \zeta_2,\}\mathbb{F}_3[\alpha_3]$

$3(J_3\oplus J_3\oplus J_3):\ \{x_1x_3\alpha_1^2,\ \lambda\mu_1,\ \alpha_1(x_3\mu_1+\alpha_1\pi),\ \alpha_1x_1x_3\omega,\ (x_3\mu_1+\alpha_1\pi)\omega,$

$\quad \zeta_1,\ x_1x_3\omega^2,\ \eta,\ x_1\theta_1\}\mathbb{F}_3[\alpha_3]$.

Lemma 2.3. $\dfrac{\mathrm{soc}_{\mathbb{F}_3[Q]}(S)}{\mathrm{rad}^2_{\mathbb{F}_3[Q]}(S)}=(1,\ \sigma_1,\ \tau_2,\ \pi,\ \sigma_1\tau_2)(1,\alpha_1,\alpha_3,\omega)\oplus x_1(1,\tau_2,\omega,\tau_2\omega)\oplus$
$x_3(1,\sigma_1,\omega,\sigma_1\omega)\oplus(1,\tau_2)(\mu_1,\kappa)\oplus(1,\sigma_1)(\nu_2,\lambda)$.
$\dfrac{\mathrm{soc}^2_{\mathbb{F}_3[Q]}(S)}{\mathrm{rad}_{\mathbb{F}_3[Q]}(S)}=(1,\ \sigma_1,\ \tau_2,\ \pi,\ \sigma_1\tau_2)(1,\alpha_4,\alpha_5,\omega)\oplus x_4(1,\tau_2,\omega,\tau_2\omega)\oplus x_5(1,\sigma_1,\omega,\sigma_1\omega)\oplus$
$(1,\tau_2)(\mu_1,\kappa)\oplus(1,\sigma_1)(\nu_2,\lambda)$.

Proof: Here as in Siegel (see 1.11, 1.12), $\mathrm{soc}_{\mathbb{F}_3[Q]}(S)=\mathrm{Ker}(a_2-1)$, $\mathrm{soc}^2_{\mathbb{F}_3[Q]}(S)=\mathrm{Ker}((a_2-1)^2), \mathrm{rad}_{\mathbb{F}_3[Q]}(S)=\mathrm{Im}(a_2-1)$ and $\mathrm{rad}^2_{\mathbb{F}_3[Q]}(S)=\mathrm{Im}((a_2-1)^2)$. In the notation $<x>, <x,y>$, or $<x,y,z>$ for the J_i given after Proposition 2.2 on page 24, bases for the various submodules of interest are:

$$\begin{array}{llll}
\mathrm{soc}(J_1)=<x> & \mathrm{soc}^2(J_1)=<x> & \mathrm{rad}(J_1)=0 & \mathrm{rad}^2(J_1)=0\\
\mathrm{soc}(J_2)=<x> & \mathrm{soc}^2(J_2)=<x,y> & \mathrm{rad}(J_2)=<x> & \mathrm{rad}^2(J_2)=0\\
\mathrm{soc}(J_3)=<x> & \mathrm{soc}^2(J_3)=<x,y> & \mathrm{rad}(J_3)=<x,y> & \mathrm{rad}^2(J_3)=<x>.
\end{array}$$

Thus, $$\begin{array}{ll}
\dfrac{\mathrm{soc}}{\mathrm{rad}^2}(J_1)=<x> & \dfrac{\mathrm{soc}^2}{\mathrm{rad}}(J_1)=<x>\\
\dfrac{\mathrm{soc}}{\mathrm{rad}^2}(J_2)=<x> & \dfrac{\mathrm{soc}^2}{\mathrm{rad}}(J_2)=<y>\\
\dfrac{\mathrm{soc}}{\mathrm{rad}^2}(J_3)=0 & \dfrac{\mathrm{soc}^2}{\mathrm{rad}}(J_3)=0.
\end{array}$$ □

Let $\gamma_4=\prod_{i=0}^{2}(\alpha_4+i\alpha_1)$ and $\gamma_5=\prod_{i=0}^{2}(\alpha_5+i\alpha_3)$. Note that both of these are Q-invariant and that $\mathbb{F}_3[\alpha_1,\alpha_3,\alpha_4,\alpha_5]\simeq\mathbb{F}_3[\gamma_4,\gamma_5]\otimes W$. So $H^*(H;\mathbb{F}_3)\simeq\wedge_{\mathbb{F}_3}(x_1,x_3,x_4,x_5)\otimes$

$\mathbb{F}_3[\alpha_1,\alpha_3,\alpha_4,\alpha_5] \simeq \wedge_{\mathbb{F}_3}(x_1,x_3,x_4,x_5)\otimes \mathbb{F}_3[\gamma_4,\gamma_5]\otimes W \simeq \mathbb{F}_3[\gamma_4,\gamma_5]\otimes S$. In addition, $\mathrm{soc}^i_{\mathbb{F}_3[Q]}(H^*(H;\mathbb{F}_3)) = \mathbb{F}_3[\gamma_4,\gamma_5]\otimes \mathrm{soc}^i_{\mathbb{F}_3[Q]}(S),\ i\geq 0$ since $\mathbb{F}_3[\gamma_4,\gamma_5]$ is a trivial $\mathbb{F}_3[Q]$-module; similarly, $\mathrm{rad}^i_{\mathbb{F}_3[Q]}(H^*(H;\mathbb{F}_3)) = \mathbb{F}_3[\gamma_4,\gamma_5]\otimes \mathrm{rad}^i_{\mathbb{F}_3[Q]}(S),\ i\geq 0$. Thus by equation 2.2 on page 22 we get isomorphisms:

$$E_2^{r,*} \simeq \begin{cases} \mathbb{F}_3[\gamma_4,\gamma_5]\otimes \mathrm{soc}_{\mathbb{F}_3[Q]}(S), & r=0 \\ \mathbb{F}_3[\gamma_4,\gamma_5]\otimes \frac{\mathrm{soc}^2_{\mathbb{F}_3[Q]}(S)}{\mathrm{rad}_{\mathbb{F}_3[Q]}(S)}, & r \text{ odd} \\ \mathbb{F}_3[\gamma_4,\gamma_5]\otimes \frac{\mathrm{soc}_{\mathbb{F}_3[Q]}(S)}{\mathrm{rad}^2_{\mathbb{F}_3[Q]}(S)}, & r>0 \text{ even.} \end{cases} \tag{2.3}$$

Write $H^*(Q;\mathbb{F}_3) = \mathbb{F}_3[\alpha_2]\otimes \wedge_{\mathbb{F}_3}(x_2)$, x_2,α_2 defined as usual. x_2 generates $E_2^{1,0}$ and α_2 generates $E_2^{2,0}$. A product on the E_2-page is given as follows: suppose $\chi\in E_2^{s,t}$ is represented by $x\in H^s((\mathbb{Z}/3\mathbb{Z})^4;\mathbb{F}_3)$ and $\psi\in E_2^{s',t'}$ is represented by $y\in H^{s'}((\mathbb{Z}/3\mathbb{Z})^4;\mathbb{F}_3)$. Then, $(-1)^{s't}\chi\psi\in E_2^{s+s',t+t'}$ is represented by: $\begin{cases} xy & \text{if s or s' is even,} \\ \sum\limits_{0\leq i<j<3} a_2^i(x)a_2^j(y) & \text{if s and s' are odd.} \end{cases}$

Let $\tau_1 = x_2x_4, \sigma_2 = x_2x_5, \nu_1 = x_2\alpha_4$ and $\mu_2 = x_2\alpha_5$.

Corollary 2.4. $E_2^{1,*} = \mathbb{F}_3[\gamma_4,\gamma_5]\otimes\{(1,\ \sigma_1,\ \tau_2,\ \pi,\ \sigma_1\tau_2)(x_2,\nu_1,\mu_2,x_2\omega)\oplus\tau_1(1,\tau_2,\omega,\tau_2\omega)\oplus \sigma_2(1,\sigma_1,\omega,\sigma_1\omega)\oplus(x_2,x_2\tau_2)(\mu_1,\kappa)\oplus(x_2,x_2\sigma_1)(\nu_2,\lambda)\}$;
$E_2^{2,*} = \mathbb{F}_3[\gamma_4,\gamma_5]\otimes\alpha_2\{(1,\ \sigma_1,\ \tau_2,\ \pi,\ \sigma_1\tau_2)(1,\alpha_1,\alpha_3,\omega)\oplus x_1(1,\tau_2,\omega,\tau_2\omega)\oplus x_3(1,\sigma_1,\omega,\sigma_1\omega)\oplus (1,\tau_2)(\mu_1,\kappa)\oplus(1,\sigma_1)(\nu_2,\lambda)\}$.

Proof: Apply equation 2.3 to Lemma 2.3. $\square$

Theorem 2.5. *The* $E_2 - \text{page}$ *is additively isomorphic to:*

$$\mathbb{F}_3[\gamma_4,\gamma_5] \otimes \begin{cases} \mathbb{F}_3[\alpha_1,\alpha_3](1,\omega,\omega^2)\{1,x_1,x_3,x_1x_3,\pi,\sigma_1,\tau_2,\mu_1,\nu_2,x_1\tau_2,x_3\sigma_1, \\ \quad x_3\mu_1,\sigma_1\tau_2,\mu_1\nu_2\} \oplus \mathbb{F}_3[\alpha_1,\alpha_3](1,\omega)\{\sigma_1\nu_2,\tau_2\mu_1\}\oplus \\ \mathbb{F}_3[\alpha_1](1,\omega,\omega^2)(\lambda)\{1,\mu_1\} \oplus \mathbb{F}_3[\alpha_1](1,\omega)\{\sigma_1\lambda\}\oplus \\ \mathbb{F}_3[\alpha_3](1,\omega,\omega^2)\{\kappa(1,\nu_2) \oplus x_1\nu_2\} \oplus \mathbb{F}_3[\alpha_3](1,\omega)\{\tau_2\kappa\}\oplus \\ \mathbb{F}_3[\alpha_2](1,\omega)\{x_2(1,\sigma_1)(1,\tau_2) \oplus \alpha_2(1,\sigma_1)(1,\tau_2) \oplus \tau_1(1,\tau_2)\oplus \\ \quad \sigma_2(1,\sigma_1) \oplus x_2\pi \oplus x_1\alpha_2(1,\tau_2) \oplus x_3\alpha_2(1,\sigma_1) \oplus \alpha_2\pi\}\oplus \\ \mathbb{F}_3[\alpha_2]\{\nu_1(1,\sigma_1)(1,\tau_2) \oplus \mu_2(1,\sigma_1)(1,\tau_2) \oplus x_2(\mu_1,\kappa)(1,\tau_2)\oplus \\ \quad x_2(\nu_2,\lambda)(1,\sigma_1) \oplus \alpha_2(\alpha_1,\alpha_3)(1,\sigma_1)(1,\tau_2) \oplus \nu_1\pi \oplus \mu_2\pi\oplus \\ \quad \alpha_2(\mu_1,\kappa)(1,\tau_2) \oplus \alpha_2(\nu_2,\lambda)(1,\sigma_1) \oplus \alpha_2\pi(\alpha_1,\alpha_3)\}\oplus \\ \zeta_1 \oplus \zeta_2 \oplus \theta_1(1,\sigma_1) \oplus \theta_2(1,\tau_2) \oplus \eta(1,\alpha_1,\alpha_3) \oplus x_1\theta_1. \end{cases}$$

Proof: Corollary 2.4 gives $E_2^{r,*}$ for $r > 0$. $E_2^{0,*}$ is the tensor product of $\mathbb{F}_3[\gamma_4,\gamma_5]$ and $\mathrm{soc}_{\mathbb{F}_3[Q]}(S)$, which is described in the proof of Lemma 2.3. The reader checking this description may find the two relations useful: $x_3\lambda = -x_1\nu_2 - \alpha_3\pi$ and $x_1\lambda = -x_3\mu_1 + \alpha_1\pi$. $\alpha_1\theta_2$ may be rewritten by noticing that $(a_2-1)^2(\alpha_1x_5\alpha_4^2\alpha_5^2) = \lambda(a_2-1)^2(\alpha_4^2\alpha_5^2) - x_3\gamma_4(a_2-1)^2(\alpha_5^2) - x_3\alpha_1^2(a_2-1)^2(\alpha_4\alpha_5^2) = \lambda(-\omega^2-\alpha_1^2\alpha_3^2) + x_3\gamma_4\alpha_3^2 + x_3\alpha_1^2\alpha_3\omega$. Similarly, α_1- and α_3-multiples of $\zeta_1,\zeta_2,\theta_1,\theta_2$ and η can be rewritten. □

Corollary 2.6. E_2 *is generated as a ring by*

$$\underbrace{x_1,x_2,x_3}_{deg\,1},\quad \underbrace{\alpha_1,\alpha_2,\alpha_3,\sigma_1,\sigma_2,\tau_1,\tau_2,\pi}_{deg\,2},\quad \underbrace{\mu_1,\mu_2,\nu_1,\nu_2,\kappa,\lambda}_{deg\,3},\quad \underbrace{\omega}_{deg\,4},$$
$$\underbrace{\gamma_4,\gamma_5}_{deg\,6},\quad \underbrace{\zeta_1,\zeta_2}_{deg\,8},\quad \underbrace{\theta_1,\theta_2}_{deg\,9},\quad \underbrace{\eta.}_{deg\,10}$$

Proof: Immediate from 2.5. □

Theorem 2.7. *In this spectral sequence $E_2 = E_\infty$.*

Corollary 2.8. *$H^*(H;\mathbb{F}_3)$ is given additively by 2.5.*

Proof (of theorem): Define: $x_1, x_2, x_3 \in H^1(H;\mathbb{F}_3) \simeq \mathrm{Hom}_{\mathbb{F}_3}(H,\mathbb{F}_3)$, by $x_i(a_5^{j_5}a_4^{j_4}a_3^{j_3}a_2^{j_2}a_1^{j_1}) = j_i$. Take $\gamma_4 = \mathcal{N}^H_{(\mathbb{Z}/3\mathbb{Z})^4}(\alpha_4)$ and $\gamma_5 = \mathcal{N}^H_{(\mathbb{Z}/3\mathbb{Z})^4}(\alpha_5)$. We define ζ_1 as the transfer (or, corestriction), $Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4x_5\alpha_4^2\alpha_5)$, ζ_2 as $Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4x_5\alpha_4\alpha_5^2)$, $\theta_1 = \beta(\zeta_1), \theta_2 = \beta(\zeta_2)$ and $\eta = Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4x_5\alpha_4^2\alpha_5^2)$.

