ADVANCED STUDY OF THE METHOD OF MOMENT DISTRIBUTION

by

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SYNOPSIS

The purpose of this report is to present seven topics which are related to the method of moment distribution. These topics are selected because the student usually is not familiar with them. The fundamental principles are first described and seven illustrative examples, one for each topic, are presented.

INTRODUCTION

The method of moment distribution presented by Hardy Cross in 1930 is one of the most useful methods developed in the last three decades for the analysis of continuous beams and frames. The reason for this is chiefly that the student, thoroughly understanding the method, can obtain a firmer grasp of the importance of structure geometry and the factors that affect this geometry. It is also for the same reason that the writer first decided to make an advanced study of this method.

The fundamental concept and general application of the moment distribution method is well known and quite familiar to the student. Thus, in this report emphasis is laid on some comparatively unusual topics which are not necessarily difficult but which certainly need special considerations when applying the method of moment distribution.

The topics selected in this report are structures with elastic supports, structures with hinges, the effects of axial forces, structures with nonprismatic members, direct moment distribution method, moment distribution as a relaxation procedure, and the use of digital computer programming. Each topic is treated in a separate section. Following the description in each section a numerical example is given to illustrate the steps of the procedure.

SIGN CONVENTIONS AND NOTATIONS

The following sign conventions will be used in this report except those especially arranged:

A moment acting on the end of a member is positive in the counterclockwise direction and negative in the clockwise direction (see Fig. A).

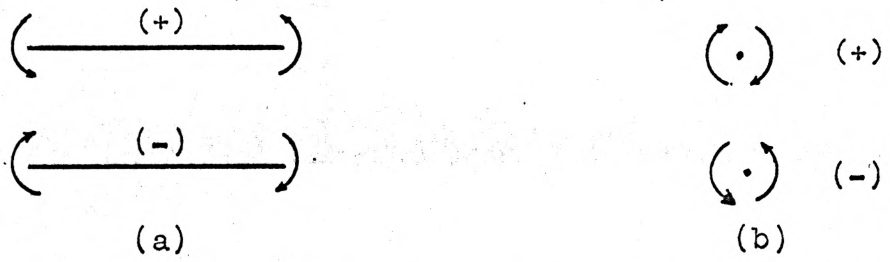


Fig. A. Sign conventions for moments, (a) acting on the ends of a beam, and (b) acting on joints.

The notations employed are as follows:

COF = Carry-over Factor

DF = Distribution Factor

E = Modulus of Elasticity

h = Height of Column

I = Moment of Inertia

K = Rotational Stiffness

L = Length of Span

M = Moment

M_F = Fixed-end Moment

R = Reactive Force

θ = Angle of Rotation

Δ = Displacement

Σ = Summation

t = Spring Constant

STRUCTURES WITH ELASTIC SUPPORTS

The main difficulty in applying the moment distribution method to the analysis of the structures with elastic supports is that the support displacement cannot be determined in advance since it depends on the magnitude of the reactive force. Thus the determination of the support displacement becomes an essential part in the solution of this kind of problem.

In the book Moment Distribution, written by J. W. Gere (1), a special procedure called the superposition method is used to obtain the support displacement. The fundamental concept of this method may be best expressed by the continuous beam shown in Fig. 1.

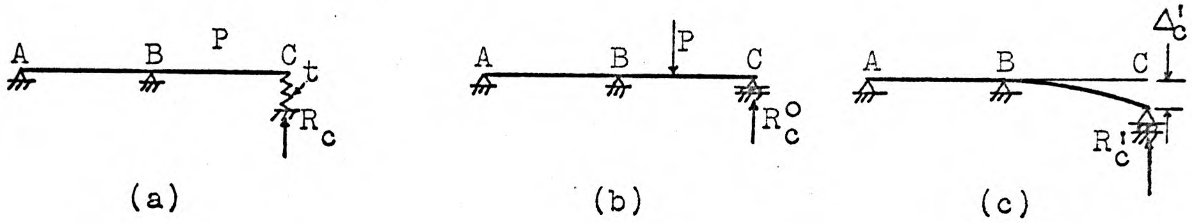


Fig. 1. Continuous beam with an elastic support.

In Fig. 1, t represents the spring constant, R_c^o is the reaction at C obtained from the assumed unyielding structure similar to the original one. R_c' is the reaction at C for the structure with an arbitrarily assumed value of joint displacement Δ_c' at the elastic support.

In order to analyze the original beam shown in Fig. 1 (a), the two beams shown in (b) and (c) should be analyzed first.

A linear superposition of the results of these two analyses will give the moments in (a). In other words, the moments in the original beam (a) are equal to those in beam (b) plus some constant of proportionality a_1 times those in beam (c). The existence of a_1 is due to the fact that Δ_c' is only an arbitrarily assumed value. If Δ_c is the real deflection at C, then

$$\Delta_c = a_1 \Delta_c',$$

$$R_c = R_c^o + a_1 R_c' . \qquad (1)$$

Since

$$R_c = t \cdot \Delta_c,$$

$$a_1 = \frac{R_c^o}{t\,\Delta_c' - R_c'} . \qquad (2)$$

Thus, the moment M of the original beam can be found from equation (3),

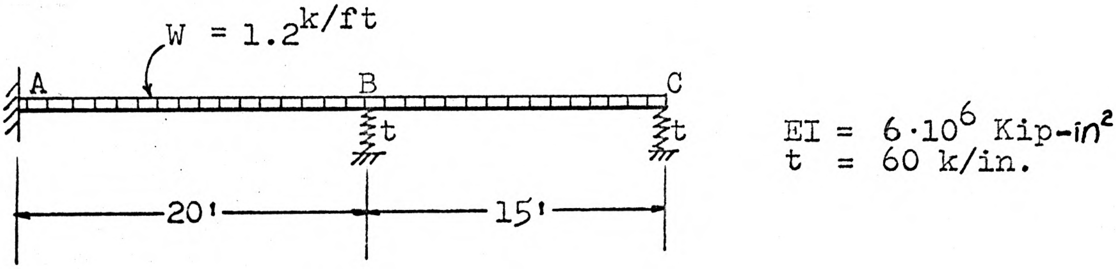
$$M = M^o + a_1 M' , \qquad (3)$$

where M^o is the moment in the beam in Fig. 1 (b), and M' is that in the beam in Fig. 1 (c).

This method can be extended and applied to beams with more than one elastic support. Equation (1) will yield a set of simultaneous equations. There will be as many equations as there are elastic supports. To illustrate the steps of the procedure the following numerical example is presented.

Numerical Example 1

The beam ABC shown in Fig. 2 (a) will be analyzed by the method



(a)

Fig. 2. Example 1. Continuous beam with two elastic supports.

described in the previous paragraphs. The first step is to calculate the stiffness and distribution factors for the members.

$$K_{ab} = \frac{4EI}{L} = \frac{4EI}{20} = \frac{EI}{5} ,$$

$$K_{bc} = \frac{3EI}{L} = \frac{3EI}{15} = \frac{EI}{5} ,$$

$$DF_{ab} = 0.5 , \qquad DF_{bc} = 0.5 .$$

Then, assume that the displacements at B and C are equal to zero, (see Fig. 2 (b)).

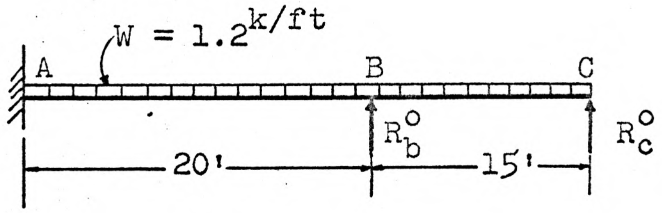


Fig. 2 (b). Beam assuming zero support displacement.

The moment distribution for the beam in Fig. 2 (b) is carried out in Fig. 2 (c).

Member	AB	BA	BC	CB
DF		0.5	0.5	
M_F	40	-40	22.5	-22.5
			11.25	22.5
	1.56	3.13	3.13	
M^o (k-ft)	41.56	-36.87	36.88	0

Fig. 2 (c) Moment distribution assuming zero displacement at supports B, C.

From the results shown in Fig. 2 (c) and by the equilibrium conditions, one obtains

$$15R_c^o - 1.2(15)\frac{15}{2} + 36.88 = 0,$$

$$\therefore R_c^o = \frac{135-36.88}{15} = 6.55\ ^k\uparrow,$$

and

$$35R_c^o + 20R_b^o - 1.2(35)\frac{35}{2} + 41.56 = 0,$$

$$R_b^o = \frac{735-229-41.56}{20} = 23.22\ ^k\uparrow.$$

The second step assumes that the displacement at B is equal to 0.1 inch, see Fig. 2 (d).

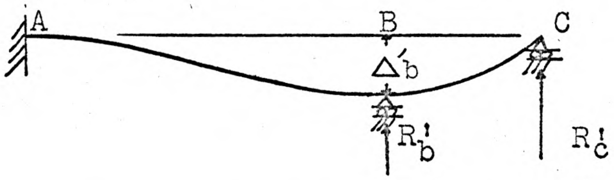


Fig. 2 (d). Beam with 0.1 inch displacement at support B.

The fixed end moments are

$$M_{Fab} = M_{Fba} = \frac{6EI\,\Delta_b'}{L^2} = \frac{6(6)10^6(0.1)}{20(20)1728} = 5.21 \text{ k-ft} ,$$

$$M_{Fbc} = -\frac{3EI\,\Delta_b'}{L^2} = -\frac{3(6)10^6(0.1)}{15(15)1728} = -4.67 \text{ k-ft}.$$

The moment distribution for the beam in Fig. 2 (d) is shown in Fig. 2 (e).

From the results in Fig. 2 (e),

$$R_c' (15) - 4.92 = 0,$$

$$R_c' = 0.328 \text{ k} \uparrow ,$$

and

$$R_b' (20) + 0.328(35) + 5.07 = 0 ,$$

$$R_b' = \frac{-11.5-5.07}{20} = -0.803 \text{ k} \downarrow ,$$

Member	AB	BA	BC	CB
DF		0.5	0.5	
M_F	5.21	5.21	-4.63	
		-0.29	-0.29	
	-0.14			
M' k-ft	5.07	4.92	-4.92	

Fig. 2 (e). Moment distribution for the beam with 0.1 inch displacement at B.

The third step is similar to that of the second step except that the assumed 0.1 inch displacement is at C instead of at B, see Fig. 2 (f).

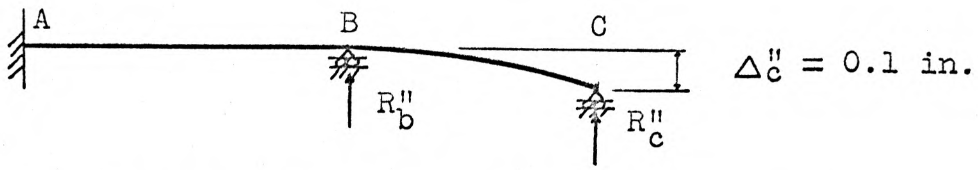


Fig. 2 (f). Beam with 0.1 inch displacement at C.

In Fig. 2 (f), Δ''_c has the same meaning of Δ'_c in Fig. 1, and the explanation of R''_b , R''_c is similar to that of R'_c.
The only fixed-end moment in beam Fig. 2 (f) is

$$M_{Fbc} = \frac{3EI\ \Delta''_c}{L^2} = \frac{3(6)10^6(0.1)}{15(15)1728} = 4.63^{\ k\text{-}ft}.$$

Member	AB	BA	BC	CB
DF		0.5	0.5	
M_F			4.63	
	-1.15	-2.31	-2.32	
M''(k-ft)	-1.15	-2.31	2.31	

Fig. 2 (g). Moment distribution for the beam with 0.1 inch displacement at C.

From the results shown in Fig. 2 (g), one obtains

$$R''_c(15) + 2.31 = 0 ,$$

$$R''_c = -0.154^{k},$$

$$R_b''(20) - 0.157 - 1.15 = 0 ,$$

$$R_b'' = \frac{1.15 + 2.31}{20} = 0.173^{k}.$$

Substituting the above results into equation (4), which is an extension of equation (1),

$$\begin{aligned} R_b^o + a_1 R_b' + a_2 R_b'' &= R_b , \\ R_c^o + a_1 R_c' + a_2 R_c'' &= R_c , \end{aligned} \qquad (4)$$

where

$$R_b = ta_1 \Delta_b' ,$$

$$R_c = ta_2 \Delta_c'' .$$

Thus, one obtains

$$23.22 + a_1(-0.803) + a_2(0.173) = 60(0.1)a_1 , \qquad (5)$$

$$6.55 + a_1(0.328) + a_2(-0.154) = 60(0.1)a_2 . \qquad (6)$$

Solving equations (5) and (6) gives

$$a_1 = 3.5, \qquad a_2 = 1.25.$$

Equation (7),

$$M = M^o + a_1 M' + a_2 M'' , \qquad (7)$$

which is an extension of equation (3), gives the final moments in the original beam

$$M_{ab} = 41.56 + 3.5(5.07) + 1.25(-1.15)$$

$$= 57.82^{\text{ k-ft}},$$

$$M_{bc} = 36.88 - 17.2 + 3.9 = 22.58^{\text{ k-ft}} .$$

The moment diagram for example 1 is shown in Fig. 2 (h).

