THE CONSTRUCTION OF A GREEN'S FUNCTION

by

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INTRODUCTION

Green's Function is the term applied to an unusual function developed by George Green, an English mathematician, early in the nineteenth century and used, primarily at that time, in the solution of problems in electricity and magnetism.

The use of the Green's Function is a powerful method for solving boundary-value problems and many physical problems involving heat flow, electricity and magnetism, aerodynamics, and others. The Green's Function may be calculated in one, two, or three dimensions and its form depends on the number of dimensions and the type of solid for which it is constructed. It has been the purpose of this paper to calculate the Green's Function for a number of systems and collect some of the work done by writers in this field.

GREEN'S FUNCTION IN ONE DIMENSION DEFINED

Green's Function (Ince, 1927) in one dimension may be defined by considering the completely-homogeneous linear differential system:

$$(1)\quad \begin{cases} p_0\dfrac{d^n u}{dx^n} + p_1\dfrac{d^{n-1}u}{dx^{n-1}} + p_2\dfrac{d^{n-2}u}{dx^{n-2}} + \; . \; . \; . \; + p_{n-1}\dfrac{du}{dx} + p_n u = 0, \\ U_i(u) = 0, \;\; \underline{i} = 1,2,3, \; . \; . \; ., \; n. \end{cases}$$

where p_i, $i = 1,2,3, \; . \; . \; ., \; n$ are functions of x and

$$U_i(u) \equiv A_i u(a) + A_i' u'(a) \; + \; . \; . \; . \; + \; A_i^{(n-1)} u^{(n-1)}(a)$$

$$+ \; B_i u(b) + B_i u'(b) \; + \; . \; . \; . \; + \; B_i^{(n-1)}(b) = C_i$$

It will be supposed that for a given interval (a, b) this system is incompatible, that is to say, it admits of no solution, not identically zero, which together with its first (n-1) derivatives, is continuous throughout the interval. Although no solution exists which satisfies these stringent conditions, there may exist another solution which formally satisfies the system but violates, at least in part, the condition of continuity.

Such a function is the Green's Function G(x,t) which has the following properties:

(A) It is continuous and possesses continuous derivatives of orders up to and including (n-2) when $a \leqslant x \leqslant b$;

(B) It is such that its derivative of order (n-1) is discontinuous at a point $\underline{t}$ within (a,b), the discontinuity being an upward jump of amount $-\dfrac{1}{p_0(t)}$;

(C) It formally satisfies the system at all points except at $x = t$.

The algebraic formula for the jump may be written as:

$$(2) \quad \left.\frac{\partial^{n-1} G(x,t)}{\partial x^{n-1}}\right]_{x=t+0} - \left.\frac{\partial^{n-1} G(x,t)}{\partial x^{n-1}}\right]_{x=t-0} = -\frac{1}{p_0(t)}.$$

THE CONSTRUCTION OF A GREEN'S FUNCTION

The method of calculating a Green's Function will be shown for two special cases.

In the first example consider the self-adjoint differential system

$$(3) \begin{cases} \dfrac{d^2u}{dx^2} = 0, \\ u_1(0) = u_2(1) = 0. \end{cases}$$

where $u_1(x)$ and $u_2(x)$ represent the solutions of the system (3) on the intervals $(0,t)$ and $(t,1)$, respectively.

The formal solution of the differential equation in (3) is $u = C'x + C''$, and hence on the separate intervals the solution must be of the same form. Since the discontinuity of the Green's Function is assumed to be at $x = t$, the Green's Function may be represented by

$$(4) \quad G(x,t) = \begin{cases} u_1 = Ax + B, \\ u_2 = Cx + D. \end{cases}$$

The use of the boundary conditions in (3) gives:

for $u_1(0) = 0$; $B = 0$,

for $u_2(1) = 0$; $C = -D$.

The Green's Function may now be rewritten as

$$(5)\quad G(x,t) = \begin{cases} u_1 = Ax, & (0,t) \\ u_2 = C(x-1), & (t,1). \end{cases}$$

The application of (2) gives

$$\left.\frac{\partial G}{\partial x}\right]_{x=t+o} - \left.\frac{\partial G}{\partial x}\right]_{x=t-o} = C - A = -1$$

Hence $C = A - 1$.

Since the function is continuous at $x = t$, it follows that

$At = (A-1)(t-1)$,

$At = \cancel{At} - A - t + 1$,

or $A = -t + 1$.

Therefore

$$(6)\quad G(x,t) \equiv \begin{cases} u_1 = -x(t-1), & (0,t) \\ u_2 = -t(x-1), & (t,1). \end{cases}$$

As a second example, consider the boundary-value problem

$$(7)\quad \begin{cases} \dfrac{d^4u}{dx^4} = 0 \\ u_1(0) = u_2(1) = u_1'(0) = u_2'(0) = 0; \end{cases}$$

where $u'(a) \equiv \left.\dfrac{\partial u}{\partial x}\right]_{x=a}$.

$u = C'x^3 + C''x^2 + C'''x + C''''$ is the formal solution.