We define the remaining generators as classes represented by particular related cocycles of Massey products and their Bocksteins. For $g \in H$, let $g = a_5^va_4^ta_3^ra_2^sa_1^u$. In the bar resolution for H consider the cochains: $u[g] = u$ $s[g] = s$ $r[g| = r$ $t[g] = t$ $v[g] = v$. Note that u represents x_1, s represents x_2, r represents x_3. Also note (see 2.9)

$\delta(t_1) = -u_1s_2$ represents $-x_1 \cup x_2$, $\delta(v_1) = -s_1r_2 = -x_2 \cup x_3$,

$\delta(u_1^2) = u_1u_2 = x_1 \cup x_1$, $\delta(s_1^2) = s_1s_2 = x_2 \cup x_2$ and $\delta(r_1^2) = r_1r_2 = x_3 \cup x_3$.

In degree two, we define $\alpha_1 = u_1u_2^2 + u_1^2u_2$, $\alpha_2 = s_1s_2^2 + s_1^2s_2$, $\alpha_3 = r_1r_2^2 + r_1^2r_2$, $\sigma_1 = u_1^2s_2 - u_1v_2$, $\tau_1 = u_1s_2^2 - v_1s_2$, $\sigma_2 = s_1^2r_2 - s_1t_2$, $\tau_2 = s_1r_2^2 - t_1r_2$ and $\pi = u_1t_2 + v_1r_2$. That these represent Massey products in shown in 2.10 on the following page. Now, let $\mu_i = \beta(\sigma_i), \nu_i = \beta(\tau_i)$.

We take κ (see 2.10) as the class represented by

$-v_1(s_2^2r_3 - s_2t_3 - r_2r_3^2 - r_2^2r_3) + u_1(t_2^2t_3 + t_2t_3^2 + (t_2+t_3)^2s_2r_3 + (t_2+t_3)s_2^2r_3^2 - s_2^2t_3)$.

Finally, we define $\lambda = \kappa + \beta(\pi)$ and $\omega = \beta(\kappa) = \beta(\lambda)$.

We have shown that all the generators of the E_2-page represent generators for $H^*(H;\mathbb{F}_3)$ and therefore, live to E_∞. □

Lemma 2.9. $\delta(t_1) = -u_1s_2$.

Proof: $\delta(t_1)[a_5^{v_1}a_4^{t_1}a_3^{r_1}a_2^{s_1}a_1^{u_1}|a_5^{v_2}a_4^{t_2}a_3^{r_2}a_2^{s_2}a_1^{u_2}] = t_1\{\partial([a_5^{v_1}a_4^{t_1}a_3^{r_1}a_2^{s_1}a_1^{u_1}|a_5^{v_2}a_4^{t_2}a_3^{r_2}a_2^{s_2}a_1^{u_2}])\} =$

$t_1\{[a_5^{v_2}a_4^{t_2}a_3^{r_2}a_2^{s_2}a_1^{u_2}] - [a_5^{v_1+v_2+r_2s_1}a_4^{t_1+t_2+u_1s_2}a_3^{r_1+r_2}a_2^{s_1+s_2}a_1^{u_1+u_2}] + [a_5^{v_1}a_4^{t_1}a_3^{r_1}a_2^{s_1}a_1^{u_1}]\} = t_2 - (t_1 + t_2 + u_1s_2) + t_1 = -u_1s_2$. The other boundaries noted above are decided similarly. □

Lemma 2.10. *i.* $\alpha_i =< x_i, x_i, x_i >$, *ii.* $\sigma_1 =< x_1, x_1, x_2 >$, *iii.* $\tau_1 =< x_1, x_2, x_2 >$,

iv. $\pi =< x_1, x_2, x_3 >$, *v.* $\sigma_2 =< x_2, x_2, x_3 >$, *vi.* $\tau_2 =< x_2, x_3, x_3 >$,

vii. $\kappa \in< x_1, x_2, \sigma_2 - \alpha_3 >$.

Proof: Immediate from the definitions. For $i. - vi.$ see also 1.19. For example, to show $vii.$ let $\rho = t_1^2t_2 + t_1t_2^2 + (t_1+t_2)^2s_1r_2 + (t_1+t_2)s_1^2r_2^2$. Then, $\delta(\rho) = (s_1s_2^2 + s_1^2s_2)r_3 - s_1(r_2^2r_3 + r_2r_3^2) = \alpha_2x_3 - x_2\alpha_3$. In addition $\delta(s_1^2t_2) = s_1(s_2t_3 - s_2^2r_3) + (s_1s_2^2 + s_1^2s_2)r_3 = -x_2\sigma_2 + \alpha_2x_3$ and so $\delta(\rho - s_1^2t_2) = x_2(\sigma_2 - \alpha_3)$.□

Proposition 2.11. *i.* $\sigma_1 = Inf_{G_{27}^h}^H(\sigma)$, *ii.* $\tau_1 = Inf_{G_{27}^h}^H(\tau)$,

iii. $\sigma_2 = Inf_{G_{27}^l}^H(\sigma)$, *iv.* $\tau_2 = Inf_{G_{27}^l}^H(\tau)$.

Proof: Easily seen by inflating the related cocycles defining the σ_i and τ_i given in 1.18.□

Now that we have a set of generators for $H^*(H;\mathbb{F}_3)$, we may consider the automorphism $\phi : H \to H$ defined by $a_1 \leftrightarrow a_3,\ a_2 \to a_2$ and $a_4 \leftrightarrow a_5^{-1}$.

The ring automorphism of $H^*(H;\mathbb{F}_3)$, Φ, induced by ϕ is given as follows:

$$x_1 \leftrightarrow x_3 \quad x_2 \to x_2 \quad \alpha_1 \leftrightarrow \alpha_3 \quad \alpha_2 \to \alpha_2 \quad \sigma_1 \leftrightarrow \tau_2 \quad \tau_1 \leftrightarrow \sigma_2 \quad \pi \to \pi$$

$$\mu_1 \leftrightarrow \nu_2 \quad \nu_1 \leftrightarrow \mu_2 \quad \kappa \leftrightarrow -\lambda \quad \omega \to -\omega \quad \gamma_4 \leftrightarrow -\gamma_5 \quad \zeta_1 \leftrightarrow \zeta_2 \quad \theta_1 \leftrightarrow \theta_2 \quad \eta \leftrightarrow -\eta.$$

Consider the collection of subgroups of H: four copies of $(\mathbb{Z}/3\mathbb{Z})^3$, $H_{111},\ H_{112},\ H_{121},\ H_{211}$ defined as $H_{ijk} =< a_5, a_4, (a_3^ia_2^ja_1^k) >$; a copy of $(\mathbb{Z}/3\mathbb{Z})^4$ generated by $a_5,\ a_4,\ a_3$ and a_1; and, two copies of $\mathbb{Z}/3\mathbb{Z} \times G_{27}$ generated by $< a_5 > \times < a_4, a_2, a_1 >$ and $< a_4 > \times < a_5, a_3, a_2 >$. With these definitions, the proof of 2.12 is straightforward albeit laborious.

Proposition 2.12. *The above subgroups form a collection of detecting subgroups for* $H^*(H;\mathbb{F}_3)$.

Proof: (We present table 2.1 on the next page for reference.) One begins by restricting to the $(\mathbb{Z}/3\mathbb{Z})^4$ since the kernel of this restriction is those elements of $E_2^{r,s}$ with $r > 0$. Next, restrict to the two copies of $\mathbb{Z}/3\mathbb{Z} \times G_{27}$ and then, to the four copies of $(\mathbb{Z}/3\mathbb{Z})^3$. $\square$

Table 2.1: Restrictions in H

	$<a_5,a_4,(a_3a_2a_1)>$ H_{111}	$<a_5,a_4,(a_3a_2a_1^2)>$ H_{112}	$<a_5,a_4,(a_3a_2^2a_1)>$ H_{121}	$<a_5,a_4,(a_3^2a_2a_1)>$ H_{211}
x_1	x	$-x$	x	x
x_2	x	x	$-x$	x
x_3	x	x	x	$-x$
α_1	α	$-\alpha$	α	α
α_2	α	α	$-\alpha$	α
α_3	α	α	α	$-\alpha$
σ_1	$\alpha - xx_4$	$\alpha + xx_4$	$-\alpha - xx_4$	$\alpha - xx_4$
τ_1	$\alpha + xx_4$	$-\alpha + xx_4$	$\alpha - xx_4$	$\alpha + xx_4$
σ_2	$\alpha - xx_5$	$\alpha - xx_5$	$\alpha + xx_5$	$-\alpha - xx_5$
τ_2	$\alpha + xx_5$	$\alpha + xx_5$	$-\alpha + xx_5$	$\alpha - xx_5$
π	$-\alpha - xx_4 + xx_5$	$\alpha - xx_4 - xx_5$	$\alpha - xx_4 + xx_5$	$\alpha + xx_4 + xx_5$
μ_1	$-\alpha x_4 + x\alpha_4$	$\alpha x_4 - x\alpha_4$	$-\alpha x_4 + x\alpha_4$	$-\alpha x_4 + x\alpha_4$
ν_1	$\alpha x_4 - x\alpha_4$	$\alpha x_4 - x\alpha_4$	$-\alpha x_4 + x\alpha_4$	$\alpha x_4 - x\alpha_4$
μ_2	$-\alpha x_5 + x\alpha_5$	$-\alpha x_5 + x\alpha_5$	$\alpha x_5 - x\alpha_5$	$-\alpha x_5 + x\alpha_5$
ν_2	$\alpha x_5 - x\alpha_5$	$\alpha x_5 - x\alpha_5$	$\alpha x_5 - x\alpha_5$	$-\alpha x_5 + x\alpha_5$
κ	$x\alpha_5 - \alpha x_5 - xx_4x_5$	$-x\alpha_5 + \alpha x_5 - xx_4x_5$	$x\alpha_5 - \alpha x_5 + xx_4x_5$	$x\alpha_5 - \alpha x_5 - xx_4x_5$
λ	$x\alpha_4 - \alpha x_4 - xx_4x_5$	$x\alpha_4 - \alpha x_4 - xx_4x_5$	$x\alpha_4 - \alpha x_4 + xx_4x_5$	$-x\alpha_4 + \alpha x_4 - xx_4x_5$
ω	$-\alpha x_4x_5 + x\alpha_4x_5$ $-xx_4\alpha_5$	$-\alpha x_4x_5 + x\alpha_4x_5$ $-xx_4\alpha_5$	$\alpha x_4x_5 - x\alpha_4x_5$ $+xx_4\alpha_5$	$-\alpha x_4x_5 + x\alpha_4x_5$ $-xx_4\alpha_5$
γ_4	$\alpha_4^3 - \alpha^2\alpha_4$	$\alpha_4^3 - \alpha^2\alpha_4$	$\alpha_4^3 - \alpha^2\alpha_4$	$\alpha_4^3 - \alpha^2\alpha_4$
γ_5	$\alpha_5^3 - \alpha^2\alpha_5$	$\alpha_5^3 - \alpha^2\alpha_5$	$\alpha_5^3 - \alpha^2\alpha_5$	$\alpha_5^3 - \alpha^2\alpha_5$
ζ_1	0	0	0	0
ζ_2	0	0	0	0
θ_1	0	0	0	0
θ_2	0	0	0	0
η	0	0	0	0

Table 2.1: Restrictions in H (continued)

