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Electromagnetic Wave Scattering by Small Impedance Particles of an Arbitrary Shape

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Abstract

Scattering of electromagnetic (EM) waves by one and many small ($ka \ll 1$) impedance particles D_m of an arbitrary shape, embedded in a homogeneous medium, is studied. Analytic formula for the field, scattered by one particle, is derived. The scattered field is of the order $O(a^{2-\kappa})$, where $\kappa \in [0, 1)$ is a number. This field is much larger than in the Rayleigh-type scattering. An equation is derived for the effective EM field scattered by many small impedance particles distributed in a bounded domain. Novel physical effects in this domain are described and discussed.

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Key words: electromagnetic waves; wave scattering by small body; boundary impedance; many-body scattering.

1 Introduction

In this paper we develop a theory of electromagnetic (EM) wave scattering by one and many small impedance particles (bodies) D_m , $1 \leq m \leq M = M(a)$, embedded in a homogeneous medium which is described by the constant permittivity $\epsilon_0 > 0$ and permeability $\mu_0 > 0$. The smallness of a particle means that $ka \ll 1$, where $a = 0.5 \text{diam} D_m$ is the characteristic dimension of a particle, and $k = \omega(\epsilon_0 \mu_0)^{1/2}$ is the wave number in the medium exterior to the particles. Although scattering of EM waves by small bodies has a long history, going back to Rayleigh (1871), see [1], [14], the results of this paper are new and useful in applications because light scattering by colloidal particles in a solution, and light scattering by small dust particles

in the air are examples of the problems to which our theory is applicable. The Mie theory deals with the scattering by a sphere, not necessarily small, and gives the solution to the scattering problem in terms of the series in the spherical harmonics. If the sphere is small, $ka \ll 1$, then the first term in the Mie series yields the main part of the solution. Our theory is applicable *only* to small particles. They can be of arbitrary shapes and orientations. The solution to the scattering problem for one small particle of an arbitrary shape is given analytically. For many such particles the solution is reduced to solving a linear algebraic system. This system is not obtained by a discretization of some boundary integral equation, and it has a clear physical meaning. Its limiting form as $a \rightarrow 0$ yields an integro-differential equation for the limiting effective field in the medium where the small particles are distributed. One-body wave scattering problems can be studied theoretically only in the limiting cases of a small body, $ka \ll 1$, or a large body, $ka \gg 1$. In the latter case the geometrical optics is applicable. This paper deals with the case $ka \ll 1$. Rayleigh (1871) understood that the scattering by a small body is given mainly by the dipole radiation. For a small body of an arbitrary shape this dipole radiation is determined by the polarization moment, which is defined by the polarizability tensor. Lord Rayleigh and other researchers, from 1871 for about a century, did not give analytical formulas for calculating this tensor for bodies of arbitrary shapes. This was done by the author in 1970, see [16] and [14]. For homogeneous bodies of arbitrary shapes analytical formulas, which allow one to calculate this tensor with any desired accuracy, were derived in [14], Chapter 5. These bodies were assumed dielectric or conducting in [14]. Under the Rayleigh assumption the scattered field is proportional to a^3 , that is, to the volume of a small body. The physically novel feature of our theory is a conclusion that for small impedance particle with impedance $\zeta = h(x)a^{-\kappa}$, where $h(x)$ is a given function, $\text{Re}h(x) \geq 0$ and $\kappa \in [0, 1)$ is a given number, the scattered field is proportional to $a^{2-\kappa}$, that is, it is much larger than in the Rayleigh case, since $a^3 \ll a^{2-\kappa}$ if $a \ll 1$ and $\kappa \geq 0$. We do not claim that the boundary impedance ζ , as $a \rightarrow 0$, has to behave like $O(a^{-(2-\kappa)})$, $\kappa \in [0, 1)$. What is done in this paper (among other things) is deriving the formulas for the scattered field in the cases of one-body and many-body EM wave scattering problems when the boundary impedances are $O(a^{-(2-\kappa)})$ as $a \rightarrow 0$. These formulas show that new physical phenomena occur:

- a) the scattered field by a single scatterer is much larger than in the Rayleigh scattering,
- and
- b) the scattered field in a medium, in which many small impedance

particles are embedded, satisfies an equation which shows that new physical phenomena occur, see Section 3.

In this paper wave scattering by small impedance particles is studied. Besides high intrinsic interest in this problem, the theory we develop allows one to get some physically interesting conclusions about the changes of the material properties of the medium in which many small particles are embedded. The results of this paper can be used to develop a method for creating materials with a desired refraction coefficient by embedding many small impedance particles into a given material. Such a theory has been developed by the author for *scalar* wave scattering, for example, acoustic wave scattering, in a series of publications [4]-[15].

The novel physical idea in this paper is to reduce solving the scattering problem to finding some constant pseudovector Q (see formula (18)), rather than a pseudovector function J (see formula (10)) on the surface of the scatterer. The quantity Q is a pseudovector which somewhat analogous to the total charge on the surface of the perfect conductor with the shape of D_m , while the quantity J is somewhat analogous to the surface charge density. We assume for simplicity that the impedance ζ (see formula (5)) for a single scatterer is a constant given in (16). A similar assumption appeared in paper [5], where the scalar wave scattering theory was developed. The result of the theory in [5] was a recipe for creating materials with a desired refraction coefficient in acoustics (see [11], [12], [13]). The boundary impedance (16) grows to infinity as $a \rightarrow 0$. One can pass to the limit in the equation for the effective (self-consistent) field in the medium, obtained by embedding many small impedance particles into a given medium. Such a theory is briefly summarized in paper [12], where the equation for the limiting (self-consistent) field in the medium is derived for the scalar wave scattering by small bodies of arbitrary shapes.