Hence the Green's Function is

$$G(x,t) \equiv \begin{cases} u_1 = Ax^3 + Bx^2 + Cx + D \\ u_2 = Ex^3 + Fx^2 + Hx + J. \end{cases}$$

The use of the boundary conditions gives the following equations:

$$u_1(0)=0; \qquad D=0$$

$$u_2(1)=0; \qquad E+F+H+J=0$$

$$u_1'(0)=0; \qquad C=0$$

$$u_2(1)=0; \qquad 3E+2F+J=0.$$

The conditions of continuity give

$$At^3+Bt^2-Et^3-Ft-Ht-J=0,$$

$$3At^2+2Bt-3Et^2-2Ft-H=0,$$

and $$6At+2B-6Et-2F=0.$$

From the discontinuity of the third derivative, we have

$$\left.\frac{\partial^3 G}{\partial x^3}\right]_{x=t+o}-\left.\frac{\partial^3 G}{\partial x^3}\right]_{x=t-o}=-6A+6E=-1.$$

Hence $A-E=1/6$.

The evaluation of the constants in the above equations leads to

$$A=\frac{t^3}{3}-\frac{t^2}{2}+\frac{1}{6}, \qquad B=-\frac{t^3}{2}+t^2-\frac{t}{2},$$

$$E=\frac{t^3}{3}-\frac{t^2}{2}, \qquad G=-\frac{t^3}{2}+t^2,$$

$$H=-\frac{t^2}{2} \qquad J=\frac{t^3}{6}.$$

Therefore, the Green's Function for this differential system is

$$(8)\quad G(x,t)\equiv\begin{cases} u_1=\dfrac{x^2}{6}(t-1)^2\left[(2t+1)\,x-3t\right] & (0,t)\\[2ex] u_2=\dfrac{t^2}{6}(x-1)^2\left[(2x+1)\,t-3x\right] & (t,1).\end{cases}$$

It may be noticed that $G(x,t)$ is symmetric in x and t. That is, $G(x,t) = G(t,x)$ (Ince, 1927).

A TABLE OF GREEN'S FUNCTIONS

Green's Functions are of sufficient importance in the solution of physical problems to warrant a study of the ones most likely to be met. Different one dimensional differential systems have been used to construct a number of these Green's Functions, according to the methods in the preceding section. These results are tabulated in table I.

Column III is applicable only to the case of second order differential equations and is obtained from a general formula for $F(x,t)$ for this case (Ince, 1927).