	$\overset{<a_5,a_4,a_3,a_1>}{(\mathbb{Z}/3\mathbb{Z})^4}$	$\overset{<a_5>\times<a_2,a_1,a_4>}{\mathbb{Z}/3\mathbb{Z}\times G_{27}}$	$\overset{<a_4>\times<a_3,a_2,a_5>}{\mathbb{Z}/3\mathbb{Z}\times G_{27}}$
x_1	x_1	x_1	0
x_2	0	x_2	x_2
x_3	x_3	0	x_3
α_1	α_1	α_1	0
α_2	0	α_2	α_2
α_3	α_3	0	α_3
σ_1	$-x_1x_4$	σ_1	0
τ_1	0	τ_1	x_2x_4
σ_2	0	$-x_2x_5$	σ_2
τ_2	x_3x_5	0	τ_2
π	$x_1x_5 - x_3x_4$	x_1x_5	$-x_3x_4$
μ_1	$-\alpha_1x_4 + x_1\alpha_4$	μ_1	0
ν_1	0	ν_1	$\alpha_2x_4 - x_2\alpha_4$
μ_2	0	$-\alpha_2x_5 + x_2\alpha_5$	μ_2
ν_2	$\alpha_3x_5 - x_3\alpha_5$	0	ν_2
κ	$\alpha_3x_4 + x_1\alpha_5$	$x_1\alpha_5 - \tau_1x_5$	$\alpha_3x_4 - \sigma_2x_4$
λ	$\alpha_1x_5 + x_3\alpha_4$	$\alpha_1x_5 - \tau_1x_5$	$\alpha_4x_3 - \sigma_2x_4$
ω	$\alpha_3\alpha_4 + \alpha_1\alpha_5$	$\alpha_1\alpha_5 - \nu_1x_5 - \tau_1\alpha_5$	$\alpha_3\alpha_4 - \mu_2x_4 - \sigma_2\alpha_4$
γ_4	$\alpha_4^3 - \alpha_1^2\alpha_4$	γ_4	$\alpha_5^3 - \alpha_2^2\alpha_4$
γ_5	$\alpha_5^3 - \alpha_3^2\alpha_5$	$\alpha_5^3 - \alpha_2^2\alpha_5$	γ_5
ζ_1	$N_2(x_4x_5\alpha_4^2\alpha_5)$ †	$(\alpha_1\mu_1 + \sigma_1\nu_1)x_5\alpha_5$	$(x_3\alpha_3 - x_3\sigma_2)x_4\alpha_4^2$
ζ_2	$N_2(x_4x_5\alpha_4\alpha_5^2)$ †	$(-x_1\alpha_1 + x_2\sigma_1)x_5\alpha_5^2$	$(\alpha_3\nu_2 + \tau_2\mu_2)x_4\alpha_4$
θ_1	$\beta(N_2(x_4x_5\alpha_4^2\alpha_5))^\dagger$	$-(\alpha_1\mu_1 + \sigma_1\nu_1)\alpha_5^2$	$(\alpha_3^2 + x_2\nu_2 - \alpha_2\tau_2)x_4\alpha_4^2$ $-(x_3\alpha_3 - x_3\sigma_2)\alpha_4^3$
θ_2	$\beta(N_2(x_4x_5\alpha_4\alpha_5^2))^\dagger$	$-(\alpha_1^2 - \alpha_2\sigma_1 + x_2\mu_1)x_5\alpha_5^2$ $-(-x_1\alpha_1 + x_2\sigma_1)\alpha_5^3$	$-(\alpha_3\nu_2 + \tau_2\mu_2)\alpha_4^2$
η	$N_2(x_4x_5\alpha_4^2\alpha_5^2)$ †	$(\alpha_1\mu_1 + \sigma_1\nu_1)x_5\alpha_5^2$	$(\alpha_3\nu_2 + \tau_2\mu_2)x_4\alpha_4^2$

† N_2 here means $(1 + a_2 + a_2^2)$.

Much work was done using extension 2.1 without using Siegel's approach. $E_2^{i,j} \simeq H^*(Q; H^*(N; \mathbb{F}_3))$. Rather than decompose the coefficients as a $\mathbb{F}_3[Q]$-module, one uses the definition of cohomology of the group G. To wit, one uses the usual minimal resolution for $\mathbb{F}_3$ over $\mathbb{F}_3[Q]$ and then applies the funtor $\mathrm{Hom}_{\mathbb{F}_3[Q]}(\_\_, H^*((\mathbb{Z}/3\mathbb{Z})^4; \mathbb{F}_3))$. The coefficients are not trivial, indeed, we obtain a cochain complex:

$$\cdots \leftarrow H^*((\mathbb{Z}/3\mathbb{Z})^4;\mathbb{F}_3) \overset{a_2-1}{\longleftarrow} H^*((\mathbb{Z}/3\mathbb{Z})^4;\mathbb{F}_3) \overset{a_2^2+a_2+1}{\longleftarrow} H^*((\mathbb{Z}/3\mathbb{Z})^4;\mathbb{F}_3) \overset{a_2-1}{\longleftarrow} H^*((\mathbb{Z}/3\mathbb{Z})^4;\mathbb{F}_3).$$

When we construct the E_2- page the vertical axis consists of those elements in the kernel of $a_2 - 1$, the elements in the first column are those in $\mathrm{Ker}(a_2-1)/\mathrm{Im}(a_2^2+a_2+1)$ and those in the second column in $\mathrm{Ker}(a_2^2+a_2+1)/\mathrm{Im}(a_2-1)$. Let x_2 and α_2 denote the basis elements in this resolution in degrees one and two, respectively. Use the same notation to indicate the dual bases in the cochain complex. Clearly, x_2 and α_2 are a_2-invariant. We may write the first column as $x_2\mathrm{Ker}(a_2-1)/\mathrm{Im}(a_2^2+a_2+1)$ and the second as $\alpha_2\mathrm{Ker}(a_2^2+a_2+1)/\mathrm{Im}(a_2-1)$. The remaining columns are α_2-multiples of these.

A Java program was written to assist with these computations. The user creates a separate file containing each element in the resolution. We need only consider elements of the form $x_1^{i_1}x_3^{i_3}x_4^{i_4}x_5^{i_5}\alpha_4^{j_4}\alpha_5^{j_5}$ for $0 \leq i_1, i_3, i_4, i_5 \leq 1$ and $0 \leq j_4, j_5 \leq 2$ since $(a_2-1)\alpha_i^3 = ((a_2-1)\alpha_i)^3 = \alpha_1^3$ and $H^*((\mathbb{Z}/3\mathbb{Z})^4;\mathbb{F}_3)$ is propagated by α_1 and α_3. Thus this task is not as big as it may seem; also, a program was written that generates all elements of this form and creates files containing them. The program then reads these files, evaluates $a_2 - 1$ and/or $a_2^2 + a_2 + 1$ at these elements then, simplifies the result. While this program makes deciding the image of an element under these maps immediate, one still has to look through the output to find elements with equal images. Some observations and results from this investigation follows. Table 2.2 on the following page is a sample output from this program for some elements.

Table 2.2: Java Output

$2w$	$(a_2-1)(w)$	w	$(a_2^2+a_2+1)(w)$
α_4	$-\alpha_1 x_2$	$\alpha_4 x_2$	0
α_5	$+\alpha_3 x_2$	$\alpha_5 x_2$	0
x_1	0	$x_1 x_2$	0
x_3	0	$x_3 x_2$	0
x_4	$-x_1 x_2$	$x_4 x_2$	0
x_5	$+x_3 x_2$	$x_5 x_2$	0
α_4^2	$+\alpha_1\alpha_4 x_2+\alpha_1^2 x_2$	$\alpha_4^2 x_2$	$-\alpha_1^2\alpha_2$
$\alpha_4\alpha_5$	$+\alpha_3\alpha_4 x_2-\alpha_1\alpha_5 x_2-\alpha_1\alpha_3 x_2$	$\alpha_4\alpha_5 x_2$	$+\alpha_1\alpha_3\alpha_2$
α_5^2	$-\alpha_3\alpha_5 x_2+\alpha_3^2 x_2$	$\alpha_5^2 x_2$	$-\alpha_3^2\alpha_2$
$x_1\alpha_4$	$-\alpha_1 x_1 x_2$	$x_1\alpha_4 x_2$	0
$x_1\alpha_5$	$+\alpha_3 x_1 x_2$	$x_1\alpha_5 x_2$	0
$x_1 x_3$	0	$x_1 x_3 x_2$	0
$x_1 x_4$	0	$x_1 x_4 x_2$	0
$x_1 x_5$	$+x_1 x_3 x_2$	$x_1 x_5 x_2$	0
$x_3\alpha_4$	$-\alpha_1 x_3 x_2$	$x_3\alpha_4 x_2$	0
$x_3\alpha_5$	$+\alpha_3 x_3 x_2$	$x_3\alpha_5 x_2$	0
$x_3 x_4$	$+x_1 x_3 x_2$	$x_3 x_4 x_2$	0
$x_3 x_5$	0	$x_3 x_5 x_2$	0
$x_4 x_5$	$-x_3 x_4 x_2-x_1 x_5 x_2-x_1 x_3 x_2$	$x_4 x_5 x_2$	$+x_1 x_3\alpha_2$
$x_1 x_3\alpha_5$	$+\alpha_3 x_1 x_3 x_2$	$x_1 x_3\alpha_5 x_2$	0
$x_1 x_3 x_4$	0	$x_1 x_3 x_4 x_2$	0
$x_1 x_3 x_5$	0	$x_1 x_3 x_5 x_2$	0
$x_1 x_4\alpha_4$	$-\alpha_1 x_1 x_4 x_2$	$x_1 x_4\alpha_4 x_2$	0
$x_1 x_4\alpha_5$	$+\alpha_3 x_1 x_4 x_2$	$x_1 x_4\alpha_5 x_2$	0
$x_1 x_4 x_5$	$-x_1 x_3 x_4 x_2$	$x_1 x_4 x_5 x_2$	0
$x_1 x_5\alpha_4$	$-\alpha_1 x_1 x_5 x_2+x_1 x_3\alpha_4 x_2-\alpha_1 x_1 x_3 x_2$	$x_1\alpha_5^2 x_2$	$-\alpha_3^2 x_1\alpha_2$

A glance at this table reveals that among others, $(a_2-1)x_1 = (a_2-1)x_3 = 0$ and so $(a_2-1)\alpha_i = (a_2-1)\beta(x_i) = \beta((a_2-1)x_i) = 0$ for $i=1,2$ as well, as expected. Note also that $(a_2-1)x_1\alpha_4 = (a_2-1)x_4\alpha_1$, both equal to $-\alpha_1 x_1$. Thus we have element $x_1\alpha_4 - x_4\alpha_1 \in E_2^{0,3}$. Continuing in this way, label the generators analogous to those of 2.6 as follows:

$x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3,$

$$
\begin{array}{llllll}
\sigma_1: & 14 & \mu_1: & x_1\alpha_4 - x_4\alpha_1 & \kappa: & x_3\alpha_4 + x_5\alpha_1 \\
\tau_1: & 24 & \nu_1: & x_2\alpha_4 & \lambda: & x_1\alpha_5 + x_4\alpha_3 \\
\sigma_2: & 25 & \mu_2: & x_2\alpha_5 & \omega: & \alpha_1\alpha_5 + \alpha_3\alpha_4 \\
\tau_2: & 35 & \nu_2: & x_3\alpha_5 - x_5\alpha_3 & \gamma_4: & \alpha_4^3 - \alpha_1^2\alpha_4 \\
\pi: & 15 - 34 & & & \gamma_5: & \alpha_5^3 - \alpha_3^2\alpha_5 \\
\zeta_1: & (a_2^2 + a_2 + 1)(x_4x_5\alpha_4^2\alpha_5) & \theta_1: & (a_2^2 + a_2 + 1)(x_4\alpha_4^2\alpha_5^2) & & \\
\zeta_2: & (a_2^2 + a_2 + 1)(x_4x_5\alpha_4\alpha_5^2) & \theta_2: & (a_2^2 + a_2 + 1)(x_5\alpha_4^2\alpha_5^2) & & \\
\eta: & (a_2^2 + a_2 + 1)(x_4x_5\alpha_4^2\alpha_5^2). & & & &
\end{array}
$$

That these elements in fact generate $E_2^{*,*}$ is not difficult to see though it does require a lot of work. Once we know these generate, we can appeal to Theorem 2.7 to see that in this spectral sequence $E_2 = E_\infty$ also. Without 2.7 one must resort to carefully describing the various kernels and images and then decide the kernels modulo images, etc. to complete this spectral sequence. This computer program lessens the burden but, a great deal of hand work is required. ♦

and $\phi' : H \to H$ by

$$a_1 \leftrightarrow a_3,\ a_2 \to a_2^{-1} \text{ and } a_4 \leftrightarrow a_5.$$

and Φ', induced by ϕ' is given as follows:

$$
\begin{array}{llllllll}
x_1 \leftrightarrow x_3 & x_2 \to -x_2 & \alpha_1 \leftrightarrow \alpha_3 & \alpha_2 \to -\alpha_2 & \sigma_1 \leftrightarrow -\tau_2 & \tau_1 \leftrightarrow \sigma_2 & \pi \to -\pi & \\
\mu_1 \leftrightarrow -\nu_2 & \nu_1 \leftrightarrow \mu_2 & \kappa \leftrightarrow \lambda & \omega \to \omega & \gamma_4 \leftrightarrow \gamma_5 & \zeta_1 \leftrightarrow -\zeta_2 & \theta_1 \leftrightarrow -\theta_2 & \eta \leftrightarrow -\eta.
\end{array}
$$

Chapter 3

A Non-cyclic Central Extension

We consider the extension

$$(\mathbb{Z}/3\mathbb{Z})^2 \longrightarrow H \longrightarrow (\mathbb{Z}/3\mathbb{Z})^3, \tag{3.1}$$

with the subgroup the one generated by a_4 and a_5. $E_2^{i,j} = H^i((\mathbb{Z}/3\mathbb{Z})^3; H^j((\mathbb{Z}/3\mathbb{Z})^2; \mathbb{F}_3))$. The coefficients are trivial as this is a central extension.

Write $H^*((\mathbb{Z}/3\mathbb{Z})^2; \mathbb{F}_3) \simeq \mathbb{F}_3[\alpha_4, \alpha_5](1, x_4, x_5, x_4x_5)$ and

$H^*((\mathbb{Z}/3\mathbb{Z})^3; \mathbb{F}_3) \simeq$

$\mathbb{F}_3[\alpha_1, \alpha_2, \alpha_3](1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3)$. Then, we may write $E_2^{i,j} as$

$\mathbb{F}_3[\alpha_1, \alpha_2, \alpha_3, \gamma_4, \gamma_5](1, \alpha_4, \alpha_5, \alpha_4^2, \alpha_5^2)(1, x_4, x_5, x_4x_5)(1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3)$,

where $\gamma_4 = \alpha_4^3$ and $\gamma_5 = \alpha_5^3$. Hereafter, we will denote products of two or more degree one classes by juxtaposition of their indices. For example, 12 denotes x_1x_2.