The aim of this paper is to develop a similar theory for EM wave scattering by many small impedance particles embedded in a given material. This paper is essentially self-contained. The author's earlier papers [3]-[14] deal mostly with scalar wave scattering. The novel points of the author's papers consist in treating scalar wave scattering by one and many small bodies of arbitrary shapes under the physical assumptions which imply that the multiple scattering effects are essential. The same novel points can be found in this paper.

For EM wave scattering by one small body D of an arbitrary shape with an impedance boundary condition an analytic formula for the electromag-

netic field in the region $|x| \gg a$, $y = x_1 \in D$, is derived:

$$E(x) = E_0(x) + [\nabla \frac{e^{ikr}}{4\pi r}, Q], \quad r \gg a, \quad g(x, y) := \frac{e^{ikr}}{4\pi r}, \quad r = |x - y|, \quad (1)$$

where E_0 is the incident field, which satisfies Maxwell's equations in the absence of the scatterer D . Formula (1) is obtained from formula (10) in the far zone if one neglects the terms of higher order of smallness as $a \rightarrow 0$ and denote $Q = \int_S J(t) dt$. In this case $[\nabla_x, g(x, x_1)Q] = [\nabla g(x, x_1), Q]$, which leads to formula (1). In the case of a single small scatterer we calculate Q analytically, see formula (2) below. For a small body the far zone starts fairly close to this body.

In equation (1) (and in similar equations below in which only the main terms of the equations are kept but the equality sign is used and not the approximation sign or the order terms indicating the error), the order term is *always* of the higher order of smallness, as $a \rightarrow 0$, than the kept one. It would be too cumbersome and not very helpful to the reader to specify the order of the error terms in each formula of the type (1) because this order depends in some cases on more than one parameter. For example, in formula (1) the order of the error term depends on the ratio a/r and also, through the quantity Q , on a .

We use the standard notations: $[A, B] = A \times B$ is the cross product of two vectors, $(Q, e_j) = Q \cdot e_j$ is the dot product, $\{e_j\}_{j=1}^3$ is an orthonormal basis in \mathbb{R}^3 . The quantity Q plays an important role: it defines the main term in the scattered field. One has

$$Q_j := (Q, e_j) = -\frac{\zeta |S|}{i\omega\mu_0} \Xi_{jp} (\nabla \times E_0(O))_p, \quad \Xi := (I + \alpha)\tau, \quad (2)$$

over the repeated index p summation is understood from 1 to 3, ζ is the boundary impedance, $|S|$ is the surface area of the particle, the matrix Ξ_{jp} is defined by the formula

$$\Xi_{jp} := (I + \alpha) \left(\delta_{jp} - |S|^{-1} \int_S N_j(s) N_p(s) ds \right) := (I + \alpha)\tau, \quad (3)$$

where $N_j(s)$ is the j -th component of the unit normal $N(s)$ to the surface S at a point $s \in S$, pointing *out of* D , $\tau := I - b$, b is a matrix,

$$b_{jp} := |S|^{-1} \int_S N_j(s) N_p(s) ds, \quad \tau_{jp} = \delta_{jp} - b_{jp},$$

$k = \omega(\epsilon_0\mu_0)^{1/2}$ is the wave number, $O \in D$ is the origin, I is the identity matrix, and α is a matrix, defined by the formula:

$$I + \alpha := (I + \beta)^{-1},$$

where the matrix β is defined in (29) (see below).

By S^2 the unit sphere in \mathbb{R}^3 will be denoted. The boundary S of the small body D is assumed smooth: it is sufficient to assume that in local coordinates the equation of S is given as $x_3 = \phi(x_1, x_2)$, where the function ϕ has first derivative satisfying a Hölder condition.

Briefly speaking, *there are three basic novel results* in this paper.

The *first result* are formulas (1)-(3) and equations (31) and (32). Equation (31) gives an analytical expression for the quantity Q in formula (1). The matrices α and τ in this formula are calculated analytically, ζ is the boundary impedance, $|S|$ is the surface area of S , and E_0 is the incident electric field at the point O at which the small particle is located. Equation (32) yields a formula for calculating the EM field scattered by a single small impedance particle (body) of an arbitrary shape.

The *second result* is the reduction of the solution to many-body scattering problem to solving a linear algebraic system (LAS), see equations (41) and (44). The LAS (41) has M vector unknowns, while LAS (44) has $P \ll M$ vector unknowns. By this reason LAS (44) is recommended for solving in practice. The LAS (44) is derived from the LAS (41).

The *third result* is a derivation of the equation for the limiting effective (self-consistent) field in the medium in which many small impedance particles are embedded, see equation (45).

The scattering problem by one small body is formulated and studied in Section 2, the reduction of the solution to the many-body EM wave scattering problem to the solution of a LAS is given in Section 3. Also in Section 3 a derivation of the equation for the limiting effective field is given as $a \rightarrow 0$ and $M = M(a) \rightarrow \infty$. In Section 4 the conclusions are formulated.

In this paper we do not solve the boundary integral equation to which the scattering problem can be reduced in a standard approach, but find asymptotically exact analytical expression for the pseudovector Q which defines the behavior of the scattered field at distances $d \gg a$.

In fact, these distances d can be very small if a is sufficiently small, and d can be much less than the wavelength $\lambda = \frac{2\pi}{k}$.

Therefore our theory is valid in the physical situations in which multiple scattering effects are dominant.

Let us point out, especially for experimentalists, some physical consequences of the presented theory:

a) For the first time an analytical formula is given for the EM waves scattered by a small impedance particle of an arbitrary shape (see Theorem 1 below).

b) For the first time the equation for the effective field in the medium in which many small particles are distributed is derived and the physical effects of the distributed particles are analysed (see Theorem 2 below).

c) A numerical method for computing the field scattered by many small impedance particles is given (see equation (42) below and the Conclusions).