Table 1. A table of Green's Functions

Non-Homogeneous System $u(x)$ finite	Homogeneous System	$F(x,t)$	$G(x,t)$	Solution of System
$\frac{d^2u}{dx^2} = r(x)$, $u_1(0)=u_2(1)=0$.	$\frac{d^2G}{dx^2} = 0$ $u_1(0)=u_2(1)=0$	$A + Bx \pm \frac{1}{2}(t-x)$	$u_1 = -x(t-1)$ $[0,t]$ $u_2 = -t(x-1)$ $[t,1]$	$u = \int_0^1 G(x,t)r(t)dt$
$\frac{d^2u}{dx^2} - n^2u = r(x)$ $u_1(0)=u_2(1)=0$	$\frac{d^2G}{dx^2} - n^2G = 0$ $u_1(0)=u_2(1)=0$	$A \cosh nx + B \sinh nx$ $\pm \frac{1}{2n} \sinh n(t-x)$	$u_1 = \frac{\sinh nx \sinh n(t-1)}{n \sinh n}$ $[0,t]$ $u_2 = \frac{\sinh nt \sinh u(x-1)}{n \sinh n}$ $[t,1]$	$u = \int_0^1 G(x,t)r(t)dt$
$\frac{d}{dx}\left[H(x)\frac{du}{dx}\right] = r(x)$ $\frac{1}{H(x)}$ is regular for $x=0$, $x=1$. $u_1(0)=u_2(1)=0$	$\frac{d}{dx}\left[H(x)\frac{dG}{dx}\right] = 0$ $\frac{1}{H(x)}$ is regular for $x=0$, $x=1$. $u_1(0)=u_2(1)=0$	$A\int_0^x \frac{dx}{H(x)} + B$ $\pm \frac{\int_0^x \frac{dx}{H(x)} - \int_0^t \frac{dt}{H(t)}}{-2}$	$u_1 = -\frac{\int_0^x \frac{dx}{H(x)}}{\int_0^1 \frac{dx}{H(x)}}\left[\int_0^t \frac{dx}{H(x)} \quad \int_0^1 \frac{dx}{H(x)}\right]$ $[0,t]$ $u_2 = -\frac{\int_0^t \frac{dx}{H(x)}}{\int_0^1 \frac{dx}{H(x)}}\left[\int_0^x \frac{dx}{H(x)} - \int_0^1 \frac{dx}{H(x)}\right]$ $[t,1]$	$u = \int_0^1 G(x,t)r(t)dt$
$\frac{d}{dx}\left[e^x \frac{du}{dx}\right] = r(x)$ $u_1(0) = u_2(1) = 0$	$\frac{d}{dx}\left[e^x \frac{dG}{dx}\right] = 0$ $u_1(0) = u_2(1) = 0$	$-\frac{A}{e^x} + B \pm \frac{1}{2} - \frac{1}{e^x} - \frac{1}{e^t}$	$u_1 = \frac{-e^{-t}+e^{-1}}{1-e^{-1}}(-e^{-x} + 1)$ $[0,t]$ $u_2 = \frac{-e^{-x}+e^{-1}}{1-e^{-1}}(-e^{-t} + 1)$ $[t,1]$	$u = \int_0^1 G(x,t)r(t)dt$
$\frac{d}{dx}\left[(x+1)\frac{du}{dx}\right] = r(x)$ $u_1(0) = u_2(1) = 0$	$\frac{d}{dx}\left[(x+1)\frac{dG}{dx}\right] = 0$ $u_1(0) = u_2(1) = 0$	$A \log(x+1) + B$ $\pm \frac{\log(x+1) - \log(t+1)}{-2}$	$u_1 = \log(x+1)\frac{\log 2 - \log(t+1)}{\log 2}$ $[0,t]$ $u_2 = \log(t+1)\frac{\log 2 - \log(x+1)}{\log 2}$ $[t,1]$	$u = \int_0^1 G(x,t)r(t)dt$
$\frac{d}{dx}\left[(\cos^2 x)\frac{du}{dx}\right] = r(x)$ $u_1(0) = u_2(\frac{\pi}{4}) = 0$	$\frac{d}{dx}\left[(\cos^2 x)\frac{dG}{dx}\right] = 0$ $u_1(0) = u_2(\frac{\pi}{4}) = 0$	$A \tan x + B$ $\pm \frac{\tan x - \tan t}{-2}$	$u_1 = -\tan x(\tan t - 1)$, $[0,t]$ $u_2 = -\tan t(\tan x - 1)$, $[t,\frac{\pi}{4}]$	$u = \int_0^{\pi/4} G(x,t)r(t)dt$
$\frac{d}{dx}\left[(x^2+1)\frac{du}{dx}\right] = r(x)$ $u_1(0) = u_2(1) = 0$	$\frac{d}{dx}\left[(x^2+1)\frac{dG}{dx}\right] = 0$ $u_1(0) = u_2(1) = 0$	$\operatorname{Arctan} x + B$ $\pm \frac{\arctan x - \arctan t}{-2}$	$u_1 = \frac{4}{\pi}(\frac{\pi}{4} - \arctan t)\arctan x$ $[0,t]$ $u_2 = \frac{4}{\pi}(\frac{\pi}{4} - \arctan x)\arctan t$ $[t,1]$	$u = \int_0^1 G(x,t)r(t)dt$
$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + 1 + \frac{n^2}{x^2}$ $\times u = r(x)$ $u_1(a) = u_2(b) = 0$	$\frac{d^2G}{dx^2} + \frac{1}{x}\frac{dG}{dx} + 1 + \frac{n^2}{x^2} = 0$ $u_1(a) = u_2(b) = 0$	$AJ_n(x) + BY_n(x)$ $\pm \frac{J_n(x)Y_n(t) - Y_n(x)J_n(t)}{2\, J_n(t)Y_n'(t) - Y_n(t)J_n'(t)}$	$u_1 = -\frac{[J(t)Y(b)-J(b)Y(t)]\,[J(x)Y(a)-J(a)Y(x)]}{t[J(b)Y(a)-J(a)Y(b)]\,[J'(t)Y(t)-J(t)Y'(t)]}$ $[a,t]$ $u_2 = -\frac{[J(x)Y(b)-J(b)Y(x)]\,[J(t)Y(a)-J(a)Y(t)]}{t[J(b)Y(a)-J(a)Y(b)]\,[J'(t)Y(t)-J(t)Y'(t)]}$ $[t,b]$	$u = \int_a^b G(x,t)r(t)dt$
$\frac{d}{dx}\left[x\frac{du}{dx}\right] = r(x)$ $u_1(0) = u_2(1) = 0$	$x\frac{d^2G}{dx^2} + \frac{dG}{dx} = 0$ $u_1(0) = u_2(1) = 0$	$\ln \frac{t}{2} \pm \frac{\ln t - \ln x}{2}$ $A \ln x + B \pm \frac{\log x - \log}{-2}$	$u_1 = -\ln t$ $[0,t]$ $u_2 = -\ln x$ $[t,1]$	$u = \int_0^1 G(x,t)r(t)dt.$

A PRACTICAL APPLICATION AS RELATED TO A GREEN'S FUNCTION

As a practical application (Bateman, 1932) of the equation $\frac{d^4y}{dx^4} = 0$, consider a weightless contilever beam of length $\underline{a}$ with a weight W concentrated at a point $\underline{t}$ units from the fixed end. Choose the axis as in figure 1; the x-axis coinciding with the elastic curve. Since there is no bending at the fixed end, $y' = 0$ at $x = 0$, and $y = 0$ at $x = 0$. If $\underline{y}$ represents the deflection from the normal, the resisting moment is, for $\underline{y}$ small, $M = B \frac{d^2y}{dx^2}$, where B is a constant depending on the elastic properties of the material of the beam.