Proposition 3.1. $d_2(x_4) = 12$

Proof: Let ι be the map of spectral sequences induced by the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d] & G_{27} \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^2 \arrow[d, "j"] \\
(\mathbb{Z}/3\mathbb{Z})^2 \arrow[r] & H \arrow[r] & (\mathbb{Z}/3\mathbb{Z})^3.
\end{tikzcd}$$

Then, $\iota(d_2(x_4)) = d_2(\iota(x_4)) = 12$. So, in the spectral sequence associated to the lower extension, $d_2(x_4) = 12 + A$ where $j^*(A) = 0$. $H^2((\mathbb{Z}/3\mathbb{Z})^3; \mathbb{F}_3)$ consists of linear combinations of the classes $\{\alpha_1, \alpha_2, \alpha_3, 12, 13, 23\}$ and $H^2((\mathbb{Z}/3\mathbb{Z})^2; \mathbb{F}_3)$ consists of linear combinations of the classes $\{\alpha_2, \alpha_3, 13, 23\}$. The map j^* is given as follows:

$$j^*(\alpha_i) = \begin{cases} 0 & i = 1 \\ \alpha_i & i \neq 1 \end{cases} \qquad j^*(rs) = \begin{cases} 0 & rs = 13 \\ 23 & rs = 23. \end{cases}$$

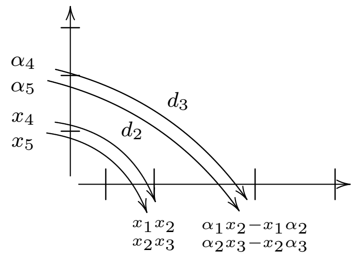
Thus, A is a linear combination of α_1 and 13. We say that α_2, α_3 and 23 are 'detected' in this case.

One finds that $d_2(x_4)$ does not involve α_1 or 13 using the diagram:

$$\begin{tikzcd}
(\mathbb{Z}/3\mathbb{Z})^2 \arrow[r] \arrow[d, equal] & (\mathbb{Z}/3\mathbb{Z})^4 \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^2 \arrow[d, "j"] \\
(\mathbb{Z}/3\mathbb{Z})^2 \arrow[r] & H \arrow[r] & (\mathbb{Z}/3\mathbb{Z})^3.
\end{tikzcd}$$

$\iota(d_2(x_4)) = d_2(\iota(x_4)) = 0$ so that $j^*(A) = 0$. $j^*(\alpha_i) = \alpha_1$ and $j^*(13) = 13$ (note that $H^2((\mathbb{Z}/3\mathbb{Z})^2; \mathbb{F}_3)$ consists of linear combinations of $\{\alpha_1, \alpha_3, 13\}$) so that α_1 and 13 are detected. Thus $A = 0$. $\square$

By Serre's transgression theorem, $d_3(\alpha_4) = \beta(12) = \alpha_1 x_2 - x_1 \alpha_2$ and $d_3(\alpha_5) = \beta(23) = \alpha_2 x_3 - x_2 \alpha_3$. The following picture depicts the differentials identified thus far appear in this spectral sequence:



Generators we have seen from Chapter 2 show up in this spectral sequence as follows: $d_2(14) = d_2(24) = d_2(25) = d_2(35) = d_2(245) = 0$. Note that $d_2(15) = 123 = d_2(34)$ so that we have a generator $15 - 34$. Also, $d_3(x_1\alpha_4) = d_3(x_2\alpha_4) = d_3(x_2\alpha_5) = d_3(x_3\alpha_5) = 0$. $d_3(x_3\alpha_4) = -\alpha_1 23 + \alpha_2 13 = \alpha_2 13$ modulo the d_2 differential and $d_3(x_1\alpha_5) = \alpha_2 13$ as well, thus we have another generator $x_3\alpha_4 - x_1\alpha_5$. Also, $d_3(24\alpha_5 - 25\alpha_4) = 0$. Both α_4^3 and α_5^3 live to the $E_\infty -$ page by an Evens norm argument (they are in the image of restriction from H to G_{27}^h and G_{27}^l, respectively. Use the Mackey formula for Evens norms.

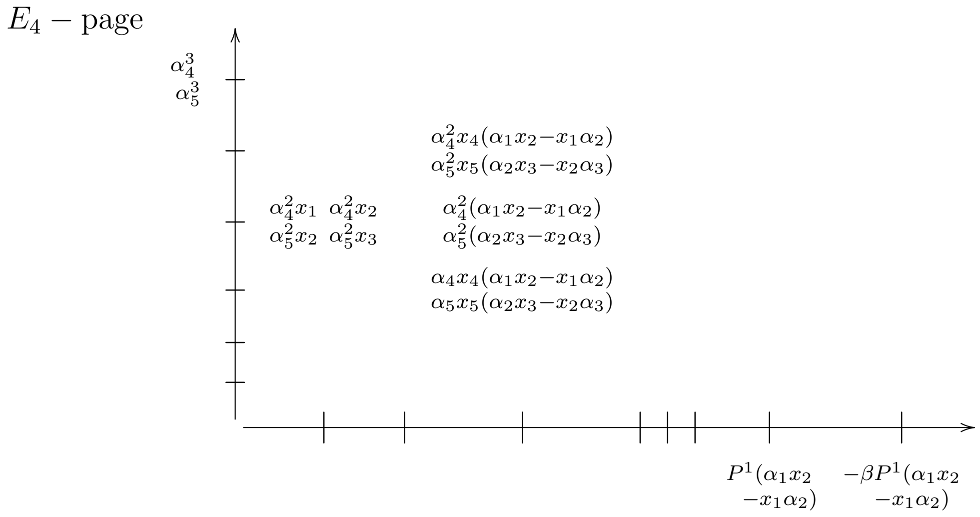
Label the generators we've identified as follows: $x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3$,

$\sigma_1 :$	14	$\mu_1 :$	$x_1\alpha_4$	$\kappa :$	245
$\tau_1 :$	24	$\nu_1 :$	$x_2\alpha_4$	$\lambda :$	$x_3\alpha_4 - x_1\alpha_5$
$\sigma_2 :$	25	$\mu_2 :$	$x_2\alpha_5$	$\omega :$	$24\alpha_5 - 25\alpha_4$
$\tau_2 :$	35	$\nu_2 :$	$x_3\alpha_5$	$\gamma_4 :$	α_4^3
$\pi :$	$15 - 34$			$\gamma_5 :$	α_5^3.

It is not clear which elements correspond to ζ_i, θ_i and η. There are several possibilities for each and there is still room for differentials that may effect these possibilities.

The d_2 and d_3 differentials described above completely determine these differentials. One thus obtains an additive description of the $E_4 -$ page. As a ring, it consists of more than thirty five generators. This description will not be noted here as it is large and unilluminating.

Some classes to be discussed next are indicated in this picture:



Now, nonzero d_4 differentials certainly exist. For example, the elements $\alpha_4x_4(\alpha_1x_2 - x_1\alpha_2), \alpha_5x_5(\alpha_2x_3 - x_2\alpha_3) \in E_4^{3,3}$ must die by comparison with our result from Chapter 2. In [9], Leary provides a result to give some of the nonzero d_4 differentials of certain elements on the E_4-page. Below are more broad results in this vein. The main difficulty in applying these results in full generality is describing the Massey products as particular classes in the groups' cohomologies.

The inspiration for these results is fairly straightforward. Suppose x generates $E_2^{0,1}$ and α generates $E_2^{0,2}$ with representatives in $E_0^{0,1}$ and $E_0^{0,2}$ of the same names. If these elements are to live to the E_4-page, then we must have $d_0(x) = 0$, $d_1(x) = d_0(\theta)$ for some $\theta \in E_0^{1,0}$ and that $d_1(\theta) = \epsilon$ is an element representing $d_2(x)$. Similarly there are elements $\eta_1 \in E_0^{1,1}$ and $\eta_2 \in E_0^{2,0}$ with $d_0(\alpha) = 0$, $d_1(\alpha) = d_0(\eta_1)$, $d_1(\eta_1) = d_0(\eta_2)$ and $d_1(\eta_2) = \zeta$, with ζ representing $d_3(t)$.

So to determine the d_4 differential of an element, say z on the E_4-page, we need construct a sequence of elements $\{\phi_i\}_{i=1}^4$ with roughly, $d_1(z) = d_0(\phi_1)$, $d_1(\phi_i) = d_0(\phi_{i+1})$ and $d_1(\phi_3)$ representing $d_4(z)$ modulo $d_0(\phi_4)$. Diagramatically,

$$
\begin{tikzcd}
0 \\
z \arrow[u, "d_0"] \arrow[r, "d_1"'] \arrow[rrrrddd, bend left, "d_4"] & {} \\
& \phi_1 \arrow[u, "d_0"] \arrow[r, "d_1"] & {} \\
& & \phi_2 \arrow[u, "d_0"] \arrow[r, "d_1"] & {} \\
& & & \phi_3 \arrow[u, "d_0"] \arrow[r, "d_1"] & {} \\
& & & & \phi_4 \arrow[u, "d_0"]
\end{tikzcd}
.
$$

Remark 3.2. *3.4 on the next page and 3.7 on page 43 hold for a general class of groups. Namely, those groups E that fit into a central extension $(\mathbb{Z}/3\mathbb{Z})^n \longrightarrow E \longrightarrow G$ whereas Leary's results in [9] are for groups that fit into this extension with $n = 1$.*

In the Lyndon-Hochschild-Serre spectral sequence associated to the central extension $(\mathbb{Z}/3\mathbb{Z})^m \longrightarrow E \longrightarrow G$, let $\{x_i\}_{i=1}^m$ generate $E_2^{0,1}$ and $\alpha_i = \beta(x_i)$ (so that $\{\alpha_i\}_{i=1}^m$ generates $E_2^{0,2}$) and let $d_2(x_j) = \epsilon_j$, and $d_3(\alpha_i) = \zeta_i$. The x_i and α_i are represented on the E_0-page by elements of the same names. We fix these identifications so that from now on $x_i, \alpha_i, \theta_i, \epsilon_i$ and ζ_i refer to these elements.

We prepare the following technical lemma, proved in [9], used in the proofs of Theorems 3.4 and its corollary and 3.7.

Lemma 3.3. *[9, Lemma 2.] For all $n \geq 1$ there exist cochains $\eta_1(n), \ldots, \eta_i(n) \in E_0^{i,2n-i}$ such that $d_1(\alpha^n) = d_0(\eta_1(n))$,*

$d_1(\eta_1(n)) = d_0(\eta_2(n))$,

$d_1(\eta_2(n)) = n\alpha^{n-1}\zeta + d_0(\eta_3(n))$,

$d_1(\eta_3(n)) = n\eta_1(n-1)\zeta + d_0(\eta_4(n))$.

The proof is done by defining these η_i carefully and using cup-1 products. Note that the existence of $\eta_1(n), \eta_2(n)$ and $\eta_3(n)$ is immediate from our knowledge of $d_3(\alpha_i^n)$ so that the content of this lemma is the existence of $\eta_4(n)$ and the equation for $d_1(\eta_3(n))$. $\square$

Theorem 3.4. *In the Lyndon-Hochschild-Serre spectral sequence with $\mathbb{F}_3$ coefficients associated to the central extension $(\mathbb{Z}/3\mathbb{Z})^m \longrightarrow E \longrightarrow G$, let $\{x_i\}_{i=1}^m$ generate $E_2^{0,1}$ and $\{\alpha_i\}_{i=1}^m$ generate $E_2^{0,2}$ and let $d_2(x_j) = \epsilon_j$, and $d_3(\alpha_i) = \zeta_i$.*
For fixed $1 \leq i, j \leq m$, if $\rho \in H^*(G; \mathbb{F}_3)$ with $\zeta_i\rho = 0$ and $\rho\epsilon_j = 0$, then $d_2(\alpha_i^n x_j \rho) = 0$ and $d_3(\alpha_i^n x_j \rho) = 0$, and $d_4(\alpha_i^n x_j \rho) = n\alpha_i^{n-1} < \zeta_i, \rho, \epsilon_j >$.

Proof: Let $d_0(\theta_j) = d_1(x_j)$,

$d_0(\eta_1^i(n)) = d_1(\alpha_i^n)$,

$d_0(\omega_{ij}) = d_0(\omega'_{ij}) = 0, d_1(\omega_{ij}) = \rho\epsilon_j, \text{and } d_1(\omega'_{ij}) = \zeta_i\rho.$

Define

$$\phi_0 = \alpha_i^n \rho x_j$$

$$\phi_1 = \eta_1^i(n)\rho x_j + \alpha_i^n \rho\theta_j - (-1)^{|\rho|}\alpha_i^n \omega'_{ij}$$

$$\phi_2 = \eta_2^i(n)\rho x_j + \eta_1^i(n)\rho\theta_j - (-1)^{|\rho|}\eta_1^i(n)\omega'_{ij} - n\alpha_i^{n-1}\omega_{ij}x_j$$

$$\phi_3 = \eta_3^i(n)\rho x_j + \eta_2^i(n)\rho\theta_j - (-1)^{|\rho|}\eta_2^i(n)\omega'_{ij} - n\eta_1^i(n-1)\omega_{ij}x_j - n\alpha_i^{n-1}\omega_{ij}\theta_j$$

$$\phi_4 = \eta_4^i(n)\rho x_j + \eta_3^i(n)\rho\theta_j - (-1)^{|\rho|}\eta_3^i(n)\omega'_{ij} - n\eta_2^i(n-1)\omega_{ij}x_j - n\eta_1^i(n-1)\omega_{ij}\theta_j.$$

By explicit calculation, $d_0(\phi_0) = 0$, $d_1(\phi_i) = d_0(\phi_{i+1})$ for $i = 1, 2,$ and

$d_1(\phi_3) = d_0(\phi_4) - (-1)^{|\rho|} n\alpha_i^{n-1}(\zeta_i\omega'_{ij} + \omega_{ij}\epsilon_j)$. Note that $-(\zeta_i\omega'_{ij} + \omega_{ij}\epsilon_j)$ is a related cocycle for $< \zeta_i, \rho, \epsilon_j >$. $\square$

Corollary 3.5. *[9, Part (a) of Theorem 3]: If $m = 1$ in 3.4, then*
$d_2(\alpha_1^n x_1 \chi) = 0, d_3(\alpha_1^n x_1 \chi) = 0$, *and* $d_4(\alpha_1^n x_1 \chi) = n\alpha_1^{n-1} < \zeta_1, \chi, \epsilon_1 >$.