2 EM wave scattering by one small impedance particle

Let us use in this Section the following notations: D is a small body, $D' := \mathbb{R}^3 \setminus D$ is the exterior domain to D , $k > 0$ is a wave number, $ka \ll 1$, $a = 0.5 \text{diam}D$, $k = \frac{2\pi}{\lambda}$, λ is the wavelength of the incident EM wave, $k^2 = \omega^2 \epsilon_0 \mu_0$, where ω is the frequency and ϵ_0, μ_0 are constant permittivity and permeability of the medium. Our arguments remain valid if one assumes that the medium has a constant conductivity $\sigma_0 \geq 0$. In this case ϵ_0 is replaced by $\epsilon_0 + i\frac{\sigma_0}{\omega}$. Denote by S the boundary of D , by $|S|$ its surface area, by V the volume of D , by $[E, H] = E \times H$ the cross product of two vectors, and by $(E, H) = E \cdot H$ the dot product of two vectors, N is the unit normal to S *pointing out of* D , ζ is the boundary impedance of the particle.

Let D be embedded in a homogeneous medium with constant parameters ϵ_0, μ_0 . Electromagnetic (EM) wave scattering problem consists of finding vectors E and H satisfying Maxwell's equations:

$$\nabla \times E = i\omega\mu_0 H, \quad \nabla \times H = -i\omega\epsilon_0 E \quad \text{in } D' := \mathbb{R}^3 \setminus D, \quad (4)$$

the impedance boundary condition:

$$[N, [E, N]] = \zeta[N, H] \quad \text{on } S \quad (5)$$

and the radiation condition:

$$E = E_0 + v_E, \quad H = H_0 + v_H, \quad (6)$$

where E_0, H_0 are the incident fields satisfying equations (4) in all of \mathbb{R}^3 , $v_E = v$ and v_H are the scattered fields. In the literature, for example in [1], the impedance boundary condition is written sometimes as $E^t = \zeta[H^t, N]$, where N is the unit normal on S pointed *into* D . Since our N is pointed *out of* D , our impedance boundary condition (5) is the same as in [1].

One often assumes that the incident wave is a plane wave, i.e., $E_0 = \mathcal{E}e^{ik\alpha \cdot x}$, \mathcal{E} is a constant vector, $\alpha \in S^2$ is a unit vector, S^2 is the unit sphere in \mathbb{R}^3 , $\alpha \cdot \mathcal{E} = 0$. This condition guarantees that $\nabla \cdot E_0 = 0$. We assume that v_E and v_H satisfy the Sommerfeld radiation condition: $r(\frac{\partial v}{\partial r} - ikv) = o(1)$ as $r := |x| \rightarrow \infty$, and, consequently, $[r^0, E] = H + o(r^{-1})$ and $[r^0, H] = -E + o(r^{-1})$ as $r := |x| \rightarrow \infty$, where $r^0 := x/r$.

It is assumed in this paper that the impedance ζ is a constant, $\text{Re } \zeta \geq 0$.

This assumption guarantees the uniqueness of the solution to Maxwell's equation satisfying the radiation condition. For completeness a proof of the uniqueness result is given in Lemma 1. The tangential component of E on S , E^t , is defined as:

$$E^t = E - N(E, N) = [N, [E, N]]. \quad (7)$$

This definition differs from the one used often in the literature, namely, from the definition $E^t = [N, E]$. Our definition (7) corresponds to the geometrical meaning of the tangential component of E and, therefore, should be used. The impedance boundary condition is written in [1] as $E^t = \zeta[H^t, N_i]$, where ζ is the boundary impedance and N_i is the unit normal to S pointing *into* D . In our paper N is the unit normal pointing *out* of D . Therefore, the impedance boundary condition in our paper is written as in equation (5). If one uses definition (7), then this condition reduces to (5), because $[[N, [H, N]], N] = [H, N]$. The assumption $\text{Re } \zeta \geq 0$ is physically justified by the fact that this assumption guarantees the uniqueness of the solution to the boundary problem (4)-(6).

Lemma 1. *Problem (4)-(6) has at most one solution.*

Proof of Lemma 1. In Lemma 1 it is not assumed that D is small. The proof is valid for an arbitrary finite domain D . To prove the lemma one assumes that $E_0 = H_0 = 0$ and has to prove that then $E = H = 0$. Let the overbar stand for the complex conjugate. The radiation condition implies $[H, x^0] = E + o(|x|^{-1})$, where $x^0 := x/r$ is the unit normal on the sphere S_R centered at the origin and of large radius R , and one has $I' := \int_{S_R} [E, \bar{H}] \cdot x^0 ds = \int_{S_R} |E|^2 ds + o(1)$ as $R \rightarrow \infty$.