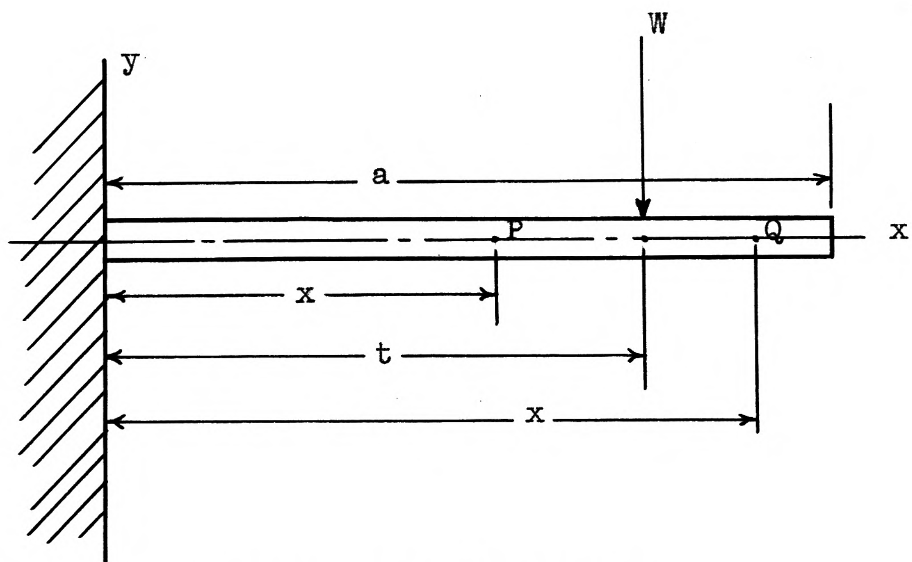

Fig. 1. Cantilever beam.

If moments are taken about any point, $0 \leq x \leq t$, the moment due to W is

$$M = -Wx + Wt.$$

Therefore

$$(9) \quad B\frac{d^2y}{dx^2} = -Wx + Wt,$$

and upon integrating

$$(10) \quad y = \frac{Wtx^2}{2B} - \frac{Wx^3}{6B} + \frac{C'x}{B} + \frac{C''}{B}.$$

The use of the boundary conditions $y(0) = 0$; $y'(0) = 0$ gives

for $y'(0) = 0$; $\quad C = 0$,

for $y(0) = 0$; $\quad C' = 0$.

Therefore

$$(11) \quad y = \frac{Wtx^2}{2B} - \frac{Wx^3}{6B}, \quad 0 \leq x \leq t.$$

But for a point where $a \geq x \geq t$, there is no shearing force. Hence

$$B\frac{d^2y}{dx^2} = 0, \quad a \geq x \geq t.$$

Two integrations give

$$(12) \quad y = \frac{C'x}{B} + \frac{C''}{B}, \quad a \geq x \geq t.$$

The continuity of the deflection $\underline{y}$ necessitates that the values of $\underline{y}$ in (11) and (12) be identical at $x = t$; whence

$$(13) \quad \frac{Wt^3}{3B} = \frac{C't}{B} + \frac{C''}{B}.$$

Using the value of $\underline{y}$ from (11), the slope of the elastic curve at $x = t$ is given by

$$y'\Big]_{x=t} = \frac{Wt^2}{2B}$$

and from (12)

$$y'\Big]_{x=t} = \frac{C'}{B}$$

Therefore

$$(14) \quad \frac{C'}{B} = \frac{Wt^2}{2B} .$$

The solution of (13) and (14) gives

$$C' = \frac{Wt^2}{2}, \text{ and } C'' = -\frac{Wt^3}{6} .$$

The deflection of $\underline{y}$ may then be written as

$$y = \frac{Wx^2}{6B}(3t-x), \quad 0 \leqslant x \leqslant t,$$

$$y = \frac{Wt^2}{6B}(3x-t), \quad a \geqslant x \geqslant t,$$

which is a type of Green's Function. The Green's Function might have been constructed for the boundary value problem given by

$$(15)\begin{cases} \dfrac{d^4u}{dx^4} = 0 \\ u_1(0) = u_1'(0) = 0, \ u_2''(a) = 0, \ u_2'''(a) = 0. \end{cases}$$

GREEN'S FORMULA

For the further development of the theory of Green's Function it will be necessary to use Green's Formula; the proof of which will now be given.

Consider the Sturm-Liouville equation (Lovitt, 1924) for which the following permanent notation will be used:

$$(16) \quad L(u) = \frac{d}{dx}\left(p\frac{du}{dx}\right) + qu$$

where p(x), q(x), and P'(x) are continuous functions in $\underline{x}$

on $a \leqslant x \leqslant b$. If $\underline{u}$ and $\underline{v}$ are any two linearly independent solutions of (16), the following identity may be established:

$$(17)\quad vL(u) - uL(v) = \frac{d}{dx}\left[p\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)\right] .$$

The relation given by (17) is known as Green's Formula. To prove this relation, write the left member of (17) in determinant form as follows:

$$vL(u) - uL(v) \equiv \begin{vmatrix} L(u) & u \\ L(v) & v \end{vmatrix} .$$

The determinant may be rewritten using a familiar property of determinants to give the following steps in the process of simplification:

$$\begin{vmatrix} \frac{d}{dx}(p\frac{du}{dx}) + qu & u \\ \frac{d}{dx}(p\frac{dv}{dx}) + qv & v \end{vmatrix} = \begin{vmatrix} \frac{d}{dx}(p\frac{du}{dx}) & u \\ \frac{d}{dx}(p\frac{dv}{dx}) & v \end{vmatrix}$$

$$= v\frac{d}{dx}(p\frac{du}{dx}) - u\frac{d}{dx}(p\frac{dv}{dx}).$$

The right member of (16) may be shown to be of the same form by the following equation:

$$\frac{d}{dx}\left| vp\frac{du}{dx} - up\frac{dv}{dx}\right| = v\frac{d}{dx}(p\frac{du}{dx}) + \cancel{p\frac{du}{dx}\frac{dv}{dx}} - u\frac{d}{dx}(p\frac{dv}{dx}) - \cancel{p\frac{dv}{dx}\frac{du}{dx}} .$$

Therefore (17) has been established.