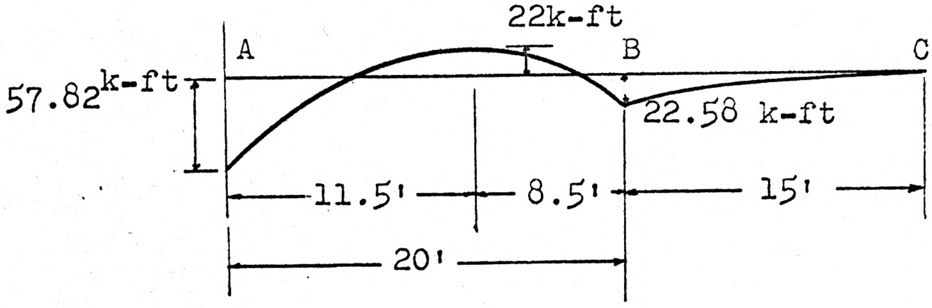
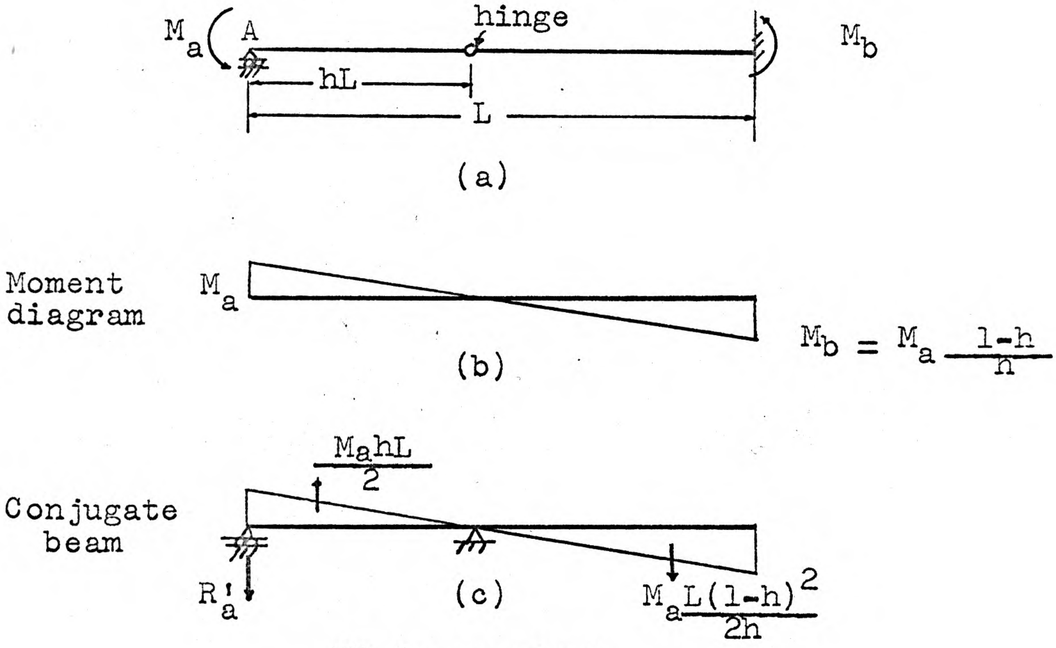


Fig. 2 (h). Moment diagram for example 1, plotted on the compression side.

STRUCTURES WITH HINGES

The usual moment distribution problems which the student encounters in structural analysis are problems with hinges at the joints of structures. It is not, however, too unusual to find structures which contain hinges at intermediate points along the members. The stiffnesses, carry-over factors and fixed-end moments for a beam containing a hinge are somewhat different from those for a beam which does not contain a hinge.

In order to obtain the formulas for the moment distribution factors, for a beam containing a hinge, the beam shown in Fig. 3 (a) will be considered. If a moment acts at the end A, then the angle of rotation θ_a can be obtained by the conjugate beam method.



EI is constant

Fig. 3. Beam with a hinge.

In Fig. 3 (c), R'_a is the reaction at A for the conjugate beam. By the equilibrium condition, the following equation is obtained:

$$R'_a\, hL - M_a \cdot \frac{hL}{2EI} \cdot \frac{2}{3} \cdot hL - M_a \frac{L(1-h)^2}{2h \cdot EI} \cdot \frac{2(L-hL)}{3} = 0 ,$$

$$\therefore\ R'_a = \frac{M_a\, L^3 \left[h^3 + (1-h)^3 \right]}{3EIh^2L^2}$$

$$= \frac{M_a L(3h^2-3h+1)}{3EIh^2} .$$

But, R'_a, the reaction on the conjugate beam, is equal to θ_a, the slope on the original beam.

Thus:

$$\theta = \frac{M_a L}{3EI} \cdot \frac{3h^2-3h+1}{h^2} ,$$

and

$$M_a = \frac{3EIh^2\theta_a}{L(3h^2-3h+1)} .$$

Setting θ equal to unity, the absolute stiffness K_{ab} is obtained:

$$K_{ab} = \frac{3EIh^2}{L(3h^2-3h+1)} . \qquad (9)$$

The carry-over factor is

$$COF_{ab} = \frac{M_b}{M_a} = \frac{1-h}{h} .$$

The formulas for fixed-end moments caused by uniform and concentrated loads can also be obtained by the conjugate beam method, but only the formulas for concentrated load will be illustrated here.

In Fig. 4, the combination of the effects of (b) and (c) will give the moment diagram for the original beam (a). The formula for the fixed-end moment at the end B can be obtained by the equilibrium

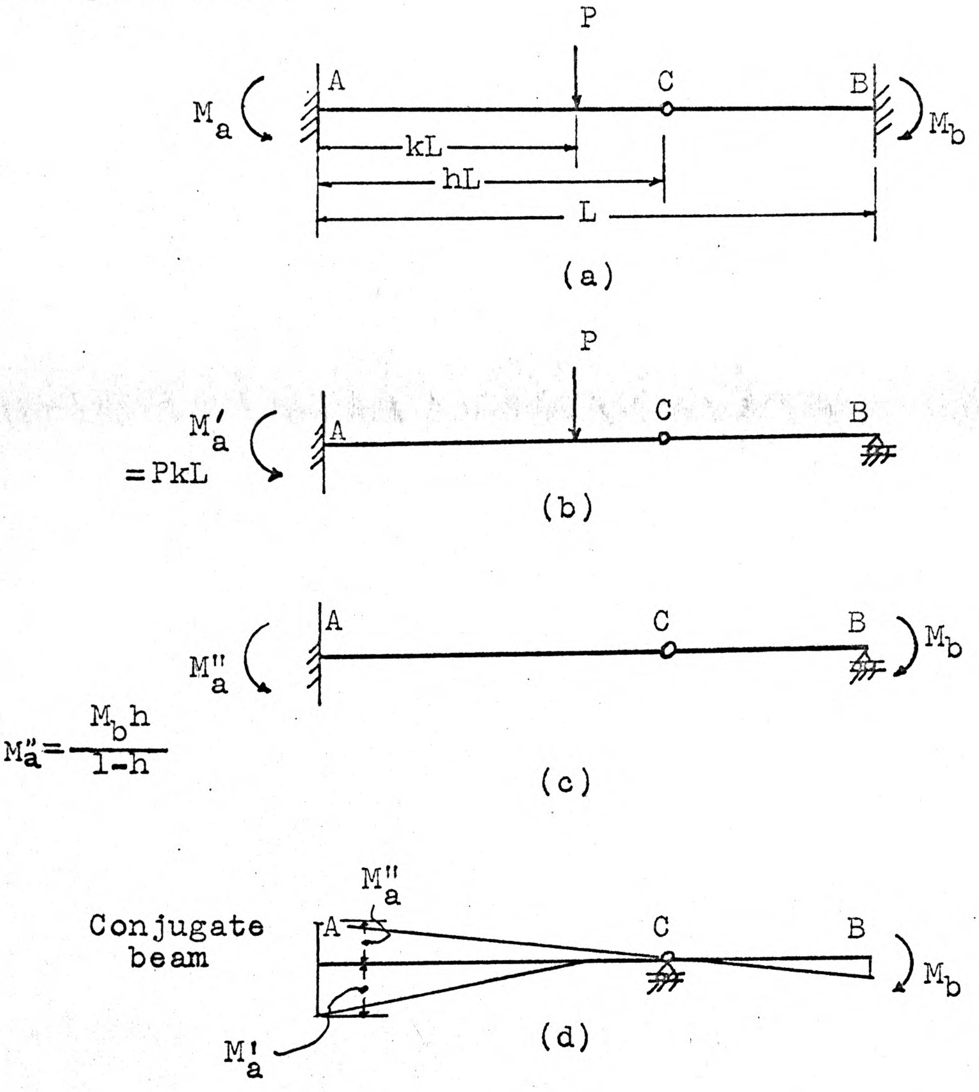


Fig. 4. Beam with hinge and fixed ends.

condition, $\Sigma M_c = 0$, for the beam and loading shown in Fig. 4 (d),

$$\frac{M_b h(hL)^2}{3EI(1-h)} + \frac{M_b(L-hL)^2}{3EI} - \frac{PkL}{EI} \cdot \frac{kL}{6}(3hL-kL) = 0.$$

But the fixed-end moment $M_{Fba} = M_b$, so that

$$M_{Fba} = -PL\frac{k^2(1-h)(3h-k)}{2(3h^2-3h+1)} \quad \text{for} \quad k \leq h.$$

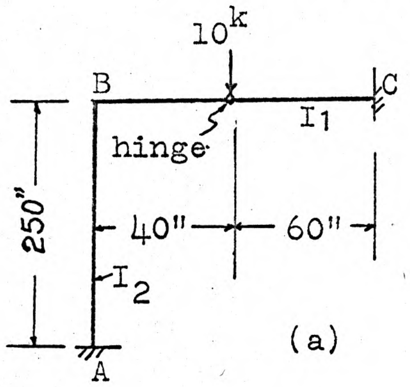
Similarly,

$$M_{Fab} = PL\left[k - \frac{k^2h(3H-k)}{2(3h^2-3h+1)}\right] \quad \text{for} \quad k \leq h. \qquad (10)$$

The formulas for concentrated load with $k \geq h$ and for uniform load can be derived by the same procedure, but they are omitted in this report for the sake of simplicity.

A numerical example of the use of equations (10) and (9) is given in Fig. 5 (a).

Numerical Example 2



E = constant.
$I_1 = 300$ in.^{4}
$I_2 = 500$ in.^{4}

Fig. 5. Example 2. Structure with hinges.

This example offers a special case, that of the load concentrated at the hinge (h=k). Thus, equation (10) simplifies to

$$M_{Fbc} = PL\frac{h(1-h)^3}{3h^2-3h+1},$$

and (11)

$$M_{Fcb} = \frac{-PLh^3(1-h)}{3h^2-3h+1},$$

where $h = k = \frac{2}{5}$, $P = 10^k$, $L = 100$ in.

Thus,

$$M_{Fbc} = \frac{(10)100(1-2/5)^3(\frac{2}{5})}{3(\frac{2}{5})^2 - 3(\frac{2}{5}) + 1} = 309 \text{ k-in.},$$

and

$$M_{Fcb} = -10(100)\frac{(\frac{2}{5})^3(1-\frac{2}{5})}{3(\frac{2}{5})^2 - 3(\frac{2}{5})+1} = -137 \text{ k-in.}$$

The stiffness factors and distribution factors are obtained from equation (9).

$$K_{ab} = \frac{4EI}{L} = \frac{4(50)E}{250} = 8E ,$$

$$K_{bc} = \frac{3EI}{L} \cdot \frac{h^2}{3h^2 - 3h + 1} = \frac{3(300)E}{100} \cdot \frac{(\frac{2}{5})^2}{3(\frac{2}{5})^2 - 3(\frac{2}{5})+1}$$

$$= 5.14E .$$

Therefore, $DF_{ab} = \frac{8}{8 + 5.14} \doteq 0.61,$

$DF_{bc} = 0.39$.

The carry-over factor from B to C is

$$COF_{bc} = \frac{1-h}{h} = 1.5 .$$

Member	AB	BA	BC	CB
DF		0.61	0.39	
COF			1.5	
M_F			309	-137
	-94	-188	-121	-181
M (k-ft)	-94	-188	188	-319

Fig. 5 (b). Moment distribution for the structure in Fig. 5 (a).

The final moments are shown in Fig. 5 (c).

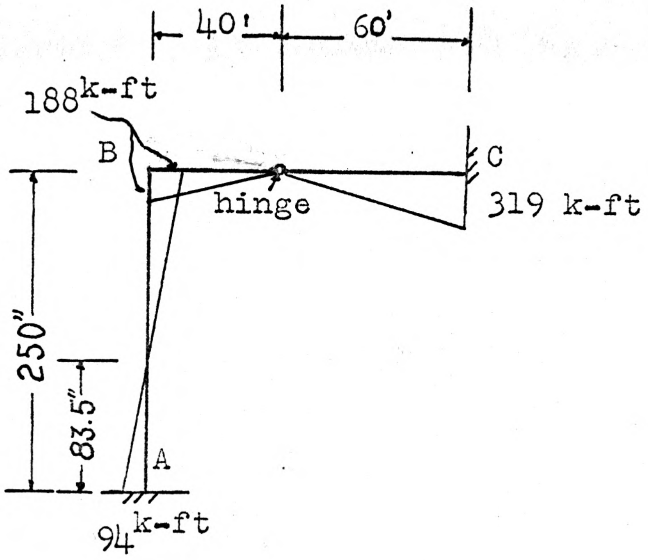


Fig. 5 (c). Moment diagram for example 2, plotted on the compression side.

EFFECTS OF AXIAL FORCES

The presence of axial forces in the members of a structure has two effects. The first is the change in the stiffness characteristics of a member because of the bending moments caused by the axial force. The second effect is that the lengths of the members will change as a result of the axial deformation. In the following paragraphs the first effect, which is also called the "beam-column effect", on the behaviors of the structure, will be considered and a moment distribution method with certain modifications to the standard procedure will be developed.

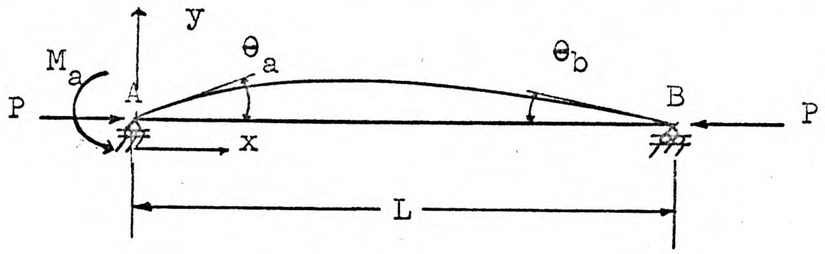


Fig. 6. Beam-column with a moment at left end.