Serre's transgression tells us $d_5(\alpha_4^3) = P^1(\zeta_4) = \alpha_1^3 x_2 - x_1 \alpha_2^3$ and $d_5(\alpha_5^3) = P^1(\zeta_5) = \alpha_2^3 x_3 - x_2 \alpha_3^3$. These elements live to the $E_\infty -$ page by an Evens norm argument. Thus some earlier differential hits these Serre differentials. [9] shows that in the spectral sequence associated to the central extension $\mathbb{Z}/3\mathbb{Z} \longrightarrow G_{27} \longrightarrow (\mathbb{Z}/3\mathbb{Z})^2$, the corresponding elements are hit by the d_4 differentials from 3.5. In fact, it is shown that $< \zeta_4, \zeta_4, \epsilon_4 >= \alpha_1^3 x_2 - x_1 \alpha_2^3$.

We thus expect that in the spectral sequence under consideration, we have a similar description, and so we do:

Proposition 3.6. *In the spectral sequence associated to 3.1,* $d_7(\alpha_4^3) = \pm d_4(\alpha_4^2 x_4 \zeta_4)$ *and* $d_7(\alpha_5^3) = \pm d_4(\alpha_5^2 x_5 \zeta_5)$.

Proof: By 3.4 we need only show that $< \zeta_4, \zeta_4, \epsilon_4 >= \alpha_1^3 x_2 - x_1 \alpha_2^3$ and $< \zeta_5, \zeta_5, \epsilon_5 >= \alpha_2^3 x_3 - x_2 \alpha_3^3$ modulo the d_2 and d_3 differentials. We calculate these products in $H^*(< \alpha_1, \alpha_2 >; \mathbb{F}_3)$ and $H^*(< \alpha_2, \alpha_3 >; \mathbb{F}_3)$, where according to an example in [9] are $\alpha_1^3 x_2 - x_1 \alpha_2^3$ and $\alpha_2^3 x_3 - x_2 \alpha_3^3$. $\square$

Theorem 3.7. *Given the hypotheses of 3.4, for fixed $1 \leq i, j \leq m$ if $\chi \in H^*(G; \mathbb{F}_3)$ with $\zeta_i \chi = 0$ and $\chi \zeta_j = 0$, then $d_i(\alpha_i \chi \alpha_j) = 0$ for $i < 5$ and $d_5(\alpha_i \chi \alpha_j) =< \zeta_i, \chi, \zeta_j >$.*

Proof: Let $d_0(\eta_1^i) = d_1(\alpha_i)$,

$d_0(\psi) = d_0(\psi') = 0$,

$d_1(\psi) = \zeta_i \chi$ and $d_1(\psi') = \chi \zeta_j$.

Define

$\phi_0 = \alpha_i \chi \alpha_j$

$\phi_1 = \eta_1^i \chi \alpha_j + \alpha_i \chi \eta_1^j$

$\phi_2 = \alpha_i \chi \eta_2^j + \eta_2^i \chi \alpha_j + \eta_1^i \chi \eta_1^j - \psi \alpha_j - (-1)^{|\chi|} \alpha_i \psi'$

$\phi_3 = \eta_1^i \chi \eta_2^j + \eta_2^i \chi \eta_1^j - \psi \eta_1^j - (-1)^{|\chi|} \eta_1^i \psi'$

$\phi_4 = \eta_2^i \chi \eta_2^j - \psi \eta_2^j - (-1)^{|\chi|} \eta_2^i \psi'$.

Then, $d_0(\phi_0) = 0$, and $d_1(\phi_i) = d_0(\phi_{i+1})$ for $i = 1, 2, 3$, and $d_1(\phi_4) = -(-1)^{|\chi|}(\zeta_i \psi' + \psi \zeta_j)$. Note that $-(\zeta_i \psi' + \psi \zeta_j)$ is a related cocycle for $< \zeta_i, \chi, \zeta_j >$. □

Using Theorem 3.7 we recover differentials given by Kudo's transgression which tells us that $d_5(\alpha_4^2(\alpha_1 x_2 - x_1 \alpha_2)) = -\beta(\mathrm{P}^1(\alpha_1 x_2 - x_1 \alpha_2))$ and $d_5(\alpha_5^2(\alpha_2 x_3 - x_2 \alpha_3)) = -\beta(\mathrm{P}^1(\alpha_2 x_3 - x_2 \alpha_3))$.

Corollary 3.8. *In the spectral sequence associated to 3.1,* $d_5(\alpha_4^2 \zeta_4) =< \zeta_4 >^3$ *and* $d_5(\alpha_5^2 \zeta_5) =< \zeta_5 >^3$.

Proof: According to [8, Theorem 14], $-\beta(\mathrm{P}^1(\alpha_1 x_2 - x_1 \alpha_2)) =< \zeta_4 >^3$. Here $\chi = \zeta_4 = \zeta_i = \zeta_j$ in the statement of 3.7. The related cocycle given in that proof is a (the) related cocycle for the restricted product $< \zeta_4 >^3$. The other differential is shown similarly. □

Remark 3.9. *Some comment should be made on the equalties given in results 3.4 to 3.7. At first one may be concerned that there are well-definedness problems with the Massey products in the conclusions. However in each case the indeterminacy is in the image of the* d_2 *and/or* d_3 *differential. Thus, these Massey products are given as single elements in* $E_4^{*,0}$*. Note that although the classes described in Theorems 3.4 and 3.7 have zero* d_2 *and* d_3*, or* d_2*,* d_3 *and* d_4 *differentials, they may be hit by some differential and so may not appear on the* $E_5 - \text{page}$*.*

Leary also proves the following which is applicable here.

Proposition 3.10. *[9, Theorem 3 part (b)] With* $m = 1$ *in the hypotheses of 3.4, if* $\chi \in H^*(G;\mathbb{F}_3)$ *with* $\zeta\chi = \epsilon\chi'$ *for some* $\chi' \in H^*(G;\mathbb{F}_3)$*, then for all* $n \geq 2$*,* $a^n\chi$ *survives until* E_4 *and* $d_4(\alpha^n\chi) = n(n-1)\alpha^{n-2}x\zeta\chi'$*.*

Remark 3.11. *'Survive' here means has zero* d_2 *and* d_3 *differentials, though may be hit by one of these differentials.*

Corollary 3.12. *In the spectral sequence associated to 3.1,*
(a). $d_4(\alpha_4^2 x_1) = \alpha_1^2\tau_1 - \alpha_1\alpha_2\sigma_1$ *(b).* $d_4(\alpha_4^2 x_2) = \alpha_1\alpha_2\tau_1 - \alpha_2^2\sigma_1$
(c). $d_4(\alpha_5^2 x_2) = \alpha_2\alpha_3\sigma_2 - \alpha_2^2\tau_2$ *(d).* $d_4(\alpha_5^2 x_3) = \alpha_3^2\sigma_2 - \alpha_2\alpha_3\tau_2$

Proof: (a). For $\chi = x_1$, $\zeta\chi = \epsilon\chi'$ with $\chi' = -\alpha_2$. Using [9], and naturality of spectral sequence applied to the diagrams

$$\begin{tikzcd} <c> \arrow[r] \arrow[d] & G_{27}^h \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^2 \arrow[d] \\ (\mathbb{Z}/3\mathbb{Z})^2 \arrow[r] & H \arrow[r] & (\mathbb{Z}/3\mathbb{Z})^3, \end{tikzcd}$$

$$\begin{tikzcd}
(\mathbb{Z}/3\mathbb{Z})^2 \arrow[r] \arrow[d, equal] & (\mathbb{Z}/3\mathbb{Z})^4 \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^3 \arrow[d] \\
(\mathbb{Z}/3\mathbb{Z})^2 \arrow[r] & H \arrow[r] & (\mathbb{Z}/3\mathbb{Z})^3,
\end{tikzcd}$$

calculations like those of 3.1 give the result. (b). is found similarly. (c). and (d). follow by applying the automorphism Φ introduced on page 30. □

Additional non-zero d_4 differentials certainly exist. The following discussion indicates some such and conjectures on what these differentials might be. These examples show how spectral sequences associated to different extensions can be compared to obtain information about them.

Proposition 3.13. *The following three elements in* $E_4^{1,4}$ *have non-zero* d_4 *differentials*

(a). $\alpha_4 x_2 \alpha_5\,,$ *(b).* $\alpha_4 x_1 \alpha_5 + \alpha_4^2 x_3\,,$ *(c).* $\alpha_4 x_3 \alpha_5 + \alpha_5^2 x_1\,.$

Proof: From chapter 2, $|H^5(H;\mathbb{F}_3)| = 44$ but, in the spectral sequence under consideration, $\sum_{i+j=5} |E_4^{i,j}| = 51$. Thus seven elements in this degree are either hit by differentials or have non-zero differentials. Since $E_4^{0,4} = 0$ none of these classes are hit by a d_4 or higher differential. Also, since $E_4^{0,5} = 0$ the only possibility is that the seven classes in $E_4^{1,4}$ all have non-zero differentials. Since the elements in $E_4^{6,0}$ all must live to E_∞, these must be d_4 differentials. Non-zero d_4 differentials for four of these entries are given in 3.12. The elements in the statement are the remaining three. □

It is not clear exactly what the differentials of the classes in Proposition 3.13 are. However, $d_4(\alpha_4 x_2 \alpha_5)$ must be invariant under the automorphism Φ of page 30, and

$d_4(\alpha_4 x_1 \alpha_5 + \alpha_4^2 x_3) = \Phi(d_4(\alpha_4 x_3 \alpha_5 + \alpha_5^2 x_1))$. By considering the elements in $E_4^{5,1}$ and known relations in $H^*(H;\mathbb{F}_3)$, I conjecture that $d_4(\alpha_4 x_2 \alpha_5) = k(\alpha_1\alpha_3\tau_1 \pm \alpha_1\alpha_3\sigma_2 \pm \alpha_2^2\pi)$ for some $k \in \mathbb{F}_3$; and that $d_4(\alpha_4 x_1 \alpha_5 + \alpha_4^2 x_3) = k(\alpha_3^2\tau_1 + \alpha_1\alpha_2\sigma_2 \pm z\pi)$ whence, $d_4(\alpha_4 x_3 \alpha_5 + \alpha_5^2 x_1) = k(\alpha_1^2\sigma_2 + \alpha_2\alpha_3\tau_1 \pm \Phi(z)\pi)$ for z of the form $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}$.

We note here that $|H^6(H;\mathbb{F}_3)| = 51$ and $\sum_{i+j=6} |E_4^{i,j}| = 57$. Thus in degree six, six entries have non-zero differentials or are hit. Kudo's transgression and 3.12 account for these and there are no more differentials in degree six.

Conceivably one could continue this analysis to find other non-zero differentials although the utility of such a pursuit is not evident. We thus leave this spectral sequence. ♦

Chapter 4

A Cyclic Central Extension

We consider the central extension

$$\mathbb{Z}/3\mathbb{Z} \longrightarrow H \longrightarrow G_{27} \times \mathbb{Z}/3\mathbb{Z}, \tag{4.1}$$

with the subgroup the one generated by a_5. As usual, $E_2^{i,j} = H^i(G_{27} \times \mathbb{Z}/3\mathbb{Z}; H^j(\mathbb{Z}/3\mathbb{Z}; \mathbb{F}_3))$. The coefficients are trivial as this is a central extension. We will use our shortened notation for products of degree one generators, e.g. 12 denotes x_1x_2, as in 3.

Let $H^*(\mathbb{Z}/3\mathbb{Z}; \mathbb{F}_3) = \mathbb{F}_3[\alpha_5](1, x_5)$. According to the Künneth theorem, we may write $H^*(G_{27} \times \mathbb{Z}/3\mathbb{Z}; \mathbb{F}_3) = \mathbb{F}_3[\alpha_1, \alpha_3, \gamma_4](1, x_3)(1, x_1, \sigma, \mu) \oplus$
$\mathbb{F}_3[\alpha_2, \alpha_3, \gamma_4](1, x_3)(x_2, \alpha_2, \tau, \nu, x_1\alpha_2, x_2\sigma, \alpha_1\alpha_2, \alpha_2\sigma, x_2\mu, \alpha_2\mu, \sigma\nu, \alpha_1\alpha_2\sigma)$.

Additively, the E_2-page is
$\mathbb{F}_3[\alpha_1, \alpha_3, \gamma_4, \gamma_5](1, x_5)(1, \alpha_5, \alpha_5^2)(1, x_3)(1, x_1, \sigma, \mu) \oplus$
$\mathbb{F}_3[\alpha_2, \alpha_3, \gamma_4, \gamma_5](1, x_5)(1, \alpha_5, \alpha_5^2)(1, x_3)(x_2, \alpha_2, \tau, \nu, x_1\alpha_2, x_2\sigma, \alpha_1\alpha_2, \alpha_2\sigma, x_2\mu, \alpha_2\mu, \sigma\nu, \alpha_1\alpha_2\sigma)$,
where $\gamma_5 = \alpha_5^2$.

Proposition 4.1. *We have* $d_2(x_5) = 23$.