From equations (4) one derives:

$$\int_{D_R} (\bar{H} \cdot \nabla \times E - E \cdot \nabla \times \bar{H}) dx = \int_{D_R} (i\omega\mu_0|H|^2 - i\omega\epsilon_0|E|^2) dx,$$

where $D_R := D' \cap B_R$, and $R > 0$ is so large that $D \subset B_R := \{x : |x| \leq R\}$. Recall that $\nabla \cdot [E, \bar{H}] = \bar{H} \cdot \nabla \times E - E \cdot \nabla \times \bar{H}$. Applying the divergence

theorem, using the radiation condition on the sphere $S_R = \partial B_R$, and taking real part, one gets

$$0 = -\operatorname{Re} \int_S [E, \overline{H}] \cdot N ds + \operatorname{Re} \int_{S_R} [E, \overline{H}] \cdot x^0 ds := I + I'.$$

The radiation condition implies $I' \geq 0$ as $R \rightarrow \infty$. The minus sign in front of the integral I comes from the assumption that N on S is directed out of D . The impedance boundary condition and the assumption $\operatorname{Re}\zeta \geq 0$ implies $I \geq 0$. One has $I + I' = 0$. One has

$$-\int_S [E, \overline{H}] \cdot N ds = \int_S E \cdot \overline{[N, H]} ds = \int_S E \cdot \overline{[N, [E, N]]/\zeta},$$

and

$$E \cdot \overline{[N, [E, N]]/\zeta} = |E^t|^2 \zeta |\zeta|^{-2}.$$

Therefore,

$$I = \operatorname{Re}\zeta |\zeta|^{-2} \int_S |E_t|^2 ds \geq 0.$$

Consequently, $I' \geq 0$, and the relation $I + I' = 0$ implies $I = 0$. Therefore, if $\operatorname{Re}\zeta > 0$, then $E_t = 0$ on S . Consequently, $E = H = 0$ in D . If $\operatorname{Re}\zeta \geq 0$, then $\lim_{R \rightarrow \infty} \int_{S_R} |E|^2 ds = 0$, and since E satisfies the Sommerfeld radiation condition it follows that $E = H = 0$ in D' .

Lemma 1 is proved. \square

Let us note that problem (4)-(6) is equivalent to the problem

$$\nabla \times \nabla \times E = k^2 E \quad \text{in } D', \quad H = \frac{\nabla \times E}{i\omega\mu_0}, \quad (8)$$

$$[N, [E, N]] = \frac{\zeta}{i\omega\mu_0} [N, \nabla \times E] \quad \text{on } S, \quad (9)$$

together with the radiation condition (6). Thus, we have reduced the scattering problem to finding one vector $E(x)$. If $E(x)$ is found, then $H = \frac{\nabla \times E}{i\omega\mu_0}$, and the pair E and H solves Maxwell's equations, satisfies the impedance boundary condition and the radiation condition (6).

Let us look for E of the form

$$E = E_0 + \nabla \times \int_S g(x, t) J(t) dt, \quad g(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad (10)$$

where E_0 is the incident field, which satisfies Maxwell's equations in the absence of the scatterer D , t is a point on the surface S , $t \in S$, dt is an

element of the area of S , and $J(t)$ is an unknown pseudovector-function on S , which is tangential to S , i.e., $N(t) \cdot J(t) = 0$, where $N(t)$ is the unit normal to S at the point $t \in S$. The claim that $J = J(t)$ is a pseudovector follows from the fact that E is a vector and $\nabla \times (gJ)$ is a vector only if J is a pseudovector, because g is a scalar. One can look for the solution in any form in which the solution can be found. One can prove that the solution E can be found in the form (10).

It is assumed that J is a smooth function on S , for example, $J \in C^2(S)$.

The right-hand side of (10) solves equation (8) in D for any continuous $J(t)$, because E_0 solves (8) and

$$\begin{aligned} \nabla \times \nabla \times \nabla \times \int_S g(x, t) J(t) dt &= g \operatorname{raddiv} \nabla \times \int_S g(x, t) J(t) dt \\ - \nabla^2 \nabla \times \int_S g(x, t) J(t) dt &= k^2 \nabla \times \int_S g(x, t) J(t) dt, \quad x \in D'. \end{aligned} \quad (11)$$

Here we have used the known identity $\operatorname{divcurl} E = 0$, valid for any smooth vector field E , and the known formula

$$-\nabla^2 g(x, y) = k^2 g(x, y) + \delta(x - y). \quad (12)$$

The integral $\int_S g(x, t) J(t) dt$ satisfies the radiation condition. Thus, formula (10) solves problem (8), (9), (6), if $J(t)$ is chosen so that boundary condition (9) is satisfied.

Let $O \in D$ be a point inside D . The following known formula (see, for example, [2]) is useful:

$$[N, \nabla \times \int_S g(x, t) J(t) dt]_{\mp} = \int_S [N_s, [\nabla_x g(x, t)|_{x=s}, J(t)]] dt \pm \frac{J(s)}{2}, \quad (13)$$

where the \pm signs denote the limiting values of the left-hand side of (13) as $x \rightarrow s$ from D , respectively, from D' . To derive an integral equation for $J = J(t)$, substitute $E(x)$ from (10) into impedance boundary condition (9), and get

$$0 = f + [N, [\nabla \times \int_S g(s, t) J(t) dt, N]] - \frac{\zeta}{i\omega\mu_0} [N, \nabla \times \nabla \times \int_S g(s, t) J(t) dt], \quad (14)$$

where

$$f := [N, [E_0(s), N]] - \frac{\zeta}{i\omega\mu_0} [N, \nabla \times E_0]. \quad (15)$$

We assume that

$$\zeta = \frac{h}{a^\kappa}, \quad (16)$$

where $\text{Re } h \geq 0$ and $\kappa \in [0, 1)$ is a constant. This assumption is physically valid because the only restriction on the boundary impedance is the requirement $\text{Re } \zeta \geq 0$, the absolute value of the boundary impedance can be arbitrary large. Note that (16) is our requirement on the boundary impedance, and not a statement that any concrete physical material has this boundary impedance. Our aim is to solve a synthesis problem: we wish to create material with a desired refraction coefficient. To do this, as it was shown in paper [12] for scalar wave scattering, one may use embedding of many small particles of an arbitrary shape with boundary impedances given in (16). We want to establish a result of this type in the case of EM wave scattering. Note that our a , although theoretically tends to zero, does not practically reach the size below, say, 30nm, and our theory is based on the classical Maxwell's equations, it does not consider quantum electrodynamics. One point that is currently not clear, is the dependence of the bulk properties of the material when the size of a particle tends to zero. Experiments show that these properties differ drastically from the properties of the macroscopic portions of the same material. This problem is not discussed in our paper.