In Fig. 6, y represents the deflection of the axis of the beam. From Strength of Materials the following equation may be obtained,

$$EI\frac{d^2y}{dx^2} = M ,$$

where M is the bending moment at any cross section, and x is the distance from left end to the section. The bending moment M is

$$M = -Py - M_a + M_a\frac{x}{L},$$

This gives,

$$EI\frac{d^2y}{dx^2} = -Py - M_a + M_a\frac{x}{L} .$$

Let $k^2 = \frac{P}{EI}$, thus one obtains

$$\frac{d^2y}{dx^2} + k^2y = \frac{M_a}{EI}(\frac{x}{L} - 1) ,$$

The general solution of this equation is

$$y = C\cos kx + D\sin kx + \frac{M_a}{P}(\frac{x}{L} - 1), \qquad (12)$$

where C and D are constants of integration. The conditions x=0, y=0 and y=0, x=L give

$$C = \frac{M_a}{P} , \qquad D = \frac{M_a}{P}\cdot\cot kL .$$

Substituting these values into equation (12)

$$y = \frac{M_a}{P}(\cos kx - \cot kL \cdot \sin kx + \frac{x}{L} - 1) . \qquad (13)$$

The first derivation of equation (13) gives

$$\frac{dy}{dx} = \frac{M_a}{PL}(-kL\cdot\sin kx - kL\cdot\cot kL . \cos kx + 1).$$

Setting x=0 in this equation, one obtains the angle rotation at A

$$(\frac{dy}{dx})_{x=0} = \theta_a = \frac{M_aL}{3EI}\cdot\frac{3}{kL}\cdot(\frac{1}{kL} - \cot kL).$$

It is seen that as kL approaches π , θ_a becomes infinite. This and $k^2 = \frac{P}{EI}$ give the critical buckling load P

$$P = \frac{\pi^2 EI}{L^2} .$$

By the same procedure,

$$\theta_b = -\frac{M_a L}{6EI} \cdot \frac{6}{KL} \cdot \left(\frac{1}{\sin kL} - \frac{1}{kL}\right).$$

Let $\phi = \frac{6}{kL}\left(\frac{1}{\sin kL} - \frac{1}{kL}\right)$,

$$\psi = \frac{3}{kL}\left(\frac{1}{kL} - \cot kL\right).$$

Then, the expressions for the angles for the beam with positive moments at both ends are

$$\theta_a = \frac{M_a L}{3EI}\psi - \frac{M_a L}{6EI}\phi, \tag{14}$$

and

$$\theta_b = -\frac{M_a L}{6EI}\phi + \frac{M_a L}{3EI}\psi. \tag{15}$$

Setting $\theta_a = 1$, $\theta_b = 0$, the absolute rotational stiffness K_{ab} should be equal to the moment M_a solved from equations (14) and (15). Thus,

$$K_{ab} = \frac{4EI}{L} \cdot \frac{3\psi}{4\psi^2 - \phi^2}. \quad \text{(Far end fixed)} \tag{16}$$

Similarly,

$$K_{ab} = \frac{3EI}{L} \cdot \frac{1}{\psi}. \quad \text{(Far end pinned)} \tag{17}$$

The carry-over factor can be found from equation (15) by setting $\theta_b = 0$,

$$0 = -\frac{M_a L}{6EI}\phi + \frac{M_b L}{3EI}\psi,$$

so that

$$COF_{ab} = \frac{M_b}{M_a} = \frac{1}{2}\frac{\phi}{\psi}. \tag{18}$$

The effect of axial force on the fixed-end moment can be determined by analyzing the beam as shown in Fig. 8.

The fixed-end moments of the beam in Fig. 8 (a) can be found by superposing the three cases shown in (b), (c) and (d). In (c) M_a' is added to rotate the axis of the beam to the horizontal, and in (d) M_b'' is added to remove the angle θ_b' caused by the lateral load in the simply supported beam in Fig. 8 (b).

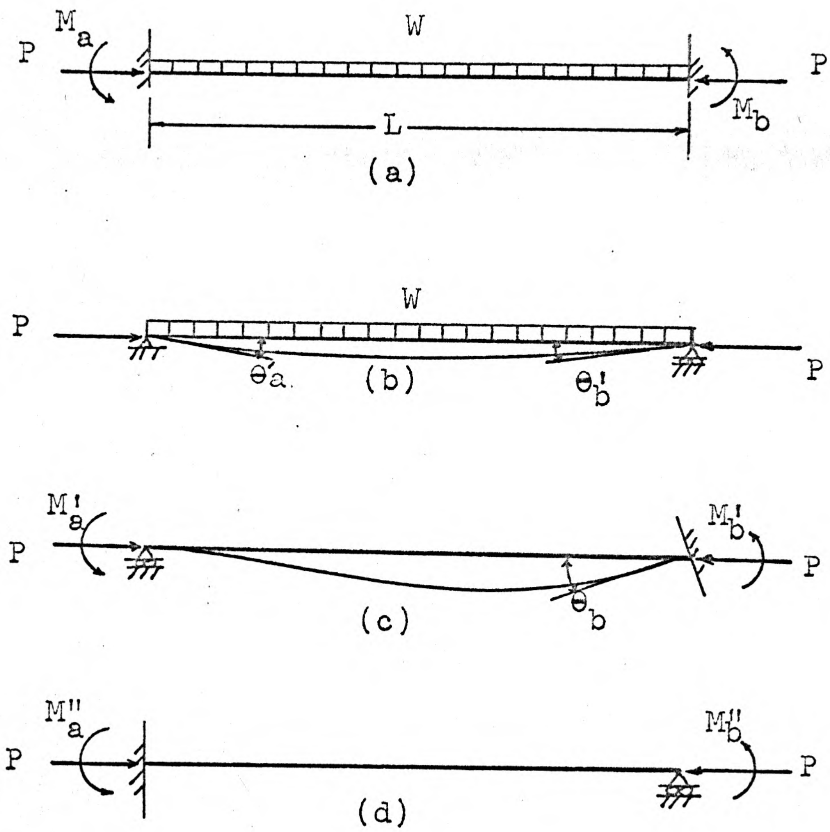


Fig. 8. Fixed end beam with axial load.

The rotational stiffness multiplied by the angle of rotation will give the required moment,

$$M_a' = -K_{ab}\theta_a' ,$$

$$M_b' = -K_{ab}\theta_a'\cdot(COF_{ab}) ,$$

and

$$M_b'' = -K_{ba}\theta_b' ,$$

$$M_a'' = -K_{ba}\theta_b'(COF_{ba}) ,$$

Because the beam is prismatic, therefore,

$$M_b'' = -K_{ab}\theta_b' ,$$

$$M_a'' = -K_{ab}\theta_b'(COF_{ab}) .$$

By the principle of superposition,

$$M_a = M_a' + M_a'' ,$$

$$M_b = M_b' + M_b'' .$$

Thus,

$$\begin{aligned} M_a &= -K_{ba}(\theta_a' + COF_{ab}\theta_b') , \\ M_b &= -K_{ab}(\theta_b' + COF_{ab}\theta_a') . \end{aligned} \qquad (19)$$

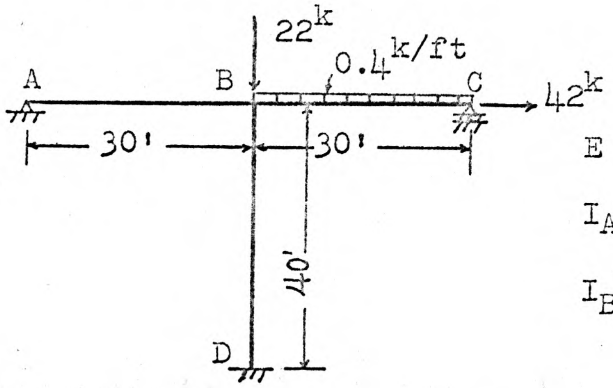
Equation (19) gives the formulas for finding the fixed-end moments. But, before the equation (19) can be used, the unknowns θ_a' and θ_b' should be solved first. This requires a good deal of mathematical derivation, for instance, in the case of a beam with a uniform load as shown in Fig. 8 (b), the expression for θ_a' and θ_b' is

$$\theta_a' = -\theta_b' = -\frac{WL^2}{24EI}\cdot\frac{24}{k^3L^3}\cdot(\tan\frac{kL}{2} - \frac{kL}{2}) . \qquad (20)$$

This equation was derived in Timoshenko's Theory of Elastic Stability, and for the sake of simplicity, the derivation is omitted in this report.

The following numerical example is presented to illustrate the steps of the procedure.

Numerical Example 3



$E = 30{,}000$ ksi

$I_{ABC} = 106.3$ in.^{4}

$I_{BD} = 53.5$ in.^{4}

Fig. 9. Example 3. Rigid frame with beam-column effect considered in the analysis.

Because the axial forces in the members cannot be determined in advance, an estimate of the values is required. Here P=29 k for member BD and for members AB, BC is assumed to be 42k. Thus, the first approximate values of kL are

$$\text{kL for BD} = L\sqrt{\frac{P}{EI}} = 40(12)\sqrt{\frac{22\frac{7}{12}(12)}{30000(53.5)}} = 2.04\ ,$$

$$\text{kL for BC} = L\sqrt{\frac{P}{EI}} = 30(12)\sqrt{\frac{42}{30(10^3)106.3}} = 1.31,$$

and

$$\text{kL for AB} = L\sqrt{\frac{P}{EI}} = 30(12)\sqrt{\frac{42}{30(10^3)106.3}} = 1.31.$$

The stiffness factors can be obtained by equations (16) and (17) or from the stiffness factors table provided on P. 181, Ref. (1),

$$K_{bd} = \frac{3.4EI}{L} = \frac{3.4(30000)53.5}{40(12)} = 11400^{k\text{-}in.} ,$$

$$K_{ba} = \frac{3.3EI}{L} = \frac{3.3(30000)106.3}{30(12)} = 29100^{k\text{-}in.} ,$$

$$K_{bc} = \frac{3.3EI}{L} = \frac{3.3(30000)106.3}{30(12)} = 20100^{k\text{-}in.} .$$

The distribution factors are

$$DF_{bd} = \frac{10400}{11400+2(29100)} = 0.164 ,$$

$$DF_{ba} = \frac{29100}{11400+2(29100)} = 0.418 ,$$

and

$$DF_{bc} = DF_{ba} = 0.418 .$$

The carry-over factors can be obtained by means of equation (18), and also can be found from the carry-over factors table provided on P. 182 Ref. (1),

$$COF_{bd} = 0.64 .$$

The fixed-end moment of member BC can be found by equations (19) and (20), but it also can be obtained by the use of the table provided on p. 187, Ref. (1), that is

$$\frac{M_{bc}}{WL^2} = 0.081 ,$$

$$M_{bc} = 0.081(0.4)(30)^2 = 29.1^{k\text{-}ft} .$$

The moment distribution for the structure shown in Fig. 9 is carried out in Fig. 10.

Member	AB	BA	BD	BC	CB	DB
DF		0.4	0.164	0.418		
COF			0.64		0.46	
M_F				29.1	-29.1	
				13.3	29.1	
		-17.9	-6.9	-17.7		-4.4
M(ft-k)		-17.9	-6.9	24.7	0	-4.4

Fig. 10. Moment distribution for the structure in Fig. 9.

After calculating the distribution of moments, it is found that the assumed axial forces are sufficiently accurate, so that a second assumption for the axial forces is not required.

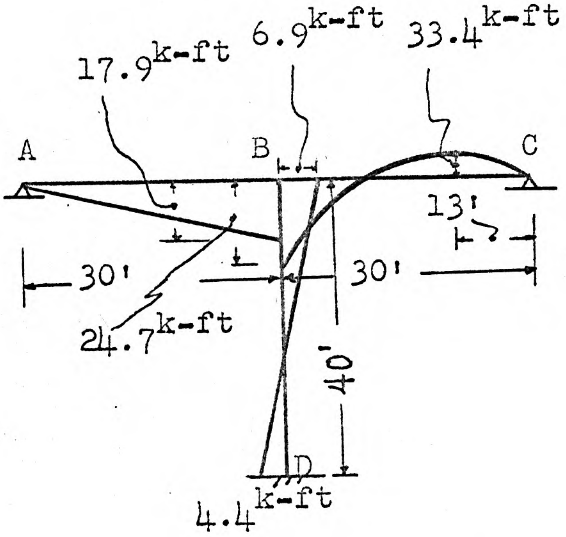


Fig. 11. Moment diagram for example 3, plotted on the compression side.

DIRECT MOMENT DISTRIBUTION

Usually, the first step in the analysis of a sidesway problem uses artificial supports to prevent the translation, then determines the moment distribution by the general procedure; after the supporting forces are determined, loads equal to the forces, but opposite in direction, are applied to the original structure to replace the supporting forces. Then carry out the moment distribution. The combination of the two sets of moments will give the final moments for the original structure.

A modified method of moment distribution, taking the translation and joint rotation into consideration at the same time, can also be used to solve the sidesway problem. This method determines the moments directly with only one distribution procedure, rather than the two or more required by the general method mentioned in the preceding paragraph.

For the rigid frame shown in Fig. 12, it is assumed that the horizontal girders are infinitely rigid when it deflects sidewise. If the joints may translate during the application of the loads, the column AB can be treated as having an elastic support, and the elastic support is provided by the stiffness of the other column CD, see Fig. 12 (b).

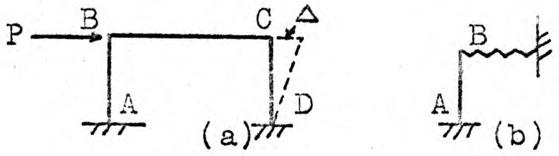


Fig. 12. Rigid frame.

The stiffness and carry-over factors for a beam on an elastic support will be obtained in the following paragraph.

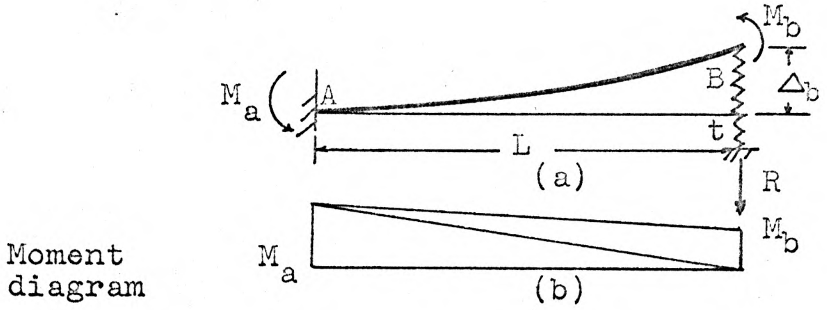


Fig. 13. Beam with an elastic support.