EQUIVALENCE BETWEEN A BOUNDARY PROBLEM AND A HOMOGENEOUS LINEAR INTEGRAL EQUATION

The relation (Lovitt, 1924) between a boundary value problem and an integral equation will now be shown.

Theorem 1. If F is continuous, together with its first two derivatives and

$$(18)\quad \begin{cases} L(F) + f = 0; \\ R_o(f) = AF(a) + BF'(a) = 0, \\ R_1(F) = CF(b) + DF'(b) = 0. \end{cases}$$

Then

$$F(x) = \int_a^b G(x,t)f(t)dt,$$

where $G(x,t)$ is the Green's Function for the boundary value problem (18).

Proof:

$L(F) = -f$ on (a,b).

$L(G) = 0$ on (a,t) and (t,b) separately.

Multiply the first by $-G$ and the last by F and add.

$$\begin{array}{l} -GL(F) = Gf \\ \underline{FL(G) \quad = 0 \quad} \\ FL(G) - GL(F) = Gf. \end{array}$$

By Green's Formula

$$FL(G) - GL(F) = \frac{d}{dx}p(FG' - GF') = Gf.$$

Integrate from $\underline{a}$ to $\underline{t}$ and from $\underline{t}$ to $\underline{b}$. Then

$$(19)\quad \Big[p(FG' - GF')\Big]_a^{t-o} = \int_a^{t-o} G(x,t)f(x)dx,$$

where

$$\phi(x)\Big]_a^{t-o} \equiv \lim_{\epsilon \to 0} \phi(x)\Big]_a^{t-\epsilon}$$

and

$$\left[p(FG'-GF')\right]_{t+o}^{b} = \int_{t+o}^{b} G(x,t)f(x)dx.$$

Therefore

$$(20)\quad p(FG'-GF')\Big]_{t+o}^{t-o} - \left[p(FG'-GF')\right]^{x=o} + \left[p(FG'-GF')\right]^{x=b}$$

$$= \int_{a}^{b} G(x,t)F(x)dx.$$

Since G and F' are continuous at $x = t$, the value of the first term is

$$-GF'\Big]_{t+o}^{t-o} = 0,$$

and

$$pFG'\Big]_{t+o}^{t-o} = p(t-o)F(t-o)G'(t-o)-p(t+o)F(t+o)G'(t+o)$$

$$= p(t)F(t)\left[G'(t-o)-G'(t+o)\right].$$

But by the definition given in (2)

$$\left[G'(t-o)-G'(t+o)\right] = \frac{1}{p(t)}.$$

Therefore

$$pFG'\Big]_{t\ o}^{t-o} = F(t)$$

The value of the second term of the left member in (20) if $A \neq 0$ is

$$\left[p(FG'-GF')\right]^{x=a} = p(a)\begin{vmatrix} F(a) & F'(a) \\ G(a) & G'(a) \end{vmatrix}$$

$$= \frac{p(a)}{A}\begin{vmatrix} AF(a)+BF'(a) & F'(a) \\ AG(a)+BG'(a) & G'(a) \end{vmatrix}$$

The use of the boundary condition $R_0(F) = 0$ gives

$$\left[p(FG'-GF')\right]^{x=a} = \frac{p(a)}{A}\begin{vmatrix} R_0(F) & F'(a) \\ R_0(G) & G'(a) \end{vmatrix} = 0.$$

If $A = 0$, then $B \neq 0$ and

$$\Big[p(FG'-GF')\Big]^{x=a} = \frac{p(a)}{B}\begin{vmatrix} F(a) & [R_0\ F(a)] \\ G(a) & [R_0\ G(a)] \end{vmatrix} = 0.$$

In like manner

$$\Big[p(FG'-GF')\Big]^{x=b} = 0.$$

Therefore

$$(21) \qquad f(t) = \int_a^b G(x,t)f(x)dx.$$

If x and t are interchanged, which is allowable since

$$G(x,t) = G(t,x),$$

then

$$f(x) = \int_a^b G(x,t)f(t)dt.$$

GREEN'S FUNCTION IN THE FLOW OF HEAT

The use of Green's Function in the theory of potential is well known. The function is most conveniently defined for the closed surface S as the potential which vanishes over the surface, and is infinite as $\frac{1}{r}$ when r is zero, at the point P inside the surface. If this solution be denoted by $G(P)$, the solution of the equation

$$\nabla^2 u = 0$$

with no infinity inside S and an arbitrary value V over the surface is given by

$$(22) \quad u = \frac{1}{4\pi}\iint \frac{\partial}{\partial n} G(P) V dS,$$

$\frac{\partial}{\partial n}$ denoting differentiation along the outward drawn normal.

We proceed (Carslaw, 1906) to show how a similar function may be employed with advantage in the mathematical theory of the conduction of heat. In this case we shall take the Green's Function as the temperature at (x,y,z) at the time t, due to an instantaneous point source of strength unity generated at the point P (x_0, y_0, z_0) at the time T, the solid being initially at zero temperature, and the surface being kept at zero temperature.