Let us write (10) as

$$E(x) = E_0(x) + [\nabla_x g(x, O), Q] + \nabla \times \int_S (g(x, t) - g(x, O)) J(t) dt, \quad (17)$$

where

$$Q := \int_S J(t) dt. \quad (18)$$

The central physical idea of the theory, developed in this paper, can now be stated:

The third term in the right-hand side of (17) is negligible compared with the second term if $ka \ll 1$. Consequently, the scattering problem is solved if Q is found.

A traditional approach requires finding an unknown function $J(t)$, which is usually found numerically by the boundary integral equations (BIE) method. The reason for the third term in the right-hand side of (17) to be negligible compared with the second one, is explained by the estimates given below. In these estimates the smallness of the body is used essentially: even if one is in the far zone, i.e., $\frac{a}{d} \ll 1$, one cannot conclude that estimate (21) (see below) holds unless one assumes that $ka \ll 1$. Thus, *the third term in (17) cannot be neglected in the far zone if the condition $ka \ll 1$ does not hold.*

We prove below that $Q = O(a^{2-\kappa})$.

To prove that the third term in the right-hand side of (17) is negligible compared with the second one, let us establish several estimates valid if

$a \rightarrow 0$ and $d := |x - O| \gg a$. Under these assumptions one has

$$j_1 := |[\nabla_x g(x, O), Q]| \leq O\left(\max\left\{\frac{1}{d^2}, \frac{k}{d}\right\}\right) O(a^{2-\kappa}), \quad (19)$$

$$j_2 := |\nabla \times \int_S (g(x, t) - g(x, O))\sigma(t)dt| \leq aO\left(\max\left\{\frac{1}{d^3}, \frac{k^2}{d}\right\}\right) O(a^{2-\kappa}), \quad (20)$$

and

$$\left|\frac{j_2}{j_1}\right| = O\left(\max\left\{\frac{a}{d}, ka\right\}\right) \rightarrow 0, \quad \frac{a}{d} = o(1), \quad a \rightarrow 0. \quad (21)$$

These estimates show that one may neglect the third term in (17), and write

$$E(x) = E_0(x) + [\nabla_x g(x, O), Q], \quad a \rightarrow 0. \quad (22)$$

The error of this formula tends to zero as $a \rightarrow 0$ under our assumptions.

Note that the inequality $|x| \gg ka^2$ is satisfied for $|x| \geq O(a)$ if $ka \ll 1$. Thus, formula (22) is applicable in a wide region.

Let us estimate Q asymptotically, as $a \rightarrow 0$.

Take cross product of (14) with N and integrate the resulting equation over S to get

$$I_0 + I_1 + I_2 = 0,$$

where I_0 is defined in (23) (see below), I_1 is defined in (26), and

$$I_2 := - \int_S \frac{\zeta}{i\omega\mu_0} [N, [N, \nabla \times \nabla \times \int_S g(s, t)J(t)dt]] ds.$$

Let us estimate the order of I_0 as $a \rightarrow 0$. One has

$$I_0 = \int_S \left([N, E_0] - \frac{\zeta}{i\omega\mu_0} [N, [N, \nabla \times E_0]] \right) ds. \quad (23)$$

In what follows we keep only the main terms as $a \rightarrow 0$, and denote by the sign \simeq the terms equivalent up to the terms of higher order of smallness as $a \rightarrow 0$. One has

$$I_{00} := \int_S [N, E_0] ds = \int_D \nabla \times E_0 dx \simeq \nabla \times E_0(O)V,$$

where dx is the element of the volume, $O \in D$ is a point chosen as the origin, and V is the volume of D . Thus, $I_{00} = O(a^3)$. Denoting by $|S|$ the surface area of S , one obtains

$$I_{01} := - \int_S \frac{\zeta}{i\omega\mu_0} [N, [N, \nabla \times E_0]] ds = \frac{\zeta|S|}{i\omega\mu_0} \tau \nabla \times E_0, \quad (24)$$

where

$$\tau := I - b, \quad b_{mj} := \frac{1}{|S|} \int_S N_m(t) N_j(t) dt, \quad (25)$$

$b = (b_{mj})$ is a matrix which depends on the shape of S , and $I := \delta_{mj}$ is the unit matrix. Since $\zeta = O(a^{-\kappa})$ one concludes that $I_{01} = O(a^{2-\kappa})$, so $|I_{00}| \ll |I_{01}|$, because $I_{00} = O(a^3)$ as $a \rightarrow 0$. Thus,

$$I_0 \simeq I_{01} = O(a^{2-\kappa}), \quad a \rightarrow 0.$$

Let us consider I_1 :

$$I_1 = \int_S \left(\int_S [N(s), [\nabla g(s, t), J(t)]] dt + \frac{J(s)}{2} \right) ds := I_{11} + \frac{Q}{2}. \quad (26)$$

One has

$$I_{11} = \int_S ds \int_S dt \left(\nabla g(s, t) N(s) \cdot J(t) - J(t) \frac{\partial g(s, t)}{\partial N(s)} \right).$$

It is well known (see [14], p.14) that $\int_S \frac{\partial g_0(s, t)}{\partial N(s)} ds = -\frac{1}{2}$, where $g_0 := \frac{1}{4\pi|s-t|}$. Since

$$g(s, t) - g_0(s, t) = \frac{ik}{4\pi} + O(|s-t|), \quad |s-t| \rightarrow 0,$$

and $|s-t| = O(a)$, one concludes that if $a \rightarrow 0$ then

$$\int_S \frac{\partial g(s, t)}{\partial N(s)} ds \simeq -\frac{1}{2}, \quad (27)$$

and

$$\int_S dt \int_S ds \nabla_s g(s, t) N(s) \cdot J(t) := \beta Q, \quad (28)$$

where the matrix β is defined by the formula:

$$\beta := (\beta_{mj}) := \int_S \frac{\partial g(s, t)}{\partial s_m} N_j(s) ds. \quad (29)$$

Therefore,

$$I_1 \simeq (I + \beta)Q. \quad (30)$$

Matrices β and b for spheres are calculated at the end of Section 1.