In Fig. 13 (a), if t is the spring constant, by use of the moment area method, the expression of the angle of rotation is found to be

$$\theta_b = \frac{L}{2EI}(M_a + M_b) = \frac{L}{2EI}(2M_b - RL). \qquad (21)$$

The deflection at the end B is

$$\Delta_b = \frac{R}{t} = \frac{1}{EI}\left[\frac{M_b L^2}{6} + \frac{2(M_b - RL)L^2}{6}\right]; \qquad (22)$$

thus,

$$R = \frac{t}{EI}\cdot\left[\frac{M_b L^2}{6} + \frac{2(M_b - RL)L^2}{6}\right]$$

$$= \frac{3M_b L^2 t}{6EI + 2L^3 t}. \qquad (23)$$

Substituting the expression for R in equation (23) into (21),

$$\theta_b = \frac{M_b L}{EI}\left(1 - \frac{3L^3 t}{12EI + 4L^3 t}\right).$$

Then, the ratio $\frac{M_b}{\theta_b}$ which is the rotational stiffness is

$$K_{ba} = \frac{EI}{L}\left(\frac{12EI+4tL^3}{12EI+tL^3}\right). \qquad \text{(Far end fixed)} \quad (24)$$

Substituting the translation stiffness $T = \frac{12EI}{L^3}$ into equation (24), the following equation is obtained

$$K_{ba} = \frac{EI}{L}\left(\frac{4t+T}{t+T}\right) . \qquad \text{(Far end fixed)} \qquad (25)$$

The carry-over factor for member BA is

$$COF_{ba} = \frac{M_a}{M_b} = \frac{M_b - RL}{M_b} = 1 - \frac{3L^3t}{6EI+2L^3t}$$

$$= \frac{6EI-L^3t}{6EI+2L^3t} = \frac{T-2t}{T+4t} . \qquad (26)$$

The formulas for a beam with pinned end at A can be obtained in the same matter, with

$$\theta_b = \frac{\Delta_b}{L} + \frac{M_b L}{3EI} ,$$

where $\Delta_b = R/t$; thus

$$\theta_b = M_b\left(\frac{L}{3EI} + \frac{1}{L^2 t}\right).$$

$$K_{ba} = \frac{M_b}{\theta_b} = \frac{3EIL^2t}{L^3t+3EI} = \frac{12EIL^2t}{4L^3t+12EI}$$

$$= \frac{3EI}{L} \cdot \frac{t}{t+T} , \qquad \text{(Far end pinned)} \qquad (27)$$

where $T = \frac{3EI}{L^3}$.

Now the formulas derived above will be applied to the case of a rectangular rigid frame by substituting the spring constant t with the corresponding translational stiffness.

In Fig. 12, the moments in CD caused by the displacement Δ are

$$M_{cd} = M_{dc} = -\frac{6EI_{cd}\Delta}{L^2_{cd}} . \qquad (28)$$

Substituting $T = \frac{12EI}{L^3}$ and K_{ab} from equation (22) into equation (28), one obtains

$$M_{cd} = M_{dc} = -\frac{M_b L_{cd}}{L_{ab}} \cdot \frac{3T_{cd}}{4T_{cd} + T_{ab}} .$$

The ratio of the moment induced in CD to the distributed moment at B is

$$TCOF_{bacd} = -\frac{L_{cd}}{L_{ab}} \cdot \frac{3T_{cd}}{4T_{cd}+T_{ab}} , \qquad \text{(End D fixed)} \qquad (29)$$

where $TCOF_{bacd}$ is the translational carry-over factor to CD from B. Similarly

$$TCOF_{bacd} = -\frac{L_{cd}}{L_{ab}} \cdot \frac{6T_{cd}}{4T_{cd}+T_{ab}} . \qquad \text{(End D pinned)} \qquad (30)$$

If the end A itself is pinned, then

$$TCOF_{bacd} = -\frac{L_{cd}}{2L_{ab}} , \qquad \text{(End D fixed)} \qquad (31)$$

$$TCOF_{bacd} = -\frac{L_{cd}}{L_{ab}} . \qquad \text{(End D pinned)} \qquad (32)$$

The following example is worked out by the translational moment distribution method discussed above.

Numerical Example 4

The frame with both vertical and horizontal loads as shown in Fig. 14 is fixed at D and pinned at A. The first step of the solution is the calculation of the translational stiffness for the members.

$$T_{ab} = \frac{3EI_{ab}}{L^3_{ab}} = \frac{6EI}{(20)^3} ,$$

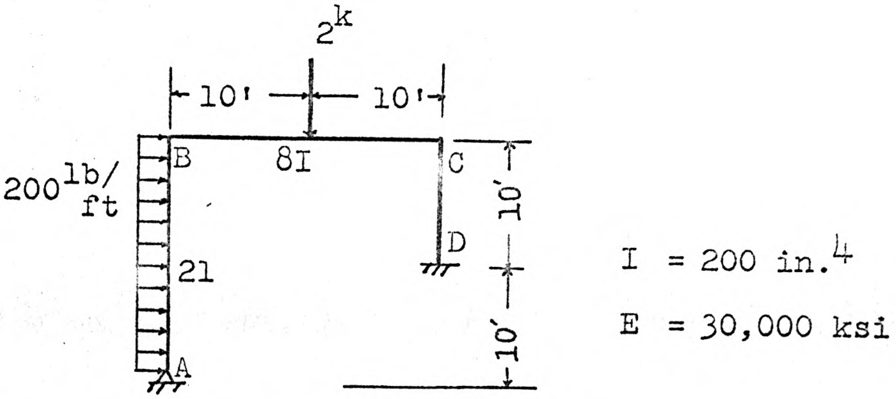


Fig. 14. Example 4. Rigid frame.

$$T_{cd} = \frac{12EI_{cd}}{L^3_{cd}} = \frac{12EI}{(10)^3} ,$$

and

$$\frac{T_{ab}}{T_{cd}} = \frac{1}{16} .$$

Thus, the stiffness factors are

$$K_{ba} = \frac{3EI_{ab}}{L_{ab}} \cdot \frac{T_{cd}}{T_{cd} + T_{ab}} = \frac{3E2I}{20} \cdot \frac{16}{17} = \frac{24}{85} EI ,$$

$$K_{bc} = \frac{4EI_{bc}}{L_{bc}} = \frac{4E8I}{20} = \frac{8}{5} EI ,$$

and

$$K_{cd} = \frac{4EI_{cd}}{L_{cd}} \cdot \frac{4T_{ab}+T_{cd}}{4(T_{ab}+T_{cd})} = \frac{4EI}{10} \cdot \frac{4(1)+16}{4(17)} = \frac{2EI}{17} .$$

Therefore,

$$K_{ba} : K_{bc} : K_{cd} = \frac{24}{85} : \frac{8}{5} : \frac{2}{17}$$

$$= 12 : 68 : 5 \quad .$$

The carry-over factors are

$$COF_{cd} = \frac{2T_{ba}-T_{cd}}{4T_{ab}+T_{cd}} = \frac{2(1) - 16}{4(1) + 16} = -0.7,$$

$$COF_{ba} = 0 \;, \qquad COF_{bc} = \tfrac{1}{2} \;.$$

From equations (31) and (30), the translational carry-over factors are obtained,

$$TCOF_{bcad} = - \frac{L_{cd}}{2L_{ab}} = - \frac{10}{2(20)} = 0.25,$$

$$TCOF_{cdab} = - \frac{L_{ab}}{L_{cd}} \cdot \frac{6T_{ab}}{4T_{ab}+T_{cd}} = - \frac{20}{10} \cdot \frac{6(1)}{4(1)+16} = -0.6 \;.$$

The fixed-end moment in BC is

$$M_{Fbc} = \frac{PL}{8} = \frac{2(20)}{8} = 5^{k\text{-}ft} \;,$$

$$M_{Fcb} = -5 \text{ k-ft} \;.$$

If translation is prevented, then

$$M_{Fab} = - \frac{WL^2{}_{ab}}{8} = - \frac{0.2(20^2)}{8} = -10 \text{ k-ft} \;.$$

The force required to prevent translation is

$$F = \frac{5}{8} WL_{ab} = \frac{5}{8}\, 0.2(20) = 2.5 \text{ k} \;.$$

The resistance of the vertical members to the translation is in proportion to their translational stiffness, therefore,

$$R_{ab} = \frac{T_{ba}}{T_{ba}+T_{cd}} F = \frac{1}{17} 2.5 = 0.147 \, ^{k} ,$$

$$R_{cd} = \frac{T_{cd}}{T_{ba}+T_{cd}} F = \frac{16}{17} 2.5 = 2.353 \, ^{k} .$$

Thus, the fixed-end moments for the vertical members are

$$M_{Fba} = -10 + 0.147(20) = -7.06 \text{ k-ft} ,$$

and

$$M_{Fdc} = M_{Fcd} = 2.354(0.5) = 11.765 \text{ k-ft} .$$

Member	BA	BC	CB	CD	DC
K	12	68	68	5	
DF	0.15	0.85	0.93	0.07	
COF	0	0.5	0.5	-0.7	
TCOF	-0.25			-0.6	
M_F	-7.06	5	-5	11.765	11.765
Bal.	0.31	1.75	0.88	-0.08	-0.08
	0.3	-3.0	-6.1	-0.46	0.35
	0.4	2.3	1.2	-0.1	-0.1
	0.1	-0.5	-1.0	-0.1	-0.07
	0.1	0.3	0.15	0	0
M(k-ft)	-5.8	5.8	-10.3	10.9	11.8

Fig. 15. Translation moment distribution for example 4.

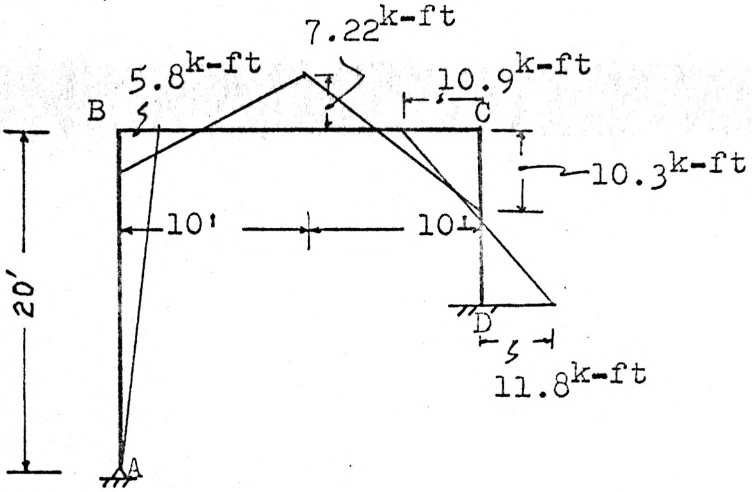


Fig. 16. Moment diagram for example 4, plotted on the compression side.

STRUCTURE WITH NONPRISMATIC MEMBERS

When the moment distribution method is applied to the analysis of rigid frame in which members are nonprismatic, the stiffness factors and carry-over factors will be different from those for members with constant cross section.

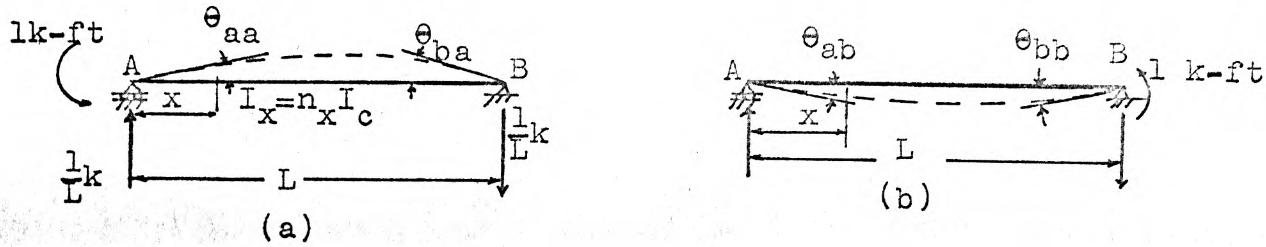


Fig. 17. Beam with variable section.

In Fig. 17 (a) is shown the member AB with moment of inertia $I_x = n_x I_c$, where I_c is a constant moment of inertia and n_x is a ratio which varies with x. If θ_{ba} represents the angle of rotation at B due to end moment $1^{k\text{-}ft}$ counterclockwise applied at end A. Then, by the moment area principle,

$$\theta_{ba} = \frac{1}{L}\int_0^L \left(\frac{M_x}{EI_x}dx\right)(x) = \frac{1}{L}\int_0^L \frac{(L-x)(x)}{EI_c L n_x}dx \;,$$

$$\theta_{aa} = \frac{1}{L}\int_0^L \left(\frac{M_x}{EI_x}dx\right)(L-x) = \frac{1}{L}\int_0^L \frac{(L-x)^2}{EI_c L n_x}dx. \qquad (33)$$

Similarly, from Fig. 17 (b),

$$\theta_{ab} = \frac{1}{L}\int_0^L \left(\frac{M_x}{EI_x}dx\right)(x) = \frac{1}{L}\int_0^L \frac{(L-x)(x)}{EI_c L n_x}dx ,$$

(34)

$$\theta_{bb} = \frac{1}{L}\int_0^L \left(\frac{M_x}{EI_x}\right)dx(x) = \frac{1}{L}\int_0^L \frac{(x^2)}{EI_c L n_x}\, dx .$$

Let

$$C_1 = \frac{1}{L^3}\int_0^L \frac{(L-x)^2}{n_x}\, dx ,$$

$$C_2 = \frac{1}{L^3}\int_0^L \frac{(L-x)(x)}{n_x}\, dx ,$$

$$C_3 = \frac{1}{L^3}\int_0^L \frac{(x)^2}{n_x}\, dx .$$

If the member AB shown in Fig. 18 (a) is to be considered, the sum of the effects in Fig. 18 (b), (c), (d), and (e) will give the same effect as in (a), and it is seen that the expression for θ_a and θ_b in Fig. 18 (a) can be obtained by substituting M'_{ab} and M'_{ba} for 1 k-ft as shown in Fig. 17, Fig. 18 (d) and (c).