This temperature at (x,y,t) may be written

$$u = F(x,y,t,x_0,y_0,t_0,t-T), \quad (t > T)$$

and u satisfies the heat equation (Carslaw, 1906)

$$\frac{\partial u}{\partial t} \; K = \nabla^2 u, \quad (t > T).$$

However since t only enters in the form t-T, we have also

$$\frac{\partial u}{\partial T} + K \nabla^2 u = 0, \quad (T < t).$$

Further

$$\lim_{t \to T} (u) = 0$$

at all points inside S except at the point (x_0, y_0, z_0) where the solution takes the form

$$\frac{1}{\left(2\sqrt{\pi k(t-T)}\right)^3} e^{-\frac{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}{4k(t-T)}}$$

To show that this is a solution we should first consider the linear flow of heat in an infinite solid. The equation of conduction then reduces to

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2},$$

since v depends only on x and t. Consider the expression

$$u = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4kt}} .$$

Since

$$\frac{\partial u}{\partial t} = -\frac{1}{2t^{3/2}} e^{-\frac{x^2}{4kt}} + \frac{x^2}{4kt^{5/2}} e^{-\frac{x^2}{4kt}}$$

and

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2kt^{3/2}} e^{-\frac{x^2}{4kt}} + \frac{x^2}{4k^2 t^{5/2}} e^{-\frac{x^2}{4kt}},$$

this expression is a particular integral of the differential equation. Therefore

$$\frac{1}{2\sqrt{\pi kt}} e^{-\frac{(x-x_0)^2}{4kt}}$$

is also an integral.

Finally at the surface S, $u = 0$, $(T < t)$.

Now let v be the temperature at time t in this solid due to the surface temperature $\phi(x,y,z,t)$ and the initial temperature $f(x,y,z)$.

Then v satisfies the equations

$$\frac{\partial v}{\partial t} = k\nabla^2 v, \quad (t > 0).$$

$v = f(x,y,z)$ initially, inside S,

$v = \phi(x,y,z,t)$ at S for $(t > 0)$;

and, since the time T of our former equations lies within the interval for t, we have also

$$\frac{\partial v}{\partial T} \quad K\nabla^2 v, \ (T < t),$$

$$v = \phi(x,y,z,t) \text{ at the surface.}$$

Therefore

$$\frac{\partial}{\partial T}(uv) = u\frac{\partial v}{\partial T} + v\frac{\partial u}{\partial T} = k\left[u\nabla^2 v - v\nabla^2 u\right],$$

and

$$(23) \int_0^{t-\epsilon}\left[\iiint \frac{\partial}{\partial T}(uv)\ dx\ dy\ dz\right] dT$$

$$= K\int_0^{t-\epsilon}\left[\iiint (u\nabla^2 v - v\nabla^2 u)\ dx\ dy\ dz\right] dT,$$

the triple integration being taken throughout the solid, and ϵ being any positive quantity less than t, as small as we please.

Interchanging the order of integration on the left hand side of this equation and applying Green's Theorem (Osgood, 1935) to the right hand side, we have

$$(24) \iiint (uv)_{T=t-\epsilon} dx\ dy\ dz - \iiint (uv)_{T=0} dx\ dy\ dz$$

$$= k\int_0^{t-\epsilon}\left[\iint u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} dS\right] dT = k\int_0^{t-\epsilon}\left[\iint v\left(\frac{\partial u}{\partial n}\right)_i\ dS\right] dT,$$

where $\frac{\partial}{\partial n}{}_i$ denotes differentiation along the inward drawn normal, and we have used the condition that u vanishes at the surface.

Now take the limit as ϵ vanishes. The left side gives

$$[V_p]_t\left[\iiint u_{T=t-0} dx\ dy\ dz\right] - \iiint u_{T=0} dx\ dy\ dz,$$

the first integral being taken through an element of volume including the point P (x_0, y_0, z_0) where the function u becomes infinite at $t = 0$, the second integral being taken through the solid, and $\left[v_p\right]_t$ stands for the value at the point P (x_o,y_o,z_o) at the time t.

But since u is the temperature at the time t due to a source at (x_o,y_o,z_o) at time T (Carslaw, 1906)

$$\iiint u_{T=t-o}\,dx\,dy\,dz = 1$$

and we have

$$\text{(25)}\quad \left[v_p\right]_t = \iiint (u)_{T=0}(v)_{t=0}\,dx\,dy\,dz + k\int_0^t\left[\iint v\left(\frac{u}{n}\right)_i\,dS\right]dT,$$

and

$$\left[v_p\right]_t = \iiint u_{T=0}\,f(x,y,z)\,dx\,dy\,dz + k\int_0^t\left[\iint \phi(x,y,z,T)\,\frac{\partial u}{\partial n}_i\,dS\right]dT$$

as the temperature at (x_o,y_o,z_o) at the time t due to the initial distribution f(x,y,z) and the surface temperature $\phi(x,y,z,t)$.

Green's Functions for the flow of heat in various solids will now be obtained. Three special cases will be given.

In the first case (Carslaw, 1906) consider the linear flow of heat in a semi-infinite solid bounded by the plane $x = 0$. The initial temperature will be f(x), $x > 0$, and the boundary will be kept at a temperature $\phi(t)$. Then the Green's Function, or the temperature at (x,y,z) at the time t due to the instantaneous source of unit strength at time T is given by

$$(26)\quad u = \frac{1}{2\sqrt{\pi k(t-T)}}\left\{ e^{-\frac{(x-x_o)^2}{4k(t-T)}} - e^{-\frac{(x+x_o)^2}{4k(t-T)}} \right\}$$

It is known (Carslaw, 1906) that

$$u_1 = \frac{1}{2\sqrt{\pi k(t-T)}}\, e^{-\frac{(x-x_o)^2}{4k(t-T)}}$$

gives the temperature at (x,y,z) due to a heat source located at $x_o,y_o,z_o)$ in the semi-infinite solid. The temperature at (x,y,t) due to a sink of unit strength at $(-x_o,y_o,z_o)$ is given by

$$u_2 = -\frac{1}{2\sqrt{\pi k(t-T)}}\, e^{-\frac{x+x_o}{4k(t-T)}}.$$

Physically, the temperature due to a source and boundary kept at a zero temperature is equivalent to the source and a sink of equal and opposite strength located at the image point. The combination of the source and sink of equal strength will keep the boundary $x = 0$, at the temperature $u = 0$ and at the same time, satisfy the heat equation. The Green's Function is the one given in (26).