Let us show that $I_2 = O(a^{3-\kappa})$ and therefore I_2 is negligible compared with I_0 as $a \rightarrow 0$. If this is done, then equation for Q is

$$Q = -\frac{\zeta|S|}{i\omega\mu_0} (I + \alpha)\tau \nabla \times E_0, \quad (31)$$

where $E_0 = E_0(O)$, the point O is located inside the small particle,

$$I + \alpha := (I + \beta)^{-1},$$

the matrix $I + \beta$ is invertible, and the matrix τ is defined in formula (25). From (31) it follows that

$$E(x) = E_0(x) - \frac{\zeta|S|}{i\omega\mu_0} [\nabla_x g(x, O), (I + \alpha)\tau\nabla \times E_0(O)]. \quad (32)$$

This equation is our first main result which gives an analytic formula for the solution of the EM wave scattering problem by a small body of an arbitrary shape, on the boundary of which an impedance boundary condition holds.

In the far zone $r := |x| \rightarrow \infty$ one has $\nabla_x g(x, O) = ikg(x, O)x^0 + O(r^{-2})$, where $x^0 := x/r$ is a unit vector in the direction of x . Consequently, for $r \rightarrow \infty$ one can rewrite formula (32) as

$$E(x) = E_0(x) - \zeta|S| \left(\frac{\epsilon_0}{\mu_0} \right)^{1/2} \frac{e^{ikr}}{4\pi r} [x^0, (I + \alpha)\tau\nabla \times E_0(O)]. \quad (33)$$

This field is orthogonal to the radius-vector x in the far zone as it should be.

Let us show that the term I_2 is negligible as $a \rightarrow 0$. Remember that $\text{curlcurl} = \text{graddiv} - \nabla^2$ and

$$-\nabla^2 \int_S g(x, t) J(t) dt = k^2 \int_S g(x, t) J(t) dt.$$

Consequently,

$$-i\omega\mu_0 I_2 \simeq \zeta \int_S ds [N, [N, \text{graddiv} \int_S g(x, t) J(t) dt]]|_{x \rightarrow s} := I_{21}.$$

Since the function $J(t)$ is assumed smooth, one has

$$\text{div} \int_S g(x, t) J(t) dt = \int_S g(x, t) \text{div} J(t) dt, \quad \text{div} J = \frac{\partial J_m}{\partial t_m},$$

summation is understood here and below over the repeated indices, and $\text{div} J$, where J is a tangential to S field, is the surface divergence. Furthermore,

$$\text{grad} \int_S g(x, t) \frac{\partial J_m}{\partial t_m} dt = e_p \int_S g(x, t) \frac{\partial^2 J_m}{\partial t_p \partial t_m} dt,$$

where the relation $\frac{\partial g(x,t)}{\partial x_p} = -\frac{\partial g(x,t)}{\partial t_p}$ was used, and an integration by parts with respect to t_p has been done over the closed surface S . Therefore

$$I_2 \leq c|\zeta| \int_S ds \int_S |g(s,t)| dt = O(a^{3-\kappa}) \ll I_0.$$

The constant $c > 0$ here is a bound on the second derivatives of J on S .

Example of calculation of matrices β and b .

Let us calculate β and b for a sphere of radius a centered at the origin. It is quite easy to calculate b :

$$b_{jm} = \frac{1}{|S|} \int_S N_j N_m dt = \frac{1}{3} \delta_{jm}.$$

Note that $|S| = 4\pi a^2$, $dt = a^2 \sin \theta d\theta d\phi$, and N_m is proportional to the spherical function $Y_{1,m}$, so the above formula for $j \neq m$ follows from the orthogonality properties of the spherical functions, and for $m = j$ this formula is a consequence of the normalization $|N| = 1$.

For small a one may set $\beta \simeq 0$.

Let us formulate the result of this Section as a theorem.

Theorem 1. *If $ka \ll 1$, then the solution to the scattering problem (4)-(6) is given by formula (32).*

In the next Section the EM wave scattering problem is studied in the case of many small bodies (particles) whose physical properties are described by the boundary impedance conditions.

3 Many-body EM wave scattering problem

Consider now EM wave scattering by many small bodies (particles) D_m , $1 \leq m \leq M = M(a)$, distributed in an arbitrary bounded domain Ω . Here a is the characteristic dimension of a particle, $M(a)$ is specified below, and $x_m \in D_m$ is a point. Assume for simplicity of formulation of the problem that the particles are of the same shape and orientation, and their physical properties are described by the boundary impedance (16), where $h = h(x_m)$, and $h(x)$, $\text{Re}h(x) \geq 0$, is a continuous function in an arbitrary large but finite domain Ω where the particles are distributed. These simplifying assumptions one can easily remove: it is possible to assume a distribution in sizes and orientations of the small particles. This, however, complicates the formulation of the results considerably, and does not contribute new ideas and techniques for a study of many-body scattering problem. By this reason we will not formulate the results which hold without the above assumptions.