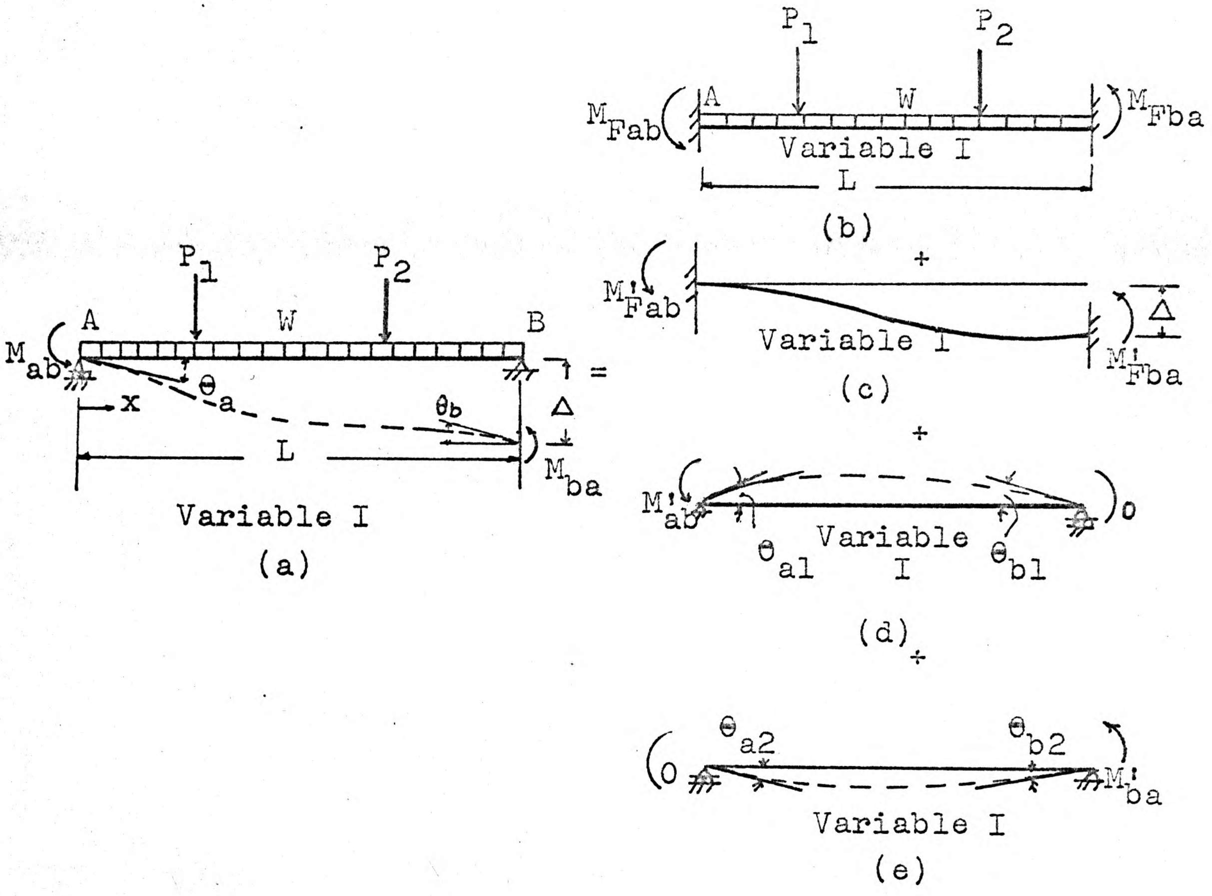


Fig. 18. Nonprismatic beam.

Referring to Fig. 18, it is found that

$$\theta_a = -\theta_{a1} + \theta_{a2} ,$$

$$\theta_b = \theta_{b1} - \theta_{b2} ,$$

where,

$$\theta_{a1} = M'_{ab}\left(\frac{C_1 L}{EI_c}\right) ,$$

$$\theta_{b1} = M'_{ab}\left(\frac{C_2 L}{EI_c}\right) ,$$

$$\theta_{a2} = M'_{ba}\left(\frac{C_2 L}{EI_c}\right) ,$$

$$\theta_{b2} = M'_{ba}\left(\frac{C_3 L}{EI_c}\right) ,$$

So that,

$$\theta_a = -M'_{ab}\left(\frac{C_1 L}{EI_c}\right) + M'_{ba}\left(\frac{C_2 L}{EI_c}\right) ,$$

$$\theta_b = M'_{ab}\left(\frac{C_2 L}{EI_c}\right) - M'_{ba}\left(\frac{C_3 L}{EI_c}\right) .$$

This gives,

$$M'_{ab} = \frac{2EI_c}{L}(-N_{aa}\theta_a - N_{ab}\theta_b) ,$$

$$M'_{ba} = \frac{2EI_c}{L}(-N_{bb}\theta_b - N_{ab}\theta_a) , \tag{35}$$

where,

$$N_{aa} = \frac{C_3}{2(C_1C_3 - C_2^2)} ,$$

$$N_{ab} = \frac{C_2}{2(C_1C_3 - C_2^2)} ,$$

$$N_{bb} = \frac{C_1}{2(C_1C_3 - C_2^2)} .$$

Referring to Fig. 18 (c) and using the moment area method to find that the deflection of B from the tangent at A is

$$\Delta = - \frac{M'_{Fab}L^2}{EI_c} C_1 + \frac{M'_{Fba}L^2}{EI_c} C_2 , \tag{36}$$

and the deflection of A from tangent at B is

$$\Delta = - \frac{M'_{Fab}L^2}{EI_c} C_2 + \frac{M'_{Fba}L^2}{EI_c} C_3 . \tag{37}$$

Substituting $\Delta = RL$, and solving equations (35) and (36), one obtains

$$\begin{aligned} M'_{Fab} &= + \frac{2EI_c}{L}(N_{aa} + N_{ab})R , \\ M'_{Fba} &= + \frac{2EI_c}{L}(N_{bb} + N_{ab})R . \end{aligned} \tag{38}$$

It is seen from Fig. 18 that

$$\begin{aligned} M_{ab} &= M_{Fab} + M'_{Fab} + M'_{ab} , \\ M_{ba} &= M_{Fba} + M'_{Fba} + M'_{ba} . \end{aligned} \tag{39}$$

Substituting M'_{ab}, M'_{ba} and Δ in equations (35) and (36) into equation (39)

$$\begin{aligned} M_{ab} &= M_{Fab} + \frac{2EI_c}{L} -N_{aa}\theta_a - N_{ab}\theta_b + (N_{aa}+N_{ab})R , \\ M_{ba} &= M_{Fba} + \frac{2EI_c}{L} -N_{bb}\theta_b - N_{ab}\theta_a + (N_{bb}+N_{ab})R . \end{aligned} \tag{40}$$

Equations (40) are the slope-deflection equations for members with variable cross section.

In equation (40), If $M_{Fab}=M_{Fba}=0$, $\theta_b=0$, and $R=0$, then according to the definition of rotational stiffness K, the following expressions can be obtained ,

$$M'_{ab} = K_a\theta_a = \frac{2EI}{L}(-N_{aa}\theta_a) \ ,$$

and

$$M'_{ba} = COF_{ab}M'_{ab} = \frac{2EI}{L}(-N_{ab}\theta_a) \ .$$

Then,

$$K_a=N_{aa}\left(\frac{2EI}{L}\right) \ , \tag{41}$$

$$COF_{ab}= \frac{N_{ab}}{N_{aa}} = \frac{C_2}{C_3} \ . \tag{42}$$

Similarly,

$$K_b = N_{bb}\left(\frac{2EI_c}{L}\right) \ , \tag{43}$$

$$COF_{ba} = \frac{K_{ab}}{K_{bb}} = \frac{C_2}{C_1} \ . \tag{44}$$

Equations (41) to (44) refer to Fig. 19. The modified stiffness factors for the simply supported beam shown in Fig. 20 are obtained in the same manner.

$$K'_a = K_a(1 - COF_{ab}COF_{ba}) \ , \tag{45}$$

$$K'_b = K_b(1 - COF_{ab}COF_{ba}) \ . \tag{46}$$

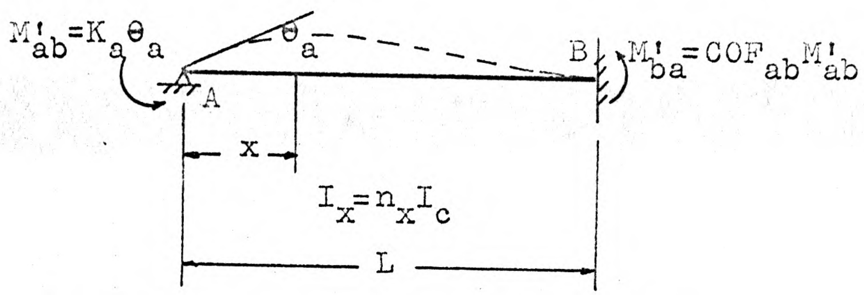


Fig. 19. Beam with one end fixed.

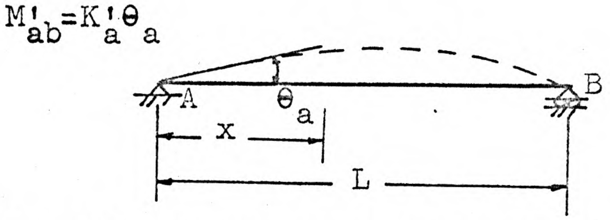


Fig. 20. Beam with simple supports.

The details of the procedure will be illustrated in the following numerical example.

Numerical Example 5

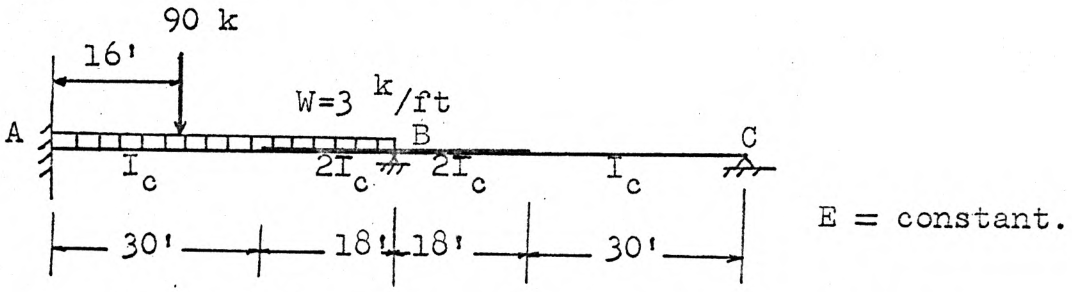


Fig. 21. Example 5. Beam with variable cross section.

The beam with variable cross section as shown in Fig. 21 is to be analyzed by the method of moment distribution. The first step is the determination of the fixed-end moment for member AB, which can be done by the use of the column analogy method.

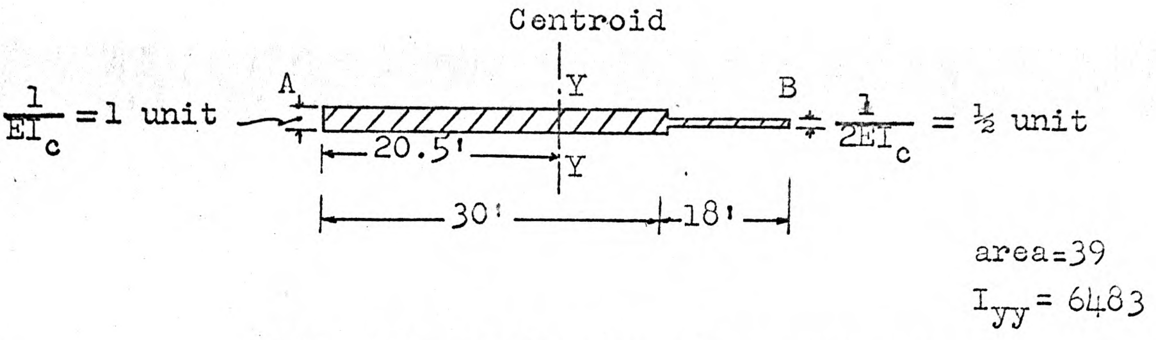


Fig. 22. Analogous-column section.

The properties of the analogous-column section are shown in Fig. 22.

The moment diagram for the basic determinate beam is shown in Fig. 23 (b), which is plotted on the compression side and named M_s. The $\frac{M_s}{EI}$ diagram is shown in Fig. 23 (c).