The application of the general formula (25) for the temperature at (x_o,y_o,z_o) at time <u>t</u>, is given by

$$(27)\quad \Big[V_p\Big]_t = \int_o^{\infty} u_{T=0}\, f(x)\,dx + k\int_o^t \phi(T)\, \frac{\partial u}{\partial x}\bigg|_{x=o} dT$$

$$= \frac{1}{2\sqrt{\pi kt}}\int_o^{\infty} f(x)\left\{ e^{-\frac{(x-x_o)^2}{4kt}} - e^{-\frac{(x+x_o)^2}{4kt}} \right\} dx$$

$$+ \frac{x_o}{2\sqrt{\pi k t}} \int_o^t \phi(T) \frac{e^{-\frac{x_o^2}{4k(t-T)}}}{\sqrt{(t-T)^3}} \, dT.$$

The result in (27) is of theoretical value but for all cases except the simple ones arising from elementary functions for $f(x)$ and $\phi(t)$ give integrals in general which must be evaluated by some method of approximation.

In the second case we study the two dimensional flow of heat (Carslaw, 1906) in a semi-infinite solid bounded by the plane $y = 0$. The initial temperature will be $v = f(x,y)$ and the boundary will be kept at $v = F(x,t)$. In a manner similar to the one for the linear case, Green's Function may be written as:

$$(28) \quad u = \frac{1}{4\pi k(t-T)} \left\{ e^{-\frac{(x-x_o)^2 + (y-y_o)^2}{4k(t-T)}} - e^{-\frac{(x-x_o)^2 + (y+y_o)^2}{4k(t-T)}} \right\}.$$

Differentiation and evaluation at $y = 0$ gives

$$\frac{\partial u}{\partial n}_i = \frac{\partial u}{\partial y}_{y=0} = \frac{y_o}{4\ \ k^2 (t-T)^2} \, e^{-\frac{(x-x_o)^2 + y_o^2}{4k(t-T)}}$$

Thus the temperature at $P(x_o, y_o)$ at time $\underline{t}$ is given by

$$(29) \quad \left[V_p\right]_t = \frac{1}{4\pi kt} \int_{-\infty}^{\infty} \int_o^{\infty} f(x,y) \left[e^{-\frac{(x-x_o)^2 + (y-y_o)^2}{4kt}} - e^{-\frac{(x-x_o)^2 + (y+y_o)^2}{4kt}} \right] dx \, dy$$

$$+ \frac{y_o}{4\ \ k} \int_o^t \int_{-\infty}^{\infty} \frac{F(x,T)}{(t-T)^2} \, e^{-\frac{(x-x_o)^2 + y_o^2}{4k(t-T)}} \, dx \, dT.$$

In the three dimensional flow of heat (Carslaw, 1906) in the semi-infinite solid $x > 0$, the initial temperature will be $v = f(x,y,z)$ and the plane $x = 0$, kept at $V = F(y,z,t)$.

In this case the Green's Function is

$$(30)\quad u = \frac{1}{\left(2\sqrt{\pi k(t-T)}\right)^3}\left\{ e^{-\frac{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}{4k(t-T)}} - e^{-\frac{(x+x_o)^2+(y-y_o)^2+(z-z_o)^2}{4k(t-T)}} \right\}$$

and

$$\frac{\partial u}{\partial n}_i = \frac{\partial u}{\partial x}_{x=o} = \frac{x_o}{8\pi^{3/2}k^{5/2}(t-T)^{5/2}}\, e^{-\frac{x_o^2+(y-y_o)^2+(z-z_o)^2}{4k(t-T)}}$$

The temperature at $P(x_o,y_o,z_o)$ at the time $\underline{t}$ is given by

$$(31)\quad \left[V_p\right]_t = \frac{1}{(2\sqrt{\pi kt})^3}\int_o^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f(x,y,t) \left[e^{-\frac{(x-x_o)^2+(y-y_o)+(t-t_o)^2}{4kt}} - e^{-\frac{(x\quad x_o)^2+(y-y_o)^2+(t-t_o)^2}{4kt}} \right] dx\,dy\,dz$$

$$+ \frac{x_o}{8\pi^{3/2}k^{3/2}}\int_o^t\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{F(x,y,T)}{(t-T)^{5/2}} \times e^{-\frac{x_o^2+(y-y_o)^2+(z-z_o)^2}{4k(t-T)}}\, dT\,dy\,dt.$$

In the case of radiation at the surface (Carslaw, 1906), the Green's Function $\underline{u}$ is taken as the temperature at (x,y,z) at time $\underline{t}$ due to an instantaneous point source at (x_o,y_o,z_o) at time $\underline{T}$, radiation taking place at the surface into a medium at zero temperature. For completeness we include some results on this type of boundary value problem.