For simplicity also let us assume that the particles are distributed in a homogeneous medium with parameters ϵ_0, μ_0 .

The distribution of small particles is given by the following law:

$$\mathcal{N}(\Delta) = a^{-(2-\kappa)} \int_{\Delta} N(x) dx (1 + o(1)), \quad a \rightarrow 0, \quad (34)$$

where Δ is an arbitrary open subset of Ω , $N(x) \geq 0$ is a given function which one can choose as one wishes, $\mathcal{N}(\Delta)$ is the number of particles in Δ , and $\kappa \in [0, 1)$ is a parameter from formula (16). The physical meaning of the quantity $a^{-(2-\kappa)}N(x)$ is the number of particles per unit volume at a point x . Thus, the dimensionality of $N(x)$ depends on the choice of the parameter κ . For example, if $\kappa = 0$ then the dimensionality of $N(x)$ is L^{-1} , where L is the dimensionality (dim) of the length. Indeed, $\dim\{N(x)a^{-2}dx\} = \dim\{N(x)L\}$ is a dimensionless quantity, so $\dim\{N(x)\} = L^{-1}$. The distance $d = d(a) \gg a$ between the neighboring particles is assumed to satisfy the relation $\lim_{a \rightarrow 0} ad^{-1}(a) = 0$. The order of magnitude of d as $a \rightarrow 0$ can be found by a simple argument: if one assumes that the particles are packed so that between each neighboring pair of particles the distance is d , then in a unit cube Δ_1 the number of particles is $O(d^{-3})$, and this amount should be equal to $\mathcal{N}(\Delta_1) = O(a^{-(2-\kappa)})$. Therefore

$$d = O(a^{(2-\kappa)/3}).$$

Denote $D := \bigcup_{m=1}^M D_m$, $D' := \mathbb{R}^3 \setminus D$.

The scattering problem consists of finding the solution to

$$\nabla \times \nabla \times E = k^2 E \text{ in } D', \quad (35)$$

$$[N, [E, N]] - \zeta_m [N, \nabla \times E] \text{ on } S_m, \quad 1 \leq m \leq M, \quad (36)$$

$$E = E_0 + v_E, \quad (37)$$

where E_0 is the incident field, which satisfies equation (35) in \mathbb{R}^3 , and v_E satisfies the radiation condition. Existence and uniqueness of the solution to the scattering problem (35)-(37) is known. Let us look for the solution of the form

$$E(x) = E_0(x) + \sum_{m=1}^M \nabla \times \int_{S_m} g(x, t) J_m(t) dt. \quad (38)$$

Define the effective (self-consistent) field, acting on the j -th particle, by the formula

$$E_e(x) = E_0(x) + \sum_{m=1, m \neq j}^M \nabla \times \int_{S_m} g(x, t) J_m(t) dt. \quad (39)$$

Because of our assumption $d \gg a$, any single particle D_j can be considered as being placed in the incident field $E_e(x)$. Therefore

$$E_e(x) = E_0(x) - \sum_{m=1, m \neq j}^M \frac{\zeta_m |S|}{i\omega\mu_0} [\nabla_x g(x, x_m), (I + \alpha)\tau \nabla \times E_e(x_m)]. \quad (40)$$

Note that $\zeta |S| = h(x_m) c_S a^{2-\kappa}$, where the constant c_S depends on the shape of S and not on its diameter. This constant is defined by the formula

$$|S| = c_S a^2.$$

For example, if S is a sphere of radius a then $c_S = 4\pi$. If S is an ellipsoid with the semiaxes a, b, c , and $b = s_1 a$, $c = s_2 a$, where s_1 and s_2 are some constants, then the volume of this ellipsoid is $4\pi s_1 s_2 a^3/3$, but the formula for its surface area is complicated, see [3].

Thus, formula (40) implies

$$E_e(x) = E_0(x) - \frac{c_S}{i\omega\mu_0} a^{2-\kappa} \sum_{m=1, m \neq j}^M h(x_m) [\nabla_x g(x, x_m), (I + \alpha)\tau \nabla \times E_e(x_m)]. \quad (41)$$

Denote $A_m := \nabla \times E_e(x_m)$. Applying operator $\nabla \times$ to (41) and then setting $x = x_j$, one obtains a linear algebraic system (LAS) for the unknown A_m :

$$A_j = A_{0j} - \frac{c_S}{i\omega\mu_0} a^{2-\kappa} \sum_{m=1, m \neq j}^M h(x_m) \nabla_x \times [\nabla_x g(x, x_m), (I + \alpha)\tau A_m]|_{x=x_j}, \quad (42)$$

where $1 \leq j \leq M$. If A_m are found, then formula (40) yields the solution to many-body EM wave scattering problem. The magnetic field is calculated by the formula $H(x) = \frac{\nabla \times E(x)}{i\omega\mu_0}$, if $E(x)$ is found.

Let us now pass to the limit in equation (41) as $a \rightarrow 0$.