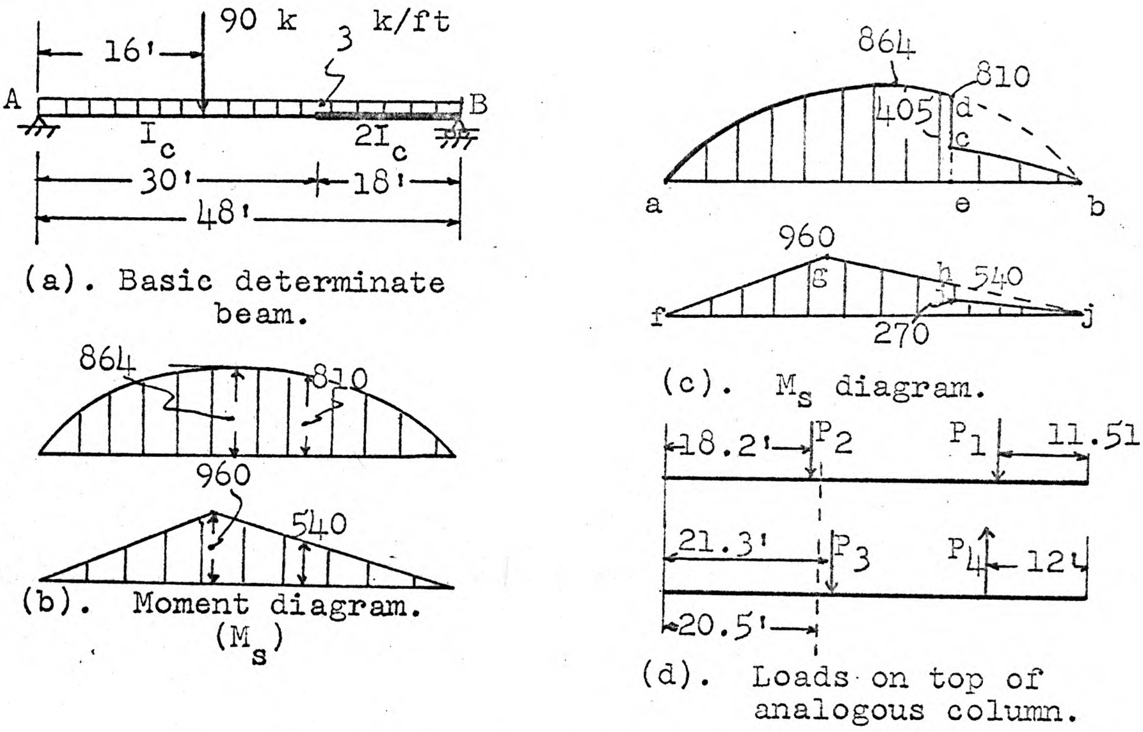


Fig. 23. Analogous-column beam AB.

The area of (bcd) in Fig. 23 (c) is

$$\int_0^{18}(72x - \frac{3}{2}x^2)dx = \left[\frac{72x^2}{2} - \tfrac{1}{2}x^3\right]_0^{18}$$

$$= 11650 - 2912 = 8738 ,$$

$$P_1 = \frac{\text{area(bcd)}}{2} = \frac{8738}{2} = 2469.$$

The moment of area (bcd) about a vertical axis through B is

$$\int_0^{18}(72x - \frac{3}{2}x^2)xdx = \left[\frac{72x^3}{3} - \frac{3x^4}{8}\right]_0^{18}$$

$$= 5832(17.24) = 100800 .$$

The distance between the centroid of area (bcd) and the right support is

$$\frac{100800}{8738} = 11.55',$$

$$P_2 = \text{area of (aed)} = \frac{2}{3}(48)(864) - 8738 = 18892 .$$

The moment of area (aed) about a vertical axis through A is

$$\int_0^{30}(72x - \frac{3}{2}x^3)\cdot xdx = \left[\frac{72x^3}{3} - \frac{3x^4}{8}\right]_0^{30}$$

$$= 30^3\left[24 - \frac{3(30)}{8}\right] = 344000 ,$$

The distance between the centroid of area (aed) and the left support is

$$\frac{344000}{18892} = 18.2' ,$$

$$P_3 = \text{area (fgj)} = \frac{960(48)}{2} = 23030 ,$$

$$P_4 = \text{area (hij)} = \tfrac{1}{2}(270)18 = 2430 \ .$$

The total load on the column is

$$P_1 + P_2 + P_3 - P_4 = 18892 + 4369 + 23030 - 2430$$

$$= 43861 \ ,$$

the total moment of the loads about centroid y-y is

$$18892(20.5-18.2) + 2430(48-20.5-12) - 4369(48-20.5-11.51)$$

$$-23030(21.3-20.5) = -7150 \ , \qquad \text{(Clockwise)}$$

M_s at $A=0$,

$$M_i \text{ at } A = \frac{P}{A} + \frac{Mc}{I}$$

$$= \frac{43861}{39} + \frac{(-7150)(20.5)}{6483} = 1097.4 \text{ k-ft} \ ,$$

$$M_a = (M_s \text{ at } A)-(M_i \text{ at } A) = 0 - 1097.4 \text{ k-ft}$$

$$= -1097.4 \text{ k-ft} \ ,$$

M_s at $B=0$,

$$M_i \text{ at } B = \frac{P}{A} - \frac{Mc}{I} = 1120 + \frac{7150}{6483}(48-20.5)$$

$$= 1150.4 \text{ k-ft} \ ,$$

$$M_b = (M_s \text{ at } B)-(M_i \text{ at } B) = 0 + 1150.4 = 1150.4 \text{ k-ft} \ .$$

For span AB :

$$C_1 = \frac{1}{L^3}\int_0^L \frac{(L-x)^{22}}{n_x}\,dx = \frac{1}{(48)^3}\left[\int_0^{L} \frac{(48-x)^2}{1}\,dx + \int_{30}^{48} \frac{(48-x)^2}{2}\,dx\right]$$

$$= \frac{1}{48^3}\left(48^2x - \frac{96x^2}{2} + \frac{x^3}{3}\right)_0^{30} + \frac{1}{2\cdot 48^3}\left(48^2x - \frac{96x^2}{2} + \frac{x^3}{3}\right)_{30}^{48}$$

$$= 0.324 \ ,$$

$$C_2 = \frac{1}{L^3}\int_0^L \frac{x(L-x)}{n_x}\,dx = \frac{1}{48^3}\left[\int_0^{30} \frac{(x)(48-x)\,dx}{1} + \int_{30}^{48} \frac{x(48-x)\,dx}{2}\right]$$

$$= \frac{1}{48^3}\left\{\left[\frac{48x^2}{2} - \frac{x^3}{3}\right]_0^{30} + \tfrac{1}{2}\left[\frac{48x^2}{2} - \frac{x^3}{3}\right]_{30}^{48}\right\}$$

$$= 0.14 \text{ ,}$$

$$C_3 = \frac{1}{L^3}\int_0^L \frac{(x)^2}{n_x}\,dx = \frac{1}{48^3}\left[\int_0^{30}\frac{(x)^2}{1}dx + \int_{30}^{48}\frac{(x)^2}{2}\,dx\right]$$

$$= \frac{1}{48^3}\left[\frac{30^3}{3} + \tfrac{1}{2}\left(\frac{48^3}{3} - \frac{30^3}{3}\right)\right] = 0.208 \text{ .}$$

And,

$$N_{aa} = \frac{C_3}{2(C_1C_3 - C_2^2)} = 2.17 \text{ ,}$$

$$N_{ab} = \frac{C_2}{2(C_1C_3 - C_2^2)} = 1.47 \text{ ,}$$

$$N_{bb} = \frac{C_1}{2(C_1C_3 - C_2^2)} = 3.38 \text{ ,}$$

$$N_{bc} = N_{ab} = 1.47 \text{ ,}$$

$$N_{cc} = N_{aa} = 2.17 \text{ .}$$

The stiffness and carry-over factors are as follows:

For span AB:

$$K_a = \frac{2EI_c}{L}N_{aa} = 2.17\left(\frac{2EI_c}{48}\right) = 2.17\frac{EI_c}{24} \text{ ,}$$

$$K_b = \frac{2EI_c}{L}N_{bb} = 3.38\frac{EI_c}{24} \text{ ,}$$

$$COF_{ab} = \frac{N_{ab}}{N_{aa}} = \frac{1.47}{2.17} = 0.673 \text{ ,}$$

$$COF_{ba} = \frac{N_{ab}}{N_{bb}} = \frac{1.47}{3.38} = 0.435.$$

For span BC:

$$K_b = \frac{2EI_c}{L} N_{bb} = 3.38\frac{EI_c}{24} \quad ,$$

$$K_c = \frac{2EI_c}{L} N_{cc} = 2.17\frac{EI_c}{24} \quad ,$$

$$COF_{bc} = \frac{N_{bc}}{N_{bb}} = \frac{1.47}{3.38} = 0.435 \ ,$$

$$COF_b = \frac{N_{bc}}{N_{cc}} = \frac{1.47}{2.17} = 0.675 \ ,$$

Modified stiffness at B of span BC is

$$K_b(1 - COF_{bc}COF_{cb})$$

$$= 3.38(\frac{EI_c}{24})\left[1-(0.432)(0.673)\right]$$

$$= 2.4\frac{EI_c}{24} \ .$$

Using the fixed-end moments, stiffness factors and carry-over factors obtained above, the moment distribution procedure is carried out in Fig. 24.

Member	AB	BA	BC	CB
DF		0.59	0.41	
COF	0.673	0.435		
M_F	1097.4	-1150.4		
	295	678	472	
M(k-ft)	1392.4	-478	478	

Fig. 24. Moment distribution for example 5.

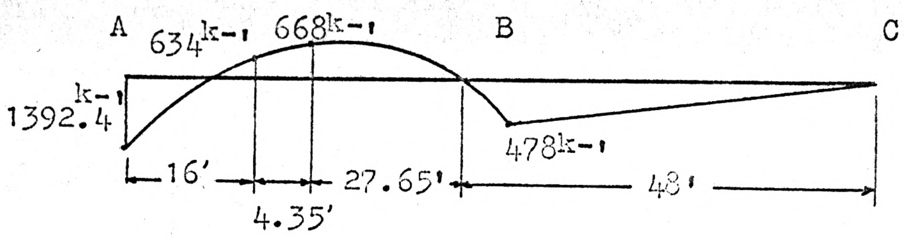


Fig. 25. Moment diagram for example 5, plotted on the compression side.

MOMENT DISTRIBUTION AS A RELAXATION PROCEDURE

The relaxation method, introduced by Southwell in 1935, is defined as the ordered iterative solution of simultaneous equations. Many such techniques have been developed, among them the moment distribution method, which obtained true results by means of successive approximations, will be presented in this report.

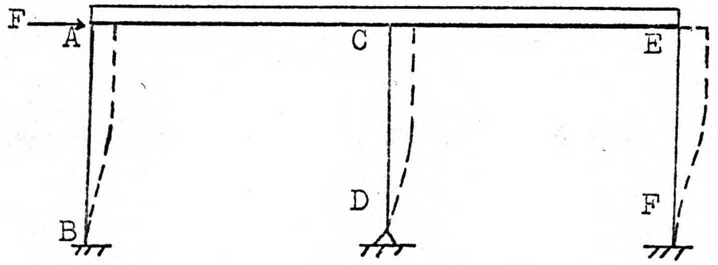


Fig. 26. Rigid frame with lateral load.

For rigid frame shown in Fig. 26, it is assumed that the girder ACE is infinitely rigid when the frame undergoes rotations and displacement at the joints. The expressions for the moments and the shears induced are

$$M_{ab} = K_{ab}\theta_a + COF_{ba}K_{ba}\theta_b + m_{ab}\Delta_{ab} ,$$

$$M_{ba} = COF_{ab}K_{ab}\theta_a + K_{ba}\theta_b + m_{ba}\Delta_{ab} ,$$

$$V_{ab} = V_{ba} = -\frac{M_{ab}+M_{ba}}{L_{ab}} ,$$

where m_{ab} is the moment induced at end A when the displacement $\Delta_{ab} = 1$, $\theta_a = 0$, $\theta_b = 0$, and can be obtained by referring to Fig. 27.

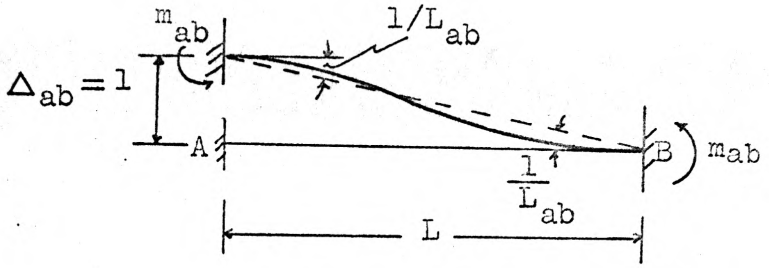


Fig. 27. Beam with unit displacement.

It can be seen that

$$m_{ab} = K_{ab}\left(-\frac{1}{L_{ab}}\right) + COF_{ba}K_{ba}\left(-\frac{1}{L_{ab}}\right) ,$$

$$m_{ba} = COF_{ab}K_{ab}\left(-\frac{1}{L_{ab}}\right) + K_{ba}\left(-\frac{1}{L_{ab}}\right) . \tag{48}$$

Substituting m_{ab} and m_{ba} from equation (48) into equation (47),

$$M_{ab} = K_{ab}\theta_a + COF_{ba}K_{ba}\theta_b - \frac{\Delta ab}{L_{ab}}(K_{ab} + COF_{ba}K_{ba}) ,$$

$$M_{ba} = COF_{ab}K_{ab}\theta_a + K_{ba}\theta_b - \frac{\Delta ab}{L_{ab}}(COF_{ab}K_{ab} + K_{ba}) ,$$

$$V_{ab} = -\frac{M_{ab} + M_{ba}}{L_{ab}} \tag{49}$$

$$= -\frac{(COF_{ab}+1)K_{ab}}{L_{ab}}\theta_a - \frac{(COF_{ba}+1)K_{ba}}{L_{ab}}\theta_b$$

$$+ \frac{(COF_{ab}+1)K_{ab} + (COF_{ba}+1)K_{ba}}{L_{ab}} \cdot \frac{\Delta ab}{L_{ab}} .$$

The relation shown in equation (49) will be used to obtain an operation table which shows the changes of all the moments and shears due to changing of one or more angles of rotation and displacement. Then a relaxation table which carries out the relaxation procedure by reducing the residual joint moments and story shears to zero can be set up.

The details of the procedure will be illustrated in example 6 shown in Fig. 28 and the operation table shown in table 1 is obtained according to equation (49). The relaxation procedure is shown in Table 2. On the first line, no rotation or displacement occurs. The unbalanced shears V_2 and V_1 are 6 kips and 18 kips respectively. By relaxing the story shear to zero the unbalanced moments at the joints are introduced as shown on line 2. The procedure is repeated until the unbalanced moments are small enough. The final results are summarized at the bottom of Table 2.