Theorem 2. The temperature at $P(x_o,y_o,z_o)$ at time $\underline{t}$ due to an initial distribution $f(x,y,z)$ and radiation at the surface into a medium at temperature $\phi(x,y,z,t)$ is given as

$$(32) \quad \left[V_p\right]_t = \iiint u_{T=0} f(x,y,z)\,dx\,dy\,dz$$

$$+ hk\int_0^t \left[\iint u\,\phi(x,y,z,T)\,dS\right] dT$$

$$\left[V_p\right]_t = \iiint u_{T=0} f(x,y,z)\,dx\,dy\,dz$$

$$k\int_0^t \left[\iint \left(\frac{\partial u}{\partial n}\right)_i \phi(x,y,z,T)\,dS\right] dT$$

For a semi-infinite solid: Radiation at the surface $x = 0$ into a medium at zero,

$$v = \frac{1}{2\sqrt{\pi kt}}\left[e^{-\frac{(x-x_o)^2}{4kt}} + e^{-\frac{(x+x_o)^2}{4kt}} - 2h\int_0^\infty e^{-h\xi}\cdot e^{-\frac{(x+x_o+\xi)^2}{4kt}}\,d\right]$$

For a semi-infinite solid: radiation into a medium at temperature $\phi(t)$,

$$v = \frac{1}{2\sqrt{\pi kt}}\int_0^\infty \left[e^{-\frac{(x-x')^2}{4kt}} + e^{-\frac{(x+x')^2}{4kt}} - 2h\int_0^\infty e^{-h\xi}\cdot e^{-\frac{(x+x'+\xi)^2}{4kt}} \times d\xi\right] f(x')\,dx'$$

$$+h\sqrt{\frac{k}{\pi}}\int_0^t\left[e^{-\frac{x^2}{4k(t-T)}}-h\int_0^\infty e^{-h}\ .e^{-\frac{(x+\xi)^2}{4k(t-T)}}d\xi\right]\frac{\phi(T)}{\sqrt{t-T}}\ dT.$$

For other types of solids involving radiation at the surface the reader is referred to Carslaw's text on the flow of heat.

GREEN'S FUNCTION FOR A MISCELLANEOUS GROUP OF SOLIDS

The Green's Function for various solids has been constructed. Smythe (Smythe, 1939) has given the following:

(A) For a cone

$$r < a \quad V_i = \sum_n \sum_{m=0}^{n} A_{mn}\left[\frac{r}{a}\right]^n P_n^m(\mu)\cos m(\phi-\phi_o)$$

$$r > a \quad V_o = \sum_n \sum_{m=0}^{n} A_{mn}\left[\frac{a}{r}\right]^{n+1} P_n^m(\mu)\cos m(\phi-\phi_o)$$

(B) For a conical box

$$r < a \quad V_i = \sum_n \sum_{m=0}^{n} A_{mn}\frac{a^{2n+1}-d^{2n+1}}{a^n(c^{2n+1}-d^{2n+1})}\quad r^n-\frac{c^{2n+1}}{r^{n+1}}$$

$$\times P_n^m(\mu)\cos m(\phi-\phi_o)$$

$$r > a \quad V_o = \sum_n \sum_{m=0}^{n} A_{mn}\frac{a^{2n+1}-c^{2n+1}}{a^n(c^{2n+1}-d^{2n+1})}\quad r^n-\frac{d^{2n+1}}{r^{n+1}}$$

$$\times P_n^m(\mu)\cos m(\phi-\phi_o)$$

(C) For a circular cylinder

$$V = \frac{2q}{a^2}\sum_{r=1}^{\infty} e^{-\mu_r z} \sum_{s=0}^{\infty}(2-\delta_s^o)\frac{J_s(\mu_r b)J_s(\mu_r p)}{\mu_r\left[J_{s+1}(\mu_r a)\right]^2}\cos S(\phi-\phi_o)$$

(D) For a cylindrical box

$$V = \frac{4q}{a^2}\sum_{r=1}^{\infty}\frac{\sinh\mu_r(L-c)\sinh\mu_r z}{\sinh\mu_r L}$$

$$\sum_{s=0}^{\infty}(2-\delta_s^o)\frac{J_s(\mu_r b)J_s(\mu_r p)}{\mu_r\left[J_{s+1}(\mu_r a)\right]^2}\cos S(\phi-\phi_o)$$

(E) For a disc

$$V_i = \frac{2q}{\pi a}\sum_{n=0}^{\infty}(4n+1)Q_{zn}(j\xi_o)Q_{2n}(j\zeta)P_{2n}(\xi)$$

(F) For a rectangular prism

$$V = 2q\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(m^2a^2+n^2b^2)^{-\frac{1}{2}} e^{-\frac{(m^2a^2+n^2b^2)^{\frac{1}{2}}\pi\,|z-z_o|}{ab}}$$

$$\times \sin\frac{n\pi x_o}{a}\sin\frac{n\pi x}{a}\sin\frac{m\pi y_o}{b}\sin\frac{m\pi y}{b}\ .$$

(G) For a sphere

$$V = q\left[(r^2+b^2-2b_r\cos\theta)^{\frac{1}{2}} - a(b^2r^2+a^4-2a^2b_r\cos\theta)^{\frac{1}{2}}\right]$$

CONCLUSION

The main results of this paper are given in Table 1. By the application of the formulas in the section on the flow of heat, the solutions to certain types of problems in the flow of heat may be found.

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