Proof: $H^2(G_{27}^l \times \mathbb{Z}/3\mathbb{Z}; \mathbb{F}_3)$ consists of linear combinations of $\{\alpha_1, \alpha_2, \sigma_1, \tau_1, 13, 23\}$. We first look at the map of spectral sequences induced by the commutative diagram where the upper extension is the inclusion of the center of G_{27}^l.

$$\begin{tikzcd}
\mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, equal] & G_{27}^l \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^2 \arrow[d] \\
\mathbb{Z}/3\mathbb{Z} \arrow[r] & H \arrow[r] & G_{27} \times \mathbb{Z}/3\mathbb{Z}.
\end{tikzcd}$$

This diagram detects α_2 and 23.

$$\begin{tikzcd}
\mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, equal] & G_{27}^h \times \mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d] & G_{27} \arrow[d] \\
\mathbb{Z}/3\mathbb{Z} \arrow[r] & H \arrow[r] & G_{27} \times \mathbb{Z}/3\mathbb{Z},
\end{tikzcd}$$

where the normal subgroup in the upper extension is the center of the copy of G_{27}^h, detects α_1, σ_1 and τ_1.

Finally, 13 is detected on the diagram:

$$\begin{tikzcd}
\mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, equal] & (\mathbb{Z}/3\mathbb{Z})^4 \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^3 \arrow[d] \\
\mathbb{Z}/3\mathbb{Z} \arrow[r] & H \arrow[r] & G_{27} \times \mathbb{Z}/3\mathbb{Z}.\Box
\end{tikzcd}$$

Using the labels $x_i, \alpha_i, \sigma_1, \tau_1, \mu_1, \nu_1, \gamma_4$ as classes in $H^*(G_{27} \times \mathbb{Z}/3\mathbb{Z}; \mathbb{F}_3)$ of the same names, the E_3- page is given additively as:

$\mathbb{F}_3[\alpha_3, \alpha_5, \gamma_4]\{15\mathbb{F}_3[\alpha_1] \oplus 25\{\mathbb{F}_3[\alpha_1](1, \alpha_2) \oplus \alpha_2^2\mathbb{F}_3[\alpha_2] \oplus \sigma_1\} \oplus$

$(\tau_1 - \alpha_1)x_5\{\mathbb{F}_3[\alpha_1](1, \alpha_2, \sigma_1, \tau_1, \mu_1) \oplus \alpha_2^2\mathbb{F}_3[\alpha_2]\} \oplus$

$(1, 35)\{\mathbb{F}_3[\alpha_1](1, s_1, x_2, \alpha_2, \sigma_1, \tau_1, \mu_1, \nu_1, x_2\alpha_2, \alpha_2^2, \alpha_2\tau_1, \alpha_2\nu_1) \oplus$

$\mathbb{F}_3[\alpha_2](x_2\alpha_2^2, \alpha_2^3, \alpha_2^2\tau_1, \alpha_2^2\nu_1) \oplus \mathbb{F}_3(x_2\sigma_1, \alpha_2\sigma_1, \tau_1\mu_1\} \oplus$

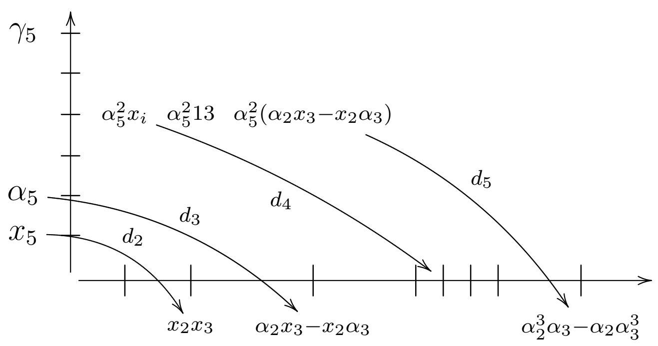
$x_3\{\mathbb{F}_3[\alpha_1 3](1, x_1, \alpha_2, \sigma_1, \mu_1, \nu_1, \alpha_2^2, \alpha_2\nu_1) \oplus \mathbb{F}_3[\alpha_2](\alpha_2^3, \alpha_2^2\nu_1) \oplus \mathbb{F}_3(\tau_1, \alpha_2\sigma_1, \tau_1\mu_1)\}.$

The ring generators we have seen from Chapter 2 show up in this spectral sequence as follows: $d_2(15) = d_2(25) = d_2(35) = 0$. By Serre's transgression theorem, $d_3(\alpha_5) = \beta(23) = \alpha_2 x_3 - x_2\alpha_3$. So, $d_3((\tau_1 - \alpha_1)x_5) = d_3(x_1\alpha_5) = d_3(x_2\alpha_5) = d_3(x_3\alpha_5) = 0$ modulo the d_2 differential. Lastly, $d_3((\tau_1 - \alpha_1)\alpha_5) = 0$. We can then label the generators we've identified corresponding to those of Chapter 2 as follows: $x_1, x_2, x_3, \alpha_1, \alpha_2, \alpha_3$,

$\sigma_1:$	σ_1	$\mu_1:$	μ_1	$\kappa:$	$(\tau_1 - \alpha_1)x_5$
$\tau_1:$	τ_1	$\nu_1:$	ν_1	$\lambda:$	$x_1\alpha_5$
$\sigma_2:$	25	$\mu_2:$	$x_2\alpha_5$	$\omega:$	$(\tau_1 - \alpha_1)\alpha_5$
$\tau_2:$	35	$\nu_2:$	$x_3\alpha_5$	$\gamma_4:$	γ_4
$\pi:$	15			$\gamma_5:$	α_5^3.

It is not clear which elements correspond to ζ_i, θ_i and η. There are several possibilities for each and there is still room for differentials that may effect these possibilities.

The following picture sums up important differentials identified above and following:



There are higher differentials. Results 3.4 to 3.12 on pages 42–45 apply here. We may use Proposition 3.10 to find some elements with d_4 differentials.

Proposition 4.2. *In the spectral sequence associated to 4.1,*

(a). $d_4(\alpha_5^2 x_1) = -\alpha_1\alpha_2\tau_2 + \alpha_1\alpha_3\sigma_2$,

(b). $d_4(\alpha_5^2 x_2) = -\alpha_2^2\tau_2 + \alpha_2\alpha_3\sigma_2$,

(c). $d_4(\alpha_5^2 x_3) = -\alpha_2\alpha_3\tau_2 + \alpha_3^2\sigma_2$,

(d). $d_4(\alpha_5^2 13) = 0$.

Proof: Apply 3.10.

(a). For $\chi = x_1$, $\zeta\chi = \epsilon\chi'$ with $\chi' = \alpha_1$.

(b). For $\chi = x_2$, $\zeta\chi = \epsilon\chi'$ with $\chi' = \alpha_2$.

(c). For $\chi = x_3$, $\zeta\chi = \epsilon\chi'$ with $\chi' = \alpha_3$,

(d). For $\chi = x_3$, $\zeta\chi = \epsilon\chi'$ with $\chi' = 0$. □

Since $|H^5(H;\mathbb{F}_3)| = 44$ but in the spectral sequence under consideration $\sum\limits_{i+j=5} |E_4^{i,j}| = 47$, these three are the only non-zero differentials of degree five elements.

Note that $d_4(\alpha_2\alpha_5^2 x_3) = d_4(\alpha_3\alpha_5^2 x_2)$ so that $\alpha_5^2(\alpha_2 x_3 - x_2\alpha_3)$ lives to the E_5-page as guaranteed by Kudo's transgression theorem.

Proposition 4.3. $d_5(\alpha_5^2(\alpha_2 x_3 - x_2\alpha_3)) = -\beta(\mathrm{P}^1(\alpha_2 x_3 - x_2\alpha_3)) = -(\alpha_2^3\alpha_3 + \alpha_2\alpha_3^3) =< \alpha_2 x_3 - x_2\alpha_3 >^3$.

Proof: The first equality is from Kudo's transgression theorem; the second is by calculation: $-\beta(P^1(\alpha_2 x_3 - x_2\alpha_3)) = -\beta(\sum\limits_{i+j=1} P^i(\alpha_2)P^j(x_3) - \sum\limits_{i+j=1} P^i(x_2)P^j(\alpha_3)) = -\beta(\alpha_2^3 x_3 - x_2\alpha_3^3) = -(\alpha_2^3\alpha_3 + \alpha_2\alpha_3^3)$, using the Cartan formulas; the third a consequence of Theorem 3.7 and Proposition 1.15. □

Theorem 3.4 gives some additional d_4 differentials.

Proposition 4.4. *(a).* $d_4(\alpha_5 x_5(\alpha_2 x_3 - x_2\alpha_3)) =< \alpha_2 x_3 - x_2\alpha_3, \alpha_2 x_3 - x_2\alpha_3, 23 >$

(b). $d_4(\alpha_5 x_5(\alpha_1 x_3 - x_1\alpha_3)) =< \alpha_2 x_3 - x_2\alpha_3, \alpha_1 x_3 - x_1\alpha_3, 23 >$

(c). $d_4(\alpha_5 x_5(\tau_1 - \alpha_1)x_3) =< \alpha_2 x_3 - x_2\alpha_3, (\tau_1 - \alpha_1)x_3, 23 >$

(d). $d_4(\alpha_5 x_5(\alpha_1 x_1 - x_2\sigma_1)) =< \alpha_2 x_3 - x_2\alpha_3, \alpha_1 x_1 - x_2\sigma_1, 23 >$

Proof: The statements are of the form $d_4(\alpha_5 x_5 \chi) =< \zeta_5, \chi, \epsilon_5 >$. So, each follows from 3.4; We should check that the hypotheses of the theorem are satisfied. That is, we need see that $\epsilon_5\chi = 0$ (which is obvious in each case) and $\zeta_5\chi = 0$. Based on the description of $H^*(G_{27}; \mathbb{F}_3)$ in Section 1.7, $\tau_1 x_2 = \alpha_1 x_2$ so that indeed, $(\alpha_2 x_3 - x_2\alpha_3)((\tau_1 - \alpha_1)x_3) = 0$; and $\alpha_1\alpha_2 x_1 = x_2\alpha_2\sigma_1$ whence $(\alpha_2 x_3 - x_2\alpha_3)(\alpha_1 x_1 - x_2\sigma_1) = 0$. □

$|H^6(H; \mathbb{F}_3)| = 53$ but, in the spectral sequence under consideration, $\sum_{i+j=6} |E_4^{i,j}| = 60$. Thus seven elements in this degree are either hit by differentials or have non-zero differentials. Three degree six elements are hit by the non-zero differentials from 4.2, and 4.4 eliminates the remaining four. There are no other non-zero differentials in degree six.

As in Chapter 3, one could conceivably continue such analysis to find other differentials. But, since $\sum_{i+j=7} |E_4^{i,j}| = 86$ whereas $|H^7(H; \mathbb{F}_3)| = 65$ it is already clear that this would be inefficient and ultimately of little value.

One may also consider applying Leary's 'circle method', as presented in his PhD thesis [10], to this extension. His approach is to fix an embedding of $\mathbb{Z}/3\mathbb{Z}$ into S^1 and consider the extension $B_{S^1} \longrightarrow B_{\tilde{H}} \longrightarrow B_{G_{27}\times\mathbb{Z}/3\mathbb{Z}}$ where $\tilde{H}$ is the central product $H \circ S^1$ and B denotes the associated classifying space. The associated Serre spectral sequence is nonzero only in even rows since B_{S^1} has nonzero cohomology only in even degrees indeed, $H^*(B_{S^1};\mathbb{F}_3) \simeq \mathbb{F}_3[\alpha_5]$ for a degree two generator α_5. This implies that the only nonzero differentials are of odd degree. After $H^*(\tilde{H};\mathbb{F}_3)$ is known, one then considers the extension $S^1/(\mathbb{Z}/3\mathbb{Z}) \simeq \tilde{H}/H \longrightarrow B_{\tilde{H}} \longrightarrow B_H$. Now construct the E_2-page of the associated Serre spectral sequence according to the formula: $E_2^{i,j} \simeq H^i(BG_{27} \times \mathbb{Z}/3\mathbb{Z}; H^j(S^1;\mathbb{F}_3))$. In this spectral sequence $E_2^{*,1}$ is the only nonzero row since $H^*(S^1;\mathbb{F}_3)$ is an exterior algebra on one generator of degree one, $H^*(S^1;\mathbb{F}_3) \simeq \wedge_{\mathbb{F}_3}(x_5)$. Thus in this second spectral sequence, we have at most a nonzero d_2 differential.

The drawback of the circle method is the lack of theorems that indicate differentials in the first spectral sequence. Leary, for example, largely argued using ranks of the various cohomology groups to show that certain non-zero differentials existed and what they were. In the case of 4.1, since the cohomology grows rapidly, these dimension arguments are not so efficient. For G_{27}, Leary was able to complete his calculation since he had a relatively small number of generators for the E_2-page, only four of them. In the case of H, however, the E_2-page has twelve generators. He then argued that based on the known rank of low dimension cohomology group certain elements in the spectral sequence died. Here this task is less tractable.

In the Lyndon-Hochschild-Serre spectral sequence associated to the extension $H^*(S^1;\mathbb{F}_3) \simeq \mathbb{F}_3[\alpha_5]$, the diagram

$$\begin{array}{ccccc} \mathbb{Z}/3\mathbb{Z} & \longrightarrow & H & \longrightarrow & G_{27} \times \mathbb{Z}/3\mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & \tilde{H} & \longrightarrow & G_{27} \times \mathbb{Z}/3\mathbb{Z}, \end{array}$$

shows $d_3(\alpha_5) = \alpha_2 x_3 - x_2\alpha_3$, which we will also write as ζ_5. The additive description of the $E_4 = E_5$-page is messy and won't be set down here.