The moments at the ends of the members can be computed according to equation (49).

$$M_{ce} = 4E\theta_c - 6E\frac{\Delta_1}{18} = 4(7.425) - 6\frac{304.4}{18} = 71.8 \text{ k-ft},$$

$$M_{ec} = \tfrac{1}{2}(4E\theta_c) - 6E\frac{\Delta_1}{18} = \tfrac{1}{2}(4)7.425 - 6\frac{304.4}{18} = 86.7 \text{ k-ft},$$

$$M_{df} = 4E\theta_d - 6E\frac{\Delta_1}{18} = 4(7.705) - 6\frac{304.4}{18} = 70.7 \text{ k-ft},$$

$$M_{fd} = \tfrac{1}{2}(4E\theta_d) - 6E\frac{\Delta_1}{18} = \tfrac{1}{2}(4)7.705 - 6\frac{304.4}{18} = 86.1 \text{ k-ft},$$

$$M_{ac} = 4E\theta_a + \tfrac{1}{2}(4E\theta_c) - 6E\frac{\Delta_2}{18}$$

$$= 4(2.355) + 14.8 - 6\frac{157.3}{18} = 28.3 \text{ k-ft},$$

$$M_{ca} = 4E\theta_c + \tfrac{1}{2}(4E\theta_a) - 6E\frac{\Delta_2}{18}$$

$$= 4(7.472) + 2(2.355) - 6\frac{157.3}{18} = 18.09 \text{ k-ft},$$

$$M_{bd} = 4E\theta_b + \tfrac{1}{2}(4E\theta_d) - 6E\frac{\Delta_2}{18}$$

$$= 4(2.425) + 2(7.705) - 6\frac{157.3}{18} = 27.4 \text{ k-ft},$$

$$M_{db} = 4E\theta_d + \tfrac{1}{2}(4E\theta_b) - 6E\frac{\Delta_2}{18}$$

$$= 4(7.705) + 2(2.425) - 6\frac{157.3}{18} = 17.85 \text{ k-ft}.$$

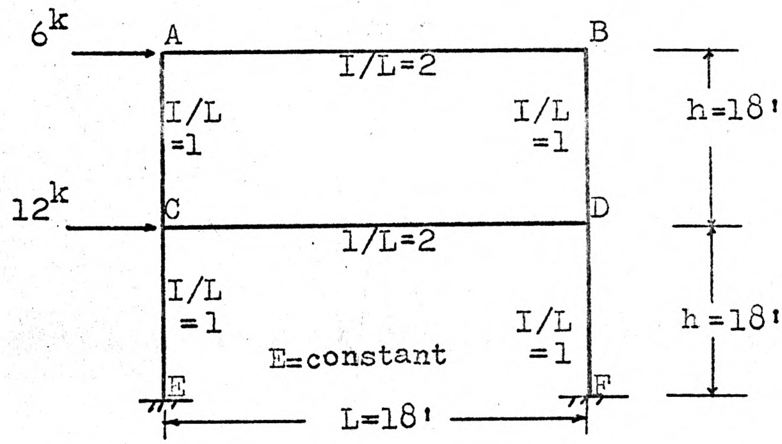


Fig. 28. Example 6. Two-story frame.

Table 1. Operation table for example 6.

Operation	Change of residuals					
	M_c	M_a	M_b	M_d	V_2 second story	V_1 first story
$\theta_c = 1$	$\Sigma 4^{EI/L}$ 16	$2^{EI/L}$ 2	 0	$2^{EI/L}$ 4	$-6^{EI/L^2}$ -0.333	$-6^{EI/L^2}$ -0.333
$\theta_a = 1$	2	12	4	0	-0.333	0
$\theta_b = 1$	0	4	12	2	-0.333	0
$\theta_d = 1$	4	0	2	16	-0.333	-0.333
$\Delta_1 = 1$	-0.333	0	0	-0.333	0	0.074
$\Delta_2 = 1$	-0.333	-0.333	-0.333	-0.333	0.074	0
$\theta_c = \theta_d = 1$	20	2	2	20	-0.667	-0.667
$\theta_c = \theta_b = 1$	16	6	12	6	-0.667	-0.333
$\theta_a = \theta_d = 1$	6	12	6	16	-0.667	-0.333
$\theta_c = \theta_b = 1$ $\theta_a = \theta_d = 2$	28	30	24	38	-2	-1
$\theta_a = \theta_b = 1$ $\theta_c = \theta_d = 1$	22	18	18	22	-1.333	-0.667

Table 2. Relaxation table for example 6.

θ_c	θ_a	θ_b	θ_d	Δ_2	Δ_1	M_c	M_a	M_b	M_d	V_2	V_1
0	0	0	0	0	0	0	0	0	0	-6	-18
					244	-81.3	0	0	-81.3	-6	0
				81		-108.3	-27	-27	-108.3	0	0
3	3	3	3	0	0	-42.3	27	27	-42.3	-4	-2
					-126	0	27	27	0	-4	-11.3
		-4				0	11	-21	-8	-2.7	-11.3
					150	-50	11	-21	-50	-2.7	0
				36.5		-62	-1	-33	-62	0	0
3			3			-2	5	-27	-2	-2	-2
		2				-2	13	-3	2	-2.67	-2
	-1					0	1	1	2	-2.33	-2.
-0.075	-0.075	-0.075	-0.075			-1.63	-0.35	-0.35	0.37	-2.33	-1.95
				30		-11.63	-10.35	-10.35	10.35	0	-1.95
					26.4	-20.43	-19.15	-19.15	1.55	0	0
			1			-24.43	-19.15	-21.15	-14.45	0.33	0.33
1.06	1.06	1.06	1.06			-1	0	-2	9	0	0
			-0.56			-3.24	0	-3.12	0	-0.7	-0.7
0.2		0.2				-0.04	1.2	-0.72	1.2	-0.83	-0.76
	-0.05		-0.05			-0.34	0.6	-1.02	0.6	-0.8	-0.75
				10.8		-4	-3	-1.38	-3	0	-0.75
					10	-7.34	-6.34	-4.72	-6.34	0	0
0.24	0.48	0.24	0.48			-0.54	0.76	1	2.76	-0.48	-0.24
			-0.15			-1.14	0.76	0.7	0.36	-0.43	-0.19
				-1		-0.8	0.76	0.7	0.7	-0.43	-0.26 ← small enough
7.425	2.355	2.425	7.705	157.3	304.4						

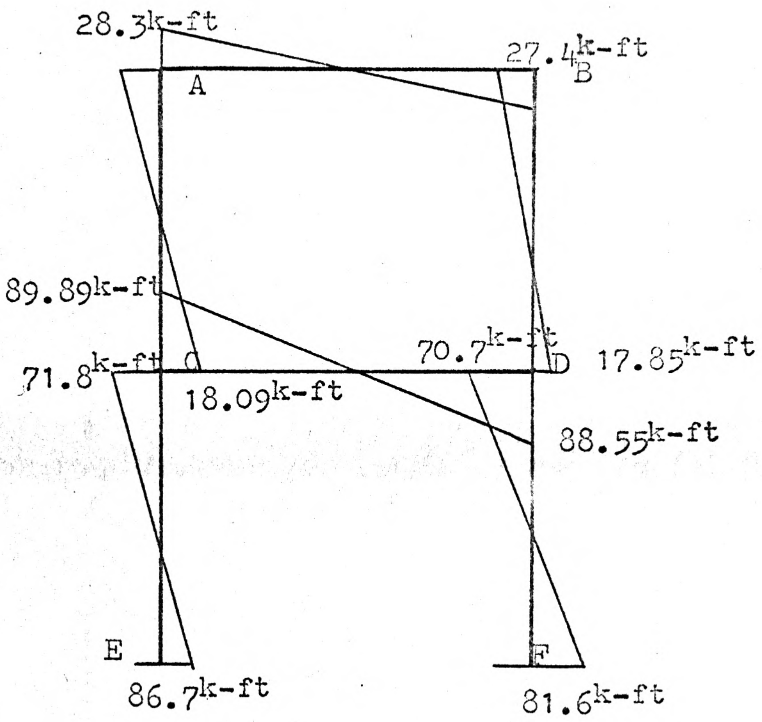


Fig. 28 (a). Moment diagram for example 6, plotted on the compression side.

APPLICATION TO DIGITAL COMPUTER SOLUTION

Depending upon the purpose to be served, there are many ways in which computer programs for the analysis of a structure by moment distribution can be written. For the purpose of this report, the program presented on page 58 applies only to the particular problem shown in Fig. 29, which is the same one used in example 6.

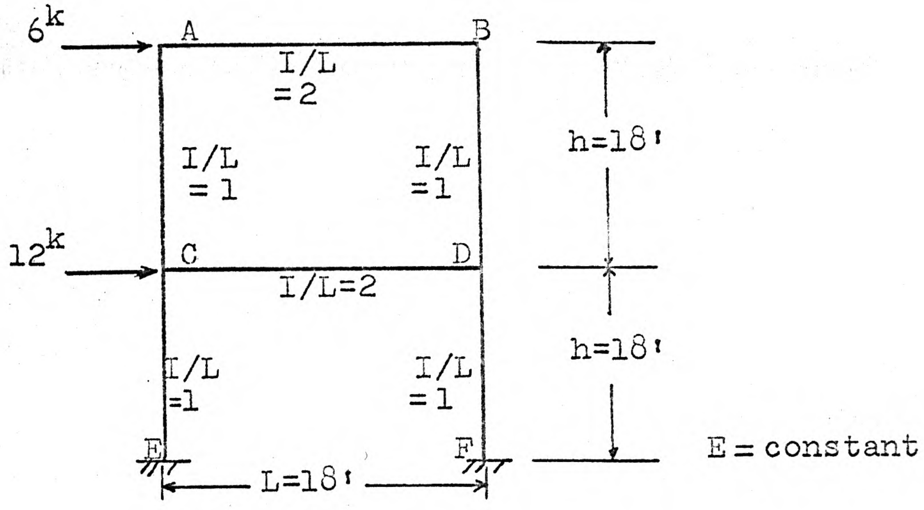


Fig. 29. Rigid frame.

Program number 1 shown on page 58 consists of the following steps.

Step 1. Input data appears in the following order:

(a) IMAX, JMAX.

(b) Span length.

(c) Column height.

(d) Moment of inertia for horizontal girders.

(e) Moment of inertia for columns.

(f) EPS, the test increment.

(g) The horizontal loads.

Step 2. Calculates the stiffness factors for the members.

Step 3. Computes the distribution factors for the members framed into a joint.

Step 4. Carries out the moment distribution procedure. This can be described as follows:

(a) All joints are held against translation and rotation, then the fixed-end moments are calculated.

(b) All joints at the top level are allowed to rotate with the frame held against translation.

(c) Shears in top-story column are balanced.

(d) Repeats steps (b) and (c) for the second story.

(e) Steps (a) to (d) are repeated until all moments converge to values that are within acceptable limits.

Step 4. Punches the answers in the following order: I,J,EMR(I,J), EML(I,J+1), VR(I,J), VL(I,J+1) for each beam from left to right starting with the top story. I,J, EMB(I,J), EMA(I+1,J), VB(I,J), VA(I+1,J) for each column from left to right starting with the top story.

The flow chart for the program is shown in Fig. 30, and the notations used in program are as follows:

I	Story number from top down (I=1 to IMAX).
J	Column number from left to right (J=1 to JMAX).
SPAN(J)	Span length between columns J and J+1.
HT(I)	Story height between stories I and I+1.
BI(I,J)	Moment of inertia of the beam on story I and to the right of column J.
CI(I,J)	Moment of inertia of the column below story I and in column J.
SKL, SKR, SKA, SKB	Stiffness factors for members located to the left of, the right of, above and below a joint.
DFL, DFR, DFA, DFB	Distribution factors for members located to the left of, the right of, above and below a joint.
SSK(I,J)	Shear stiffness factor of a column below story I and in column J.
SUMSK(I)	Sum of SSK(I,J) between story I and I+1.
SDF(I,J)	Shear distribution factor of a column below story I and in column J.
EML, EMR, EMA, EMB	End moments for members to the left of, the right of, above and below a joint.
PH(I)	Horizontal load at level I (positive from left to right).
EPS	Small number to set the criterion of tolerable unbalanced moment.
M	A counter.
Thv	Sum of the horizontal forces above level I.
SUMM2	Change in column moments due to translation.