Consider a partition of Ω into a union of cubes Δ_p , $1 \leq p \leq P$. These cubes have no common interior points, and their side $\ell \gg d$, $\ell = \ell(a) \rightarrow 0$ as $a \rightarrow 0$. Choose a point $x_p \in \Delta_p$, for example, let x_p be the center of Δ_p . Rewrite (41) as

$$E_e(x_q) = E_0(x_q) - \frac{c_S}{i\omega\mu_0} \sum_{p=1, p \neq q}^P h(x_p) [\nabla_x g(x, x_p)|_{x=x_q}, (I + \alpha)\tau \nabla \times E_e(x_p)] \\ \cdot a^{2-\kappa} \sum_{x_m \in \Delta_p} 1. \quad (43)$$

It follows from equation (34) that

$$a^{2-\kappa} \sum_{x_m \in \Delta_p} 1 \simeq N(x_p) |\Delta_p|, \quad a \rightarrow 0,$$

where $|\Delta_p|$ denotes the volume of the cube Δ_p . Thus, (43) can be written as:

$$E_e(x_q) = E_0(x_q) - \frac{c_S}{i\omega\mu_0} \sum_{p=1, p \neq q}^P h(x_p) N(x_p) [\nabla_x g(x, x_p)|_{x=x_q}, (I + \alpha)\tau \nabla \times E_e(x_p)] |\Delta_p|. \quad (44)$$

This equation is a discretized version of the integral equation:

$$E(x) = E_0(x) - \frac{c_S}{i\omega\mu_0} \nabla \times \int_{\Omega} g(x, y) (I + \alpha)\tau \nabla \times E(y) N(y) h(y) dy. \quad (45)$$

Equation (45) is the equation for the limiting, as $a \rightarrow 0$, effective field in the medium in which $M(a)$ small impedance particles are embedded according to the distribution law (34).

Let us summarize our result under the simplifying assumptions stated in the beginning of Section 3.

Theorem 2. *If small particles are distributed according to (34) and their boundary impedances are defined in (16), then the solution to a many-body EM wave scattering problem (35)-(37) is given by formula (40), where the quantities $\nabla \times E(x_m) := A_m$ are found from the linear algebraic system (42), and the limiting, as $a \rightarrow 0$, electric field $E(x)$ in Ω solves integral equation (45).*

Let us discuss the *novel physical consequences of our theory*. Applying the operator *curlcurl* to equation (45) and taking into account that *curlcurl* = *graddiv* - Δ , *divcurl* $\equiv 0$, and $-\Delta g(x, y) = k^2 g(x, y) + \delta(x - y)$ one obtains

$$\nabla \times \nabla \times E = k^2 E - \nabla \times \left(\frac{c_S}{i\omega\mu_0} (I + \alpha)\tau \nabla \times E(x) N(x) h(x) \right). \quad (46)$$

The novel term, which is due to the limiting distribution of the small particles, is

$$T := -\nabla \times \left(\frac{c_S}{i\omega\mu_0} (I + \alpha)\tau \nabla \times E(x) N(x) h(x) \right).$$

Let us make a simple assumption that $N(x)h(x) = \text{const}$. Physically this means that the distribution of the small particles is uniform in the space

and the boundary impedances of the particles do not vary in the space. Denote $\frac{cS}{i\omega\mu_0}Nh := c_1$. Assume also that α and τ are proportional to the identity matrix. This, for example, happens if all S_m are spheres. Then $T = -c_2\nabla \times \nabla \times E(x)$, where $c_2 = \text{const}$.

In this case the novel term T can be interpreted physically in a simple way: it results to a change in the refraction coefficient of the medium in Ω .

Indeed, by taking the term T to the left side of (46) and dividing both sides of this equation by $1 + c_2$, one sees that the coefficient k^2 is replaced by $k_1^2 := \frac{k^2}{1+c_2}$. Since c_2 is a complex number, the new medium is absorptive. The assumption $\text{Re}h \geq 0$ implies that $\text{Im}k_1^2 \geq 0$. Since $h(x)$ and $N(x)$ can be chosen by the experimentalist as he wishes, their dependence on x can be chosen as he wishes. Therefore, distributing many small particles in Ω one can change the refraction coefficient of the medium in a desirable direction.

Formula (46) allows one to derive a formula for the magnetic permeability $\mu(x)$ of the limiting medium. This formula can be used for creating material with a desired magnetic permeability $\mu(x)$. Let us derive this formula.

From Maxwell's equations (4) with μ_0 replaced by a $\mu(x)$, one gets

$$\nabla \times \nabla \times E = i\omega\mu(x)\nabla \times H + i\omega[\nabla\mu(x), H] = \omega^2\epsilon_0\mu(x)E + \left[\frac{\nabla\mu(x)}{\mu(x)}, \nabla \times E\right]. \quad (47)$$

Suppose that the tensors α and τ are proportional to the identity. This is the case, for example, when the small bodies are all balls of radius a . In this case equation (46) yields

$$\nabla \times \nabla \times E = k^2E - c_1[\nabla(N(x)h(x)), \nabla \times E] - c_1N(x)h(x)\nabla \times \nabla \times E,$$

where $c_1 = \text{const}$ and $k^2 = \omega^2\epsilon_0\mu_0$. This equation can be rewritten as

$$\nabla \times \nabla \times E = \frac{k^2}{1 + c_1N(x)h(x)}E - \frac{c_1}{1 + c_1N(x)h(x)}[\nabla(N(x)h(x)), \nabla \times E]. \quad (48)$$

Comparing equations (47) and (48), one concludes that the new $\mu(x)$ in the limiting medium is given by the formula

$$\mu(x) = \frac{\mu_0}{1 + c_1N(x)h(x)}, \quad (49)$$

$$\text{and } \frac{\nabla\mu(x)}{\mu(x)} = -\frac{c_1\nabla(N(x)h(x))}{1+c_1N(x)h(x)}.$$

4 Conclusions

The main results of this paper are:

1. Equations (32) and (31) for the solution of the EM wave scattering problem (4)-(6) by one small body of an arbitrary shape.
2. A numerical method, based on linear algebraic system (LAS) (42), see formulas (41) and (44), for solving many-body EM wave scattering problem in the case of small bodies of an arbitrary shape.
3. Equation (45) for the electric field in the limiting medium obtained by embedding $M(a)$ small impedance particles of an arbitrary shape, distributed according to (34), as $a \rightarrow 0$.
4. Formula (49) for the magnetic permeability of the limiting medium is derived.

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