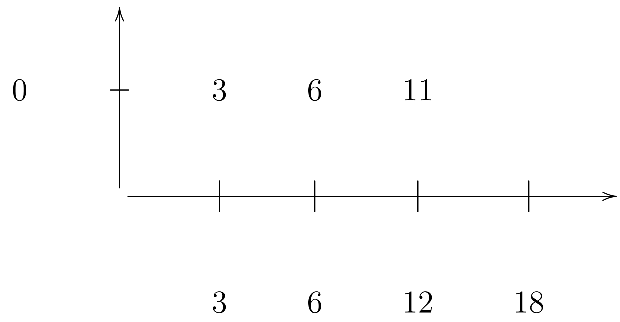
It is not clear where the expected generators for the $E_\infty -$ page live in this spectral sequence. None of the results from Chapter 3 apply here. Nonetheless, it is possible to do some analysis on this spectral sequence. Since we know the only d_3 differential, we can make some conclusions in dimensions ≤ 4. Relations arising from differentials that hit the bottom row are actual relations in $H^*(\tilde{H};\mathbb{F}_3)$ so that we may make the following identifications by rewriting the classes on the left with those on the right: $\alpha_2 x_3 = \alpha_3 x_2$, $\alpha_2\sigma_1 x_3 = x_2\alpha_3\sigma_1$ and $x_3\alpha_2\tau_1 = x_2\alpha_1\alpha_3$.

To begin, $H^2(H;\mathbb{F}_3)$ has rank 9. The seven elements in $E_4^{2,0}$ of the first spectral sequence all must live to E_∞. Thus on the E_2-page of the second spectral sequence, there are seven elements in $E_2^{2,0}$ and three in $E_2^{1,1}$ making $\sum\limits_{i+j=2}|E_2^{i,j}| = 10$. The d_2 differential lands on the bottom row and so represents actual relations in cohomology. Since $x_2x_3 = 0$ is an actual relation in $H^*(H;\mathbb{F}_3)$, $x_2x_3 \in E_2^{2,0}$ must die. Therefore, $d_2(x_5) = x_2x_3$. Thus, $\sum\limits_{i+j=2}|E_\infty^{i,j}| = 9$, as desired.

$H^3(H;\mathbb{F}_3)$ has rank 18. The twelve elements in $E_4^{3,0}$ of the first spectral sequence all must live to E_∞. Thus on the $E_2 -$ page of the second spectral sequence, there are twelve elements in $E_2^{2,0}$ and seven in $E_2^{1,1}$ making $\sum\limits_{i+j=3}|E_2^{i,j}| = 19$. The only element whose d_2 differential is non-zero is labeled $\sigma_1 x_5$ and $d_2(\sigma_1 x_5) = \sigma_1 x_2 x_3 \neq 0$. The d_2 differentials of the elements in $E_2^{1,1}$ are all zero and so $\sum\limits_{i+j=3}|E_\infty^{i,j}| = 18$, as desired.

$|H^4(H;\mathbb{F}_3)| = 29$. The nineteen elements in $E_4^{4,0}$ of the first spectral sequence all must live to E_∞. Thus on the $E_2 -$ page of the second spectral sequence, there are nineteen elements in $E_2^{4,0}$ and twelve in $E_2^{3,1}$ making $\sum\limits_{i+j=4}|E_2^{i,j}| = 31$. Thus, one element must have a non-zero d_2 differential (the element $\sigma_1 x_2 x_3$ is hit by $d_2(\sigma_1 x_5)$, from the paragraph preceding). This is $d_2(\mu_1 x_5) = \mu_1 x_2 x_3 = -\alpha_1\tau_1 x_3 - x_2\sigma_1\alpha_3 \neq 0$. Thus $\sum\limits_{i+j=3}|E_\infty^{i,j}| = 29$, as desired.

The graph below shows the ranks on the $E_\infty -$ page of the second spectral sequence in some low degrees.



We could continue such analysis however, the task quickly becomes overbearing. In addition, without knowledge of the higher differentials that certainly must exist (for example, the element $\alpha_5^2(\alpha_2 x_3 - x_2\alpha_3) \in E_4^{3,4}$ in the first spectral sequence must die) completing this second spectral sequence seems impossible. ♦

Chapter 5

Future Considerations

The initial goal was to find $H^*(UT_4(\mathbb{F}_p);\mathbb{F}_3)$. We can begin by considering the extension

$$\mathbb{Z}/3\mathbb{Z} \longrightarrow U \longrightarrow H. \tag{5.1}$$

We use the notation for elements in this group as in the previous chapters and additionally, the 1,4-entry a_6. Take $H^*(< a_6 >;\mathbb{F}_3) = \mathbb{F}_3[\alpha_6](1, x_6)$, x_6, α_6 as usual. By the UCT $E_2^{i,j} \simeq H^i(H; H^j(< a_6 >;\mathbb{F}_3)) \simeq H^*(H;\mathbb{F}_3) \otimes H^*(< a_6 >;\mathbb{F}_3)$, since this central extension implies trivial coefficients.

Theorem 5.1. $d_2(x_6) = \pi$.

Corollary 5.2. $d_3(\alpha_6) = \lambda - \kappa$.

Proof (of corollary): Immediate from 5.1 and Serre's transgression theorem.

Proof (of theorem): Using the naturality of spectral sequences and the following two diagrams, up to elements in the kernels of j^*, we have $d_2(x_6) = 0$:

$$\begin{tikzcd}
\mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^2 \arrow[d] \\
\mathbb{Z}/3\mathbb{Z} \arrow[r] & U \arrow[r] & H
\end{tikzcd}$$

where in each case $\mathbb{Z}/3\mathbb{Z} =< a_6 >$, and T is one of $< a_6, a_4, a_1 >$, $< a_6, a_5, a_3 >$, $< a_6, a_3, a_1 >$, or $< a_6, a_5, a_3 >$; and

$$\begin{tikzcd}
\mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^3 \arrow[d] \\
\mathbb{Z}/3\mathbb{Z} \arrow[r] & U \arrow[r] & H
\end{tikzcd}$$

where $T =< a_6, a_5, a_4, a_2 >$.

Also note that $< a_6, a_4, a_3 >\simeq G_{27}$ and so according to the diagram

$$\begin{tikzcd}
\mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, equal] & G_{27} \arrow[r] \arrow[d] & (\mathbb{Z}/3\mathbb{Z})^2 \arrow[d] \\
\mathbb{Z}/3\mathbb{Z} \arrow[r] & U \arrow[r] & H
\end{tikzcd}$$

we have $d_2(x_6) \neq 0$. Indeed, in the spectral sequence from the upper extension, $d_2(x_6) = x_3x_4$.

This d_2 differential lands in $H^2(H; \mathbb{F}_3)$. The degree two classes in $H^*(H; \mathbb{F}_3)$ are $x_1x_3, \alpha_1, \alpha_2, \alpha_3, \sigma_i, \tau_i, \pi$. Restricting these classes to $H \cap T$, for each subgroup T above:

Table 5.1: Restrictions of degree two generators

	$< a_4, a_1 >$	$< a_5, a_3 >$	$< a_3, a_1 >$	$< a_5, a_4, a_2 >$
x_1x_3	0	0	x_1x_3	0
α_1	0	α_1	0	α_1
α_2	0	0	0	α_2
α_3	0	α_3	α_3	0
σ_1	x_1x_4	0	0	0
τ_1	0	0	0	x_2x_4
σ_2	0	x_3x_5	0	0
τ_2	0	0	0	x_2x_5
π	0	0	0	0

Thus, $d_2(x_6) = z\pi$ for some $z \in \mathbb{F}_3$. $\square$

From Appendix A, $x_1x_3\pi = 0$ in $H^*(H;\mathbb{F}_3)$. Using this relation and Theorem 5.1 in small degrees, we find a new generator labeled $136 \in E_3^{2,1}$. Additional information will require a better understanding of the relations in $H^*(H;\mathbb{F}_3)$. We may employ the results of Chapter 3 to obtain some higher differentials. It may be that Leary's circle method works better here than in the case of Chapter 4. There is a certain symmetry to the present extension that is not in the one used in 4. Although, we will still have the difficulty of higher differentials. Siegel's approach doesn't apply here and although we may use the circle method it seems that we will have the same trouble as in the last part of Chapter 4. One then must resort to using the more standard techniques (say like those of Chapters 3 and 4). We may also use the long exact sequence $\cdots \to H^*(H;\mathbb{F}_3) \to H^*(H;\mathbb{Z}) \xrightarrow{p} H^*(H;\mathbb{Z}) \to \cdots$ induced by the sequence of coefficients $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

We would also like to find $H^*(H;\mathbb{F}_p)$ for $p > 3$. Siegel [17] was able to give $H^*(G_{27};\mathbb{F}_p)$ for p arbitrary. The main difficulty in the case of H appears to be the additional indecomposable $\mathbb{F}_3[Q]$-modules needed to describe W.

The closing pages of Chapter 3 contain some ad hoc arguments for non-zero differentials of some elements on the E_4-page of that spectral sequence. A theorem similar in statement to 3.4 and 3.7 is desired that would yield precise descriptions of elements of the form, for example, $\alpha_i\chi\alpha_j$ with $\zeta_i\chi$ and $\chi\zeta_j$ not necessarily zero. Ultimately a theorem of the form $d_n(\alpha_i^r\chi\alpha_j^s)$ for arbitrary r, s and χ subject to weak conditions is sought. It does not seem that the calculational approach used to prove those theorems are sufficient for more general statements.

Based on the known results of $UT_2(\mathbb{F}_3)$, and $UT_3(\mathbb{F}_3)$, and as suggested by the results of Chapters 3 and 4, I propose the conjecture: Given the hypotheses of 3.4, $< \zeta_i, \zeta_i, \epsilon_i >=$

$P^1(\zeta_i)$. Such a result implies that $d_7(\alpha_i^3) = d_4(\alpha_i x_i \zeta_i)$ since Serre's transgression theorem gives the d_7 differential and α_i^3 lives to E_∞ by an Evens norm argument.

These two results will allow us to begin considerations of groups G which fit into extensions of the form $(\mathbb{Z}/3\mathbb{Z})^n \longrightarrow G \longrightarrow Q$. Most of the references cited in the discussion of G_{27} concerned the more general class of groups that fit into extensions
$\mathbb{Z}/3\mathbb{Z} \longrightarrow G \longrightarrow (\mathbb{Z}/3\mathbb{Z})^n$. A natural next step would be those with non-cyclic elementary abelian kernels. ♦

Bibliography

[1] Alejandro Adem and Dikran Karagueuzian. Essential cohomology of finite groups. *Comment. Math. Helv.*, 72:101–109, 1997.

[2] J.L. Alperin. *Local Representation Theory.* Number 11 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, first edition, 1986.

[3] D. J. Benson. *Representations and Cohomology I: Cohomology of Groups and Modules.* Number 31 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, first (paperback) edition, 1998.

[4] D. J. Benson and J.F. Carlson. The cohomology of extraspecial groups. *Bulletin of the American Mathematical Society*, 24(1992):209–235, 1991.

[5] Kenneth S. Brown. *Cohomology of Groups.* Number 58 in Graduate Texts in Mathematics. Springer-Verlag, first edition, 1982.

[6] L. Evens. A generalization of the transfer map in the cohomology of groups. *Trans. of the Amer. Math. Society*, 108(1):54–65, July 1963.

[7] Leonard Evens. *The Cohomology of Groups.* Oxford Mathematical Monographs. Oxford University Press, first edition, 1991.

[8] David Kraines. Massey higher products. *Trans. of the Amer. Math. Society*, 124:431–449, Sep 1966.

[9] Ian Leary. A differential in the Lyndon–Hochschild–Serre spectral sequence. *Journal of Pure and Applied Algebra*, 88:155–168, 1993.

[10] Ian James Leary. *The Cohomology of Certain Groups.* PhD thesis, University of Cambridge, January 1992.

[11] G. Lewis. The integral cohomology rings of groups of order p^3. *Trans. of the Amer. Math. Society*, 132:501–529, Jul 1968.

[12] J. Peter May. Matric Massey products. *Journal of Algebra*, 12:533–568, 1969.

[13] John McCleary. *A User's Guide to Spectral Sequences.* Number 58 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, second edition, 2001.

[14] R. James Milgram and Michishige Tezuka. The geometry and cohomology of M_{12}: II. *Bol. Soc. Mat. Mexicana*, 1(2):91–108, 1995.

[15] Pham Anh Mimh. On the mod p cohomology groups of extra-special p-groups. *Japanese Journal of Mathematics*, 18(1):139–154, 1992.

[16] Daniel G. Quillen. The mod 2 cohomology rings of extra-special 2-groups and the Spinor group. *Math. Ann.*, 194:197–223, 1974.

[17] Stephen F. Siegel. The spectral sequence of a split extension and the cohomology of an extraspecial group of order p^3 and exponent p. *Journal of Pure and Applied Algebra*, 106:185–198, 1996.

[18] A.J. Weir. Sylow p-subgroups of the general linear group over finite fields of characteristic p. *Proceedings of the American Mathematical Society*, 6:454–464, June 1955.

Appendix A

Relations in $H^*(H;\mathbb{F}_3)$

The results of Chapter 2 only give the additive structure of $H^*(H;\mathbb{F}_3)$. The 25 classes given in 2.6 are ring generators for this cohomology. There are many relations amongst the generators.