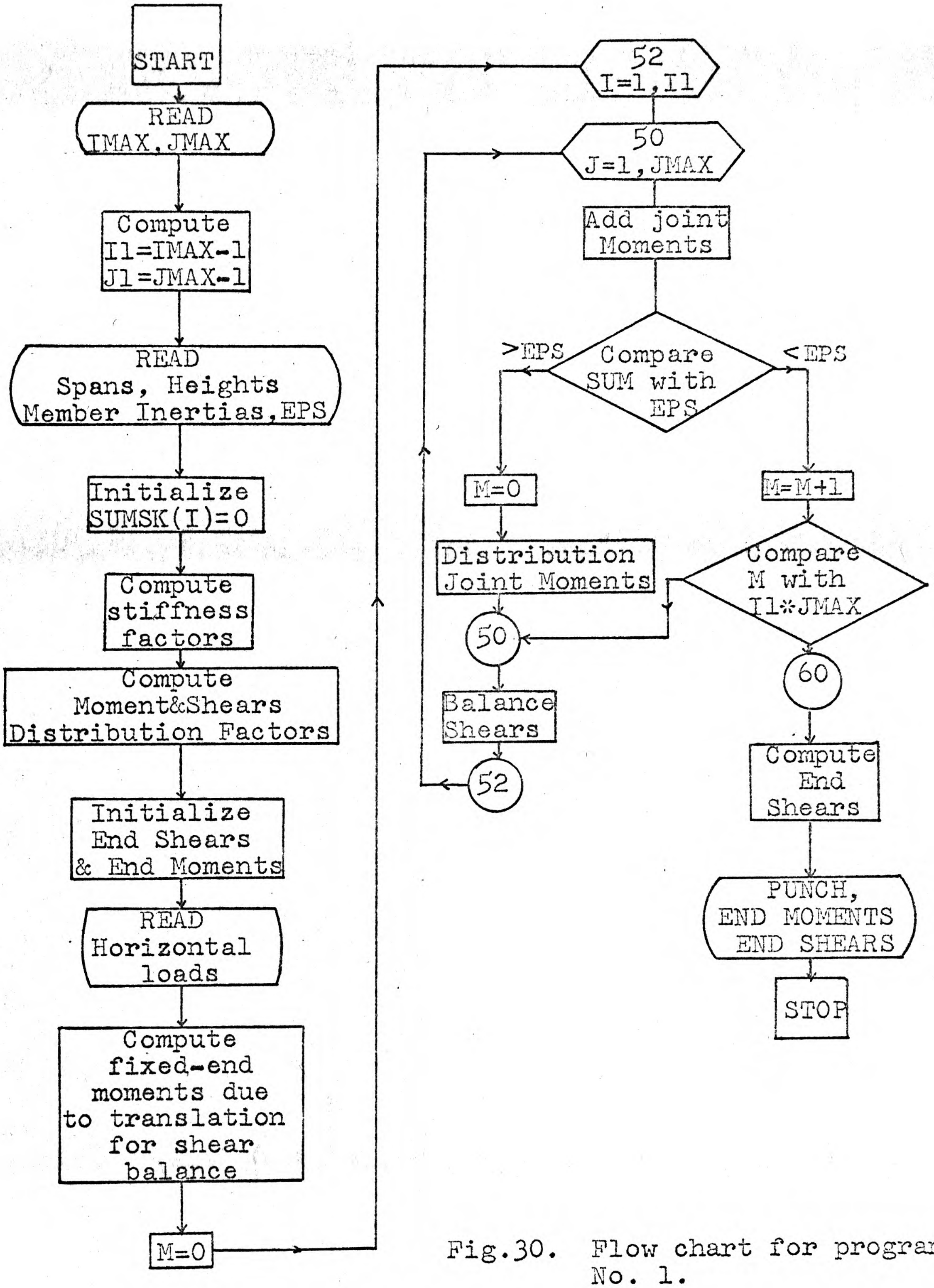


Fig.30. Flow chart for program No. 1.

```
C     PROGRAM NO 1
C     MOMENT DISTRIBUTION OF MULTI-STORY FRAMES
      DIMENSION HT(3),SPAN(3),SSK(3,4),SDF(3,4),SUMSK(3)
      DIMENSION DFL(3,4),DFA(3,4),DFR(3,4),DFB(3,4),BI(3,3)
      DIMENSION EMA(4,4),EMB(3,4),EML(3,4),EMR(3,4),CI(3,4)
      DIMENSION PH(3),VR(3,3),VL(3,4),VA(4,4),VB(3,4)
    1 FORMAT(2I5)
    2 FORMAT(F10.5)
    3 FORMAT(2I5)
    4 FORMAT(F14.8)
C     THIS PART COMPUTES STIFFNESS FACTORS
   10 READ1,IMAX,JMAX
      I1=IMAX-1
      J1=JMAX-1
      DO11J=1,J1
   11 READ2,SPAN(J)
      DO12I=1,I1
   12 READ2,HT(I)
      DO13I=1,I1
      DO13J=1,J1
   13 READ2,BI(I,J)
      DO14 I=1,I1
      DO14J=1,JMAX
   14 READ2,CI(I,J)
      READ2,EPS
      DO30I=1,I1
      SUMSK(I)=0
      DO30J=1,JMAX
      IF(J-1)16,16,15
   15 SKL=BI(I,J-1)/SPAN(J-1)
      GO TO 17
   16 SKL=0
   17 IF(I-1)19,19,18
   18 SKA=CI(I-1,J)/HT(I-1)
      GO TO 20
   19 SKA=0
   20 IF(J-JMAX)21,22,22
   21 SKR=BI(I,J)/SPAN(J)
      GO TO 23
   22 SKR=0
   23 SKB=CI(I,J)/HT(I)
C     THIS PART COMPUTES DISTRIBUTION FACTORS
      SUMK=SKL+SKA+SKR+SKB
      DFL(I,J)=SKL/SUMK
      DFA(I,J)=SKA/SUMK
      DFR(I,J)=SKR/SUMK
      DFB(I,J)=SKB/SUMK
      SSK(I,J)=SKB/HT(I)
      SUMSK(I)=SUMSK(I)+SSK(I,J)
   30 CONTINUE
      DO31 I=1,I1
      DO31J=1,JMAX
```

```
   31 SDF(I,J)=SSK(I,J)/SUMSK(I)
C     THIS PART COMPUTES MOMENTS CAUSED BY HORIZONTAL FORCES
      DO32 I=1,I1
      DO32J=1,JMAX
      VR(I,J)=0
      VL(I,J)=0
      EML(I,J)=0
      EMR(I,J)=0
      EMA(I,J)=0
   32 EMB(I,J)=0
      DO33I=1,I1
   33 READ2,PH(I)
      THV=0
      DO34I=1,I1
      THV=THV+PH(I)
      DO34J=1,JMAX
      EMB(I,J)=0.5*SDF(I,J)*THV*HT(I)
   34 EMA(I+1,J)=EMB(I,J)
C     UP TO 52 MOMENT DISTRIBUTION IS CARRIED OUT
      M=0
  334 THV=0
      DO52I=1,I1
      DO50 J=1,JMAX
      IF(I-1)35,35,36
   35 DFA(I,J)=0.
      EMA(I,J)=0.
   36 IF(J-1)37,37,38
   37 DFL(I,J)=0.
      EML(I,J)=0.
      GO TO 40
   38 IF(J-JMAX)40,39,39
   39 DFR(I,J)=0.
      EMR(I,J)=0.
   40 SUMM1=EML(I,J)+EMA(I,J)+EMR(I,J)+EMB(I,J)
      IF(SUMM1-EPS)41,41,43
   41 IF(-SUMM1-EPS)42,42,43
   42 M=M+1
      IF(M-I1*JMAX)50,60,60
C     THIS PART DISTRIBUTES THE UNBALANCED MOMENTS
   43 M=0
      EML(I,J)=EML(I,J)-DFL(I,J)*SUMM1
      EMA(I,J)=EMA(I,J)-DFA(I,J)*SUMM1
      EMR(I,J)=EMR(I,J)-DFR(I,J)*SUMM1
      EMB(I,J)=EMB(I,J)-DFB(I,J)*SUMM1
      IF(J-1)45,45,44
C     THIS PART COMPUTES THE CARRY-OVER MOMENTS
   44 EMR(I,J-1)=EMR(I,J-1)-.5*DFL(I,J)*SUMM1
      IF(J-JMAX)45,46,46
   45 EML(I,J+1)=EML(I,J+1)-.5*DFR(I,J)*SUMM1
      IF(I-1)47,47,46
   46 EMB(I-1,J)=EMB(I-1,J)-.5*DFA(I,J)*SUMM1
   47 EMA(I+1,J)=EMA(I+1,J)-.5*DFB(I,J)*SUMM1
```

```
50 CONTINUE
   SUMM2=0
   THV=THV+PH(I)
   DO51 J=1,JMAX
51 SUMM2=SUMM2+EMB(I,J)+EMA(I+1,J)
   SUMM2=SUMM2-THV*HT(I)
   DO52 J=1,JMAX
   EMB(I,J)=EMB(I,J)-.5*SDF(I,J)*SUMM2
   EMA(I+1,J)=EMA(I+1,J)-.5*SDF(I,J)*SUMM2
52 CONTINUE
   GO TO 334
C     THIS PART COMPUTES SHEARS AT THE ENDS OF MEMBERS
60 DO61I=1,I1
   DO 61 J=1,J1
   VR(I,J)=VR(I,J)-(EMR(I,J)+EML(I,J+1))/SPAN(J)
61 VL(I,J+1)=VL(I,J+1)-(EMR(I,J)+EML(I,J+1))/SPAN(J)
   DO62I=1,I1
   DO62 J=1,JMAX
   VB(I,J)=-(EMB(I,J)+EMA(I+1,J))/HT(I)
62 VA(I+1,J)=VB(I,J)
   DO63 I=1,I1
   DO63J=1,J1
   PUNCH3,I,J
63 PUNCH4,EMR(I,J),EML(I,J+1),VR(I,J),VL(I,J+1)
   DO64 I=1,I1
   DO64 J=1,JMAX
   PUNCH3,I,J
64 PUNCH4,EMB(I,J),EMA(I+1,J),VB(I,J),VA(I+1,J)
   STOP
   END
```

```
C        EXAMPLE 7
C        DATA
     3     2
  18.00000
  18.00000
  18.00000
  32.00000
  32.00000
  18.00000
  18.00000
  18.00000
  18.00000
   0.01000
   6.00000
  12.00000
C        ANSWERS
     1     1
 -32.70620700
 -32.70158600
   3.63376620
   3.63376620
     2     1
 -93.52398200
 -93.52349700
  10.39152600
  10.39152600
     1     1
  32.69912800
  21.29656800
  -2.99976080
  -2.99976080
     1     2
  32.70459700
  21.29971200
  -3.00023930
  -3.00023930
     2     1
  72.23187900
  89.76771900
  -8.99997720
  -8.99997720
     2     2
  72.23242200
  89.76799000
  -9.00002270
  -9.00002270
```

Table 3. Input and output data for example 7.

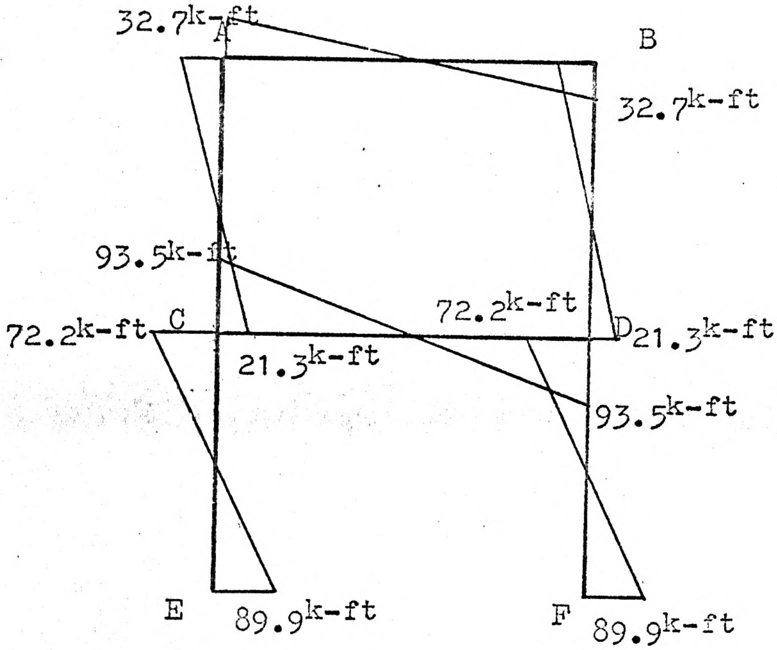


Fig. 31. Moment diagram for example 7, plotted on the compression side.

CONCLUSION

It is seen from the discussion in the report that the moment distribution method, which starts with fixed-end moments and obtains final results by correcting the moments from the effect of allowing the joints to rotate in succession, can be applied to numerous types of indeterminate structures provided the constants (a) fixed-end moment at each end, (b) stiffness at each end, and (c) carry-over factor at each end, are known or can be determined. Besides the cases mentioned in this report, there are other additional subjects, such as the effect of temperature changes and the plastic analysis, in which the method may be applied.

It is also seen that the method is not restricted to only one standard procedure. The relaxation method and the direct distribution method discussed in this report are two of the possible variants worth recording. By comparison, however, the standard method itself is found to be so simple and easy to remember that one is inclined to discard the variants.

The program used for example 7 showed that the method is also suited to digital computer programming, and the question pertaining to the loss of numerical accuracy can be safely neglected because of the nature of the calculation.

The solution of the example problems can be improved by repeating the procedure if a higher degree of accuracy is desired. However, for the purpose of practical use it is the approximate effect that is desired rather than an exact analysis.

ACKNOWLEDGMENT

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ADVANCED STUDY OF THE METHOD OF MOMENT DISTRIBUTION

by

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To the study of structural analysis, the establishment of the moment diagram is pre-requisite. Among the many methods for the solution of the moment diagram, the moment distribution method is one of the most useful because it is an easy-to-learn method in which each step of the calculations may be interpreted readily in terms of the physical behavior of the structure.

The fundamental concept of the moment distribution method is well known to the student. Thus, the report is mainly concerned with some comparatively unusual topics which need special consideration when using the method. These topics are (1) structures with elastic supports, (2) structures with hinges, (3) the effects of axial forces, (4) structures with nonprismatic members, (5) direct moment distribution method, (6) moment distribution as a relaxation procedure, and (7) the use of digital computer programming.

Brief descriptions of the seven topics are as follows:

(1) In general, for beam on n elastic supports, (n+1) separate moment distribution analyses should be carried out. The first one takes account of the effect of the loads on the beam assumed unyielding supports, and the others are for the effect of an arbitrary amount of joint displacement at the elastic support. A linear superposition of the results will give the moments in the original beam.

(2) For the structures which contain hinges at intermediate points along the members, the stiffness and carry-over

factors and fixed-end moments are modified as compared to similar members without hinges.

(3) If beam-column effect is considered, it is necessary to recalculate values for the stiffness factors, carry-over factors, and fixed-end moment coefficients. It is seen, as a result, that a compressive force will decrease the rotational stiffness of a beam as compared to the same beam with no axial force. However, the basic procedures used in moment distribution are not altered.

(4) In the practical field, the stiffness factors, carry-over factors, and fixed-end-moment coefficients for nonprismatic members can be found from Tables and Figures such as the one provided by Portland Cement Association. However, the theoretical derivation of the general formulas for these factors is treated in the report in spite of the time and space required.

(5) The direct distribution method, which takes the translation and joint rotation into consideration at the same time, is a modified method which determines the moments directly with one distribution procedure, rather than the two or more required by the standard one.

(6) In reality, the moment distribution method is a variation of the relaxation method by which the true results are obtained by successive approximations. For convenience, the relaxation procedure is carried out in the report by employing a relaxation table which starts by assuming values for joint rotations and obtains the final results by successively reducing the unbalanced moments.

(7) The program is presented to illustrate that the method is suited for a high speed digital computer. The result of example 7 obtained from the IBM 1620 computer is identical to that of example 6.

Following the description for each topic a numerical example is presented to demonstrate the steps of the procedure.