Some relations are immediate for degree reasons:

$$x_i^2 = \mu_i^2 = \nu_i^2 = \kappa^2 = \lambda^2 = \theta_i^2 = 0.$$

By restriction to G_{27}^h and G_{27}^l, we can get some others:

$$\begin{array}{ccc}
 & x_1\tau_1 = x_2\sigma_1 \qquad x_1\sigma_1 = x_2\tau_1 = x_2\alpha_1 = x_1\alpha_2 & \\
 & x_3\sigma_2 = x_2\tau_2 \qquad x_3\tau_2 = x_2\sigma_2 = x_2\alpha_3 = x_3\alpha_2 & \\
\sigma_1\tau_1 = \alpha_1\alpha_2 & \sigma_1^2 = \alpha_1\tau_1 = x_2\mu_1 + \alpha_2\sigma_1 & \tau_1^2 = \alpha_2\sigma_1 \\
\tau_2\sigma_2 = \alpha_3\alpha_2 & \tau_2^2 = \alpha_3\sigma_2 = x_2\nu_2 + \alpha_2\tau_2 & \sigma_2^2 = \alpha_2\tau_2 \\
x_1\mu_1 = \alpha_1\sigma_1 - \alpha_1\alpha_2 & x_1\nu_1 = -x_2\mu_1 & x_2\nu_1 = \alpha_2\tau_1 - \alpha_1\alpha_2 \\
x_3\nu_2 = \alpha_3\tau_2 - \alpha_2\alpha_3 & x_3\mu_2 = -x_2\nu_2 & x_2\mu_2 = \alpha_2\sigma_2 - \alpha_2\alpha_3 \\
\tau_1\mu_1 = -\sigma_1\nu_1 & \sigma_1\mu_1 = -\alpha_1\nu_1 = -\tau_1\nu_1 = \alpha_2\mu_1 & x_1\alpha_1\alpha_2 = x_2\alpha_2\sigma_1 \\
\sigma_2\nu_2 = -\tau_2\mu_2 & \tau_2\nu_2 = -\alpha_3\mu_2 = -\sigma_2\mu_2 = \alpha_2\nu_2 & x_3\alpha_2\alpha_3 = x_2\alpha_2\tau_2 \\
 & \alpha_1^2\alpha_2 = \alpha_2^2\sigma_1 - x_2\alpha_2\mu_1 \qquad \alpha_1^2\tau_1 = \alpha_1\alpha_2\sigma_1 - \alpha_1\alpha_2^2 & \\
 & \alpha_2\alpha_3^2 = \alpha_2^2\tau_2 - x_2\alpha_2\nu_2 \qquad \alpha_3^2\tau_2 = \alpha_2\alpha_3\tau_2 - \alpha_2^2\alpha_3 & \\
 & \alpha_1\alpha_2\mu_1 = \alpha_2\sigma_1\nu_1 \qquad \alpha_2\alpha_3\nu_2 = \alpha_2\tau_2\mu_2 & \\
 & \alpha_1^2\alpha_2\sigma_1 = \alpha_2^3\sigma_1 \qquad \alpha_2\alpha_3^2\tau_2 = \alpha_2^3\tau_2. &
\end{array}$$

Note that every other line is the images under the automorphism Φ of the line above it.

Some low degree relations are easy to find using table 2.1 on page 32:

$$x_1\pi = -x_3\sigma_1 \qquad x_1\sigma_2 = x_3\tau_1 = -x_2\pi \qquad x_3\pi = -x_1\tau_2$$
$$13\alpha_2 = 13\sigma_1 = 13\tau_1 = 13\sigma_2 = 13\tau_2 = 13\pi = 0$$
$$x_3\mu_2 = -x_2\nu_2$$
$$\underline{\sigma_1\pi = -\alpha_1\sigma_2} \qquad \pi^2 = \sigma_1\tau_2 \qquad \sigma_2\pi = -\tau_1\tau_2$$
$$\tau_1\pi = -\sigma_1\sigma_2 \qquad \sigma_2\tau_1 = \alpha_2\pi \qquad \underline{\tau_2\pi = -\alpha_3\tau_1}$$
$$x_3\mu_1 = \alpha_1\pi + \sigma_1\sigma_2 - x_1\lambda \qquad x_1\mu_2 = \alpha_2\pi - x_2(\lambda-\kappa) + \alpha_1\sigma_2$$
$$x_3\nu_1 = \alpha_2\pi - x_2(\lambda-\kappa) + \alpha_3\tau_1 \qquad x_1\nu_2 = \alpha_3\pi + \tau_1\tau_2 + x_3\kappa$$
$$x_1\kappa = \sigma_1\sigma_2 - \alpha_3\sigma_1 \qquad x_3\lambda = -\tau_1\tau_2 + \alpha_1\tau_2$$
$$13\nu_1 = 13\mu_2 = 0$$
$$\pi\mu_1 = -\sigma_1\lambda \qquad \pi\nu_2 = \tau_2\kappa$$
$$\underline{\pi^3 = -\alpha_1\alpha_2\alpha_3} \qquad \pi\omega = \kappa\lambda = \mu_1\nu_2$$
$$\pi\zeta_1 = \pi\zeta_2 = 0$$
$$\pi\eta = 0$$

Proposition A.1. $Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4\alpha_5) = x_1\alpha_3 - x_3\tau_1$

Proof: We prove this statement as an example of the steps necessary to produce Table A.2. See [5] or [7] for the double coset (Mackey formula) and transfer of a restricted class formulas used below. We have:

$$\mathrm{Res}^H_T(Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4\alpha_5)) = \sum_{g\in T\backslash H/(\mathbb{Z}/3\mathbb{Z})^4} Tr^T_{T\cap g(\mathbb{Z}/3\mathbb{Z})^4g^{-1}}(\mathrm{Res}^{g(\mathbb{Z}/3\mathbb{Z})^4g^{-1}}_{T\cap g(\mathbb{Z}/3\mathbb{Z})^4g^{-1}}(c_{g^{-1}}(x_4\alpha_5))).$$

Recall that $Tr^{(\mathbb{Z}/3\mathbb{Z})^m}_{(\mathbb{Z}/3\mathbb{Z})^n}(z) = 0$ for $m > n$ so that for $T = H_{ijk}$, this sum is zero. When $T = (\mathbb{Z}/3\mathbb{Z})^4$ (note that T is normal) we have

$\mathrm{Res}^H_{(\mathbb{Z}/3\mathbb{Z})^4}(Tr^H_T(x_4\alpha_5)) = \sum_{g\in H/T\simeq\mathbb{Z}/3\mathbb{Z}} Tr^T_T(\mathrm{Res}^T_T(c_{g^{-1}}(x_4\alpha_5))) = (1 + a_2^{-1} + a_2^{-2})(x_4\alpha_5) = x_4\alpha_5 + (x_4+x_1)(\alpha_5-\alpha_3) + (x_4-x_1)(\alpha_5+\alpha_3) = x_1\alpha_3.$

For $T =< a_5 > \times G^h_{27}$ there is only one double coset, and

$\mathrm{Res}^H_T(Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4\alpha_5)) = Tr^T_{<a_5>\times<a_4,a_1>}(\mathrm{Res}^{(\mathbb{Z}/3\mathbb{Z})^4}_{<a_5>\times<a_4,a_1>}(x_4\alpha_5)) =$

$\alpha_5 Tr^{<a_5>\times G_{27}}_{<a_5>\times<a_4,a_1>}(x_4) = 0;$

and, for $T =< a_4 > \times G^l_{27}$ there is also only one double coset, so

$\mathrm{Res}^H_T(Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4\alpha_5)) = Tr^T_{<a_4>\times<a_5,a_3>}(\mathrm{Res}^{(\mathbb{Z}/3\mathbb{Z})^4}_{<a_4>\times<a_5,a_3>}(x_4\alpha_5)) =$

$x_4 Tr^{<a_5>\times G_{27}}_{<a_4>\times<a_5,a_3>}(\alpha_5) = 0.$

Thus $Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4\alpha_4) = x_1\alpha_3 + A$ for some A with $\mathrm{Res}^H_{H_{111}}(A) = \mathrm{Res}^H_{H_{121}}(A) = -x\alpha$, $\mathrm{Res}^H_{H_{112}}(A) = \mathrm{Res}^H_{H_{211}}(A) = x\alpha$, $\mathrm{Res}^H_{(\mathbb{Z}/3\mathbb{Z})^4}(A) = 0$, $\mathrm{Res}^H_{<a_5>\times G^h_{27}}(A) = 0$ and $\mathrm{Res}^H_{<a_4>\times G^l_{27}}(A) = 0$. According to table 2.1, $A = -x_3\tau_1$. We conclude $Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(x_4\alpha_4) = x_1\alpha_3 - x_3\tau_1$. □

Table A.2: Transfers in H

	$Tr^H_{(\mathbb{Z}/3\mathbb{Z})^4}(\quad)$
x_4	0
$x_4\alpha_4$	$-x_1\alpha_1 + x_2\sigma_1$
$x_4\alpha_4^2$	$\alpha_1\mu_1 + \sigma_1\nu_1$
$x_4\alpha_5$	$x_1\alpha_3 - x_2\pi$
$x_4\alpha_5^2$	$\pi\mu_2 - \alpha_3\kappa$
$x_4\alpha_4\alpha_5$	$\underline{\alpha_1\kappa} - \sigma_1\mu_2 + x_1\omega$
$x_4\alpha_4^2\alpha_5$	$\mu_1\omega + x_1\alpha_1^2\alpha_3 + x_2\alpha_2^2\pi$
$x_4\alpha_4\alpha_5^2$	$-\kappa\omega - x_1\alpha_1\alpha_3^2 + x_2\alpha_2\sigma_1\tau_2$
x_5	0
$x_5\alpha_4$	$x_3\alpha_1 - x_3\tau_1$
$x_5\alpha_4^2$	$\pi\nu_1 - \alpha_1\lambda$
$x_5\alpha_5$	$-x_3\alpha_3 + x_2\tau_2$
$x_5\alpha_5^2$	$-\alpha_3\nu_2 + \sigma_2\nu_2$
$x_5\alpha_4\alpha_5$	$\alpha_1\nu_2 - x_3\omega + \pi\mu_2$
$x_5\alpha_4^2\alpha_5$	$-\lambda\omega - x_3\alpha_1^2\alpha_3 + x_2\alpha_2\sigma_1\tau_2$
$x_5\alpha_4\alpha_5^2$	$-\nu_2\omega + x_3\alpha_1\alpha_3^2 + x_2\alpha_2^2\pi$
x_4x_5	x_1x_3
$x_4x_5\alpha_4$	$-\alpha_1\pi - x_3\mu_1 - \sigma_1\sigma_2$
$x_4x_5\alpha_4^2$	$\mu_1\lambda + x_1x_3\alpha_1^2$
$x_4x_5\alpha_5$	$-\alpha_3\pi - x_1\nu_2 - \tau_1\tau_2$
$x_4x_5\alpha_5^2$	$\nu_2\kappa + x_1x_3\alpha_3^2$
$x_4x_5\alpha_4\alpha_5$	$\omega\pi + \mu_1\nu_2 - x_1x_3\alpha_1\alpha_3$
α_4	0
α_4^2	$-\alpha_1^2 - \alpha_1\tau_1 - \alpha_2\sigma_1$
α_5	0
α_5^2	$-\alpha_3^2 - x_2\nu_2 + \alpha_2\tau_2$

continued on next page

Table A.2: Transfers in H (continued)

$\alpha_4\alpha_5$	$\alpha_1\alpha_3 - \alpha_2\pi - \alpha_3\tau_1 - \underline{\alpha_1\sigma_2}$
$\alpha_4^2\alpha_5$	$-\alpha_1\omega - \underline{\mu_1\mu_2}$
$\alpha_4\alpha_5^2$	$-\alpha_3\omega - \underline{\nu_1\nu_2}$
$\alpha_4^2\alpha_5^2$	$-\alpha_1^2\alpha_3^2 - \alpha_2 x_2\nu_2\sigma_1 - \alpha_2 x_2\mu_1\tau_2 + \alpha_2^2\sigma_1\tau_2 - \omega^2$

Table A.2 is useful for finding relations in $H^*(H;\mathbb{F}_3)$. For example: $x_1\zeta_1 = x_1Tr(45\alpha_4^2\alpha_5) = Tr((14)x_5\alpha_4^2\alpha_5) = \sigma_1 Tr(x_5\alpha_4^2\alpha_5) = \sigma_1(-\lambda\omega - x_3\alpha_1^2\alpha_3 + x_2\alpha_2\sigma_1\tau_2) = -(\lambda\sigma_1)\omega - (x_3\sigma_1)\alpha_1^2\alpha_3 + x_2\alpha_2(x_2\mu_1 + \alpha_2\sigma_1)\tau_2 = -(\lambda\sigma_1)\omega - (x_3\sigma_1)\alpha_1^2\alpha_3 + x_2\alpha_2^2\sigma_1\tau_2 = -(\lambda\sigma_1)\omega - (x_3\sigma_1)\alpha_1^2\alpha_3 + \alpha_2^3\alpha_3\sigma_1$. We immediately get other relations from this information. Using the automorphism Φ discussed in 2, $-x_1\zeta_2 = -(\kappa\tau_2)\omega - (x_1\tau_2)\alpha_3^2\alpha_1 + \alpha_2^3\alpha_1\tau_2$. Since the Bockstein is natural with respect to group homomorphisms, applying the Bockstein gives $\alpha_1\zeta_1 = x_1\theta_1 - \sigma_1\omega^2 + \lambda\mu_1\omega - \alpha_1^2\alpha_3^2\sigma_1 + x_3\alpha_1^2\alpha_3\mu_1 + \underline{\alpha_2^3\alpha_3\mu_1}$; and Φ to this Bockstein, $\alpha_3\zeta_2 = -x_3\theta_2 - \tau_2\omega^2 + \kappa\nu_2\omega - \alpha_1^2\alpha_3^2\tau_2 + x_1\alpha_1\alpha_3^2\nu_2 + \underline{\alpha_2^3\alpha_1\nu_2}$. ♦